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Research Article

Reliability of Modules with Load-Sharing Components

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To increase the reliability of modules, and thus of systems assembled from them, they are frequently constructed using parallel load-sharing components. Examples include jet engines, electrical power networks, and telecommunications networks. We consider the situation when the components operate independently, but when any one of them fails, the load of the failed component is instantaneously distributed among the working components. The entire module fails when the last working component fails. We analyze the survival probability and residual life expectancy of such modules. An obvious application is to the case of the 1998 Auckland power supply failure in New Zealand.

1. Introduction and motivation

Reliability systems often consist of several subsystems, which may be called modules. In practical reliability analysis, one often considers first the reliability of each module, and derives the reliability of the system as a whole. A classical example of such a system is a combination of data transmission routers. Since, in many areas, the continuity of data flow is of utmost importance, the system's reliability is increased by incorporating redundancy in the form of parallel components or subsystems. For instance, data transfers between two points may be accomplished by multiple (identical or not) parallel routers, with electricity supplied to each of the routers by several (identical or not) power units.

In general, we are interested in a module consisting $K \geq 2$ parallel components. We denote the lifetimes of the components by T_k, $1 \leq k \leq K$, with survival functions $S_k(t) = \mathbf{P}\{T_k > t\}$, and hazard rate (HR) functions $h_k(t) = -S_k'(t)/S_k(t)$, respectively. When one of the components fails, its load is distributed among the working components. The entire module fails when the last working component fails; denote the module's failure time by $M(K)$. The corresponding survival and mean residual life (MRL) functions are, respectively,

$$S_{M(K)}(t) = \mathbf{P}\{M(K) > t\},$$
$$\mu_{M(K)}(t) = \frac{IS_{M(K)}(t)}{S_{M(K)}(t)}, \tag{1.1}$$

where $IS_{M(K)}(t) = \int_t^\infty S_{M(K)}(x)dx$. We next give a couple of illustrative examples, where the need to estimate the above two functions is a natural one.

> "Consider jet engines functioning under full load on a commercial airplane. One functioning jet engine is enough for a small airplane, while 2 engines are necessary for a big airplane. But for higher reliability, 2 engines are functioning for the small airplane and 4 for the big airplane. An engine controller manages the load sharing. When 2 engines function in a small airplane, the load on each is much less than when they function alone. From the test data, the failure rate of the engines is reduced to 45% under half load. Similarly, if 4 engines are functioning for a big airplane, the failure rate for each engine is reduced to 45%, while if three engines are functioning, the failure rate is reduced to 75% ... for how long can the small and big airplanes fly before the reliability drops below 0.9?" [1].

We see from this excerpt that it is natural to aim at estimating the airplane's survival function $S_{M(K)}(t)$. We may also want to know for how long, on average, the airplane can still stay in the air, for which we need to estimate the MRL function $\mu_{M(K)}(t)$. Of course, the above questions are more mathematical idealizations than reflections of reality, but they serve as conceptual examples of some of the types of problems in the area. In practice, even large jets can land relatively safely without a single functioning engine [2, 3].

> "The 1998 Auckland power crisis was an event that occurred in the Auckland, New Zealand, Central Business District. The area suffered a five-week-long power outage in 1998. At the beginning of 1998, almost all of downtown Auckland received electricity from the supplier Mercury Energy via only four power cables, two of them were 40-year-old oil-filled cables past their replacement date. One of the cables failed on 20 January, possibly due to the unusually hot and dry conditions, another on 9 February, and due to the increased load from the failure of the first cables, the remaining two failed on 19 and 20 February, leaving the central business district (except parts of a few streets) without power" [4].

For a detailed account and analysis of the power crisis, see [5]. In this case, estimation of the mean residual life is of utmost importance in deciding what emergency repair or replacement activities may be (more) effective.

To get an initial feel about the module's survival, HR, and MRL functions, we note that if the failure of any one of the K components does not influence the HR functionsof the functioning components, then the module's survival function $S_{M(K)}(t)$ can be written in terms of the individual survival functions as $1 - \prod_{k=1}^{K}(1 - S_k(t))$. The individual survival functions $S_k(t)$ can in turn be expressed using the corresponding HR functions $h_k(t)$ as $S_k(t) = \exp\{-\int_0^t h_k(y)dy\}$. In the context of the present paper, due to the load-sharing scenario, the dynamics of the entire module and thus of its survival and MRL functions are quite different from those in the case of non-interacting parallel components.

There are a few closely related references on this topic. The reliability of load sharing systems may be studied through positively dependent multivariate life distributions [6]; for positively dependent bivariate life distributions, we refer to [7, Section 9.2]. Another approach of studying dependency among parallel components is by using interaction schemes. For example, Murthy and Nguyen [8], and Murthy and Wilson [9] propose and analyze an interaction scheme where, in a two-component system, the failure of one component provokes the failure of another component with probability p, and thus does not provoke with $1 - p$. Another failure interaction scheme in various generalities—we follow a similar line of thought in the present paper—is where the failure of a component modifies the HR function of the other components by not provoking its failure instantaneously but modifying its conditional time to failure [10–13]. These papers assume piecewise constant failure rates, or various degrees of interchangeability and symmetry in their components and/or redistribution schemes, whereas our results are presented in complete generality, and include estimators for the MRL. Perhaps more importantly, our work starts with the notion that there might be too few observations of failing entire modules in order to derive desired statistical inferential results, but failure times of individual module's components might be more readily available (e.g., from laboratory-type testing). Hence assuming the availability of such data, we then aim at deriving formulae for the survival function—and thus, in turn, failure, MRL, and other functions—of the entire module. In contrast, the aforementioned papers are concerned with estimating the component failure rate function given the observed failure times of entire systems. Note also that this problem can be considered [1, 12] in the context of a more general system, the k-out-of-K:G, which, by definition, functions as long as there are at least k ($1 \le k \le K$) components working. These papers consider specific distributions and load sharing rules, with less generality than our results.

The remainder of the paper is organized as follows. In Section 2, we present the model, notational conventions, and other mathematical formalities. Section 5 contains expressions for the survival and MRL functions, $S_{M(K)}(t)$ and $\mu_{M(K)}(t)$, in terms of individual components that work under the original or increased loads. The general results, Theorems 5.1 and 5.2, are preceded in Section 3 by a detailed analysis of the case $K = 2$, which is of interest in its own right, as well as for a more easily comprehended example of the general theorems. Explicit examples of the $K = 2$ case are given in Section 4, where the performance of parametric and nonparametric estimators of the survival and MRL functions are examined.

Two of us (M.B. and C.D.L.) were fortunate enough to be colleagues of Jeff Hunter when he occupied the Chair in Statistics at Massey University. Jeff's inaugural address was on the subject of reliability and warranty analysis, and we hope he enjoys this sequel. The many visits of the third author (R.Z.) to Massey University in PalmerstonNorth did not pass by without Jeff flying in from Auckland either to give an inspiring seminar on Generalized Inverses and Stochastic Processes, or to enliven morning and afternoon teas.

2. Mathematical formalism

We assume that the failure times $T_1,\dots,T_K$ are independent, though not necessarily identically distributed, random variables. We work with continuous life-time distributions, and hence assume that there are no multiple failures at any time as multiple failures can occur only with zero probabilities. The first failure occurs at the time $T_{1:K} = \min_{1\le k\le K} T_k$, which is the first order statistic of $T_1,\dots,T_K$. Let D be the first antirank of $T_1,\dots,T_K$, which is (uniquely) defined by $T_D = T_{1:K}$. Hence the pair $(D,T_{1:K})$ tells us which of the components $\{1,\dots,K\}$ fails first and at what time.

At the time $T_{1:K}$, the load of the failed component D is instantaneously distributed among the remaining $K-1$ components, whose set we denote by $\Delta^{(1)} = \{1,\dots,K\}\setminus\{D\}$. Specifically, for every $k\in\Delta^{(1)}$, the failure of the Dth component increases the HR function $h_k(t)$ of the kth (working) component by a function $a^{(1)}_{D,k}(t)$, where the superscript (1) indicates that the redistribution has occurred (immediately) after the 1st failure. Hence for every $k\in\Delta^{(1)}$, we have the conditional-on-$\{T_1,\dots,T_K\}$ HR function $h^{(1)}_k(t) = (h_k(t)+a^{(1)}_{D,k}(t))\mathbf{1}_{\{T_{1:K}\le t\}}$, where the indicator $\mathbf{1}_{\{T_{1:K}\le t\}}$ is equal to 1 when the statement $T_{1:K}\le t$ is true and is 0 otherwise. Let $T^{(1)}_k$, $k\in\Delta^{(1)}$ be conditionally-on-$\{T_1,\dots,T_K\}$ independent random variables whose conditional-on-$\{T_1,\dots,T_K\}$ distributions have the HR functions $h^{(1)}_k(t)$.

Before proceeding further, let us discuss intuitively what we have introduced so far. First, note that $h^{(1)}_k(t)=0$ for all $t<T_{1:K}$, which implies that the random variables $T^{(1)}_k$, $k\in\Delta^{(1)}$ do not take on any value in the interval $[0,T_{1:K}]$. Hence in addition to the 'original' situation with K random variables $T_1,\dots,T_K$, we have constructed an "artifact" with $K-1$ random variables $T^{(1)}_k$, $k\in\Delta^{(1)}$, which are "activated" at the moment $t=T_{1:K}$ and governed by the HR functions $h_k(t)+a^{(1)}_{D,k}(t)$. When one of the $\Delta^{(1)}$ components fails, we create new $K-2$ "artificial" components. Proceeding in a similar fashion, we specify the mechanism that governs the life of the entire module and allows us, via a conditioning technique, to determine its survival, HR, and MRL functions. We next describe this procedure rigorously and also introduce additional notation to be used throughout the rest of the paper.

To begin, we find it convenient to use the notation $T^{(0)}_1,\dots,T^{(0)}_K$ instead of $T_1,\dots,T_K$, respectively, and $D^{(0)}$ instead of D. Next, starting with the "initial" random variables $T^{(0)}_k$, we recursively, for all $i=1,\dots,K-2$, define the following quantities.

(i) The random variables $D^{(i)}$ and $T^{(i)}_{1:(K-i)}$, which respectively specify the $(i+1)$st failed component and its failure time, which are related via (or defined by) the equations $T^{(i)}_{D^{(i)}} = T^{(i)}_{1:(K-i)} \equiv \min_{k\in\Delta^{(i)}} T^{(i)}_k$, where $\Delta^{(0)}=\{1,\dots,K\}$ and, for any $i\ge 1$, the set $\Delta^{(i)}=\Delta^{(i-1)}\setminus\{D^{(i-1)}\}$ consists of all working components immediately before the $(i+1)$st failure.

(ii) Conditionally-on-$\{D^{(0)},\dots,D^{(i)},T^{(i)}_{1:(K-i)}\}$ independent random variables $T^{(i+1)}_k$, $k\in\Delta^{(i)}$, whose conditional-on-$\{D^{(0)},\dots,D^{(i)},T^{(i)}_{1:(K-i)}\}$ distributions have the HR functions

$$h_k^{(i+1)}(t) = \left(h_k(t) + \sum_{m=1}^{i+1} a_{D^{(m-1)},k}^{(m)}(t) \right) \mathbf{1}_{\{T_{1:(K-i)}^{(i)} \le t\}}. \tag{2.1}$$

Hence, $T_k^{(i+1)}$ is the lifetime of the kth component after $i+1$ failed components, which are $D^{(0)},\dots,D^{(i)}$. The random variable $T_k^{(i+1)}$ starts its life at the time $t = T_{1:(K-i)}^{(i)}$.

Note that, since there are K components in the module, the largest value of i is $K-1$ as there are no functioning components after the Kth failure. When $i = K-2$, then there is only one "surviving" random variable $T_k^{(K-1)}$, whose index k is the (only) member of the singleton set $\{1,\dots,K\} \setminus \{D^{(0)},\dots,D^{(K-2)}\}$; denote the member by $\kappa(K-1)$. Hence we have $M(K) = T_{\kappa(K-1)}^{(K-1)}$, and so the module's survival function $S_{M(K)}(t)$ can be written as $S_{M(K)}(t) = \mathbf{P}\{T_{\kappa(K-1)}^{(K-1)} > t\}$. With the help of the latter equation, the corresponding formula for the MRL function $\mu_{M(K)}(t)$ can be expressed in terms of the survival function of the random variable $T_{\kappa(K-1)}^{(K-1)}$ using (1.1). Of course, one can also derive an analogous expression for the HR function via the equation $h_{M(K)}(t) = -S'_{M(K)}(t)/S_{M(K)}(t)$. Section 3 provides a detailed analysis of the survival and MRL functions when $K = 2$.

3. Survival and MRL functions for two components

In this section, we give a detailed analysis of the survival function $S_{M(2)}(t)$ of a module with *two* (possibly different) components whose independent lifetime variables are T_1 and T_2 with (possibly different) survival functions $S_1(t)$ and $S_2(t)$, respectively. At the time $T_{1:2} = \min(T_1, T_2)$, one of the two components fails; let it be i. As a result of the failure, the HR function of the working component $k = \text{NOT}(i)$ increases by a function $a_{i,k}^{(1)}(t)$, for all $t \ge T_{1:2}$. (Note that $\text{NOT}(i) = 3 - i$ as we consider the $K = 2$ case.) Let $S_k^{+i}(t)$ be the survival function of the component k when it is working under its own load plus the load of the failed component i, which in our current two-component situation means that the component k takes on the whole module's load.

There is a possibility that we might have a sufficiently large number of failure times of such modules, in which case we estimate $S_{M(2)}(t)$ using the empirical survival function, or fit a parametric distribution to the failure times. Failing a sufficiently large number of modules may not, however, be feasible, due to time and/or cost considerations. However, assessing the reliability of individual components under normal and/or increased loads can be quite a feasible task, say, in a laboratory environment. Quantitative accelerated life testing techniques can be used to speed up the process (cf., e.g., Nelson [14]). For the reasons noted above, in the next theorem, we express $S_{M(2)}(t)$ in terms of the "individual" survival functions $S_i(t)$ and $S_{\text{NOT}(i)}^{+i}(t)$, for $i = 1$ and 2.

Theorem 3.1. *We have that*

$$S_{M(2)}(t) = -\sum_{i=1}^{2} S_{\text{NOT}(i)}^{+i}(t) \int \mathbf{1}_{\{y \le t\}} \frac{S_{\text{NOT}(i)}(y)}{S_{\text{NOT}(i)}^{+i}(y)} dS_i(y) + S_1(t)S_2(t). \tag{3.1}$$

We can estimate the survival functions $S_1(t)$ and $S_2(t)$ on the right-hand side of (3.1) by exposing (e.g., in a laboratory environment) the two components to their "normal" loads, and we can also estimate the survival functions $S_1^{+2}(t)$ and $S_2^{+1}(t)$ by exposing the corresponding components to the load of the entire module. In the nonparametric approach, we estimate the survival functions $S_i(t)$, $i = 1,2$ as $\widehat{S}_i(t) = (1/n_i)\sum_{\ell=1}^{n_i} \mathbf{1}_{\{T_i(\ell)>t\}}$, where $T_i(1),\ldots,T_i(n_i)$ are independent copies of the random variable $T_i \sim S_i$. (For a given random variable X, it is customary to use the notation $X_1,\ldots,X_n$ for copies of X. Since we already use subscripts for other good reasons, throughout the paper, we use $X(1),\ldots,X(n)$ to denote copies of X.) Next, we use independent copies $T_j^{+i}(1),\ldots,T_j^{+i}(m_j)$ of the random variable $T_j^{+i} \sim S_j^{+i}$ to construct an estimator for $S_j^{+i}(t)$, which is $\widehat{S}_j^{+i}(t) = (1/m_j)\sum_{\ell=1}^{m_j} \mathbf{1}_{\{T_j^{+i}(\ell)>t\}}$. Thus, we have the nonparametric estimator of the module's survival function

$$\widehat{S}_{M(2)}(t) = \sum_{i=1}^{2} \widehat{S}_{\text{NOT}(i)}^{+i}(t)\frac{1}{n_i}\sum_{\ell=1}^{n_i} \mathbf{1}_{\{T_i(\ell)\le t\}} \frac{\widehat{S}_{\text{NOT}(i)}(T_i(\ell))}{\widehat{S}_{\text{NOT}(i)}^{+i}(T_i(\ell))} + \widehat{S}_1(t)\widehat{S}_2(t). \tag{3.2}$$

To derive an analogous expression for the MRL function $\mu_{M(2)}(t)$ in terms of the four "individual" survival functions, we need to derive an analogous expression for the integral $IS_{M(2)}(t)$, which can be done by either integrating the right-hand side of (3.1) or by using general Theorem 5.2 with $K = 2$. This gives us the following corollary.

Corollary 3.2. *We have that*

$$\begin{aligned} IS_{M(2)}(t) = {} & \sum_{i=1}^{2} \iint (x - \max(y,t))_+ dS_{\text{NOT}(i)}^{+i}(x) \frac{S_{\text{NOT}(i)}(y)}{S_{\text{NOT}(i)}^{+i}(y)} dS_i(y) \\ & - \sum_{i=1}^{2} \int (y-t)_+ S_{\text{NOT}(i)}(y) dS_i(y), \end{aligned} \tag{3.3}$$

where $c_+ = c$ if $c > 0$ and $c_+ = 0$ otherwise.

Equations (3.1) and (3.3) can be used for constructing parametric estimators for the MRL function $\mu_{M(2)}(t)$. If, however, we want to use a nonparametric estimator, then we can construct it with the help of the non-parametric estimator for the integral $IS_{M(2)}(t)$,

$$\begin{aligned} \widehat{IS}_{M(2)}(t) = {} & \sum_{i=1}^{2} \frac{1}{n_i m_{\text{NOT}(i)}} \sum_{\ell=1}^{n_i} \sum_{\nu=1}^{m_{\text{NOT}(i)}} \left(T_{\text{NOT}(i)}^{+i}(\nu) - \max\left(T_i(\ell),t\right)\right)_+ \frac{\widehat{S}_{\text{NOT}(i)}(T_i(\ell))}{\widehat{S}_{\text{NOT}(i)}^{+i}(T_i(\ell))} \\ & + \sum_{i=1}^{2} \frac{1}{n_i} \sum_{\ell=1}^{n_i} \left(T_i(\ell) - t\right)_+ \widehat{S}_{\text{NOT}(i)}(T_i(\ell)). \end{aligned} \tag{3.4}$$

We now define a nonparametric estimator for the MRL function $\mu_{M(2)}(t)$ as

$$\widehat{\mu}_{M(2)}(t) = \frac{\widehat{IS}_{M(2)}(t)}{\widehat{S}_{M(2)}(t)}. \tag{3.5}$$

The above expressions for the module's survival and MRL functions are based on the survival functions of individual components under their original and increased loads. If desired, however (and we will find it convenient in Section 4), the expressions can easily be rewritten in terms of the corresponding HR functions. This can be done using the equations $S_k(t) = \exp\{-\int_0^t h_k(x)dx\}$, $S_k^{+i}(t) = \exp\{-\int_0^t h_k(x) + a_{i,k}^{(1)}(x)dx\}$, and so forth, or simply using (A.6) derived in the appendix. (Indeed, the proof of general Theorem 5.1 is based on HR functions.) Clearly now, we have $S_k(t)/S_k^{+i}(t) = \exp\{\int_0^t a_{i,k}^{(1)}(x)dx\}$, which is convenient when dealing with the right-hand sides of (3.1) and (3.3). (Of course, we have $i \neq k$.)

4. Examples

As an example, consider the simple but important case when the module's two components have *exponential* lifetimes. (For a recent discussion of tests for exponentiality, we refer to Mimoto and Zitikis [15] and references therein.) That is, we assume the survival function $S_k(t) = \exp(-\lambda_k t)$ and, consequently, the HR function $h_k(t) = \lambda_k$. (We will later find it also convenient to use the notation $S(t;\lambda_k)$ instead of $S_k(t)$, and the notation $f(t;\lambda_k)$ for the corresponding density function.) Since the exponential HR function is constant, it leaps to mind to choose the redistribution function also as a constant; hence we assume that $a_{i,k}^{(1)}(t) \equiv \alpha_{i,k}$. Under this assumption and using (3.1), we obtain the survival function

$$S_{M(2)}(t) = \left(1 + t\sum_{i=1}^{2} \lambda_i \Delta(t;\lambda_i - \alpha_{i,k}) e^{(\lambda_i - \alpha_{i,k})t}\right) e^{-(\lambda_1+\lambda_2)t}, \tag{4.1}$$

where

$$\Delta(t;c) = \begin{cases} \dfrac{1}{ct}(1 - e^{-ct}) & \text{if } c \neq 0, \\ 1 & \text{if } c = 0. \end{cases} \tag{4.2}$$

Irrespectively of the sign of c, the quantity $\Delta(t;c)$ is nonnegative, and so we have the bound $S_{M(2)}(t) \geq e^{-(\lambda_1+\lambda_2)t}$, which can be rewritten as $S_{M(2)}(t) \geq \mathbf{P}\{\min(T_1, T_2) > t\}$; hence the obvious fact is that the module functions at least until the time of the first failure.

We next derive the HR function, which is

$$h_{M(2)}(t) = \frac{(\lambda_1 + \lambda_2) - \sum_{i=1}^{2} \lambda_i e^{(\lambda_i - \alpha_{i,k})t} + t(\lambda_1 + \lambda_2)\sum_{i=1}^{2} \lambda_i \Delta(t;\lambda_i - \alpha_{i,k}) e^{(\lambda_i - \alpha_{i,k})t}}{1 + t\sum_{i=1}^{2} \lambda_i \Delta(t;\lambda_i - \alpha_{i,k}) e^{(\lambda_i - \alpha_{i,k})t}}. \tag{4.3}$$

Integrating (4.1), we obtain an expression for $IS_{M(2)}(t)$ and, in turn, for the MRL function:

$$\mu_{M(2)}(t) = \frac{1 + \sum_{i=1}^{2} \lambda_i/(\lambda_k + \alpha_{i,k}) e^{(\lambda_i - \alpha_{i,k})t} + t\sum_{i=1}^{2} \lambda_i \Delta(t;\lambda_i - \alpha_{i,k}) e^{(\lambda_i - \alpha_{i,k})t}}{\left(1 + t\sum_{i=1}^{2} \lambda_i \Delta(t;\lambda_i - \alpha_{i,k}) e^{(\lambda_i - \alpha_{i,k})t}\right)(\lambda_1 + \lambda_2)}. \tag{4.4}$$

We will next further examine two special cases.

4.1. Scenario A. If we suppose that the components are functionally identical but the HR functions differ because the load is shared unequally, then we can have $a_{i,k}^{(1)}(t) \equiv \lambda_i$.

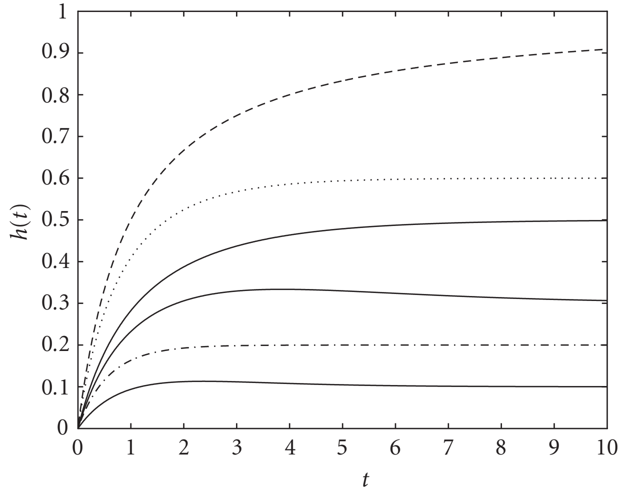

FIGURE 4.1. Representative shapes of the failure rate of a module consisting of two exponential components in parallel with failure rates $\lambda_1,\lambda_2 : \lambda_1+\lambda_2 = 1$. The solid lines are for the independent case [16, Equation 2.2] with the upper, middle, and lower curves having $\lambda_1 = 0.5, 0.3, 0.1$, respectively. The dashed curve is Scenario A ($\lambda_1 = 0.5, 0.3, 0.1$) and Scenario B with $\lambda_1 = 0.5$, the dotted curve is Scenario B with $\lambda_1 = 0.3$, and the dot-dashed curve is Scenario B with $\lambda_1 = 0.1$.

(As a special case, we may have $\lambda_1 = \lambda_2 =$, say, λ.) Equations (4.1) and (4.3) yield the survival and HR functions

$$S_{M(2)}(t) = \left(1+t(\lambda_1+\lambda_2)\right)e^{-(\lambda_1+\lambda_2)t}, \qquad h_{M(2)}(t) = \frac{t(\lambda_1+\lambda_2)^2}{1+t(\lambda_1+\lambda_2)}, \tag{4.5}$$

while (4.4) gives the MRL function

$$\mu_{M(2)}(t) = \frac{2+t(\lambda_1+\lambda_2)}{\left(1+t(\lambda_1+\lambda_2)\right)(\lambda_1+\lambda_2)}. \tag{4.6}$$

4.2. Scenario B. As an alternative to Scenario A, we might suppose that the components are sharing the load equally but the component reliabilities differ. In this case, we set $a_{i,k}^{(1)}(t) \equiv \lambda_k$, assuming without loss of generality that $\lambda_1 \neq \lambda_2$, as the case of equality (i.e., $\lambda_1 = \lambda_2 =$, say, λ) is covered by Scenario A. As before, (4.1) and (4.3) give

$$S_{M(2)}(t) = \frac{\lambda_1 e^{-2\lambda_2 t} - \lambda_2 e^{-2\lambda_1 t}}{\lambda_1 - \lambda_2}, \qquad h_{M(2)}(t) = \frac{2\lambda_1\lambda_2\left(e^{-2\lambda_2 t} - e^{-2\lambda_1 t}\right)}{\lambda_1 e^{-2\lambda_2 t} - \lambda_2 e^{-2\lambda_1 t}}. \tag{4.7}$$

Finally, from (4.4), we have the MRL function

$$\mu_{M(2)}(t) = \frac{\lambda_2^2 e^{-2\lambda_1 t} - \lambda_1^2 e^{-2\lambda_2 t}}{2\lambda_1\lambda_2\left(\lambda_2 e^{-2\lambda_1 t} - \lambda_1 e^{-2\lambda_2 t}\right)}. \tag{4.8}$$

Figure 4.1 shows the behaviour of the HR function for various combinations of λ_1, λ_2, normalized so that $\lambda_1+\lambda_2 = 1$.

In both scenarios, the survival, HR, and MRL functions depend only on λ_1 and λ_2, which are parameters of individual components and can, therefore, be estimated by failing the components under, for example, their "usual" loads a number of times in a laboratory environment. Assuming that we have such data

$$\begin{aligned} &t_1(1),\dots,t_1(n_1) - \text{ observations of } T_1 \sim S(\bullet;\lambda_1),\\ &t_2(1),\dots,t_2(n_2) - \text{ observations of } T_2 \sim S(\bullet;\lambda_2), \end{aligned} \tag{4.9}$$

the MLEs of λ_i, $i = 1,2$ are the standard ones: $\widehat{\lambda}_i = n_i/s_i$, where $s_i = \sum_{\ell=1}^{n_i} t_i(\ell)$. However, we may have more information about failures under the original and redistributed load.

First, consider the case of individual components. Suppose that the reliability of individual components can be determined in a laboratory environment, providing n_i observations of T_i, and m_i observations of $T^{+i}_{\text{NOT}(i)}$. Hence in addition to data (4.9), we now also have

$$\begin{aligned} &t_1^+(1),\dots,t_1^+(m_1) - \text{ observations of } T_1^{+2} \sim \begin{cases} S(\bullet;\lambda_1+\lambda_2) & \text{Scenario A},\\ S(\bullet;2\lambda_1) & \text{Scenario B}, \end{cases}\\ &t_2^+(1),\dots,t_2^+(m_2) - \text{ observations of } T_2^{+1} \sim \begin{cases} S(\bullet;\lambda_1+\lambda_2) & \text{Scenario A},\\ S(\bullet;2\lambda_2) & \text{Scenario B}. \end{cases} \end{aligned} \tag{4.10}$$

(It would be more precise to write $t_i^{+\text{not}(i)}(\ell)$ instead of $t_i^+(\ell)$, but the latter is simpler and we expect no confusion.) The likelihood is the product of the $n_1 + n_2 + m_1 + m_2$ individual likelihoods. Denote $s_i^+ = \sum_{\ell=1}^{m_i} t_i^+(\ell)$. Then in Scenario A, the loglikelihood function is

$$\begin{aligned} \log L(\boldsymbol{\lambda}) &= n_1\log\lambda_1 - \lambda_1 s_1 + n_2\log\lambda_2 - \lambda_2 s_2\\ &\quad + (m_1+m_2)\log(\lambda_1+\lambda_2) - (\lambda_1+\lambda_2)(s_1^+ + s_2^+). \end{aligned} \tag{4.11}$$

Solving the system of equations $(\partial/\partial\lambda_i)\log L(\boldsymbol{\lambda}) = 0$, $i = 1,2$ yields the MLEs for $i = 1,2$,

$$\widehat{\lambda}_i = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{4.12}$$

where $a = (s_i - s_{3-i})(s_i + s_1^+ + s_2^+)$, $b = (s_i - s_{3-i})(n_i + m_1 + m_2) + (n_1 + n_2)(s_i + s_1^+ + s_2^+)$, and $c = n_i(n_1 + n_2 + m_1 + m_2)$. In Scenario B, we have the loglikelihood function

$$\begin{aligned} \log L(\boldsymbol{\lambda}) &= n_1\log\lambda_1 - \lambda_1 s_1 + n_2\log\lambda_2 - \lambda_2 s_2\\ &\quad + m_1\log(2\lambda_1) + m_2\log(2\lambda_2) - 2\lambda_1 s_1^+ - 2\lambda_2 s_2^+, \end{aligned} \tag{4.13}$$

which yields the MLEs for $i = 1,2$,

$$\widehat{\lambda}_i = \frac{n_i + m_i}{s_i + 2s_i^+}. \tag{4.14}$$

Now, consider the case where we have data on failures of the entire module. We have already noted the "trivial" situation when the module's survival, HR, and MRL function can be estimated using modules' observed failures, provided that the number of such observations is sufficiently large. If, however, the sample size is not large, then in order to increase the reliability of statistical inference, we want to use every possible bit of information. Hence assume that we have n independent observations of the random vector $(D, T_{1:2}, T^{+D}_{\text{not}(D)})$, where D is the first failed component, $T_{1:2}$ is the time of the first failure, and $T^{+D}_{\text{not}(D)}$ is the time of module's failure. (Note that $\text{not}(D) = 3 - D$.) Our data are the three-dimensional vectors $(d(\ell), t(\ell), t^{+}(\ell))$, $\ell = 1, \dots, n$, which are independent observations of the random vector $(D, T_{1:2}, T^{+D}_{\text{not}(D)})$. (It would be more precise to write $t^{+d(\ell)}_{\text{not}(d(\ell))}(\ell)$ instead of $t^{+}(\ell)$, but the latter is less cumbersome and we expect no confusion.) In addition, we assume that we also know $n_1 = \sum_{\ell=1}^{n} \mathbf{1}_{\{d(\ell)=1\}}$, the number of times component 1 has failed first. The frequency of component 2 failing first is, therefore, $n_2 = n - n_1$. Whether we are dealing with Scenario A or B, the (unknown) parameter is $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, and we need to estimate it. In Scenario A, we have the likelihood function

$$\begin{aligned} L(\boldsymbol{\lambda}) &= \prod_{\ell=1}^{n} f\left(t(\ell); \lambda_{d(\ell)}\right) S\left(t(\ell); \lambda_{3-d(\ell)}\right) f\left(t^{+}(\ell) - t(\ell); \lambda_1 + \lambda_2\right) \\ &= (\lambda_1 + \lambda_2)^{n} \lambda_1^{n_1} \lambda_2^{n_2} \exp\left\{ -(\lambda_1 + \lambda_2) \sum_{\ell=1}^{n} t^{+}(\ell) \right\}. \end{aligned} \tag{4.15}$$

Solving the system of equations $(\partial/\partial\lambda_i) \log L(\boldsymbol{\lambda}) = 0$, $i = 1, 2$ yields the MLEs for $i = 1, 2$, $\widehat{\lambda}_i = 2n_i / \sum_{\ell=1}^{n} t^{+}(\ell)$. In Scenario B, the likelihood function is

$$\begin{aligned} L(\boldsymbol{\lambda}) &= \prod_{\ell=1}^{n} f\left(t(\ell); \lambda_{d(\ell)}\right) S\left(t(\ell); \lambda_{3-d(\ell)}\right) f\left(t^{+}(\ell) - t(\ell); 2\lambda_{3-d(\ell)}\right) \\ &= (2\lambda_1\lambda_2)^{n} \exp\left\{ -(\lambda_1 + \lambda_2) \sum_{\ell=1}^{n} t(\ell) \right\} \exp\left\{ -2 \sum_{\ell=1}^{n} \lambda_{3-d(\ell)} \left(t^{+}(\ell) - t(\ell)\right) \right\}, \end{aligned} \tag{4.16}$$

which gives the MLEs, for $i = 1, 2$,

$$\widehat{\lambda}_i = \frac{n}{\sum_{\ell=1}^{n} t(\ell) + 2\sum_{\ell=1}^{n} \mathbf{1}_{\{d(\ell)=3-i\}} \left(t^{+}(\ell) - t(\ell)\right)}. \tag{4.17}$$

We are now able to compare the performance of the parametric estimators obtained from (3.1) and (3.3), and the nonparametric estimators (3.2) and (3.5), using a small simulation study. We suppose that $\lambda_1 = 0.001$, $\lambda_2 = 0.002$, and that we have $n_1 = n_2 = m_1 = m_2$ observations of failure times of individual components in a laboratory setting, allowing us to estimate the parameters from (4.12) or (4.14). Figure 4.2 compares the estimated survival and MRL functions for Scenario A, while Figure 4.3 shows the same for Scenario B. We can see in both examples that the estimators appear to be unbiased, except, possibly, the nonparametric estimator of the MRL, where there may be underestimation. The variation is larger, as expected, for the nonparametric estimators, and increases over time, except in the case of the parametric estimate of the MRL, where the 90-percentile band appears to be of approximately constant width.

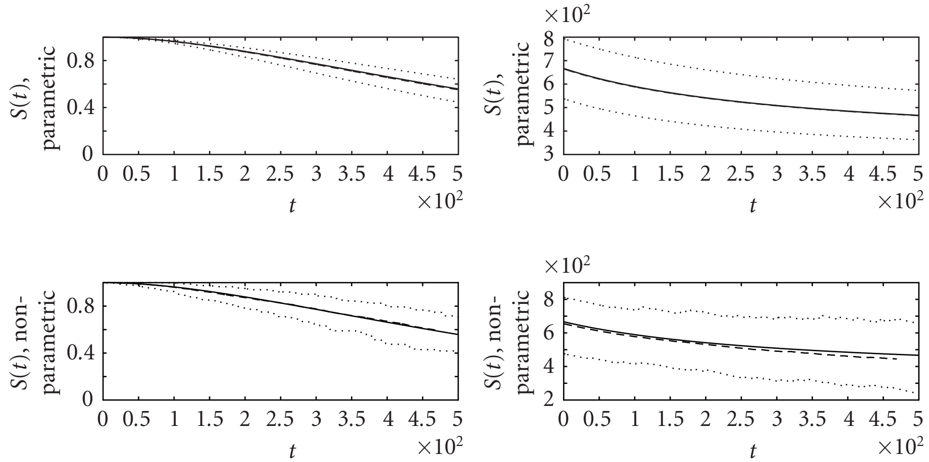

Figure 4.2. The estimated survival (left) and MRL (right) functions for Scenario A. Parametric estimates are shown in the top panel, nonparametric in the bottom. The true curve is a solid line. The mean of 100 repetitions is shown as a dashed line, while the dotted lines are the 5th and 95th percentiles.

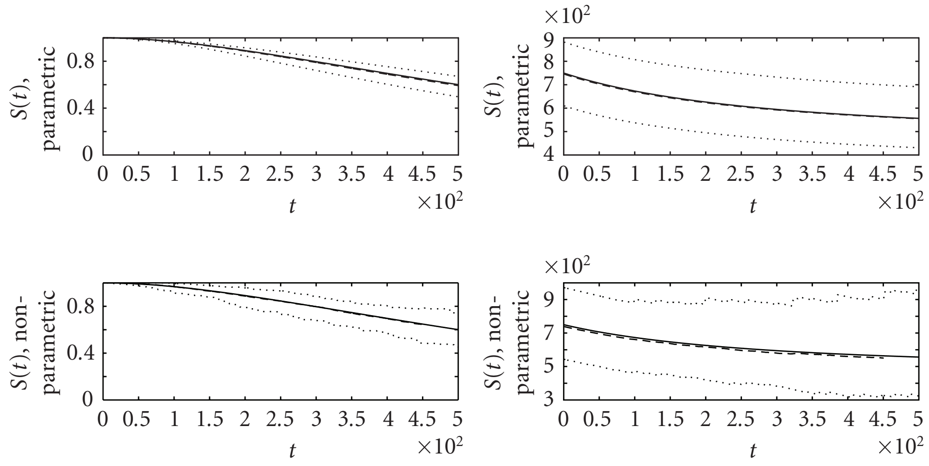

Figure 4.3. The estimated survival (left) and MRL (right) functions for Scenario B. Parametric estimates are shown in the top panel, nonparametric in the bottom. The true curve is a solid line. The mean of 100 repetitions is shown as a dashed line, while the dotted lines are the 5th and 95th percentiles.

5. Survival and MRL functions for more than two components

In this section, we consider the survival and MRL functions of modules with arbitrarily, $K \geq 2$, many components. We will need additional notation. Let $S_k^{+(i,j)}(t)$ denote the survival function of a working component k when two other components, i and j, have failed. Likewise, we interpret the survival functions $S_{i_{K-1}}^{+(i_1,\dots,i_{K-2})}(t)$, $S_{\text{NOT}(i_1,\dots,i_{K-1})}^{+(i_1,\dots,i_{K-1})}(t)$, and so forth.

Theorem 5.1. *For every $K \geq 2$, we have $S_{M(K)}(t) = S_{M(K)}^{*}(t) + S_{M(K)}^{**}(t)$, where*

$$\begin{aligned}
S_{M(K)}^{*}(t) = (-1)^{K-1} \sum_{i_1 \in \{1,\dots,K\}} \cdots \sum_{i_{K-1} \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-2}\}} & S_{\text{NOT}(i_1,\dots,i_{K-1})}^{+(i_1,\dots,i_{K-1})}(t) \int \cdots \int \mathbf{1}_{\{y_{K-1} \leq t\}} \\
& \times \prod_{q \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-1}\}} \frac{S_q^{+(i_1,\dots,i_{K-2})}(y_{K-1})}{S_q^{+(i_1,\dots,i_{K-1})}(y_{K-1})} \mathbf{1}_{\{y_{K-1} > y_{K-2}\}} dS_{i_{K-1}}^{+(i_1,\dots,i_{K-2})}(y_{K-1}) \\
& \cdots \\
& \times \prod_{q \in \{1,\dots,K\} \setminus \{i_1\}} \frac{S_q(y_1)}{S_q^{+i_1}(y_1)} \mathbf{1}_{\{y_1 > 0\}} dS_{i_1}(y_1), \\
S_{M(K)}^{**}(t) = (-1)^{K-1} \sum_{i_1 \in \{1,\dots,K\}} \cdots \sum_{i_{K-1} \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-2}\}} & \int \cdots \int \mathbf{1}_{\{y_{K-1} > \max(y_{K-2},t)\}} \\
& \times S_{\text{NOT}(i_1,\dots,i_{K-1})}^{+(i_1,\dots,i_{K-2})}(y_{K-1}) \mathbf{1}_{\{y_{K-1} > y_{K-2}\}} dS_{i_{K-1}}^{+(i_1,\dots,i_{K-2})}(y_{K-1}) \\
& \times \prod_{q \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-2}\}} \frac{S_q^{+(i_1,\dots,i_{K-3})}(y_{K-2})}{S_q^{+(i_1,\dots,i_{K-2})}(y_{K-2})} \mathbf{1}_{\{y_{K-2} > y_{K-3}\}} dS_{i_{K-1}}^{+(i_1,\dots,i_{K-3})}(y_{K-2}) \\
& \cdots \\
& \times \prod_{q \in \{1,\dots,K\} \setminus \{i_1\}} \frac{S_q(y_1)}{S_q^{+i_1}(y_1)} \mathbf{1}_{\{y_1 > 0\}} dS_{i_1}(y_1).
\end{aligned} \tag{5.1}$$

The proof of Theorem 5.1 is deferred from the appendix.

In the following theorem, we consider the integral $IS_{M(K)}(t)$ for arbitrary $K \geq 2$, from which we can arrive at the MRL function $\mu_{M(K)}(t)$ via the equation $\mu_{M(K)}(t) = IS_{M(K)}(t)/S_{M(K)}(t)$.

Theorem 5.2. *For every $K \geq 2$, we have $IS_{M(K)}(t) = IS_{M(K)}^{*}(t) + IS_{M(K)}^{**}(t)$, where*

$$\begin{aligned}
IS_{M(K)}^{*}(t) = (-1)^{K} \sum_{i_1 \in \{1,\dots,K\}} \cdots \sum_{i_{K-1} \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-2}\}} & \\
& \int \cdots \int (x - \max(y_{K-1}, t))_{+} dS_{\text{NOT}(i_1,\dots,i_{K-1})}^{+(i_1,\dots,i_{K-1})}(x) \\
& \times \prod_{q \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-1}\}} \frac{S_q^{+(i_1,\dots,i_{K-2})}(y_{K-1})}{S_q^{+(i_1,\dots,i_{K-1})}(y_{K-1})} \mathbf{1}_{\{y_{K-1} > y_{K-2}\}} dS_{i_{K-1}}^{+(i_1,\dots,i_{K-2})}(y_{K-1}) \\
& \cdots \\
& \times \prod_{q \in \{1,\dots,K\} \setminus \{i_1\}} \frac{S_q(y_1)}{S_q^{+i_1}(y_1)} \mathbf{1}_{\{y_1 > 0\}} dS_{i_1}(y_1),
\end{aligned}$$

$$IS^{**}_{M(K)}(t) = (-1)^{K-1} \sum_{i_1 \in \{1,\dots,K\}} \cdots \sum_{i_{K-1} \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-2}\}}$$
$$\int \cdots \int (y_{K-1} - t)_+ S^{+(i_1,\dots,i_{K-2})}_{\text{NOT}(i_1,\dots,i_{K-1})}(y_{K-1}) \mathbf{1}_{\{y_{K-1} > y_{K-2}\}} dS^{+(i_1,\dots,i_{K-2})}_{i_{K-1}}(y_{K-1})$$
$$\times \prod_{q \in \{1,\dots,K\} \setminus \{i_1,\dots,i_{K-2}\}} \frac{S^{+(i_1,\dots,i_{K-3})}_q(y_{K-2})}{S^{+(i_1,\dots,i_{K-2})}_q(y_{K-2})} \mathbf{1}_{\{y_{K-2} > y_{K-3}\}} dS^{+(i_1,\dots,i_{K-3})}_{i_{K-1}}(y_{K-2})$$
$$\cdots$$
$$\times \prod_{q \in \{1,\dots,K\} \setminus \{i_1\}} \frac{S_q(y_1)}{S^{+i_1}_q(y_1)} \mathbf{1}_{\{y_1 > 0\}} dS_{i_1}(y_1). \tag{5.2}$$

The proof of Theorem 5.2 is again deferred to the appendix. We have by now established all the necessary formulas to derive the MRL function $\mu_{M(K)}(t)$ via original and increased loads of individual components.

Explicit formulae for Theorems 5.1 and 5.2 in the case of three and four components are available from the authors. The case $K = 4$ features prominently in our motivating examples in Section 1.

6. Summary

In this paper, we argue that reliability of modules with load-sharing components can be expressed in terms of the reliabilities of individual components exposed to various levels of load (normal and increased). This is of practical interest since the reliability of individual components can be conveniently estimated in a laboratory environment using either a natural aging regime (if time permits) or employing, for example, a quantitative accelerated life testing technique (cf., e.g., Nelson [14]). Hence we have derived equations expressing the module's survival, and thus HR and MRL, functions in terms of the survival functions of individual components. We have also discussed parametric and nonparametric inference for the latter functions, or their parameters if a parametric model has been assumed, under various load-sharing scenarios and data gathering regimes.

Appendix

A. Proofs

Proof of Theorem 5.1. We start calculating the survival function $S_{M(K)}(t)$ using first conditioning and then the formula of total probability. Hence

$$S_{M(K)}(t) = \mathbf{E}\Big[\mathbf{P}\Big\{T^{(K-1)}_{\kappa(K-1)} > t \mid D^{(0)},\dots,D^{(K-2)}, T^{(K-2)}_{1:2}\Big\}\Big]$$
$$= \mathbf{E}\left[\exp\left\{-\mathbf{1}_{\{T^{(K-2)}_{D(K-2)} \le t\}} \int_{T^{(K-2)}_{D(K-2)}}^{t} \left(h_{\kappa(K-1)}(x) + \sum_{m=1}^{K-1} a^{(m)}_{D^{(m-1)},\kappa(K-1)}(x)\right) dx\right\}\right]$$

$$
\begin{aligned}
&= \sum_{i_1\in\{1,\dots,K\}} \sum_{i_2\in\{1,\dots,K\}\setminus\{i_1\}} \cdots \sum_{i_{K-1}\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}} \\
&\quad \mathbf{E}\Bigg[\exp\Bigg\{-\mathbf{1}_{\{T_{i_{K-1}}^{(K-2)}\le t\}}\int_{T_{i_{K-1}}^{(K-2)}}^{t}\Bigg(h_{i_K}(x)+\sum_{m=1}^{K-1}a_{i_m,i_K}^{(m)}(x)\Bigg)dx\Bigg\} \\
&\qquad \times \mathbf{1}_{\{D^{(0)}=i_1\}}\cdots\mathbf{1}_{\{D^{(K-3)}=i_{K-2}\}}\mathbf{1}_{\{D^{(K-2)}=i_{K-1}\}}\Bigg],
\end{aligned}
\tag{A.1}
$$

where i_K is the (only) member of the singleton set $\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-1}\}$. Given $D^{(0)}=i_1,\dots,D^{(K-3)}=i_{K-2}$, the event $D^{(K-2)}=i_{K-1}$ is equivalent to $T_{i_{K-1}}^{(K-2)}<T_{i_K}^{(K-2)}$. By construction, the latter two random variables are independent. Hence we calculate the conditional expectation of $\mathbf{1}_{\{D^{(K-2)}=i_{K-1}\}}$ by first writing

$$
\begin{aligned}
&\mathbf{P}\Big\{T_{i_K}^{(K-2)}>t \mid D^{(0)}=i_1,\dots,D^{(K-3)}=i_{K-2}\Big\} \\
&\quad = \exp\Bigg\{-\mathbf{1}_{\{T_{i_{K-2}}^{(K-3)}\le t\}}\int_{T_{i_{K-2}}^{(K-3)}}^{t}\Bigg(h_{i_K}(x)+\sum_{m=1}^{K-2}a_{i_m,i_K}^{(m)}(x)\Bigg)dx\Bigg\}.
\end{aligned}
\tag{A.2}
$$

Next, we use (A.2) with $t=T_{i_{K-1}}^{(K-2)}$ to get the desired probability of the event $T_{i_{K-1}}^{(K-2)}<T_{i_K}^{(K-2)}$. This, together with (A.1), gives

$$
\begin{aligned}
S_{M(K)}(t) &= \sum_{i_1\in\{1,\dots,K\}} \sum_{i_2\in\{1,\dots,K\}\setminus\{i_1\}} \cdots \sum_{i_{K-1}\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}} \\
&\quad \mathbf{E}\Bigg[\exp\Bigg\{-\mathbf{1}_{\{T_{i_{K-1}}^{(K-2)}\le t\}}\int_{T_{i_{K-1}}^{(K-2)}}^{t}\Bigg(h_{i_K}(x)+\sum_{m=1}^{K-1}a_{i_m,i_K}^{(m)}(x)\Bigg)dx\Bigg\} \\
&\qquad \times \exp\Bigg\{-\mathbf{1}_{\{T_{i_{K-2}}^{(K-3)}\le T_{i_{K-1}}^{(K-2)}\}}\int_{T_{i_{K-2}}^{(K-3)}}^{T_{i_{K-1}}^{(K-2)}}\Bigg(h_{i_K}(x)+\sum_{m=1}^{K-2}a_{i_m,i_K}^{(m)}(x)\Bigg)dx\Bigg\} \\
&\qquad \times \mathbf{1}_{\{D^{(0)}=i_1\}}\cdots\mathbf{1}_{\{D^{(K-3)}=i_{K-2}\}}\Bigg].
\end{aligned}
\tag{A.3}
$$

Our next step is to integrate the expression inside $\mathbf{E}[\cdots]$ on the right-hand side of (A.3) with respect to the random variable $T_{i_{K-1}}^{(K-2)}$, for which we need to derive the survival function. Analogously to (A.2), we have that

$$
\begin{aligned}
&\mathbf{P}\Big\{T_{i_{K-1}}^{(K-2)}>t \mid D^{(0)}=i_1,\dots,D^{(K-3)}=i_{K-2}\Big\} \\
&\quad = \exp\Bigg\{-\int_{0}^{t}\Bigg(h_{i_{K-1}}(x)+\sum_{m=1}^{K-2}a_{i_m,i_{K-1}}^{(m)}(x)\Bigg)\mathbf{1}_{\{T_{i_{K-2}}^{(K-3)}\le x\}}dx\Bigg\}.
\end{aligned}
\tag{A.4}
$$

Using the latter equation on the right-hand side of (A.3), we have that

$$S_{M(K)}(t) = \sum_{i_1\in\{1,\dots,K\}} \sum_{i_2\in\{1,\dots,K\}\setminus\{i_1\}} \cdots \sum_{i_{K-1}\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}}$$
$$\mathbf{E}\Bigg[\int_{T^{(K-3)}_{i_{K-2}}}^{\infty} \exp\bigg\{-\mathbf{1}_{\{y_{K-1}\le t\}}\int_{y_{K-1}}^{t}\Big(h_{i_K}(x)+\sum_{m=1}^{K-1}a^{(m)}_{i_m,i_K}(x)\Big)dx\bigg\}$$
$$\times\exp\bigg\{-\sum_{q\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}}\int_{T^{(K-3)}_{i_{K-2}}}^{y_{K-1}}\Big(h_q(x)+\sum_{m=1}^{K-2}a^{(m)}_{i_m,q}(x)\Big)dx\bigg\}$$
$$\times\Big(h_{i_{K-1}}(y_{K-1})+\sum_{m=1}^{K-2}a^{(m)}_{i_m,i_{K-1}}(y_{K-1})\Big)dy_{K-1}\mathbf{1}_{\{D^{(0)}=i_1\}}\cdots\mathbf{1}_{\{D^{(K-3)}=i_{K-2}\}}\Bigg]. \tag{A.5}$$

Comparing the latter equation with (A.1), we see that we have "eliminated" the indicator $\mathbf{1}_{\{D^{(K-2)}=i_{K-1}\}}$. Continuing the above arguments until the last indicator $\mathbf{1}_{\{D^{(0)}=i_1\}}$ is "eliminated," we arrive at

$$S_{M(K)}(t) = \sum_{i_1\in\{1,\dots,K\}} \sum_{i_2\in\{1,\dots,K\}\setminus\{i_1\}} \cdots \sum_{i_{K-1}\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}} \int_0^\infty\int_{y_1}^\infty\cdots\int_{y_{K-2}}^\infty$$
$$\exp\bigg\{-\mathbf{1}_{\{y_{K-1}\le t\}}\sum_{q\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-1}\}}\int_{y_{K-1}}^{t}\Big(h_q(x)+\sum_{m=1}^{K-1}a^{(m)}_{i_m,q}(x)\Big)dx\bigg\}$$
$$\times\exp\bigg\{-\sum_{q\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}}\int_{y_{K-2}}^{y_{K-1}}\Big(h_q(x)+\sum_{m=1}^{K-2}a^{(m)}_{i_m,q}(x)\Big)dx\bigg\}$$
$$\times\Big(h_{i_{K-1}}(y_{K-1})+\sum_{m=1}^{K-2}a^{(m)}_{i_m,i_{K-1}}(y_{K-1})\Big)dy_{K-1}$$
$$\cdots$$
$$\times\exp\bigg\{-\sum_{q\in\{1,\dots,K\}\setminus\{i_1\}}\int_{y_1}^{y_2}\Big(h_q(x)+a^{(1)}_{i_1,q}(x)\Big)dx\bigg\}\Big(h_{i_2}(y_2)+a^{(1)}_{i_1,i_2}(y_2)\Big)dy_2$$
$$\times\exp\bigg\{-\sum_{q\in\{1,\dots,K\}}\int_0^{y_1}h_q(x)dx\bigg\}h_{i_1}(y_1)dy_1. \tag{A.6}$$

We will next modify the last $K-1$ exponents in (A.6). We start with

$$\exp\bigg\{-\sum_{q\in\{1,\dots,K\}}\int_0^{y_1}h_q(x)dx\bigg\}h_{i_1}(y_1)dy_1 = -\exp\bigg\{-\sum_{q\in\{1,\dots,K\}\setminus\{i_1\}}\int_0^{y_1}h_q(x)dx\bigg\}dS_{i_1}(y_1). \tag{A.7}$$

We now combine the exponent on the right-hand side of (A.7) with the penultimate exponent in (A.6). The last two lines of (A.6) become

$$\begin{aligned}&\cdots\times\exp\left\{-\sum_{q\in\{1,\dots,K\}\setminus\{i_1\}}\int_0^{y_2}\left(h_q(x)+a_{i_1,q}^{(1)}(x)\right)dx\right\}\left(h_{i_2}(y_2)+a_{i_1,i_2}^{(1)}(y_2)\right)dy_2\\&\times(-1)\exp\left\{\sum_{q\in\{1,\dots,K\}\setminus\{i_1\}}\int_0^{y_1}a_{i_1,q}^{(1)}(x)dx\right\}dS_{i_1}(y_1),\end{aligned}\tag{A.8}$$

which can be rewritten as

$$\begin{aligned}&\cdots\times(-1)\exp\left\{-\sum_{q\in\{1,\dots,K\}\setminus\{i_1,i_2\}}\int_0^{y_2}\left(h_q(x)+a_{i_1,q}^{(1)}(x)\right)dx\right\}dS_{i_2}^{+i_1}(y_2)\\&\times(-1)\exp\left\{\sum_{q\in\{1,\dots,K\}\setminus\{i_1\}}\int_0^{y_1}a_{i_1,q}^{(1)}(x)dx\right\}dS_{i_1}(y_1).\end{aligned}\tag{A.9}$$

We continue with these arguments and arrive at

$$\begin{aligned}S_{M(K)}(t)=&\sum_{i_1\in\{1,\dots,K\}}\cdots\sum_{i_{K-1}\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}}\int_0^\infty\cdots\int_{y_{K-2}}^\infty\\&\exp\left\{-\mathbf{1}_{\{y_{K-1}\le t\}}\sum_{q\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-1}\}}\int_{y_{K-1}}^{t}\left(h_q(x)+\sum_{m=1}^{K-1}a_{i_m,q}^{(m)}(x)\right)dx\right\}\\&\times(-1)\exp\left\{-\sum_{q\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-1}\}}\int_0^{y_{K-1}}\left(h_q(x)+\sum_{m=1}^{K-2}a_{i_m,q}^{(m)}(x)\right)dx\right\}\\&\times dS_{i_{K-1}}^{+(i_1,\dots,i_{K-2})}(y_{K-1})\\&\cdots\\&\times(-1)\exp\left\{\sum_{q\in\{1,\dots,K\}\setminus\{i_1\}}\int_0^{y_1}a_{i_1,q}^{(1)}(x)dx\right\}dS_{i_1}(y_1).\end{aligned}\tag{A.10}$$

Next, we write $S_{M(K)}(t)=S_{M(K)}^{*}(t)+S_{M(K)}^{**}(t)$, where

$$\begin{aligned}S_{M(K)}^{*}(t)=(-1)^{K-1}&\sum_{i_1\in\{1,\dots,K\}}\cdots\sum_{i_{K-1}\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}}S_{\mathrm{NOT}(i_1,\dots,i_{K-1})}^{+(i_1,\dots,i_{K-1})}(t)\int_0^\infty\cdots\int_{y_{K-2}}^\infty\mathbf{1}_{\{y_{K-1}\le t\}}\\&\times\exp\left\{\sum_{q\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-1}\}}\int_0^{y_{K-1}}a_{i_m,q}^{(K-1)}(x)dx\right\}dS_{i_{K-1}}^{+(i_1,\dots,i_{K-2})}(y_{K-1})\\&\cdots\\&\times\exp\left\{\sum_{q\in\{1,\dots,K\}\setminus\{i_1\}}\int_0^{y_1}a_{i_1,q}^{(1)}(x)dx\right\}dS_{i_1}(y_1)\end{aligned}$$

$$S^{**}_{M(K)}(t) = (-1)^{K-1} \sum_{i_1\in\{1,\dots,K\}} \cdots \sum_{i_{K-1}\in\{1,\dots,K\}\setminus\{i_1,\dots,i_{K-2}\}}$$
$$\int_0^\infty \int_{y_1}^\infty \cdots \int_{y_{K-2}}^\infty \mathbf{1}_{\{y_{K-1}>t\}} S^{+(i_1,\dots,i_{K-2})}_{\mathrm{NOT}(i_1,\dots,i_{K-1})}(y_{K-1})\, dS^{+(i_1,\dots,i_{K-2})}_{i_{K-1}}(y_{K-1})$$
$$\cdots$$
$$\times \exp\left\{ \sum_{q\in\{1,\dots,K\}\setminus\{i_1\}} \int_0^{y_1} a^{(1)}_{i_1,q}(x)dx \right\} dS_{i_1}(y_1). \tag{A.11}$$

Write $a^{(1)}_{i_1,q}(x)$ as the sum of $h_q(x) + a^{(1)}_{i_1,q}(x)$ and $-h_g(x)$, which shows that the rightmost exponent in (A.11) can be written as the ratio $S_q(y_1)/S_q^{+i_1}(y_1)$. Similarly, we have the equations

$$\exp\left\{ \int_0^{y_2} a^{(2)}_{i_2,q}(x)dx \right\} = \frac{S_q^{+i_1}(y_2)}{S_q^{+(i_1,i_2)}(y_2)}, \dots, \exp\left\{ \int_0^{y_{K-1}} a^{(K-1)}_{i_2,q}(x)dx \right\} = \frac{S_q^{+(i_1,\dots,i_{K-2})}(y_{K-1})}{S_q^{+(i_1,\dots,i_{K-1})}(y_{K-1})}. \tag{A.12}$$

Theorem 5.1 follows. □

Proof of Theorem 3.1. This is a consequence of Theorem 5.1 and the observation that the product $S_1(t)S_2(t)$ is equal to $-\sum_{i=1}^2 \int_t^\infty S_{\mathrm{NOT}(i)}(y)dS_i(y)$, which appears in the result of Theorem 5.1 when $K = 2$. □

Proof of Theorem 5.2. For any random variable X, whose survival function we denote by $S_X(t)$, the integral $\int_t^\infty S_X(x)\mathbf{1}_{\{z\le x\}}dx$ is equal to the expectation $\mathbf{E}[(X - \max(z,t))_+]$, which is of course equal to $-\int_0^\infty (x - \max(z,t))_+ dS_X(x)$. Furthermore, $\int_t^\infty \mathbf{1}_{\{y>x\}}dx$ is equal to $(y - t)_+$. These observations and (3.1) complete the proof of Theorem 5.2. □

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