Model Fitting of a Two-Factor Arbitrage-Free Model for the Term Structure of Interest Rates using Markov Chain Monte Carlo: Part 2: Applications with UK Data

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Abstract

In this paper we apply Markov chain Monte Carlo (MCMC) algorithms developed by Leuwattanachotinan et al. (2012) to calibrate the two-factor Cairns term structure model (Cairns, 2004) with monthly UK Strips data. We first estimate the model parameters and latent state variables and then assess the goodness of fit of the model. Consequently, the model is used for forecasting the yield curves and annuity prices where the impact of parameter uncertainty is also investigated. Additionally, the two-factor Vasicek term structure model is also fitted for comparison. We conclude that our algorithms work reasonably well for estimating both models with the UK market data. The models are found to produce reasonable fits for medium- and long-term yields, but we also conclude that some improvement may be required for the short-end of the yield curves.

Keywords: term structure model; two-factor Cairns model; two-factor Vasicek model; MCMC; Metropolis-Hastings; UK Strips; parameter uncertainty; yield curve forecasting; annuity pricing.

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1 Introduction

Following Leuwattanachotinan et al. (2012), this paper is intended to be its second part in which we continue on employing the developed MCMC algorithm (i.e. the adaptive Metropolis-Hastings algorithm with the reparameterised posterior and a blocking strategy) to fit the two-factor Cairns term structure model (Cairns, 2004) with real UK market data. The detail regarding the model, estimation framework and methodology can be referred to the first part of this paper. Here, we briefly recall that the two-factor Cairns bond price is given by

$$C(\tau, x, \theta) = \frac{\int_\tau^\infty H(u,x)du}{\int_0^\infty H(u,x)du},$$

where

$$H(u,x) = \exp\Big[-\beta u + \sum_{i=1}^{2} \sigma_i x_i \, e^{-\alpha_i u} - \frac{1}{2}\sum_{i,j=1}^{2} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j} e^{-(\alpha_i+\alpha_j)u} \Big],$$

$\theta = (\beta, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho, \gamma_1, \gamma_2)'$ is the model parameter vector and $X(t) = (X_1(t), X_2(t))'$, for $t = 1, \ldots, M$, are the latent variables which follow

$$dX_i(t) = \alpha_i(\gamma_i - X_i(t))dt + \sum_{j=1}^{2} \sigma_{ij} dW_j(t),$$

where $W_1(t)$ and $W_2(t)$ are two independent Wiener processes with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ under the real world probability P.

Let $Y(t) = (Y_1(t), Y_2(t))'$, where $Y_1(t) = \sigma_1 X_1(t)$ and $Y_2(t) = \sigma_2 X_2(t)$ and $\gamma_y = (\gamma_{y_1}, \gamma_{y_2})'$, where $\gamma_{y_1} = \sigma_1\gamma_1$ and $\gamma_{y_2} = \sigma_2\gamma_2$. Then, the log posterior of the Cairns bond price can be can be achieved at

$$\begin{aligned}\log f(\Theta|\mathbb{P}) \;\propto\; & -\frac{MN_t}{2}\log(2\pi\sigma_\varepsilon^2) - \frac{1}{2\sigma_\varepsilon^2}\sum_{t=1}^{M}\sum_{j=1}^{N_t}\left(P(t,\tau_{tj}) - C_Y(\tau_{tj}; Y(t), \theta)\right)^2 \\ & -M\log(2\pi) - \frac{1}{2}\log|\Omega_y| - \frac{(M-1)}{2}\log|\Sigma_Y| \\ & -\frac{1}{2}(Y(1)-\gamma_y)'\Omega_y^{-1}(Y(1)-\gamma_y) \\ & -\frac{1}{2}\sum_{t=2}^{M}\hat{Z}_Y(t)'\Sigma^{-1}\hat{Z}_Y(t) + \log f_0(\theta), \qquad (1)\end{aligned}$$

where

$$C_Y(\tau, y, \theta) = \frac{\int_\tau^\infty H(u,y)du}{\int_0^\infty H(u,y)du},$$

$$H(u,y) = \exp\Big[-\beta u + y_1\, e^{-\alpha_1 u} + y_2\, e^{-\alpha_2 u} - \frac{1}{2}\sum_{i,j=1}^{2} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j} e^{-(\alpha_i+\alpha_j)u} \Big]$$

$$\hat{Z}_Y(t) = Y(t) - \gamma_y - K(Y(t-1) - \gamma_y),$$

$$\Sigma_Y = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \frac{\sigma_1^2}{2\alpha_1}(1-e^{-2\alpha_1\Delta t}) & \frac{\rho\sigma_1\sigma_2}{\alpha_1+\alpha_2}(1-e^{-(\alpha_1+\alpha_2)\Delta t}) \\ \frac{\rho\sigma_1\sigma_2}{\alpha_1+\alpha_2}(1-e^{-(\alpha_1+\alpha_2)\Delta t}) & \frac{\sigma_2^2}{2\alpha_2}(1-e^{-2\alpha_2\Delta t}) \end{pmatrix},$$

$$\Omega_y = \begin{pmatrix} \frac{\sigma_{11}}{1-k_1^2} & \frac{\sigma_{12}}{1-k_1k_2} \\ \frac{\sigma_{21}}{1-k_1k_2} & \frac{\sigma_{22}}{1-k_2^2} \end{pmatrix}, K = \begin{pmatrix} e^{-\alpha_1\Delta t} & 0 \\ 0 & e^{-\alpha_2\Delta t} \end{pmatrix}.$$

Once the model was fitted to the data, applications to the Cairns model are subsequently carried out. In particular, we use the model for forecasting the yield curves and annuity prices where the impact of parameter uncertainty is also particularly addressed.

The remainder of this paper is as follows. In Section 2, we describe UK Strips which are the market data used for calibrating the Cairns model. Section 3 discusses the estimation results and Section 4 investigates the goodness of fit of the model. In Section 5 and 6, we provide a comparison of fitting and forecasting yield curves and annuity prices with parameter uncertainty of two-factor Cairns and Vasicek term structure models. Section 7 concludes.

2 Market Data

For the market data, we consider monthly UK Strips from November 2002 to June 2008 (68 months) where the prices are taken on the last business day of each month and pooled into fixed 20 maturities: 0.25, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 12.5, 15.0, 17.5, 20.0, 22.5, 25.0, 27.5, 30.0 years. Note that we simply use a linear interpolation in order to obtain the data for constant maturities. The plots for UK Strips yields are shown in Figure 1.

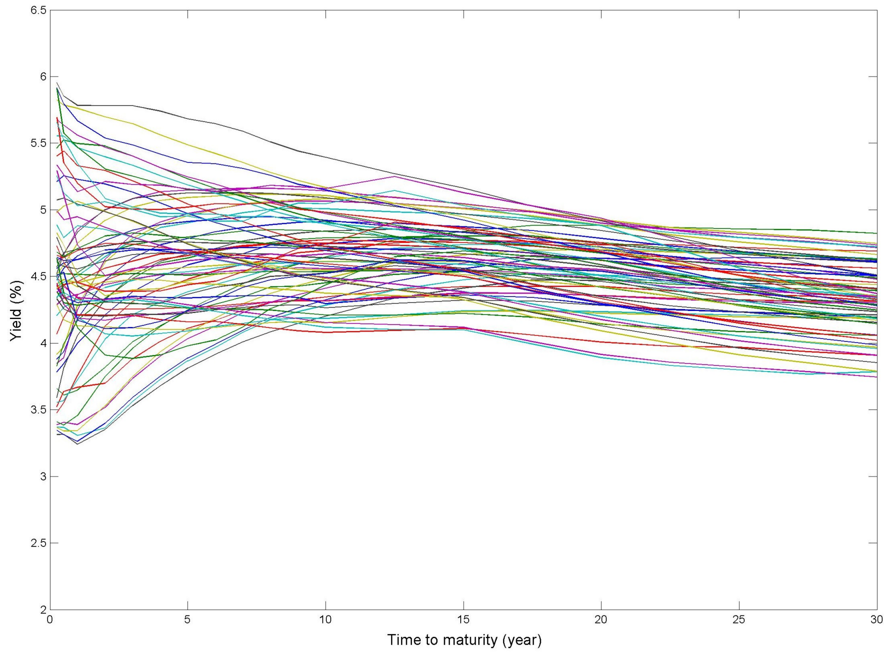

Figure 1: Monthly UK Strips yields from November 2002 to June 2008 (68 months).

3 Estimation Results

We are estimating the two-factor Cairns term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data (68 months from November 2002 to June 2008 as described in Section 2). In general, the algorithm and procedure used are exactly the same as in Leuwattanachotinan et al. (2012, Section 6) but the dataset is smaller (due to the availability of the market data). Also, the parameter σ_ε will now be estimated in terms of the precision parameter $\tau_\varepsilon = 1/\sigma_\varepsilon^2$ with prior distribution

$$f_0(\tau_\varepsilon) = \Gamma(0.01, 1.0 \times 10^8).$$

We note that the above prior has mean and standard deviation of 1.0×10^6 (equivalent to 0.001 for σ_ε) and 1.0×10^7 respectively. For the MH algorithm, τ_ε will be updated individually and a candidate point at j-th iteration will be drawn from the constant variance proposal distribution

$$q_{\tau_\varepsilon} \sim N(\tau_\varepsilon(j-1), vol_{\tau_\varepsilon}^2),$$

where vol_{τ_ε} is set equal to 750 in the simulation. It is worth mentioning that we initially tried to use an adaptive proposal variance for τ_ε, but it turned out to be inefficient as the variance computed from previous 200 sample values tends to go to zero since the chain moves very slowly. Hence, we use a constant proposal variance for τ_ε.

In the current circumstances, we do need to run the chain much longer since the true parameter values are not known. In particular, we found that it took very long for the parameters β and ρ to start converging. After long simulations, we achieve the following results.

- We first note that all the following results are with 4,000 values by recording every 20th iteration out of 80,000 iterations. This is known as a thinning procedure which can mitigate the degree of autocorrelation in the individual simulation paths of each parameter and latent variable. Furthermore, when mentioning the results of using the simulated data, we refer to those in Section 6, Leuwattanachotinan et al. (2012).

- Figure 2 shows the simulated chains of the model parameters and the corresponding posterior estimates are given in Table 1. As can be seen, all parameters converge reasonably well (though, rather slowly for parameter σ_ε). Unsurprisingly, the convergence of γ_{y_1} and γ_{y_2} is far better than for the other parameters since these two parameters appear only in the likelihood of the latent variables (not in part of the pricing formula). With the UK market data, we can observe that the estimated α_1 turns to be relatively low (mean of $\alpha_1 = 0.113$) and the latent variables Y_1 and Y_2 are strongly negatively correlated (mean of $\rho = -0.807$).

- Sample paths of the latent variables $Y_1(t)$ and $Y_2(t)$, for $t = 1, 20, 30, 40, 50$, and 68, are demonstrated in Figure 3. We can see that the movements of each pair are likely to be in opposite direction. This is consistent with the estimating result for ρ in which the mean is about -0.807.

- Figure 4 shows plots of 95% credible intervals constructed from the sample paths with the mean values of $Y(t)$ for all t. The MH acceptance rates are between around 6% to 8%. It is obvious that the intervals in the figure are much wider than those when estimated with the simulated data, reflecting the higher level of uncertainty of the estimation.

- Figure 5 provides the scatter plots of pairs of the model parameters with the corresponding correlation matrix given in Table 2. As can be seen, the high positive correlations among α_2, σ_2 and β still exist (as with the simulated data), and also σ_1 becomes more correlated with other parameters. Furthermore, it turns out that γ_{y_1} is strongly negatively correlated to γ_{y_2}, whereas the additional parameter σ_ε does not appear to be correlated with any other parameters.

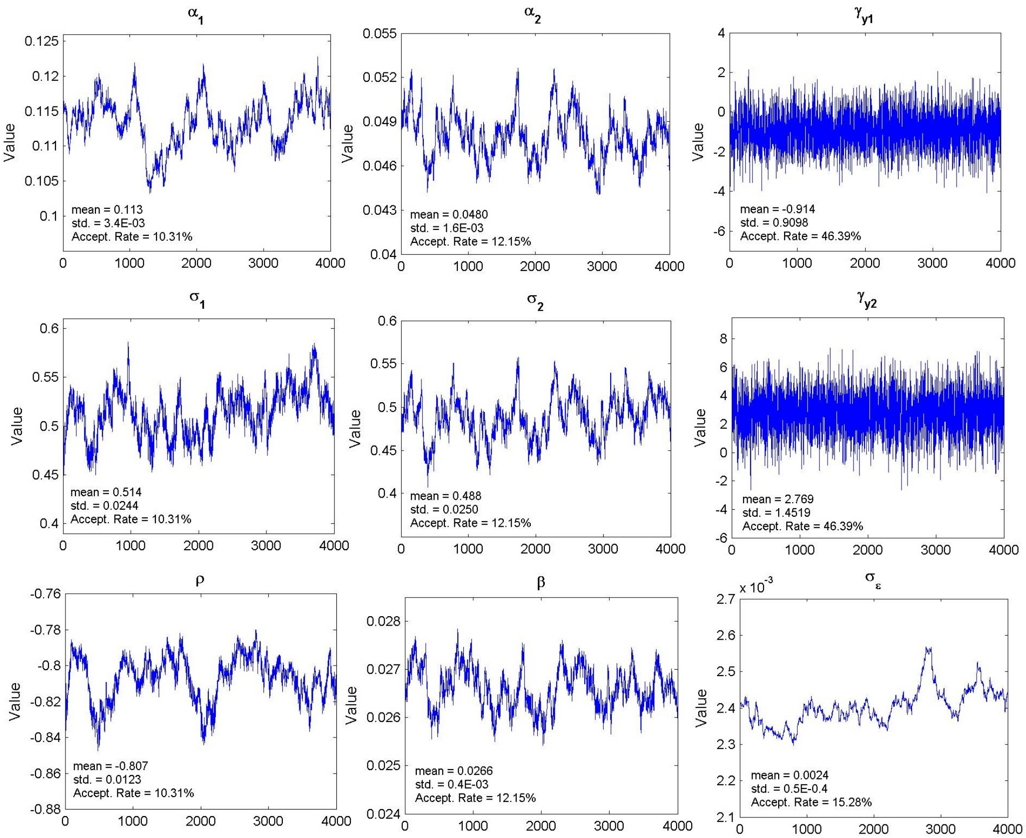

Figure 2: Sample paths of model parameters of the two-factor Cairns term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. Plots are of 4,000 values (every 20th iteration out of 80,000 iterations).

	Mean	Std.	95% Credible Interval	Acceptance rate	Scaling of the proposal std.
α_1	0.113	0.0034	(0.1055, 0.1195)	10.31%	2.0
σ_1	0.514	0.0244	(0.4682, 0.5615)	10.31%	2.0
ρ	-0.807	0.0123	(-0.8347, -0.7876)	10.31%	2.0
α_2	0.0480	0.00157	(0.04514, 0.05133)	12.15%	1.8
σ_2	0.488	0.0250	(0.4385, 0.4385)	12.15%	1.8
β	0.0266	0.00043	(0.02582, 0.02742)	12.15%	1.8
γ_{y_1}	-0.914	0.9098	(-2.6730, 0.9299)	46.39%	1.0
γ_{y_2}	2.769	1.4519	(-0.1392, 5.6265)	46.39%	1.0
σ_ε	0.0024	0.00005	(0.00232, 0.00254)	15.28%	1.0

Table 1: Summary statistics of parameter posterior estimates of the two-factor Cairns term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. The inference is made from 4,000 values (every 20th iteration out of 80,000 iterations).

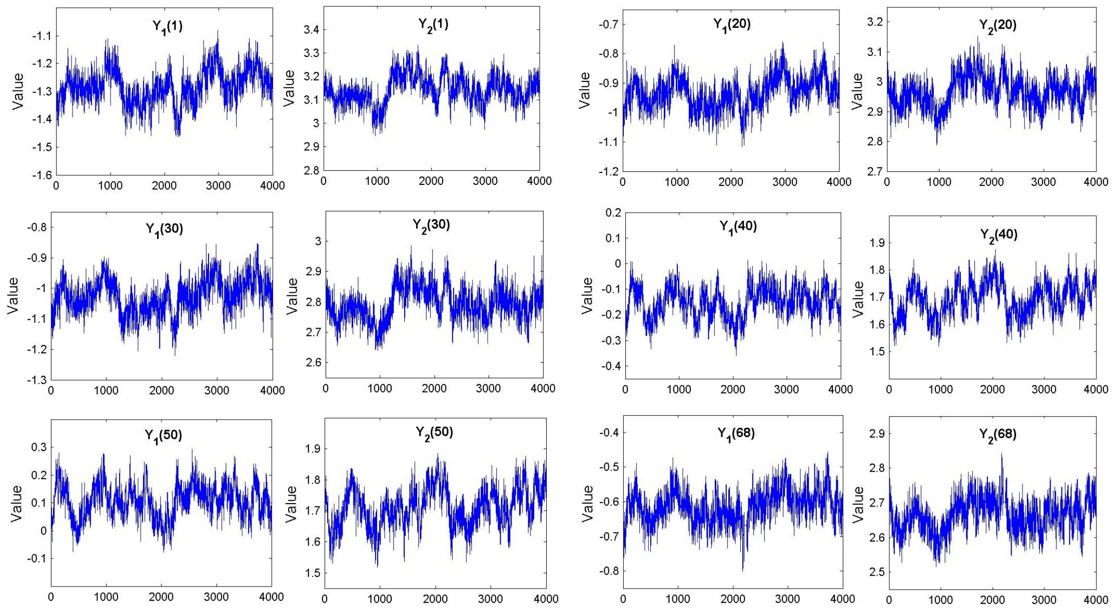

Figure 3: Sample paths of latent variables (for $t = 1, 20, 30, 40, 50$ and 68) of the two-factor Cairns term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. Plots are of 4,000 values (every 20th iteration out of 80,000 iterations).

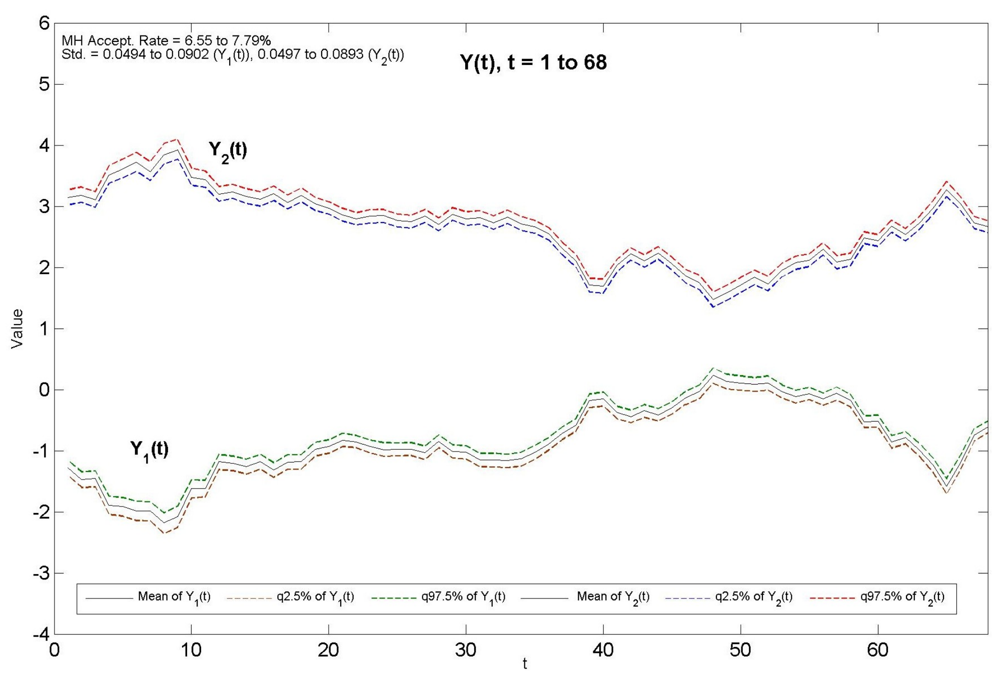

Figure 4: Plots of 95% credible interval constructed from the sample paths with the mean values of $Y_1(t)$ and $Y_2(t)$ for $t = 1, \ldots, 68$, of the two-factor Cairns term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. The inference is made from 4,000 values (every 20th iteration out of 80,000 iterations).

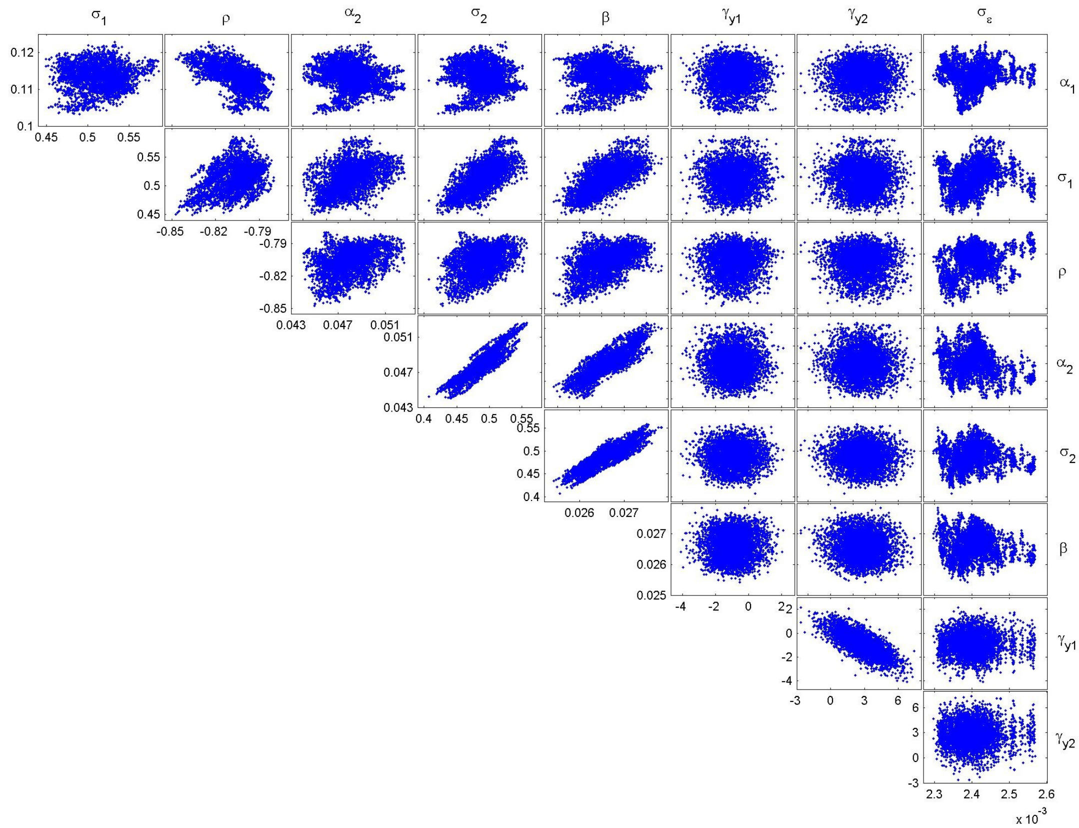

Figure 5: Scatter plots of model parameters of the two-factor Cairns term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. Plots are of 4,000 values (every 20th iteration out of 80,000 iterations).

	α_1	σ_1	ρ	α_2	σ_2	β	γ_{y_1}	γ_{y_2}	σ_ε	$Y_1(t)$	$Y_2(t)$
α_1	1.00	0.04	-0.58	-0.14	-0.06	-0.06	0.01	0.01	0.08	-0.39 to 0.68	-0.76 to 0.47
σ_1		1.00	0.38	0.43	0.69	0.67	0.04	-0.03	0.24	0.06 to 0.64	-0.49 to 0.25
ρ			1.00	0.39	0.39	0.48	0.03	-0.02	0.31	-0.32 to 0.71	-0.64 to 0.39
α_2				1.00	0.90	0.85	0.01	-0.04	-0.17	-0.59 to 0.56	-0.56 to 0.44
σ_2					1.00	0.87	0.02	-0.03	0.07	-0.40 to 0.63	-0.50 to 0.36
β						1.00	0.04	-0.04	-0.04	-0.28 to 0.71	-0.73 to 0.15
γ_{y_1}							1.00	-0.74	0.03	0.02 to 0.08	-0.07 to 0.05
γ_{y_2}								1.00	0.02	-0.06 to 0.03	-0.02 to 0.06
σ_ε									1.00	0.16 to 0.39	-0.14 to 0.33

Table 2: Correlation matrix of the simulation using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. The inference is made from 4,000 values (every 20th iteration out of 80,000 iterations).

4 Goodness of Fit

In this section, we investigate the goodness of fit of the two-factor Cairns term structure model by considering bond price residuals or the differences between the market UK Strips and the estimated Cairns bond prices. Let $P(t, \tau_{tj})$ and $C(\tau_{tj}; \bar{Y}(t), \bar{\theta})$ be respectively market and estimated bond prices at time t maturing at $t + \tau_{tj}$, where $\bar{Y}(t)$ and $\bar{\theta}$ are the mean values of the latent variables and model parameters from the MCMC results. Hence, the residuals can be defined by

$$\hat{\varepsilon}(t, \tau_{tj}) = P(t, \tau_{tj}) - C(\tau_{tj}; \bar{Y}(t), \bar{\theta}), \tag{2}$$

where $\hat{\varepsilon}(t, \tau_{tj}) \sim$ i.i.d. $N(0, \sigma_\varepsilon^2)$ for each time t.

Figure 6 shows the bond price residual surface from November 2002 to June 2008. From the figure, we generally cannot observe any particular pattern of the residuals except that there is graphical evidence that the residuals are not independent. However, the residuals are relatively higher for the last 8 months which is the same period as the Northern Rock bank crisis started and the financial credit crisis loomed. In Figure 7, it can be noticed that the means of the residuals (left) are fairly in line with the model assumption that they are assumed to be zero on average (even though there is serial correlation with a trough and peaks at around the middle and the two ends respectively). For the standard deviation (right), it generally can remain the values within about $\hat{\sigma_\varepsilon}$ (the estimated value of $\sigma_\varepsilon = 0.0024$) for the first 60 months, and then dramatically increases from around the 61st month (the end of 2007) onwards.

The bond price residuals of 3-month, 5-year and 30-year maturities are particularly considered in Figure 8. It can be found that in all cases the bond residuals of 3-month maturity can have values in the ranges of $\pm 1.0 \times \hat{\sigma_\varepsilon}$. For the 5-year maturity, the bond residuals are out of the range for the last 5 months while the residuals of 30-year maturity are relatively higher than those of the other two maturities. We may infer from the figure that the two-factor Cairns model can fairly capture the dynamics of UK Strips prices for all three maturities during the normal market condition but poorly in the volatile market period, especially for the medium- and long-term maturities.

Figure 9 demonstrates normal QQ-plots of the bond price residuals for the selected months from November 2002 to June 2008. As can be seen, the distribution of the residuals may not be normal for some t. Also, most are correlated with one another over time (the correlation matrix not shown here). These suggest that our assumptions that the residuals are independent and normally distributed may not be always valid.

Next, we compare the market UK Strips to the fitted spot rate curves for the selected months from November 2002 to June 2008. In Figures 10a to 10c, black solid lines represent the fitted spot rates using the means of parameter and latent variable values, whereas green bands are the fan charts constructed from the fitted spot rates using all 4,000 sets of parameters and latent variables from the MCMC output (the outer limits of a fan chart are the 5% and 95% quantile range). The yields are converted from the interpolated UK Strips and the fitted Cairns bond prices. From the figures, we can find that if the yield curves are in a rather simple shape (e.g. October 2004), the two-factor Cairns model fits the data reasonably well but for the more complex shapes (e.g. July 2005, March 2008 and so on), the fitting is rather poor. Particularly, in many cases, the two-factor model is unlikely to produce a steeply humped or a kink shape at the short-end of the yield curves. Also, during the volatile market period (Figure 10c), we can observe that the model fitting is problematic. Clearly, the time-homogeneous two-factor Cairns model cannot generate several humps in a yield curve. According to the fan charts, we found that parameter uncertainty affects the fitting of the model in many cases. For instance, in October 2004, several market yields are out of the black line but they can still remain in the green fan.

We make a final remark that for a single date of the yield curve, with suitable parameter values, one may find the two-factor Cairns model be able to produce one steeply humped shape. However, in this case since we estimate the parameters for a whole bond price surface and the data are complex (very high variation at the short-end, particular for the last 8 months), there is a trade-off of the estimated parameter values so that they cannot be fitted as well as expected in some months. Moreover, we note that here we use the mean values of estimated parameters and latent variables from the MCMC simulation results in which we observed that the parameter uncertainty can have an impact on the fitting to an extent.

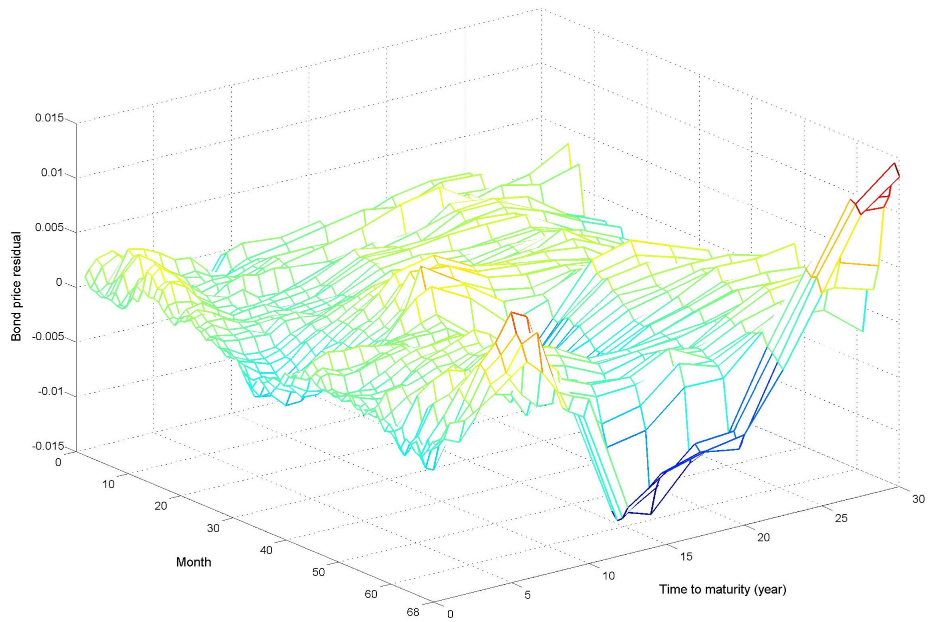

Figure 6: Bond price residual surface of the two-factor Cairns term structure model fitted with monthly UK Strips data from November 2002 to June 2008 (68 months).

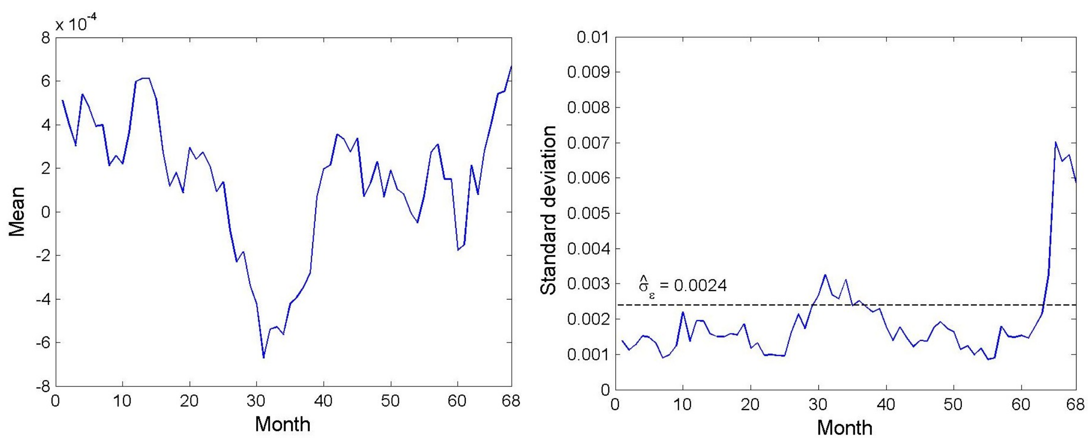

Figure 7: Means (left) and standard deviations (right) of the bond price residuals of the two-factor Cairns term structure model fitted with monthly UK Strips data from November 2002 to June 2008 (68 months). Dash line: the estimated $\hat{\sigma_\varepsilon} = 0.0024$.

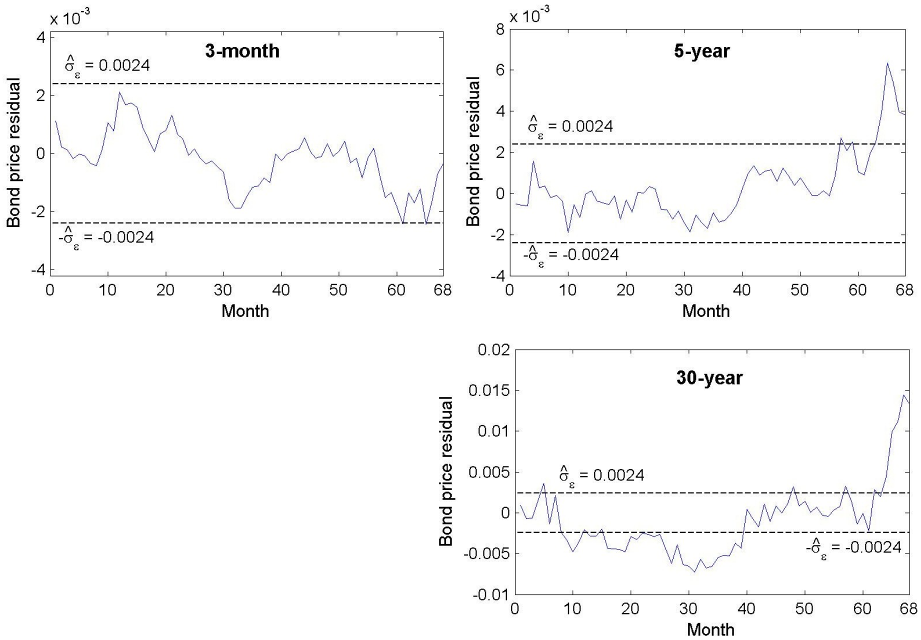

Figure 8: Bond price residuals of 3-month (top left), 5-year (top right) and 30-year (bottom right) maturities of the two-factor Cairns term structure model fitted with monthly UK Strips data from November 2002 to June 2008 (68 months). Dash lines: the intervals of $\pm 1.0 \times \hat{\sigma_\varepsilon}$, where $\hat{\sigma_\varepsilon} = 0.0024$.

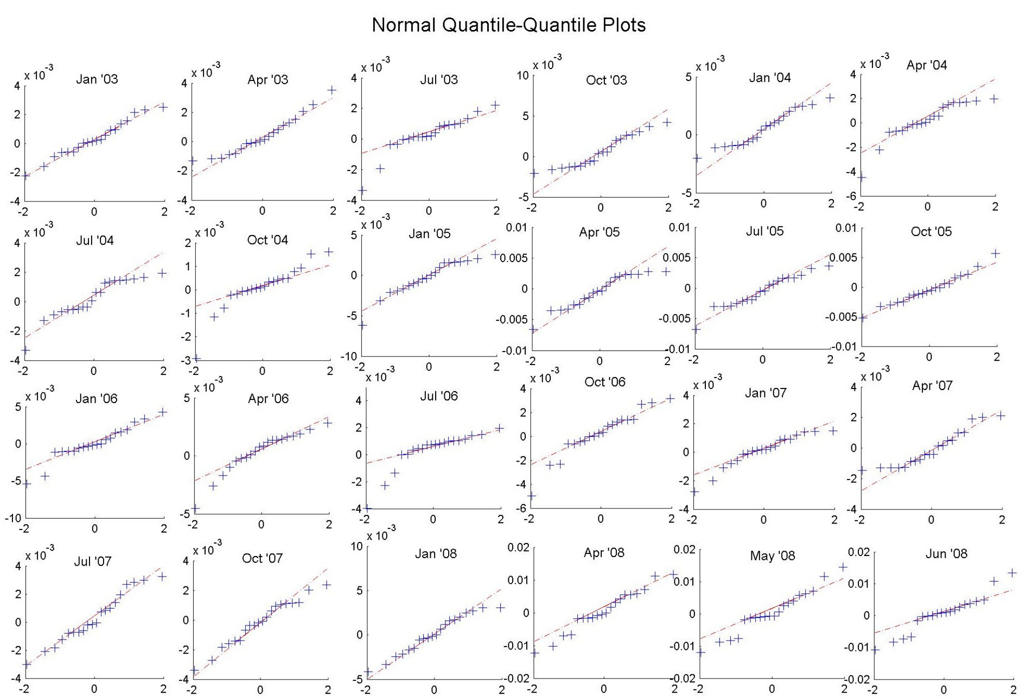

Figure 9: Normal QQ-plots of bond price residuals of the two-factor Cairns term structure model fitted with monthly UK Strips data for the selected months from November 2002 to June 2008.

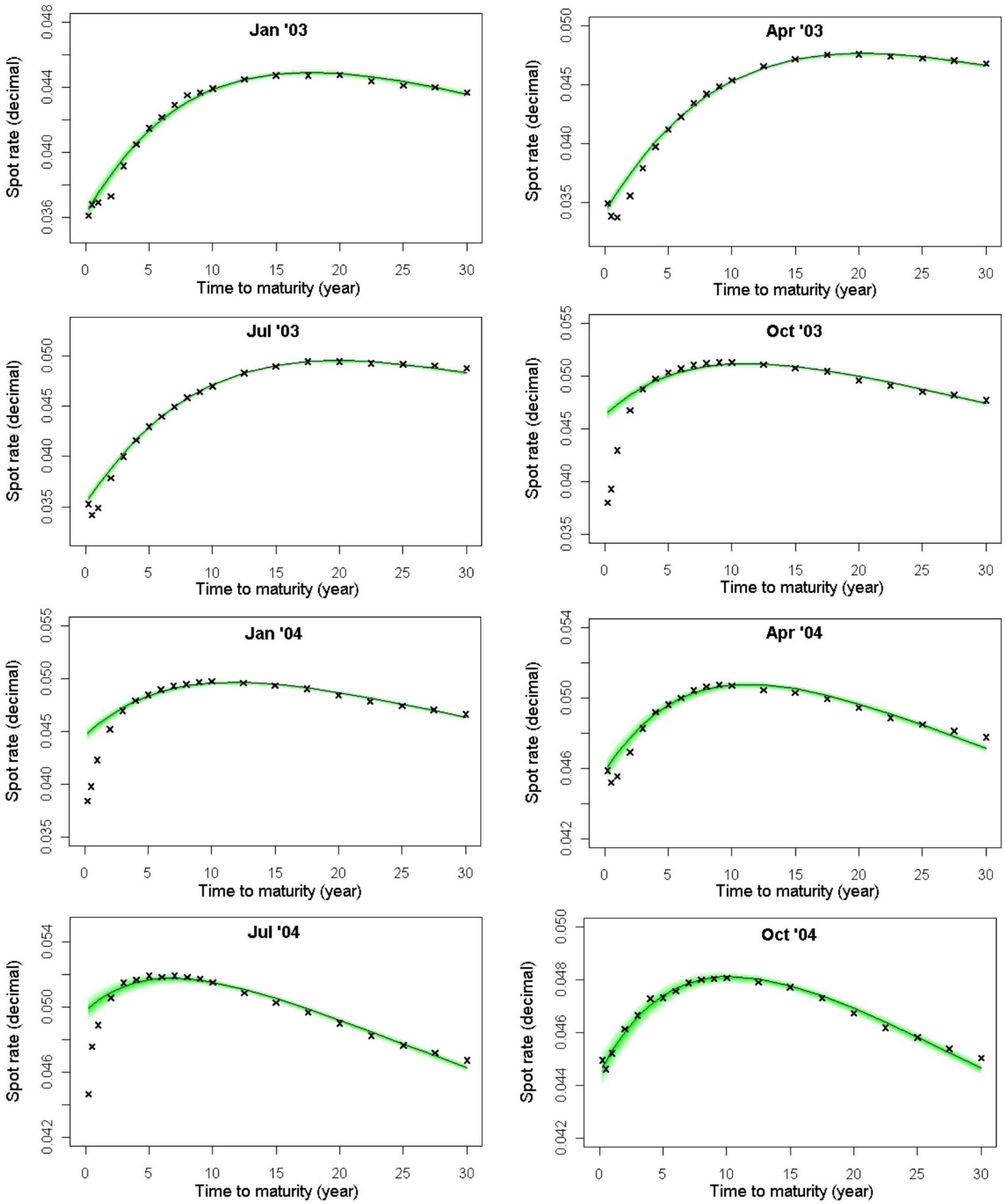

Figure 10a: UK Strips yields (cross mark) compared with the fitted spot rates (solid) of the two-factor Cairns term structure model using the means of parameters and latent variables for the selected months from November 2002 to October 2004. Green bands: fan charts constructed from the the fitted spot rates using all 4,000 sets of parameters and latent variables from the MCMC output.

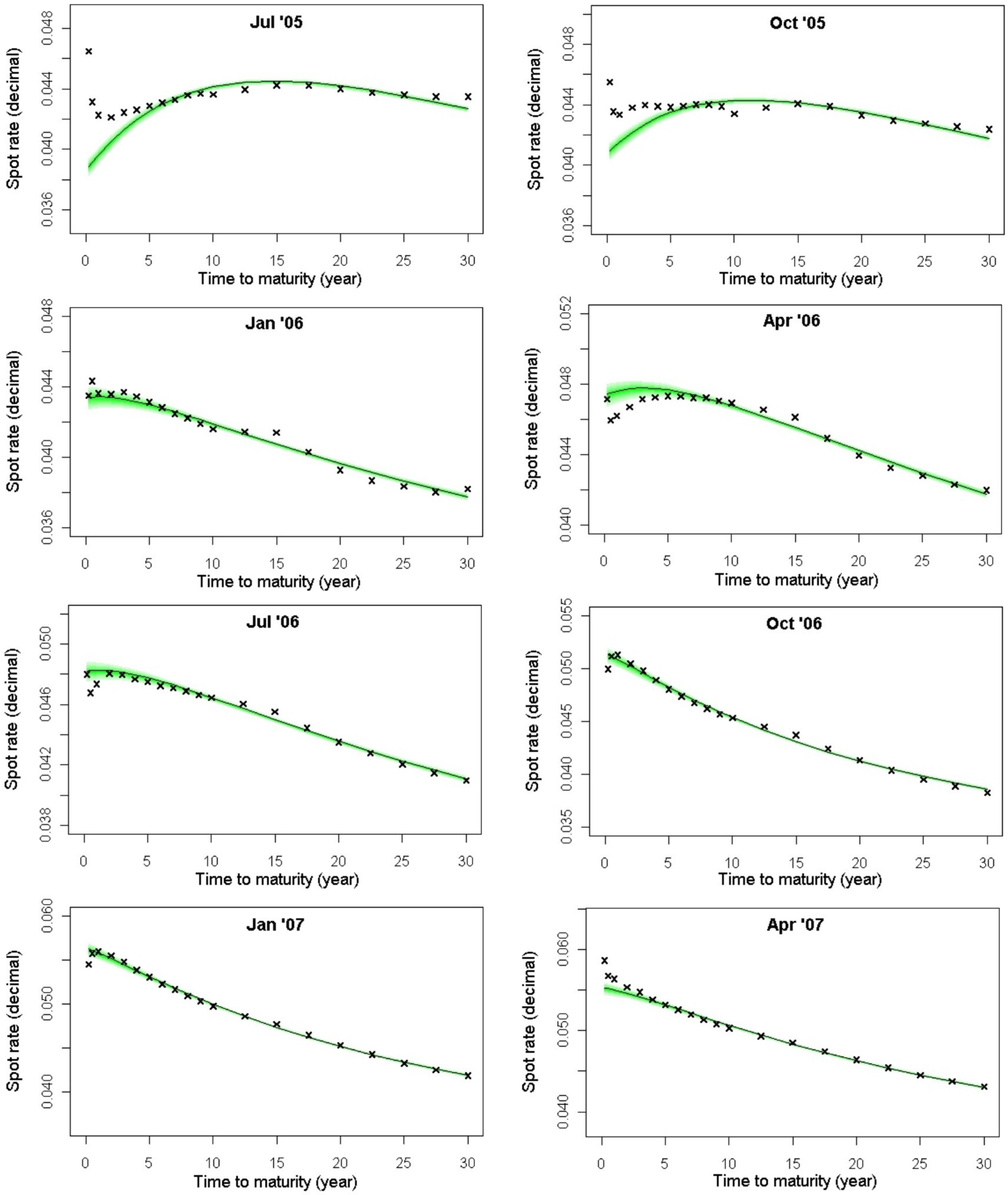

Figure 10b: UK Strips yields (cross mark) compared with the fitted spot rates (solid) of the two-factor Cairns term structure model using the means of parameters and latent variables for the selected months from May 2005 to April 2007. Green bands: fan charts constructed from the the fitted spot rates using all 4,000 sets of parameters and latent variables from the MCMC output.

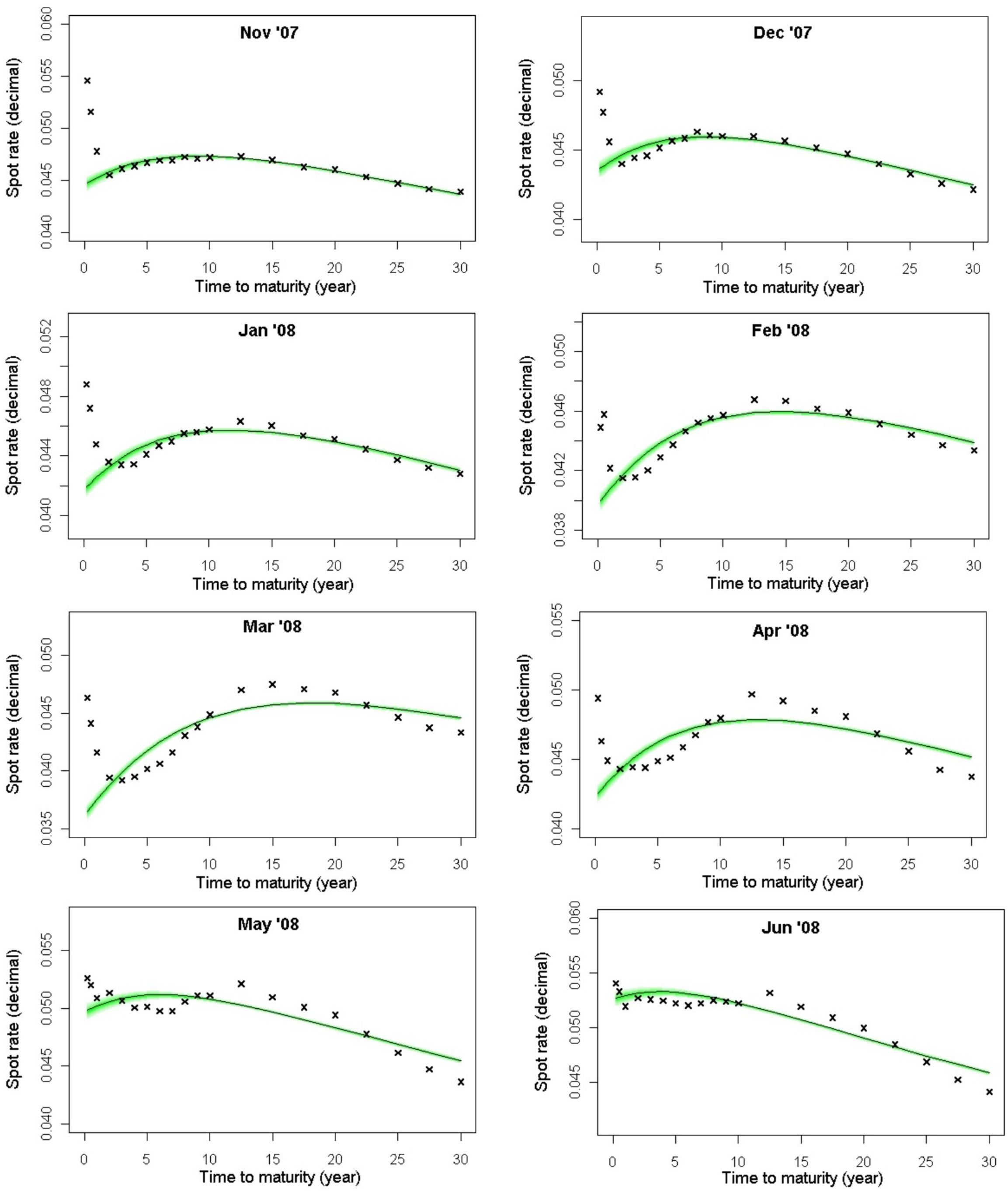

Figure 10c: UK Strips yields (cross mark) compared with the fitted spot rates (solid) of the two-factor Cairns term structure model using the means of parameters and latent variables for the last 8 months from November 2007 to June 2008. Green bands: fan charts constructed from the the fitted spot rates using all 4,000 sets of parameters and latent variables from the MCMC output.

5 A Comparison of Fitting Two-Factor Cairns and Vasicek Term Structure Models

In this section, the two-factor Vasicek term structure model is first described and then it will be estimated using the same framework, algorithm and UK Strips data as with estimating the Cairns model in previous section. Eventually, the goodness of fit of both models will also be investigated for comparison.

5.1 Two-Factor Vasicek Term Structure Model

The model discussed below is adapted from that proposed by Babbs and Nowman (1999). (Babbs and Nowman's model starts with the real world measure P and moves implicitly to the risk-neutral measure Q, whereas here we start with Q and move to P. This results in a simpler pricing formula.) The model is not the most general two-factor Vasicek model but the small number of restrictions result in a model that has similar elements in its structure to the two-factor Cairns model.

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and $W(t)$ is a two-dimensional Wiener process adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. For the two-factor Vasicek model, the short rate under the risk-neutral measure Q (equivalent to the real world measure P) can be defined by

$$r(t) = \mu + X_1(t) + X_2(t), \tag{3}$$

where $X_1(t)$ and $X_2(t)$ are the latent state variables follow

$$\begin{aligned} dX_1(t) &= -\alpha_1 X_1(t)dt + \sigma_1 d\tilde{W}_1(t) \\ dX_2(t) &= -\alpha_2 X_2(t)dt + \sigma_2 \rho d\tilde{W}_1(t) + \sigma_2\sqrt{1-\rho^2}d\tilde{W}_2(t), \end{aligned}$$

where $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ are two independent standard Wiener processes under the risk-neutral pricing measure Q and $\mu, \alpha_1, \alpha_2, \sigma_1, \sigma_2, \rho$ are constants.

Under this model, the price at t for £1 payable at $t+\tau$ is given by

$$V(\tau, X(t), \theta) = \exp[A(\tau) - \tau B(\alpha_1\tau)X_1(t) - \tau B(\alpha_2\tau)X_2(t)] \tag{4}$$

where $B(x) = (1-e^x)/x$ and

$$\begin{aligned} A(\tau) &= -\tau\mu + \frac{\tau\sigma_1^2}{2\alpha_1^2}(1 + B(2\alpha_1\tau) - 2B(\alpha_1\tau)) \\ &+ \frac{\tau\sigma_2^2}{2\alpha_2^2}(1 + B(2\alpha_2\tau) - 2B(\alpha_2\tau)) \\ &+ \frac{\tau\sigma_1\sigma_2\rho}{\alpha_1\alpha_2}(1 - B(\alpha_1\tau) - B(\alpha_2\tau) + B((\alpha_1+\alpha_2)\tau)). \end{aligned}$$

Note that the long-term spot rate $R(t, t+\tau)$ as $\tau \to \infty$ (which we shall denote by $R(t,\infty)$ and is equivalent to β in the Cairns model) is

$$R(t,\infty) = \mu - \frac{\sigma_1^2}{2\alpha_1^2} - \frac{\sigma_2^2}{2\alpha_2^2} - \frac{\sigma_1\sigma_2\rho}{\alpha_1\alpha_2}.$$

Dynamics under P are governed by

$$\begin{aligned} dX_1(t) &= -\alpha_1 X_1(t)dt + \sigma_1(dW_1(t) + \delta_1 dt) \\ &= -\alpha_1(X_1(t) - \frac{\sigma_1\delta_1}{\alpha_1})dt + \sigma_1 dW_1(t) \end{aligned}$$

and

$$dX_2(t) = -\alpha_2\left(X_2(t) - \frac{\sigma_2}{\alpha_2}(\delta_1\rho + \delta_2\sqrt{1-\rho^2})\right)dt + \sigma_1(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)),$$

where $W_1(t)$ and $W_2(t)$ are standard Wiener processes under the real world measure P and δ_1 and δ_2 are the corresponding market prices of risk.

When estimating the Cairns model, γ_1 and γ_2 are known as the mean reversion levels for X_1 and X_2. Hence, we define

$$\begin{aligned} \gamma_1 &= \frac{\sigma_1\delta_1}{\alpha_1} \\ \gamma_2 &= \frac{\sigma_2}{\alpha_2}(\delta_1\rho + \delta_2\sqrt{1-\rho^2}). \end{aligned}$$

More generally, a key qualitative difference between the two-factor Vasicek model and the Cairns model is that the former model allows interest rates to become negative. Further, as we will see in next section, future spot rates under the Vasicek model are normally distributed whereas future spot rates under the Cairns model have a positively skewed distribution.

5.2 Estimation Results of Two-Factor Vasicek Model on Monthly UK Strips Data

Similar to the estimation framework for the Cairns model, we suppose that interest rates in the market follow the two-factor Vasicek model in (4), i.e. the observations

$$P(t, \tau_{tj}) = V(\tau_{tj}; X(t), \theta) + \varepsilon(t, j), \tag{5}$$

where $\varepsilon(t, j) \sim$ i.i.d. $N(0, \sigma_\varepsilon^2)$. Accordingly, the full posterior distribution of the Vasicek bond price can be obtained in a similar form as that of the Cairns bond price merely by replacing the function $C(\tau_{tj}; X(t), \theta)$ by $V(\tau_{tj}; X(t), \theta)$. In order to get the best comparison of estimation with the achieved results in previous section, we also define the prior distributions for $\mu, \alpha_1, \alpha_2, \sigma_1, \sigma_2$ in the same way such that prior means are specified based on the posterior means from earlier simulations with about the same coefficient of variation.

In the following results, we run MCMC for 240,000 iterations by recording values for every 20th iteration using adaptive Metropolis-Hastings with a blocking strategy (exactly same algorithm as developed previously). More precisely, we update groups of $(\alpha_1, \sigma_1, \rho), (\alpha_2, \sigma_2, \mu), (\gamma_1, \gamma_2)$ and $(X_1(t), X_2(t))$ for each time t where σ_ε is updated individually with a constant proposal variance. This is the same blocking as was used for the Cairns model and seems to be just as effective.

Figure 11 shows sample paths of all model parameters and the corresponding posterior estimates are provided in Table 4. As can be observed, all parameters generally converge very well, especially γ_1, γ_2, ρ and σ_ε. Comparing to the Cairns model, ρ turns to be much easier to estimate since Vasicek bond prices appear in a linear (affine) function of latent variables. The estimated ρ from the Vasicek model is slightly higher than Cairns model, but both are still strongly negative.

Figure 12 demonstrates sample paths of latent variables for some selected months. As expected, all the chains converge reasonably well. Furthermore, when considering plots of 95% credible intervals for all t in Figure 13, we can find that they also behave similar to those from the Cairns model in Figure 4.

Finally, we consider the correlation structure for model parameters (Figures 14 and Table 5). For the Vasicek model, we can only notice strongly positive correlation for (α_1, σ_1), and moderately negative correlations for (σ_1, σ_2) and (γ_1, γ_2). Apart from these, strong correlations are not observed.

5.3 Goodness of Fit: Two-Factor Cairns versus Vasicek Term Structure Models

We here compare model fitting of the two-factor Cairns and Vasicek models using the estimated means of parameters and latent variables from the MCMC output. In terms of bond price and spot rate residuals (referring to equation (2)), it can be seen from Table 3 that, overall, the Cairns model fits the data slightly better than the two-factor Vasicek model. For the sum of squared yield residuals by maturity, we found that those by Cairns model are lower than Vasicek model. Furthermore, the total difference in terms of average yield residual per maturity is mainly from the first three maturities (0.25, 0.5, 1 years).

	Total Sum of Squared Residuals (in decimal)		Average Residual per maturity (in bps)	
	Cairns	Vasicek	Cairns	Vasicek
Bond Price	0.007219	0.007227	23.04	23.05
Spot Rate	0.002282	0.002408	12.95	13.31

Table 3: Total sum of squared residuals and average residual from fitting the two-factor Cairns and Vasicek term structure model using the estimated means of parameters and latent variables form the MCMC output.

	Mean	Std.	95% Credible Interval	Acceptance rate	Scaling of the proposal std.
α_1	0.0386	0.00243	(0.03383, 0.04310)	11.31%	2.0
σ_1	0.0081	0.00024	(0.00768, 0.00858)	11.31%	2.0
ρ	-0.718	0.0449	(-0.7948, -0.6182)	11.31%	2.0
α_2	0.132	0.0037	(0.1258, 0.14039)	10.71%	1.8
σ_2	0.0136	0.00065	(0.01242, 0.01485)	10.71%	1.8
μ	0.0491	0.00045	(0.04826, 0.05006)	10.71%	1.8
γ_1	0.0109	0.02701	(-0.04223, 0.06335)	51.20%	1.0
γ_2	-0.0146	0.02176	(-0.05769, 0.02837)	51.20%	1.0
σ_ε	0.0024	0.00005	(0.00233, 0.00253)	44.35%	1.0
$R(t,\infty)$	0.0372	0.00182	(0.03323, 0.04035)	-	-

Table 4: Summary statistics of parameter posterior estimates of the two-factor Vasicek term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. The inference is made from 12,000 values (every 20th iteration out of 240,000 iterations).

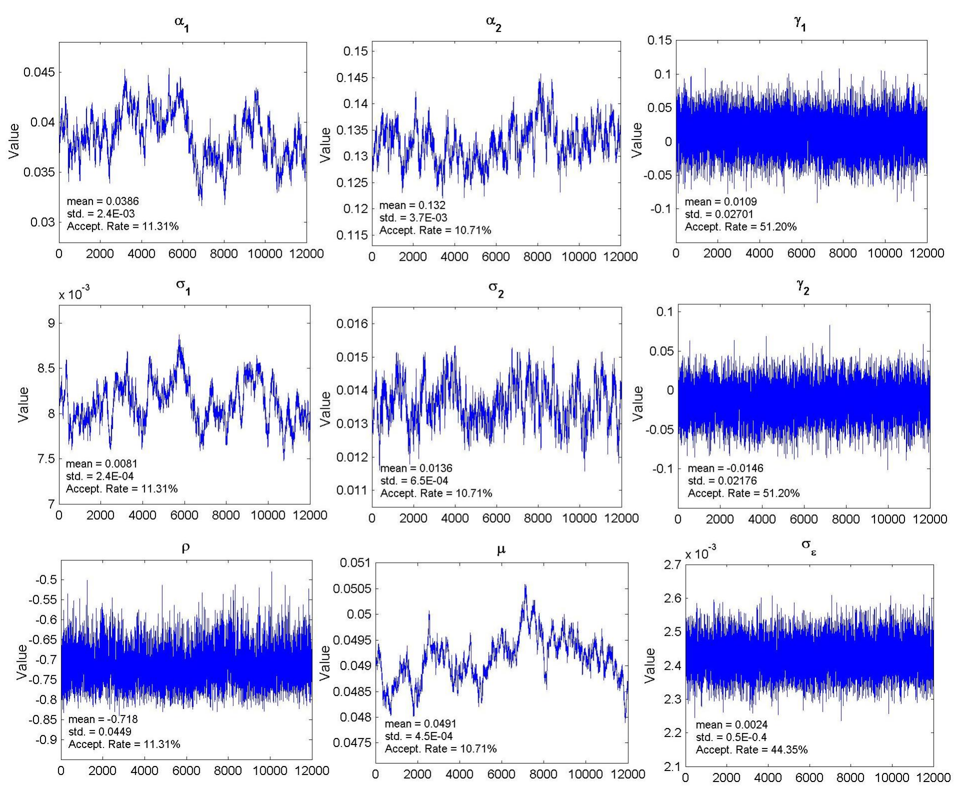

Figure 11: Sample paths of model parameters of the two-factor Vasicek term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. Plots are of 12,000 values (every 20th iteration out of 240,000 iterations).

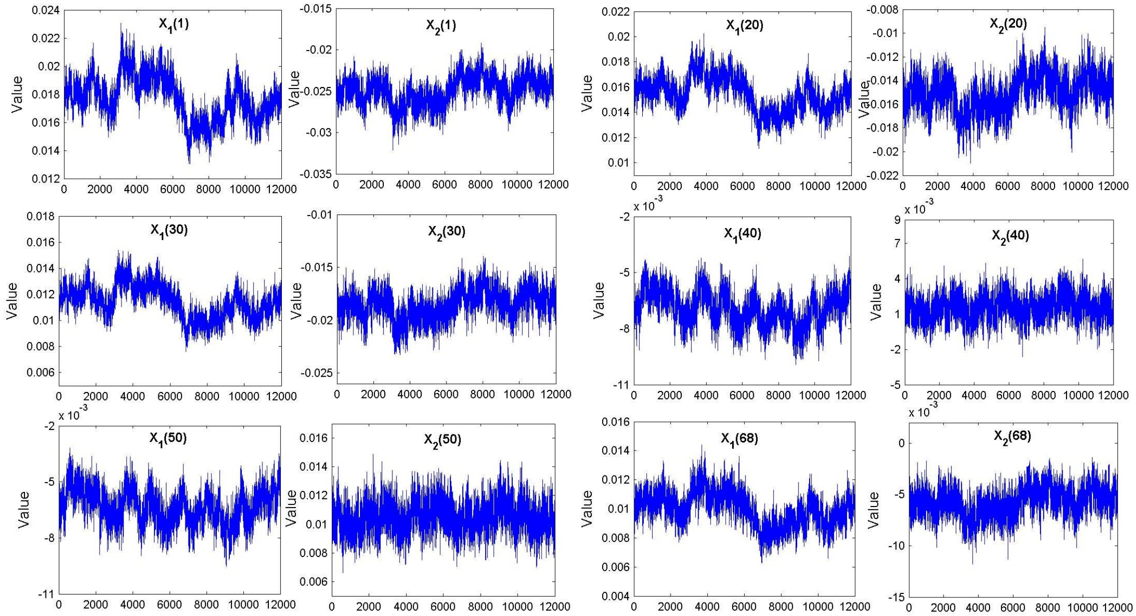

Figure 12: Sample paths of latent variables (for $t = 1, 20, 30, 40, 50$ and 68) of the two-factor Vasicek term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. Plots are of 12,000 values (every 20th iteration out of 240,000 iterations).

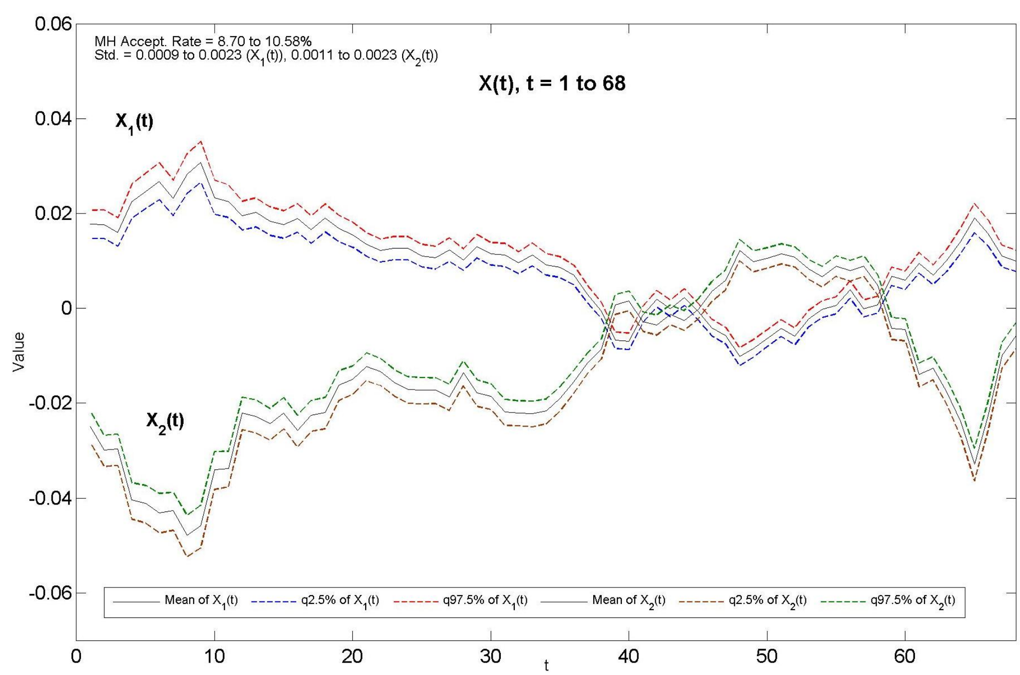

Figure 13: Plots of 95% credible interval constructed from the sample paths with the mean values of $X_1(t)$ and $X_2(t)$ for $t = 1, \ldots, 68$, of the two-factor Vasicek term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. The inference is made from 12,000 values (every 20th iteration out of 240,000 iterations).

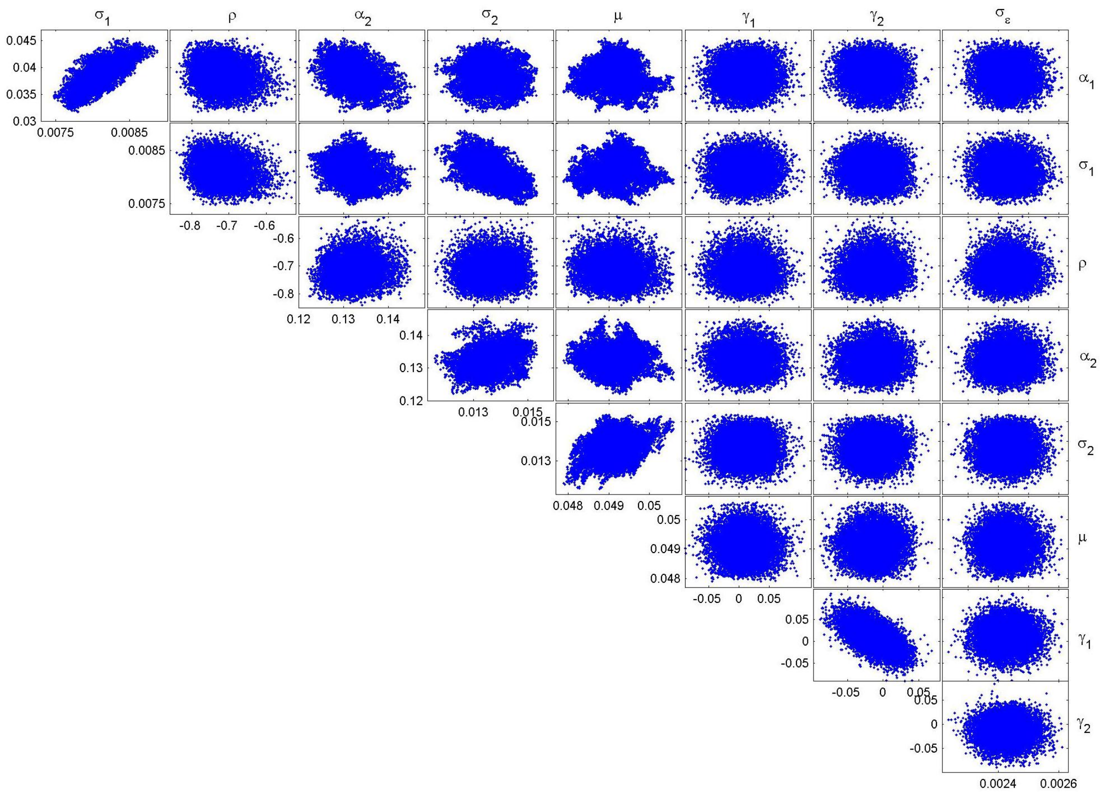

Figure 14: Scatter plots of model parameters of the two-factor Vasicek term structure model using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. Plots are of 12,000 values (every 20th iteration out of 240,000 iterations).

	α_1	σ_1	ρ	α_2	σ_2	μ	γ_1	γ_2	σ_ε	$X_1(t)$	$X_2(t)$
α_1	1.00	0.75	-0.15	-0.39	-0.13	-0.07	0.05	-0.05	-0.04	-0.45 to 0.86	-0.85 to 0.76
σ_1		1.00	-0.18	-0.26	-0.52	0.12	0.02	-0.03	-0.04	-0.61 to 0.52	-0.55 to 0.50
ρ			1.00	0.14	0.03	-0.07	-0.02	0.01	0.02	-0.14 to 0.13	-0.12 to 0.16
α_2				1.00	0.31	-0.03	-0.03	0.04	0.04	-0.65 to 0.32	-0.54 to 0.57
σ_2					1.00	0.33	0.01	0.00	0.02	-0.11 to 0.33	-0.45 to 0.01
μ						1.00	-0.03	0.03	0.00	-0.58 to -0.26	-0.37 to 0.09
γ_1							1.00	-0.59	0.00	0.01 to 0.07	-0.06 to 0.06
γ_2								1.00	0.00	-0.07 to 0.00	-0.07 to 0.07
σ_ε									1.00	-0.05 to 0.05	-0.05 to 0.06

Table 5: Correlation matrix of the simulation using the adaptive MH algorithm with the reparameterised posterior and a blocking strategy, with monthly UK Strips data. The inference is made from 12,000 values (every 20th iteration out of 240,000 iterations).

6 Forecasting Yield Curves and Annuity Prices with Parameter Uncertainty

In this section, we assess the impact of parameter uncertainty on the forecasting of yield curves using the MCMC output from previous sections. To begin with, an introduction to the uncertainty that may arise from modelling is presented. Next, we define the predictive density of the Cairns bond prices and then describe the forecasting simulation procedure. In the end, the results of forecasts with parameter uncertainty will eventually be discussed and compared with those from the Vasicek model.

6.1 Introduction

Uncertainty naturally occurs in most estimation problems even if a good model and technique are being used. With reference to Cairns (2000), uncertainty may arise from three main sources: process, model and parameters. Process risk is meant to be the randomness inherent in the underlying structural stochastic process. Model uncertainty refers to the choice of model in situations where the true model is unknown. What is of particular interest here is parameter uncertainty, a risk that is often ignored when a model is implemented.

By parameter uncertainty, we typically mean the uncertainty in the parameter values in a selected model. Given the availability of large data, we can still never know the parameter values with certainty. For example, the maximum likelihood method provides us the parameter values that are "most likely" from the data. Accordingly, the impact of parameter uncertainty should be taken into an account for use of any model, particularly in a long-term horizon where uncertainty is generally magnified.

In Bayesian paradigm, parameters are treated as random variables so that it explicitly gives us a coherent framework to quantify the additional impact of parameter uncertainty on the particular financial quantities that we are interested in. In the following section, we first consider the predictive density of the Cairns bond prices.

6.2 Forecasting Cairns Bond Prices

Denote $\mathbb{P}_M$ as all historical bond prices from time $t_1, \ldots, t_M$. Given the bond posterior distribution in (1), the h-year ahead conditional predictive density of the Cairns bond price for the maturities τ_j, for $j = 1, \ldots, N$, can be defined by

$$\begin{aligned} f(P(t_M+h,\tau_j)|\mathbb{P}_M,\Theta(t_M)) \;=\; \int\!\!\int & f_1(P(t_M+h,\tau_j)|Y(t_M+h),\theta) \\ & \times f_{2c}(Y(t_M+h)|Y(t_M),\theta_2) \\ & \times f_{2u}(Y(t_M)|\theta,\mathbb{P}_M) \\ & \times f_0(\theta)dY(t_M+h)d\theta, \end{aligned} \tag{6}$$

where $\Theta(t_M) = (Y(t_M), \theta)$, f_1 is the normal density function of the bond prices, f_{2c} and f_{2u} are respectively the conditional and unconditional densities of the latent variables, and f_0 is the prior.

6.2.1 Forecasting Bond Prices with Parameter Uncertainty

Having obtained the MCMC output, simulating the forecast bond price $P(t_M+h, \tau_j)$ according to (6) is straightforward. Specifically, the sample values of $\Theta(t_M)$ for all iterations can be used to simulate the latent variable $Y(t_M+h)$ and hence compute the bond price $P(t_M+h, \tau_j)$ (which inherently includes the effect of parameter uncertainty). Initially, we present a rough procedure for simulating the forecasting of quantity of interest as follows.

1. Simulate Θ from the posterior distribution (already completed in Section 3).
2. Extract $Y(t_M), \theta$ from Θ.
3. Simulate $Y(t_M+h)$ given $Y(t_M), \theta$.
4. Calculate quantity of interest at t_M+h given the values of $Y(t_M+h)$ and θ.

The detailed procedure to simulate the h-year ahead forecasting Cairns bond prices is described below.

1. Select the index $k = k(i)$, for $i = 1, \ldots, I$, from K iterations of MCMC at random. Therefore, we have $\theta^{(k(i))}, Y^{(i)}(t_M)$, where $Y^{(i)}(t_M) := Y^{k(i)}(t_M)$.
2. Simulate the latent variables $Y^{(i)}(t_M+h)$ from

$$\begin{pmatrix} Y_1^{(i)}(t_M+h) \\ Y_2^{(i)}(t_M+h) \end{pmatrix} = \begin{pmatrix} \gamma_{y_1}^{(k(i))} + (Y_1^{(i)}(t_M) - \gamma_{y_1}^{(k(i))})e^{-\alpha_1^{(k(i))}h} \\ \gamma_{y_2}^{(k(i))} + (Y_2^{(i)}(t_M) - \gamma_{y_2}^{(k(i))})e^{-\alpha_2^{(k(i))}h} \end{pmatrix} + B^{(i)}A^{(i)}Z(t_M+h),$$

where $Z = (Z_1, Z_2)$ with $Z_1(t_M+h), Z_2(t_M+h) \sim i.i.d.\ N(0,1)$ and

$$B^{(i)} = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ 0 & \sqrt{\sigma_{22}} \end{pmatrix}, \quad A^{(i)} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix},$$

$$\Rightarrow B^{(i)}C^{(i)}Z(t_M+h) = \begin{pmatrix} \sqrt{\sigma_{11}}Z_1(t_M+h) \\ \rho\sqrt{\sigma_{22}}Z_1(t_M+h) + \sqrt{1-\rho^2}\sqrt{\sigma_{22}}Z_2(t_M+h) \end{pmatrix}$$

such that the covariance matrix is

$$\Sigma^{(i)} \;=\; A^{(i)}A'^{(i)} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix},$$

where

$$\begin{aligned}
\sigma_{11} &= \frac{\sigma_1^{(k(i))^2}}{2\alpha_1^{(k(i))}}(1 - e^{-2\alpha_1^{(k(i))}h}), \\
\sigma_{12} = \sigma_{21} &= \frac{\rho^{(k(i))}\sigma_1^{(k(i))}\sigma_2^{(k(i))}}{\alpha_1^{(k(i))} + \alpha_2^{(k(i))}}(1 - e^{-(\alpha_1^{(k(i))}+\alpha_2^{(k(i))})h}), \\
\sigma_{22} &= \frac{\sigma_2^{(k(i))^2}}{2\alpha_2^{(k(i))}}(1 - e^{-2\alpha_2^{(k(i))}h})
\end{aligned}$$

and hence $\rho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$.

3. Then, compute the forecast bond price for the maturities τ_j, for $j = 1, \ldots, N$

$$P^{(i)}(t_M + h, \tau_j)|\, Y^{(i)}(t_M + h), \theta^{(k(i))} = C_Y(\tau_j, Y^{(i)}(t_M + h), \theta^{(k(i))}),$$

where $C_Y(\tau_j, Y^{(i)}(t_M + h), \theta^{(k(i))})$ is the bond price by the two-factor Cairns model defined by

$$C_Y(\tau, y, \theta) = \frac{\int_\tau^\infty H(u, y)du}{\int_0^\infty H(u, y)du},$$

$$H(u, y) = \exp\Big[-\beta u + \sum_{i=1}^{2} y_i e^{-\alpha_i u} - \frac{1}{2}\sum_{i,j=1}^{2} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i+\alpha_j)u}\Big].$$

6.2.2 Forecasting Bond Prices with Parameter Certainty

We here define "parameter certainty" as the point estimates of the MCMC output. The forecast bond prices with parameter certainty therefore can be simulated using the same procedure as with the parameter uncertainty described earlier where the means of the parameter and latent variable values will be used instead of the selected $\theta^{(k(i))}, Y^{(i)}(t_M)$ for each $k(i)$. The rough procedure is outlined below.

1. Let $\bar{\Theta}$ be the mean of the posterior distribution for Θ.
2. Extract $Y(t_M), \theta$ from $\bar{\Theta}$.
3. Simulate $Y(t_M + h)$ given $Y(t_M), \theta$.
4. Calculate quantity of interest at $t_M + h$ given the values of $Y(t_M + h)$ and θ.

6.3 Forecast Spot Rates and Annuity Prices: Two-Factor Cairns versus Vasicek Term Structure Models

According to the simulation procedures described earlier, this section compares forecast spot rates and annuity prices with parameter uncertainty (PU) and parameter certainty (PC) using achieved MCMC output from estimating the two-factor Cairns and Vasicek term structure models on monthly UK Strips data from November 2002 to June 2008. For the PU case, 100 sets of parameter and latent variable values from MCMC output will be selected at random and incorporated with 100 fresh pairs of future normal randomness Z_1 and Z_2. For the PC case, the posterior means will be employed with 10,000 pairs of Z_1 and Z_2.

6.3.1 Forecasting Results: Spot Rates

Figures 15 and 16 compare forecast 3-month, 5-year and 30-year spot rates from the Cairns (blue line) and Vasicek (red line) models with PC (left column) and PU (right column) cases for 5 and 20 years ahead respectively. For 30-year maturity, the par yield curves (dash line) are also provided since in practice, long-dated par yields are referred to as much as spot rates. From the figures, observations can be made as follows.

- Despite achieving a very similar picture of model fitting, forecast distributions from both models are noticeably different, especially for the short-term rates and longer time horizons. Most importantly, it can be observed that Vasicek model can generate negative rates for all three maturities which is an undesirable characteristic for term structure models.

- More consistent with historical data, the Cairns model can produce more realistic forecast spot rates in the sense that the higher rates can be obtained, albeit with a small probability.

- For 30-year maturity, the difference of forecast par yields between two models is clearer than spot rates, particularly on the right tails of distributions.

- Parameter uncertainty does have an impact on forecast spot rates in all cases, particularly when time horizon is longer.

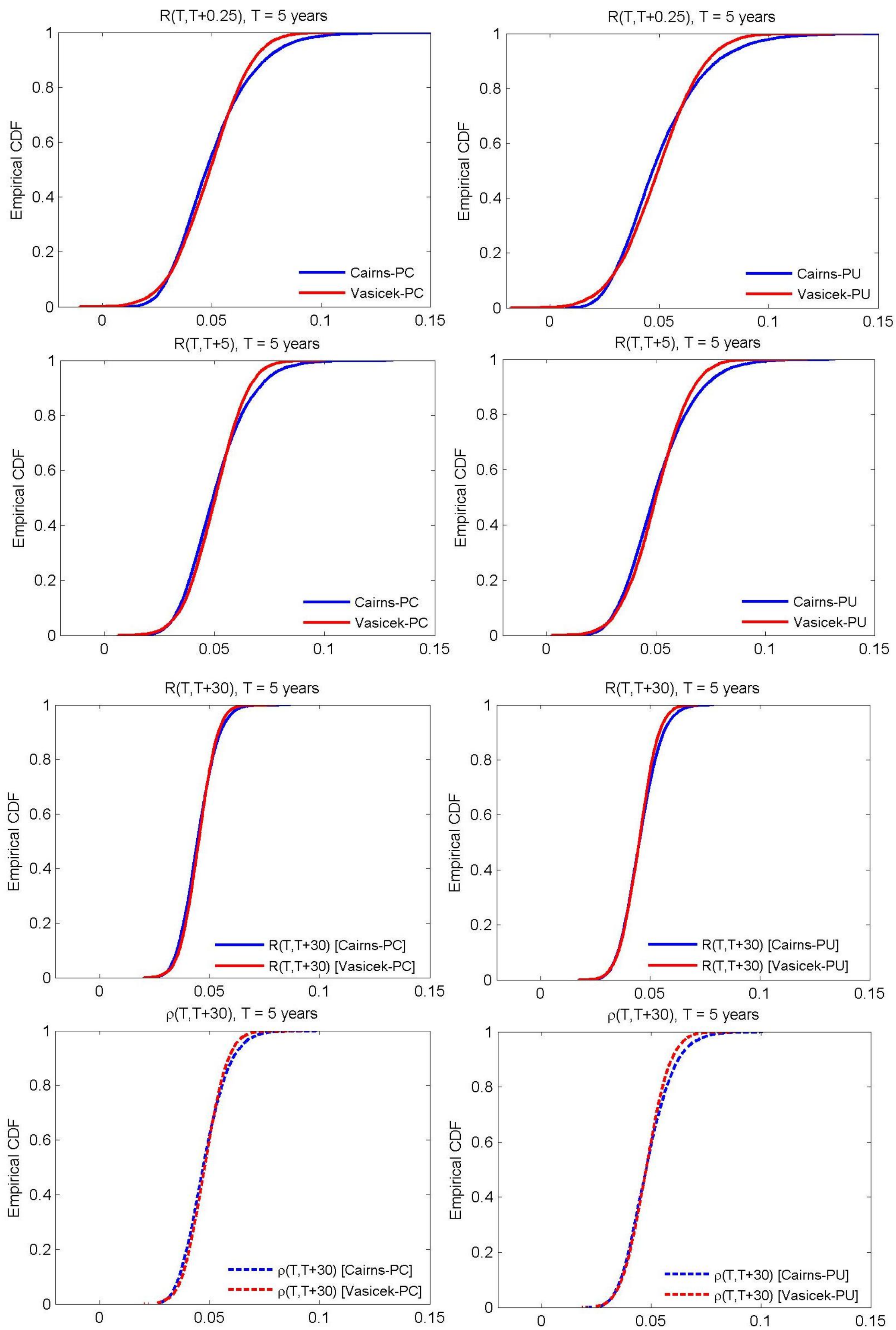

Figure 15: Distributions of the forecast 3-month, 5-year and 30-year spot rates (including 30-year par yields) for 5 years ahead with parameter certainty (PC) (left column) and parameter uncertainty (PU) (right column). Blue line: the two-factor Cairns model. Red line: the two-factor Vasicek model.

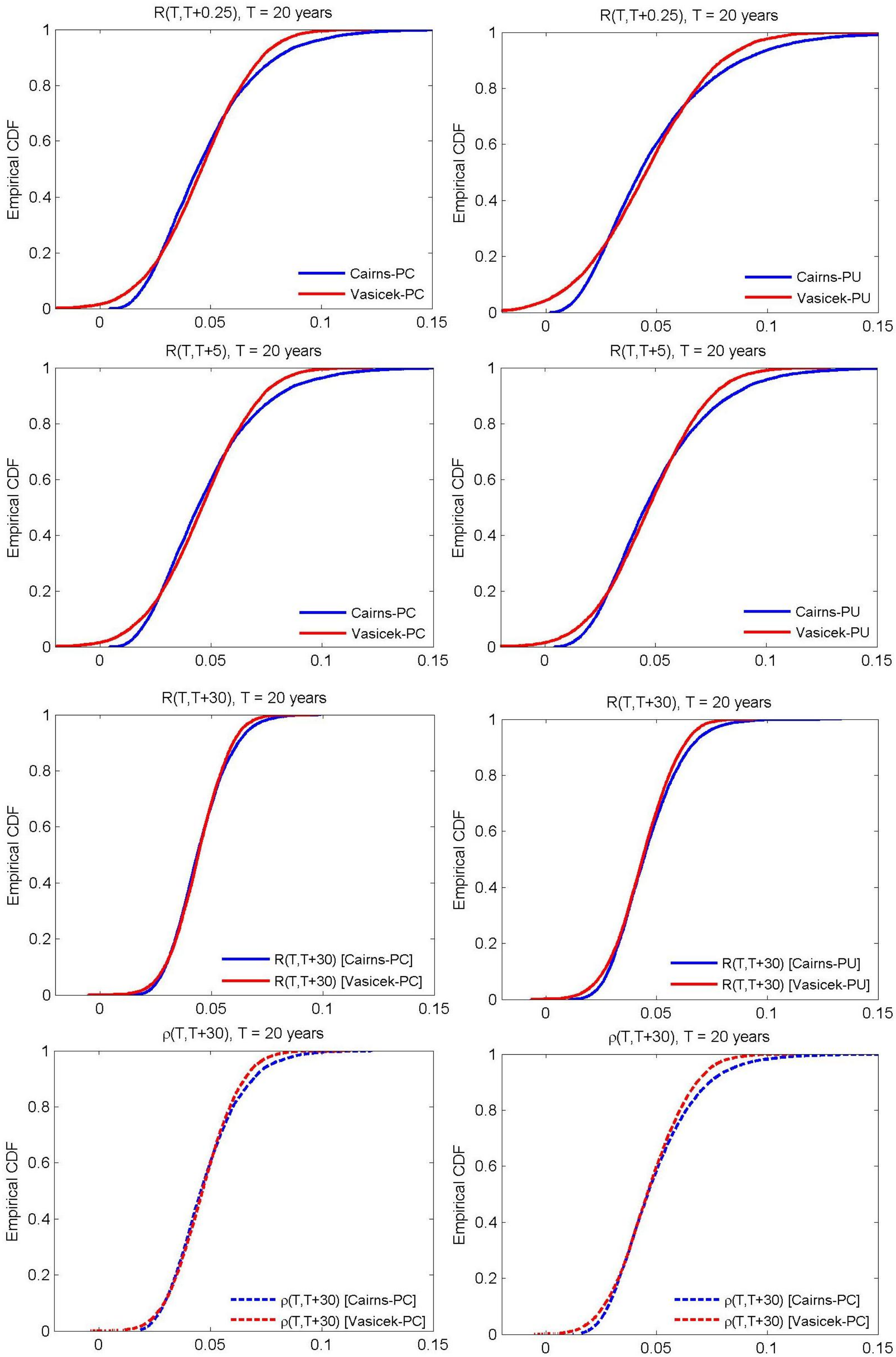

Figure 16: Distributions of the forecast 3-month, 5-year and 30-year spot rates (including 30-year par yields) for 20 years ahead with parameter certainty (PC) (left column) and parameter uncertainty (PU) (right column). Blue line: the two-factor Cairns model. Red line: the two-factor Vasicek model.

6.3.2 Forecasting Results: Annuity Prices

To elaborate the impact of the forecast rates on a more tangible financial contract, we consider forecast annuity values at age 65 for T years ahead which can be defined by

$$a_{65}(T) = \sum_{\tau=1}^{\infty} {}_{\tau}p_{65} P(T,\tau) \tag{7}$$

where ${}_{\tau}p_{65}$ is the probability of survival from age 65 to $65+\tau$ (taken from the PMA92C20, the Faculty of Actuaries and the Institute of Actuaries, 2002, page 112) and $P(T,\tau)$ is a forecast zero-coupon bond price at time T maturing at $T+\tau$. Hence, the forecast bond prices (i.e. spot rates) from the previous section can be directly used to compute forecast annuity prices.

Figures 17 and 18 present empirical distributions and kernel densities of the forecast annuity values for 5, 10, 20 and 40 years ahead with PC and PU cases from Cairns (blue line) and Vasicek (red line) models in comparison. The corresponding summary statistics are also provided in Table 6. According to the results, we can make the following points:

- In both PC and PU cases, empirical cumulative distributions of the forecast annuity values from the two models are more different on both tails when time horizon is longer, representing higher model risk. The closeness of the distributions from the two models for 5 years ahead is as expected since annuity prices are calculated using a large number of long-maturity bond prices in which we can see from the previous section that model selection has least impact on the forecast long-term rates.

- According to the kernel densities, when time horizon is longer, the forecast prices from the Vasicek model tend to be skewed to the right (more low/negative interest rates), while those from the Cairns model are skewed to the left (high interest rates; the skewness is slightly more obvious in the PU case than in the PC case). The results are intuitive according to the corresponding forecast spot rate distributions which are skewed in opposite directions, and indicate more upside and downside risks from the Cairns and Vasicek models respectively. Additionally, it can be noticed that, in all cases, forecast values from Vasicek model are more concentrated around their means (more peaked) than those from the Cairns model particularly for the shorter forecast time horizons and when comparing PC to PU case.

- From Table 6, it can be seen that, in all cases, means of the annuity values are relatively close. Despite having rather similar means and standard deviations, the differences are certainly not negligible, with further differences revealed in the shapes of the distributions in Figure 16. In both PC and PU cases, the Cairns model gives rise to higher standard deviations than the Vasicek model for short forecast time horizon. However, when the time horizon is longer, the difference is diminishing and eventually the standard deviations under the Vasicek model turns out to be larger for 40 years ahead. For each model, it is clear that the PU case provides higher standard deviations than the PC case for all time horizons. Moreover, the difference is also higher provided longer time horizons (more obvious for Vasicek model).

Forecast Annuity Means and Standard Deviations

Forecast Time Horizon	Cairns Model		Vasicek Model	
	PC	PU	PC	PU
5 years ahead	11.641 (1.1586)	11.598 (1.2530)	11.632 (1.0078)	11.647 (1.0929)
10 years ahead	11.741 (1.4559)	11.725 (1.6460)	11.769 (1.3391)	11.832 (1.5459)
20 years ahead	11.827 (1.7935)	11.800 (2.0683)	11.915 (1.7181)	12.036 (2.0673)
40 years ahead	11.752 (1.9842)	11.794 (2.3933)	11.995 (1.9947)	12.261 (2.6289)

Table 6: Forecast annuity means and standard deviations for 5, 10, 20 and 40 years ahead by the two-factor Cairns and Vasicek term structure models with parameter certainty (PC) and parameter uncertainty (PU).

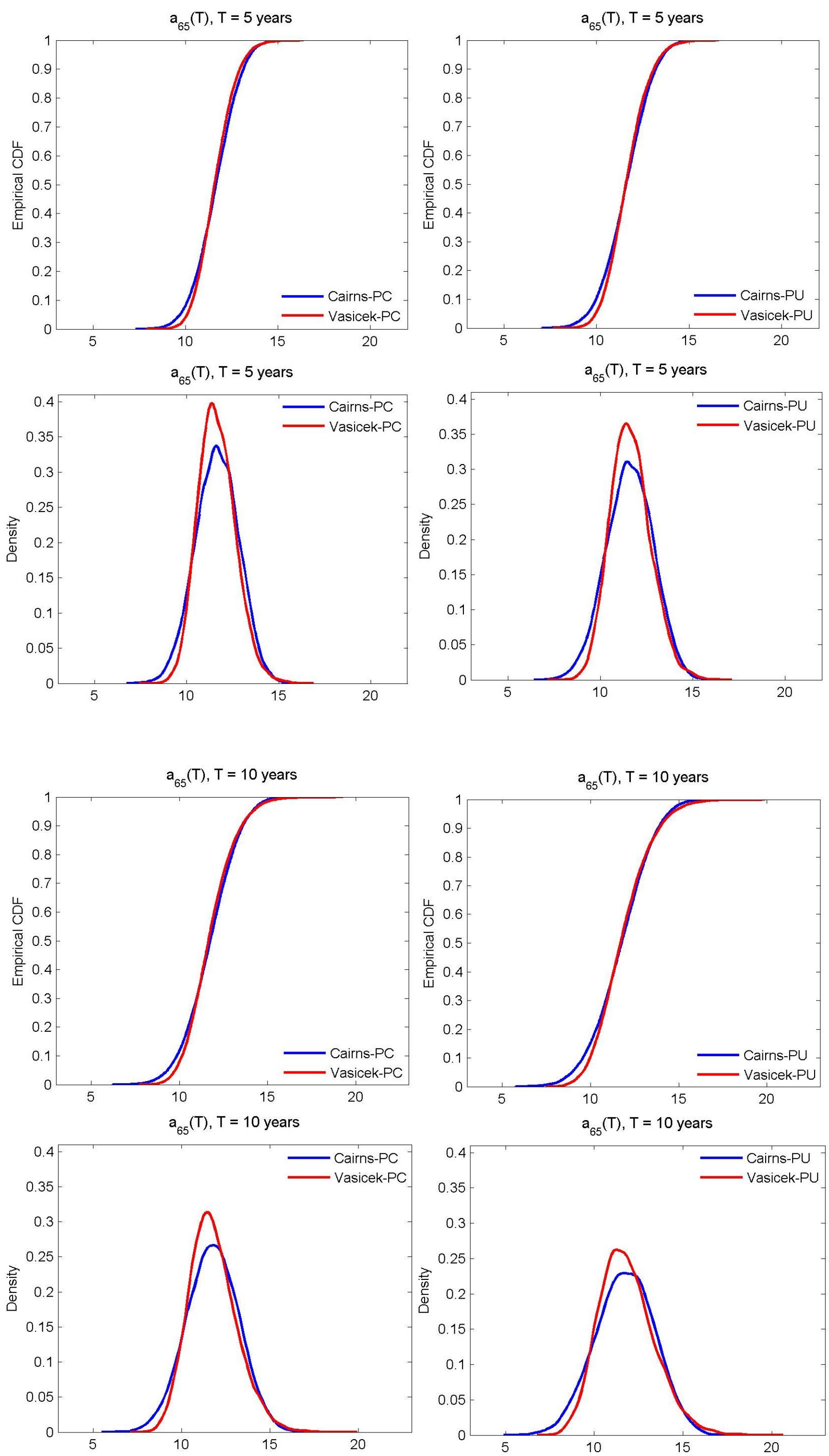

Figure 17: Distributions and kernel densities of the forecast annuity values for 5 and 10 years ahead with parameter certainty (PC) (left column) and parameter uncertainty (PU) (right column). Blue (solid) line: the two-factor Cairns model. Red (solid) line: the two-factor Vasicek model.

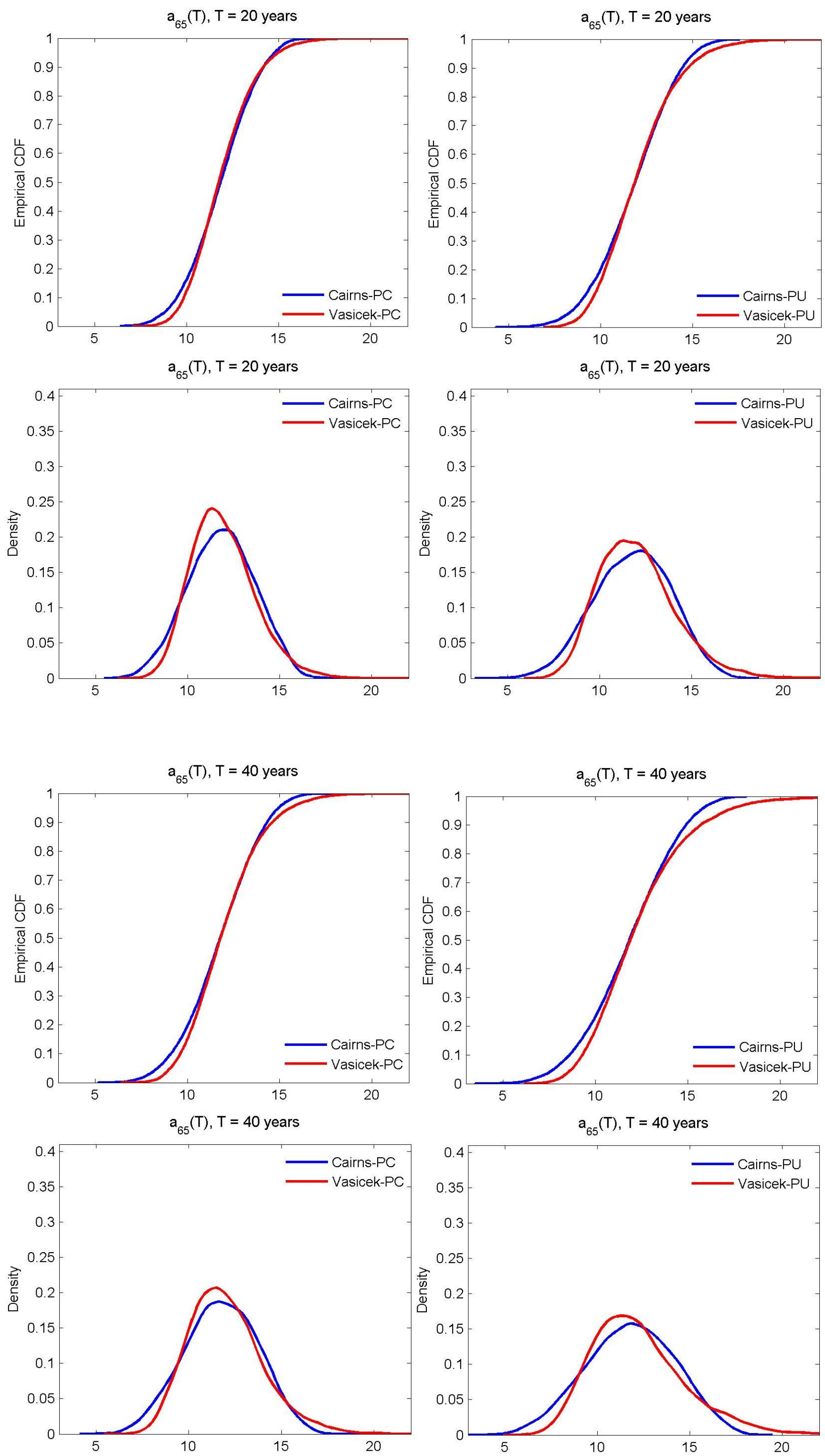

Figure 18: Distributions and kernel densities of the forecast annuity values for 20 and 40 years ahead with parameter certainty (PC) (left column) and parameter uncertainty (PU) (right column). Blue (solid) line: the two-factor Cairns model. Red (solid) line: the two-factor Vasicek model.

7 Conclusions

We have implemented the two-factor Cairns term structure model with applications to yield curve forecasting and annuity pricing for the UK market using the adaptive Metropolis-Hastings algorithm with the repameterised posterior and a blocking strategy. By using the mean values of the parameters and latent variables from the MCMC output, the two-factor Cairns model is generally fitted to the UK Strips data fairly well for the medium- and long-term yields (except during the turmoil period after 2008), but is rather poorly fitted for the short-end of the yield curves. Despite knowing that discrepancies between the estimated and market yields is a common drawback of time-homogeneous arbitrage-free models, two factors may be sufficient in the Cairns model to capture the dynamics of the data in some cases. It is obvious though that at least one additional factor may be required in order to be able to capture the dynamics of UK Strips at the short-end in particular.

The two-factor Vasicek model is also estimated and then compared with the two-factor Cairns model in terms of model fitting, yield curve forecasting and annuity pricing with parameter uncertainty and certainty given the same methodology and market data. Evidently, the developed MCMC algorithm was also found to be very efficient to estimate the two-factor Vasicek model. The Cairns model fits the short-end of yield curves better than the Vasicek model in terms of sum of squared residuals when using estimated posterior mean values of parameter and latent variables. Comparing their forecast spot rates, we can observe distinct differences on tails of the distributions where the Vasicek model discloses a substantial drawback of producing negative rates. Furthermore, model risk also reveals when considering the distributions of forecast annuity prices from the two models (negative skewness for the Cairns model and positive skewness for the Vasicek model).

Parameter uncertainty clearly does have an impact, particularly for the short-end, on forecasting the yield curves from both models. It becomes essentially important when we consider the distributions of the forecast interest rates at both tails (e.g. Value-at-Risk). Of all model parameters, the market prices of risk are likely to be influential parameters for the forecasting. Eventually, we may conclude that allowance for parameter uncertainty should not be neglected when using any model.

References

Babbs, S.H., and Nowman, K.B. (1999). Kalman Filtering of Generailized Vasicek Term Structure Models, *Journal of Financial and Quantitative Analysis*, 1: 115-130.

Cairns, A.J.G. (2000). A discussion of parameter and model uncertainty in insurance. *Insurance: Mathematics and Economics*, 27: 313-330.

Cairns, A.J.G. (2004). A Family of term-structure models for long-term risk management and derivative pricing. *Mathematical Finance*, 14: 415-444.

Leuwattanachotinan, C., Cairns, A.J.G, and Streftaris, G. (2012). Model fitting of a two-factor arbitrage-free model for the term structure of interest rates using Markov chain Monte Carlo: Part 1: theory and methodology.

The Faculty of Actuaries and The Institute of Actuaries (2002). Pensioner Mortality Tables, Formulae and Tables for Examinations.