On a Conic Approach to Convex Analysis

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Abstract

The aim of this paper is to make an attempt to justify the main results from Convex Analysis by one elegant tool, the conification method, which consists of three steps: *conify, work with convex cones, deconify.* It is based on the fact that the standard operations and facts ('the calculi') are much simpler for special convex sets, *convex cones.* By considering suitable classes of convex cones, we get the standard operations and facts for all situations in the complete generality that is required. The main advantages of this conification method are that the standard operations—linear image, inverse linear image, closure, the duality operator, the binary operations and the inf-operator—are defined on *all* objects of each class of convex objects—convex sets, convex functions, convex cones and sublinear functions—and that moreover the standard facts—such as the duality theorem—hold for*all closed* convex objects. This requires that the analysis is carried out in the context of convex objects over *cosmic space*, the space that is obtained from ordinary space by adding a *horizon*, representing the directions of ordinary space.

1 Introduction

Reduction to conic calculus. This paper presents some progress on joint work with V.M.Tikhomirov that was published in [7]. It is an attempt to develop a general and universal tool for justifying the main results of Convex Analysis. The idea is as follows. Many standard operations and facts are much simpler and more complete for special convex objects, *convex cones*. This suggests to *conify* all situations in *advance*. Then a simple 'conic calculus'—consisting of two formulae—can be applied, and after that one can translate the results into the language of the initial problem. This could be seen as an illustration of Plato's 'Allegory of the Cave', which explains that everything in the visible world is like a shadow from the real ideal world (book VII of [21]). The situations of convex sets and functions in our vision are like shadows from the world of convex cones. Fixing, with initial effort, ones eyes upon the world of convex cones, reveals that these are the source of truth.

Conification method. The novelty of the present paper is that we try to give a consistent and exhaustive development of the idea 'conify, work, deconify', to be called *the conification method.* The standard operations—linear image, inverse linear image, closure, the duality operator, binary operations and inf-operations—on each class of convex objects—convex sets, convex functions, convex cones and sublinear functions—are defined on *all* convex objects. The standard facts, such as the duality theorem, turn out to hold for *all closed* convex objects. It will be convenient to work in the general framework of convex objects over *cosmic space.* Then each class of convex objects consists of the proper ones and one or more types of improper ones. To construct the binary operations and to prove their duality formulae, the work on the level of convex cones is done in terms of two special linear transformations, the sum operator and the diagonal operator. Attention is restricted in the present paper to *linear* images and inverse *linear* images of convex sets. In particular, the lists of operations considered in the present paper are not complete. For example, the composition of a convex non-decreasing function and a convex function is not considered.

Why the usual duality theorem for convex functions is not satisfactory. Let us consider for example the question what is the need for the novel version of the duality theorem given in the present paper, $f^{\circ\circ} = f$, which holds for all closed convex functions f on cosmic space—where the duality operator $f \mapsto f^{\circ}$ is given by the conification method ('conify, apply the polar operator, deconify'). The usual version is $f^{**} = f$, which holds only for all *proper* closed convex functions f on $\mathbb{R}^n$—where the duality operator $f \mapsto f^*$ is the Legendre-Fenchel transform. This usual version seems satisfactory at first sight, as "proper convex functions are the real object of study" ([22]), and as the formula $f^{**} = f$ does not hold for any improper closed convex function (apart from the functions $\equiv +\infty$ and $\equiv -\infty$). However, to continue the quotation from [22], "but improper functions do arise from proper ones in many natural situations, and it is more convenient to admit them than to exclude them laboriously from consideration," or, from [24], "While proper functions are our central concern, improper functions may arise indirectly and can't always be excluded from consideration". To be more specific, linear images and inverse linear images of proper convex functions need not be proper.

Illustration of the inconvenience of restriction to proper convex functions. Now we give a simple illustration of the inconvenience of restricting to proper convex functions. Suppose we want to prove the other main calculus rule, $(\Lambda^{-1}\bar{f})^* = \mathrm{cl}(\Lambda'(\bar{f}^*))$, for closed convex functions $\bar{f}$ on $\mathbb{R}^m$ and all linear transformations $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$. A natural strategy would be to derive this as an immediate consequence of the 'obvious' rule $(\Lambda' h)^* = \Lambda^{-1}(h^*)$, by choosing $h = \bar{f}^*$, then applying the duality operator to both sides, and, finally, using the formula $g^{**} = \mathrm{cl}g$ twice. However, this nice strategy of proof can not be realized with the usual version of the duality theorem, not even for the case that f is proper: indeed, the linear image and the inverse linear image of a proper convex function are not always proper. This failure appears to reveal some flaw in the usual version of the duality theorem for convex functions. The novel version of the duality theorem, given in the present paper, allows to realize this strategy of proof in full generality.

Current state of the problem. In [22], these problems are addressed by the introduction of a different *closure* operator for improper convex functions. This makes the formula $f^{**} = f$ valid for all convex functions f that are closed with respect to this closure operator. The closure operator for improper convex functions is defined in [22] in an ad hoc way, and then the only improper closed convex functions are the two function $\equiv +\infty$ and $\equiv -\infty$. The other main rule $(\Lambda^{-1}f)^* = \mathrm{cl}(\Lambda' f^*)$ is verified in [22] by a direct verification, based on the defining formula for the Legendre-Fenchel transform. In [24] and in other accounts of Convex Analysis, the duality theorem is only given for *proper* convex functions, and it is pointed out that the formula $f^{**} = f$ does not hold for improper convex functions (apart from $\equiv +\infty$ and $\equiv -\infty$).

Solution proposed in the present paper. In the present paper, an other approach is offered: a different *duality* operator $f \to f^\circ$ is taken—the one defined by the conification method; for proper convex functions, it coincides with the Legendre-Fenchel transform—and the usual closure operator ('closure of epigraphs') is used; then the formula $f^{\circ\circ} = f$ turns out to hold for all improper convex functions that are closed in the usual sense, that is, the functions that take the value $-\infty$ inside a closed convex set, and the value $+\infty$ outside it. Now the other main calculus rule, $(\Lambda^{-1}\bar{f})^\circ = \mathrm{cl}(\Lambda'(\bar{f}^\circ))$, can be proved in full generality by realizing the nice strategy described above. The obtained rule is stronger than the usual rule in some cases.

How the conification method leads to convex functions on cosmic space. The conification method suggests to complete the class of proper convex functions not only with the usual improper convex functions, but even further, considering the slightly larger class of convex functions on *cosmic space*, to be defined in the present paper. The duality theorem $f^{\circ\circ} = f$, and so the other main calculus formula, $(\Lambda^{-1}\bar{f})^\circ = \mathrm{cl}(\Lambda'(\bar{f}^\circ))$, turn out to hold for *all* closed convex functions on cosmic n-space, if the duality operator is defined by the conification method. An explicit formula is offered in proposition 4.3 for this duality operator on convex functions on cosmic space. This formula expresses this operator in terms of the usual duality operators of the four main classes of convex objects: the Legendre-Fenchel transform (for convex functions), the support operator (for convex sets), the polar operator (for convex cones), the subdifferential operator (for sublinear functions). The concept 'convex function on cosmic space' appears to be considered for the first time in the present paper. This natural concept is suggested by the concept 'convex set in cosmic space' considered in [24]. To give an explicit description of the duality for closed convex functions, four types of convex functions on cosmic space are distinguished, each type having its own character: *proper ones, improper ones of sublinear function type, improper ones of convex set type*, and *improper ones of convex cone type.*

Other calculus rules and other convex objects. We will see in the present paper that if all operations on all convex objects are defined by the conification method, then all calculus rules are valid for all closed objects of all classes of convex objects.

An example showing the need to consider convex sets in cosmic space. Now we explain how the conification method leads to the present treatment of the calculi of convex objects. The crucial point is that attempting to realize the conification method, forces one to address the following issue. Suppose that the initial situation involves an unbounded closed convex set. Then the first step, 'conification', gives a convex cone that is not closed. The second step, 'work', might require taking the closure of this convex cone. Then it is not clear in advance how to carry out the last step, 'deconification'. To overcome this obstacle to the deconification step, *convex sets in cosmic space* are considered instead of just ordinary convex sets. Convex sets in cosmic space contain points on the *horizon*, representing some of the recession directions of the convex set. All convex cones that arise from conifications of convex sets by 'working', can be deconified into *convex sets in cosmic space*. For convex functions and sublinear functions, the obstacle is solved in a similar way: the concept 'convex function on cosmic space' is introduced and the most general definition of sublinear functions is used. Then again deconification of the convex cones that arise, after 'working' with conifications of convex functions or sublinear functions, is always possible.

Historic perspective. To put the present work in historic perspective, we begin by recalling the words from Fenchel's lecture notes [9] that "most of the basic concepts and results can be traced back in one form or another to the very first papers on the subject". What follows is an attempt to mention some of the decisive developments from the vast literature, based on the list of references given at the end of the paper.

The study of the duality and calculus of convex sets started with the work of Minkowski [17]; for convex functions, the originator of such a study is Fenchel [8]. The general understanding that the calculi of convex sets, convex functions, sublinear functions and convex cones are identical in the sense that all formulae of one of them can be obtained from suitable formulae of the other can be traced back to the very dawn of convex analysis. Hörmander [14] explained that the study of convex sets can be reduced to convex cones. The analogy between convex sets and functions was one of the driving ideas in works of Moreau [18] and especially Rockafellar (for example [22, 23]). Hörmander's approach, in particular its extension to convex cones and sublinear functions was powerfully presented by Kutateladze and Rubinov [15]. The creation of Convex Analysis as a subject is due to Rockafellar [22]. Accounts of the calculi of convex objects are given by Tikhomirov in [26, 27], and by Magaril-Il'yaev and Tikhomirov in [16]. There are extremely many applications of the calculi for convex objects, especially to optimization (see for example [16, 27]).

The idea behind convex sets in cosmic space, to portray the directions in ordinary n-dimensional space as abstract points, is already given in [22], and is in fact due to Steinitz [25]. A formal mathematical development is given in Rockafellar-Wets [24] (see chapter 3, Cones and Cosmic Closure). The device of using diagonal operators has been used many times in the literature. The construction of binary operations in [22] by means of 'partial addition' is an informal variant of this device: for the four 'natural' commutative, associative binary operations on convex sets this construction is given on p. 20; for the eight ones on proper convex functions on p. 39. The use of the diagonal operators in the construction of binary operations is for example made explicit in Hiriart-Urruty and Lemaréchal [11].

In the last fifteen years, there has been a renewal of interest in convexity, stimulated by progress in convex optimization algorithms by Nesterov and Nemirovski [19], and many books on convexity and in particular convex optimization have been published, for example [1–4, 6, 10–13, 16, 19, 20, 24].

Comparison with the previous paper on this work. Finally, we indicate the progress that this paper represents compared to [7]. In the first place, improved concepts of conification and deconification are given in the present paper. This requires the consideration of cosmic space. *Conification is now not always unique*, and deconification is always possible. All operations and facts are extended to the right level of generality, that of cosmic space. In the second place, the presentation has been simplified.

Further developments. The treatment of the deconification method in the present paper is restricted to the fundamentals. It can be developed further, to cover for example the subdifferential calculus, the duality theory of optimization problems, and to derive many applications in an exhaustive way. Moreover, it is hoped that it can be fruitfully applied to derive the 'convex calculus' in other situations than just the space $\mathbb{R}^n$.

Organization of the paper. The organization of the present paper is as follows. In the second section, the conic calculus is given, consisting of two formulae and two assumptions (polyhedrality or existence of a suitable relative interior point) under which the closure operations in the second formula can be omitted. In the third section, the calculi for convex sets (in cosmic space) and (extended real valued) sublinear functions are derived by means of the conification method. In the fourth section, the same is done for convex functions (on cosmic space). In the fifth section, binary operations are constructed on the level of convex cones; then the conification method is used to construct all standard binary operations for convex sets, convex functions and sublinear functions, and to derive the duality formulae, again in the context of cosmic space. In the sixth section, the calculi are given for the subclasses of convex sets and sublinear functions that are self-dual: *zero-sets* and *gauges*. For all results in the present paper that require non-routine proofs, detailed verifications are given. However, in order to save space, a number of routine proofs have not been displayed. At the end of each section, we state precisely what is novel in that section, we motivate these new results, and make comparisons to the literature, mainly to [22] and [24]

2 Conic calculus.

The aim of this section is to recall the standard operations and facts for convex cones. A *cone* in $\mathbb{R}^n$ is a—possibly empty—subset that is closed under taking positive scalar multiples; a *convex cone* $\mathbb{R}^n$ is a cone that is closed under taking sums. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation; the *linear image* of a subset $S \subset \mathbb{R}^n$ under Λ is defined to be the subset $\Lambda S = \{y \in \mathbb{R}^m | \exists x \in S : y = \Lambda x\}$ of $\mathbb{R}^m$, the *inverse linear image* of a subset $\bar{S} \subset \mathbb{R}^m$ under Λ is defined to be the subset $\bar{S}\Lambda = \{x \in \mathbb{R}^n | \exists y \in \bar{S} : y = \Lambda x\}$. These two operations preserve the convex cone property. The *relative interior* $\mathrm{ri}S$ of a subset $S \subset \mathbb{R}^n$ is the interior of S regarded as a subset of its affine hull, and the *closure* $\mathrm{cl}S$ of a set $S \subset \mathbb{R}^n$ is the topological closure of S in $\mathbb{R}^n$. Both operations preserve the convex cone property. As the duality operator on convex cones in $\mathbb{R}^n$ we use the *polar operator* $C \mapsto C^\circ$, where $C^\circ = \{y \in \mathbb{R}^n | x^T y \leq 0 \text{ for all } x \in C\}$ (another possible choice would have been the operator $C \mapsto -C^\circ = \{y \in \mathbb{R}^n | x^T y \geq 0 \text{ for all } x \in C\}$; in the present paper, the polar cone is used to avoid minus signs in certain formulae). The relations between the polar operator and the linear image and inverse linear image operators are given by the following result. The transpose of a linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by $\Lambda' : \mathbb{R}^m \to \mathbb{R}^n$, that is, $(\Lambda' y)^T x = (\Lambda x)^T y$ for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. The linear transformations $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ and $\Theta : \mathbb{R}^m \to \mathbb{R}^n$ are said *to be in duality* if $\Lambda' = \Theta$ or, equivalently, $\Theta' = \Lambda$.

Theorem 2.1 *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ and $\Theta : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations in duality, and let $C, D \subset \mathbb{R}^n$ be nonempty closed convex cones. Then the following statements hold true:*

1. *$C^\circ = D$ precisely if $D^\circ = C$ (then C and D are said to be in duality);*

2. *if C and D are in duality, then ΛC and $D\Theta$ are in duality, after taking closures.*

If one of the following two assumptions holds true, then ΛC is closed:

a. $\mathrm{ri}C \cap \Theta(\mathbb{R}^m) \neq \emptyset$;

b. *the convex cones C or D—and so both—are polyhedral.*

Convex cones containing the origin and the subclass of subspaces. The first of the four main classes of convex objects to be considered in the present paper is *convex cones containing the origin.* For this class, the operations linear image, inverse linear image, closure, duality operator, are always defined and the duality theorem, '$C^\circ = D$ precisely if $D^\circ = C$', holds for all closed objects C, D. The *subspaces* L of $\mathbb{R}^n$ form a subclass of the class of convex cones. For subspaces, the polar operator is the *orthogonal complement* $L \mapsto L^\perp = \{y \in \mathbb{R}^n | x^T y = 0 \text{ for all } x \in L\}$; moreover, all subspaces are closed.

Justification. The statements on convex cones in theorem 2.1 are included in the present paper, as these are used as the base for deriving complete versions of the standard facts for other convex objects. Because of this fundamental role in the present paper, it is of importance that short and direct proofs of these statements are possible. We assume that this is generally known and so we do not display such proofs. The 'symmetric' formulation of theorem 2.1 is chosen with the aim of making explicit that this result expresses properties of pairs of convex cones in duality. However, we emphasize one essential asymmetry: $D\Theta$ is always closed, but ΛC is not always closed. Theorem 2.1 is stated for nonempty *closed* convex cones, as this gives a simpler formulation, and as this implies the usual formulation for arbitrary nonempty convex cones (as a convex cone and its closure have the same polar cone). However, in the present paper, arbitrary, not necessarily closed, convex cones are considered; these do not have to contain the origin, and are even allowed to be the empty set. All these choices are forced by the conification method. To begin with, we do not require cones and convex cones to contain the origin, as it is convenient to view the strict epigraph of a sublinear function, which does not contain the origin if it takes value zero at the origin, as a convex cone. Moreover, it is convenient to allow cones and convex cones to be empty in advance; then the inverse linear image of a convex cone is always a convex cone. However, the role of convex cones in the present paper is that they arise by conification of convex objects; the conification operators will be defined in such a way that the resulting convex cones are always nonempty, either because these contain the origin (this is the case for convex sets and convex functions) or because these contain a specific open ray (this is the case for sublinear functions and convex functions). The reason that the conifications need to be nonempty is that the duality theorem (statement 1 of theorem 2.1) for convex cones does not hold for empty convex cones.

Comparison with the literature. The usual formulation of the statements 1 and 2 of the theorem, for example in [22], is as three formulas for—not necessarily closed—nonempty convex cones: $C^{\circ\circ} = \mathrm{cl}C$, $(\Lambda C)^\circ = (C^\circ)\Lambda'$ and $((\mathrm{cl}\bar{C})\Lambda)^\circ = \mathrm{cl}(\Lambda'(\bar{C}^\circ))$. The statements in theorem 2.1 are all standard, and can for example be found in [22]: theorem 14.1 in [22] gives statement 1; applying cor.16.3.2 in [22] to closed convex cones gives statement 2, and, moreover, the closedness of ΛC under assumption a; the similar statement but under assumption b follows by applying theorem 19.1 $(a) \Leftrightarrow (b)$ in [22] to convex cones. Some of the statements of theorem 2.1 are derived in [22] from the corresponding properties for other convex objects. For example, cor.16.3.2 in [22] is derived from the similar statement for convex functions. The convention to allow cones and convex cones not to contain the origin agrees with the convention in [22]; in [24] cones and convex cones are required to contain the origin.

3 The standard operations and facts for convex sets and sublinear functions.

The aim of this section is to construct in complete generality the standard operations and facts for convex sets—and so for sublinear functions—by means of the conification method—'conify, work with convex cones, deconify'. In particular we want to achieve that the operations linear image, inverse linear image, closure and the duality operator are defined for all convex sets and all sublinear functions. This requires that recession directions are allowed as points of convex sets, that is, 'convex sets in cosmic space' have to be considered. Indeed, working with conifications of convex sets in $\mathbb{R}^n$ can lead to a convex cone that is the conification of a convex set in cosmic space, but not of a convex set in $\mathbb{R}^n$. Thus in the present paper, convex sets in cosmic space are considered rather than just convex sets in $\mathbb{R}^n$. For convex sets in cosmic space—and for extended real valued sublinear functions—the complete version of the 'standard operations and facts' given in this section appears to be novel.

Convex sets in cosmic space as deconifications of suitable convex cones. A—possibly empty—set S in $\mathbb{R}^n$ is called *convex* if $(1-\alpha)x+\alpha y \in S$ for all $x, y \in S$, $0 \leq \alpha \leq 1$. We recall the concept convex sets in cosmic space. The n-dimensional *cosmic space* $\mathrm{csm}\mathbb{R}^n$ is defined to be $\mathbb{R}^n \cup \mathrm{hzn}\mathbb{R}^n$, where $\mathrm{hzn}\mathbb{R}^n$ is the set of directions in $\mathbb{R}^n$—that is, nonzero vectors in $\mathbb{R}^n$ considered up to positive scalar multiples—which can be viewed as the points on the *horizon*. A subset A of $\mathrm{csm}\mathbb{R}^n$—written as $A = S \cup \mathrm{dir}K$ for a set $S \subset \mathbb{R}^n$ and a cone $K \subset \mathbb{R}^n$ that contains the origin, where $\mathrm{dir}K \subset \mathrm{hzn}\mathbb{R}^n$ is the set of directions determined by the nonzero elements of K—is said to be *convex* if S and K are convex and $S + K \subset S$. For each convex set S in $\mathbb{R}^n$, the set of vectors $x \in \mathbb{R}^n$ for which $S + x \subset S$ will be called the *recession cone* of S and it is denoted by 0^+S. A convex cone C that contains the origin can be viewed as a special case of a convex set in cosmic n-space, $C \cup \mathrm{dir}C$. We will need to distinguish two types of convex sets $A = S \cup \mathrm{dir}K$ in cosmic n-space: A is called *proper* if $S \neq \emptyset$, and A is called *improper* if $S = \emptyset$. The following result gives a well-known equivalent description of convex sets in $\mathrm{csm}\mathbb{R}^n$. We write $\mathbb{R}_{++} = (0, +\infty)$.

Proposition 3.1 *The following mapping from the collection of convex cones C in $\mathbb{R}^{n+1}$ that contain the origin and that are contained in the upper halfspace $x_{n+1} \geq 0$ to the collection of convex sets A in $\mathrm{csm}\mathbb{R}^n$ is a bijection: $C \mapsto A(C) = S(C) \cup \mathrm{dir}K(C)$, where $S(C)$ consists of all points $x \in \mathbb{R}^n$ for which $(x, 1) \in C$ and where $K(C)$ consists of all points $y \in \mathbb{R}^n$ for which $(y, 0) \in C$. The inverse mapping of this bijection associates to each convex set $A = S \cup \mathrm{dir}K$ in $\mathrm{csm}\mathbb{R}^n$, the convex cone $C(A) = (\mathbb{R}_{++}(S \times 1)) \cup (K \times 0)$.*

In the situation of proposition 3.1, $C(A)$ will be called the *conification* of A, and $A(C)$ will be called the *deconification* of C. We note that conifications of convex sets are always nonempty. We will often call convex sets in $\text{csm}\mathbb{R}^n$ just (n-dimensional) convex sets and then we call convex sets in $\mathbb{R}^n$ ordinary (n-dimensional) convex sets. The conification of an ordinary n-dimensional convex set $A = S$ is the convex cone $\mathbb{R}_{++}(S \times 1) \cup \{0_{n+1}\}$. The following criterion for properness of a convex set $A = S \cup \text{dir}K$ in cosmic n-space in terms of its conification C holds true: A is proper precisely if C is not contained in the horizontal hyperplane $\mathbb{R}^n \times 0$.

Linear images, inverse linear images and closure of convex sets. The conification method gives definitions for the concepts linear image, inverse linear image and closure of convex sets in cosmic space. Now we will give direct descriptions of these concepts in terms of the usual concepts linear image, inverse linear image and closure of subsets of $\mathbb{R}^n$. A linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ induces a linear transformation $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$, again denoted by Λ, by means of the trivial action on the last coordinate, $(x, \alpha) \mapsto (\Lambda x, \alpha)$.

Proposition 3.2 *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $A = S \cup \text{dir}K$ be a convex set in $\text{csm}\mathbb{R}^n$ and $\bar{A} = \bar{S} \cup \text{dir}\bar{K}$ a convex set in $\text{csm}\mathbb{R}^m$.*

1. *Let $C = C(A)$ be the conification of A. Then ΛC is a convex cone in $\mathbb{R}^{m+1}$ that contains the origin and is contained in $x_{m+1} \geq 0$. Its deconification is the convex set $\Lambda S \cup \text{dir}(\Lambda K)$ in $\text{csm}\mathbb{R}^m$.*
2. *Let $\bar{C} = C(\bar{A})$ be the conification of $\bar{A}$. Then $\bar{C}\Lambda$ is a convex cone in $\mathbb{R}^{n+1}$ that contains the origin and is contained in $x_{n+1} \geq 0$. Its deconification is the convex set $\bar{S}\Lambda \cup \text{dir}(\bar{K}\Lambda)$ in $\text{csm}\mathbb{R}^n$.*
3. *Let $C = C(A)$ be the conification of A. Then $\text{cl}C$ is a convex cone in $\mathbb{R}^n$ that contains the origin and is contained in $x_{n+1} \geq 0$. Its deconification is the convex set $\text{cl}S \cup \text{dir}(0^+\text{cl}S)$ if A is proper, and it is equal to $\text{dir}(\text{cl}K)$ if A is improper.*

This proposition gives the following explicit formulae for the operations linear image, inverse linear image and closure, that are defined on convex sets by the conification method.

1. linear image: $A = S \cup \text{dir}K \mapsto \Lambda A = \Lambda S \cup \text{dir}(\Lambda K)$;
2. inverse linear image: $\bar{A} = \bar{S} \cup \text{dir}\bar{K} \mapsto \bar{A}\Lambda = \bar{S}\Lambda \cup \text{dir}(\bar{K}\Lambda)$;
3. closure: $A = S \cup \text{dir}K \mapsto \text{cl}A$, where $\text{cl}A$ is equal to $\text{cl}S \cup \text{dir}(0^+\text{cl}S)$ if A is proper and equal to $\text{dir}(\text{cl}K)$ if A is improper.

A convex set A in cosmic space is called *closed* if $A = \text{cl}A$. There are two types of convex sets in cosmic n-space: proper ones—these are given by a nonempty closed convex set S in $\mathbb{R}^n$: this determines the closed convex set in cosmic n-space $S \cup \text{dir}(0^+S)$—and improper ones—these are given by a closed convex cone K in $\mathbb{R}^n$ containing the origin: this determines the closed convex set in cosmic n-space $\mathrm{dir}K$.

Proof. The first two statements of the proposition follow immediately from the definitions. We only give the proof for the third statement. Let $A = S \cup \mathrm{dir}K$ be a convex set. It suffices to check that $\mathrm{cl}C(A)$ is equal to $C(\mathrm{cl}S \cup 0^+\mathrm{cl}S)$ if $S \neq \emptyset$ and that it equals $C(\emptyset \cup \mathrm{cl}K)$ if $S = \emptyset$.

- The case $S = \emptyset$. Then $\mathrm{cl}C(S \cup \mathrm{dir}K) = \mathrm{cl}(K \times 0) = (\mathrm{cl}K) \times 0 = C(\emptyset \cup \mathrm{cl}K)$, as required.
- The case $S \neq \emptyset$.
  - We check the inclusion $\mathrm{cl}C(A) \subset C(\mathrm{cl}S \cup \mathrm{dir}0^+\mathrm{cl}S)$. We take an arbitrary element $(x, \rho) \in \mathrm{cl}C(A)$. We choose an infinite sequence $(x_k, \rho_k)_k$ in $C(A)$ that converges to (x, ρ). If $\rho > 0$, then it suffices to show that $\rho^{-1}x \in \mathrm{cl}S$, as this implies $(x, \rho) = \rho(\rho^{-1}x, 1) \in \mathbb{R}_{++}(\mathrm{cl}S \times 1) \subset C(\mathrm{cl}S \cup 0^+\mathrm{cl}S)$. We assume that $\rho_k > 0$ for all k as we may by going over to a suitable subsequence. Then, for each k we have $(x_k, \rho_k) \in \mathbb{R}_{++}(S \times 1)$, and so $\rho_k^{-1}x_k \in S$. Taking the limit $k \to +\infty$ gives $\rho^{-1}x \in \mathrm{cl}S$, as required. If $\rho = 0$, then it suffices to show that $x \in 0^+\mathrm{cl}S$, as this implies $(x, \rho) = (x, 0) \in 0^+\mathrm{cl}S \times 1 \subset C(\mathrm{cl}S \cup 0^+\mathrm{cl}S)$. We distinguish two cases:
    - $\rho_k = 0$ for at most finitely many k; we assume that $\rho_k > 0$ for all k, as we may by going over to a suitable subsequence. Then for all k one has $(\rho_k^{-1}x_k, 1) \in C(A) \cap ((\mathbb{R}^n \times 1) = S \times 1$, that is $\rho_k^{-1}x_k \in S$. We take an arbitrary $a \in S$, and choose an infinite sequence $(a_k)_k$ in S that converges to a. Then $a + x = \lim_{k \to +\infty}(\frac{1}{1+\rho_k}a_k + \frac{\rho_k}{1+\rho_k}(\rho_k^{-1}x_k)) \in \mathrm{cl}S$. Therefore, $x \in 0^+S$, as required.
    - $\rho_k = 0$ for infnitely many k; we assume that $\rho_k = 0$ for all k, as we may by going over to a suitable subsequence. Then for all k one has $x_k \in K \subset 0^+S$. We take an arbitrary $a \in \mathrm{cl}S$ and choose an infinite sequence $(a_k)_k$ in S that converges to a. Then $a + x = \lim_{k \to +\infty} a_k + x_k$. One has for each k that $a_k + x_k \in S$ as $a_k \in S$ and $x_k \in 0^+S$. Therefore, $a + x \in \mathrm{cl}S$.
  - We check the inclusion $C(\mathrm{cl}S \cup 0^+\mathrm{cl}S) \subset \mathrm{cl}C(A)$. We take an arbitrary element $(x, \rho) \in C(\mathrm{cl}S \cup 0^+\mathrm{cl}S)$. If $\rho > 0$, then $(\rho^{-1}x, 1) \in (\mathbb{R}^n \times 1) \cap C(\mathrm{cl}S \cup 0^+\mathrm{cl}S) = \mathrm{cl}S \times 1$, and so $\rho^{-1}x \in \mathrm{cl}S$. We choose an infinite sequence $(y_k)_k$ in S that converges to $\rho^{-1}x$. Then $(x, \rho) = \lim_{k \to +\infty}(\rho y_k, \rho)$ and so $(x, \rho) \in \mathrm{cl}C(A)$. If $\rho = 0$, then we choose $a \in S$. We have $(x, \rho) = (x, 0) = \lim_{k \to +\infty}(k^{-1}a + x, k^{-1}) = \lim_{k \to +\infty} k^{-1}(a + kx, 1)$. As $a \in S$ and $x \in 0^+\mathrm{cl}S$, we have for all k that $a + kx \in \mathrm{cl}(\mathbb{R}_{++}(\mathrm{cl}S \times 1)) \subset \mathrm{cl}(\mathbb{R}_{++}(S \times 1)) \subset \mathrm{cl}C(A)$. It follows that $(x, \rho) \in \mathrm{cl}C(A)$.

Thus prepared, one can apply the conification method—in the following restricted sense—to convex sets. One can conify the convex sets of a given situation. Then one can 'work', taking linear images, inverse linear images and closures. Finally, one can always translate into the language of the original problem, that of convex sets. In order to apply the conification method in the full sense to convex sets, the dual of a convex set has to be defined. This requires the concept sublinear function.

Extended real valued sublinear functions as conifications of suitable convex cones. We have to consider *extended real valued sublinear functions* and not just real valued sublinear functions. A function h on $\mathbb{R}^n$ that takes values in $\mathbb{R} \cup \{\pm\infty\}$ is called an *extended real valued function.* For example, functions of the following type will arise: let T be a subset of $\mathbb{R}^n$ and let the function i_T on $\mathbb{R}^n$ be defined by $i_T(x) = -\infty$ for all $x \in T$ and $i_T(x) = +\infty$ for all $x \notin T$. The *epigraph* of such a function h is the set $\mathrm{epi}h = \{(x,\alpha) |\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{R},\ \alpha \geq h(x)\}$ in $\mathbb{R}^{n+1}$ and its *strict epigraph* is the set $\mathrm{sepi}h = \{(x,\alpha) |\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{R},\ \alpha > h(x)\}$ in $\mathbb{R}^{n+1}$. An extended real valued function p on $\mathbb{R}^n$ is called *sublinear* if its epigraph is a convex cone—or equivalently if its strict epigraph is a convex cone—and if, moreover, $p(0) \neq +\infty$. We recall the following fact. We need to distinguish two types of extended real valued sublinear functions p on $\mathbb{R}^n$: p is called *proper* if it does not assume the value $-\infty$, and it is called *improper* otherwise. The following result gives a well-known equivalent description of extended real valued sublinear functions.

Proposition 3.3 *The following mapping from the collection of convex cones C in $\mathbb{R}^{n+1}$ that contain the open ray generated by $(0_n, 1)$, $\mathbb{R}_{++}(0_n, 1)$, to the collection of all extended real valued sublinear functions p on $\mathbb{R}^n$ is surjective: $C \mapsto p(C)$, where $p(C)(x) = \inf\{\alpha | (x,\alpha) \in C\}$ for all $x \in \mathbb{R}^n$. For each extended real valued sublinear function p on $\mathbb{R}^n$, its inverse image under this mapping, consists of the convex cones C that are intermediate between the epigraph and the strict epigraph of p, that is,* $\mathrm{sepi}p \subset C \subset \mathrm{epi}p$.

In the situation of proposition 3.3, each one of the convex cones C for which $\mathrm{sepi}p \subset C \subset \mathrm{epi}p$ will be called a *conification* of p; the extended real valued sublinear function $p(C)$ will be called the *deconification* of C. We note that the conification of an extended real valued sublinear function is always nonempty. We will often call extended real valued sublinear functions on $\mathbb{R}^n$ just sublinear functions (on $\mathbb{R}^n$) and then we call real valued sublinear functions on $\mathbb{R}^n$ ordinary sublinear functions (on $\mathbb{R}^n$). The following criterion for properness of an extended real valued sublinear function p in terms of any one of its conifications C holds true: p is proper precisely if $(0_n, -1) \notin \mathrm{cl}C$.

Linear images, inverse linear images and closure of sublinear functions. We recall the usual definition of linear images, inverse linear images and the closure of arbitrary extended real valued functions. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let h be an extended real valued function on $\mathbb{R}^n$, and $\bar{h}$ an extended real valued function on $\mathbb{R}^m$. Then the *linear image* Λh is the extended real valued function on $\mathbb{R}^m$ defined by $\Lambda h(y) = \inf\{h(x) | \Lambda x = y\}$ for all $y \in \mathbb{R}^m$. The *inverse linear image* $\bar{h}\Lambda$ is the extended real valued function on $\mathbb{R}^n$ defined by $(\bar{h}\Lambda)(x) = \bar{h}(\Lambda x)$ for all $x \in \mathbb{R}^n$. The *closure* $\mathrm{cl}h$ is the extended real valued function on $\mathbb{R}^n$ defined by the equality $\mathrm{epi}(\mathrm{cl}h) = \mathrm{cl}(\mathrm{epi}h)$.

The conification method gives definitions for the concepts linear image, inverse linear image and closure of sublinear functions. The following proposition states formally the easy observation that these operations are just the usual ones. As before, a linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ induces a linear transformation $\Lambda : \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$, again denoted by Λ, by means of the trivial action on the last coordinate, $(x, \alpha) \mapsto (\Lambda x, \alpha)$.

Proposition 3.4 *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let p be an extended real valued sublinear function on $\mathbb{R}^n$ and $\bar{p}$ an extended real valued sublinear function on $\mathbb{R}^m$.*

1. *Let C be a conification of p. Then ΛC is a convex cone in $\mathbb{R}^{m+1}$ that contains the open ray generated by $(0_m, 1)$. Its deconification is the sublinear function Λp on $\mathbb{R}^m$.*
2. *Let $\bar{C}$ be a conification of $\bar{p}$. Then $\bar{C}\Lambda$ is a convex cone in $\mathbb{R}^{n+1}$ that contains the open ray generated by $(0_n, 1)$. Its deconification is the sublinear function $\bar{p}\Lambda$ on $\mathbb{R}^n$.*
3. *Let C be a conification of p. Then $\mathrm{cl}C$ is a convex cone in $\mathbb{R}^{n+1}$ that contains the open ray generated by $(0_n, 1)$. Its deconification is the sublinear function $\mathrm{cl}p$ on $\mathbb{R}^n$.*

A sublinear function p is called *closed* if $p = \mathrm{cl}p$. There are two types of closed sublinear functions: proper ones—these are given by a real valued closed sublinear function on a convex cone K in $\mathbb{R}^n$: this determines a function on $\mathbb{R}^n$ by assigning the value $+\infty$ to points outside K—and improper sublinear functions—these are given by a closed convex cone K in $\mathbb{R}^n$ containing the origin: this determines the function i_K.

Proof. Immediate from the definitions.

Thus prepared, one can apply the conification method to sublinear functions in the following restricted sense. One can conify the sublinear functions of a given situation. Then one can work, taking linear images, inverse linear images and closures. Finally, one can translate into the language of the original problem, that of sublinear functions. In order to apply the conification method in a full sense to convex sets and sublinear functions, we have to define the dual objects for convex sets and sublinear functions.

The duality operator for convex sets in cosmic space. We will take the polar cone of convex cones C in $\mathbb{R}^{n+1}$ with respect to the following symmetric nondegenerate bilinear form on $\mathbb{R}^{n+1}$:

$$\langle (x, \alpha), (y, \beta) \rangle = x^T y - \alpha\beta \qquad (*)$$

for all $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. That is, $C^\circ = \{w \in \mathbb{R}^{n+1} | \langle w, v \rangle \leq 0 \text{ for all } v \in C\}$. The *support function* of an ordinary convex set S in $\mathbb{R}^n$ is the sublinear function on $\mathbb{R}^n$ defined by $sS(y) = \sup\{y^T x | x \in S\}$ for all $y \in \mathbb{R}^n$. For an extended real valued function h on $\mathbb{R}^n$ and a subset $T \subset \mathbb{R}^n$, the restriction of h to T is defined to be the extended real valued function $h|_T$ on $\mathbb{R}^n$ for which $h|_T(x) = h(x)$ for all $x \in T$ and $h|_T(x) = +\infty$ for all $x \notin T$. The motivation for this definition is that convex functions often arise in the context of minimization problems and that the problem to minimize $h(x)$ under the constraint $x \in T$ is equivalent to the problem to minimize $h|_T(x)$ without constraint.

Proposition 3.5 ***Dual object of a convex set in cosmic space.*** *Let $A = S \cup \mathrm{dir}K$ be a convex set in $\mathrm{csm}\mathbb{R}^n$. Let $C = C(A)$ be the conification of A. Then the polar cone C° contains the open ray generated by $(0_n, 1)$. Moreover, the deconification of C° is the extended real valued sublinear function $p(A) = sS|_{K^\circ}$ on $\mathbb{R}^n$.*

Thus the conification method for constructing the duality operator—conify, apply the polar operator, deconify— gives the following duality operator from convex sets in cosmic space to extended real valued sublinear functions:

$$A = S \cup \mathrm{dir}K \mapsto p(A) = sS|_{K^\circ}.$$

This formula covers two distinct cases. To see this, we restrict attention to the dual object of *closed* convex sets in cosmic space. This does not restrict the generality in the sense that a convex set and its closure have the same dual object. For a proper closed convex set $A = S$, the dual object is the proper sublinear function $p(A) = sS$ and for an improper closed convex set $A = \mathrm{dir}K$, the dual object is the improper sublinear function i_{K°.

Proof. As C is the conification of A, it is contained in the half space $x_{n+1} \geq 0$. The polar cone of this half space is the closed ray $\mathbb{R}_+(0_n, 1)$ generated by $(0_n, 1)$. Therefore, C° contains the open ray generated by $(0_n, 1)$. To prove the last statement of the proposition, it suffices to check the equality $C^\circ = \mathrm{epi}(sS|_{K^\circ})$. For an element $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ to belong to C° means that $\langle (y, \beta), (x, 1) \rangle = y^T x - \beta \leq 0$ for all $x \in S$, and $\langle (y, \beta), (x, 0) \rangle = y^T x \leq 0$ for all $x \in K$. This can be formulated as follows: $(y, \beta) \in \mathrm{epi}(sS)$ and $y \in K^\circ$. That is, (y, β) is an element of $\mathrm{epi}(sS|_{K^\circ})$. This completes the verification of the equality $C^\circ = \mathrm{epi}(sS|_{K^\circ})$.

Now the conification method for convex sets is fully developed: we conify the convex sets of a given situation, then we work, taking linear images, inverse linear images, closures and dual objects, and finally we translate into the language of the original problem, that of convex sets, or—if we have gone over to a dual object—sublinear functions.

The duality operators for extended real valued sublinear functions. The *subdifferential* of a sublinear function p is defined to be the ordinary convex set $\partial p = \{y \in \mathbb{R}^n | y^T x \leq p(x) \text{ for all } x \in \mathbb{R}^n\}$. For each extended real valued function h on $\mathbb{R}^n$, its *effective domain* is defined to be $\mathrm{dom}h = \{x \in \mathbb{R}^n | h(x) < +\infty\}$. The effective domain of a sublinear function is a convex cone.

Proposition 3.6 ***Dual object of an extended real valued sublinear function.*** *Let p be an extended real valued sublinear function on $\mathbb{R}^n$. Let C be a conification of p. Then the polar cone C° contains the origin and is contained in the half space $x_{n+1} \geq 0$. Moreover, its deconification is the convex set in cosmic space $A(p) = \partial p \cup \mathrm{dir}((\mathrm{dom} p)^\circ)$.*

Thus the conification method for constructing duality operators—conify, apply the polar operator, deconify—gives the following duality operator from extended real valued sublinear functions to convex sets in cosmic space:

$$p \mapsto A(p) = \partial p \cup \mathrm{dir}((\mathrm{dom} p)^\circ).$$

This formula covers two distinct cases. To see this, we now restrict attention to the dual objects of *closed* extended valued sublinear functions. This does not restrict the generality in the sense that a sublinear function and its closure have the same dual object. For a proper closed sublinear function p, the dual object is the proper convex set $\partial p \cup \mathrm{dir}(0^+(\partial p))$; for an improper closed sublinear function i_K it is the improper convex set $\mathrm{dir}(K^\circ)$.

Proof. As C is a conification of p, it contains the open ray generated by $(0_n, 1)$. The polar cone of this ray is the half space $x_{n+1} \geq 0$. Therefore, C° contains the origin and is contained in the half space $x_{n+1} \geq 0$. To prove the last statement of the proposition, it suffices to check the equality $C^\circ = C(\partial p \cup \mathrm{dir}((\mathrm{dom} p)^\circ))$. By proposition 3.3, $\mathrm{epi} p \subset C \subset \mathrm{sepi} p$. This implies $(\mathrm{epi} p)^\circ \supset C^\circ \supset (\mathrm{sepi} p)^\circ$. Clearly $\mathrm{epi} p$ and $\mathrm{sepi} p$ have the same closure, and so their polar cones are equal. It follows that $(\mathrm{epi} p)^\circ = C^\circ = (\mathrm{sepi} p)^\circ$. Therefore, to prove the last statement of the proposition, it suffices to check the equality $(\mathrm{epi} p)^\circ = C(\partial p \cup \mathrm{dir}((\mathrm{dom} p)^\circ))$. For an element $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}$ to belong to $(\mathrm{epi} p)^\circ$ means that $\langle (y, \beta), (x, \alpha) \rangle \leq 0$ for all $(x, \alpha) \in \mathrm{epi} p$. That is, $x^T y - \beta\alpha \leq 0$ for all $(x, \alpha) \in \mathrm{epi} p$. For $\beta = 1$ this reduces to $y^T x \leq 0$ for all $(x, \alpha) \in \mathrm{epi} p$, that is, $y \in (\mathrm{dom} p)^\circ$. That is, (y, β) is an element of $C(\partial p \cup \mathrm{dir}((\mathrm{dom} p)^\circ))$. This completes the verification of the equality $(\mathrm{epi} p)^\circ = C(\partial p \cup \mathrm{dir}((\mathrm{dom} p)^\circ))$.

Now the conification method for sublinear functions is fully developed: we conify the sublinear functions of a given situation, then we work, taking linear images, inverse linear images, closures and dual objects, and finally we translate into the language of the original problem, that of sublinear functions, or—if we have gone over to a dual object—convex sets.

Standard facts for convex sets and sublinear functions. Now we are ready to give the standard facts for convex sets and sublinear functions. We call a convex set $A = S \cup \mathrm{dir} K$ or a sublinear function p *polyhedral* if it has a conification that is polyhedral. This gives for A that S and K are intersections of finite collections of closed half spaces, and for p that its epigraph is the intersection of a finite collection of closed half spaces.

Theorem 3.7 *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ and $\Theta : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations in duality. Let*

$A = S \cup \mathrm{dir} K$ be a closed convex set in $\mathrm{csm}\mathbb{R}^n$ and let p be a closed extended real valued sublinear function on $\mathbb{R}^n$. Then the following two statements hold true.

1. *$A = \partial p \cup \mathrm{dir}(\mathrm{dom} p)^\circ$ precisely if $p = sS|_{K^\circ}$ (then A and p are said to be in duality). Under this correspondence, proper convex sets correspond to proper sublinear functions, and improper convex sets to improper sublinear functions.*
2. *If p and A are in duality, then*
   - *Λp and $A\Theta$ are in duality, after taking closures,*
   - *ΛA and $p\Theta$ are in duality, after taking closures.*

   Moreover, each one of the following two assumptions implies that Λp (resp. ΛA) is closed:

   - *if $S \neq \emptyset$, then $\mathrm{ri} S \cap \Theta(\mathbb{R}^m) \neq \emptyset$; if $S = \emptyset$, then $\mathrm{ri} K \cap \Theta(\mathbb{R}^m) \neq \emptyset$ (resp. $\mathrm{ridom} p \cap \Theta(\mathbb{R}^m) \neq \emptyset$);*
   - *A or p—and so both—are polyhedral.*

Proof. Conify, apply the statements of theorem 2.1, and deconify.

Novelty. The novelty of this section is the complete development of the conification method for convex sets and sublinear functions. The convex cone point of view shows how to make the 'tricky' deconification step always possible: by working with convex sets in cosmic space rather than with ordinary convex sets, and by working with the 'right' concept of sublinear functions. To give an explicit description of the duality between convex sets and sublinear functions, the distinction in proper and improper objects is introduced in the present paper. The propositions 3.5, 3.6 and theorem 3.7 appear to be new in the generality that is required for the conification method: that of convex sets in cosmic space and extended real valued sublinear functions. For the classes of convex sets in cosmic space and extended real valued sublinear functions, the operations linear image, inverse linear image, closure, the duality operators support function and subdifferential, are always defined and the duality theorem '$p(A) = p$ precisely if $A(p) = A$' holds for all closed convex sets A in cosmic space and all closed extended real valued sublinear functions p.

Why convex sets in cosmic space are needed. It is well known how to conify convex sets in $\mathbb{R}^n$. The resulting collection of conifications is in some sense not complete. To repair this 'imperfection', it is natural to complete the collection of conifications, and to view the added convex cones also as conifications, of convex sets in some extended sense: convex sets in cosmic space. In fact there is a more compelling reason for working with convex sets in cosmic space. If one conifies ordinary convex sets and then works with convex cones, this might lead to one of the added convex cones, so deconification in the usual sense is not possible. To be more precise about the obstacle, conifications of ordinary convex sets in a vector space V are convex cones in the product space $V \times \mathbb{R}$ that are contained in the upper half space $H = V \times [0, +\infty)$ and that have no nonzero points in common with the hyperplane that bounds H, $L = V \times 0$. Therefore, from the convex cone point of view, it is natural to consider the collection of *all* convex cones that contain the origin and that are contained in H. Then deconification leads to convex sets in cosmic space, as is shown above. The inclusion of the origin in the conification is forced by the requirement that conifications are nonempty sets. This in its turn is dictated by the fact that the duality theorem (statement 1 from theorem 2.1) does not hold for the empty convex cone.

How the convex cone point of view leads to the 'right' concept of sublinear functions. To define the duality operator for convex sets by the conification method, one starts with a convex set in cosmic n-space, and then one conifies it. This is a convex cone contained in the closed upper half space $H = \mathbb{R}^n \times [0, +\infty)$. Therefore, its polar cone is a closed convex cone that contains the ray $R = H^\circ = \mathbb{R}_+(0_n, 1)$. This cone is the epigraph of an extended real valued function on $\mathbb{R}^n$. It is natural to define this function as the dual object of the given convex set in cosmic n-space. This suggests to define a sublinear function on $\mathbb{R}^n$ to be an extended real valued function on $\mathbb{R}^n$ for which the graph is a convex cone, and that does not take the value $+\infty$ at the origin.

Why the conification of a sublinear function is not unique. At first sight, it might seem to be a good idea to define the conification of a sublinear function to be its epigraph. However, then working with these conifications might lead to convex cones that are not epigraphs. For example, $\Lambda(\mathrm{epi}p)$, the image of the epigraph of a sublinear function p on $\mathbb{R}^n$ under a linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$, need not be an epigraph of a function. The solution to this problem is to consider all convex cones that are intermediate between the strict epigraph and the epigraph of p to be conifications of p.

Comparison with the literature. The idea of viewing recession directions of convex sets as points of these sets is due to Steinitz [25]. This idea is developed further for example in [22] and in the treatment of the cosmic space in [24]. The propositions 3.1, 3.2, 3.3 and 3.4 are essentially contained in [24]. For example, proposition 3.2 follows readily from the statement in [24] on p.87 that a function is sublinear precisely if its epigraph is a convex cone. Note that there is a sign difference between the convention in the present paper to let a point $x \in \mathbb{R}^n$ be represented by the ray generated by $(x, 1)$ and the convention in [24] to let it be represented by the ray generated by $(x, -1)$; the present choice is made to avoid minus signs in certain formulae. Proposition 3.5, 3.6 and theorem 3.7 appear to be novel. In the special case of ordinary convex sets and sublinear functions, these results are well-known and are for example given in [22]. Indeed, then propositions 3.5, 3.6 and statement 1 of theorem 3.7 are consequences of theorem 13.2 in [22], which gives the duality between the indicator function and the support function of a closed convex set; statement 2 of theorem 3.7 and the first closedness statement from cor.16.3.1 in [22] on the indicator function and the support function of a convex set; the closedness statement in theorem 3.7 under polyhedrality assumptions follows then from theorem 19.3 and cor.19.3.1 in [22].

Often a stricter concept of sublinear function is given than the one used in the present paper: in Bourbaki (II p.20) [5] it is required to take only finite values, in Hiriart-Urruty and Lemaréchal (I p.197) [12] it is required to be proper but it can take the value $+\infty$ at 0, in Borwein and Lewis (p.33) [4] it is not allowed to take the value $-\infty$, and it has to take value 0 at 0; the definition in the present paper is the same as in [24](p. 87). The convex cone point of view leaves no room for doubt on the choice of definition: it leads automatically to this latter definition.

4 The standard operations and facts for convex functions.

The aim of this section is to construct in complete generality the standard operations and facts for convex functions—by means of the conification method. We will show that the operations linear image, inverse linear image, closure and the duality operator are defined for all convex functions. The duality theorem will be shown to hold for *all* closed convex functions. This requires that recession directions are allowed as points of epigraphs of convex functions, that is, *convex functions on cosmic space*, to be defined below, have to be considered. For convex functions on cosmic space, the complete version of the 'standard operations and facts', given in this section, appears to be novel.

Convex functions on cosmic space as deconification of suitable convex cones. An *extended real valued function* f on $\mathrm{csm}\mathbb{R}^n$ is defined to be a pair $f = (g, q)$ consisting of a convex function g on $\mathbb{R}^n$ and a sublinear function q on $\mathbb{R}^n$ for which $\mathrm{epi}g + \mathrm{epi}q \subset \mathrm{epi}g$. For a convex function g on $\mathbb{R}^n$, its *recession function* $g0^+$ is the sublinear function defined by $\mathrm{epi}(g0^+) = 0^+(\mathrm{epi}g)$. The terminology 'convex function on cosmic space' is suggested by the possibility to view the pair $f = (g, q)$ as a function on cosmic n-space: as the function that take the same value as g at points of $\mathbb{R}^n$ and that takes the same value as q at points on the horizon, if we let these points be represented by unit vectors. However, we will work with the representation of convex functions on cosmic space as pairs (g, q), as this is more convenient. An extended real valued sublinear function p is a special case of a convex function on cosmic n-space: to wit, (p, p). We need to distinguish the following types of convex functions $f = (g, q)$ on cosmic n-space:

1. f is called *proper* if $g \not\equiv +\infty$ and, moreover, g does not assume the value $-\infty$;
2. f is called *improper of convex set type* if $g \not\equiv +\infty$ and, moreover, g assumes the value $-\infty$;
3. f is called *improper of sublinear function type* if $g \equiv +\infty$ and, moreover, q does not assume the value $-\infty$;

4. f is called *improper of convex cone type* if $g \equiv +\infty$ and, moreover, q assumes the value $-\infty$.

The motivation for introducing these concepts and for this terminology will be given below. The following result can be verified in the same way as the corresponding results for convex sets (proposition 3.1) and sublinear functions (proposition 3.3).

Proposition 4.1 *The following mapping from the collection of convex cones C in $\mathbb{R}^{n+2}$ that are contained in the upper half space $x_{n+2} \geq 0$ and that contain the open ray generated by $(0_n, 1, 0)$ to the collection of convex functions on* $\mathrm{csm}\mathbb{R}^n$ *is surjective: $C \mapsto f(C) = (g(C), q(C))$, where $g(C)(x) = \inf\{\alpha | (x, \alpha, 1) \in C\}$ for all $x \in \mathbb{R}^n$ and $q(C)(y) = \inf\{\beta | (y, \beta, 0) \in C\}$ for all $y \in \mathbb{R}^n$. For each convex function $f = (g, q)$, its inverse image under this mapping consists of all convex cones C that are intermediate between the convex cones in $\mathbb{R}^{n+2}$ that are naturally associated to the pairs* $(\mathrm{sepi}g, \mathrm{sepi}q)$ *and* $(\mathrm{epi}g, \mathrm{epi}q)$*: $C_l(f) = \mathbb{R}_{++}((\mathrm{sepi}g) \times 1) \cup \mathbb{R}_{++}((\mathrm{sepi}q) \times 0)$ and $C_u(f) = \mathbb{R}_{++}((\mathrm{epi}g) \times 1) \cup \mathbb{R}_{++}((\mathrm{epi}q) \times 0)$. That is, $C_l(f) \subset C \subset C_u(f)$.*

In the situation of proposition 4.1, the convex cones C for which $C_l(f) \subset C \subset C_u(f)$ are called *conifications* of f, and $f(C)$ is called the *deconification* of C. We note that the conification of a convex function on cosmic space is always nonempty. We will often call convex functions on cosmic n-space just convex functions (on $\mathbb{R}^n$) and then we call convex functions on $\mathbb{R}^n$ ordinary convex functions (on $\mathbb{R}^n$). The following criteria for the properness of a convex function $f = (g, q)$ in terms of any one of its conifications C hold true:

1. f is proper precisely if C is not contained in the horizontal hyperplane $\mathbb{R}^{n+1} \times 0$ and, moreover, $\mathrm{cl}C$ does not contain the open ray generated by $(0_n, -1, 0)$;

2. f is improper of convex set type precisely if C is not contained in the horizontal hyperplane $\mathbb{R}^{n+1} \times 0$ and, moreover, $\mathrm{cl}C$ contains the open ray generated by $(0_n, -1, 0)$;

3. f is improper of sublinear function type precisely if C is contained in the horizontal hyperplane $\mathbb{R}^{n+1} \times 0$ and, moreover, $\mathrm{cl}C$ does not contain the open ray generated by $(0_n, -1, 0)$;

4. f is improper of convex cone type precisely if C is contained in the horizontal hyperplane $\mathbb{R}^{n+1} \times 0$ and, moreover, $\mathrm{cl}C$ contains the open ray generated by $(0_n, -1, 0)$.

Linear images, inverse linear images and closure of convex functions. We want to give direct descriptions of linear images, inverse linear images and the closure of convex functions, defined by the conification method. A linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ induces a linear transformation $\mathbb{R}^{n+2} \to \mathbb{R}^{m+2}$, again denoted by Λ, by means of the trivial action on the last two coordinates, $(x, \alpha, \beta) \mapsto (\Lambda x, \alpha, \beta)$.

Proposition 4.2 *Let* $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation. Let* $f = (g, q)$ *be a convex function on* $\mathrm{csm}\mathbb{R}^n$ *and* $\bar{f} = (\bar{g}, \bar{q})$ *a convex function on* $\mathrm{csm}\mathbb{R}^m$*.*

1. *Let* C *be a conification of* f*. Then* ΛC *is a convex cone in* $\mathbb{R}^{m+2}$ *that is contained in the upper half space* $x_{m+2} \geq 0$ *and that contains the open ray generated by* $(0_m, 1, 0)$*. Its deconification is the convex function* $(\Lambda g, \Lambda q)$ *on* $\mathrm{csm}\mathbb{R}^m$*.*

2. *Let* $\bar{C}$ *be a conification of* $\bar{f}$*. Then* $\bar{C}\Lambda$ *is a convex cone in* $\mathbb{R}^{n+2}$ *that is contained in the upper half space* $x_{n+2} \geq 0$ *and that contains the open ray generated by* $(0_n, 1, 0)$*. Its deconification is the convex function* $(\bar{g}\Lambda, \bar{q}\Lambda)$ *on* $\mathrm{csm}\mathbb{R}^n$*.*

3. *Let* C *be a conification of* f*. Then* $\mathrm{cl}C$ *is a convex cone in* $\mathbb{R}^{n+2}$ *that is contained in the upper half space* $x_{n+2} \geq 0$ *and that contains the open ray generated by* $(0_n, 1, 0)$*. Its deconification is the convex function* $(\mathrm{cl}g, (\mathrm{cl}g)0^+)$ *if* f *is not improper of type* $+\infty$ *and it equals* $(+\infty, \mathrm{cl}q)$ *if* f *is improper of type* $+\infty$*.*

Proof. These statements can be verified in a similar way as the corresponding statements for convex sets and sublinear functions, the propositions 3.2 and 3.4.

This proposition gives the following explicit formulae for the operations linear image, inverse linear image and closure, that are defined on convex functions by the conification method:

1. linear image: $f = (g, q) \mapsto \Lambda f = (\Lambda g, \Lambda q)$;

2. inverse linear image: $\bar{f} = (\bar{g}, \bar{q}) \mapsto \bar{f}\Lambda = (\bar{g}\Lambda, \bar{q}\Lambda)$;

3. closure: $f = (g, q) \mapsto \mathrm{cl}f$, where $\mathrm{cl}f = (\mathrm{cl}g, (\mathrm{cl}g)0^+)$ if f is not improper of type $+\infty$ and it equals $(+\infty, \mathrm{cl}q)$ if f is improper of type $+\infty$.

A convex function f is called *closed* if $f = \mathrm{cl}f$. The following types of closed convex functions on cosmic n-space can be distinguished:

1. proper ones: these are given by a real valued closed convex function g on a nonempty convex set S; this determines a closed convex function on cosmic n-space $(g, g0^+)$, where in g one assigns the value $+\infty$ outside S;

2. improper ones of convex set type: these are given by a nonempty closed convex set S in $\mathbb{R}^n$; this determines the function $(i_S, (i_S)0^+)$;

3. improper ones of sublinear function type: these are given by a real valued closed sublinear function p on a convex cone K in $\mathbb{R}^n$ containing the origin; this determines the function $(+\infty, p)$, where in p one assigns the value $+\infty$ outside K;

4. improper one of convex cone type: these are given by a closed convex cone K in $\mathbb{R}^n$ containing the origin; this determines the function $(+\infty, i_K)$.

Thus prepared, one can apply the conification method in the following restricted sense to convex functions. One can conify the convex functions of a given situation. Then one can work, taking linear images, inverse linear images and closures. Finally, one can translate into the language of the original problem, that of convex functions. In order to apply the conification method in a full sense, we need to define the duality operator for convex functions.

The duality operator for convex functions on cosmic spaces. We will take the duality of convex cones in $\mathbb{R}^{n+2}$ with respect to the following symmetric nondegenerate bilinear form on $\mathbb{R}^{n+2}$

$$\langle (x,\alpha,\alpha'),(y,\beta,\beta')\rangle = \langle x,y\rangle - \alpha\beta' - \alpha'\beta,$$

for all $x, y \in \mathbb{R}^n$ and all $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$ (we emphasize the permutation of the last two coordinates that is involved in the formula defining the form). That is, $C^\circ = \{w \in \mathbb{R}^{n+2} | \langle w, v\rangle \le 0 \text{ for all } v \in C\}$. The *Legendre-Fenchel transform* of an ordinary convex function h on $\mathbb{R}^n$ is the ordinary convex function h^* on $\mathbb{R}^n$ defined by $h^*(y) = \sup_y(y^T x - h(x))$ for all $y \in \mathbb{R}^n$. The effective domain $\mathrm{dom}h$ of an ordinary convex function h on $\mathbb{R}^n$ is a convex set in $\mathbb{R}^n$.

Proposition 4.3 *Let $f = (g, q)$ be a convex function on cosmic n-space. Let C be a conification of f. Then the polar cone C° is contained in the upper half space $x_{n+2} \ge 0$ and it contains the open ray generated by $(0_n, 1, 0)$. Its deconification is the convex function on the cosmic n-space $(g^*|_{\partial q}, s(\mathrm{dom}g)|_{(\mathrm{dom}q)^\circ})$.*

Thus the conification method for constructing duality operators—conify, apply the polar operator, deconify—gives the following duality operator from convex functions on cosmic n-space to itself:

$$f = (g, q) \mapsto f^\circ = (g^*|_{\partial q}, s(\mathrm{dom}g)|_{(\mathrm{dom}q)^\circ}).$$

This formula covers the following distinct cases:

1. from proper closed convex functions to proper closed convex functions: for a real valued closed convex function g on a nonempty convex set S, the dual object of $f = (g, g0^+)$ is $f^\circ = (g^*, (g^*)0^+)$;

2. from improper closed convex functions of convex set type to improper closed convex functions of sublinear function type: for a nonempty closed convex set S in $\mathbb{R}^n$, the dual object of $f = (i_S, (i_S)0^+)$ is $f^\circ = (+\infty, sS)$;

3. from improper closed convex functions of sublinear function type to improper closed convex functions of convex set type: for a real valued closed sublinear function p on a convex cone K in $\mathbb{R}^n$ containing the origin, the dual object of $f = (+\infty, p)$ is $f^\circ = (i_{\partial p}, (i_{\partial p})0^+)$.

4. from improper convex functions of convex cone type to improper convex functions of convex cone type: for a closed convex cone K in $\mathbb{R}^n$, the dual object of $f = (+\infty, i_K)$ is $f^\circ = (+\infty, i_{K^\circ})$.

Now we give the proof of proposition 4.3.

Proof. It suffices to check the equality $C_u(f)^\circ = C(g^*|_{\partial q}, s(\text{dom}g)|_{\text{dom}q})$. For an element $(y, \beta, \delta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ to belong to $C_u(f)^\circ$ means that $\langle (y, \beta, \delta), (x, \alpha, \gamma) \rangle \leq 0$ for all $(x, \alpha, \gamma) \in C_u(f)$. By the definition of $C(f)$, this amounts to the conditions $\langle (y, \beta, \delta), (x, \alpha, 1) \rangle = \langle y, x \rangle_n - \beta - \delta\alpha \leq 0$ for all $(x, \alpha) \in \text{epi}g$, and $\langle (y, \beta, \delta), (x, \alpha, 0) = \langle y, x \rangle_n - \delta\alpha \leq 0$ for all $(x, \alpha) \in \text{epi}q$. For $\delta = 1$ this reduces to $\beta \geq \langle y, x \rangle_n - \alpha$ for all $(x, \alpha) \in \text{epi}g$, that is, $(y, \beta) \in \text{epi}g^*$, and $\alpha \geq \langle y, x \rangle_n$ for all $(x, \alpha) \in \text{epi}q$, that is, $y \in \partial q$; these conditions can be written as $(y, \beta) \in \text{epi}(g^*|_{\partial q})$. For $\delta = 0$ this reduces to $\beta \geq \langle y, x \rangle_n$ for all $(x, \alpha) \in \text{epi}g$, that is, $(y, \beta) \in \text{epi}s(\text{dom}g)$, and $\langle y, x \rangle_n \leq 0$ for all $(x, \alpha) \in \text{epi}q$, that is, $y \in (\text{dom}q)^\circ$; these conditions can be written as $(y, \beta) \in \text{epi}(s(\text{dom}g)|_{(\text{dom}q)^\circ})$. In all, we obtain that $(y, \beta, \delta) \in C(g^*|_{\partial q}, s(\text{dom}g)|_{(\text{dom}q)^\circ})$.

Now the conification method for convex functions is fully developed: we conify the convex functions of a given situation, then we work, taking linear images, inverse linear images, closures and dual objects, and finally we translate into the language of the original problem, that of convex functions.

The standard facts for convex functions. Now we are ready to give the standard facts for convex functions. A convex function $f = (g, q)$ on $\text{csm}\mathbb{R}^n$ is called *polyhedral* if it has a conification that is polyhedral. Explicitly, the function is polyhedral if the epigraphs of g and q are intersections of finite collections of closed half spaces.

Theorem 4.4 *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ and $\Theta : \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations in duality. Let f and h be closed convex functions on* $\text{csm}\mathbb{R}^n$*.*

1. *$f = h^\circ$ precisely if $h = f^\circ$ (then f and h are said to be in duality). Under this correspondence, a proper f corresponds to a proper h, an improper f of convex set type to an improper h of sublinear function type, and an improper f of convex cone type to an improper h of convex cone type.*

2. *If f and h are in duality, then Λf and $h\Theta$ are in duality, after taking closures.*

   If one of the following two assumptions holds true, then Λf is closed:

   (a) $\Lambda(\mathbb{R}^n) \cap \text{ridom}h \neq \emptyset$.

(b) f or h—and so both—are polyhedral.

Proof. Conify, apply the statements of theorem 2.1 and deconify.

Remark. The important inf-operator $f(x, y) \mapsto \inf_x f(x, y)$ is a special case of the considered operations (and so the second formula from theorem 4.4 can be applied). Indeed, consider the linear transformation $\Lambda : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. Then for each convex function f on $\mathbb{R}^n \times \mathbb{R}^m$, the image Λf is precisely the function $y \mapsto \inf_x f(x, y)$. In a similar way inf-operators on other types of convex objects can be reduced to the linear image operator.

Novelty. The novelty of this section is the complete development of the conification method for convex functions. The convex cone point of view shows how to make the 'tricky' deconification step always possible: by working with convex functions on cosmic space rather than with ordinary convex functions. The concept convex function on cosmic n-space, and its distinction in four types, appears to be novel. Propositions 4.2 and theorem 4.4 appear to be new in the generality that is required for the conification method: convex functions on cosmic space. For the class of convex functions on cosmic space and extended real valued sublinear functions, the operations linear image, inverse linear image, closure, the duality operator $f \mapsto f^\circ$ are always defined and the duality theorem '$f^\circ = h$ precisely if $h^\circ = f$' holds for all closed convex functions f and h on cosmic space.

Why convex functions on cosmic space are needed. It is well-known how to conify a convex function on $\mathbb{R}^n$. The resulting collection of conifications is in some sense not complete. To repair this 'imperfection', it is again natural to complete the collection of conifications, and to view the added convex cones also as conifications, of convex functions in some extended sense: this leads to the concept convex functions on cosmic space. Again there is a more compelling reason for working with convex functions on cosmic space. If one conifies ordinary convex functions, then works with convex cones, this might lead to one of the added convex cones, so deconification in the usual sense is not possible. To be more precise, conifications of ordinary convex functions in a vector space V are convex cones in the product space $V \times \mathbb{R}^2$ that contain the ray R generated by $(0_V, 1, 0)$, are contained in the upper half space $H = V \times \mathbb{R} \times [0, +\infty)$, and that have no nonzero points in common with the hyperplane L that bounds H, $L = V \times \mathbb{R} \times 0$. Therefore, from the convex cone point of view, it is natural to consider *all* convex cones in the product space $V \times \mathbb{R}^2$ that contain the ray R generated by $(0_V, 1, 0)$, and are contained in the upper half space $H = V \times \mathbb{R} \times [0, +\infty)$. Then deconification leads to convex functions on cosmic space, as is shown above.

Why the conification of a convex function is not unique. This runs parallel to the situation for sublinear functions.

Comparison with the literature. The statements of the propositions 4.2, 4.3 and theorem 4.4 are well-known for the special case of ordinary convex functions. For example, these can be found in [22]: proposition 4.3 is then the statement that the duality operator on proper convex functions that is defined by the conification method is equal to the Legendre-Fenchel transform: throughout [22], the compatibility of duality operators for different types of convex objects is emphasized and used; theorem 12.2 of [22] gives statement 1 of theorem 4.4. for the case of ordinary closed convex functions; theorem 16.3 of [22] gives the other statements of theorem 4.4 apart from the closedness statement under assumption (b): this follows from theorem 19.1 (a) $\Leftrightarrow$ (c) from [22]. In [22], along with ordinary convex functions, recession directions of their epigraphs are investigated. The concept convex function on cosmic space, the distinction of four types, proposition 4.2 and theorem 4.4 represent further developments. In [22] the closure of improper convex functions is not defined by means of taking the closure of the epigraph; it is always defined to be the function $-\infty$. Therefore, the results in the present paper represent sharper results. More generally, there is no need in the present paper to pay any special attention to improper convex functions: all results are derived in a straightforward way for proper and improper convex functions.

5 Binary operations and their duality

The aim of this section is to construct in complete generality the standard binary operations and their duality formulae by means of the conification method for the four main classes of convex objects, considered on the right level of generality: convex cones containing the origin, convex sets in cosmic space, extended real valued sublinear functions, and convex functions on cosmic space. For these, the standard binary operations and their duality formulae are novel.

Working with convex cones. To begin with, we work on the level of convex cones. We will define, for each finite sequence of natural numbers $n_1, \ldots, n_r$ and each sequence ω of length r, consisting of plus and minus symbols, a binary operation $(C_1, C_2) \mapsto C_1 \odot_\omega C_2$ on the convex cones in $\mathbb{R}^n$, where $n = \sum_{i=1}^r n_i$. In terms of the two actions of linear transformations on convex cones—taking the image and taking the inverse image—this binary operation can be defined as follows. We consider the special linear transformations 'sum' $+_k : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k : (x_1, x_2) \mapsto x_1 + x_2$ and 'diagonal' $\Delta_k : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k : x \mapsto (x, x)$. We have the factorization $\mathbb{R}^n = \prod_{i=1}^r \mathbb{R}^{n_i}$. We define

$$C_1 \odot_\omega C_2 = (\Lambda_1, \ldots, \Lambda_r)(C_1 \times C_2)$$

where $\Lambda_i = +_{n_i}$ if $\omega_i = +$ and $\Lambda_i = \Delta_{n_i}^{-1}$ if $\omega_i = -$; we let Λ_i act on the factor $\mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$ of $\mathbb{R}^n \times \mathbb{R}^n$ for all $i \in \{1, \ldots, r\}$. This binary operation $\odot_\omega$ is commutative and associative, clearly.

Now we give an equivalent definition of the binary operation $(C_1, C_2) \mapsto C_1 \odot_\omega C_2$. We denote for each $x \in \mathbb{R}^n$ its projection on the i-th factor of $\mathbb{R}^n$ by $x^{(i)} \in \mathbb{R}^{n_i}$ $(1 \le i \le r)$. Let $P(\omega)$ be the collection of pairs of vectors $x(j),\ j = 1, 2$ in $\mathbb{R}^n$ for which $x(1)^{(i)} = x(2)^{(i)}$ for all $i \in \{1, \ldots, r\}$ for which $\omega_i = -$. For each such pair, we define the vector $x(1) \odot_\omega x(2) \in \mathbb{R}^n$ by $(x(1) \odot_\omega x(2))^{(i)} = x(1)^{(i)} + x(2)^{(i)}$ if

$\omega_i = +$ and $(x(1) \odot_\omega x(2))^{(i)} = x(1)^{(i)} = x(2)^{(i)}$ if $\omega_i = -$. Then one has

$$C_1 \odot_\omega C_2 = \{x(1) \odot_\omega x(2) | x(j) \in C_j, j = 1, 2, \ (x(1), x(2)) \in P(\omega)\}$$

for each pair of convex cones $C_j, \ j = 1, 2$ in $\mathbb{R}^n$.

If we replace in a finite sequence ω of plus and minus symbols, each $+$ by $-$ and conversely, then the resulting sequence is written as ω^d.

Proposition 5.1 *Let ω be a sequence of r plus and minus symbols. If the closed convex cones C_j and D_j in $\mathbb{R}^n$ are in duality ($j = 1, 2$), then the convex cones $C_1 \odot_\omega C_2$ and $D_1 \odot_{\omega^d} D_2$ are in duality, after taking closures.*

We need the following lemma.

Lemma 5.2 *The linear operators $+_n$ and Δ_n are dual to each other: $(+_n)' = \Delta_n$ and $(\Delta_n)' = +_n$.*

Proof. To prove this lemma, it suffices to show that (x_1, x_2) has the same inner product with $(+_n)'y$ as with $\Delta_n y$: for all $x_1, x_2, y \in \mathbb{R}^n$: indeed, $\langle (x_1, x_2), (+_n)'y \rangle$ equals $\langle \Delta_n(x_1, x_2), y \rangle$, by definition of the dual linear transformation; this equals $\langle x_1 + x_2, y \rangle$; moreover, $\langle (x_1, x_2), \Delta_n y \rangle = \langle (x_1, x_2), (y, y) \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$.

No we are ready to prove proposition 5.1.

Proof. The statement of this proposition follows from theorem 2.1. and lemma 5.2.

Construction of binary operations on convex objects by the conification method.

The conification method in the restricted sense generates the following extensions of well-known binary operations on convex sets and functions to the cosmic space context (as well as some other binary operations). We recall that the *infimal convolution of two ordinary convex functions* $h_j, \ j = 1, 2$ on $\mathbb{R}^n$ is defined by $(h_1 \oplus h_2)(x) = \inf_y \{h_1(x - y) + h_2(y)\}$ for all $x \in \mathbb{R}^n$. Here, and below, the convention $\infty + (-\infty) = (-\infty) + \infty = \infty$, which is usual in the context of minimization, is observed.

1. **Convex sets in cosmic space** $S_j \cup \mathrm{dir} K_j, \ j = 1, 2$
   (a) intersection $\cap$: $(S_1 \cap S_2) \cup \mathrm{dir}(K_1 \cap K_2)$;
   (b) sum $+$: $(S_1 + S_2) \cup \mathrm{dir}(K_1 + K_2)$;
   (c) convex hull of the union co$\cup$: $((S_1 \mathrm{co} \cup S_2) \cup (S_1 + K_2) \cup (K_1 + S_2)) \cup \mathrm{dir}(K_1 + K_2)$.
2. **Convex functions on cosmic space** $(g_j, q_j), \ j = 1, 2$

(a) maximum $\vee$: $(g_1 \vee g_2, q_1 \vee q_2)$;

(b) sum $+$: $(g_1 + g_2, q_1 + q_2)$;

(c) infimal convolution $\oplus$: $(g_1 \oplus g_2, q_1 \oplus q_2)$;

(d) convex hull of the minimum co$\wedge$: $((g_1 \mathrm{co} \wedge g_2) \wedge (g_1 \oplus q_2) \wedge (q_1 \oplus g_2), q_1 \oplus q_2)$.

The conification method to generate these binary operations works in the following way. One starts with two convex objects from one of the four classes of convex objects. Then conification gives two convex cones in a vector space where a specific factorization into r factors is singled out: for convex functions on cosmic n-space, this is $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ (so $r = 3$), for convex sets in n-cosmic space and extended real valued sublinear functions on $\mathbb{R}^n$, this is $\mathbb{R}^n \times \mathbb{R}$ (so $r = 2$), for convex cones in $\mathbb{R}^n$ containing the origin, this is just $\mathbb{R}^n$ (so $r = 1$). Then one applies one of the 2^r binary operations $\odot_\omega$ that are determined by the factorization, and finally one deconifies the result. The resulting binary operation is again denoted by $\odot_\omega$. We recall that $\vee$ is used as shorthand for 'maximum' and $\wedge$ as shorthand for 'minimum'. The following proposition displays the results.

Proposition 5.3 *The conification method generates the following commutative and associative binary operations in the cosmic space context:* $\cap, +, \mathrm{co}\cup$ *on convex sets in cosmic space, and* $\vee, +, \oplus, \mathrm{co}\wedge$ *on convex functions on cosmic space:*

1. *convex sets in cosmic space:* $\odot_{(--)} = \cap,\ \odot_{(+-)} = +,\ \odot_{(++)} = \mathrm{co}\cup$;
2. *convex functions on cosmic space:* $\odot_{(---)} = \vee,\ \odot_{(-+-)} = +,\ \odot_{(++-)} = \oplus,\ \odot_{(+++)} = \mathrm{co}\wedge$

Proof.

Now we display the calculations, without comment, for the verification of some of the formulas; the verification of the other ones is similar.

1. Convex sets in cosmic space $A = S \cup \mathrm{dir} K$. We recall that $C(A) = \mathbb{R}_{++}(S \times 1) \cup (K \times 0) = \{\rho(x, 1) | \rho > 0, x \in S\} \cup \{(x, 0) | x \in K\}$. For each ω, and each pair $u_j,\ j = 1, 2$ in $P(\omega)$ for which $u_j \in C(A_j),\ j = 1, 2$, we want to compute $u_1 \odot_\omega u_2$. We have to distinguish four cases:

   (a) $u_1 = \rho_1(x_1, 1)$ and $u_2 = \rho_2(x_2, 1)$;

   (b) $u_1 = \rho_1(x_1, 1)$ and $u_2 = (x_2, 0)$;

   (c) $u_1 = (x_1, 0)$ and $u_2 = \rho_2(x_2, 1)$;

   (d) $u_1 = (x_1, 0)$ and $u_2 = (x_2, 0)$.

   - $\omega = (--)$.

(a) $\rho_1 x_1 = \rho_2 x_2$ and $\rho_1 = \rho_2$. This gives $\{\rho(x,1)|x \in S_1 \cap S_2\}$

(b) $\rho_1 x_1 = x_2$ and $\rho_1 = 0$. This leads to contradiction as $\rho_1 > 0$.

(c) $x_1 = \rho_2 x_2$ and $0 = \rho_2$. This leads to contradiction as well.

(d) $x_1 = x_2$ and $0 = 0$. This gives $\{(x,0)|x \in K_1 \cap K_2\}$.

- $\omega = (++)$.

(a) $\rho_1 x_1 + \rho_2 x_2$ and $\rho_1 + \rho_2$. This gives $\{\rho((1-\alpha)x_1 + \alpha x_2, 1)|\rho > 0, 0 < \alpha < 1, x_j \in S_j, j = 1,2\}$.

(b) $\rho_1 x_1 + x_2$ and ρ_1. This gives, after replacing x_2 by $\rho^{-1}x_2$, $\{\rho(x_1 + x_2, 1)|\rho > 0, x_1 \in S_1, x_2 \in K_2\}$.

(c) The same as the previous case with the roles of 1 and 2 interchanged.

(d) $x_1 + x_2$ and 0. This gives $\{(x,0)|x \in K_1 + K_2\}$.

2. Convex functions on cosmic space $f = (g,q)$. We recall that $C_u(f) = (\mathbb{R}_{++}(\text{epi}g \times 1)) \cup (\text{epi}q \times 0) = \{\rho(x,\alpha,1)|\rho > 0, \alpha \geq g(x)\} \cup \{(x,\alpha,0)|\alpha \geq q(x)\}$. For each ω, and each pair $u_j,\ j = 1,2$ in $P(\omega)$ for which $u_j \in C(f_j),\ j = 1,2$, we want to compute $u_1 \odot_\omega u_2$. We have to distinguish four cases:

(a) $u_1 = \rho_1(x_1, \alpha_1, 1)$ and $u_2 = \rho_2(x_2, \alpha_2, 1)$;

(b) $u_1 = \rho_1(x_1, \alpha_1, 1)$ and $u_2 = (x_2, \alpha_2, 0)$;

(c) $u_1 = (x_1, \alpha_1, 0)$ and $u_2 = \rho_2(x_2, \alpha_2, 1)$;

(d) $u_1 = (x_1, \alpha_1, 0)$ and $u_2 = (x_2, \alpha_2, 0)$.

- $(---)$.

(a) $\rho_1 x_1 = \rho_2 x_2$ and $\rho_1 \alpha_1 = \rho_2 \alpha_2$ and $\rho_1 = \rho_2$. This gives

$$\{\rho(x,\alpha,1)|\alpha \geq \max(g_1,g_2)(x)\}.$$

(b) $\rho_1 x_1 = x_2$ and $\rho_1 \alpha_1 = \alpha_2$ and $\rho_1 = 0$. This leads to contradiction as $\rho_1 > 0$.

(c) $x_1 = \rho_2 x_2$ and $\alpha_1 = \rho_2 \alpha_2$ and $0 = \rho_2$. This leads again to contradiction.

(d) $x_1 = x_2$ and $\alpha_1 = \alpha_2$ and $0 = 0$. This gives $\{\rho(x,\alpha,0)|\alpha \geq \max(q_1,q_2)\}$.

- $(+++)$.

(a) $\rho_1 x_1 + \rho_2 x_2$ and $\rho_1 \alpha_1 + \rho_2 \alpha_2$ and $\rho_1 + \rho_2$. This gives

$$\{(\rho_1+\rho_2)(\frac{\rho_1}{\rho_1+\rho_2}x_1 + \frac{\rho_2}{\rho_1+\rho_2}x_2, \frac{\rho_1}{\rho_1+\rho_2}\alpha_1 + \frac{\rho_2}{\rho_1+\rho_2}\alpha_2, 1)|\rho_i > 0, \alpha_i \geq g_j(x_j),\ j = 1,2\}.$$

(b) $\rho_1 x_1 + x_2$ and $\rho_1 \alpha_1 + \alpha_2$ and ρ_1. This gives

$$\{\rho(x_1 + x_2, \alpha_1 + \alpha_2, 1)|\rho > 0,\ \alpha_1 \geq g_1(x_1),\ \alpha_2 \geq q_1(x_2)\}.$$

(c) Similar to the previous case, with the roles of 1 and 2 interchanged.

(d) $x_1 + x_2$ and $\alpha_1 + \alpha_2$ and 0. This gives $(\text{epi}q_1(1) + \text{epi}q_2(2)) \times 0$.

Additional binary operations. For convex sets, one more binary operation is generated: $\odot_{(-+)} = \#$ where $A_1 \# A_2 = ((S_1 \# S_2) \cup (S_1 + K_2) \cup (K_1 + S_2)) \cup \text{dir}(K_1 + K_2)$, and where $S_1 \# S_2 = \cup\{(1-\lambda)S_1 \cap \lambda S_2 | 0 \leq \lambda \leq 1\}$ (this is called the *inverse sum* or *Kelley's sum*). For convex functions, four more binary operations are generated. For later use, we introduce the following notation for these extended binary operations:

$$\boxplus = \odot_{(+-+)}, \ \heartsuit = \odot_{(+-+)}, \ \diamondsuit = \odot_{(-++)}, \ \spadesuit = \odot_{(-++)}.$$

For ordinary convex functions, the first one is called Kelley's sum. For convex cones containing the origin, two binary operations are generated: $\odot_- = \cap$ and $\odot_+ = +$. For extended real valued sublinear functions, four binary operations are generated: $\odot_{(--)} = \vee$, $\odot_{(+-)} = \text{co}\wedge$, $\odot_{(-+)} = +$, $\odot_{(++)} = \oplus$.

Collapsing of binary operations. The eight binary operations on convex functions collapse to four on sublinear functions: $\vee = \spadesuit, \boxplus = \heartsuit, + = \diamondsuit, \oplus = \text{co}\vee$. The four binary operations on convex sets collapse to two on convex cones containing the origin: $\cap = \#, \oplus = \text{co}\cup$.

Now we display the duality formulae for the binary operations on convex objects.

Theorem 5.4 *1. Convex sets in cosmic space and extended real valued sublinear functions: if the convex sets in cosmic n-space A_j and extended real valued sublinear functions p_j on $\mathbb{R}^n$ are in duality ($j = 1, 2$), then the following pairs are in duality after taking closures:*

(a) $A_1 \cap A_2$ and $p_1 \oplus p_2$;

(b) $A_1 \oplus A_2$ and $p_1 + p_2$;

(c) $A_1 \text{co} \cup A_2$ and $p_1 \vee p_2$.

2. Convex functions on cosmic space: if the convex functions on cosmic n-space f_j and h_j are in duality ($j = 1, 2$), then the following pairs are in duality after taking closures:

(a) $f_1 \vee f_2$ and $h_1 \text{co} \wedge h_2$;

(b) $f_1 + f_2$ and $h_1 \oplus h_2$.

Proof. Conify, apply the proposition 5.1 and 5.3, using lemma 5.2, and deconify.

Additional dualities. Moreover, one gets the following additional dualities. For convex sets and sublinear functions: between $A_1 \# A_2$ and $p_1 \boxplus p_2$, after taking closures. For convex functions: between $f_1 \boxplus f_2$ and $h_1 \diamondsuit h_2$, and between $f_1 \spadesuit f_2$ and $h_1 \heartsuit h_2$, in both cases after taking closures. For convex cones containing the origin: between $C_1 \cap C_2$ and $D_1 + D_2$, after taking closures.

Remarks. Often, the closure operators can be omitted in these dualities, as the outcome $O_1 \odot_\omega O_2$ of taking the binary operation is already closed. This is the case if the convex objects O_1 and O_2 are polyhedral or if the first sign of ω is $-$. It is also the case under the following assumption of general position: for convex sets in cosmic space $S_j \cup \mathrm{dir} K_j,\ j = 1, 2$, the assumption is $\mathrm{ri} S_1 \cap \mathrm{ri} S_2 \neq \emptyset$; for convex functions on cosmic space $(g_j, q_j),\ j = 1, 2$, the assumption is $\mathrm{ridom} g_1 \cap \mathrm{ridom} g_2 \neq \emptyset$.

Novelty. The novelty of this section is the development of the conification method for binary operations on convex objects. Again, here the deconification step is made possible by working with convex objects in extended sense: convex sets in cosmic space, convex functions on cosmic space, and extended real valued sublinear functions. Proposition 5.3 and theorem 5.4 are new in the generality that is required for the conification method.

Comparison with the literature. In [22] all binary operations on convex objects that are constructed in this section are constructed for ordinary convex objects, under the assumption of properness, and in essentially the same way. For convex sets, this construction is given on pp 20-21 of [22], for convex functions on p. 39 of [22]. This amounts to the statements in proposition 5.3 for ordinary convex sets and functions. The duality of the operations $+$ and $\oplus$ for ordinary proper convex functions is given in theorem 16.4 of [22]; the duality of the operations $\vee$ and $\mathrm{co}\wedge$ for ordinary proper convex functions is given in theorem 16.5 of [22]; the result that the closure operator can be omitted is given in [22] under a stronger assumption than in the present paper. For some other binary operations, the duality [R] from these two results. For the four less known binary operations on ordinary convex functions, the duality appears to be given for the first time in [7].

6 Self dual types of convex sets and sublinear functions.

We have seen that the classes 'convex sets in cosmic space' and 'extended real valued sublinear functions' are each others dual. The following subclasses are self dual, that is, one can choose duality operators from this class to itself.

1. **Zero sets.** A *zero set* in cosmic n-space is a convex set in cosmic n-space, $A = S \cup \mathrm{dir} K$, that contains the origin, $0 \in S$.
2. **Gauges.** A *gauge* on $\mathbb{R}^n$ is an extended real valued sublinear functions n on $\mathbb{R}^n$ that assume only nonnegative values.

For example, each *norm* on $\mathbb{R}^n$ is a gauge, and the *unit ball of a norm on* $\mathbb{R}^n$ is a zero set. We display the calculi for zero sets and gauges in the following theorem. The *polar set* of a convex set S in $\mathbb{R}^n$ containing the origin, is the convex set S° in $\mathbb{R}^n$ containing the origin defined by $S^\circ = \{y \in$

$\mathbb{R}^n | \langle x, y \rangle \leq 1$ for all $x \in S\}$. The *dual gauge* of a gauge n is the gauge n^* defined by $n \mapsto n^*$, where $n^*(y)$ is, for each $y \in \mathbb{R}^n$, the smallest element in $[0, +\infty]$ for which $\langle y, x \rangle \leq n^*(y)n(x)$ for all $x \in \mathbb{R}^n$.

Theorem 6.1 *The zero sets and the gauges have the same collection of conifications, the convex cones in* $\mathbb{R}^{n+1}$ *that contain the open ray generated by* $(0_n, 1)$ *and that are contained in* $x_{n+1} \geq 0$.

Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ *and* $\Theta : \mathbb{R}^m \to \mathbb{R}^n$ *be linear transformations in duality. Let* A *and* B *be zero sets in* $\mathbb{R}^n$*. Then the following statements hold true.*

1. ***Zero sets.*** *Let* $A, B, A_j, B_j,\ j = 1, 2$ *be zero sets in* $\mathrm{csm}\mathbb{R}^n$.
   - *(a) The dual object of* $A = S \cup \mathrm{dir}K$*, as defined by the conification method, is the zero set* A° *defined to be* $S^\circ \cup \mathrm{dir}(0^+S)^\circ$.
   - *(b)* $A^\circ = B$ *precisely if* $B^\circ = A$ *(then* A *and* B *are said to be in duality).*
   - *(c) If* A *and* B *are in duality, then* ΛA *and* $B\Theta$ *are in duality, after taking closures.*
   - *(d) If* A_j *and* B_j *are in duality* $(j = 1, 2)$*, then the following pairs are in duality after taking closures:*
     - $A_1 \cap A_2$ *and* $B_1 \mathrm{co} \cup B_2$,
     - $A_1 \oplus A_2$ *and* $B_1 \boxplus B_2$.
2. ***Gauges.*** *Let* $m, n, m_j, n_j,\ j = 1, 2$ *be gauges on* $\mathbb{R}^n$.
   - *(a) The dual object of* n*, as defined by the conification method, is the gauge* n^*.
   - *(b)* $m^* = n$ *precisely if* $n^* = m$ *(then* m *and* n *are said to be in duality).*
   - *(c) If* m *and* n *are in duality, then* Λm *and* $n\Theta$ *are in duality, after taking closures.*
   - *(d) If* m_j *and* n_j *are closed gauges in duality* $(j = 1, 2)$*, then the following pairs are in duality after taking closures:*
     - $m_1 \vee m_2$ *and* $n_1 \oplus n_2$,
     - $m_1 + m_2$ *and* $n_1 \boxplus n_2$.

Proof. The proof is similar to the proofs of the calculi for convex sets and sublinear functions.

Remark. The well-known *Minkowski function* of a zero set S in $\mathbb{R}^n$ is the gauge defined by $\mu_S(x) = \inf\{\alpha | \alpha > 0,\ \alpha^{-1}x \in S\}$ for all $x \in \mathbb{R}^n$. The fact that zero sets and gauges have the same collection of conifications, leads to the following equivalent definition: take an arbitrary zero set S in $\mathbb{R}^n$, conify, view the result as the conification of a gauge; this gauge is precisely μ_S, the Minkowski function of S.

Novelty. Again, the development of the calculi in the context of cosmic space for the self dual subclasses gauges and zero sets, is novel. For these classes, the operations linear image, inverse linear image, closure and the duality operators are always defined and the duality theorems hold for all closed objects.

Comparison with the literature. For ordinary zero sets, the selfduality is given in theorem 14.5 of [22]; for ordinary gauges, the selfduality is given in theorem 15.1 of [22]. The duality results for the binary operations on these self-dual classes is given for proper objects in [16].

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