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Additive Utility in Prospect Theory

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Prospect theory is currently the main descriptive theory of decision under uncertainty. It generalizes expected utility by introducing nonlinear decision weighting and loss aversion. A difficulty in the study of multiattribute utility under prospect theory is to determine when an attribute yields a gain or a loss. One possibility, adopted in the theoretical literature on multiattribute utility under prospect theory, is to assume that a decision maker determines whether the complete outcome is a gain or a loss. In this *holistic evaluation*, decision weighting and loss aversion are general and attribute-independent. Another possibility, more common in the empirical literature, is to assume that a decision maker has a reference point for each attribute. We give preference foundations for this *attribute-specific evaluation* where decision weighting and loss aversion are depending on the attributes.

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1. Introduction

Many decision situations involve outcomes that consist of several attributes. In applied decision analyses, it is useful to decompose the utility function over these multiattribute outcomes into separate utility functions over the different attributes to reduce the number of preference elicitations. This is only justified if the decision-maker's preferences satisfy particular assumptions. Several authors have identified the preference conditions that allow decomposing multiattribute utility functions into additive, multiplicative, and related decompositions (e.g., Farquhar 1975, Fishburn 1965, Keeney and Raiffa 1976).

Most of these decomposition results have been derived under expected utility. Abundant evidence exists, however, that (subjective) expected utility is not valid as a descriptive theory of decision under (uncertainty) risk. The descriptive deficiencies of expected utility complicate the empirical assessment of the preference conditions underlying decompositions: it cannot be excluded that observed violations of preference conditions are due to violations of expected utility rather than to violations of a decomposition. To obtain robust tests of the appropriateness of decompositions, it is desirable to derive conditions that are valid even when expected utility is violated.

In this paper, we study multiattribute utility theory under prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992). Prospect theory is currently the most influential theory of decision making under uncertainty. It models two major deviations from expected utility: nonlinear decision weighting and loss aversion, i.e., the tendency that people treat outcomes as deviations from a reference point and are more sensitive to losses than to gains of the same magnitude. Both nonlinear decision weighting and loss aversion are widely documented in the empirical literature. Despite its popularity, some evidence has been accumulated recently revealing limitations of the theory (see the summaries of Marley and Luce 2005 and Birnbaum 2008).

Fishburn (1984), Miyamoto (1988), Dyckerhoff (1994), and Miyamoto and Wakker (1996) also studied multiattribute utility under nonexpected utility, but only considered outcomes of the same sign. Like us, Zank (2001) and Bleichrodt and Miyamoto (2003) studied multiattribute utility theory under prospect theory but their approach was different than the approach of this paper, as we explain next.

A central issue in multiattribute prospect theory is to determine when an attribute yields a gain or a loss. Consider, for example, a research associate

(RA) who contemplates changing jobs. In evaluating different jobs, the RA has to consider several aspects, e.g., salary, commuting time, cost of living, amount of research time, etc. How does the RA determine whether a particular job offer is an improvement (a gain) compared with her reference point (presumably her current job)? One possibility is that she first determines whether the job offer, as a whole, is a gain or a loss compared to her reference point, and then applies the decomposition to determine how attractive the job offer is compared with other offers. This *holistic evaluation* was used by Zank (2001) and by Bleichrodt and Miyamoto (2003).

Another approach, the focus of this paper, is that the RA determines a reference point for each attribute and evaluates job offers as gains and losses on each attribute. This *attribute-specific evaluation* seems plausible when the number of attributes is large and the choice is complex. A decision context where the attribute-specific evaluation is particularly intuitive is welfare theory: there we are interested in whether each individual's welfare is above some reference level. The attribute-specific evaluation is commonly assumed in empirical studies on loss aversion for trade-offs under certainty and was found to be descriptively accurate by some studies (Bateman et al. 1997, Bleichrodt and Pinto 2002, Tversky and Kahneman 1991). Also empirical studies for decision under risk relied on the attribute-specific evaluation. See, for example, Payne et al. (1984) who study managers' choices among capital budget proposals involving cash flows at two points in time, and Fischer et al. (1986) who consider both risky, multiperiod cash flows and risky job alternatives. Both studies use attribute-specific reference points. No preference foundation of the attribute-specific evaluation existed until now. Providing such a foundation is the topic of this paper.

The difference between the holistic and the attribute-specific evaluation is that in the former, loss aversion and decision weighting are attribute-independent, whereas in the latter they depend on the attributes. As we show in §5, the holistic and the attribute-specific evaluation are in general equivalent only when people behave according to expected utility, i.e., when loss aversion does not affect people's preferences and there is no decision weighting. An example to further clarify the difference between the holistic and the attribute-specific evaluation is in §3.

This paper gives preference foundations for additive utility under prospect theory and the attribute-specific evaluation. We restrict our attention to the additive decomposition for two reasons: First, it is commonly applied in many areas of economics and decision analysis. Second, other decompositions, such as multiplicative and multilinear utility, raise special problems under the attribute-specific evaluation. Solving these problems is beyond the scope of this paper.

The remainder of this paper is organized as follows. Section 2 gives notation and explains prospect theory for single-attribute outcomes. In §3, we move to multiattribute utility where we first assume, for ease of exposition, that there are just two attributes, both numerical. Section 4 gives preference foundations for prospect theory with additive utility under the attribute-specific evaluation. As mentioned, weighting functions are defined per attribute and they may differ across attributes in the attribute-specific evaluation. To force them to be equal across attributes requires additional conditions. We will characterize this special case in §5. We extend our results to the case where there are more than two attributes in §6 and to the case of nonnumeric outcomes in §7. Section 8 concludes the paper with some observations on the empirical measurement of additive utility in prospect theory under the attribute-specific evaluation. All proofs are in the appendix.

2. Prospect Theory for Single-Attribute Outcomes

We consider a decision maker in a situation where there is a finite number, $n \geq 2$ of distinct states of nature, exactly one of which obtains. $S = \{1, \ldots, n\}$ denotes the finite set of *states of nature*. Subsets of S are called *events*. In a medical decision problem, the states of nature can, for example, be mutually exclusive diseases, and the decision maker has to choose between different treatments before knowing what the actual disease is. We consider decision under uncertainty where the probabilities for the states of nature may, but need not, be given. The assumption of a finite number of states of nature is made for expositional purposes. The results of this paper can be extended to an infinite state space using tools from Wakker (1993). The extension to decision under risk, i.e., the case where probabilities are objectively given, is as in Köbberling and Wakker (2003, §5.3).

The decision-maker's problem is to choose between prospects. Each *prospect* is an n-tuple of *outcomes*, one for each state of nature. Formally, a prospect is a function from the set of states of nature to the set of outcomes C. We denote the set of prospects as $P = C^n$. We shall write $(f_1, \ldots, f_n)$ for the prospect f that gives f_j if state j occurs. A *constant prospect* gives the same outcome for each state of nature. For ease of exposition, we assume in this section that outcomes are one-dimensional. The set of outcomes C is a nondegenerate convex subset of $\mathbb{R}$. Outcomes are defined with respect to a *reference point*. The reference point is a constant prospect, that we will denote as r. We assume that the reference point is fixed, i.e., we restrict attention to preferences with respect to one reference point. Variations in the reference point are analyzed by Schmidt (2003).

Let $\succcurlyeq$ denote a preference relation on the set of prospects. As usual, $\succ$ denotes the asymmetric part of $\succcurlyeq$ (strict preference) and $\sim$ denotes the symmetric part of $\succcurlyeq$ (indifference), and $\preccurlyeq$ and $\prec$ denote reversed preferences. We shall use the same notation for the binary relations on C, derived through constant prospects. An outcome $x \succ r$ is a *gain* and an outcome $x \prec r$ is a *loss*.

A prospect f is *rank-ordered* if $f_1 \succcurlyeq \cdots \succcurlyeq f_n$. For each prospect, there exists a permutation ρ, such that $f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}$. For each permutation ρ, let $P_\rho = \{f \in P\colon f_{\rho(1)} \succcurlyeq \cdots \succcurlyeq f_{\rho(n)}\}$. That is, P_ρ is the set of all prospects that are rank-ordered by ρ. If two prospects can be rank-ordered by a common ρ, then they are *comonotonic*. For each event $A \subset S$, the set P^A contains those prospects that yield gains for states in A, and no gains for states not in A. We define the set P^A_ρ as the intersection of P^A and P_ρ. Subsets of sets P^A_ρ are *sign-comonotonic*.

A real-valued function $V\colon P \to \mathbb{R}$ *represents* $\succcurlyeq$ on P if for all $f, g \in P$ we have $f \succcurlyeq g$ if and only if (iff) $V(f) \geq V(g)$. A function V is a *ratio scale* if it is unique up to unit, i.e., if V can be replaced by U if and only if $U = \sigma V$ for positive σ. A *weighting function* or *capacity* W is a function on 2^S, such that $W(\varnothing) = 0$, $W(S) = 1$, and for any two events A and B, if $B \subset A$ then $W(B) \leq W(A)$. W is *strictly increasing* if $W(B) < W(A)$ whenever B is a proper subset of A.

Prospect theory holds if there exists a utility function $U\colon C \to \mathbb{R}$ with $U(r) = 0$ such that prospects $f \in P^A_\rho$ with $A = \{\rho(1), \ldots, \rho(k)\}$ for some $k \leq n$ are evaluated by

$$\mathrm{PT}(f) = \sum_{j=1}^{k} \pi^+_{\rho(j)} U(f_{\rho(j)}) + \sum_{j=k+1}^{n} \pi^-_{\rho(j)} U(f_{\rho(j)}) \qquad (1)$$

with

$$\pi^+_{\rho(j)} = W^+(\rho(1), \ldots, \rho(j)) - W^+(\rho(1), \ldots, \rho(j-1)) \qquad (2a)$$

and

$$\pi^-_{\rho(j)} = W^-(\rho(j), \ldots, \rho(n)) - W^-(\rho(j+1), \ldots, \rho(n)), \qquad (2b)$$

and choices and preferences correspond with this evaluation. $\mathrm{PT}(f)$ denotes the *prospect theory value*, or PT value for short, of f, and W^+ and W^- are weighting functions for gains and losses, respectively. We will assume throughout that U is *strictly increasing*, i.e., for all $x, y \in C$, $U(x) \geq U(y)$ iff $x \geq y$, and *continuous*. If prospect theory holds, then utility is a ratio scale and the weighting functions are uniquely determined.

3. Prospect Theory for Two-Attribute Outcomes

From now on $C = C_1 \times C_2$ is a product of two nondegenerate convex subsets of $\mathbb{R}$. Hence we now deal with two product structures: the two-dimensional structure of C and the n-dimensional structure C^n. In what follows, the index i will refer to the attributes, and the index j to the states of nature. Hence f_{ji} denotes the ith attribute of the outcome that is obtained under state j. *Outcomes* in C will be denoted as $x = (x_1, x_2)$ or as x_1x_2 for short. Note that although we assumed x_1 and x_2 to be numerical, the notation x_1x_2 should not be interpreted as multiplication.

Let P_1 denote the set of prospects on C_1^n and P_2 the set of prospects on C_2^n. For a fixed $f_2 \in P_2$, we define a preference relation $\succcurlyeq_1$ on P_1 by $f_1 \succcurlyeq_1 g_1$ iff $f_1 f_2 \succcurlyeq g_1 f_2$. In §4 we impose a condition that implies that the choice of f_2 is immaterial. By restricting attention to constant prospects in P_1, we can define a preference relation $\succcurlyeq_1$ on C_1. In a similar fashion we can define $\succcurlyeq_2$ on P_2 and on C_2.

A function $U\colon C \to \mathbb{R}$ is *additive* if $U\colon x \mapsto U_1(x_1) + U_2(x_2)$ where U_i is a real-valued function on C_i, $i = 1, 2$. The functions U_1 and U_2 are *joint ratio scales* if U_1 and U_2 can be replaced by V_1 and V_2 if and only if $V_i = \sigma U_i$, $\sigma > 0$. That is, any common change in unit is allowed.

In the holistic evaluation, any outcome x that is indifferent to r can also be interpreted as a reference point. Hence, it does not make sense to consider gains or losses on any separate dimension in the holistic evaluation. What matters in the holistic evaluation is whether an outcome x is a gain or a loss compared to r (i.e., whether $x \succ r$ or $x \prec r$, respectively).

Under the *holistic evaluation*, additive decomposability means that a prospect $f \in P^A_\rho$ with $A = \{\rho(1), \ldots, \rho(k)\}$ for some $k \leq n$ is evaluated as

$$\mathrm{PT}(f) = \sum_{j=1}^{k} \pi^+_{\rho(j)} (U_1(f_{\rho(j)1}) + U_2(f_{\rho(j)2})) + \sum_{j=k+1}^{n} \pi^-_{\rho(j)} (U_1(f_{\rho(j)1}) + U_2(f_{\rho(j)2})), \qquad (3)$$

where the decision weights are defined as in Equations (2a) and (2b). The uniqueness results of prospect theory apply, which implies that the attribute utility functions are joint ratio scales and the weighting function is unique. There is only one permutation function that applies to both attributes. In this representation, the decision weight that is assigned to a single-attribute utility function U_i, $i = 1, 2$, depends on whether the entire outcome is a gain or a loss. If an outcome x is a gain then the decision weight π^+ is applied, if it is a loss then π^- is applied. Preference foundations for Equation (3) were given by Zank (2001) and Bleichrodt and Miyamoto (2003).

The attribute-specific evaluation assesses for each attribute separately whether it yields a gain or a loss and the magnitude of each. That is, the attribute-specific evaluation interprets reference-dependence for each attribute separately. We will denote the *reference point on the first attribute* by r_1 and the *reference point on the second attribute* by r_2. $x_1 \in C_1$ is a gain if $x_1 \succ_1 r_1$, and a loss if $x_1 \prec_1 r_1$, and $x_2 \in C_2$ is a gain if $x_2 \succ_2 r_2$, and a loss if $x_2 \prec_2 r_2$. We assume that preferences are monotonic in each attribute. Then, unlike in the holistic evaluation, the reference point will be unique. We further assume that r_1 is an interior point of C_1 and that r_2 is an interior point of C_2. This ensures that C_1 and C_2 both contain outcomes that are gains and outcomes that are losses, and that genuine trade-offs between gains and losses exist for both attributes.

For each prospect f, there exist permutations ρ_1 and ρ_2 such that $f_{\rho_1(1)1} \succcurlyeq \cdots \succcurlyeq f_{\rho_1(n)1}$ and $f_{\rho_2(1)2} \succcurlyeq \cdots \succcurlyeq f_{\rho_2(n)2}$. Let $P_{\rho_1} = \{f \in P\colon f_{\rho_1(1)1} \succcurlyeq \cdots \succcurlyeq f_{\rho_1(n)1}\}$. That is, P_{ρ_1} is the set of all prospects where outcomes are rank-ordered by ρ_1. P_{ρ_2} is defined similarly. For each event $A_1 \subset S$, the set P^{A_1} contains those prospects that yield gains on the first attribute for states in A_1 and no gains on the first attribute for states not in A_1. Similarly, P^{A_2} contains those prospects that yield gains on the second attribute for states in A_2 and no gains on the second attribute for states not in A_2. We define $P^{A_1}_{\rho_1} = P^{A_1} \cap P_{\rho_1}$ and $P^{A_2}_{\rho_2} = P^{A_2} \cap P_{\rho_2}$. Subsets of $P^{A_1}_{\rho_1}$ are said to be *sign-comonotonic on* C_1 and subsets of $P^{A_2}_{\rho_2}$ are said to be *sign-comonotonic on* C_2. Under the *attribute-specific evaluation*, a prospect $f \in P^{A_1}_{\rho_1} \cap P^{A_2}_{\rho_2}$ with $A_1 = \{\rho_1(1), \ldots, \rho_1(k_1)\}$ and $A_2 = \{\rho_2(1), \ldots, \rho_2(k_2)\}$ for some $k_1, k_2 \le n$ is evaluated as

$$\begin{aligned}\mathrm{PT}(f) = {}& \sum_{j=1}^{k_1} \pi^+_{\rho_1(j)1} U_1(f_{\rho_1(j)1}) + \sum_{j=k_1+1}^{n} \pi^-_{\rho_1(j)1} U_1(f_{\rho_1(j)1}) \\ &+ \sum_{j=1}^{k_2} \pi^+_{\rho_2(j)2} U_2(f_{\rho_2(j)2}) + \sum_{j=k_2+1}^{n} \pi^-_{\rho_2(j)2} U_2(f_{\rho_2(j)2}) \end{aligned} \tag{4}$$

with

$$\begin{aligned}\pi^+_{\rho_i(j)i} = {}& W_i^+(\rho_i(1), \ldots, \rho_i(j)) \\ &- W_i^+(\rho_i(1), \ldots, \rho_i(j-1)), \quad i = 1, 2 \end{aligned} \tag{5a}$$

and

$$\begin{aligned}\pi^-_{\rho_i(j)i} = {}& W_i^-(\rho_i(j), \ldots, \rho_i(n)) \\ &- W_i^-(\rho_i(j+1), \ldots, \rho_i(n)), \quad i = 1, 2, \end{aligned} \tag{5b}$$

and preferences and choices correspond with this evaluation. The functions U_1 and U_2 are strictly increasing and continuous and satisfy $U_1(r_1) = U_2(r_2) = 0$. The decision weights $\pi^+_{\rho_1(\cdot)1}$ and $\pi^-_{\rho_1(\cdot)1}$ are the decision weights for gains and losses for the first attribute, $\pi^+_{\rho_2(\cdot)2}$ and $\pi^-_{\rho_2(\cdot)2}$ are the decision weights for gains and losses for the second attribute, W_1^+ and W_1^- are the weighting functions for gains and losses for the first attribute, and W_2^+ and W_2^- are the weighting functions for gains and losses for the second attribute. The utility functions are joint ratio scales and the attribute weighting functions are unique. A comparison between Equations (3) and (4) reveals that the holistic evaluation and the attribute-specific evaluation differ both in loss aversion and in decision weighting.

An example may further clarify the difference between the holistic and the attribute-specific evaluation of additive utility. Suppose that the RA considers a job offer from a university in a different town. The uncertainty she faces is whether her husband will be able to find a suitable job in the new town. If he does, their combined annual income will be \$80K but she will only have 15 hours research time per week because she will have to take over some domestic activities from her husband (e.g., taking care of the children). If he does not find a job, their combined annual income will be \$40K but she will have 30 hours research time per week because her husband will take care of all domestic activities. In the example, there is only one source of uncertainty (whether or not the RA's husband finds a suitable job). In real-life applications, there may be different sources of uncertainty affecting the attributes separately. For example, the RA's research time may not be affected by her husband finding a job (because she can hire someone to take care of her domestic activities) but it is affected by whether or not she will be able to find a suitable home near the university (if not, commuting will negatively affect the time available for research). To model such situations we have to refine the state space (events are "husband finds job and home near the university," "husband finds job but home far from university," etc.). For simplicity of exposition, we only consider one source of uncertainty.

The RA's preferences are monotonic both in money (more money is preferred) and in research time (more research time is preferred). Suppose that currently the RA and her husband earn \$50K per year and she has 20 hours research time per week. Suppose also that (\$80K, 15 h) $\succ$ (\$50K, 20 h) $\succ$ (\$40K, 30 h). The RA's reference point is (\$50K, 20 h) in the holistic evaluation. In the attribute-specific evaluation, the RA's reference point for annual income is \$50K and for research time it is 20 hours per week.

In the holistic evaluation, where we determine first the sign of an outcome and then apply the decompositions, we assume that the RA's utility function for gains is $u(x_1x_2) - u(r_1r_2)$ and her utility function for losses is $\lambda(u(x_1x_2) - u(r_1r_2))$, where λ is a coefficient reflecting loss aversion and u is a *basic utility function*, expressing the RA's attitude toward outcomes, which is reference independent (Tversky and Kahneman 1991, Köbberling and Wakker 2005). We assume that the holistic basic utility is additive such that $u(x_1x_2) = u_1(x_1) + u_2(x_2)$.

In the attribute-specific evaluation, where first the decomposition is applied and then it is determined whether attributes yield gains or losses, the utility for gains is $u_i(x_i) - u_i(r_i)$ and for losses it is $\lambda_i(u_i(x_i) - u_i(r_i))$; $i = 1, 2$, where the λ_i are attribute-specific loss aversion coefficients.

The RA does not care about job aspects other than wage rate and available research time. If event 1 is, "her husband finds a job" and event 2 is, "her husband does not find a job," then, according to the holistic evaluation (Equation (3)), the PT value of the new job is equal to

$$\pi_1^+\big((u_1(80)+u_2(15)) - (u_1(50)+u_2(20))\big) + \pi_2^-\lambda\big((u_1(40)+u_2(30)) - (u_1(50)+u_2(20))\big), \quad (6)$$

and according to the attribute-specific evaluation (Equation (4)), it is equal to

$$\pi_{11}^+\big(u_1(80)-u_1(50)\big) + \pi_{12}^-\lambda_2\big(u_2^-(15)-u_2(20)\big) + \pi_{21}^-\lambda_1\big(u_1(40)-u_1(50)\big) + \pi_{22}^+\big(u_2(30)-u_2(20)\big). \quad (7)$$

A comparison between Equations (6) and (7) shows that both decision weighting and loss aversion differ between the two evaluations. Loss aversion and decision weighting are common for all individual attributes in the holistic evaluation; the attribute-specific evaluation, in general, allows for different degrees of loss aversion and different weighting functions for each of the individual attributes.

The possibility of attribute-dependent weighting functions can be realistic in applications. Rottenstreich and Hsee (2001) showed that decision weighting depends on the outcome domain with people deviating more from expected utility for affect-rich outcomes, outcomes that arouse strong emotions. Examples of such outcomes are health states and environmental effects. For example, Smith and Keeney (2005) studied trade-offs between consumption and health. In such a setting it might well be that people weight health risks differently than consumption risks. Dyer et al. (1998) compared different alternatives for disposing surplus weapons-grade plutonium. Here decision makers may weight risks to the environment differently from risk regarding the costs of the alternatives. In §5, we characterize the special case of the attribute-specific evaluation where the weighting functions are the same across different attributes. There is no empirical evidence to conclude that loss aversion differs across different attributes, but intuitively this seems to make sense.

4. Preference Foundations

This section develops preference foundations for additive prospect theory under the attribute-specific evaluation, i.e., Equation (4). We continue to assume that $C = C_1 \times C_2$ with C_1 and C_2 nondegenerate convex subsets of $\mathbb{R}$.

4.1. General Preference Conditions

This subsection presents the standard preference conditions that are used in both the holistic and attribute-specific approaches. The preference relation $\succcurlyeq$ on the set of prospects P is a weak order if it is *complete* (for all prospects f, g, $f \succcurlyeq g$ or $g \succcurlyeq f$) and transitive.

Any prospect $f \in P$ yields both a prospect $f_1 \in P_1$ and a prospect $f_2 \in P_2$ and, hence, each prospect f may be viewed as an element of the product $P_1 \times P_2$. Hence, we can denote prospects as f_1f_2. *Weak separability* holds when for all $f_1, g_1 \in P_1$ and for all $f_2, g_2 \in P_2$, $f_1f_2 \succcurlyeq g_1f_2$ iff $f_1g_2 \succcurlyeq g_1g_2$ and when for all $f_1, g_1 \in P_1$ and for all $f_2, g_2 \in P_2$, $f_1f_2 \succcurlyeq f_1g_2$ iff $g_1f_2 \succcurlyeq g_1g_2$. Weak separability entails that the relations $\succcurlyeq_1$ on P_1 and $\succcurlyeq_2$ on P_2 are well-defined.

Further standard properties are monotonicity for outcomes and continuity: *outcome monotonicity* holds if for $i = 1, 2$, $f_{ji} \geq g_{ji}$ for all j implies $f_i \succcurlyeq_i g_i$ with strict preference holding if one of the antecedent inequalities is strict; *continuity* holds if for all prospects f_i, the sets $\{g_i \in P_i\colon g_i \succcurlyeq_i f_i\}$ and $\{g_i \in P_i\colon g_i \preccurlyeq_i f_i\}$ are both closed in C_i^n, $i = 1, 2$.

4.2. Trade-off Consistency

To define trade-off consistency we introduce some notation. For $x \in C_i$, $f_i \in P_i$, $i = 1, 2$, and $j \in S$ define

$$x_{(j)}f_i = (f_{1i}, \ldots, f_{j-1i}, x, f_{j+1i}, \ldots, f_{ni}),$$

that is, $x_{(j)}f_i$ is the prospect f_i with f_{ji} replaced by x. Let $a, b, c, d \in C_1$. We write

$$ab \sim_1^* cd$$

if (i) there exist $f_1, g_1 \in P_1$, and $f_2 \in P_2$ and a state j such that

$$(a_{(j)}f_1, f_2) \sim (b_{(j)}g_1, f_2) \quad \text{and} \quad (c_{(j)}f_1, f_2) \sim (d_{(j)}g_1, f_2),$$

where $a_{(j)}f_1$, $b_{(j)}g_1$, $c_{(j)}f_1$, and $d_{(j)}g_1$ are sign-comonotonic on C_1, or (ii) there exist v, $w \in C_2$, and $f_1 \in P_1$ such that

$$(a_{(1)}f_1, v_{(1)}f_2) \sim (b_{(1)}f_1, w_{(1)}f_2) \quad \text{and}$$
$$(c_{(1)}f_1, v_{(1)}f_2) \sim (d_{(1)}f_1, w_{(1)}f_2),$$

where $a_{(1)}f_1$, $b_{(1)}f_1$, $c_{(1)}f_1$, and $d_{(1)}f_1$ are rank-ordered prospects in P_1 and $v_{(1)}f_2$, and $w_{(1)}f_2$ are rank-ordered prospects in P_2.

In the first two indifferences, the prospect on the second attribute is kept fixed, in the final two indifferences, everything outside state of nature 1 is kept fixed. The $\sim_1^*$ relationship may be interpreted as measuring strength of preference. For example, if $a > b$, from the indifferences $(a_{(j)}f_1, f_2) \sim (b_{(j)}g_1, f_2)$ and $(c_{(j)}f_1, f_2) \sim (d_{(j)}g_1, f_2)$, we can see that $ab \sim_1^* cd$ means that, in the presence of f_2, a trade-off of a for b is an equally good improvement as a trade-off of c for d: both exactly offset receiving f_1 instead of g_1 for all other states of nature. A similar interpretation can be assigned to the indifferences $(a_{(1)}f_1, v_{(1)}f_2) \sim (b_{(1)}f_1, w_{(1)}f_2)$ and $(c_{(1)}f_1, v_{(1)}f_2) \sim (d_{(1)}f_1, w_{(1)}f_2)$. The $\sim_1^*$ relations are defined entirely in terms of observed indifferences and no new primitives beyond observed choice are assumed in their definition. Hence, we stay entirely within the revealed preference paradigm when using the $\sim_1^*$ relations.

Let $w, x, y, z \in C_2$. We define

$$wx \sim_2^* yz$$

if (i) there exist $f_2, g_2 \in P_2$, and $f_1 \in P_1$ and a state j such that

$$(f_1, w_{(j)}f_2) \sim (f_1, x_{(j)}g_2) \quad \text{and} \quad (f_1, y_{(j)}f_2) \sim (f_1, z_{(j)}g_2),$$

where $w_{(j)}f_2$, $x_{(j)}g_2$, $y_{(j)}f_2$, and $z_{(j)}g_2$ are sign-comonotonic on C_2, or (ii) there exist $a, b \in C_1$, and $f_2 \in P_2$ such that

$$(a_{(1)}f_1, w_{(1)}f_2) \sim (b_{(1)}f_1, x_{(1)}f_2) \quad \text{and}$$
$$(a_{(1)}f_1, y_{(1)}f_2) \sim (b_{(1)}f_1, z_{(1)}f_2),$$

where $w_{(1)}f_2$, $x_{(1)}f_2$, $y_{(1)}f_2$, and $z_{(1)}f_2$ are rank-ordered prospects in P_2 and $a_{(1)}f_1$, and $b_{(1)}f_1$ are rank-ordered prospects in P_1.

We say that $\succcurlyeq$ satisfies *trade-off consistency* on C_1 if improving the first attribute of an outcome in any $\sim_1^*$ relationship breaks that relationship. That is, if $ab \sim_1^* cd$ and $a' \succ_1 a$ then it cannot be that $a'b \sim_1^* cd$. Loosely speaking, trade-off consistency on C_1 ensures that the $\sim_1^*$ relationship is well-behaved when interpreted as a strength of preference relationship. If the strength of preference of a over b is equal to the strength of preference of c over d, then the strength of preference of a' over b cannot be equal to the strength of preference of c over d, when a' is strictly better than a.

Similarly, $\succcurlyeq$ satisfies *trade-off consistency on* C_2 if improving the second attribute of an outcome in any $\sim_2^*$ relationship breaks that relationship. That is, if $wx \sim_2^* yz$ and $y' \succ_2 y$ then it cannot be that $wx \sim_2^* y'z$. *Trade-off consistency* holds if trade-off consistency holds both on C_1 and on C_2. An important advantage of trade-off consistency as a preference condition is that it is closely related to measurements of utility by the trade-off method (Wakker and Deneffe 1996). This makes it easy to test trade-off consistency empirically. Empirical studies that have used the trade-off method include Abdellaoui (2000), Etchart-Vincent (2004), Schunk and Betsch (2006), and Abdellaoui et al. (2007) amongst others.

Trade-off consistency is a powerful condition. It has two effects. First, it ensures that we can define prospect theory functionals for both attributes and second, it ensures that the overall evaluation is additive in these two prospect theory functionals.

4.3. Representation for Two Attributes

To derive the representing functional for preferences, we need an additional assumption. *Solvability* holds if for any two prospects $f, g \in P$ there exists outcomes α and β such that $(\alpha_{(1)}f_1, f_2) \sim g$ and $(f_1, \beta_{(1)}f_2) \sim g$. Solvability implies that the attribute utility functions U_1 and U_2 are unbounded.

The next theorem characterizes Equation (4).

Theorem 1. *The following two statements are equivalent*:

(i) $\succcurlyeq$ *is represented by the functional in Equation* (4) *with strictly increasing weighting functions* W_1^+, W_1^-, W_2^+, *and* W_2^- *and continuous, strictly increasing utility functions* U_1 *and* U_2.

(ii) $\succcurlyeq$ *satisfies* (1) *weak ordering*, (2) *continuity*, (3) *weak separability*, (4) *outcome monotonicity*, (5) *solvability, and* (6) *trade-off consistency*.

The uniqueness results of prospect theory apply, that is, the weighting functions W_i^+ *and* W_i^-, $i = 1, 2$, *are uniquely determined, and the utility functions* U_1 *and* U_2 *are joint ratio scales*.

5. Common Weighting Functions

In the attribute-specific evaluation, the weighting functions may differ across the two attributes. In some cases, however, it might be reasonable to take the weighting functions independent of the attributes. Empirical evidence suggests, for example, that decision weights for money and for health are close (Abdellaoui 2000 compared with Bleichrodt and Pinto 2000). Using common weighting functions facilitates the use of prospect theory in practical applications, because fewer elicitations are required. In this section we will give a preference foundation for the special case of Equation (4) where the weighting functions do not depend on the attributes.

By continuity and connectedness of C_1 and C_2, there exist gains $x_1 \in C_1$ and $x_2 \in C_2$ and losses $y_1 \in C_1$ and $y_2 \in C_2$, such that $(x_1, r_2) \sim (r_1, x_2)$ and $(y_1, r_2) \sim (r_1, y_2)$ and, hence, such that $U_1(x_1) = U_2(x_2)$ and $U_1(y_1) = U_2(y_2)$. Recall that r is the constant prospect that gives (r_1, r_2) in every state of nature. For any event B, let $x_B f$ denote the prospect f with f_j replaced by x for all j in B. We can now define a condition that ensures attribute independence of the weighting functions for gains and for losses.

We say that $\succcurlyeq$ satisfies *attribute-independence for states*, if for all $x_1 \in C_1$ and $x_2 \in C_2$ for which $(x_1, r_2) \sim (r_1, x_2)$ and for all events B, $(x_1, r_2)_B r \sim (r_1, x_2)_B r$. Note that the condition holds for all $x_1 \in C_1$ and $x_2 \in C_2$, but x_1 and x_2 must be either both gains or both losses for otherwise the indifference $(x_1, r_2) \sim (r_1, x_2)$ cannot obtain.

We will now explain the idea behind the condition. As mentioned before, if $(x_1, r_2) \sim (r_1, x_2)$ then $U_1(x_1) = U_2(x_2)$. If Equation (4) holds and x_1 and x_2 are both gains, the indifference $(x_1, r_2)_B r \sim (r_1, x_2)_B r$ implies that $W_1^+(B)U_1(x_1) = W_2^+(B)U_2(x_2)$ and $W_1^+(B) = W_2^+(B)$ follows from $U_1(x_1) = U_2(x_2)$. A similar line of argument shows that $W_1^-(B) = W_2^-(B)$ whenever x_1 and x_2 are losses. Because these equalities hold for all events B, we obtain the following result:

Corollary 2. *If we add attribute-independence for states to statement* (ii) *of Theorem* 1, *then the weighting functions in statement* (i) *of Theorem* 1 *are attribute-independent, i.e.,* $W_1^+ = W_2^+ = W^+$ *and* $W_1^- = W_2^- = W^-$.

If $\rho_1 = \rho_2$ and $A_1 = A_2$ then Corollary 2 also implies that the decision weights π^+ and π^- are attribute-independent. This follows from the definition of the decision weights, Equations (5a) and (5b). Having the weighting functions independent of the attributes does not make the attribute-specific evaluation equal to the holistic evaluation. This is easily seen by referring back to the example of the RA considering the new job. Under the attribute-specific evaluation with common weighting functions, Equation (7) becomes

$$\pi_1^+(u_1(80) - u_1(50)) + \pi_1^- \lambda_2(u_2(15) - u_2(20)) + \pi_2^- \lambda_1(u_1(40) - u_1(50)) + \pi_2^+(u_2(30) - u_2(15)),$$

showing that the attribute-specific evaluation clearly differs from the holistic evaluation, Equation (6).

Note that it is not only the presence of the loss aversion parameter that distinguishes the holistic from the attribute-specific evaluation. In general, the two evaluations differ even if a prospect yields only gains or only losses. Consider again the job offer example but suppose now that the RA's reference point for annual earnings is \$30K and for research time it is 10 hours per week. The preference (\$80K, 15 h) $\succ$ (\$40K, 30 h) still holds. Let E_1 denote the event "husband finds a job" and E_2 the event "husband does not find a job." Under the holistic evaluation, the job's value is

$$W^+(E_1)\big((u_1(80)+u_2(15)) - (u_1(30)+u_2(10))\big) + \big(1-W^+(E_1)\big)\big((u_1(40)+u_2(30)) - (u_1(30)+u_2(10))\big),$$

and under the attribute-specific evaluation it is

$$W^+(E_1)\big(u_1(80) - u_1(30)\big) + \big(1-W^+(E_2)\big)\big(u_2(15) - u_2(10)\big) + \big(1-W^+(E_1)\big)\big(u_1(40) - u_1(30)\big) + W^+\big(E_2\big)\big(u_2(30) - u_2(10)\big).$$

Equality only holds if $W^+(E_1) = (1 - W^+(E_2))$, i.e., if $W^+(E_1) + W^+(E_2) = 1$. This must hold for all events E_1 and E_2, and for the attribute-specific evaluation this can only be the case if W^+ is a probability measure. A similar argument can be used to derive that W^- must be a probability measure. Hence, for outcomes of the same sign and attribute independent weighting, the attribute-specific evaluation agrees with the holistic evaluation only when both representations reduce to subjective expected utility.

6. More Than Two Attributes

We will now extend our results to more than two attributes. Let $C = C_1 \times \cdots \times C_m$, $m > 2$. Each C_i is a nondegenerate convex subset of $\mathbb{R}$. The reference point on the ith attribute is denoted r_i and is assumed to be an interior point of C_i. We will denote the set of prospects on C_i^n as P_i and write prospects as $f_1, \ldots, f_m$. Let $g_i f$ denote the prospect $f \in P$ with f_i replaced by g_i, and let $g_i h_k f$ denote the prospect $f \in P$ with f_i replaced by g_i and f_k replaced by h_k. Weak separability is now defined as follows: for all $i \in \{1, \ldots, m\}$, $f_i, g_i \in P_i$, $f', g' \in P$, $f_i f' \succcurlyeq g_i f'$ iff $f_i g' \succcurlyeq g_i g'$. The definitions of outcome monotonicity, continuity, and solvability easily generalize to the case of more than two attributes. For trade-off consistency we define

$$ab \sim_i^* cd$$

if (i) there exist $f_i, g_i \in P_i$, $f \in P$, and a state j such that

$$(a_{(j)} f_i)_i f \sim (b_{(j)} g_i)_i f \quad \text{and} \quad (c_{(j)} f_i)_i f \sim (d_{(j)} g_i)_i f,$$

where $a_{(j)} f_i$, $b_{(j)} g_i$, $c_{(j)} f_i$, and $d_{(j)} g_i$ are sign-comonotonic on C_i, or (ii) there exist $v, w \in C_k$, and $f \in P$ such that

$$(a_{(1)} f_i)_i (v_{(1)} f_k)_k f \sim (b_{(1)} f_i)_i (w_{(1)} f_k)_k f \quad \text{and} \quad (c_{(1)} f_i)_i (v_{(1)} f_k)_k f \sim (d_{(1)} f_i)_i (w_{(1)} f_k)_k f,$$

where $a_{(1)} f_i$, $b_{(1)} f_i$, $c_{(1)} f_i$, and $d_{(1)} f_i$ are rank-ordered prospects in P_i and $v_{(1)} f_k$, and $w_{(1)} f_k$ are rank-ordered prospects in P_k.

Trade-off consistency holds if each $\sim_i^*$-relationship satisfies trade-off consistency on C_i.

We are now in a position to extend Theorem 1 to the case of more than two attributes.

Theorem 3. *The following two statements are equivalent*:

(i) $\succcurlyeq$ *is represented by* $V = \sum_{i=1}^m V_i(f_i)$ *where the* V_i *are prospect theory functionals with strictly increasing weighting functions* W_i^+ *and* W_i^- *and continuous, strictly increasing utility functions* U_i.

(ii) $\succcurlyeq$ *satisfies* (1) *weak ordering*, (2) *continuity*, (3) *weak separability*, (4) *outcome monotonicity*, (5) *solvability, and* (6) *trade-off consistency*.

The uniqueness results of prospect theory apply, that is, the weighting functions W_i^+ and W_i^- are uniquely determined, and the utility functions U_i are joint ratio scales.

Attribute independence can easily be extended to the case of more than two attributes, so that the arguments preceding Corollary 2 can still be used to ensure that the weighting functions are attribute-independent.

7. General Outcomes

For ease of exposition, we have assumed thus far that all attributes are numerical. In many real-world decisions, this assumption is too restrictive. An example is health, the area in which decision analysis is most frequently applied (Keller and Kleinmuntz 1998, Smith and von Winterfeldt 2004). Health consists of two dimensions, survival duration and health quality, and health quality is a nonnumeric attribute. The extension of our analysis to nonnumeric attributes is as follows.

Assume that the C_i are connected topological spaces. $C = C_1 \times \cdots \times C_m$ is endowed with the product topology and so is C^n. The reference points r_i are in the interior of C_i for each i. Redefine outcome monotonicity as follows: for all i, if $f_{ji} \succcurlyeq g_{ji}$ for all j then $f_i \succcurlyeq_i g_i$. The strict version of outcome monotonicity is not necessary here as it follows from the version with weak preferences and trade-off consistency (Köbberling and Wakker 2003, Lemma 26). We can now state the extension of our results to nonnumeric attributes.

Corollary 4. *If the C_i, $i = 1, \ldots, m$, are connected topological spaces, then Theorems* 1 *and* 3 *still hold if we drop in* (i), *the requirement that the attribute-wise utility functions are strictly increasing*.

The proof of this claim follows easily from the proofs of Theorems 1 and 3. Corollary 2 can still be used to ensure that the weighting functions are attribute-independent.

8. Empirical Measurement

A few comments concerning the empirical implementability of additive prospect theory under the attribute-specific evaluation are worth mentioning. For empirical purposes a first step is obviously the verification of the preference conditions that have been identified in this paper. When these are satisfied, the elicitation of the attribute-specific evaluation is simpler than that of the holistic evaluation because we do not need to know the ranking of outcomes. Essentially, we can apply the known elicitation techniques for single-dimensional prospect theory to each of the attributes. When attribute-independence for states holds, the weighting functions have to be assessed only once.

A procedure to measure utility under prospect theory was recently proposed by Abdellaoui et al. (2007). Their method uses various elicitation techniques (probability equivalence, certainty equivalence, and Wakker and Deneffe's 1996 trade-off method). A simpler procedure was proposed by Abdellaoui et al. (2008). Their method only uses certainty equivalence questions but is less general than the procedure of Abdellaoui et al. (2007) in that it assumes that the utility functions are power functions.

The weighting functions W_i^+ and W_i^-, $i \in \{1, \ldots, m\}$ can be measured either through the nonchoice-based methods of Tversky and Fox (1995), Fox and Tversky (1998), Wu and Gonzalez (1999), or Kilka and Weber (2001) or through the choice-based method of Abdellaoui et al. (2005). If probabilities are known then the methods of Abdellaoui (2000) or Bleichrodt and Pinto (2000) can be applied.

We can also use the representation results for single-dimensional prospect theory (Prelec 1998, Wakker and Tversky 1993, Wakker and Zank 2002) to restrict the functional forms of the utility functions and the weighting functions. If preferences do not change when we multiply all levels of an attribute by a common constant (while holding the other attribute constant), then the attribute utility function must be a power function. If preferences are invariant to adding a constant to all levels of an attribute such that the sign of the attribute levels is preserved (and the other attribute is held constant), then the attribute utility function is exponential. When probabilities are known and preferences satisfy Prelec's (1998) compound invariance conditions, then the weighting functions must have the form $w^i(p) = \exp(-\beta^i \ln(-p)^\alpha)$, $i = +, -$. Conditions for deriving exponential or power weighting functions are presented in Diecidue et al. (2009).

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Appendix. Proofs

Proof of Theorem 1. That (i) implies (ii) is routine. Hence we assume (ii) and derive (i).

By weak order, weak separability, outcome monotonicity, and continuity, $\succcurlyeq$ on P can be represented by $V(V_1(f_1), V_2(f_2))$ with V strictly increasing in V_1 and V_2. V_1 represents $\succcurlyeq_1$ and V_2 represents $\succcurlyeq_2$. By continuity, V_1 and V_2 are continuous, by outcome monotonicity, they are strictly increasing.

We will now show that V_1 and V_2 are prospect theory functionals. For a prospect $f_1 \in P_1$, define the prospect f_1^+

by $f_{1j}^+ = f_{1j}$ if $f_{1j} \succ_1 r_1$ and by $f_{1j}^+ = r_1$ otherwise, and the prospect f_1^- by $f_{1j}^- = f_{1j}$ if $f_{1j} \prec_1 r_1$ and by $f_{1j}^- = r_1$ otherwise. That is, f_1^+ is the positive part of f_1 and f_1^- is its negative part. In a similar fashion, we define f_2^+ and f_2^-. Consider $\succcurlyeq_1$ on P_1. Because $\succcurlyeq$ satisfies outcome monotonicity and C_1 is nondegenerate, all states of nature are nonnull (a state is null if replacing any outcomes in that state does not affect the preference). Also, because r_1 lies in the interior of C_1, $\succcurlyeq_1$ is truly mixed ($\succcurlyeq_1$ is *truly mixed* if there exists a prospect f_1 such that $f_1^+ \succ r_1$ and $f_1^+ \prec r_1$, that is, genuine trade-offs between gains and losses occur). By Theorem 12 in Köbberling and Wakker (2003) there exists a prospect theory representation for $\succcurlyeq_1$ with U_1, the continuous utility function over C_1, $U_1(r_1) = 0$, and W_1^+, and W_1^-, the weighting functions over gains and losses on the first attribute, respectively. Köbberling and Wakker's (2003) weak monotonicity follows from outcome monotonicity and sign-comonotonic trade-off consistency follows from trade-off consistency on C_1. By Proposition 8.2 in Wakker and Tversky (1993), gain-loss consistency can be dropped from Köbberling and Wakker's (2003) conditions when the number of states of nature exceeds two. This is the case in our analysis if we interpret attributes as events (Sarin and Wakker 1998, Corollary B.3). U_1 is strictly increasing because V_1 is strictly increasing. W_1^+ and W_1^- are strictly increasing by outcome monotonicity. By Observation 13 in Köbberling and Wakker (2003), U_1 is a ratio scale and W_1^+ and W_1^- are unique. By solvability, U_1 is unbounded.

By a similar line of argument, there exists a prospect theory representation for $\succcurlyeq_2$, with U_2 the continuous and strictly increasing utility function on C_2, $U_2(r_2) = 0$, U_2 a ratio scale, and W_2^+ and W_2^- the unique and strictly increasing weighting functions over gains and losses on the second attribute, respectively. By solvability, U_2 is unbounded.

So far we have shown that $V(\mathrm{PT}_1(f_1), \mathrm{PT}_2(f_2))$ represents $\succcurlyeq$. It remains to show that V is additive. We will do so by showing that the rate of trade-offs between PT_1 and PT_2 is everywhere constant. Take $f_1 \in P_1$, and let f_2 be a rank-ordered prospect in P_2. Take $\alpha_0^1 \in C_2$ such that $\alpha_0^1 \succcurlyeq f_{22}$. Then $(\alpha_0^1)_{(1)} f_2$ is a rank-ordered prospect in P_2. Let g_2 be such that $f_{2j} \succcurlyeq g_{2j}$ for all j with at least one of these preferences strict. By solvability there exists an outcome α_1^1 such that $(f_1, (\alpha_0^1)_{(1)} f_2) \sim (f_1, (\alpha_1^1)_{(1)} g_2)$. By outcome monotonicity $\alpha_1^1 \succ_2 \alpha_0^1$. Next we consider the prospect $(f_1, (\alpha_1^1)_{(1)} f_2)$. By solvability we can find an outcome α_2^1 such that $(f_1, (\alpha_1^1)_{(1)} f_2) \sim (f_1, (\alpha_2^1)_{(1)} g_2)$. Hence, $\alpha_2^1 \alpha_1^1 \sim_2^* \alpha_1^1 \alpha_0^1$. We proceed in this manner to construct a *standard sequence* $\alpha_0^1, \alpha_1^1, \ldots$ on the second attribute for which $\alpha_s^1 \alpha_{s-1}^1 \sim_2^* \alpha_1^1 \alpha_0^1$ for all natural s. It is easily verified that this implies that $\mathrm{PT}_2((\alpha_s^1)_{(1)} f_2) - \mathrm{PT}_2((\alpha_{s-1}^1)_{(1)} f_2) = \mathrm{PT}_2((\alpha_1^1)_{(1)} f_2) - \mathrm{PT}_2((\alpha_0^1)_{(1)} f_2)$. Suppose without loss of generality that $\mathrm{PT}_2((\alpha_1^1)_{(1)} f_2) - \mathrm{PT}_2((\alpha_0^1)_{(1)} f_2) = 1$.

Next we construct a standard sequence $\beta_0^1, \beta_1^1, \ldots$ on the first attribute by eliciting indifferences $((\beta_t^1)_{(1)} f_1, (\alpha_0^1)_{(1)} f_2) \sim ((\beta_{t-1}^1)_{(1)} f_1, (\alpha_1^1)_{(1)} f_2)$, $t = 1, 2, \ldots,$ such that all prospects involved are rank-ordered. These indifferences imply that $\beta_t^1 \beta_{t-1}^1 \sim_1^* \beta_1^1 \beta_0^1$ for all natural t and, thus that $\mathrm{PT}_1((\beta_t^1)_{(1)} f_1) - \mathrm{PT}_1((\beta_{t-1}^1)_{(1)} f_1) = \mathrm{PT}_1((\beta_1^1)_{(1)} f_1) - \mathrm{PT}_1((\beta_0^1)_{(1)} f_1)$. The indifferences also define a rate of trade-off between PT_1 and PT_2. Let $\mathrm{PT}_1((\beta_1^1)_{(1)} f_1) - \mathrm{PT}_1((\beta_0^1)_{(1)} f_1) = c$. Then the rate of trade-off between PT_1 and PT_2 is constant for all the points we have elicited thus far. This claim follows from trade-off consistency. By trade-off consistency, we must have $((\beta_1^1)_{(1)} f_1, (\alpha_{s-1}^1)_{(1)} f_2) \sim ((\beta_0^1)_{(1)} f_1, (\alpha_s^1)_{(1)} f_2)$ for any $s = 1, 2, \ldots$. Applying trade-off consistency again implies that we must have $((\beta_t^1)_{(1)} f_1, (\alpha_{s-1}^1)_{(1)} f_2) \sim ((\beta_{t-1}^1)_{(1)} f_1, (\alpha_s^1)_{(1)} f_2)$ for any $s = 1, 2, \ldots; t = 1, 2, \ldots$. Hence the rate of trade-off between PT_1 and PT_2 is everywhere c.

Next we double the density of the grid $\{\beta_0^1, \beta_1^1, \ldots\} \times \{\alpha_0^1, \alpha_1^1, \ldots\}$ that we constructed above. By continuity of U_2 and connectedness of C_2 we can find an outcome $\alpha_1^{1/2}$ such that $\mathrm{PT}_2((\alpha_1^{1/2})_{(1)} f_2) - \mathrm{PT}_2((\alpha_0^1)_{(1)} f_2) = 1/2$. Let $\alpha_0^1 = \alpha_0^{1/2}$ and construct a new standard sequence $\alpha_0^{1/2}, \alpha_1^{1/2}, \ldots$ by eliciting indifferences $(f_1, (\alpha_{s-1}^{1/2})_{(1)} f_2) \sim (f_1, (\alpha_s^{1/2})_{(1)} g_2')$. It follows from outcome monotonicity that $\alpha_2^{1/2} = \alpha_1^1$ and, hence, in general $\alpha_{2s}^{1/2} = \alpha_s^1$, $s = 0, 1, \ldots$.

We construct a new standard sequence $\beta_0^{1/2}, \beta_1^{1/2}, \ldots$ on the first attribute by setting $\beta_0^{1/2} = \beta_0^1$ and eliciting indifferences $((\beta_t^{1/2})_{(1)} f_1, (\alpha_0^{1/2})_{(1)} f_2) \sim ((\beta_{t-1}^{1/2})_{(1)} f_1, (\alpha_1^{1/2})_{(1)} f_2)$, $t = 1, 2, \ldots$. We have to show that the rate of trade-off between PT_1 and PT_2 in this new grid $\{\beta_0^{1/2}, \beta_1^{1/2}, \ldots\} \times \{\alpha_0^{1/2}, \alpha_1^{1/2}, \ldots\}$ is still constant. For this we have to show that $\beta_{2t}^{1/2} = \beta_t^1$, $t = 0, 1, \ldots$. We will show that $\beta_2^{1/2} = \beta_1^1$. Then $\beta_{2j}^{1/2} = \beta_j^1$, for all $j = 0, 1, \ldots$ follows from the construction of the standard sequence. By the construction of the standard sequence, $((\beta_2^{1/2})_{(1)} f_1, (\alpha_0^{1/2})_{(1)} f_2) \sim ((\beta_1^{1/2})_{(1)} f_1, (\alpha_1^{1/2})_{(1)} f_2)$. By trade-off consistency $((\beta_1^{1/2})_{(1)} f_1, (\alpha_1^{1/2})_{(1)} f_2) \sim ((\beta_0^{1/2})_{(1)} f_1, (\alpha_2^{1/2})_{(1)} f_2) = ((\beta_0^1)_{(1)} f_1, (\alpha_1^1)_{(1)} f_2) \sim ((\beta_1^1)_{(1)} f_1, (\alpha_0^1)_{(1)} f_2)$. By transitivity and because $\alpha_0^1 = \alpha_0^{1/2}$, $((\beta_2^{1/2})_{(1)} f_1, (\alpha_0^1)_{(1)} f_2) \sim ((\beta_1^1)_{(1)} f_1, (\alpha_0^1)_{(1)} f_2)$. By outcome monotonicity $\beta_2^{1/2} = \beta_1^1$. Hence, the rate of trade-off between PT_1 and PT_2 is still constant when we double the density of the grid.

We continue this doubling of density infinitely, creating increasingly fine standard sequences $\alpha_0^{2^{-m}}, \alpha_1^{2^{-m}}, \ldots$ and $\beta_0^{2^{-m}}, \beta_1^{2^{-m}}, \ldots, m = 2, \ldots$. On the resulting increasingly fine grids the rate of trade-off between PT_1 and PT_2 remains constant by a similar proof as for the case where the density of the grid was doubled.

Because U_1 and U_2 are unbounded, for any natural m there can be no $x_1 \in C_1$ and no $x_2 \in C_2$ such that $x_1 \succ \beta_t^{2^{-m}}$ for all t or $x_2 \succ \alpha_s^{2^{-m}}$ for all s. There can also be no outcomes infinitely close to β_0^1 and α_0^1 in the sense that there is always an outcome from the grid that lies between an outcome $x_1 \in C_1$ and β_0^1, and an outcome from the grid that lies between an outcome $x_2 \in C_2$ and α_0^1 when x_1 is unequal to β_0^1 and x_2 is unequal to α_0^1. If $x_2 \neq \alpha_0^1$ then $U_2(x_2) - U_2(\alpha_0^1) = c > 0$ and, hence, there exists a natural number m such that $2^{-m} < c$. By construction, there is an element $\alpha_1^{2^{-m}}$ of the grid such that $\alpha_0^1 \prec \alpha_1^{2^{-m}} \prec x_2$. A similar argument shows that the grid interferes everywhere. Let x_2 and y_2 be two outcomes such that $x_2 \succ y_2$. Suppose that $U_2(x_2) - U_2(y_2) = d > 0$. Then there exists a natural number m such that $2^{-m} < d$ and by construction there is an element $\alpha_s^{2^{-m}}$ of the grid such that $y_2 \prec \alpha_s^{2^{-m}} \prec x_2$.

Finally, because U_1 and U_2 are unbounded there cannot be elements x_1 and x_2 that are so bad that they are never included in any grid. Consider an outcome x_2. Then we can construct a prospect f_2 with $f_{2j} = x_2$ for all j. Because U_2 is unbounded, we can construct a prospect g_2 such that $f_{2j} \succ g_{2j}$ for all j. Let $\alpha_0^1 = x_2$ and construct a new grid by eliciting indifferences $(f_1, (\alpha_0^1)_{(1)} f_2) \sim$

$(f_1, (\alpha_1^1)_{(1)}g_2), (f_1, (\alpha_1^1)_{(1)}f_2) \sim (f_1, (\alpha_2^1)_{(1)}g_2)$ etc. This produces a dense grid that includes x_2.

By continuity we can extend the dense grid to all outcomes. Hence, we have shown that on the whole domain the rate of trade-off between PT_1 and PT_2 is constant for rank-ordered prospects. Hence, for rank-ordered prospects $V(\text{PT}_1(f_1), \text{PT}_2(f_2))$ is additive: $V(f) = \text{PT}_1(f_1) + \text{PT}_2(f_2)$. Because U_1 and U_2 are continuous and unbounded and C_1 and C_2 are connected, we can for any prospects f_1 and f_2 find rank-ordered prospects g_1 and g_2 such that $f_1 \sim_1 g_1$ and $f_2 \sim_2 g_2$. We set $V(\text{PT}_1(f_1), \text{PT}_2(f_2)) = \text{PT}_1(g_1) + \text{PT}_2(g_2)$. Finally, we show that $\text{PT}_1(f_1) + \text{PT}_2(f_2)$ represents $\succcurlyeq$. Suppose that $f \succcurlyeq g$. There are rank-ordered prospects f' and g' such that $f' \sim f$ and $g' \sim g$. By transitivity, $f' \succcurlyeq g'$. Hence, $\text{PT}_1(f_1) + \text{PT}_2(f_2) = \text{PT}_1(f_1') + \text{PT}_2(f_2') \geq \text{PT}_1(g_1') + \text{PT}_2(g_2') = \text{PT}_1(g_1) + \text{PT}_2(g_2)$ which completes the proof of statement (i).

The uniqueness results follow from the uniqueness results for PT_1 and PT_2 combined with the fact that on each grid the rate of trade-off between PT_1 and PT_2 must be constant. This completes the proof of Theorem 1. □

Proof of Theorem 3. That (i) implies (ii) is routine. Hence, we assume (ii) and derive (i). The proof is very similar to the proof of Theorem 1 and will not be elaborated here. By weak separability $V(V_1(f_1), \ldots, V_m(f_m))$ with V strictly increasing in each of the V_i represents $\succcurlyeq$. We then use the results of Köbberling and Wakker (2003) to show that each V_i has a prospect theory representation. Finally, we show, exactly as in the proof of Theorem 1, that for all $i, k \in \{1, \ldots, m\}$, the rate of trade-off between any PT_i and PT_k is constant. This establishes the proof. □

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