Goodness-of-fit tests for a heavy tailed distribution

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Abstract

For testing whether a distribution function is heavy tailed, we study the Kolmogorov test, Berk-Jones test, score test and their integrated versions. A comparison is conducted via Bahadur efficiency and simulations.

The score test and the integrated score test show the best performance. Although the Berk-Jones test is more powerful than the Kolmogorov-Smirnov test, this does not hold true for their integrated versions; this differs from results in Einmahl and McKeague (2003), which shows the difference of Berk-Jones test in testing distributions and tails.

Keywords. Bahadur efficiency, heavy tail, tail index.

Contents

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1 Introduction

We say that a distribution has a heavy tail with tail index if

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha} \quad \text{for all } x>0 \tag{1.1}$$

holds for some $\alpha > 0$ (notation: $1 - F \in RV_\alpha$). Heavy tailed distributions have been applied in many different areas such as population size (Zipf (1949)), random graph (Reittu and Norros (2004)), internet traffic (Resnick (1997a)), hydrology (Katz et al. (2002)), finance (Danielsson and de Vries (1997)).

Most of the theoretical work on heavy tailed distributions concentrates on estimating the tail index α. One of the best known estimators is the Hill estimator

$$\hat{\alpha} = \left\{ k^{-1} \sum_{j=1}^{k} \log X_{n,n-k+j} - \log X_{n,n-k} \right\}^{-1} \tag{1.2}$$

(Hill (1975)), which employs a fraction of upper order statistics $X_{n,n-k}, X_{n,n-k+1}, \ldots, X_{n,n}$. Several data-driven methods for choosing the sample fraction k/n are proposed in the literature; see Drees and Kaufmann (1998), Danielsson et al. (2001) and Guillou and Hall (2001).

As far as we know, goodness-of-fit testing has not received as much attention as tail index estimation. An interesting aspect of testing goodness-of-fit of the heavy tail distribution is that the null hypothesis provides a description of the heavy tail distribution which is incomplete in the following two aspects:

(i) The tail index is unknown under the null hypothesis, and hence should be estimated. It is well-known that estimation of unknown parameters has an non-neglible effect on the distribution of the test statistics, see Durbin (1973a,b).

(ii) The tail index only describes the tail behaviour of the heavy tail distribution, and thus only gives a partial description of the underlying distribution.

By fitting a generalized Pareto distribution to exceedances over a high threshold, Davison and Smith (1990) employed Kolmogorov and Anderson-Darling statistics to test the fit, and compared these statistics to 5% critical values for testing an exponential distribution with unknown mean. However, using these critical values in this context “is suspect since the exponential distribution is only a submodel of the generalized Pareto distribution, so the true critical points are smaller”, see Davison and Smith (1990, p. 414).

In Choulakian and Stephens (2001) critical values for Cramer-von Mises and Anderson-Darling statistics for testing a generalized Pareto distribution with unknown shape parameter are given.

If X has a generalized Pareto distribution with unknown shape parameter, then $\ln X$ has an exponential distribution with unknown mean, and thus tests of exponentiality may be used to test the generalized Pareto distribution, see Marohn (2002).

Recently, Drees et al. (2004) employed the Cramer-von Mises statistic to test the null hypothesis that a distribution is in the domain of attraction with extreme value index larger than $-1/2$. This may be thought as a generalization of Choulakian and Stephens (2001), since a deterministic threshold is replaced by a random threshold in Drees et al. (2004)).

As far as the present authors know, no work has been done on comparing the performance of goodness-of-fit tests for heavy tailed distributions under the alternative hypothesis. In this paper we compare three tests, the Kolmogorov test, the Berk-Jones test and the "estimated score" test.

The Kolmogorov test, Kolmogorov (1933); Smirnov (1948); Durbin (1973a), has a long tradition in statistics, and thus is an obvious choice as long as there are no other tests which clearly perform better.

The Berk-Jones test, Berk and Jones (1978, 1979), may be viewed as a non-parametric likelihood test, and was derived for the situation where the null hypothesis completely specifies the distribution of the observations. In this particular situation, the Berk-Jones test was shown to be more efficient, in the sense of Bahadur efficiency, than any weighted Kolmogorov test at any alternative. In Li (2003) it is argued that the Berk-Jones test should also perform better than the Kolmogorov test in situations where the null hypothesis does not completely specify the distribution of the observations.

In contrast to the Berk-Jones test, the "estimated score" test in Hjort and Koning (2002) was specifically proposed for the situation where parameters are unknown.

We organize this paper as follows. In section 2 we present Kolmogorov test, Berk-Jones test, score test and their integrated versions. Large deviations results for tail empirical processes are obtained as a byproduct of studying Bahadur efficiency. A simulation study and real applications are given in section 3. All proofs are put in section 4.

2 Methodologies

2.1 The KS, BJ and SC supremum tests

In order to motivate our methods, we shall first restrict ourselves to the quintessential example of a heavy tailed distribution, the Pareto distribution. The Pareto distribution is defined by

$$G(x;\alpha,\beta) = 1-(x/\beta)^{-\alpha} \quad \text{for all } x>\beta \tag{2.1}$$

where $\alpha>0$ is a shape parameter, and $\beta>0$ is a scale parameter. For convenience, let $G(x;\alpha)$ denote $G(x;\alpha,1)=1-x^{-\alpha}$.

Let $X_{n,1}\le X_{n,2}\le\cdots\le X_{n,n}$ denote an ordered sample of size n drawn from the Pareto distribution with unknown shape and scale parameters α and β. Choose $1\le k\le n$. Let $R_{k,j}$ denote $X_{n,n-k+j}/X_{n,n-k}$ for $j\le k$, and note that $R_{k,1}\le R_{k,2}\le\cdots\le R_{k,k}$. One may show that the random variables $R_{k,1},R_{k,2},\ldots,R_{k,k}$ are jointly equal in distribution to an ordered sample of size k drawn from the Pareto distribution with shape parameter α and scale parameter 1.

Let $F_k(r)$ denote the empirical distribution function of $R_{k,1},R_{k,2},\ldots,R_{k,k}$, defined by

$$F_k(r) = k^{-1}\sum_{j=1}^{k} I(R_{k,j}\le r).$$

Observe that for $r>1$

$$1-F_k(r) = \frac{1}{k}\sum_{i=1}^{n} I(X_i > rX_{n,n-k}).$$

The Kolmogorov test is based on the statistic $\sup_{r>1}|\mathrm{KS}(r;\hat\alpha)|$, where

$$\mathrm{KS}(r;\alpha) = 1-F_k(r)-\{1-G(r;\alpha)\} = 1-F_k(r)-r^{-\alpha},$$

and $\hat\alpha = \left\{k^{-1}\sum_{j=1}^{k}\log R_{k,j}\right\}^{-1}$ is the Hill estimator.

The Berk-Jones test is based on the statistic $\sup_{r>1} k\mathrm{BJ}(r;\hat\alpha)$, where

$$\mathrm{BJ}(r;\alpha) = 2K\left(F_k(r),G(r;\alpha)\right) = 2K\left(F_k(r),1-r^{-\alpha}\right),$$

and the function K is defined by

$$K(\hat p,p) = \hat p\ln\frac{\hat p}{p} + (1-\hat p)\ln\frac{1-\hat p}{1-p}. \tag{2.2}$$

The estimated score test is based on the statistic $\sup_{r>1}|\text{SC}(r;\hat{\alpha})|$, where

$$\text{SC}(r;\alpha)=F_k(r)-\int_1^r(1-F_k(s))\,d\Lambda(s;\alpha)=F_k(r)-\alpha\int_1^r\frac{1-F_k(s)}{s}\,ds,$$

and $\Lambda(r;\alpha)=-\ln(1-G(r;\alpha))=\alpha\ln r$ is the cumulative hazard function belonging to $G(r;\alpha)$ defined by (2.1).

Suppose that $X_1,\cdots,X_n$ are i.i.d. observations with distribution function F. We intend to test whether F has a heavy tailed distribution, see (1.1). Let $U(x)$ denote the inverse function of $1/(1-F(x))$. Then (1.1) implies $U\in RV_{1/\alpha}$. In order to study the limiting behaviors of KS$(r;\alpha)$, BJ$(r;\alpha)$ and SC$(r;\alpha)$, we further assume that there exists a function $A(t)\to 0$, as $t\to\infty$, such that

$$\lim_{t\to\infty}\frac{U(tx)/U(t)-x^{1/\alpha}}{A(t)}=x^{1/\alpha}\frac{x^\rho-1}{\rho} \tag{2.3}$$

for all $x>0$, where $\rho\le 0$; see de Haan and Stadtmüller (1996) for details. Our first result is as follows.

Theorem 1. *Suppose F satisfies (2.3) and $k=k(n)\to\infty$, $k/n\to 0$, $\sqrt{k}A(n/k)\to 0$ as $n\to\infty$. Then*

$$\sup_{x>1}\left|\sqrt{k}KS(x;\hat{\alpha})\right| \xrightarrow{d} \sup_{0<x<1}\left|W(x)-xW(1)+x\log(x)\int_0^1\frac{W(s)-sW(1)}{s}\,ds\right|$$

and

$$\sup_{x>1}\left|\sqrt{k}SC(x;\hat{\alpha})\right| \xrightarrow{d} \sup_{0<x<1}\left|W(x)-xW(1)-\int_0^x\frac{W(s)-sW(1)}{s}\,ds+x\int_0^1\frac{W(s)-sW(1)}{s}\,ds\right|,$$

where $W(x)$ is a Wiener process.

Remark 1. *For any fixed $x^{-1/\alpha}\in(0,1)$, we can show by Taylor expansions that*

$$BJ(x^{-1/\alpha};\hat{\alpha})\xrightarrow{d}\frac{\{W(x)-xW(1)+x\log(x)\int_0^1\frac{W(s)-sW(1)}{s}\,ds\}^2}{x(1-x)}$$

and

$$E\frac{\{W(x)-xW(1)+x\log(x)\int_0^1\frac{W(s)-sW(1)}{s}\,ds\}^2}{x(1-x)}=\frac{x(1-x)-x^2\{\log(x)\}^2}{x(1-x)},$$

i.e., the expectation of the limiting of $BJ(x;\hat{\alpha})$ depends on location x. This is different from Berk & Jones test for testing distribution functions. But it may still be interesting to find the limiting distribution of $a_n \sup_{x>1} BJ(x;\hat{\alpha}) - b_n$ for some normalizing constants $a_n > 0$ and b_n. When $F(x) = \exp\{-x^{-\alpha}\}$, we have

$$\begin{aligned}
&\frac{1}{k}\sum_{i=1}^{n} I(X_i > y^{-1/\alpha} X_{n,n-k}) \\
&\stackrel{d}{=} \frac{1}{k}\sum_{i=1}^{k} I(-\alpha \log \frac{U(Y_{n,n-i+1})}{U(Y_{n,n-k})} < \log y) \\
&= \frac{1}{k}\sum_{i=1}^{k} I(\log \frac{\log(1 - Y_{n,n-i+1}^{-1})}{\log(1 - Y_{n,n-k}^{-1})} < \log y),
\end{aligned}$$

where $Y_{n,i}'s$ are defined in the proof of Theorem 1 below. That is, the distributions of

$$\sup_{x>1} \left|\sqrt{k} KS(x;\hat{\alpha})\right|, \quad \sup_{x>1} BS(x;\hat{\alpha}) \quad \textit{and} \quad \sup_{x>1} \left|\sqrt{k} SC(x;\hat{\alpha})\right|$$

are independent of α when $F(x) = \exp\{-x^{-\alpha}\}$. This property is employed to simulate critical values from test statistics themselves; see Section 3 for details.

2.2 Bahadur efficiency

It is known that when the null hypothesis is simple, that is, completely specifies the distribution of the observations, the Berk-Jones test is more efficient, in Bahadurs sense, than any weighted KolmogorovSmirnov test at any alternative, see Berk and Jones (1978, 1979).

Extending this result to the situation where the null hypothesis is composite, that is, does not completely specify the distribution of the observations, requires hard work, which is typically avoided. Instead, it is argued that "one may expect similar optimal properties for our test for the composite hypothesis", see for instance Li (2003, p. 178).

In this subsection, we adopt a similar approach. That is, we study the Bahadur efficiency of the "simple null hypothesis tests" based on statistics

$$\sup_{r>1} k\mathrm{BJ}(r;\alpha_0) \quad \text{and} \quad \sup_{r>1} \left|\sqrt{k}\mathrm{KS}(r;\alpha_0)\right|$$

rather than the Bahadur efficiency of "composite null hypothesis tests" based on the statistics

$$\sup_{r>1} k\mathrm{BJ}(r;\hat{\alpha}) \quad \text{and} \quad \sup_{r>1} |\sqrt{k}\mathrm{KS}(r;\hat{\alpha})|.$$

First we state the definition of exact slope of a test statistic. Let $(S, \mathcal{A})$ be the sample space of infinitely many independent and identically distributed observations $s = (X_1, X_2, \cdots)$ on an abstract random variable X. Let $\{P_\theta : \theta \in \Omega\}$ be a collection of probability distributions of X, where Ω is a parameter space, and let Ω_0 be a proper subset of Ω. Consider the hypothesis $H_0 : \theta \in \Omega_0$. Let $\{T_n(X_1, \cdots, X_n)\}$ be a sequence of test statistics. Assume that there exists an $F_n(t)$ such that

$$P_\theta(T_n \leq t) = F_n(t) \tag{2.4}$$

for all $\theta \in \Omega_0$ and all $t \in R$. Then the level obtained by T_n is defined to be

$$L_n(X_1, \cdots, X_n) = 1 - F_n(T_n(X_1, \cdots, X_n)).$$

If

$$\lim_{n\to\infty} n^{-1} \log L_n(X_1, \cdots, X_n) = -\frac{1}{2}c(\theta) \quad a.s. P_\theta, \tag{2.5}$$

then we say the sequence $\{T_n\}$ has exact slope $c(\theta)$ when θ obtains.

The following proposition describes a useful method of finding the exact slope of a given sequence $\{T_n\}$ for which (2.4) holds; see Bahadur (1971) for a detailed proof.

Proposition 1. *Suppose that*

$$\lim_{n\to\infty} n^{-1/2} T_n(X_1, \cdots, X_n) = b(\theta) \quad a.s. P_\theta$$

for each $\theta \in \Omega_1 = \Omega \setminus \Omega_0$*, where* $-\infty < b(\theta) < \infty$*, and that*

$$\lim_{n\to\infty} n^{-1} \log\{1 - F_n(n^{1/2}t)\} = -f(t)$$

for each t *in an open interval* A*, where* f *is a continuous function on* A*, and* $\{b(\theta) : \theta \in \Omega_1\} \subset A$*. Then (2.5) holds with* $c(\theta) = 2f(b(\theta))$ *for each* $\theta \in \Omega_1$*.*

Now we are ready to give our results on Bahadur efficiency. Let $\Omega_0 = \{F : 1 - F \in RV_{\alpha_0}\}$, $\Omega_1 = \{F : 1 - F \in RV_\alpha, \alpha > 0, \alpha \neq \alpha_0\}$ and $\Omega = \Omega_0 \cup \Omega_1$, where $\alpha_0 > 0$ is given. Then we have the following result.

Theorem 2. *Suppose* $k = k(n) \to \infty$, $k/n \to 0$ *and* $k/\log\log n \to \infty$ *as* $n \to \infty$*. Then*

$$\lim_{n\to\infty} \sup_{r>1} |\mathit{KS}(r; \alpha_0)| = |(\frac{\alpha_0}{\alpha})^{-\frac{\alpha}{\alpha_0-\alpha}} - (\frac{\alpha_0}{\alpha})^{-\frac{\alpha_0}{\alpha_0-\alpha}}| \quad \textit{a.s.} \tag{2.6}$$

for each $F \in \Omega_1$ *and that, for* $a \in (0, 1)$*,*

$$\lim_{n\to\infty} k^{-1} \log P_{\Omega_0}\{\sup_{r>1} |\mathit{KS}(r; \alpha_0)| \geq a\} = -f(a), \tag{2.7}$$

where $f(a) = \inf\{f_1(a,t) : 0 \le t \le 1\}$ *and*

$$f_1(a,t) = \begin{cases} (a+t)\log\frac{a+t}{t} + (1-a-t)\log\frac{1-a-t}{1-t} & \text{for } 0 \le t \le 1-a, \\ \infty & \text{for } t > 1-a. \end{cases}$$

Remark 2. *Using the same arguments as in Berk and Jones (1979), we conclude by Theorem 2 that* $\sup_{x>1}|kBJ(r;\alpha_0)|$ *is more Bahadur efficient than* $\sup_{r>1}|\sqrt{k}KS(r;\alpha_0)|$ *for testing* $H_0 : F \in \Omega_0$ *against* $H_a : F \in \Omega_1$.

Remark 3. *It would be interesting to show that (2.6) holds for a larger* Ω_1. *If we replace* $\sup_{r>1}|\sqrt{k}KS(r;\alpha_0)|$ *by* $\sup_{r>b}|\sqrt{k}KS(r;\alpha_0)|$, *where* $b > 1$ *is given, then* Ω_1 *includes all distributions* F *such that* $\lim_{t\to\infty}\sup_{x>b}\frac{1-F(tx)}{1-F(t)} = 0$. *In this case, the left hand side of (2.6) becomes* $\sup_{x>b} x^{-\alpha_0} = b^{-\alpha_0}$.

2.3 The KSI, BJI and SCI quadratic tests

The KS, BJ and SC tests introduced in paragraph 2.1 are supremum tests, which are based on statistics obtained by taking the supremum over some underlying goodness-of-fit process. Quadratic tests are popular alternatives to supremum tests, and are based on statistics obtained by integrating the squared goodness-of-fit process with respect to some appropriate measure. In this subsection we consider "quadratic" variants of the KS, BJ and SC tests.

The Cramer-von Mises test statistic KSI is defined by

$$\text{KSI} = \int_1^\infty \{\sqrt{k}\text{KS}(r;\hat{\alpha})\}^2 \, dG(r;\hat{\alpha})$$

As in Einmahl and McKeague (2003), see also Wellner and Koltchinskii (2003), define the integrated BJ test statistic BJI by

$$\text{BJI} = \int_1^\infty k\text{BJ}(r;\hat{\alpha}) \, dG(r;\hat{\alpha})$$

Finally, define the integrated score test SCI by

$$\text{SCI} = \int_1^\infty \{\sqrt{k}\text{SC}(r;\hat{\alpha})\}^2 \, dG(r;\hat{\alpha}).$$

The following theorem gives the limiting distributions of these three test statistics.

Theorem 3. *Suppose* F *satisfies (2.3) and* $k = k(n) \to \infty$, $k/n \to 0$, $\sqrt{k}A(n/k) \to 0$ *as* $n \to \infty$. *Then*

$$KSI \xrightarrow{d} \int_0^1 \{W(x) - xW(1) + x\log(x)\int_0^1 \frac{W(s) - sW(1)}{s}\, ds\}^2 \, dx, \qquad (2.8)$$

$$BJI \xrightarrow{d} \int_0^1 \frac{\{W(x) - xW(1) + x\log(x)\int_0^1 \frac{W(s)-sW(1)}{s}\,ds\}^2}{x(1-x)}\,dx \tag{2.9}$$

and

$$SCI \xrightarrow{d} \int_0^1 \{W(x) - xW(1) - \int_0^x \frac{W(s)-sW(1)}{s}\,ds + x\int_0^1 \frac{W(s)-sW(1)}{s}\,ds\}^2\,dx, \tag{2.10}$$

where $W(x)$ and $\hat{\alpha}$ are defined in Theorem 1.

3 Simulation study and real applications

3.1 Simulation study

First we simulate 100,000 random samples from Frechet distribution $F(x) = \exp\{-x^{-1}\}$ with sample size $n = 1000$, and then compute the test statistics KS $= \sup_{r>1} |\sqrt{k}\text{KS}(r;\hat{\alpha})|$, BJ $= \sup_{r>1} k\text{BJ}(r;\hat{\alpha})$, SC $= \sup_{r>1} |\sqrt{k}SC(r;\hat{\alpha})|$, KSI, BJI and SCI for $k = 20, 30, \cdots, 200$. Based on these computed test statistics, we obtain the 0.95 level critical values; see Table 1.

Using Table 1, we compute the powers of these six test statistics by simulating 10,000 random samples from distributions $1 - F(x) = \{1 + \frac{\alpha \log x}{\delta^{-1}}\}^{-\delta^{-1}}$, ($x > 1$), with sample size $n = 1000$, $\alpha = 1$, $k = 20, 30, \cdots, 200$ and $\delta\sqrt{k} = 0.1, 0.5, \cdots, 6$; see Tables 2–7. Since the limiting distribution of BJ does not exist, we compare the other five tests by simulating critical values from their corresponding limiting distributions. More specifically, we first simulated 100,000 random samples of Wiener Processes on $[0, 1]$ with 1000 equally spaced grid points, and then compute the limitings of the test statistics. The critical values with level 0.95 are 1.338, 0.456, 1.076, 0.220 and 1.313 for SC, SCI, KS, KSI and BJI, respectively. Using these numbers, we compute the powers of those five tests as above, which are reported in Tables 8–12.

We summarize our observations from Tables 2–12 as follows:

1. When the exact critical values are employed, the SCI test is most powerful for most values of k and δ. And the BJ test is more powerful than the KS test, which coincides with the Bahadur efficiency study in section 2.

2. When the asymptotic critical values are employed, the SCI test is still most powerful for most values of k and δ. But the BJI test is less powerful than the KSI test. This seems contradicting Einmahl and McKeague (2003), where an integrated empirical likelihood test is more powerful than the Cramer-von Mises test. Note that the Berk-Jones test is an empirical likelihood

test when testing distributions. Since the limiting of $k\text{BJ}(r;\hat{\alpha})$ is not a chi-squared distribution (see Remark 1), the BJI test is not an exact integrated empirical likelihood test. This is a difference of Berk-Jones test in testing distribution and tails.

3. The tests with asymptotic critical values are comparable to the corresponding tests with exact critical values when k is small, but are more powerful when k is large. This may be because the asymptotic critical values are obtained from the limiting without taking the bias, introduced by a large k, into account.

3.2 Real applications.

The first data set we analyzed consists of 2156 Danish fire loss over one million Danish krone from the years 1980 to 1990 inclusive (see Figure 1). The loss figure is a total loss for the event concerned and includes damage to buildings, furnishings and personal property as well as loss of profits. This Danish fire loss data set was analyzed by McNeil (1997) and Resnick (1997b), where the right tail index was confirmed to be between 1 and 2. Here we apply the five tests SC, SCI, KS, KSI and BJI to test whether this data set has a heavy tailed distribution. The critical values obtained from the limiting distributions were employed to compute the test statistics for $k = 20, 21, \cdots, 2000$ by step 1; see Figure 2. The Hill estimators are plotted against k in Figure 2 as well. For a large range of k, all tests do not reject the heavy tailed hypothesis. However, there is a bit strange pattern around $k = 150$.

Ideally, when the heavy tailed hypothesis is true, tests should not reject the null hypothesis for small values of k, and reject it for large values of k since the critical values are obtained by ignoring the bias. Moreover there should be one region where decision varies due to the fact that the bias and variance are comparable in this range of k. We suspect that this strange pattern in Figure 2 may be due to weak dependence inside the danish fire loss data. Next, we use the first 2150 data points and divide into 215 blocks with 10 points in each block. Then we apply those five tests to the maxima in these blocks, see Figure 3. This figure clear shows the ideal pattern. So, it would be very interesting to investigate how the bias, i.e., $\lim \sqrt{k}A(n/k) \neq 0$, affects the tests and to propose tests for heavy tailed time series.

The second data set we analyzed is the internet traffic data. The Ethernet series used here are part of a data set collected at Bellcore in August of 1989. They correspond to one ”normal” hour’s worth of traffic, collected every 10 milliseconds, thus resulting in a length of 360,000. This data set measures the number of bytes per unit time; see Figure 4. This data set was first analyzed in Leland et al.

(1994). We apply those five tests to this data set, which clearly reject the heavy tailed hypothesis; see Figure 5. Next we divide this data set into 600 blocks with 600 points in each block, and apply those five tests the maxima in each blocks; see Figure 6. Figure 6 still rejects the heavy tailed hypothesis. This may be due to the long-range dependence of the internet traffic data. How to test heavy tail distributions for long-range dependent data is quite interesting both theoretically and practically since there exist debates on fitting a heavy tailed distribution or log-normal distribution to internet traffic data.

4 Proofs

Proof of Theorem 1. Let $Y_1, \cdots, Y_n$ be i.i.d. random variables with distribution $1 - y^{-1}$, $(y > 1)$, and $Y_{n,1} \le \cdots \le Y_{n,n}$ denote the order statistics of $Y_1, \cdots, Y_n$. Then, it follows from de Haan and Resnick (1998) that

$$\sqrt{k}\{\frac{1}{k}\sum_{i=1}^{n} I(Y_i > \frac{n}{kx}) - x\} \xrightarrow{d} W(x),$$

$$\sqrt{k}\{\frac{k}{n}Y_{n,n-[kx]} - x^{-1}\} \xrightarrow{d} x^{-2}W(x),$$

$$\sqrt{k}\{\frac{1}{k}\sum_{i=1}^{n} I(X_i > xX_{n,n-k}) - x^{-\alpha}\} \xrightarrow{d} W(x^{-\alpha}) - x^{-\alpha}W(1),$$

$$\sqrt{k}\{\hat{\alpha} - \alpha\} \xrightarrow{d} -\alpha^2 \int_1^\infty s^{-1}W(s^{-\alpha})\,ds + \alpha W(1) \tag{4.1}$$

in $D[1, \infty)$, where $W(x)$ is a Wiener process. So

$$\begin{aligned} &\sqrt{k}KS(x;\hat{\alpha}) \\ &= \sqrt{k}\{\frac{1}{k}\sum_{i=1}^{n} I(X_i > xX_{n,n-k}) - x^{-\alpha}\} \\ &+\sqrt{k}\{x^{-\alpha} - x^{-\hat{\alpha}}\} \\ &\xrightarrow{d} W(x^{-\alpha}) - x^{-\alpha}W(1) - x^{-\alpha}\log x\{\alpha^2\int_1^\infty s^{-1}W(s^{-\alpha})\,ds - \alpha W(1)\} \end{aligned} \tag{4.2}$$

in $D[1, \infty)$. Hence

$$\sup_{x>1} |\sqrt{k}KS(x;\hat{\alpha})| \xrightarrow{d} \sup_{0<x<1} |W(x) - xW(1) + x\log(x)\int_0^1 \frac{W(s) - sW(1)}{s}\,ds|.$$

Similarly, we have

$$\begin{aligned}
&\sqrt{k}SC(x;\hat{\alpha})\\
&= -\sqrt{k}\{\frac{1}{k}\sum_{i=1}^{n} I(X_i > xX_{n,n-k}) - x^{-\alpha}\}\\
&-\hat{\alpha}\int_1^x s^{-1}\sqrt{k}\{\frac{1}{k}\sum_{i=1}^{n} I(X_i > xX_{n,n-k}) - s^{-\alpha}\}\,ds\\
&-\sqrt{k}\frac{\hat{\alpha}-\alpha}{\alpha}\{x^{-\alpha}-1\}\\
&\xrightarrow{d} -W(x^{-\alpha}) + x^{-\alpha}W(1) + \alpha\int_x^\infty \frac{W(s^{-\alpha}) - s^{-\alpha}W(1)}{s}\,ds\\
&\quad -\alpha x^{-\alpha}\int_1^\infty \frac{W(s^{-\alpha} - s^{-\alpha}W(1)}{s}\,ds \qquad (4.3)
\end{aligned}$$

in $D[1,\infty)$. Hence

$$\begin{aligned}
&\sup_{x>1}|\sqrt{k}SC(x;\hat{\alpha})|\\
&\xrightarrow{d} \sup_{0<x<1}|W(x) - xW(1) - \int_0^x \frac{W(s)-sW(1)}{s}\,ds + x\int_0^1 \frac{W(s)-sW(1)}{s}\,ds|.
\end{aligned}$$

Before we prove theorem 2, we give the following Lemma on large deviations for tail processes, which may be of independent interest. Note that Cheng (1992) gave large deviations results for Hill's estimator.

Lemma 1. Under the condition of Theorem 2, we have, for $0 < a < 1$,

$$k^{-1}\log P_{\Omega_0}(\sup_{x>1} KS(x;\alpha_0) \ge a) \to -f(a),$$

$$k^{-1}\log P_{\Omega_0}(\sup_{x>1}\{-KS(x;\alpha_0)\} \ge a) \to -f(a)$$

and

$$k^{-1}\log P_{\Omega_0}(\sup_{x>1}|KS(x;\alpha_0)| \ge a) \to -f(a).$$

Proof. Here we only show the first limit. The other two can be shown in a similar way as proving the first limit and Example 5.3 of Bahadur (1971). Let $U_1,\cdots,U_n$ be i.i.d. random variables with uniform distribution on $[0,1]$ and $U_{n,1}\le\cdots\le U_{n,n}$ denote the order statistics of $U_1,\cdots,U_n$. Put $G_{n,k+1}(u) = P(U_{n,k+1}\le u)$. By Potters' inequality (see Geluk and de Haan (1987)), for any $\epsilon > 0$, there exists $u_0 > 0$ such that, for all $0 < u \le u_0$ and $0 < s \le 1$,

$$(1-\epsilon)s^{-1/\alpha_0+\epsilon} \le (1-F)^-(us)/(1-F)^-(u) \le (1+\epsilon)s^{-1/\alpha_0-\epsilon}. \qquad (4.4)$$

Write

$$\begin{aligned}
&P_{\Omega_0}(\sup_{x>1} KS(x;\alpha_0) \geq a) \\
&= P_{\Omega_0}(\sup_{x>1}\{\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(U_{n,k+1})} > x) - x^{-\alpha_0}\} \geq a) \\
&= \int_0^{u_0} P_{\Omega_0}(\sup_{x>1}\{\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > x) - x^{-\alpha_0}\} \geq a | U_{n,k+1} = u)\, dG_{n,k+1}(u) \\
&+ \int_{u_0}^{1} P_{\Omega_0}(\sup_{x>1}\{\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > x) - x^{-\alpha_0}\} \geq a | U_{n,k+1} = u)\, dG_{n,k+1}(u) \\
&= I_1 + I_2. \qquad (4.5)
\end{aligned}$$

Let $V_1, \cdots, V_k$ be i.i.d. random variables with uniform distribution on $[0,1]$. It follows from (4.4) that, for any $x > 1$ and $\epsilon \in (0,1)$,

$$\begin{aligned}
I_1 &\geq \int_0^{u_0} P_{\Omega_0}(\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > x) - x^{-\alpha_0} \geq a | U_{n,k+1} = u)\, dG_{n,k+1}(u) \\
&\geq \int_0^{u_0} P_{\Omega_0}(\frac{1}{k}\sum_{i=1}^{k} I((1-\epsilon)(\frac{U_{n,i}}{u})^{-1/\alpha_0+\epsilon} > x) - x^{-\alpha_0} \geq a | U_{n,k+1} = u)\, dG_{n,k+1}(u) \\
&\geq \int_0^{u_0} P_{\Omega_0}(\frac{1}{k}\sum_{i=1}^{k} I(\frac{U_{n,i}}{u} < (\frac{x}{1-\epsilon})^{-\frac{\alpha_0}{1-\alpha_0\epsilon}}) - x^{-\alpha_0} - a \geq 0 | U_{n,k+1} = u)\, dG_{n,k+1}(u) \\
&= \int_0^{u_0} P_{\Omega_0}(\frac{1}{k}\sum_{i=1}^{k} I(V_i < (\frac{x}{1-\epsilon})^{-\frac{\alpha_0}{1-\alpha_0\epsilon}}) - x^{-\alpha_0} - a \geq 0)\, dG_{n,k+1}(u).
\end{aligned}$$

By Theorem 3.1 of Bahadur (1971),

$$\begin{aligned}
&\liminf_{n\to\infty} k^{-1}\log(I_1) \\
&\geq \liminf_{n\to\infty} k^{-1}\log P_{\Omega_0}(\frac{1}{k}\sum_{i=1}^{k} I(V_i < (\frac{x}{1-\epsilon})^{-\frac{\alpha_0}{1-\alpha_0\epsilon}}) - x^{-\alpha_0} - a \geq 0) \\
&+ \liminf_{n\to\infty} k^{-1}\log G_{n,k+1}(u_0) \\
&\geq -f_1(x^{-\alpha_0} - x^{-\frac{\alpha_0}{1-\alpha_0\epsilon}} + a, (\frac{x}{1+\epsilon})^{-\frac{\alpha_0}{1-\alpha_0\epsilon}}) + \liminf_{n\to\infty} k^{-1}\log G_{n,k+1}(u_0).
\end{aligned}$$

Let $\epsilon \to 0$ and take infinimum over $x > 1$, we get

$$\liminf_{n\to\infty} k^{-1}\log(I_1) \geq -f(a) + \liminf_{n\to\infty} k^{-1}\log G_{n,k+1}(u_0).$$

Since $U_{n,k+1} \xrightarrow{p} 0$, i.e., $G_{n,k+1}(u_0) \to 1$, we have

$$\liminf_{n\to\infty} k^{-1} \log P_{\Omega_0}(\sup_{x>1} KS(x;\alpha_0) \geq a) \geq \liminf_{n\to\infty} k^{-1} \log(I_1) \geq -f(a). \quad (4.6)$$

For any positive integer m, define $\Delta(m) = \inf\{\frac{(j-1)^{\alpha_0}-j^{\alpha_0}}{m^{\alpha_0}} : j = 1, \cdots, m\}$. Then $\Delta(m) \to 0$ as $m \to \infty$. Since

$$\sup_{m/j \leq x \leq m/(j-1)} \{\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > x) - x^{-\alpha_0}\}$$
$$\leq \frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > \frac{m}{j}) - (\frac{m}{j-1})^{-\alpha_0}$$

for $j = 1, \cdots, m$, we have, for $0 < u \leq u_0$

$$P(\sup_{x>1}\{\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > x) - x^{-\alpha_0}\} \geq a | U_{n,k+1} = u)$$
$$\leq \sum_{j=1}^{m} P(\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > \frac{m}{j}) - (\frac{m}{j-1})^{-\alpha_0} \geq a | U_{n,k+1} = u)$$
$$\leq \sum_{j=1}^{m} P(\frac{1}{k}\sum_{i=1}^{k} I(\frac{U_{n,i}}{u} < (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}}) - (\frac{m}{j-1})^{-\alpha_0} - a \geq 0 | U_{n,k+1} = u)$$
$$= \sum_{j=1}^{m} P(\frac{1}{k}\sum_{i=1}^{k} I(V_i < (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}}) - (\frac{m}{j-1})^{-\alpha_0} - a \geq 0).$$

Choose m large enough such that $0 < (\frac{m}{j-1})^{-\alpha_0} - (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}} < 1$ for all $0 < \epsilon < 1$ and $j = 1, \cdots, m$. By Theorem 2.1 of Bahadur (1971), we have

$$P(\frac{1}{k}\sum_{i=1}^{k} I(V_i < (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}}) - (\frac{m}{j-1})^{-\alpha_0} - a \geq 0)$$
$$\leq \exp\{-kf_1((\frac{m}{j-1})^{-\alpha_0} - (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}} + a, (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}})\},$$

i.e.,

$$P(\sup_{x>1}\{\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > x) - x^{-\alpha_0}\} \geq a | U_{n,k+1} = u)$$
$$\leq \sum_{j=1}^{m} \exp\{-kf_1((\frac{m}{j-1})^{-\alpha_0} - (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}} + a, (\frac{m}{j(1+\epsilon)})^{-\frac{\alpha_0}{1+\alpha_0\epsilon}})\}.$$

Letting $\epsilon \to 0$, we obtain that

$$
\begin{aligned}
&P(\sup_{x>1}\{\frac{1}{k}\sum_{i=1}^{k} I(\frac{(1-F)^{-}(U_{n,i})}{(1-F)^{-}(u)} > x) - x^{-\alpha_0}\} \geq a | U_{n,k+1} = u) \\
&\leq \sum_{j=1}^{m} \exp\{-kf_1((\frac{m}{j-1})^{-\alpha_0} - (\frac{m}{j})^{-\alpha_0} + a, (\frac{m}{j})^{-\alpha_0})\} \\
&\leq m \exp\{-kf_1(a + \Delta(m), (\frac{m}{j})^{-\alpha_0})\} \\
&\leq m \exp\{-kf(a + \Delta(m))\},
\end{aligned}
$$

i.e.,

$$
I_1 \leq m \exp\{-kf(a + \Delta(m))\}. \tag{4.7}
$$

By Hoeffding's inequality (Hoeffding (1963)) we have

$$
\begin{aligned}
&P(U_{n,k+1} > u_0) \\
&\quad = P(\sum_{i=1}^{n} I(U_i > u_0) \geq n - k - 1) \\
&\quad \leq \exp\{-2n(u_0 - \frac{k+1}{n})^2\}.
\end{aligned}
$$

Hence

$$
\limsup_{n\to\infty} \frac{1}{k} P(U_{n,k+1} > u_0) m^{-1} \exp\{kf(a + \Delta(m))\} = 0,
$$

i.e,

$$
\limsup_{n\to\infty} k^{-1} \log(1 + I_2/I_1) \leq \limsup_{n\to\infty} k^{-1} \log\{1 + (1 - G_{n,k+1}(u_0))/I_1\} = 0. \tag{4.8}
$$

By (4.7) and (4.8),

$$
\limsup_{n\to\infty} k^{-1} \log P_{\Omega_0}(\sup_{x>1} KS(x;\alpha_0) \geq a) \leq -f(a). \tag{4.9}
$$

Hence, it follows from (4.6) and (4.9) that

$$
\lim_{n\to\infty} k^{-1} \log P_{\Omega_0}(\sup_{x>1} KS(x;\alpha_0) \geq a) = -f(a).
$$

Proof of Theorem 2. (2.6) follows from the fact that

$$
\lim_{n\to\infty} \sup_{x>1} |KS(x;\alpha_0)| = \sup_{x>1} |x^{-\alpha} - x^{-\alpha_0}| \text{ a.s.}
$$

for $F \in \Omega_1$, and (2.7) follows from Lemma 1.

Proof of Theorem 3. Obviously, (2.8) and (2.10) follow from (4.2) and (4.3), respectively. To prove (2.9), we follow the lines of Einmahl and McKeague (2003). For $\epsilon \in (0,1)$, write

$$\begin{aligned}
&-\int_1^\infty BJ(x;\hat\alpha)\,dx^{-\hat\alpha}\\
&=\int_0^\epsilon BJ(x^{-1/\hat\alpha};\hat\alpha)\,dx+\int_\epsilon^{1-\epsilon} BJ(x^{-1/\hat\alpha};\hat\alpha)\,dx+\int_{1-\epsilon}^1 BJ(x^{-1/\hat\alpha};\hat\alpha)\,dx\\
&=I_1+I_2+I_3.
\end{aligned}$$

By (4.1), (4.2) and Taylor expansions, we can show that

$$I_2 \xrightarrow{d} \int_\epsilon^{1-\epsilon} \frac{\{W(x)-xW(1)+x\log(x)\int_0^1 \frac{W(s)-sW(1)}{s}\,ds\}^2}{x(1-x)}\,dx \tag{4.10}$$

for any $\epsilon \in (0,1)$. Thus, we only need to show that, as $n\to\infty$,

$$I_1 = O_p(\sqrt{\epsilon}) \quad \text{and} \quad I_3 = O_p(\sqrt{\epsilon}) \tag{4.11}$$

uniformly in $\epsilon \in (0,1/2)$. Put $\Delta_n(x) = \frac{1}{k}\sum_{i=1}^n I(X_i > xX_{n,n-k})$. Write I_1 as $II_1+\cdots+II_7$, where

$$II_1 = -\int_0^{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha}} k\log(1-x)\,dx$$

$$\begin{aligned}
II_2 = -\int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha}}^{\epsilon} & I(x\ge\Delta_n(x^{-1/\hat\alpha}))2k\{1-\Delta_n(x^{-1/\hat\alpha})\}\\
&\times\{\log(1+\frac{x-\Delta_n(x^{-1/\hat\alpha})}{1-x})-\frac{x-\Delta_n(x^{-1/\hat\alpha})}{1-x}\}\,dx,
\end{aligned}$$

$$\begin{aligned}
II_3 = \int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha_n}}^{\epsilon} & I(x<\Delta_n(x^{-1/\hat\alpha}))2k\{1-\Delta_n(x^{-1/\hat\alpha})\}\\
&\times\{\log(1+\frac{\Delta_n(x^{-1/\hat\alpha})-x}{1-\Delta_n(x^{-1/\hat\alpha})})-\frac{\Delta_n(x^{-1/\hat\alpha_n})-x}{1-\Delta_n(x^{-1/\hat\alpha})}\}\,dx,
\end{aligned}$$

$$\begin{aligned}
II_4 = \int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha}}^{\epsilon} & I(x\ge\Delta_n(x^{-1/\hat\alpha}))2k\Delta_n(x^{-1/\hat\alpha})\\
&\times\{\log(1+\frac{x-\Delta_n(x^{-1/\hat\alpha})}{\Delta_n(x^{-1/\hat\alpha})})-\frac{x-\Delta_n(x^{-1/\hat\alpha})}{\Delta_n(x^{-1/\hat\alpha})}\}\,dx,
\end{aligned}$$

$$II_5 = -\int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha}}^{\epsilon} I(x < \Delta_n(x^{-1/\hat\alpha}))2k\Delta_n(x^{-1/\hat\alpha}) \times\{\log(1+\frac{\Delta_n(x^{-1/\hat\alpha})-x}{x}) - \frac{\Delta_n(x^{-1/\hat\alpha})-x}{x}\}\,dx,$$

$$II_6 = -\int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha}}^{\epsilon} I(x \ge \Delta_n(x^{-1/\hat\alpha})2k\frac{(x-\Delta_n(x^{-1/\hat\alpha}))^2}{1-x}\,dx,$$

$$II_7 = -\int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha}}^{\epsilon} I(x < \Delta_n(x^{-1/\hat\alpha}))2k\frac{(\Delta_n(x^{-1/\hat\alpha})-x)^2}{x}\,dx.$$

Let $U_1,\cdots,U_n$ be i.i.d. random variables with uniform distribution on $[0,1]$, and $U_{n,1}\le\cdots\le U_{n,n}$ denote the order statistics. It follows from Einmahl (1997) that there exists a sequence of Wiener processes W_n such that for any $\delta>0$

$$\sup_{t>0} t^{-1/2}e^{-\delta|\log t|}|\sqrt{k}\{\frac{1}{k}\sum_{i=1}^{n} I(U_i \le \frac{k}{n}t) - t\} - W_n(t)| \xrightarrow{p} 0 \tag{4.12}$$

as $n\to\infty$, where $k=k(n)\to\infty$ and $k/n\to 0$ as $n\to\infty$. Using (4.12) and similar arguments in Drees et al. (2004), we can show that, for any $\delta>0$,

$$\sup_{0<x\le\frac{1}{2}} x^{-1/2+\delta}|\sqrt{k}\{\Delta_n(x^{-1/\hat\alpha})-x\} - W_n(x) + xW_n(1) +x\log(x)\int_0^1 \frac{W_n(s)-sW_n(1)}{s}\,ds - \sqrt{k}A(n/k)\alpha x\frac{x^{-\rho}-1}{\rho}| \xrightarrow{p} 0 \tag{4.13}$$

as $n\to\infty$. Note that

$$|\log(1+y)-y| \le 2y^2 \text{ for } y\ge 0. \tag{4.14}$$

It is easy to check that for any $\delta_1>0$ and $\delta_2>0$

$$P(k^{-1-\delta_1} \le (\frac{X_{n,n}}{X_{n,n-k}})^{-\hat\alpha} \le k^{-1+\delta_2}) \to 1 \tag{4.15}$$

and

$$P(k^{-1-\delta_1} \le 1-(\frac{X_{n,n-k+1}}{X_{n,n-k}})^{-\hat\alpha} \le k^{-1+\delta_2}) \to 1 \tag{4.16}$$

as $n\to\infty$. By (4.15), we have

$$II_1 = O_p(\sqrt{\epsilon}) \text{ uniformly in } \epsilon\in(0,1/2).$$

Using (4.13), (4.14) and (4.15), we have

$$
\begin{aligned}
|II_5| &\leq \int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat{\alpha}}}^{\epsilon} 4k\{1-\Delta_n(x^{-1/\hat{\alpha}})\}\{\frac{x-\Delta_n(x^{-1/\hat{\alpha}})}{x}\}^2\,dx \\
&= \int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat{\alpha}}}^{\epsilon} 4k\frac{(\Delta_n(x^{-1/\hat{\alpha}})-x)^3}{x^2}\,dx \\
&\quad + \int_{\epsilon\wedge(\frac{X_{n,n}}{X_{n,n-k}})^{-\hat{\alpha}}}^{\epsilon} 4k\frac{(\Delta_n(x^{-1/\hat{\alpha}})-x)^2}{x}\,dx \\
&= O_p(I(\epsilon > k^{-1-\delta_1})\int_0^{\epsilon}\frac{1}{\sqrt{k}}\frac{(x^{1/2-\delta})^3}{x^2}\,dx) \\
&\quad + O_p(\int_0^{\epsilon}\frac{(x^{1/2-\delta})^2}{x}\,dx) \text{ uniformly in } \epsilon\in(0,1/2) \\
&= O_p(\sqrt{\epsilon}) \text{ uniformly in } \epsilon\in(0,1/2).
\end{aligned}
$$

Similarly, we can show that

$$II_i = O_p(\sqrt{\epsilon}) \text{ uniformly in } \epsilon\in(0,1/2)$$

for $i = 2, 3, 4, 6, 7$. Hence

$$I_1 = O_p(\sqrt{\epsilon}) \text{ uniformly in } \epsilon\in(0,1/2).$$

Similarly, we can show that

$$I_3 = O_p(\sqrt{\epsilon}) \text{ uniformly in } \epsilon\in(0,1/2).$$

Thus, the theorem follows.

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Table 1: 0.95-level critical values based on test statistics

k	SC	SCI	KS	KSI	BJ	BJI
20	1.335	0.436	1.047	0.210	6.865	1.133
30	1.340	0.448	1.057	0.215	7.307	1.198
40	1.339	0.448	1.062	0.217	7.575	1.230
50	1.343	0.453	1.066	0.218	7.734	1.257
60	1.344	0.453	1.070	0.219	7.879	1.267
70	1.346	0.454	1.071	0.220	8.042	1.278
80	1.353	0.458	1.077	0.222	8.166	1.297
90	1.357	0.460	1.079	0.224	8.249	1.306
100	1.358	0.464	1.079	0.223	8.348	1.310
110	1.361	0.466	1.082	0.223	8.425	1.320
120	1.369	0.470	1.086	0.226	8.499	1.338
130	1.375	0.473	1.090	0.228	8.586	1.347
140	1.378	0.478	1.094	0.229	8.607	1.366
150	1.386	0.483	1.097	0.232	8.644	1.383
160	1.392	0.489	1.100	0.234	8.746	1.394
170	1.340	0.496	1.106	0.237	8.776	1.416
180	1.409	0.509	1.114	0.241	8.891	1.441
190	1.421	0.516	1.119	0.245	8.943	1.459
200	1.423	0.528	1.125	0.248	9.019	1.484

Table 2: Powers for SC with level 0.95 and exact critical values given in Table 1.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0498	0.0621	0.1119	0.2009	0.3057	0.4179	0.5219	0.6992	0.8179	0.8912
30	0.0478	0.0587	0.1166	0.2138	0.3338	0.4535	0.5693	0.7523	0.8663	0.9298
40	0.0496	0.0645	0.1183	0.2229	0.3481	0.4789	0.5955	0.7829	0.8945	0.9552
50	0.0458	0.0617	0.1196	0.2258	0.3525	0.4882	0.6096	0.7991	0.9098	0.9635
60	0.0499	0.0638	0.1205	0.2215	0.3582	0.4997	0.6220	0.8144	0.9218	0.9707
70	0.0460	0.0608	0.1207	0.2198	0.3563	0.5006	0.6298	0.8210	0.9273	0.9760
80	0.0448	0.0604	0.1196	0.2141	0.3524	0.5002	0.6319	0.8287	0.9344	0.9792
90	0.0438	0.0615	0.1180	0.2172	0.3555	0.5044	0.6389	0.8350	0.9391	0.9803
100	0.0460	0.0631	0.1195	0.2178	0.3520	0.5033	0.6421	0.8401	0.9425	0.9805
110	0.0442	0.0652	0.1169	0.2164	0.3531	0.5028	0.6392	0.8414	0.9462	0.9826
120	0.0431	0.0598	0.1132	0.2148	0.3466	0.4953	0.6361	0.8417	0.9472	0.9833
130	0.0429	0.0588	0.1121	0.2054	0.3404	0.4945	0.6343	0.8423	0.9497	0.9845
140	0.0398	0.0553	0.1092	0.2058	0.3402	0.4907	0.6354	0.8449	0.9511	0.9861
150	0.0396	0.0529	0.1030	0.2043	0.3369	0.4818	0.6338	0.8434	0.9503	0.9866
160	0.0374	0.0533	0.1053	0.1996	0.3288	0.4785	0.6324	0.8441	0.9490	0.9878
170	0.0373	0.0528	0.1019	0.1931	0.3252	0.4741	0.6223	0.8443	0.9482	0.9879
180	0.0362	0.0502	0.0982	0.1883	0.3151	0.4643	0.6168	0.8421	0.9496	0.9877
190	0.0349	0.0468	0.0918	0.1801	0.3087	0.4576	0.6104	0.8391	0.9469	0.9870
200	0.0336	0.0446	0.0904	0.1777	0.3033	0.4471	0.6041	0.8366	0.9473	0.9873

Table 3: Powers for SCI with level 0.95 and exact critical values given in Table 1.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0517	0.0898	0.1762	0.2864	0.4077	0.5258	0.6299	0.7840	0.8757	0.9297
30	0.0512	0.0854	0.1718	0.2924	0.4270	0.5524	0.6657	0.8214	0.9123	0.9576
40	0.0552	0.0862	0.1717	0.2984	0.4406	0.5725	0.6809	0.8454	0.9330	0.9739
50	0.0526	0.0844	0.1704	0.3021	0.4433	0.5792	0.6911	0.8597	0.9430	0.9793
60	0.0537	0.0824	0.1674	0.2973	0.4475	0.5885	0.7065	0.8700	0.9519	0.9825
70	0.0491	0.0784	0.1648	0.2891	0.4468	0.5891	0.7074	0.8774	0.9583	0.9876
80	0.0520	0.0784	0.1605	0.2860	0.4425	0.5909	0.7168	0.8844	0.9599	0.9882
90	0.0542	0.0791	0.1631	0.2886	0.4423	0.5907	0.7172	0.8901	0.9631	0.9888
100	0.0552	0.0825	0.1577	0.2814	0.4409	0.5895	0.7194	0.8909	0.9640	0.9892
110	0.0515	0.0785	0.1568	0.2809	0.4348	0.5877	0.7221	0.8920	0.9685	0.9912
120	0.0518	0.0782	0.1553	0.2784	0.4297	0.5862	0.7194	0.8949	0.9689	0.9917
130	0.0514	0.0775	0.1494	0.2743	0.4289	0.5819	0.7195	0.8980	0.9702	0.9922
140	0.0466	0.0713	0.1482	0.2723	0.4274	0.5825	0.7187	0.8990	0.9720	0.9928
150	0.0470	0.0699	0.1436	0.2650	0.4205	0.5793	0.7164	0.9001	0.9715	0.9935
160	0.0463	0.0719	0.1463	0.2633	0.4128	0.5755	0.7159	0.8962	0.9738	0.9940
170	0.0432	0.0690	0.1393	0.2586	0.4065	0.5674	0.7136	0.8983	0.9723	0.9942
180	0.0416	0.0636	0.1319	0.2451	0.3997	0.5593	0.7063	0.8941	0.9716	0.9942
190	0.0405	0.0610	0.1270	0.2413	0.3922	0.5518	0.7066	0.8948	0.9720	0.9939
200	0.0370	0.0578	0.1229	0.2393	0.3849	0.5453	0.7034	0.8917	0.9707	0.9949

Table 4: Powers for KS with level 0.95 and exact critical values given in Table 1.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0521	0.0692	0.1290	0.2180	0.3257	0.4307	0.5434	0.7132	0.8318	0.9364
30	0.0510	0.0662	0.1206	0.2125	0.3258	0.4443	0.5639	0.7468	0.8627	0.9299
40	0.0519	0.0699	0.1195	0.2097	0.3267	0.4570	0.5747	0.7636	0.8832	0.9469
50	0.0490	0.0689	0.1220	0.2136	0.3293	0.4558	0.5774	0.7756	0.8944	0.9562
60	0.0537	0.0692	0.1213	0.2105	0.3240	0.4568	0.5890	0.7842	0.9008	0.9649
70	0.0508	0.0664	0.1148	0.2063	0.3188	0.4516	0.5871	0.7904	0.9100	0.9682
80	0.0493	0.0658	0.1126	0.1957	0.3140	0.4487	0.5842	0.7957	0.9156	0.9698
90	0.0480	0.0657	0.1080	0.1972	0.3146	0.4496	0.5827	0.7979	0.9199	0.9716
100	0.0459	0.0626	0.1119	0.1929	0.3154	0.4483	0.5821	0.8000	0.9239	0.9717
110	0.0462	0.0614	0.1120	0.1933	0.3088	0.4431	0.5833	0.7974	0.9214	0.9746
120	0.0480	0.0632	0.1071	0.1879	0.3035	0.4375	0.5768	0.7930	0.9233	0.9763
130	0.0454	0.0605	0.1061	0.1814	0.2967	0.4319	0.5705	0.7965	0.9244	0.9767
140	0.0428	0.0557	0.0993	0.1770	0.2903	0.4238	0.5638	0.7917	0.9241	0.9789
150	0.0416	0.0536	0.0990	0.1771	0.2893	0.4224	0.5610	0.7923	0.9237	0.9771
160	0.0406	0.0535	0.0964	0.1762	0.2844	0.4202	0.5563	0.7941	0.9247	0.9781
170	0.0378	0.0526	0.0939	0.1694	0.2802	0.4072	0.5488	0.7911	0.9245	0.9772
180	0.0380	0.0510	0.0896	0.1618	0.2676	0.4032	0.5439	0.7861	0.9216	0.9764
190	0.0378	0.0494	0.0864	0.1570	0.2666	0.3937	0.5337	0.7835	0.9190	0.9771
200	0.0378	0.0496	0.0852	0.1511	0.2616	0.3925	0.5296	0.7780	0.9189	0.9763

Table 5: Powers for KSI with level 0.95 and exact critical values given in Table 1.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0512	0.0798	0.1529	0.2527	0.3720	0.4893	0.5940	0.7608	0.8649	0.9222
30	0.0520	0.0778	0.1463	0.2519	0.3815	0.5056	0.6254	0.7920	0.8960	0.9493
40	0.0553	0.0777	0.1398	0.2522	0.3832	0.5200	0.6350	0.8132	0.9127	0.9652
50	0.0497	0.0747	0.1397	0.2545	0.3867	0.5204	0.6447	0.8226	0.9257	0.9734
60	0.0535	0.0728	0.1351	0.2486	0.3817	0.5267	0.6521	0.8360	0.9339	0.9766
70	0.0463	0.0696	0.1325	0.2373	0.3782	0.5213	0.6508	0.8428	0.9422	0.9809
80	0.0498	0.0683	0.1291	0.2330	0.3709	0.5196	0.6536	0.8494	0.9439	0.9830
90	0.0504	0.0670	0.1280	0.2341	0.3664	0.5191	0.6524	0.8495	0.9475	0.9830
100	0.0496	0.0710	0.1280	0.2317	0.3611	0.5191	0.6552	0.8557	0.9494	0.9837
110	0.0522	0.0713	0.1299	0.2265	0.3629	0.5140	0.6574	0.8533	0.9531	0.9854
120	0.0497	0.0672	0.1224	0.2239	0.3581	0.5102	0.6538	0.8546	0.9537	0.9873
130	0.0480	0.0648	0.1187	0.2156	0.3519	0.5055	0.6469	0.8528	0.9541	0.9878
140	0.0428	0.0600	0.1164	0.2156	0.3526	0.5002	0.6438	0.8571	0.9551	0.9888
150	0.0428	0.0566	0.1129	0.2106	0.3445	0.4958	0.6456	0.8577	0.9552	0.9889
160	0.0425	0.0599	0.1139	0.2059	0.3388	0.4910	0.6352	0.8523	0.9545	0.9894
170	0.0409	0.0577	0.1119	0.2026	0.3316	0.4824	0.6299	0.8545	0.9563	0.9896
180	0.0416	0.0549	0.1039	0.1951	0.3230	0.4744	0.6241	0.8484	0.9556	0.9886
190	0.0395	0.0525	0.1015	0.1883	0.3157	0.4650	0.6195	0.8493	0.9542	0.9889
200	0.0375	0.0526	0.0983	0.1859	0.3125	0.4629	0.6166	0.8463	0.9526	0.9901

Table 6: Powers for BJ with level 0.95 and exact critical values given in Table 1.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0517	0.0760	0.1371	0.2250	0.3337	0.4364	0.5414	0.7257	0.9268	0.9962
30	0.0531	0.0802	0.1379	0.2256	0.3343	0.4512	0.5602	0.7395	0.8561	0.9428
40	0.0556	0.0807	0.1414	0.2266	0.3403	0.4575	0.5726	0.7573	0.8745	0.9396
50	0.0559	0.0792	0.1421	0.2317	0.3463	0.4663	0.5813	0.7704	0.8885	0.9486
60	0.0603	0.0842	0.1433	0.2301	0.3467	0.4710	0.5875	0.7763	0.8950	0.9544
70	0.0544	0.0787	0.1357	0.2254	0.3393	0.4669	0.5858	0.7803	0.9028	0.9570
80	0.0542	0.0789	0.1362	0.2237	0.3382	0.4664	0.5895	0.7872	0.9070	0.9617
90	0.0533	0.0747	0.1332	0.2225	0.3375	0.4647	0.5830	0.7869	0.9068	0.9642
100	0.0544	0.0761	0.1351	0.2182	0.3367	0.4598	0.5864	0.7887	0.9098	0.9643
110	0.0520	0.0749	0.1310	0.2174	0.3345	0.4605	0.5865	0.7911	0.9130	0.9659
120	0.0566	0.0794	0.1362	0.2207	0.3336	0.4648	0.5875	0.7903	0.9156	0.9691
130	0.0504	0.0726	0.1297	0.2156	0.3303	0.4583	0.5857	0.7885	0.9147	0.9699
140	0.0503	0.0719	0.1290	0.2185	0.3299	0.4627	0.5907	0.7928	0.9192	0.9714
150	0.0509	0.0723	0.1277	0.2160	0.3325	0.4615	0.5897	0.7937	0.9197	0.9732
160	0.0528	0.0722	0.1263	0.2163	0.3293	0.4580	0.5865	0.7919	0.9169	0.9733
170	0.0525	0.0730	0.1295	0.2165	0.3290	0.4566	0.5843	0.7939	0.9209	0.9733
180	0.0510	0.0694	0.1228	0.2079	0.3199	0.4468	0.5804	0.7927	0.9206	0.9735
190	0.0524	0.0712	0.1228	0.2056	0.3180	0.4498	0.5789	0.7934	0.9180	0.9738
200	0.0481	0.0665	0.1186	0.2022	0.3097	0.4429	0.5727	0.7872	0.9182	0.9736

Table 7: Powers for BJI with level 0.95 and exact critical values given in Table 1.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0512	0.0803	0.1534	0.2531	0.3711	0.4896	0.5973	0.7670	0.8664	0.9261
30	0.0521	0.0734	0.1418	0.2433	0.3737	0.4989	0.6156	0.7902	0.8962	0.9487
40	0.0547	0.0733	0.1325	0.2406	0.3731	0.5065	0.6238	0.8054	0.9094	0.9642
50	0.0506	0.0703	0.1310	0.2351	0.3668	0.5016	0.6284	0.8162	0.9201	0.9692
60	0.0524	0.0685	0.1254	0.2300	0.3592	0.5077	0.6353	0.8237	0.9286	0.9746
70	0.0486	0.0656	0.1196	0.2203	0.3541	0.4996	0.6308	0.8282	0.9348	0.9795
80	0.0488	0.0668	0.1179	0.2137	0.3450	0.4978	0.6318	0.8360	0.9372	0.9806
90	0.0487	0.0644	0.1185	0.2148	0.3469	0.4954	0.6314	0.8364	0.9422	0.9811
100	0.0507	0.0663	0.1154	0.2087	0.3378	0.4907	0.6296	0.8384	0.9410	0.9807
110	0.0482	0.0649	0.1166	0.2064	0.3332	0.4843	0.6285	0.8352	0.9458	0.9836
120	0.0466	0.0618	0.1109	0.2018	0.3302	0.4744	0.6243	0.8387	0.9449	0.9854
130	0.0472	0.0598	0.1048	0.1956	0.3263	0.4756	0.6156	0.8397	0.9485	0.9859
140	0.0394	0.0524	0.1000	0.1885	0.3198	0.4664	0.6171	0.8398	0.9480	0.9869
150	0.0395	0.0497	0.0963	0.1873	0.3116	0.4614	0.6129	0.8358	0.9479	0.9863
160	0.0389	0.0535	0.1010	0.1866	0.3059	0.4549	0.6057	0.8348	0.9463	0.9874
170	0.0407	0.0526	0.0969	0.1785	0.2962	0.4460	0.5940	0.8332	0.9466	0.9875
180	0.0371	0.0488	0.0884	0.1669	0.2902	0.4377	0.5881	0.8291	0.9468	0.9857
190	0.0378	0.0456	0.0879	0.1649	0.2846	0.4284	0.5834	0.8284	0.9449	0.9862
200	0.0334	0.0457	0.0843	0.1612	0.2788	0.4238	0.5767	0.8219	0.9428	0.9864

Table 8: Powers for SC with level 0.95 and exact critical values given in Table 1.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0493	0.0580	0.1058	0.1888	0.3015	0.4150	0.5160	0.6934	0.8101	0.8859
30	0.0470	0.0601	0.1130	0.2089	0.3253	0.4526	0.5673	0.7517	0.8679	0.9280
40	0.0501	0.0621	0.1189	0.2149	0.3442	0.4753	0.5947	0.7812	0.8919	0.9525
50	0.0490	0.0635	0.1210	0.2253	0.3527	0.4921	0.6097	0.8025	0.9140	0.9645
60	0.0512	0.0670	0.1246	0.2238	0.3584	0.4942	0.6259	0.8186	0.9234	0.9722
70	0.0526	0.0692	0.1259	0.2317	0.3625	0.5035	0.6398	0.8303	0.9315	0.9773
80	0.0533	0.0683	0.1280	0.2332	0.3664	0.5122	0.6488	0.8392	0.9402	0.9811
90	0.0521	0.0683	0.1276	0.2342	0.3703	0.5162	0.6537	0.8486	0.9440	0.9828
100	0.0537	0.0687	0.1292	0.2370	0.3750	0.5207	0.6561	0.8557	0.9474	0.9844
110	0.0550	0.0709	0.1314	0.2362	0.3796	0.5278	0.6608	0.8595	0.9506	0.9864
120	0.0521	0.0704	0.1321	0.2377	0.3787	0.5246	0.6673	0.8665	0.9524	0.9871
130	0.0528	0.0733	0.1319	0.2336	0.3791	0.5273	0.6693	0.8686	0.9552	0.9882
140	0.0543	0.0721	0.1327	0.2314	0.3764	0.5287	0.6736	0.8699	0.9586	0.9885
150	0.0529	0.0710	0.1317	0.2319	0.3765	0.5293	0.6745	0.8721	0.9619	0.9900
160	0.0533	0.0723	0.1334	0.2325	0.3779	0.5303	0.6760	0.8765	0.9630	0.9908
170	0.0552	0.0725	0.1354	0.2330	0.3771	0.5337	0.6742	0.8775	0.9644	0.9920
180	0.0535	0.0698	0.1353	0.2341	0.3793	0.5341	0.6771	0.8775	0.9661	0.9931
190	0.0555	0.0712	0.1338	0.2357	0.3815	0.5358	0.6777	0.8777	0.9665	0.9938
200	0.0540	0.0740	0.1322	0.2380	0.3792	0.5431	0.6792	0.8826	0.9688	0.9936

Table 9: Powers for SCI with level 0.95 and asymptotic critical value 0.456.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0456	0.0802	0.1568	0.2693	0.3972	0.5077	0.6110	0.7711	0.8651	0.9226
30	0.0503	0.0803	0.1631	0.2812	0.4161	0.5464	0.6530	0.8170	0.9088	0.9546
40	0.0522	0.0804	0.1623	0.2883	0.4310	0.5640	0.6802	0.8429	0.9276	0.9708
50	0.0511	0.0825	0.1665	0.2900	0.4368	0.5774	0.6924	0.8583	0.9442	0.9793
60	0.0530	0.0818	0.1637	0.2933	0.4385	0.5846	0.7038	0.8709	0.9510	0.9846
70	0.0549	0.0822	0.1702	0.2953	0.4425	0.5922	0.7150	0.8814	0.9587	0.9860
80	0.0541	0.0838	0.1690	0.2943	0.4489	0.5964	0.7258	0.8880	0.9625	0.9887
90	0.0511	0.0841	0.1647	0.2941	0.4475	0.5998	0.7297	0.8963	0.9662	0.9900
100	0.0549	0.0837	0.1660	0.2935	0.4514	0.5983	0.7348	0.8989	0.9690	0.9906
110	0.0574	0.0809	0.1624	0.2978	0.4517	0.6050	0.7372	0.9024	0.9697	0.9922
120	0.0553	0.0837	0.1664	0.2969	0.4484	0.6088	0.7389	0.9070	0.9711	0.9930
130	0.0558	0.0874	0.1658	0.2869	0.4517	0.6113	0.7409	0.9083	0.9730	0.9930
140	0.0535	0.0850	0.1635	0.2873	0.4512	0.6069	0.7427	0.9110	0.9773	0.9942
150	0.0545	0.0848	0.1655	0.2877	0.4467	0.6070	0.7460	0.9136	0.9772	0.9945
160	0.0549	0.0806	0.1643	0.2890	0.4470	0.6109	0.7429	0.9144	0.9786	0.9946
170	0.0554	0.0814	0.1639	0.2895	0.4484	0.6070	0.7464	0.9168	0.9803	0.9957
180	0.0546	0.0833	0.1629	0.2918	0.4502	0.6083	0.7451	0.9168	0.9802	0.9961
190	0.0533	0.0827	0.1647	0.2915	0.4522	0.6148	0.7428	0.9196	0.9814	0.9967
200	0.0544	0.0813	0.1620	0.2937	0.4530	0.6118	0.7442	0.9216	0.9822	0.9960

Table 10: Powers for KS with level 0.95 and asymptotic critical value 1.067.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0425	0.0560	0.1046	0.1903	0.2965	0.4049	0.5132	0.6883	0.8136	0.9313
30	0.0414	0.0581	0.1089	0.1976	0.3016	0.4278	0.5437	0.7326	0.8591	0.9230
40	0.0467	0.0628	0.1128	0.1995	0.3084	0.4402	0.5590	0.7511	0.8791	0.9462
50	0.0476	0.0631	0.1146	0.1992	0.3143	0.4416	0.5683	0.7681	0.8949	0.9568
60	0.0481	0.0643	0.1151	0.1968	0.3167	0.4487	0.5746	0.7812	0.9023	0.9639
70	0.0504	0.0652	0.1152	0.2016	0.3220	0.4514	0.5824	0.7901	0.9139	0.9688
80	0.0512	0.0683	0.1148	0.1977	0.3226	0.4558	0.5896	0.7987	0.9203	0.9727
90	0.0508	0.0671	0.1138	0.2001	0.3194	0.4557	0.5940	0.8095	0.9229	0.9754
100	0.0524	0.0691	0.1149	0.2056	0.3209	0.4534	0.5945	0.8127	0.9268	0.9768
110	0.0526	0.0684	0.1134	0.2010	0.3215	0.4523	0.5947	0.8168	0.9285	0.9772
120	0.0534	0.0665	0.1160	0.1996	0.3191	0.4570	0.5949	0.8205	0.9330	0.9793
130	0.0508	0.0682	0.1179	0.2006	0.3156	0.4535	0.5958	0.8189	0.9342	0.9798
140	0.0504	0.0672	0.1145	0.1949	0.3121	0.4529	0.5938	0.8161	0.9345	0.9817
150	0.0505	0.0658	0.1123	0.1969	0.3073	0.4495	0.5911	0.8148	0.9381	0.9836
160	0.0526	0.0676	0.1130	0.1954	0.3115	0.4485	0.5928	0.8196	0.9387	0.9844
170	0.0530	0.0692	0.1129	0.1961	0.3100	0.4472	0.5934	0.8191	0.9418	0.9848
180	0.0526	0.0663	0.1139	0.1963	0.3127	0.4489	0.5918	0.8183	0.9427	0.9857
190	0.0524	0.0673	0.1139	0.1967	0.3142	0.4508	0.5939	0.8172	0.9410	0.9858
200	0.0531	0.0680	0.1127	0.1974	0.3131	0.4513	0.5924	0.8155	0.9430	0.9856

Table 11: Powers for KSI with level 0.95 and asymptotic critical value 0.220.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0431	0.0655	0.1326	0.2302	0.3546	0.4717	0.5736	0.7432	0.8473	0.9154
30	0.0471	0.0667	0.1345	0.2372	0.3659	0.4935	0.6098	0.7876	0.8908	0.9464
40	0.0514	0.0690	0.1316	0.2389	0.3708	0.5074	0.6305	0.8068	0.9120	0.9628
50	0.0497	0.0710	0.1346	0.2415	0.3734	0.5140	0.6396	0.8236	0.9265	0.9738
60	0.0521	0.0733	0.1310	0.2407	0.3779	0.5204	0.6482	0.8359	0.9330	0.9784
70	0.0516	0.0735	0.1362	0.2429	0.3788	0.5211	0.6544	0.8473	0.9432	0.9804
80	0.0517	0.0733	0.1350	0.2386	0.3789	0.5296	0.6621	0.8547	0.9477	0.9849
90	0.0490	0.0698	0.1341	0.2422	0.3784	0.5297	0.6675	0.8616	0.9497	0.9861
100	0.0516	0.0725	0.1353	0.2414	0.3798	0.5285	0.6649	0.8636	0.9539	0.9864
110	0.0546	0.0723	0.1315	0.2408	0.3774	0.5282	0.6674	0.8652	0.9564	0.9872
120	0.0495	0.0721	0.1340	0.2357	0.3764	0.5285	0.6718	0.8711	0.9579	0.9875
130	0.0520	0.0758	0.1345	0.2317	0.3742	0.5331	0.6716	0.8719	0.9603	0.9891
140	0.0500	0.0745	0.1321	0.2312	0.3684	0.5288	0.6722	0.8722	0.9629	0.9902
150	0.0498	0.0725	0.1319	0.2304	0.3690	0.5254	0.6744	0.8756	0.9656	0.9914
160	0.0502	0.0711	0.1305	0.2327	0.3685	0.5273	0.6752	0.8797	0.9646	0.9918
170	0.0524	0.0692	0.1329	0.2320	0.3721	0.5265	0.6712	0.8756	0.9657	0.9930
180	0.0526	0.0706	0.1313	0.2319	0.3734	0.5270	0.6718	0.8772	0.9667	0.9942
190	0.0514	0.0725	0.1310	0.2361	0.3696	0.5298	0.6731	0.8776	0.9678	0.9944
200	0.0505	0.0732	0.1295	0.2366	0.3697	0.5275	0.6705	0.8772	0.9698	0.9933

Table 12: Powers for BJI with level 0.95 and asymptotic critical value 1.313.

	$\delta\sqrt{k}$									
k	0.1	0.5	1	1.5	2	2.5	3	4	5	6
20	0.0285	0.0471	0.1079	0.2006	0.3180	0.4374	0.5436	0.7191	0.8370	0.9084
30	0.0380	0.0563	0.1141	0.2070	0.3325	0.4597	0.5829	0.7668	0.8791	0.9398
40	0.0425	0.0550	0.1109	0.2098	0.3382	0.4743	0.6000	0.7875	0.9015	0.9579
50	0.0433	0.0612	0.1114	0.2137	0.3388	0.4791	0.6106	0.8044	0.9171	0.9688
60	0.0463	0.0611	0.1139	0.2091	0.3451	0.4863	0.6190	0.8141	0.9249	0.9730
70	0.0484	0.0642	0.1196	0.2183	0.3462	0.4892	0.6289	0.8285	0.9365	0.9768
80	0.0463	0.0620	0.1180	0.2142	0.3456	0.4948	0.6345	0.8400	0.9406	0.9816
90	0.0466	0.0604	0.1177	0.2122	0.3445	0.4945	0.6359	0.8438	0.9434	0.9830
100	0.0497	0.0667	0.1159	0.2132	0.3453	0.4953	0.6325	0.8487	0.9478	0.9832
110	0.0506	0.0664	0.1145	0.2107	0.3479	0.4971	0.6400	0.8453	0.9493	0.9850
120	0.0477	0.0641	0.1196	0.2124	0.3435	0.4910	0.6421	0.8556	0.9507	0.9860
130	0.0506	0.0672	0.1193	0.2069	0.3403	0.4984	0.6400	0.8550	0.9533	0.9868
140	0.0488	0.0650	0.1166	0.2072	0.3361	0.4948	0.6398	0.8541	0.9575	0.9889
150	0.0490	0.0632	0.1173	0.2057	0.3354	0.4914	0.6429	0.8569	0.9582	0.9900
160	0.0498	0.0667	0.1170	0.2103	0.3415	0.4901	0.6443	0.8603	0.9564	0.9901
170	0.0508	0.0652	0.1188	0.2129	0.3404	0.4944	0.6423	0.8611	0.9582	0.9914
180	0.0509	0.0656	0.1175	0.2120	0.3430	0.4968	0.6405	0.8600	0.9623	0.9929
190	0.0510	0.0666	0.1178	0.2118	0.3437	0.4966	0.6459	0.8603	0.9629	0.9927
200	0.0490	0.0690	0.1183	0.2148	0.3396	0.4964	0.6441	0.8611	0.9638	0.9910

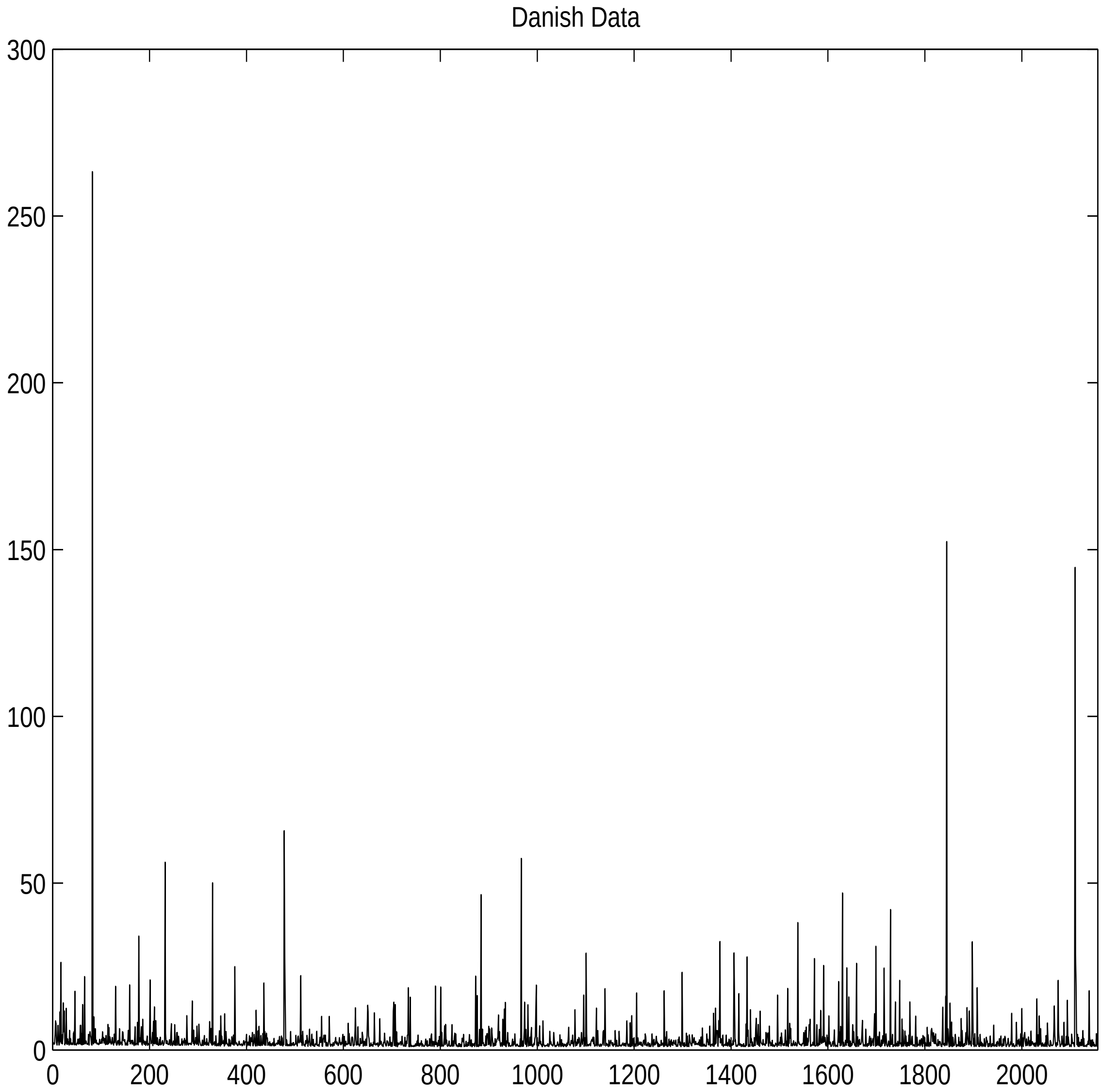

Figure 1: *Danish fire loss data.* This consists of 2156 losses over one million Danish Krone (DKK) from the years 1980 to 1990, inclusive.

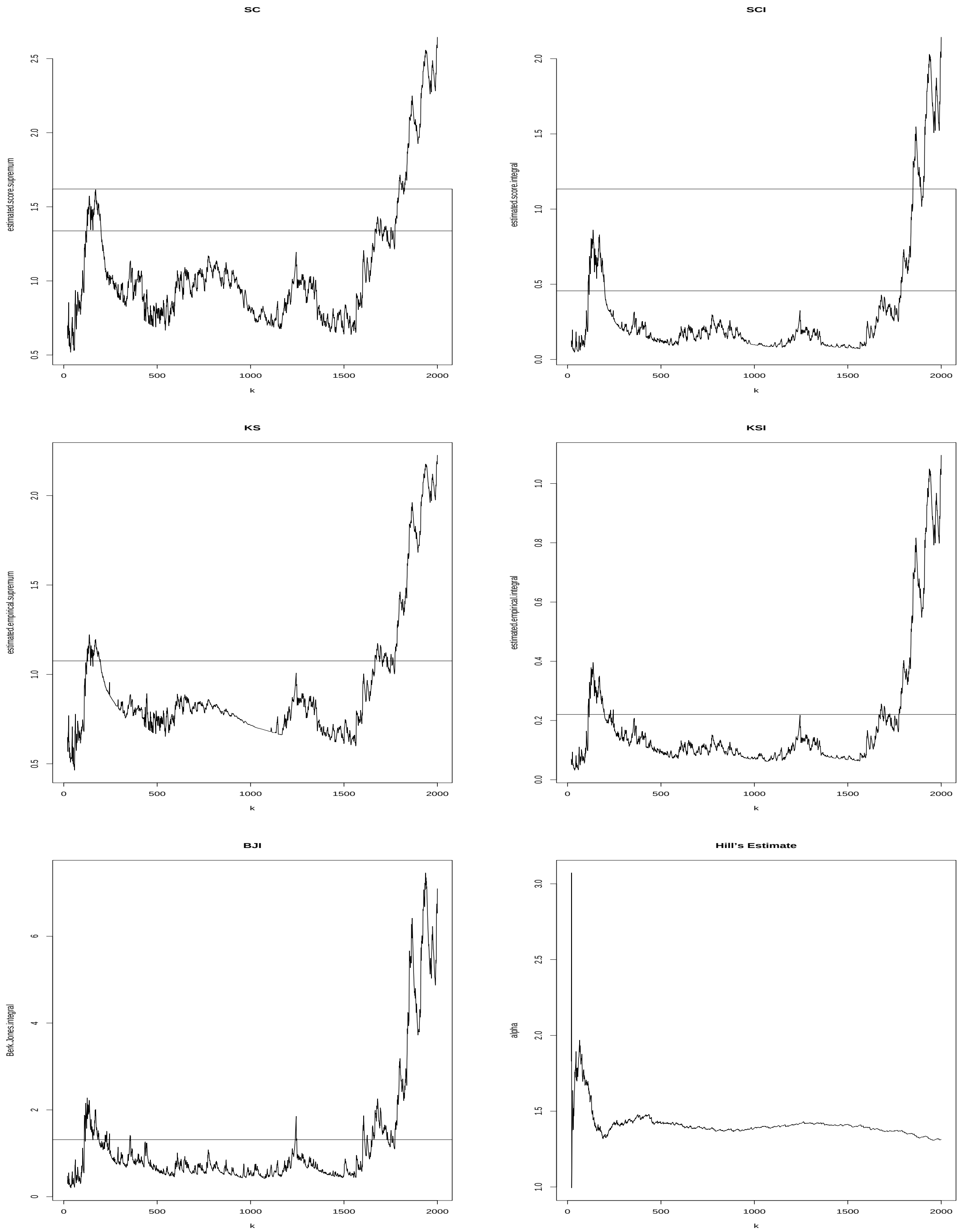

Figure 2: *Analysis of the Danish fire loss data.*

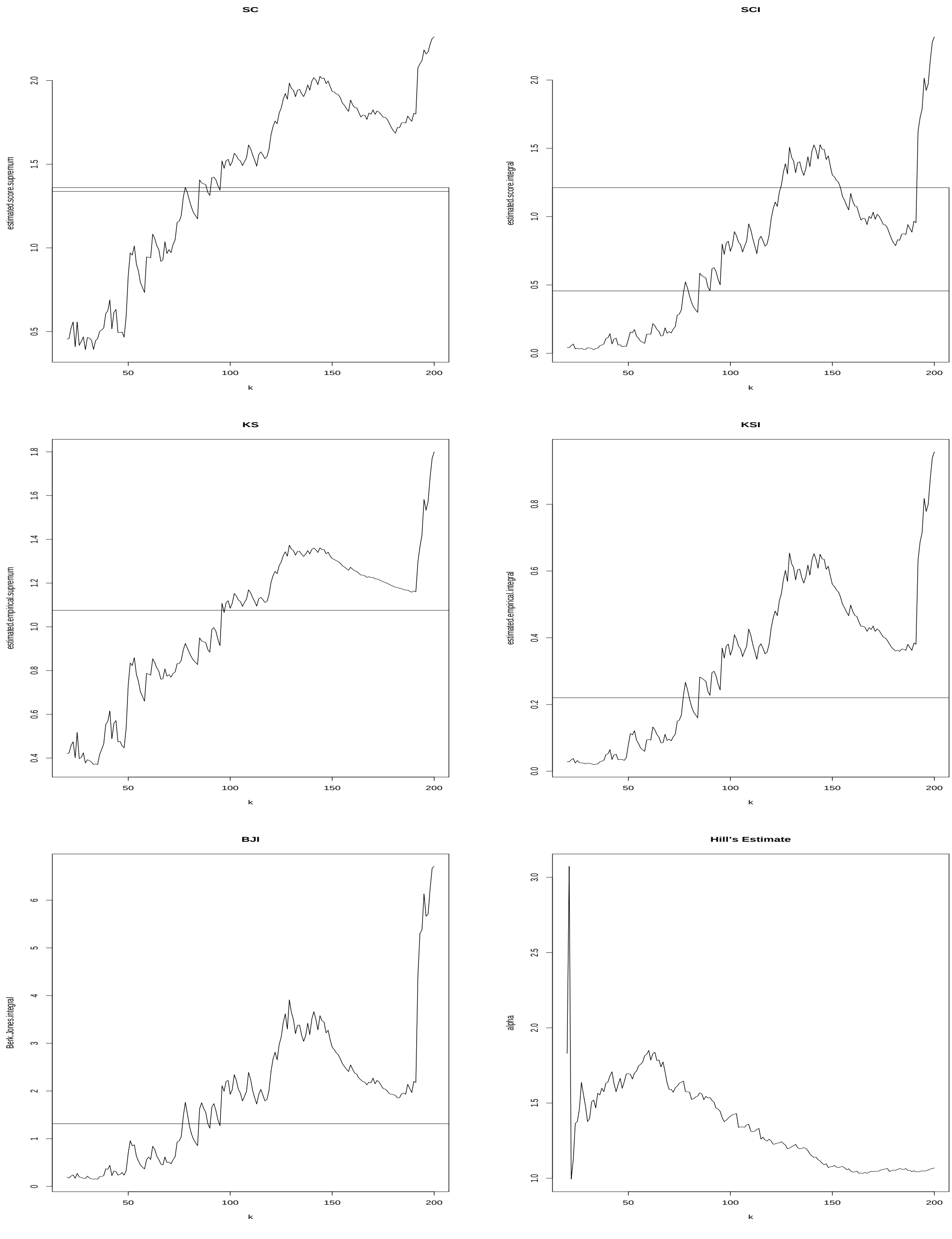

Figure 3: *Analysis of the block maxima of the Danish fire loss data.*

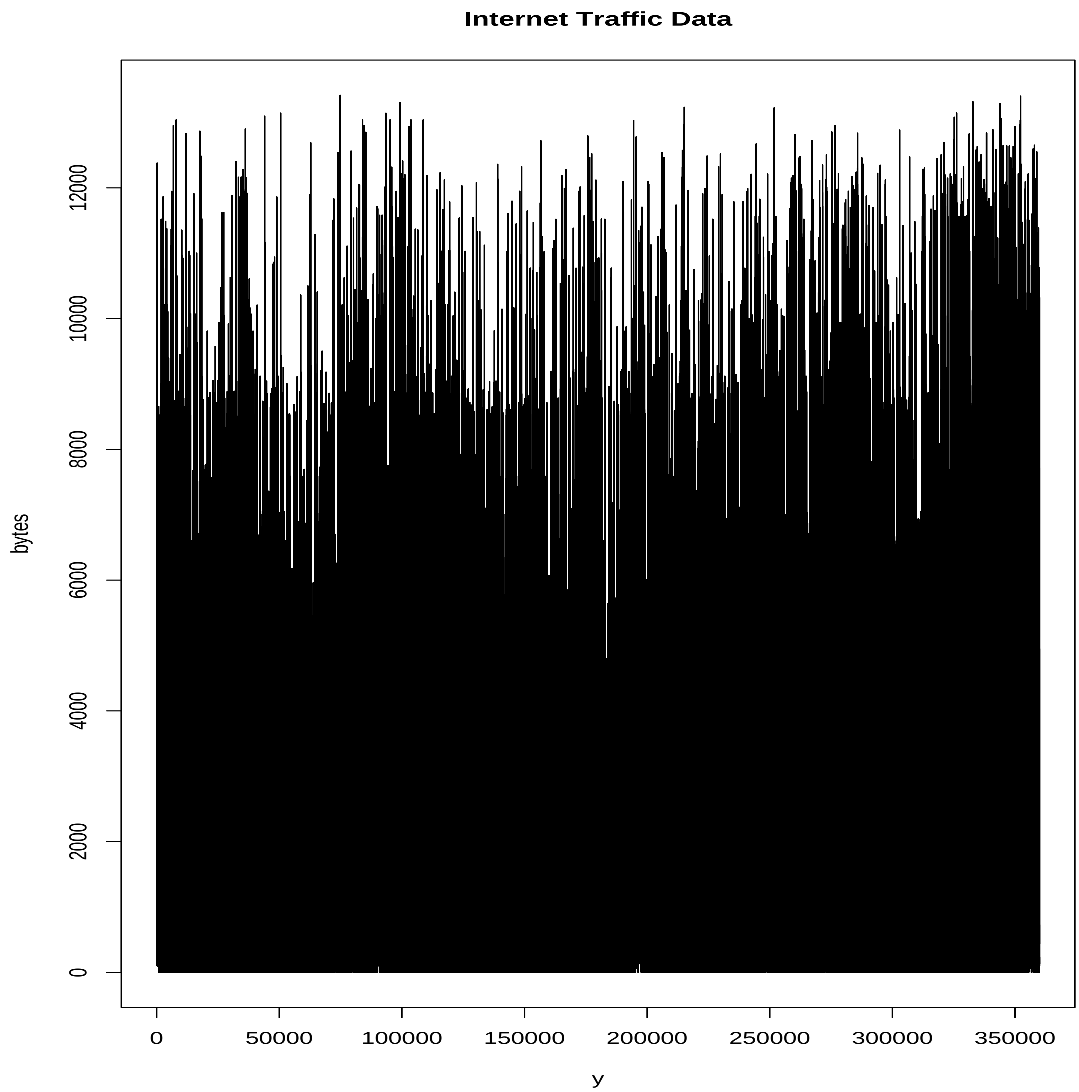

Figure 4: *Internet traffic data.* The Ethernet series used here are part of a data set collected at Bellcore in August of 1989. They correspond to one ”normal” hour’s worth of traffic, collected every 10 milliseconds, thus resulting in a length of 360,000. This data set measures the number of bytes per unit time.

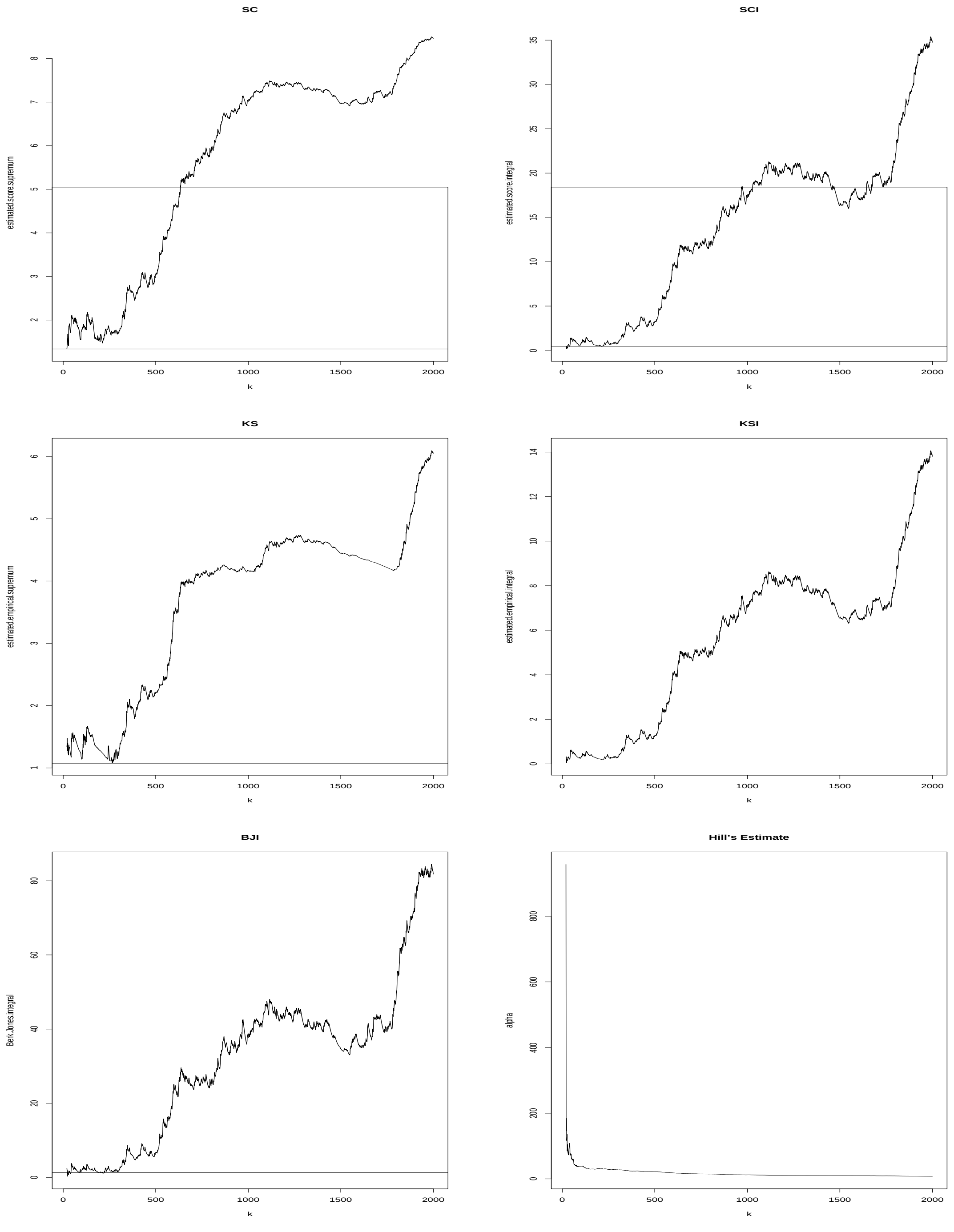

Figure 5: *Analysis of the internet traffic data data.*

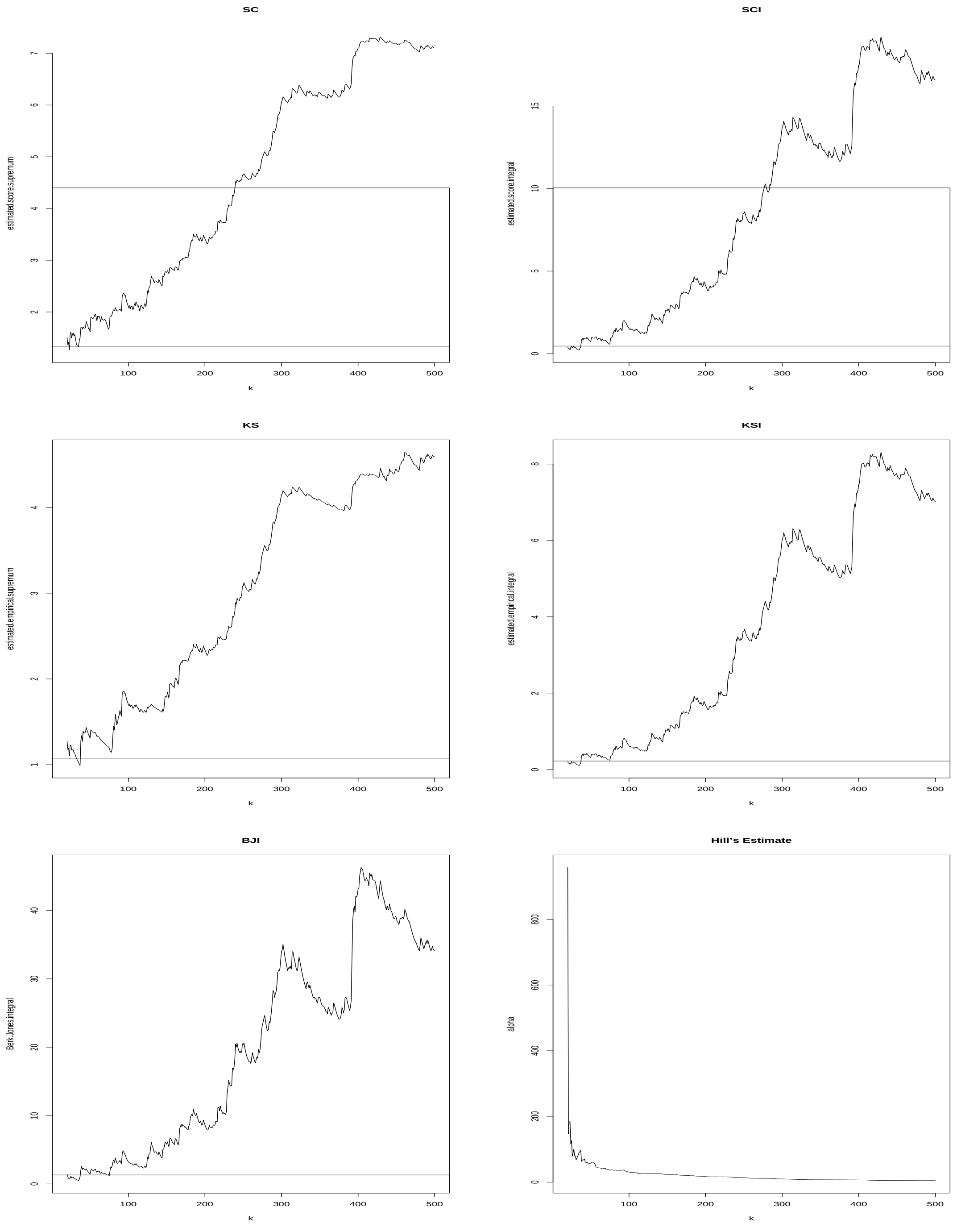

Figure 6: *Analysis of the block maxima of the internet traffic data.*