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Research Article

Inclusion Properties of Certain Subclasses of p-Valent Functions Associated with the Integral Operator

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The purpose of the present paper is to introduce two subclasses of p-valent functions by using the integral operator and to investigate various properties for these subclasses.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the following form:

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j}, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \quad (1)$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{P}_k(p, \gamma)$ be the class of functions g analytic in $\mathbb{U}$ satisfying $g(0) = p$ and

$$\int_0^{2\pi} \left| \frac{\Re\{g(z)\} - \gamma}{p - \gamma} \right| d\theta \leq k\pi, \quad (z = re^{i\theta}; k \geq 2; 0 \leq \gamma < p). \quad (2)$$

The class $\mathcal{P}_k(p, \gamma)$ was introduced by Aouf [1] and we note the following:

(i) the class $\mathcal{P}_k(1, \gamma) = \mathcal{P}_k(\gamma)$ was introduced by Padmanabhan and Parvatham [2];

(ii) the class $\mathcal{P}_k(1, 0) = \mathcal{P}_k$ was introduced by Pinchuk [3];

(iii) $\mathcal{P}_2(p, \gamma) = \mathcal{P}(p, \gamma)$ is the class of functions with positive real part greater than γ $(0 \leq \gamma < p)$;

(iv) $\mathcal{P}_2(1, \gamma) = \mathcal{P}(\gamma)$ is the class of functions with positive real part greater than γ $(0 \leq \gamma < 1)$;

(v) $\mathcal{P}_2(1, 0) = \mathcal{P}$ is the class of functions with positive real part.

From (1), we have $g \in \mathcal{P}_k(p, \gamma)$ if and only if there exists $g_1, g_2 \in \mathcal{P}(p, \gamma)$ such that

$$g(z) = \left(\frac{k}{4} + \frac{1}{2}\right) g_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) g_2(z), \quad (z \in \mathbb{U}). \quad (3)$$

It is known that [4] the class $\mathcal{P}_k(\gamma)$ is a convex set.

Motivated essentially by Jung et al. [5], Liu and Owa [6] introduced the integral operator $Q^{\alpha}_{\beta,p} : \mathcal{A}(p) \to \mathcal{A}(p)$ $(\alpha \geq 0; \beta > -p; p \in \mathbb{N})$ as follows:

$$Q^{\alpha}_{\beta,p} f(z) = \begin{cases} \dbinom{p+\alpha+\beta-1}{p+\beta-1} \dfrac{\alpha}{z^{\beta}} \displaystyle\int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt & (\alpha > 0), \\ f(z) & (\alpha = 0). \end{cases} \quad (4)$$

For $f \in \mathcal{A}(p)$ given by (1) and then from (4), we deduce that

$$Q^{\alpha}_{\beta,p} f(z) = z^p + \frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)} \sum_{j=1}^{\infty} \frac{\Gamma(\beta+p+j)}{\Gamma(\alpha+\beta+p+j)} a_{p+j} z^{p+j} \quad (\alpha \geq 0; \beta > -p). \quad (5)$$

It is easily verified from (5) that (see [6])

$$z\left(Q^{\alpha+1}_{\beta,p} f(z)\right)' = (\alpha + \beta + p)\, Q^{\alpha}_{\beta,p} f(z) - (\alpha + \beta)\, Q^{\alpha+1}_{\beta,p} f(z). \quad (6)$$

We note that (i) the one-parameter family of integral operator $Q^{\alpha}_{\beta,1} = Q^{\alpha}_{\beta}$ was defined by Jung et al. [5] and studied by Aouf [7] and Gao et al. [8].

(ii) Consider

$$Q^{1}_{c,p} f(z) = F_{c,p}(f)(z) = \frac{c+p}{z^c} \int t^{c-1} f(z)\, dt, \quad (c > -p), \tag{7}$$

where the operator $F_{c,p}$ is the generalized Bernardi-Libera-Livingston integral operator (see [9]).

We have the following known subclasses $\mathcal{S}_k(p,\gamma)$ and $\mathcal{C}_k(p,\gamma)$ of the class $\mathcal{A}(p)$ for $0 \le \gamma, \eta < p$, and $k \ge 2$ which are defined by

$$\mathcal{S}_k(p,\gamma) = \left\{ f \in \mathcal{A}(p) : \frac{zf'(z)}{f(z)} \in \mathcal{P}_k(p,\gamma), z \in \mathbb{U} \right\},$$
$$\mathcal{C}_k(p,\gamma) = \left\{ f \in \mathcal{A}(p) : \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_k(p,\gamma), z \in \mathbb{U} \right\}. \tag{8}$$

Next, by using the integral operator $Q^{\alpha}_{\beta,p}$, we introduce the following classes of analytic functions for $0 \le \gamma < p$ and $k \ge 2$:

$$\mathcal{S}_k(p,\alpha;\gamma) = \left\{ f \in \mathcal{A}(p) : Q^{\alpha}_{\beta,p} f(z) \in \mathcal{S}_k(p,\gamma) \right\},$$
$$\mathcal{C}_k(p,\alpha;\gamma) = \left\{ f \in \mathcal{A}(p) : Q^{\alpha}_{\beta,p} f(z) \in \mathcal{C}_k(p,\gamma) \right\}. \tag{9}$$

We also note that

$$f \in \mathcal{C}_k(p,\alpha;\gamma) \Longleftrightarrow \frac{zf'}{p} \in \mathcal{S}_k(p,\alpha;\gamma). \tag{10}$$

In particular, we set $\mathcal{S}_k(1,\alpha;\gamma) = \mathcal{S}_k(\alpha;\gamma)$ and $\mathcal{C}_k(1,\alpha;\gamma) = \mathcal{C}_k(\alpha;\gamma)$.

The following lemma will be required in our investigation.

Lemma 1 (see [10]). *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\Psi(u,v)$ be a complex-valued function satisfying the following conditions:*

(i) *$\Psi(u,v)$ is continuous in a domain $D \in \mathbb{C}^2$;*

(ii) *$(0,1) \in D$ and $\Psi(1,0) > 0$;*

(iii) *$\Re\{\Psi(iu_2,v_1)\} > 0$ whenever $(iu_2,v_1) \in D$ and $v_1 \le -(1/2)(1+u_2^2)$.*

If $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ such that $(h(z), zh'(z)) \in D$ and $\Re\{\Psi(h(z), zh'(z))\} > 0$ for $z \in \mathbb{U}$, then $\Re\{\Psi(h(z), zh'(z))\} > 0$ in $\mathbb{U}$.

Lemma 2 (see [11]). *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = a$ and $\Re\{p(z)\} > 0$, $z \in \mathbb{U}$. Then, for $s > 0$ and $\mu \in \mathbb{C} \setminus \{-1\}$,*

$$\Re\left\{ p(z) + \frac{szp'(z)}{p(z)+\mu} \right\} > 0, \quad (|z| < r_0), \tag{11}$$

where r_0 is given by

$$r_0 = \frac{|\mu+1|}{\sqrt{A + (A^2 - |\mu^2 - 1|)^{1/2}}}, \quad A = 2(s+1)^2 + |\mu|^2 - 1, \tag{12}$$

and this radius is the best possible.

Lemma 3 (see [12]). *Let ψ be convex and let g be starlike in $\mathbb{U}$. Then, for F analytic in $\mathbb{U}$ with $F(0) = 1$, $((\psi * Fg)/(\psi * g))$ is contained in the convex hull of $F(\mathbb{U})$.*

In this paper, we obtain several inclusion properties of the classes $\mathcal{S}_k(p,\alpha;\gamma)$ and $\mathcal{C}_k(p,\alpha;\gamma)$ associated with the operator $Q^{\alpha}_{\beta,p}$.

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $k \ge 2$, $\alpha \ge 0$, $\beta > 0$, $0 \le \gamma < p$, and $p \in \mathbb{N}$.

Theorem 4. *One has*

$$\mathcal{S}_k(p,\alpha+1;\gamma) \subset \mathcal{S}_k(p,\alpha;\gamma). \tag{13}$$

Proof. We begin by setting

$$\begin{aligned} \frac{z\left(Q^{\alpha+1}_{\beta,p} f(z)\right)'}{Q^{\alpha+1}_{\beta,p} f(z)} &= (p-\gamma)h(z) + \gamma \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\{(p-\gamma)h_1(z) + \gamma\} \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\{(p-\gamma)h_2(z) + \gamma\}, \end{aligned} \tag{14}$$

where h_i is analytic in $\mathbb{U}$ with $h_i(0) = 1$, $i = 1,2$. Using the identity (6) in (14) and differentiating the resulting equation with respect to z, we obtain

$$\frac{z\left(Q^{\alpha}_{\beta,p} f(z)\right)'}{Q^{\alpha}_{\beta,p} f(z)} = \left\{ \gamma + (p-\gamma)h(z) + \frac{(p-\gamma)zh'(z)}{(p-\gamma)h(z) + \gamma + \alpha + \beta} \right\} \in \mathcal{P}_k(p,\gamma). \tag{15}$$

This implies that

$$h_i(z) + \frac{zh_i'(z)}{(p-\gamma)h_i(z) + \gamma + \alpha + \beta} \in \mathcal{P}, \quad (z \in \mathbb{U}; i = 1,2). \tag{16}$$

We form the functional $\Psi(u,v)$ by choosing $u = h_i(z)$ and $v = zh_i'(z)$:

$$\Psi(u,v) = u + \frac{v}{(p-\gamma)u + \gamma + \alpha + \beta}. \tag{17}$$

Clearly, the first two conditions of Lemma 1 are satisfied. Now, we verify condition (iii) as follows:

$$\Re\{\Psi(iu_2,v_1)\} = \Re\left\{\frac{v_1}{(p-\gamma)iu_2+\gamma+\alpha+\beta}\right\} \le -\frac{(\gamma+\alpha+\beta)\left(1+u_2^2\right)}{2\left[(p-\gamma)^2u_2^2+(\gamma+\alpha+\beta)^2\right]} < 0. \quad (18)$$

Therefore applying Lemma 1, $h_i \in \mathscr{P}$ $(i = 1,2)$ and consequently $h \in \mathscr{P}_k$ for $z \in \mathbb{U}$. This completes the proof of Theorem 4. □

Theorem 5. *One has*

$$\mathscr{C}_k(p,\alpha+1;\gamma) \subset \mathscr{C}_k(p,\alpha;\gamma). \quad (19)$$

Proof. Applying (10) and Theorem 4, we observe that

$$\begin{aligned} & f \in \mathscr{C}_k(p,\alpha+1;\gamma) \\ & \Longleftrightarrow \frac{zf'}{p} \in \mathscr{S}_k(p,\alpha+1;\gamma) \Longrightarrow \frac{zf'}{p} \in \mathscr{S}_k(p,\alpha;\gamma) \\ & \Longleftrightarrow f \in \mathscr{C}_k(p,\alpha;\gamma), \end{aligned} \quad (20)$$

which evidently proves Theorem 5. □

Theorem 6. *If* $f \in \mathscr{S}_k(p,\alpha;\gamma)$, *then* $F_{c,p}(f) \in \mathscr{S}_k(p,\alpha;\gamma)$ $(c \ge 0)$, *where the generalized Libera integral operator* $F_{c,p}$ *is defined by* (7).

Proof. Let $f \in \mathscr{S}_k(p,\alpha;\gamma)$ and set

$$\begin{aligned} \frac{z\left(Q^{\alpha}_{\beta,p}F_{c,p}(f)(z)\right)'}{Q^{\alpha}_{\beta,p}F_{c,p}(f)(z)} &= (p-\gamma)h(z)+\gamma \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\{(p-\gamma)h_1(z)+\gamma\} \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\{(p-\gamma)h_2(z)+\gamma\}, \end{aligned} \quad (21)$$

where h is analytic in $\mathbb{U}$ with $h(0) = 1$. From (21), we have

$$z\left(Q^{\alpha}_{\beta,p}F_{c,p}(f)(z)\right)' = (c+p)Q^{\alpha}_{\beta,p}f(z) - cQ^{\alpha}_{\beta,p}F_{c,p}(f)(z). \quad (22)$$

Then, by using (21) and (22), we obtain

$$(c+p)\frac{Q^{\alpha}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}F_{c,p}(f)(z)} = (p-\gamma)h(z)+\gamma+c. \quad (23)$$

Taking the logarithmic differentiation on both sides of (23) with respect to z and multiplying by z, we have

$$\frac{1}{p-\gamma}\left(\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{Q^{\alpha}_{\beta,p}f(z)}-\gamma\right) = h(z)+\frac{zh'(z)}{(p-\gamma)h(z)+\gamma+c} \in \mathscr{P}_k. \quad (24)$$

This implies that

$$\left\{h_i(z)+\frac{zh_i'(z)}{(p-\gamma)h_i(z)+\gamma+c}\right\} \in \mathscr{P}, \quad (z \in \mathbb{U}; i = 1,2). \quad (25)$$

We form the functional $\Psi(u,v)$ by choosing $u = h_i(z)$ and $v = zh_i'(z)$:

$$\Psi(u,v) = u+\frac{v}{(p-\gamma)u+\gamma+c}. \quad (26)$$

Then clearly $\Psi(u,v)$ satisfies all the properties of Lemma 1. Hence, $h_i \in \mathscr{P}$ $(i = 1,2)$ and consequently $h \in \mathscr{P}_k$ for $z \in \mathbb{U}$, which implies that $F_{c,p}(f) \in \mathscr{S}_k(p,\alpha;\gamma)$. □

Next, we derive an inclusion property for the subclass $\mathscr{C}_k(\alpha;\gamma)$ involving $F_{c,p}(f)$, which is given by the following theorem.

Theorem 7. *If* $f \in \mathscr{C}_k(p,\alpha;\gamma)$, *then* $F_{c,p}(f) \in \mathscr{C}_k(p,\alpha;\gamma)$ $(c \ge 0)$, *where* $F_{c,p}$ *is defined by* (7).

Proof. By applying Theorem 6, it follows that

$$\begin{aligned} f \in \mathscr{C}_k(p,\alpha;\gamma) &\Longleftrightarrow \frac{zf'}{p} \in \mathscr{S}_k(p,\alpha;\gamma) \\ &\Longrightarrow F_{c,p}\left(\frac{zf'}{p}\right) \in \mathscr{S}_k(p,\alpha;\gamma) \\ &\qquad \text{(by Theorem 5)} \\ &\Longleftrightarrow \frac{z\left(F_{c,p}(f)\right)'}{p} \in \mathscr{S}_k(p,\alpha;\gamma) \\ &\Longleftrightarrow F_{c,p}(f) \in \mathscr{C}_k(p,\alpha;\gamma), \end{aligned} \quad (27)$$

which proves Theorem 7. □

Theorem 8. *If* $f \in \mathscr{C}_k(p,\alpha+1;\gamma)$, *for* $z \in \mathbb{U}$, *then* $f \in \mathscr{C}_k(p,\alpha;\gamma)$ *for*

$$|z| < r_0 = \frac{|\mu+1|}{\sqrt{A+\left(A^2-\left|\mu^2-1\right|\right)^{1/2}}}, \quad (28)$$

where $A = 2(s+1)^2+|\mu|^2-1$, *with* $\mu = ((\gamma+\alpha+\beta)/(p-\gamma)) \ne -1$ *and* $s = (1/(p-\gamma))$. *This radius is the best possible.*

Proof. Let $f \in \mathscr{C}_k(p,\alpha+1;\gamma)$ for $z \in \mathbb{U}$ and let

$$\begin{aligned} \frac{z\left(Q^{\alpha+1}_{\beta,p}f(z)\right)'}{Q^{\alpha+1}_{\beta,p}f(z)} &= (p-\gamma)h(z)+\gamma \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\{(p-\gamma)h_1(z)+\gamma\} \\ &\quad -\left(\frac{k}{4}-\frac{1}{2}\right)\{(p-\gamma)h_2(z)+\gamma\}, \end{aligned} \quad (29)$$

where h_i is analytic in $\mathbb{U}$ with $h_i(0) = 1$ and $\Re\{h_i(z)\} > 0$ for $i = 1, 2$. Using the identity (6) in (29) and differentiating the resulting equation with respect to z, we obtain

$$\frac{1}{p-\gamma}\left\{\frac{z\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{Q^{\alpha}_{\beta,p}f(z)}-\gamma\right\}$$
$$= h(z) + \frac{(1/(p-\gamma))\, zh'(z)}{h(z) + ((\gamma+\alpha+\beta)/(p-\gamma))}$$
$$= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{(1/(p-\gamma))\, zh_1'(z)}{h_1(z) + ((\gamma+\alpha+\beta)/(p-\gamma))}\right\}$$
$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{(1/(p-\gamma))\, zh_2'(z)}{h_2(z) + ((\gamma+\alpha+\beta)/(p-\gamma))}\right\}, \tag{30}$$

where $\Re\{h_i(z)\} > 0$ for $i = 1, 2$. Applying Lemma 2 with $s = ((\gamma+\alpha+\beta)/(p-\gamma))$ and $\mu = ((\gamma+\alpha+\beta)/(p-\gamma)) \neq -1$, we get

$$\Re\left\{h_i(z) + \frac{(1/(p-\gamma))\, zh_i'(z)}{h_i(z) + ((\gamma+\alpha+\beta)/(p-\gamma))}\right\} > 0 \quad \text{for } |z| < r_0, \tag{31}$$

where r_0 is given by (28). This completes the proof of Theorem 8. □

Theorem 9. *Let ϕ be a convex function and $f \in \mathcal{S}_2(\alpha;\gamma)$. Then $G \in \mathcal{S}_2(\alpha;\gamma)$, where $G = \phi * f$.*

Proof. Let $= \phi * f$. Then

$$Q^{\alpha}_{\beta,p}G(z) = Q^{\alpha}_{\beta,p}(\phi * f)(z) = \phi(z) * Q^{\alpha}_{\beta,p}f(z). \tag{32}$$

Also, $f \in \mathcal{S}_2(\alpha;\gamma)$. Therefore, $Q^{\alpha}_{\beta,p}f \in \mathcal{S}_2(\gamma)$. By logarithmic differentiation of (32) and after some simplification, we obtain

$$\frac{z\left(Q^{\alpha}_{\beta,p}G(z)\right)'}{pQ^{\alpha}_{\beta,p}G(z)} = \frac{\phi(z) * F(z)\, Q^{\alpha}_{\beta,p}f(z)}{\phi(z) * Q^{\alpha}_{\beta,p}f(z)}, \tag{33}$$

where $F = z(Q^{\alpha}_{\beta,p}f(z))'/pQ^{\alpha}_{\beta,p}f(z)$ is analytic in $\mathbb{U}$ and $F(0) = 1$. From Lemma 3, we can see that $z(Q^{\alpha}_{\beta,p}G(z))'/pQ^{\alpha}_{\beta,p}G(z)$ is contained in the convex hull of $F(\mathbb{U})$. Since $z(Q^{\alpha}_{\beta,p}G(z))'/pQ^{\alpha}_{\beta,p}G(z)$ is analytic in $\mathbb{U}$ and

$$F(\mathbb{U}) = \Omega = \left\{w : \frac{z\left(Q^{\alpha}_{\beta,p}w(z)\right)'}{pQ^{\alpha}_{\beta,p}w(z)} \in \mathcal{P}(\gamma)\right\}, \tag{34}$$

then $z(Q^{\alpha}_{\beta,p}G(z))'/pQ^{\alpha}_{\beta,p}G(z)$ lies in Ω; this implies that $G = \phi * f \in \mathcal{S}_2(\alpha;\gamma)$. □

Remark 10. Putting $p = 1$ in the above results, we obtain corresponding results for the operator Q^{α}_{β}.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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