AGGEGRATION METHODS IN INTERNATIONAL COMPARISONS:

WHAT HAVE WE LEARNED?

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ERIM REPORT SERIES *RESEARCH IN MANAGEMENT*	
ERIM Report Series reference number	ERS-2001-41-MKT
Publication	July 2001
Number of pages	42
Email address corresponding author	bblk@cbs.nl
Address	Erasmus Research Institute of Management (ERIM)
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REPORT SERIES
RESEARCH IN MANAGEMENT

BIBLIOGRAPHIC DATA AND CLASSIFICATIONS		
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Library of Congress Classification (LCC)	5001-6182	Business
	5410-5417.5	Marketing
	HG 229+	Purchasing Power
Journal of Economic Literature (JEL)	M	Business Administration and Business Economics
	M 31	Marketing
	C 44	Statistical Decision Theory
	C 43	Index Numbers and Aggregation
	F 01	International Economics: Global Outlook
European Business Schools Library Group (EBSLG)	85 A	Business General
	280 G	Managing the marketing function
	255 A	Decision theory (general)
	195 A	International economics: General theory
Gemeenschappelijke Onderwerpsontsluiting (GOO)		
Classification GOO	85.00	Bedrijfskunde, Organisatiekunde: algemeen
	85.40	Marketing
	85.03	Methoden en technieken, operations research
	83.40	Internationale economie
Keywords GOO	Bedrijfskunde / Bedrijfseconomie	
	Marketing / Besliskunde	
	Koopkrachtpariteit, Vergelijkend onderzoek	
Free keywords	International comparison, purchasing power parity, index number theory	
Other information		

Aggregation Methods in International Comparisons: What Have We Learned?

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June 2001

Abstract

This paper reviews the progress that has been made over the past decade in understanding the nature of the various multilateral international comparison methods. Fifteen methods are discussed and subjected to a system of ten tests. In addition, attention is paid to recently developed methods based on the principle of chaining, model based approaches, and the economic approach to international comparison.

Keywords: International comparison, purchasing power parity, index number theory.

1 Introduction

In this era of globalization it is of utmost importance to have methods that enable one to compare the economic situation of countries or regions. Such a comparison can be directed at (1) the 'level' of welfare or some measure of economic potential of a geographical entity, or (2) its structure of consumption or production. Key statistics which play a role are Gross Domestic Product (and its components), per capita income, industrial production, or (labor) productivity. The comparability of those statistics is to a large extent guaranteed by adherence to international guidelines like the *System of National Accounts 1993*. The actual comparison, however, is frequently complicated by the fact that the value figures involved read in different currencies. Thus the first task seems to be to convert all those value figures, where necessary, into a single numéraire currency. The instrument that springs to mind here is a set of (market) exchange rates.

The comparison exercise does however not stop here. The second, and more important task is to discern to what extent the value differences are 'real' or 'monetary', that is, to what extent they are determined by more or less physical factors (different levels of consumption or production) or by different price systems. Put otherwise, the task at hand is to split a value difference, conventionally stated in the form of a value ratio, into a quantity index number (reflecting 'real' differences between countries or regions) and a price index number (reflecting 'monetary' differences between countries or regions).

Historically seen, the problem of international comparisons is cast in terms of finding appropriate currency convertors, that is, convertors which are regarded as more adequate than exchange rates. As is wellknown, exchange rates are to a large extent determined by international flows of financial capital and can exhibit very volatile behavior. Lurking in the background here is the old idea that each currency has its own 'purchasing power'.

This wording of the problem, however, is becoming less and less significant in view of the process of monetary unification. Though a large part of Europe will soon possess a common currency, the problem of comparing economic aggregates of different countries or regions will remain with us. Put otherwise, whether or not the countries involved in a comparison study share a certain currency, has become a quite insignificant part of the problem.

In the course of time a large number of methods for solving this problem have been developed. When one only wishes to compare two countries at

a time (a so-called bilateral comparison), one can simply borrow methods familiar from the field of intertemporal comparisons.[1] However, multilateral comparisons, that is comparisons in which more than two countries at a time are involved, constitute a subject *sui generis*.

Multilateral international comparisons are not a simple translation of multilateral intertemporal comparisons. Some important differences between both types of comparisons are:

- Time proceeds continuously whereas the number of countries involved in a certain comparison study stays fixed.

- Unlike time periods, countries do not exhibit a natural ordering.

- In an intertemporal comparison, the time periods considered are usually of the same size (one compares months with months, years with years, etc.). Countries, however, are by nature not equally 'important' (with respect to area, population, economic potential, etc.).

- More than in intertemporal comparisons, there is in international comparisons a strong desire to aggregate the geographical entities and use such aggregates also in comparisons. One *e.g.* wishes to compare the European countries to the European Community as a whole, or the European Community to the United States.

Basically this paper reviews the progress that has been made over the past decade in understanding the nature of the various methods proposed. Some of these are in current use by international organizations.

Hill (1997) developed an interesting and virtually complete taxonomy of multilateral methods for international comparisons. This taxonomy provides insight into the structural similarities and dissimilarities of the various methods. However, a taxonomy as such is not enough to discriminate between competing methods. In addition we need a set of criteria, in the spirit of Fisher (1922) called tests, which a multilateral method ideally should satisfy. A fairly satisfactory set of tests has been developed for the first time by Diewert (1986). But there are also other points of view possible.

The architecture of this paper is as follows. After having done with the necessary definitional footwork, section 2 discusses an important implication

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[1] See Eichhorn and Voeller (1983) for a parallel treatment of intertemporal and interspatial comparisons within the axiomatic approach.

of the requirement of transitivity. Section 3 discusses a number of methods, all of which are related by the fact that they can be regarded as generalizations of a bilateral comparison. Section 4 is devoted to the class of additive methods. Section 5 outlines the test approach and the insights obtained from it. Section 6 reviews two recently developed methods which are based on the procedure of chaining. Section 7 turns to model based approaches, that is, approaches based on an assumption concerning the probability distribution of all the individual prices. Section 8 considers the economic approach. The characteristic feature of this approach is that the aggregate values are conceived as outcomes of optimization procedures. Section 9, finally, concludes.

2 The requirement of transitivity

We assume that we must compare countries[2] $1, ..., I$ $(I \geq 3)$ with respect to a well-defined economic aggregate that involves commodities labelled $1, ..., N$ $(N \geq 2)$. The price vector for country i, expressed in its own currency, will be denoted by $p^i \equiv (p^i_1, ..., p^i_N) \in \Re^N_{++}$, and the corresponding quantity vector will be denoted by $x^i \equiv (x^i_1, ..., x^i_N) \in \Re^N$ $(i = 1, ..., I)$. Both vectors, of course, pertain to a certain period of time. Some quantities could be negative, for instance in the case of imports. However, we will assume that[3] $p^i \cdot x^j > 0$ for all $i, j = 1, ..., I$. For all $i = 1, ..., I$, $p^i \cdot x^i$ represents the value of country i's aggregate.

The price index of country j relative to country i will be denoted by the ratio P^j/P^i, and the quantity index will be denoted by the ratio Q^j/Q^i $(i, j = 1, ..., I)$.[4] This notation expresses the requirement that price and quantity indices be transitive. A second, equally important requirement is that price index and quantity index satisfy the Product Test; that is, their product must exhaust the value ratio:

$$\frac{P^j}{P^i} \frac{Q^j}{Q^i} = \frac{p^j \cdot x^j}{p^i \cdot x^i} \ (i, j = 1, ..., I). \tag{1}$$

As indicated, the ultimate purpose of most international comparisons is to compare 'real' values of countries. The following definitions serve to

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[2]The word 'country' is used as a shorthand for any kind of geographical entity.

[3]Notation: $p^i \cdot x^j \equiv \sum_{n=1}^{N} p^i_n x^j_n$.

[4]The ratio P^j/P^i is usually called the purchasing power parity (PPP) of country j (or country j's currency) relative to country i (or country i's currency).

make this notion precise. The volume of country i is defined as $p^i \cdot x^i / P^i$ $(i = 1, ..., I)$[5] and the volume share of country i in the aggregate volume of all I countries is defined as

$$
\begin{aligned}
Q^i &\equiv \frac{p^i \cdot x^i / P^i}{\sum_{k=1}^{I} p^k \cdot x^k / P^k} \\
&= \left(\sum_{k=1}^{I} (p^k \cdot x^k / p^i \cdot x^i)(P^k / P^i)^{-1} \right)^{-1} \\
&= \left(\sum_{k=1}^{I} (Q^i / Q^k)^{-1} \right)^{-1} \quad (i = 1, ..., I). \quad (2)
\end{aligned}
$$

Notice that the volume shares add up to 1. The second and third lines are added to make clear how volume shares can be calculated from a set of price index numbers or a set of quantity index numbers respectively.

Notice further that the additive nature of the country-specific volume shares makes it possible to define volume shares for aggregates of countries as well. For example, the volume share of the union of countries i and j is simply given by $Q^i + Q^j$.

The requirement of transitivity appears to have a farreaching consequence, which can be seen as follows. Suppose that the quantity index Q^j / Q^i does not depend on prices and quantities of countries other than i and j, that is, there exists a function $f(.)$ such that[6]

$$
Q^j / Q^i = f(p^j, x^j, p^i, x^i) \; (i, j = 1, ..., I). \quad (3)
$$

In fact, this expression could be regarded as the formalization of the requirement of (maximal) characteristicity, which goes back to Drechsler (1973)[7]. The requirement of transitivity then implies that

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[5]In practice this is calculated as $p^i \cdot x^i / (P^i / P^k)$ for some choice of the numéraire country k.

[6]The fact that $f(.)$ is assumed to be the same for all pairs of countries reflects the natural but hidden assumption that all countries be treated symmetrically.

[7]According to Drechsler (1973) "... the characteristicity requirement is satisfied if in the computation of indices the weights of the given two countries are used. In a Netherlands-Belgium quantity comparison, for instance, this requirement is completely satisfied if Dutch prices, Belgian prices or average Dutch-Belgian prices are used as weights. Average EEC weights are not fully characteristic for a Netherlands-Belgium comparison, and average European weights even less. To use Indian weights in a Netherlands-Belgium comparison would be considered wrong by everybody just as if in an Indian-Pakistan

$$f(p^j, x^j, p^i, x^i) = \frac{f(p^j, x^j, p^k, x^k)}{f(p^i, x^i, p^k, x^k)} \; (i, j, k = 1, ..., I). \tag{4}$$

Since the left hand side of this equation does not depend on (p^k, x^k), the right hand side cannot depend on it either. But this implies that there exists a function $g(p, x)$ such that the quantity index can be written as

$$Q^j/Q^i = \frac{g(p^j, x^j)}{g(p^i, x^i)} \; (i, j = 1, ..., I). \tag{5}$$

A very modest, even minimal, requirement on quantity indices is that whenever all country j quantities equal country i quantities, that is $x^j = x^i$, then the quantity index Q^j/Q^i takes on the value 1. Put otherwise, quantity indices are required to satisfy the identity test. But this implies that

$$g(p^j, x^i) = g(p^i, x^i) \; (i, j = 1, ..., I), \tag{6}$$

which means that the function $g(p, x)$ is actually a function $g(x)$. Substituting this into expression (5), we must conclude that the quantity index Q^j/Q^i does not depend on any prices.

This would be an undesirable state of affairs. Thus, if we wish prices to play a role in the quantity index and if we wish to uphold the requirements of transitivity and identity, then we must sacrifice the requirement of (maximal) characteristicity.[8] Put otherwise, we must accept that each quantity index Q^j/Q^i is a function of all the prices and all the quantities, $p^1, ..., p^I, x^1, ..., x^I$ $(i, j = 1, ..., I)$.

3 Generalizations of a bilateral comparison

Let us first consider two countries i and j. The price level of country j relative to country i could be measured by the Laspeyres price index

$$P^L(p^j, x^j, p^i, x^i) \equiv p^j \cdot x^i / p^i \cdot x^i, \tag{7}$$

comparison Dutch weights were used. In the latter cases, the weights would be very uncharacteristic; their use would amount to the same as if in the case of the computation of a 1971-1970 inter-temporal index 1920 (or 2020) prices were used."

[8]This corresponds to the impossibility theorem of Van Veelen (forthcoming), which uses a slightly more general setting (using ordinal rather than cardinal relations).

or by the Paasche price index

$$P^P(p^j, x^j, p^i, x^i) \equiv p^j \cdot x^j / p^i \cdot x^j. \tag{8}$$

The interpretation of these indices depends of course on the aggregate under study. For instance, when we are studying household consumption, the Laspeyres price index compares the value of country i's household consumption at country j's prices to the value of this consumption at its own prices. Similarly, the Paasche price index compares the value of country j's household consumption at its own prices to the value at country i's prices. When the currencies of the two countries differ, the dimension of both indices is: the number of country j currency units per unit of country i's currency.

In general the two price indices (7) and (8) will yield different outcomes. Wanting a single outcome, we seek for a price index P^j / P^i that lies 'between' the Laspeyres and the Paasche price index. In particular we require that

$$P^L(p^j, x^j, p^i, x^i) = t P^j / P^i \text{ and } P^j / P^i = t P^P(p^j, x^j, p^i, x^i) \ (t > 0). \tag{9}$$

By employing the relation $1/P^P(p^j, x^j, p^i, x^i) = P^L(p^i, x^i, p^j, x^j)$, these two equations can obviously be reduced to a single one, namely

$$P^L(p^j, x^j, p^i, x^i) P^i / P^j = P^L(p^i, x^i, p^j, x^j) P^j / P^i. \tag{10}$$

One verifies easily that the solution to (10) is the Fisher price index

$$P^j / P^i = P^F(p^j, x^j, p^i, x^i) \equiv [P^L(p^j, x^j, p^i, x^i) P^P(p^j, x^j, p^i, x^i)]^{1/2}, \tag{11}$$

the geometric average of the Laspeyres and the Paasche price index. Notice that the Fisher index has the country reversal property, that is

$$P^F(p^i, x^i, p^j, x^j) = 1/P^F(p^j, x^j, p^i, x^i), \tag{12}$$

which reflects the symmetrical position of the two countries i and j.

We now turn to a truly multilateral comparison, that is a comparison of all I countries simultaneously. We will assume that we have a set of positive country weights g_i $(i = 1, ..., I)$. These weights can be regarded as initial measures of country importance. Normalized weights, adding up to 1, are defined by $f_i \equiv g_i / \sum_{i=1}^{I} g_i$ $(i = 1, ..., I)$.

We proceed by generalizing (10) in the following way

$$\prod_{i=1}^{I}[P^L(p^j,x^j,p^i,x^i)P^i/P^j]^{f_i} = \prod_{i=1}^{I}[P^L(p^i,x^i,p^j,x^j)P^j/P^i]^{f_i} \ (j=1,...,I).$$
(13)

At both sides of these equations we see weighted geometric averages. It is straightforward to check that the solution of this system of equations is

$$\left(\frac{P^j}{P^i}\right)_{GEKS} \equiv \prod_{k=1}^{I}[P^F(p^j,x^j,p^k,x^k)P^F(p^k,x^k,p^i,x^i)]^{f_k} \ (i,j=1,...,I).$$
(14)

If all the weights are the same, which implies that $f_k = 1/I$ $(k = 1, ..., I)$, then expression (14) reduces to the wellknown formula proposed independently by Eltetö and Köves (1964) and Szulc (1964).[9] This formula was, however, already proposed by Gini (1924).[10]

An other way of deriving (14) is by solving the following minimization problem

$$\min_{P^1,...,P^I} \sum_{i=1}^{I} \sum_{j=1}^{I} g_i g_j [\ln P^F(p^j,x^j,p^i,x^i) - \ln(P^j/P^i)]^2.$$
(15)

Here one seeks to determine transitive, multilateral price indices which approximate as best as possible the intransitive, bilateral Fisher price indices.[11] One could say that the impossible requirement of (maximal) characteristicity is replaced by the objective of obtaining 'optimal' characteristicity. This objective is to be attained by minimizing a sum of weighted squared residuals. The weights $g_i g_j$ are used to discriminate between all the pairwise

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[9]According to Köves (1999) the EKS formula appeared for the first time in an appendix, written by Eltetö, of a 1962 book by L. Drechsler. Actually, Drechsler (1973) introduced the EKS method to the Western world.

[10]This fact was acknowledged by Szulc (1964), who however referred to Gini (1931). On Gini's contributions, see Biggeri, Ferrari and Lemmi (1987).

[11]Replacing in (15) the Fisher price index by the Laspeyres or the Paasche price index leads to the same outcome, (14), as shown by Van IJzeren (1987). By replacing in (15) the Fisher price index by the Törnqvist price index and setting all weights equal, one obtains the multilateral Törnqvist price index proposed by Caves, Christensen and Diewert (1982a). See also Section 7.

comparisons.[12]

Still another way of interpreting (14) is to notice that each of the factors $P^F(p^j, x^j, p^k, x^k) P^F(p^k, x^k, p^i, x^i)$ provides a 'price index' for country j relative to country i, calculated via a 'bridge' country k. Since there are I choices for k, it is rather natural to take an average of all of those factors as the final index.

We notice that, according to (14), the price indices $(P^j/P^i)_{GEKS}$ depend not only on the prices and quantities of the countries i and j, but also on the prices and quantities of all the other countries involved. Thus if we extend the set of countries, all price index numbers must be recalculated. This is one of the features distinguishing multilateral price indices from bilateral price indices.

The quantity index associated with (14) can be obtained easily from (1) by using the fact that the weights f_k add up to 1 and the Factor Reversal property of Fisher's price and quantity indices. The index reads

$$\left(\frac{Q^j}{Q^i}\right)_{GEKS} \equiv \prod_{k=1}^{I} [Q^F(p^j, x^j, p^k, x^k) Q^F(p^k, x^k, p^i, x^i)]^{f_k} \ (i, j = 1, ..., I). \ (16)$$

where

$$Q^F(p^j, x^j, p^i, x^i) \equiv [(p^i \cdot x^j / p^i \cdot x^i)(p^j \cdot x^j / p^j \cdot x^i)]^{1/2} \qquad (17)$$

is the Fisher quantity index. Notice that the GEKS quantity index can also be obtained by interchanging prices and quantities in the GEKS price index (14). Thus the GEKS indices satisfy the Factor Reversal Test.

A second way of generalizing (10) is by employing arithmetic averages instead of geometric averages. We then obtain, instead of (13), the following system of equations:

$$\sum_{i=1}^{I} g_i P^L(p^j, x^j, p^i, x^i) P^i / P^j = \sum_{i=1}^{I} g_i P^L(p^i, x^i, p^j, x^j) P^j / P^i \ (j = 1, ..., I).$$
$$(18)$$

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[12]Rao (2001) considered a generalization of this procedure, whereby the weights $g_i g_j$ are replaced by weights g_{ij}. This implies that there is no closed-form solution available. His experiments, based on data from the 1993 OECD survey, suggest that the multilateral price index numbers obtained are not very sensitive to the particular choice of the weights.

One can prove that this system of equations has a unique, positive solution $P_Y^1, ..., P_Y^I$, which is determined up to a scalar factor. Thus, although not expressible in explicit form, the price indices P_Y^j/P_Y^i $(i, j = 1, ..., I)$ are completely determined, as are the quantity indices via (1). They depend on the prices and quantities of all the I countries. The method defined by (18) is known as the third, balanced method of Van IJzeren (1955), (1956).

There are other ways of deriving (18). In his original publications, Van IJzeren used a so-called "tourist model". But he also noticed that the equations (18) can be conceived as being the first-order conditions for the following minimization problem

$$\min_{P^1, ..., P^I} \sum_{i=1}^{I} \sum_{j=1}^{I} g_i g_j P^L(p^j, x^j, p^i, x^i) P^i / P^j. \tag{19}$$

Notice that $P^L(p^j, x^j, p^i, x^i) P^i / P^j$ is the discrepancy, in the form of a ratio, between the bilateral Laspeyres price index and the desired multilateral price index for country j relative to country i. Thus (19) is the same kind of problem as (15), the objective function being different.

Still another way of deriving (18) was provided by Balk (1989), (1996a). The naming of the method suggests that there are at least two other methods. Since they are now of mere historical interest, we will not discuss them here. Their definitions as well as interesting details on their genesis can be found in Balk (1999).

Using relationship (1), we find that expression (18) can be rewritten as

$$\sum_{i=1}^{I} g_i Q^L(p^j, x^j, p^i, x^i) Q^i / Q^j = \sum_{i=1}^{I} g_i Q^L(p^i, x^i, p^j, x^j) Q^j / Q^i \ (j = 1, ..., I). \tag{20}$$

where $Q^L(p^j, x^j, p^i, x^i) \equiv p^i \cdot x^j / p^i \cdot x^i$ is the Laspeyres quantity index of country j relative to country i. Notice that expression (20) can also be obtained by interchanging prices and quantities in expression (18). Thus, although not expressible in explicit form, the Van IJzeren price and quantity indices can be said to satisfy the Factor Reversal Test.

The third way of generalizing (10) is the polar opposite of the second way, namely by employing harmonic averages. We then obtain

$$[\sum_{i=1}^{I} g_i[P^L(p^j,x^j,p^i,x^i)P^i/P^j]^{-1}]^{-1} =$$

$$[\sum_{i=1}^{I} g_i[P^L(p^i,x^i,p^j,x^j)P^j/P^i]^{-1}]^{-1} \ (j=1,...,I). \qquad (21)$$

As one verifies immediately, this system of equations can be written as

$$\sum_{i=1}^{I} g_i P^P(p^j,x^j,p^i,x^i)P^i/P^j = \sum_{i=1}^{I} g_i P^P(p^i,x^i,p^j,x^j)P^j/P^i \ (j=1,...,I).$$
$$(22)$$

This (relatively unknown) system of equations was proposed by Gerardi (1974). He derived it from a so-called "immigrant model", a variation of Van IJzeren's "tourist model". But (22) can also be conceived as being the first-order conditions for the following minimization problem

$$\min_{P^1,...,P^I} \sum_{i=1}^{I} \sum_{j=1}^{I} g_i g_j P^P(p^j,x^j,p^i,x^i)P^i/P^j, \qquad (23)$$

which differs from (19) in that Paasche indices are used instead of Laspeyres indices.

Using relationship (1), we find that expression (22) can be rewritten as

$$\sum_{i=1}^{I} g_i Q^P(p^j,x^j,p^i,x^i)Q^i/Q^j = \sum_{i=1}^{I} g_i Q^P(p^i,x^i,p^j,x^j)Q^j/Q^i \ (j=1,...,I).$$
$$(24)$$

where $Q^P(p^j,x^j,p^i,x^i) \equiv p^j{\cdot}x^j/p^j{\cdot}x^i$ is the Paasche quantity index of country j relative to country i. Thus also Gerardi's price and quantity indices can be said to satisfy the Factor Reversal Test.

Our next set of generalizations departs from expressions (14) and (16). Using the country reversal property of Fisher's indices, these expressions can be rewritten as

$$\left(\frac{P^j}{P^i}\right)_{GEKS} = \frac{\prod_{k=1}^{I}[P^F(p^j,x^j,p^k,x^k)]^{f_k}}{\prod_{k=1}^{I}[P^F(p^i,x^i,p^k,x^k)]^{f_k}} \ (i,j=1,...,I) \qquad (25)$$

$$\left(\frac{Q^j}{Q^i}\right)_{GEKS} = \frac{\prod_{k=1}^I [Q^F(p^j, x^j, p^k, x^k)]^{f_k}}{\prod_{k=1}^I [Q^F(p^i, x^i, p^k, x^k)]^{f_k}} \quad (i, j = 1, ..., I). \qquad (26)$$

Two alternatives emerge when we replace in the first expression the geometric averages in numerator and denominator by arithmetic or harmonic averages. The associated quantity indices are then defined residually by (1). Two other alternatives emerge when we replace in the second expression the geometric averages by arithmetic or harmonic averages, and define the associated price indices residually.

For instance, when we replace in expression (25) the geometric averages by arithmetic averages, we obtain

$$\left(\frac{P^j}{P^i}\right)_{WFBS} \equiv \frac{\sum_{k=1}^I g_k P^F(p^j, x^j, p^k, x^k)}{\sum_{k=1}^I g_k P^F(p^i, x^i, p^k, x^k)} \quad (i, j = 1, ..., I), \qquad (27)$$

where we used the definition of normalized country weights f_k to return to unnormalized weights g_k. This system of multilateral price indices can be said to go back to Fisher (1922). When we set all weights equal, expression (27) reduces to his so-called "blended system".

When we replace in expression (26) the geometric averages by harmonic averages, we obtain in the same way

$$\begin{aligned}\left(\frac{Q^j}{Q^i}\right)_{WDOS} &\equiv \frac{\left(\sum_{k=1}^I g_k [Q^F(p^j, x^j, p^k, x^k)]^{-1}\right)^{-1}}{\left(\sum_{k=1}^I g_k [Q^F(p^i, x^i, p^k, x^k)]^{-1}\right)^{-1}} \\ &= \frac{\sum_{k=1}^I g_k Q^F(p^k, x^k, p^i, x^i)}{\sum_{k=1}^I g_k Q^F(p^k, x^k, p^j, x^j)} \quad (i, j = 1, ..., I), \qquad (28)\end{aligned}$$

where the last line was obtained by using the country reversal property of the Fisher quantity indices. Expression (28) defines multilateral quantity indices as a generalization of Diewert's (1986) Own Share system (WDOS). Setting all weights equal, expression (28) reduces to the original DOS.

The two other possible systems have, as far as I know, not been discussed in the literature.

Still other methods, not being discussed here, are those called YKS and Q-YKS (Kurabayashi and Sakuma 1982, 1990). Both are related to the Van IJzeren method which was defined by expression (18). See Balk (1996b) for details.

All the methods discussed in this section effectively provide a mapping from a vector of initial country weights $(f_1, ..., f_I)$ to a vector of volume shares $(Q^1, ..., Q^I)$. This is a continuous mapping from the I-dimensional unit simplex into itself. According to Brouwer's Fixed Point Theorem (see Green and Heller 1981) this mapping has a fixed point. Thus in all the formulas one can replace f_k (or g_k) by Q^k $(k = 1, ..., I)$. The solution vector must of course then be obtained by a suitable numerical iteration method.

4   Additive methods

A multilateral comparison method is called *additive* when

$$Q^i \propto \pi \cdot x^i \ (i = 1, ..., I), \tag{29}$$

where $\pi \equiv (\pi_1, ..., \pi_N)$ is some price vector. Expression (29) says that the volume share of country i is proportional to the aggregate value of this country's quantities at prices π. Since $\sum_{i=1}^{I} Q^i = 1$, this implies that

$$Q^i = \frac{\pi \cdot x^i}{\sum_{i=1}^{I} \pi \cdot x^i} \ (i = 1, ..., I). \tag{30}$$

Using the product relation (1), this in turn implies that

$$P^i \propto \frac{p^i \cdot x^i}{\pi \cdot x^i} \ (i = 1, ..., I). \tag{31}$$

Thus each purchasing power parity P^i is proportional to a Paasche-type price index, comparing country i's price vector p^i to the price vector π. When the proportionality factor in the last expression equals 1, the method is called *strongly additive*.

The virtue of an additive method is its simple interpretation, as evidenced by the foregoing expressions. The use of a common price vector enables us to compare the quantity structures of an aggregate across countries in a very straightforward way. The intertemporal analogue is to express the value of an aggregate through time in 'constant prices'.

The basic problem, of course, is how to pick the price vector π. The symmetric treatment of countries suggests that π must be some average of the country-specific price vectors p^i. Accordingly, π is called a vector of 'international prices'.

By far the best known member of the class of strongly additive methods was proposed by Geary (1958) and popularized by Khamis (1972). This method consists of the following set of definitions

$$
\begin{aligned}
\pi_n &= \frac{\sum_{i=1}^{I} p_n^i x_n^i / P^i}{\sum_{i=1}^{I} x_n^i} \ (n = 1, ..., N) \\
P^i &= p^i \cdot x^i / \pi \cdot x^i \ (i = 1, ..., I),
\end{aligned}
\tag{32}
$$

which actually is a system of equations that must be solved. Each international price π_n can be regarded as the unit value of commodity n, after having converted the country-specific values $p_n^i x_n^i$ to a common currency. This, however, is old-fashioned language that looses its significance when there are no currency differences between the countries involved in the comparison exercise. It is better to say that each country-specific price p_n^i is deflated by the purchasing power parity P^i, and that a weighted average of the p_n^i / P^i is taken, the weights being quantity shares $x_n^i / \sum_{j=1}^{I} x_n^j$. Put otherwise, π is not a vector of average prices, but a vector expressing some average price structure.

Using (2) we obtain the following, equivalent system of equations

$$
\begin{aligned}
\pi_n &= \alpha \frac{\sum_{i=1}^{I} w_n^i Q^i}{\sum_{i=1}^{I} x_n^i} \ (n = 1, ..., N) \\
\alpha Q^i &= \pi \cdot x^i \ (i = 1, ..., I).
\end{aligned}
\tag{33}
$$

where $w_n^i \equiv p_n^i x_n^i / p^i \cdot x^i$ is the commodity n value share in country i ($n = 1, ..., N; i = 1, ..., I$), and α is a certain scalar (normalizing) factor. Substituting (33a) into (33b) the system of equations can be reduced to

$$
\sum_{i=1}^{I} \left(\sum_{n=1}^{N} \left(w_n^i x_n^j / \sum_{i=1}^{I} x_n^i \right) \right) Q^i = Q^j \ (j = 1, ..., I).
\tag{34}
$$

Let M be the $I \times I$ matrix with elements $m^{ij} \equiv \sum_{n=1}^{N} (w_n^i x_n^j / \sum_{i=1}^{I} x_n^i)$ and let E be the $I \times I$ unit matrix. Then (34) can be written in matrix notation as

$$
(Q^1, ..., Q^I)(M - E) = (0, ..., 0).
\tag{35}
$$

Notice that the matrix $M - E$ is singular since $\sum_{j=1}^{I} m^{ij} = 1$ ($i = 1, ..., I$). Following Collier (1999), the constraint $\sum_{i=1}^{I} Q^i = 1$ can be expressed as

14

$$(Q^1, ..., Q^I)R = (1, 0, ..., 0), \tag{36}$$

where R is an $I \times I$ matrix with the first column consisting of 1's and the remaining elements being 0. Adding the equations (35) and (36), we obtain

$$(Q^1, ..., Q^I)(M - E + R) = (1, 0, ..., 0). \tag{37}$$

The Geary-Khamis (GK) volume shares are thus given by

$$(Q^1_{GK}, ..., Q^I_{GK}) = (1, 0, ..., 0)(M - E + R)^{-1}, \tag{38}$$

the price indices can be obtained via relation (1), and the solution vector π can be obtained via (33a).

The particular definition of the international prices is the point of much criticism levelled against the GK method. By virtue of its definition the vector π tends to resemble the price structure of the largest country involved in the comparison exercise[13], say ℓ. But then, as one verifies easily,

$$\frac{Q^i_{GK}}{Q^\ell_{GK}} = \frac{\pi \cdot x^i}{\pi \cdot x^\ell} \approx \frac{p^\ell \cdot x^i}{p^\ell \cdot x^\ell}, \tag{39}$$

which is the Laspeyres quantity index of country i relative to country ℓ. Depending on the market orientation of the aggregate (producer or consumer), this index is generally felt to be an under- or over-statement of the 'true' quantity index. This alleged bias is called the Gerschenkron effect, after its discovery by Gerschenkron (1951). In intertemporal comparisons it finds its parallel in the neglect of the substitution effect.

Remaining within the framework of additive methods the obvious remedy is to look for alternative definitions of the international prices. One can generalize the GK definition (32a) to

$$\pi_n = \sum_{i=1}^{I} \alpha_n^i p_n^i / P^i \ (n = 1, ..., N) \tag{40}$$

where the weights α_n^i are positive and $\sum_{i=1}^{I} \alpha_n^i = 1 \ (n = 1, ..., N)$. The explicit solution for the volume shares is given by the right hand side of expression (38) where M is now the matrix with elements $m^{ij} \equiv \sum_{n=1}^{N}(\alpha_n^i p_n^i x_n^j / (p^i \cdot x^i)) \ (i, j = 1, ..., I)$.

---

[13]Khamis (1998) denies this: "No country is large enough [with respect to all commodities] to produce such an effect."

Cuthbert (1999) proved that if each α_n^i is a function of all the quantities x_n^i $(n = 1, ..., N; i = 1, ..., I)$, then α_n^i must necessarily be of the form

$$\alpha_n^i = \beta^i x_n^i / \sum_{i=1}^{I} \beta^i x_n^i \tag{41}$$

where $\beta^i > 0$ $(i = 1, ..., I)$. He called the method defined by (32b), (40) and (41) the Generalized Geary-Khamis (GGK) method. Indeed, by taking all β^i's to be the same, the GGK method reduces to the GK method.

When $\beta^i = 1/Q^i$ $(i = 1, ..., I)$ the GGK method reduces to the Iklé (1972) method.[14] Balk (1996b) showed that this method has a unique positive solution, although this solution is not expressible in an explicit form. We will return to this method in the sequel.

An other method is obtained by choosing instead of (41)

$$\alpha_n^i = w_n^i / \sum_{i=1}^{I} w_n^i. \tag{42}$$

Hill (2000) called this method the "equally weighted GK method" (EWGK). In order to ease the comparison with the GK method (32), the complete system of equations defining the EWGK method is stated here:

$$\begin{aligned}
\pi_n &= \frac{\sum_{i=1}^{I} w_n^i p_n^i / P^i}{\sum_{i=1}^{I} w_n^i} \ (n = 1, ..., N) \\
P^i &= p^i \cdot x^i / \pi \cdot x^i \ (i = 1, ..., I).
\end{aligned} \tag{43}$$

As one sees, the difference with the GK method is that quantities x_n^i are replaced by value shares w_n^i. The explicit solution for the volume shares according to the EWGK method is given by the right hand side of expression (38) where M is now the matrix with elements $m^{ij} \equiv \sum_{n=1}^{N}(w_n^i p_n^i x_n^j / (p^i \cdot x^i \sum_{i=1}^{I} w_n^i))$ $(i, j = 1, ..., I)$.

Hill (2000) compared both methods and found that the EWGK method is less affected by the Gerschenkron effect than the GK method.[15]

---

[14]Cuthbert (2000) considers $\beta^i = 1/(Q^i)^\alpha$ where $0 \leq \alpha \leq 1$. The case $\alpha = 0$ corresponds to the GK method while the case $\alpha = 1$ corresponds to the Iklé method. Using price and expenditure data for 199 commodities and 24 countries coming from the 1993 OECD survey, price index numbers and volumes were calculated for various values of α.

[15]He used price and expenditure data for 139 commodities and 64 countries coming from the 1985 ICP survey.

A related method, differing from the GK method by the definition of international prices, is the KS-S method proposed by Kurabayashi and Sakuma (1981), (1990). See Balk (1996b) for details on this method. Recently, Sakuma, Rao and Kurabayashi (2000) developed an interesting variant of the KS-S method. The defining system of equations of the new (SRK) method is

$$
\begin{aligned}
\pi_n &= \alpha \sum_{i=1}^{I} (p_n^i/p^i \cdot \sum_{j=1}^{I} x^j) Q^i \ (n = 1, ..., N) \\
\alpha Q^i &= \pi \cdot x^i \ (i = 1, ..., I),
\end{aligned}
\tag{44}
$$

or, using price indices rather than volume shares,

$$
\begin{aligned}
\pi_n &= \sum_{i=1}^{I} (p_n^i/P^i)(p^i \cdot x^i/p^i \cdot \sum_{j=1}^{I} x^j) \ (n = 1, ..., N) \\
P^i &= p^i \cdot x^i/\pi \cdot x^i \ (i = 1, ..., I).
\end{aligned}
\tag{45}
$$

It is interesting to compare this system also to the GK system (32). In the GK system the country-specific deflated prices p_n^i/P^i are weighted with commodity-specific quantity shares $x_n^i/\sum_{j=1}^{I} x_n^j$. These quantity shares can also be expressed as value shares $p_n^i x_n^i/p_n^i \sum_{j=1}^{I} x_n^j$. In the SRK system the country-specific deflated prices are weighted with aggregate value shares $p^i \cdot x^i/p^i \cdot \sum_{j=1}^{I} x^j$. These weights are the same for every commodity. Each of these weights can be interpreted as the volume share of country i based on its own price vector, or as the Paasche-type quantity index for country i relative to the aggregate of all countries.

Although empirical evidence is as yet lacking, it could very well be that due to the fact that these weights are the same for every commodity, the SRK system is more prone to the Gerschenkron effect than the GK system.

Again, the explicit solution for the volume shares from (44) is given by the right hand side of expression (38) where M is now the matrix with elements $m^{ij} \equiv p^i \cdot x^j/p^i \cdot \sum_{j=1}^{I} x^j \ (i, j = 1, ..., I)$. Notice that $\sum_{j=1}^{I} m^{ij} = 1 \ (i = 1, ..., I)$.

It is easily seen that (45a) is an instance of the more general definition

$$
\pi_n = \sum_{i=1}^{I} g_i(p_n^i/P^i) \ (n = 1, ..., N)
\tag{46}
$$

where g_i $(i = 1, ..., I)$ are positive country weights. Expressions (29) and (46) together define Van IJzeren's second method, alluded to in the previous section. Choosing now

$$g_i = P^i/p^i \cdot \sum_{j=1}^{I} x^j \ (i = 1, ..., I) \tag{47}$$

leads us to the (Standardised Structure) method as proposed by Sergueev (2001).[16] It is straightforward to infer that this method leads to volume shares of the form

$$Q_S^i = \frac{\sum_{k=1}^{I} p^k \cdot x^i / p^k \cdot \bar{x}}{\sum_{i=1}^{I} \sum_{k=1}^{I} p^k \cdot x^i / p^k \cdot \bar{x}} \ (i = 1, ..., I) \tag{48}$$

where $\bar{x} \equiv \sum_{j=1}^{I} x^j$. Since the unknown P^i's in the numerator of (47) and the denominator of (46) cancel, there is no need to solve a system of equations. The corresponding quantity indices can be expressed as

$$\frac{Q_S^i}{Q_S^j} = \frac{\sum_{k=1}^{I} Q^L(p^i, x^i, p^k, \bar{x})}{\sum_{k=1}^{I} Q^L(p^j, x^j, p^k, \bar{x})} \ (i, j = 1, ..., I). \tag{49}$$

Hitherto the international prices were defined as (weighted) arithmetic averages of country-specific deflated prices p_n^i/P^i. Gerardi (1974) proposed to define the international prices as unweighted geometric averages

$$\pi_n = (\prod_{i=1}^{I} p_n^i)^{1/I} \ (n = 1, ..., N). \tag{50}$$

The method defined by (29) and (50) together constitutes a multilateral generalization of a bilateral quantity index proposed by Walsh (1901). In view of expression (30) for the volume shares it is clear that deflation of the country-specific prices can be dispensed with.

Let us return to the EWGK method, which was defined by expressions (29), (40), and (42). Replacing the arithmetic averages by harmonic averages, that is, replacing (40) by

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[16]According to the interpretation of Cuthbert in a letter to Sergueev dated 4 October 2000.

$$\pi_n = \left(\sum_{i=1}^{I} \alpha_n^i (p_n^i / P^i)^{-1} \right)^{-1} \ (n = 1, ..., N), \tag{51}$$

we meet again the Iklé (1972) method. A geometric variant of this method was developed by Rao (1990). The defining system of equations is

$$
\begin{aligned}
\ln \pi_n &= \sum_{i=1}^{I} w_n^i \ln(p_n^i / P^i) / \sum_{i=1}^{I} w_n^i \ (n = 1, ..., N) \\
\ln P^i &= \sum_{n=1}^{N} w_n^i \ln(p_n^i / \pi_n) \ (i = 1, ..., I).
\end{aligned}
\tag{52}
$$

This system has a unique (up to a scalar factor) positive solution $P^1, ..., P^I$ (see Balk 1996b). We will return to this method in Section 7.

5   The test approach

How do we discriminate between all these methods? The classical approach – the landmark in the area of intertemporal comparisons being Fisher (1922) – is to set up a system of tests or desirable properties and to find out which method fails which tests. This is the approach followed by Kravis, Kenessey, Heston and Summers (1975) in their pathbreaking work on international comparisons. The properties they thought most important were (formulated in our jargon) that price and quantity indices be transitive, that these indices satisfy the Product Test, that all countries be treated symmetrically, and that the method of comparison exhibits additive consistency.

Notice that the first two properties are maintained by us from the outset (see section 2). Furthermore, a quick perusal of the methods discussed in sections 3 and 4 leads to the conclusion that in all of these methods the countries are indeed treated symmetrically. The fourth property, however, is debatable. On the one hand it seems to favour only additive methods but on the other hand it is not able to discriminate between all the additive methods.

Diewert (1986), (1987) more rigidly formulated a set of tests for multilateral comparisons. Balk (1989) modified these tests by incorporating country weights. The tests emphasize the fact that the primary purpose

of any international comparison is to make volume comparisons. Price indices play only an intermediary role. Consequently, the tests are framed in terms of volume shares, which are here understood to be functions of all prices, all quantities, and all country weights (if any). Thus, formally, $Q^i = Q^i(p^1, ..., p^I, x^1, ..., x^I, g_1, ..., g_I)$ for $i = 1, ..., I$.[17]

MT1. Positivity and continuity test. The functions Q^i are continuous in all arguments, $Q^i > 0$ $(i = 1, ..., I)$, and $\sum_{i=1}^{I} Q^i = 1$.

MT2. Weak proportionality test or identity test. If there exists a price vector p and a quantity vector x such that $p^j = \alpha^j p$ and $x^j = \beta^j x$ where $\alpha^j, \beta^j > 0$ $(j = 1, ..., I)$ and $\sum_{j=1}^{I} \beta^j = 1$, then $Q^i = \beta^i$ $(i = 1, ..., I)$.

MT3. Proportionality test. Let $\lambda > 0$ and replace for country k the quantity vector x^k by λx^k and the scalar weight g_k by λg_k. Then the relation between the new volume shares $\tilde{Q}^i$ and the old volume shares Q^i is

$$
\begin{aligned}
\tilde{Q}^i &= \frac{\lambda Q^k}{1 + (\lambda - 1)Q^k} \text{ for } i = k \\
&= \frac{Q^i}{1 + (\lambda - 1)Q^k} \text{ for } i \neq k.
\end{aligned}
$$

MT4. Monetary unit test or invariance to changes in scale test. Replace p^j by $\alpha^j p^j$ and x^j by βx^j where $\alpha^j, \beta > 0$ $(j = 1, ..., I)$. Then the new volume shares are identically equal to the old volume shares.

MT5. Invariance to changes in units of measurement test. The volume shares are invariant to changes in the units of measurement of the commodities.

---

[17]Armstrong (2000) pursued a slightly different approach. Instead of quantity vectors x^i he considered vectors of per-household quantities $\bar{x}^i$ and scalar numbers of households h^i $(i = 1, ..., I)$. Price indices P^j/P^i defined as functions of $(p^j, p^i, \bar{x}^1, ..., \bar{x}^I, h^1, ..., h^I)$ were called restricted-domain indices whereas price indices defined as functions of $(p^1, ..., p^I, \bar{x}^1, ..., \bar{x}^I, h^1, ..., h^I)$ were called unrestricted-domain indices. Armstrong devised a set of tests for restricted-domain indices and modified and extended Diewert's (1986) set of tests to apply to unrestricted-domain indices. For a particular class of restricted-domain indices, namely those that are transitive, both systems of tests turned out to be equivalent.

MT6. Symmetric treatment of countries test. The volume shares are invariant to a permutation of the countries.

MT7. Symmetric treatment of commodities test. The volume shares are invariant to a permutation of the commodities.

MT8. Country partitioning test. Let country k be partitioned into two provinces, denoted by k and $I + 1$ respectively, with the same price vector p^k but quantity vectors λx^k and $(1 - \lambda)x^k$ respectively, and let the scalar country weights be λg_k and $(1-\lambda)g_k$ respectively $(0 < \lambda < 1)$. Then the relation between the new volume shares $\tilde{Q}^i$ and the old volume shares Q^i is

$$
\begin{aligned}
\tilde{Q}^i &= Q^i \text{ for } i = 1, ..., k - 1, k + 1, ..., I \\
\tilde{Q}^{I+1} &= (1 - \lambda)Q^k \\
\tilde{Q}^k &= \lambda Q^k.
\end{aligned}
$$

MT9. Irrelevance of tiny countries test. Let $\lambda > 0$ and replace for country k the quantity vector x^k by λx^k and the scalar weight g_k by λg_k. Denote the resulting volume shares by $Q^i(\lambda)$ $(i = 1, ..., I)$. Delete country k and denote the resulting volume shares by $\tilde{Q}^i$ $(i = 1, ..., k - 1, k + 1, ..., I)$. Then

$$
\lim_{\lambda \to 0} Q^i(\lambda) = \tilde{Q}^i \ (i = 1, ..., k - 1, k + 1, ..., I).
$$

We add some explanatory remarks. MT1 is an obvious test: volume shares must be positive, must add up to 1, and must exhibit continuous behavior in all variables. MT2 suggests that if all the price vectors are proportional to each other and all the quantity vectors are also proportional to each other, then the volume shares are equal to the factors of proportionality of the quantity vectors. A specific case is obtained when all price vectors are equal, that is $p^1 = ... = p^I$, and all quantity vectors are equal, that is $x^1 = ... = x^I$. Then all volume shares must be equal, that is $Q^i = 1/I$ $(i = 1, ..., I)$. MT3 suggests that if the prices remain unchanged but country k expands with a certain factor, then the volume shares behave accordingly. MT4 suggests that differing inflation rates but equal quantity growth rates leave the volume

shares invariant. MT5 - MT7 formulate obvious invariance requirements. MT8 considers the situation where we want to disaggregate one or more countries and requires consistency of the volume shares. MT9 stipulates that 'small' countries do not influence the volume shares of 'large' countries unduly.

Diewert (1999) generalized and expanded his original system of tests. In particular the test MT2 was split into two separate tests:[18]

MT2x. Proportionality w.r.t. quantities test. If there exists a quantity vector x such that $x^j = \beta^j x$ where $\beta^j > 0$ $(j = 1, ..., I)$ and $\sum_{j=1}^{I} \beta^j = 1$, then $Q^i = \beta^i$ $(i = 1, ..., I)$.

MT2p. Proportionality w.r.t. prices test. If there exists a price vector p such that $p^j = \alpha^j p$ where $\alpha^j > 0$ $(j = 1, ..., I)$, then $Q^i = p \cdot x^i / \sum_{j=1}^{I} p \cdot x^j$ $(i = 1, ..., I)$.

It is straightforward to check that if MT2x or MT2p is satisfied then MT2 is satisfied. Notice further that all additive methods, that is all methods for which (29) holds, satisfy MT2x (due to the fact that the volume shares add up to 1).

Balk (1996b) subjected ten methods to the original (modified by country weights) system of tests, *viz.* those of Van IJzeren, YKS, Q-YKS, GEKS, WDOS, GK, Iklé, Gerardi (29)+(50), KS-S, and Rao.[19] It turns out that all these methods, except the Rao method, also satisfy MT2x and MT2p. The Rao method appears to satisfy MT2p, but fails to satisfy MT2x.

With respect to the expanded system of tests MT1, MT2x, MT2p, MT3 - MT9 a number of new results could be established:

- The Gerardi (22) method fails to satisfy only MT3.[20]

- The WFBS method fails to satisfy only MT3 and MT4.[21]

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[18]The remaining modification of Diewert's original system of tests consists in a generalization of MT8, in the sense that the country k is partitioned into more than two provinces with price vectors which are proportional instead of identical. I disregard here two new tests: the bilateral consistency-in-aggregation test (which is biased towards the bilateral Fisher quantity index) and the additivity test (which would rule out all non-additive methods).

[19]See Balk (1996b) for (references to) proofs.

[20]The proof is almost a replication of the proof of Balk (1989) for the Van IJzeren method.

[21]The proof is by straightforward checking.

Table 1: Test performance of the various methods

Method	Definition	Tests violated
GEKS	(14), (16)	MT3, MT8
Van IJzeren	(18), (20)	MT3
Gerardi	(22), (24)	MT3
WFBS	(27)	MT3, MT4
WDOS	(28)	MT3, MT8
YKS		MT3, MT4
Q-YKS		MT8, MT9
GK	(32), (33)	MT3
EWGK	(43)	MT8, MT9
KS-S		MT3, MT5
SRK	(44), (45)	MT3
Sergueev	(48)	MT3, MT8,MT9
Gerardi	(29)+(50)	MT8, MT9
Iklé	(29)+(42)+(51)	MT8, MT9
Rao	(52)	MT2x, MT8, MT9

- The EWGK method fails to satisfy only MT8 and MT9.[22]

- The SRK method fails to satisfy only MT3.[23]

- Sergueev's method fails to satisfy MT3, MT8 and MT9.[24]

All the results are summarized in Table 1. It appears that there is no method which satisfies all the tests. The methods of Van IJzeren, Gerardi (22), GK, and SRK turn out to violate only MT3. The last two of these methods are in the class of additive methods.

A novel, quite natural test proposed by Diewert (1999) is:

MT10. Monotonicity test. The functions Q^i are increasing in the components of x^i ($i = 1, ..., I$).

---

[22] The proof runs parallel to the proof of Proposition 5 (Iklé method) in Balk (1996b).

[23] The proof is a replication, with obvious modifications, of the proof of Proposition 7 (KS-S method) in Balk (1996b).

[24] The proof is by straightforward checking.

Diewert (1999) was able to show that the Van IJzeren method satisfies this test.[25] By analogy, the Gerardi (22) method also satisfies this test. Diewert also showed that, when $I = 2$, the GK method does not satisfy the monotonicity test. However, it remains to be seen what happens when $I \geq 3$. Finally, whether the SRK method satisfies this test is still an open question. Thus the evidence here is, for the time being, inconclusive.

6   Methods based on chaining

The second main approach to construct transitive price and quantity indices is based on the procedure of chaining. The procedure as such is familiar in the realm of intertemporal comparisons. However, as noticed in the Introduction, unlike time periods countries don't exhibit a natural ordering. Given I countries, there appear to be $I(I - 1)/2$ bilateral index numbers (provided that the country reversal test holds) and I^{I-2} possible ways to link the countries together without creating any cycles. Put otherwise, there exist I^{I-2} spanning trees. How could we choose the 'optimal' spanning tree?

The natural approach is to use some measure of proximity and to order the countries according to this measure. In the time series context this measure is simply given by the length of the time span separating any two periods. Due to the unidirectional flow of time, this leads to a unique ordering, independent of the data. In the spatial context the ideal of data-independency of the ordering must be given up.

Hill (1999a), (1999b) developed two methods that allow the data to determine the 'optimal' spanning tree. Multilateral indices are then obtained by linking together bilateral indices as specified by the spanning tree. To give an example, suppose that we are to compare country j to country i. If, according to the spanning tree, the countries appear to be adjacent, then the price index of j relative to i is defined as $P(p^j, x^j, p^i, x^i)$ for some bilateral index satisfying the country reversal test. But if the countries j and i are connected via, say, countries k and l respectively, then the price index of j relative to i is defined as the chained index $P(p^j, x^j, p^k, x^k)P(p^k, x^k, p^l, x^l)P(p^l, x^l, p^i, x^i)$.

The device used in both methods for obtaining an 'optimal' spanning tree is the so-called Paasche-Laspeyres spread, defined as

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[25]He actually proved it for the unweighted case, that is, where $g^i = 1/I$ $(i = 1, ..., I)$, but there is no reason to suppose that this proof does not hold in the general case.

$$PLS_{ji} \equiv | \ln P^L(p^j, x^j, p^i, x^i) - \ln P^P(p^j, x^j, p^i, x^i) | \qquad (53)$$
$$= | \ln Q^L(p^j, x^j, p^i, x^i) - \ln Q^P(p^j, x^j, p^i, x^i) | \ (i, j = 1, ..., I).$$

Its properties are easily checked: $PLS_{ji} \geq 0$, $PLS_{jj} = 0$, and $PLS_{ji} = PLS_{ij}$ $(i, j = 1, ..., I)$.

The first method is called the shortest path method. It starts with selecting a base country. Then it finds the path between this country and any other country exhibiting the smallest sum of PLS_{ji} values. The union of all these $I - 1$ (bilateral) shortest paths is the spanning tree sought, called the shortest path spanning tree. It is clear that, since every country can be selected to act as base country, there are I shortest path spanning trees available, given the data. So there is no unique solution here.

The second method selects from all possible spanning trees the one with the smallest sum of $I - 1$ PLS_{ji} values, the so-called minimum-spanning tree. Given the data, this provides a unique, truly multilateral solution.[26]

A practical motive for finding and using 'optimal' spanning trees is the prospect of economizing on data. Notice that a multilateral method such as GEKS requires knowledge of all $I(I-1)/2$ bilateral index numbers. Suppose now that, based on a full data set for a certain period, we have obtained an 'optimal' spanning tree. Under the assumption that this structure remains stable over a certain time span, for later periods it is sufficient to compute only the $I - 1$ bilateral index numbers which are required by the spanning tree. This could save on the amount of data as well as lead to an appreciable gain in accuracy of the bilateral comparisons, since the data can be chosen such as to make the bilateral comparisons as accurate as possible without the need of imposing excessive data requirements on 'far off' countries.

Examples and robustness results can be found in Hill (1999a), (1999b) and (2001b). The first uses price and expenditure data coming from the ICP 1980 and 1985 surveys, for 30 countries and 151 and 139 commodities respectively. The second uses data coming from the OECD 1990 survey, for 24 countries and 198 commodities. The third uses, in addition, data coming from the OECD 1993 and 1996 surveys, for 34 countries and 147 and 162 commodities respectively.

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[26]In the time series context, using three different sets of annual data, Hill (2001a) found that the minimum-spanning tree always closely resembled the chronological order.

It appears that although the minimum-spanning trees are not stable over time, they generate similar clusters of countries. It also appears that the multilateral index numbers are less sensitive to the choice of the underlying spanning tree than one might suspect.

7 Model based approaches

The approach considered in this section assumes that all the individual prices are generated according to the following (superpopulation) model

$$\ln p_n^i = \ln P^i + \ln \pi_n + \varepsilon_n^i \ (i = 1, ..., I; n = 1, ..., N), \tag{54}$$

where ε_n^i is a residual with expectation 0. The interpretation of this model is rather straightforward: each country-specific price vector p^i is, apart from residuals, proportional to the international prices vector π, the factor of proportionality being the purchasing power parity P^i. Of course, this model identifies only price indices P^i/P^j. This model was proposed by Summers (1973), although in the context of dealing with missing price observations. In the time series context Balk (1980) used a similar model for dealing with seasonal commodities.[27]

A rather natural way of estimating the price indices P^i/P^j $(i, j = 1, ..., N)$ and the international prices π_n $(n = 1, ..., N)$ is by minimizing a sum of weighted squares of residuals

$$\min_{P^1, ..., P^I, \pi_1, ..., \pi_N} \sum_{i=1}^{I} \sum_{n=1}^{N} w_n^i (\ln p_n^i - \ln P^i - \ln \pi_n)^2, \tag{55}$$

where the weights w_n^i are the individual commodity value shares. Thus more important commodities get a larger weight in the sum of squares.

It is straightforward to check that the first-order necessary conditions for a minimum are precisely equal to the equations defining the Rao method (52). Thus at first sight it seems that there is nothing new here.

The virtue of this approach is, however, that the model (54) as well as the weights used in (55) suggest several potentially useful generalizations. One of these is to extend the model with quality characteristics. Thus the commodities involved in the comparison exercise need not be precisely the

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[27]Both proposals were surveyed by Selvanathan and Rao (1994). Summers' model is known as the Country-Product-Dummy (CPD) method.

same across countries. As long as there is sufficient information on their quality characteristics, the commodities might be different.

Another generalization is given by the fact that estimation of the model via (55) is only efficient under the assumption that the covariance matrix of the residuals is proportional to the unit matrix. Relaxing this assumption, by introducing for instance spatial autocorrelation, can lead to markedly different results (see Rao 2001).

It is interesting to make another connection between the model (54) and the index approach of the previous sections. Subtracting the equation for p_n^j from the equation for p_n^i leads to

$$\ln(p_n^i/p_n^j) = \ln(P^i/P^j) + (\varepsilon_n^i - \varepsilon_n^j) \ (i,j = 1,...,I; n = 1,...,N). \qquad (56)$$

Weighting each logarithmic price relative by its average value share $(w_n^i + w_n^j)/2$ and summing across commodities one obtains

$$\frac{1}{2}\sum_{n=1}^{N}(w_n^i+w_n^j)\ln(p_n^i/p_n^j) = \ln(P^i/P^j) + \frac{1}{2}\sum_{n=1}^{N}(w_n^i+w_n^j)(\varepsilon_n^i-\varepsilon_n^j)(i,j=1,...,I). \qquad (57)$$

This can be simplified to

$$\ln P^T(p^i,x^i,p^j,x^j) = \ln(P^i/P^j) + \varepsilon^{ij} \ (i,j = 1,...,I), \qquad (58)$$

where $P^T(p^i,x^i,p^j,x^j)$ is the Törnqvist price index of country i relative to country j. Estimating the price indices P^i/P^j via minimization of the sum of squared residuals

$$\min_{P^1,...,P^I} \sum_{i=1}^{I}\sum_{j=1}^{I}(\ln P^T(p^i,x^i,p^j,x^j) - \ln(P^i/P^j))^2 \qquad (59)$$

leads to the multilateral Törnqvist price index proposed by Caves, Christensen and Diewert (1982a):[28]

$$\left(\frac{P^i}{P^j}\right)_{CCD} \equiv \prod_{k=1}^{I}[P^T(p^i,x^i,p^k,x^k)P^T(p^k,x^k,p^j,x^j)]^{1/I} \ (i,j = 1,...,I). \qquad (60)$$

[28]A different derivation, using Weighted Least Squares on equations (56), was given by Selvanathan and Rao (1994).

Again, we obtained a multilateral price index as estimator of the model parameters under fairly simple assumptions on the residuals. Relaxing these assumptions leads us from the realm of indices to the realm of econometrics. See for example Selvanathan and Rao (1992).

An interesting feature of the use of superpopulation models such as discussed in this section is that they enable us to adjoin index numbers with estimates of their precision.

8   The economic approach

In this section we review the economic approach to the comparison of economic aggregates. The characteristic feature of the economic approach is that the value of any such aggregate is conceived as being the outcome of an optimization procedure, and that the objects of interest for the comparison are not the resulting quantity vectors as such, but the indifference curves or production possibility frontiers to which these vectors belong. Put otherwise, the objects for the comparison are sets of quantity vectors between which substitution is allowed, either from the consumer or the producer point of view.

Let the aggregate under consideration be household consumption and suppose that each country's preference structure can be represented by a utility function $U^i(x)$ $(i = 1, ..., I)$. The dual cost (or expenditure) functions are defined by $C^i(p, u) \equiv \min_x \{p \cdot x \mid U^i(x) \geq u\}$, where $u \in \text{Range } U^i(x)$ indicates a standard of living. Notice that $C^i(p, u) = p \cdot x^i(p, u)$, where $x^i(p, u) \equiv \arg\min_x \{p \cdot x \mid U^i(x) \geq u\}$ denotes the vector of cost minimizing quantities. If each cost function is continuously differentiable, then $x^i(p, u) = \nabla_p C^i(p, u)$.[29] The usual regularity conditions are supposed to hold.

The basic assumption of the economic approach is that each actual country i quantity vector x^i is optimal at the country i price vector p^i, that is, for some value u^i the following equations hold:

$$x^i = x^i(p^i, u^i) \ (i = 1, ..., I),\tag{61}$$

and thus

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[29]Notation: $\nabla_x f(x)$ denotes the vector of first-order derivatives of $f(x)$ with respect to x.

$$p^i \cdot x^i = C^i(p^i, u^i) \ (i = 1, ..., I). \tag{62}$$

If each cost function is continuously differentiable, then the foregoing equations imply that

$$w^i = \nabla_{\ln p} \ln C^i(p^i, u^i) \ (i = 1, ..., I), \tag{63}$$

where w^i is the actual country i vector of commodity value shares ($i = 1, ..., I$).

There are now two rather natural ways of measuring the price level of country j relative to country i. The first is by the Laspeyres-perspective cost of living index

$$C^i(p^j, u^i)/C^i(p^i, u^i), \tag{64}$$

which measures the relative cost of achieving country i's standard of living at the prices of country j and i respectively. The second is by the Paasche-perspective cost of living index

$$C^j(p^j, u^j)/C^j(p^i, u^j), \tag{65}$$

which measures the relative cost of achieving country j's standard of living at the prices of country j and i respectively. Both measures are equally plausible but will in general yield different outcomes.

Parallel to the argument in section 3 we can look for some intermediate index; that is, an index P^j/P^i satisfying

$$C^i(p^j, u^i)/C^i(p^i, u^i) = (P^j/P^i)t \text{ and } tC^j(p^j, u^j)/C^j(p^i, u^j) = P^j/P^i \ (t > 0). \tag{66}$$

Generalizing this to all I countries means that we must solve the following equation

$$\prod_{i=1}^{I} \left(\frac{C^i(p^j, u^i)}{C^i(p^i, u^i)} \frac{P^i}{P^j} \right)^{f_i} = \prod_{i=1}^{I} \left(\frac{C^j(p^i, u^j)}{C^j(p^j, u^j)} \frac{P^j}{P^i} \right)^{f_i} \ (j = 1, ..., I), \tag{67}$$

where $f_i \ (i = 1, ..., I)$ are normalized country weights. The solution appears to be

$$\frac{P^j}{P^i} = \prod_{k=1}^{I} \left[\left(\frac{C^k(p^j, u^k)}{C^k(p^k, u^k)} \frac{C^j(p^j, u^j)}{C^j(p^k, u^j)} \right)^{1/2} \left(\frac{C^k(p^k, u^k)}{C^k(p^i, u^k)} \frac{C^i(p^k, u^i)}{C^i(p^i, u^i)} \right)^{1/2} \right]^{f_k} \quad (68)$$

$$(i, j = 1, ..., I).$$

Each term of this product consists of the Fisher-perspective cost of living index of country j relative to country k times the Fisher-perspective cost of living index of country k relative to country i. Thus the structure of expression (68) is similar to the structure of expression (14), and we could call (68) the economic GEKS price index.

In order to make this expression operational we assume that each country-specific cost function has the translog functional form with second-order coefficients which are the same across countries; that is, we assume that

$$\ln C^i(p, u) = \alpha_0^i + \sum_{n=1}^{N} \alpha_n^i \ln p_n + \beta_1^i \ln u + \qquad (69)$$

$$\frac{1}{2} \sum_{n=1}^{N} \sum_{n'=1}^{N} \alpha_{nn'} \ln p_n \ln p_{n'} + \frac{1}{2} \beta_{11} (\ln u)^2 + \sum_{n=1}^{N} \gamma_n \ln p_n \ln u \ (u > 0) \ (i = 1, ..., I)$$

with the usual restrictions to ensure linear homogeneity of the cost function in prices. Applying the Translog Identity, due to Caves, Christensen and Diewert (1982b), we then obtain the following identity:

$$\frac{1}{2} \left[\ln \frac{C^k(p^j, u^k)}{C^k(p^k, u^k)} + \ln \frac{C^j(p^j, u^j)}{C^j(p^k, u^j)} \right] = \qquad (70)$$

$$\frac{1}{2} \left[\nabla_{\ln p} \ln C^k(p^k, u^k) + \nabla_{\ln p} \ln C^j(p^j, u^j) \right] \cdot \left[\ln p^j - \ln p^k \right] \ (j, k = 1, ..., I)$$

where $\ln p$ denotes the vector $(\ln p_1, ..., \ln p_N)$. Using (63), this equation reduces to

$$\frac{1}{2} \left[\ln \frac{C^k(p^j, u^k)}{C^k(p^k, u^k)} + \ln \frac{C^j(p^j, u^j)}{C^j(p^k, u^j)} \right] = \frac{1}{2} \sum_{n=1}^{N} (w_n^k + w_n^j) \ln(p_n^j / p_n^k) \qquad (71)$$

$$= \ln P^T(p^j, x^j, p^k, x^k) \ (j, k = 1, ..., I).$$

But this means that expression (68) reduces to

$$\frac{P^j}{P^i} = \prod_{k=1}^{I} \left(P^T(p^j, x^j, p^k, x^k) P^T(p^k, x^k, p^i, x^i) \right)^{f_k} (i, j = 1, ..., I). \qquad (72)$$

This expression is the same as (14), except that Fisher indices are replaced by Törnqvist indices. The corresponding economic quantity index Q^j/Q^i is obtained by applying the product relation (1).

The foregoing derivation basically generalizes the result of Caves, Christensen and Diewert (1982a). This result was obtained in the context of the input side of production, departing from a transformation function and assuming constant returns to scale. Notice that we did not need such an assumption.

A different approach was followed by Rao and Salazar-Carrillo (1988). Conditional on a certain price vector π, the economic purchasing power parity of country i was defined by

$$P^i \equiv \frac{C^i(p^i, u^i)}{C^i(\pi, u^i)} (i = 1, ..., I), \qquad (73)$$

which has the form of a Paasche-perspective cost of living index. The vector π was determined by requiring that for each commodity the sum of the quantities which are optimal at π equals the sum of the actual quantities, that is,

$$\sum_{i=1}^{I} x_n^i(\pi, u^i) = \sum_{i=1}^{I} x_n^i (n = 1, ..., N). \qquad (74)$$

The next step is to notice that every country-specific pair (p^i, x^i) can be rationalized by a country-specific Cobb-Douglas cost function

$$C^i(p, u) \equiv F(u) \prod_{n=1}^{N} p_n^{w_n^i} (i = 1, ..., I), \qquad (75)$$

where w_n^i are the actual commodity n value shares of country i ($n = 1, ..., N$; $i = 1, ..., I$) and $F(u)$ is monotonously increasing in u. Under this assumption expression (73) reduces to

$$P^i = \prod_{n=1}^{N} (p_n^i/\pi_n)^{w_n^i} (i = 1, ..., I). \qquad (76)$$

Now the Cobb-Douglas cost function implies that the cost minimizing quantities are given by $x_n^i(p, u) = (w_n^i/p_n)C^i(p, u)$ $(n = 1, ..., N)$. Substituting this into equation (74), rearranging terms, and making use of definition (73) together with the rationality assumption (62), one obtains finally that

$$\pi_n = \frac{\sum_{i=1}^I w_n^i C^i(\pi, u^i)}{\sum_{i=1}^I x_n^i} = \frac{\sum_{i=1}^I p_n^i x_n^i/P^i}{\sum_{i=1}^I x_n^i} \ (n = 1, ..., N). \tag{77}$$

Notice that expression (76) coincides with (52b) and that expression (77) coincides with (32a). Thus this approach leads us to a mixture of the Rao and GK methods. The system (76)-(77) must be solved numerically to obtain the purchasing power parities.

The next approach simplifies things by assuming that all countries have the same (international) preference structure, that is

$$C^i(p, u) = C(p, u) \ (i = 1, ..., I). \tag{78}$$

Under this assumption, expression (73) reduces to

$$P^i = \frac{C(p^i, u^i)}{C(\pi, u^i)} = \frac{p^i \cdot x^i}{C(\pi, u^i)} \ (i = 1, ..., I), \tag{79}$$

where the second equality is based on the basic assumption (62). Then, using the product relation (1), we find that

$$\frac{Q^j}{Q^i} = \frac{C(\pi, u^j)}{C(\pi, u^i)} \ (i, j = 1, ..., I), \tag{80}$$

that is, the economic quantity index of country j relative to country i is given by the minimum cost to achieve the standard of living u^j relative to the minimum cost to achieve the standard of living u^i, conditional on a certain price vector π. The right hand side of this equation is known as being a money metric standard of living index. For a fixed π the index is indeed transitive. The basic issue, of course, is how to pick the price vector π.

Suppose for a start that the preference structure exhibits homotheticity. This is equivalent to the supposition that the cost function can be decomposed as

$$C(p, u) = F(u)C(p, 1), \tag{81}$$

where $F(u)$ is monotonously increasing in u. Under homotheticity the economic quantity index (80) reduces to

$$\frac{Q^j}{Q^i} = \frac{F(u^j)}{F(u^i)} \ (i, j = 1, ..., I),$$ (82)

which is independent of prices. Then, for any set of country weights f_k $(k = 1, ..., I)$, adding up to 1,

$$\begin{aligned}
\frac{Q^j}{Q^i} &= \prod_{k=1}^{I} \left(\frac{F(u^j)}{F(u^k)} \frac{F(u^k)}{F(u^i)} \right)^{f_k} \\
&= \prod_{k=1}^{I} \left(\frac{F(u^j)C(p^k,1)}{F(u^k)C(p^k,1)} \frac{F(u^k)C(p^i,1)}{F(u^i)C(p^i,1)} \right)^{f_k} \\
&= \prod_{k=1}^{I} \left(\frac{C(p^k,u^j)}{C(p^k,u^k)} \frac{C(p^i,u^k)}{C(p^i,u^i)} \right)^{f_k} \ (i, j = 1, ..., I),
\end{aligned}$$ (83)

where the last line was obtained by using the homotheticity assumption (81) again. Our optimality assumption (61) implies that x^j attains the standard of living u^j and therefore, by the definition of the cost function, $C(p^k, u^j) \leq p^k \cdot x^j$. Similarly, $C(p^i, u^k) \leq p^i \cdot x^k$. Applying these inequalities to the numerators in expression (83) and using (62) for the denominators, we obtain the following inequality for the economic quantity index:

$$\begin{aligned}
\frac{Q^j}{Q^i} &\leq \prod_{k=1}^{I} \left(\frac{p^k \cdot x^j}{p^k \cdot x^k} \frac{p^i \cdot x^k}{p^i \cdot x^i} \right)^{f_k} \\
&= \prod_{k=1}^{I} \left(Q^L(p^j, x^j, p^k, x^k) Q^L(p^k, x^k, p^i, x^i) \right)^{f_k} \ (i, j = 1, ..., I).
\end{aligned}$$ (84)

By a similar reasoning we find that

$$\frac{Q^j}{Q^i} \geq \prod_{k=1}^{I} \left(Q^P(p^j, x^j, p^k, x^k) Q^P(p^k, x^k, p^i, x^i) \right)^{f_k} \ (i, j = 1, ..., I).$$ (85)

We have thus obtained an upper bound and a lower bound for the same economic quantity index. A reasonable approximation for this quantity index,

retaining transitivity, is then provided by the unweighted geometric average of the right hand sides of (84) and (85); that is, we set

$$\frac{Q^j}{Q^i} \approx \prod_{k=1}^{I} \left(Q^F(p^j, x^j, p^k, x^k) Q^F(p^k, x^k, p^i, x^i) \right)^{f_k} \ (i, j = 1, ..., I). \qquad (86)$$

But this is the GEKS index (16). We have thus obtained the result that under identical homothetic preferences the GEKS quantity index provides a reasonable approximation to the economic quantity index which was defined by (80).

A slight modification of the foregoing reasoning leads to a result that sheds light on the method of chaining as discussed in section 6.

Suppose that all the countries are connected by a spanning tree, and that countries i and j are connected to each other via countries $k_1, ..., k_L$. We can then write, instead of (83),

$$\begin{aligned}
\frac{Q^j}{Q^i} &= \frac{F(u^{k_1})}{F(u^i)} \frac{F(u^{k_2})}{F(u^{k_1})} \cdots \frac{F(u^j)}{F(u^{k_L})} \\
&= \frac{C(p^i, u^{k_1})}{C(p^i, u^i)} \frac{C(p^{k_1}, u^{k_2})}{C(p^{k_1}, u^{k_1})} \cdots \frac{C(p^{k_L}, u^j)}{C(p^{k_L}, u^{k_L})} \ (i, j = 1, ..., I), \qquad (87)
\end{aligned}$$

where the last line was again obtained by using the homotheticity assumption. As in the foregoing this leads to the following upper bound for the economic quantity index:

$$\frac{Q^j}{Q^i} \leq Q^L(p^{k_1}, x^{k_1}, p^i, x^i) Q^L(p^{k_2}, x^{k_2}, p^{k_1}, x^{k_1}) \cdots Q^L(p^j, x^j, p^{k_L}, x^{k_L}) \qquad (88)$$

$$(i, j = 1, ..., I).$$

But we can also write, by virtue of the homotheticity assumption,

$$\begin{aligned}
\frac{Q^j}{Q^i} &= \frac{F(u^{k_1})}{F(u^i)} \frac{F(u^{k_2})}{F(u^{k_1})} \cdots \frac{F(u^j)}{F(u^{k_L})} \\
&= \frac{C(p^{k_1}, u^{k_1})}{C(p^{k_1}, u^i)} \frac{C(p^{k_2}, u^{k_2})}{C(p^{k_2}, u^{k_1})} \cdots \frac{C(p^j, u^j)}{C(p^j, u^{k_L})} \ (i, j = 1, ..., I), \qquad (89)
\end{aligned}$$

which leads to the lower bound

$$\frac{Q^j}{Q^i} \geq Q^P(p^{k_1}, x^{k_1}, p^i, x^i) Q^P(p^{k_2}, x^{k_2}, p^{k_1}, x^{k_1}) \cdots Q^P(p^j, x^j, p^{k_L}, x^{k_L}) \quad (90)$$

$$(i, j = 1, ..., I).$$

A reasonable approximation for the economic quantity index is then provided by

$$\frac{Q^j}{Q^i} \approx Q^F(p^{k_1}, x^{k_1}, p^i, x^i) Q^F(p^{k_2}, x^{k_2}, p^{k_1}, x^{k_1}) \cdots Q^F(p^j, x^j, p^{k_L}, x^{k_L}) \quad (91)$$

$$(i, j = 1, ..., I).$$

Thus, under the assumption of identical homothetic preferences the chained Fisher quantity index provides a reasonable approximation to the economic quantity index which was defined by expression (80).

It is wellknown that homotheticity is a very restrictive assumption. It means that, given a vector of country-specific prices, increasing the standard of living would lead to an equi-proportionate increase of all quantities consumed. This is patently unrealistic. Afriat (1972, p. 28) remarks that

> "... it is an overwhelmingly significant fact of experience that the rich, whether individuals or countries, have things that the poor do not have at all, let alone in corresponding proportions. Deliberately to overlook this in a system of calculation that seeks to make general comparisons leaves the significance of such calculation quite obscure, even as to the locus of injustice."

It is clear that under nonhomotheticity the choice of the vector π becomes a matter of importance. Using the definition of the cost function, expression (79) can be rewritten as

$$\sum_{n=1}^{N} \pi_n x_n(\pi, u^i) = \sum_{n=1}^{N} (p_n^i / P^i) x_n^i \ (i = 1, ..., I). \quad (92)$$

In Neary and Gleeson's (1997) approach the vector π is determined by adjoining this system of equalities across countries by a similar system of equalities across commodities:

$$\sum_{i=1}^{I} \pi_n x_n(\pi, u^i) = \sum_{i=1}^{I} (p_n^i/P^i) x_n^i \; (n = 1, ..., N). \tag{93}$$

Rearranging terms, the complete system of equations appears to be

$$\begin{aligned} \pi_n &= \frac{\sum_{i=1}^{I} p_n^i x_n^i / P^i}{\sum_{i=1}^{I} x_n(\pi, u^i)} \; (n = 1, ..., N) \\ P^i &= C(p^i, u^i)/C(\pi, u^i) \; (i = 1, ..., I). \end{aligned} \tag{94}$$

Neary and Gleeson (1997) called this the Geary-Konüs system.[30] Given a functional form for the cost function, this system of equations must be solved numerically to obtain the purchasing power parities.

The problem, of course, is how to obtain an appropriate functional form for the cost function. A flexible functional form requires at least $1 + (N + 1) + (N + 1)(N + 2)/2$ parameters to be estimated, whereas the number of data points is $I \times N$. Since in any realistic comparison exercise the number of commodities N will exceed by far the number of countries I, the estimation of a flexible form is a mission impossible unless we are given more data points per country.

Neary and Gleeson (1997) therefore assumed that the international preference structure could be represented by the Stone-Geary utility function (Linear Expenditure System), which leaves only a modest number of $2N - 1$ parameters to be estimated. Using price and expenditure data for 11 commodities and 16 countries coming from the ICP 1970 survey, they calculated price and quantity index numbers and compared these to the corresponding GK and GEKS results.

9   Conclusion

Reverting to the question posed in the title of this survey, one can say that although we have learned quite a lot, the lessons are not all pointing in the same direction. Put otherwise, there appears to be no unique, award-winning

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[30] It is straightforward to verify that if the international preference structure is of the Leontief fixed coefficients type, which means that $C(p, u) = F(u)p \cdot a$ for some quantity vector a, then the system (94) reduces to the Geary-Khamis system (32).

method. However, some methods have better credentials than others. A brief recapitulation may here be sufficient.

The center stage among the methods discussed in section 3 was occupied by the GEKS-Fisher price and quantity indices (expressions (14) and (16)). From the economic viewpoint this pair of indices can be rationalized by assuming identical homothetic preferences across all the countries (see expression (86)). Assuming non-homothetic country-specific preferences which are not 'too' different – in mathematical form this is expressed by (69) – leads us to the GEKS-Törnqvist price index (72).[31] From the empirical viewpoint it can be expected that the GEKS-Fisher and the GEKS-Törnqvist indices closely approximate each other. Section 5 shows that the GEKS-Fisher volume shares only violate the tests MT3 and MT8. However, as documented in section 3, the GEKS-Fisher indices are bracketed by the Van IJzeren and the Gerardi indices, both of which do satisfy the test MT8. Thus, there is reason to expect that the GEKS-Fisher's failure of satisfying MT8 is not 'too' bad. Moreover, as demonstrated by Van IJzeren (1987) on a numerical example, the weights are not particularly influential, so that it is virtually harmless to set all (normalized) weights equal to $1/I$.

Using a data-driven spanning tree as basis for the construction of a system of chained indices seems to be an area for further research. As indicated in the previous section, chained Fisher quantity indices can be defended from the economic angle, provided that one is willing to assume identical homothetic preferences.

A comparison of the structural features of an economic aggregate is best served by employing an additive method. Economically seen, such a method does not allow substitution behavior. Judged from table 1, the choice seems to be between the GK and SRK methods. The ussue here is to find a method which suffers least from the Gerschenkron effect. As indicated, there is reason to expect that in this respect the GK method is to be preferred to the SRK method. Notice that the EWGK method, although shown to be less affected by the Gerschenkron effect than the GK method, exhibits a less trustworthy test performance.

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[31]Notice that the GEKS-Törnqvist price index (72) also emerges when one replaces expressions (7) and (8) by $\prod_{n=1}^{N}(p_n^j/p_n^i)^{w_n^i}$ and $\prod_{n=1}^{N}(p_n^j/p_n^i)^{w_n^j}$ respectively.

References

Afriat, S. N., 1972, "The Theory of International Comparisons of Real Income and Prices," in *International Comparisons of Prices and Output*, edited by D. J. Daley, Studies in Income and Wealth Volume 37 (National Bureau of Economic Research, New York).

Armstrong, K. G., 2000, A Restricted-Domain Multilateral Test Approach to the Theory of International Comparisons, mimeo (Department of Economics, Carleton University, Ottawa).

Balk, B. M., 1980, "A Method for Constructing Price Indices for Seasonal Commodities," Journal of the Royal Statistical Society A 143, 68-75.

Balk, B. M., 1989, "On Van IJzeren's Approach to International Comparisons and its Properties," Statistical Papers / Statistische Hefte 30, 295-315.

Balk, B. M., 1996a, "Van IJzeren's Method of International Price and Volume Comparison: An Exposition," in *International Comparisons of Prices, Output, and Productivity*, edited by D. S. P. Rao and J. Salazar-Carrillo (North-Holland, Amsterdam).

Balk, B. M., 1996b, "A Comparison of Ten Methods for Multilateral International Price and Volume Comparison," Journal of Official Statistics 12, 199-222.

Balk, B. M., 1999, "Contributions from Statistics Netherlands to the Axiomatic Theory of Price Indices," in *A Century of Statistics*, edited by J. G. S. J. van Maarseveen and M. B. G. Gircour (Statistics Netherlands, Voorburg; Stichting Beheer IISG, Amsterdam); also as Research paper no. 9923 (Department of Statistical Methods, Statistics Netherlands).

Biggeri, L., G. Ferrari and A. Lemmi, 1987, "The Contribution of Italian Statisticians to the Theory of Price Index Numbers," in *Italian Contributions to the Methodology of Statistics*, edited by A. Naddeo (Società Italiana di Statistica; Cleup, Padova).

Caves, D. W., L. R. Christensen and W. E. Diewert, 1982a, "Multilateral Comparisons of Output, Input, and Productivity using Superlative Index Numbers," The Economic Journal 92, 73-86.

Caves, D. W., L. R. Christensen and W. E. Diewert, 1982b, "The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity," Econometrica 50, 1393-1414.

Collier, I. L., 1999, "Comment," in *International and Interarea Comparisons of Income, Output, and Prices*, edited by A. Heston and R. E. Lipsey, Studies in Income and Wealth, Volume 61 (University of Chicago Press, Chicago and London).

Cuthbert, J. R., 1999, "Categorisation of Additive Purchasing Power Parities," The Review of Income and Wealth 45, 235-249.

Cuthbert, J. R., 2000, "Theoretical and Practical Issues in Purchasing Power Parities Illustrated with Reference to the 1993 Organization for Economic Cooperation and Development Data," Journal of the Royal Statistical Society A 163, 421-444.

Diewert, W. E., 1986, Microeconomic Approaches to the Theory of International Comparisons, Technical Working Paper No. 53 (National Bureau of Economic Research, Cambridge MA). Abridged version, entitled "Test Approaches to International Comparisons," in *Essays in Index Number Theory, Volume 1*, edited by W. E. Diewert and A. O. Nakamura (North-Holland, Amsterdam, 1993).

Diewert, W. E., 1987, "Index Numbers," in *The New Palgrave: A Dictionary of Economics, Volume 2*, edited by J. Eatwell *et al.* (Macmillan, London). Reprinted in Diewert and Nakamura (1993).

Diewert, W. E., 1999, "Axiomatic and Economic Approaches to International Comparisons," in *International and Interarea Comparisons of Income, Output, and Prices*, edited by A. Heston and R. E. Lipsey, Studies in Income and Wealth, Volume 61 (University of Chicago Press, Chicago and London).

Drechsler, L., 1973, "Weighting of Index Numbers in Multilateral International Comparisons," The Review of Income and Wealth 19, 17-34.

Eichhorn, W. and J. Voeller, 1983, "Axiomatic Foundation of Price Indexes and Purchasing Power Parities," in *Price Level Measurement*, edited by W. E. Diewert and C. Montmarquette (Statistics Canada, Ottawa); also reprinted in *Price Level Measurement*, edited by W. E. Diewert (North-Holland, Amsterdam etc., 1990).

Eltetö, Ö. and P. Köves, 1964, "On an Index Computation Problem in International Comparisons," (in Hungarian), Statiztikai Szemle 42, 507-518.

Fisher, I., 1922, *The Making of Index Numbers* (Houghton Mifflin, Boston).

Geary, R. C., 1958, "A Note on the Comparison of Exchange Rates and Purchasing Power between Countries," Journal of the Royal Statistical Society A 121, 97-99.

Gerardi, D., 1974, Sul Problema della Comparazione dei Poteri d'Aquisto della Valute (Istituto di Statistica, Universita degli Studi di Padova).

Gerschenkron, A., 1951, *A Dollar Index of Soviet Machinery Output, 1927-28 to 1937* (Rand Corporation, Santa Monica CA).

Gini, C., 1924, "Quelques Considérations au Sujet de la Construction des Nombres Indices des Prix et des Questions Analogues," Metron 4, 3-162.

Gini, C., 1931, "On the Circular Test of Index Numbers," International Review of Statistics 9, no. 2, 3-25.

Green, J. and W. P. Heller, 1981, "Mathematical Analysis and Convexity with Applications to Economics," in *Handbook of Mathematical Economics, Volume 1*, edited by K. J. Arrow and M. D. Intriligator (North-Holland, Amsterdam).

Hill, R. J., 1997, "A Taxonomy of Multilateral Methods for Making International Comparisons of Prices and Quantities," The Review of Income and Wealth 43, 49-69.

Hill, R. J., 1999a, "Comparing Price Levels Across Countries Using Minimum-Spanning Trees," The Review of Economics and Statistics 81, 135-142.

Hill, R. J., 1999b, "International Comparisons Using Spanning Trees," in *International and Interarea Comparisons of Income, Output, and Prices*, edited by A. Heston and R. E. Lipsey, Studies in Income and Wealth, Volume 61 (University of Chicago Press, Chicago and London).

Hill, R. J., 2000, "Measuring Substitution Bias in International Comparisons Based on Additive Purchasing Power Parity Methods," European Economic Review 44, 145-162.

Hill, R. J., 2001a, "Measuring Inflation and Growth Using Spanning Trees," International Economic Review 42, 167-185.

Hill, R. J., 2001b, Linking Countries and Regions using Chaining Methods and Spanning Trees, Paper presented at the Joint World Bank - OECD Seminar on Purchasing Power Parities, 30 January - 2 February 2001, Washington DC.

Iklé, D. M., 1972, "A New Approach to the Index Number Problem," Quarterly Journal of Economics 86, 188-211.

Khamis, S. H., 1972, "A New System of Index Numbers for National and International Purposes," Journal of the Royal Statistical Society A 135, 96-121.

Khamis, S. H., 1998, Measurement of Real Product: Some Index Number Aspects, Paper presented at the 25th General Conference of the IARIW, Cambridge, U. K..

Köves, P., 1999, "EKS Index and International Comparisons," Hungarian Statistical Review (Statisztikai Szemle) 77, 3-14.

Kravis, I. B., Z. Kenessey, A. Heston and R. Summers, 1975, *A System of International Comparisons of Gross Product and Purchasing Power* (The Johns Hopkins University Press, Baltimore and London).

Kurabayashi, Y. and I. Sakuma, 1981, An Alternative Method of Multilateral Comparisons of Real Product Constrained with Matrix Consistency, Paper presented at the 17th General Conference of the IARIW, Gouvieux, France.

Kurabayashi, Y. and I. Sakuma, 1982, Transitivity, Factor Reversal Test and Matrix Consistency in the International Comparisons of Real Product, Discussion Paper 54 (Institute of Economic Research, Hitotsubashi University, Tokyo).

Kurabayashi, Y. and I. Sakuma, 1990, *Studies in International Comparisons of Real Product and Prices* (Kinokuniya Company, Tokyo; Oxford University Press, Oxford).

Neary, J. P. and B. Gleeson, 1997, "Comparing the Wealth of Nations: Reference Prices and Multilateral Real Income Indexes," The Economic and Social Review 28, 401-421.

Rao, D. S. P., 1990, "A System of Log-Change Index Numbers for Multilateral Comparisons," in *Comparisons of Prices and Real Products in Latin America*, edited by J. Salazar-Carrillo and D. S. P. Rao (North-Holland, Amsterdam).

Rao, D. S. P., 2001, Weighted EKS and Generalised CPD Methods for Aggregation at Basic Heading Level and Above Basic Heading Level, Paper presented at the Joint World Bank - OECD Seminar on Purchasing Power Parities, 30 January - 2 February 2001, Washington DC.

Rao, D. S. P. and J. Salazar-Carrillo, 1988, "A General Equilibrium Approach to the Construction of Multilateral Index Numbers," in *World Comparison of Incomes, Prices and Product*, edited by J. Salazar-Carrillo and D. S. P. Rao (North-Holland, Amsterdam).

Selvanathan, E. A. and D. S. P. Rao, 1992, "An Econometric Approach to the Construction of Generalized Theil-Törnqvist Indices for Multilateral Comparisons," Journal of Econometrics 54, 335-346.

Selvanathan, E. A. and D. S. P. Rao, 1994, *Index Numbers: A Stochastic Approach* (Macmillan, Basingstoke and London).

Sergueev, S., 2001, Aggregation Methods on the Basis of Structural International Prices, Paper presented at the Joint World Bank - OECD Seminar on Purchasing Power Parities, 30 January - 2 February 2001, Washington DC.

Sakuma, I., D. S. P. Rao and Y. Kurabayashi, 2000, Additivity, Matrix Consistency and a New Method for International Comparisons of Real Income and Purchasing Power Parities, Paper presented at the 26th General Conference of the IARIW, Cracow, Poland.

Summers, R., 1973, "International Price Comparisons Based Upon Incomplete Data," The Review of Income and Wealth 19, 1-16.

Szulc, B., 1964, "Index Numbers of Multilateral Regional Comparisons," (in Polish), Przeglad Statystyczny 11, 239-254.

Veelen, M. van, forthcoming, "An Impossibility Theorem Concerning Multilateral International Comparison of Volumes," Econometrica.

Walsh, C. M., 1901, *The Measurement of General Exchange Value* (Macmillan, New York).

IJzeren, J. van, 1955, "Over Verschillende Methoden ter Berekening van Pariteiten ten behoeve van Internationale Koopkrachtvergelijking," Statistische en Econometrische Onderzoekingen (C. B. S.), Nieuwe Reeks 10, 101-132.

IJzeren, J. van, 1956, *Three Methods of Comparing the Purchasing Power of Currencies*. Netherlands Central Bureau of Statistics, Statistical Studies, No. 7 (De Haan, Zeist).

IJzeren, J. van, 1987, *Bias in International Index Numbers: A Mathematical Elucidation*, Dissertation, Hungarian Academy of Sciences (Private Edition).

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