Theory and Methodology

A dual algorithm for the economic lot-sizing problem

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Abstract: A linear description for the economic lot-sizing problem consisting of exponentially many linear inequalities was given by Barany, Van Roy and Wolsey in 1984. Using this formulation we present a dual algorithm for the economic lot-sizing problem, which is of the same complexity as the Wagner and Whitin dynamic programming algorithm. Besides its use in sensitivity analysis the dual algorithm also provides an alternative proof of the fact that the linear description is complete.

1. Introduction

We consider a manufacturer who needs to meet the known demands for a given product over a finite discrete planning horizon. In *lot-sizing problems* we have to decide when, i.e., in which time periods, and how much to produce so as to minimize total costs consisting of production and inventory costs. The inventory costs are linear in the number of items in stock at the end of each time period. The production costs decompose into two parts: a fixed set-up cost is incurred whenever a non-zero production occurs in a period in addition to a cost linear in the number of items produced. Producing in large (small) lots will decrease (increase) the set-up costs but will increase (decrease) the inventory costs. The lot-sizing problem is to find the right balance between these costs so as to minimize the total costs. We can introduce complicating factors in the lot-sizing problem by allowing backlogging (i.e., we can satisfy the demand of a period in a later period), by introducing capacities (i.e., the production level in each period is limited) or product structures (i.e., the production process includes several components that are assembled according to a specified bill of material).

We briefly discuss some results obtained for lot-sizing problems. The first and simplest model is the *economic lot-sizing model*. In this model we produce one single item, the production is uncapacitated and backlogging is not allowed. This is the lot-sizing problem we are considering in this paper. A dynamic programming algorithm was given by Wagner and Whitin (1959). Barany et al. (1984) gave a linear

description of the convex hull of the set feasible solution. This linear programming formulation, described in more detail in Section 2, will be used in this paper.

Zangwill (1969) solves the backlogging extension of the economic lot-sizing model and Pochet (1987) gives a partial linear description of the convex hull of the set of feasible solutions in this case. Zangwill (1966) and Love (1972) solve the multi-echelon extension (i.e., a number of facilities arranged in series where demand occurs only at the last facility and it is possible to stock items at any stage of the production structure). Florian and Klein (1971) solve the constant capacitated production extension both in the case of with and without backlogging. Extensions to concave costs have been considered by Erickson et al. (1985). The problem of computing optimal lot-sizes in complex production structures is NP-complete even for relatively simple production structures Florian et al., 1980). Algorithms have been developed by Crowston et al. (1973), Crowston and Wagner (1973), Afentakis et al. (1984), Afentakis and Gavish (1986), and Maxwell and Muckstadt (1985).

In Section 2 we will describe the linear programming formulation of the economic lot-sizing problem given by Barany et al. (1984).

In Sections 3 and 4 we present a dual algorithm for the economic lot-sizing problem based on this linear description. Section 3 deals with the construction of a feasible solution to the dual of the linear description, and in Section 4 we show how to construct a feasible solution to the economic lot-sizing problem having the same objective value as the dual solution. This not only implies the optimality of both solutions, but also provides an algorithmic proof of the fact that the constraints of the linear programming formulation describe the convex hull of the set of feasible solutions of the economic lot-sizing problem.

We believe that this type of constructive proof for the completeness of a linear description can be applied to other problems as well. The application to the linear description of the economic lot-sizing problem with start-up costs given by Wolsey (1988) resulted in an example which showed that the description was not complete. In Van Hoesel et al. (1990) a complete linear description is given as well as a dual algorithm along the lines described here.

The dual algorithm for the economic lot-sizing problem was developed in the context of a research program devoted to sensitivity analysis for combinatorial optimization problems. However it turned out that sensitivity analysis is more efficient using the simple plant location formulation of the economic lot-sizing problem. The sensitivity analysis results are described in Van Hoesel et al. (1990).

While obtaining the sensitivity results for the economic lot-sizing problem we also develop a new $O(n \log n)$ algorithm which runs in $O(n)$ in the Wagner-Whitin case, i.e., whenever $p_i = p$, $i = 1, \ldots, n$ Wagelmans et al., 1989).

2. Linear description economic lot-sizing problem

The economic lot-sizing problem can be formulated as

(1)   minimize $\sum_{i=1}^{n} p_i x_i + f_i y_i + h_i s_i$

      subject to $x_i + s_{i-1} = d_i + s_i, \quad i = 1, \ldots, n,$ (2.1)

                    $s_0 = s_n = 0,$ (2.2)

                    $x_i > 0 \Longrightarrow y_i = 1, \quad i = 1, \ldots, n,$ (2.3)

                    $x_i \geqslant 0, \quad s_i \geqslant 0, \quad y_i \in \{0, 1\}, \qquad i = 1, \ldots, n,$ (2.4)

where n is the number of time periods in the planning horizon, f_i the set-up cost in period i, p_i the production cost per item in period i, h_i the inventory cost per item in stock at the end of period i and d_i the positive demand for the item in period i, $i = 1, \ldots, n$. The variable y_i indicates whether we set up production in period i ($y_i = 1$) or not ($y_i = 0$), the variable x_i is the amount produced in period i and s_i is the amount in inventory at the end of period i, $i = 1, \ldots, n$. Constraint (2.3) garantuees that a set-up cost
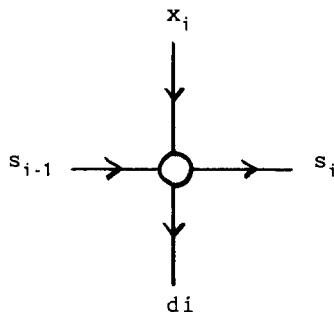
Figure 1. Balancing situation for period i

is incurred whenever production is started in a period. Note that we may decide to set up production without starting production. This occurs for example when $f_i < 0$. Constraint (2.1) is a balancing constraint which tells us that the total amount of items coming into period i (production in period i and inventory from period $i - 1$) equals the amount of items leaving period i (demand in period i and inventory at the end of period i). This is indicated in Figure 1.

Barany et al. (1984) derived a class of valid inequalities for the economic lot-sizing problem, the so called (S, m)-inequalities. Let $m \in (1, \ldots, n)$ and $S \subseteq \{1, \ldots, m\}$. Then the (S, m)-inequality is given by

$$\sum_{i \in \{1, \ldots, m\} \setminus S} x_i \leqslant \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im} y_i + s_m,$$

where d_{ij} is defined to be the sum of demands in period $i, i + 1, \ldots, j$, i.e. $d_{ij} = \sum_{k=i}^{j} d_k$. To see that this is indeed a valid inequality consider the balancing constraint $\sum_{k=i}^{m} x_i = \sum_{i=1}^{m} d_i + s_m$ for the first m periods (see Figure 2.)

This balancing equation can be obtained by adding the first m equation of (2.1) taking into account that $s_0 = 0$. Since backlogging is not allowed the production x_i can only contribute to the demand in periods $i, i + 1, \ldots, m$ (i.e., d_{im}) and to s_m. Let α_i be the fraction of the demand d_{im} and β_i the fraction of s_m supplied by x_i. Then we have

$$x_i = d_{im} \alpha_i + s_m \beta_i, \quad i = 1, \ldots, m,$$

with $0 \leqslant \alpha_i \leqslant 1$, $0 \leqslant \beta_i \leqslant 1$, $i = 1, \ldots, m$ and $\sum_{i=1}^{m} \beta_i = 1$.

Hence

$$\sum_{i \in \{1, \ldots, m\} \setminus S} x_i = \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im} \alpha_i + \sum_{i \in \{1, \ldots, m\} \setminus S} s_m \beta i.$$
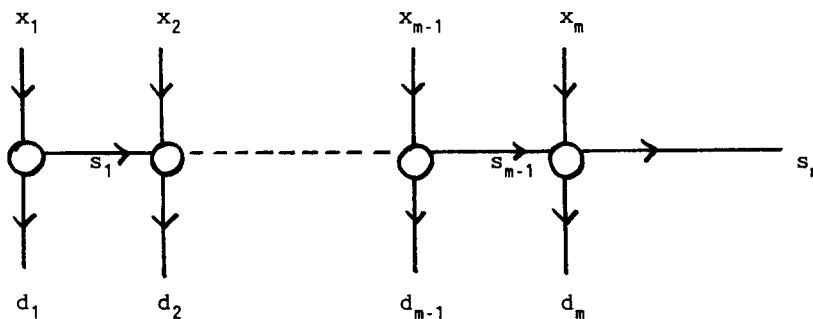

Figure 2. Balancing situation for periods 1 to m

Since $\sum_{i=1}^{m} \beta_i = 1$, $0 \leqslant \beta_i \leqslant 1$, $i = 1, \ldots, m$,

$$\sum_{i \in \{1, \ldots, m\} \setminus S} x_i \leqslant \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im}\alpha_i + s_m.$$

If $x_i = 0$, then $\alpha_i = 0 \leqslant y_i$. If $x_i > 0$, then $y_i = 1$ according to (2.3) and hence $\alpha_i \leqslant y_i$. We conclude that for every feasible solution x, y of problem formulation (1) $x_i \geqslant 0$, then $\alpha_i \leqslant y_i$. Therefore the (S, m) inequality

$$\sum_{i \in \{1, \ldots, m\} \setminus S} x_i \leqslant \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im}y_i + s_m,$$

is valid.

Note that when $m = n$ and $S = \{1, \ldots, n\} \setminus \{i\}$ the (S, m)-inequality becomes $x_i \leqslant d_{in}y_i$. This constraint garantuees that $y_i = 1$ whenever $x_i > 0$. Therefore, when the (S, m)-inequalities are added to formulation (1), constraints (2.3) are redundant. We can eliminate the inventory variables s_i, $i = 1, \ldots, n$, from the model using the balancing constraints (2.1) by substituting $s_i = \sum_{k=1}^{i} x_k - \sum_{k=1}^{i} d_k$. The (S, m)-inequality is transformed into

$$\sum_{i \in S} x_i + \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im}y_i \geqslant d_{1m}.$$

The equality $s_n = 0$ is transformed into

$$\sum_{i=1}^{n} x_i = d_{1n}. \tag{2.5}$$

The inequality $s_t \geqslant 0$ is transformed into

$$\sum_{i=1}^{t} x_i \geqslant d_{1t}. \tag{2.6}$$

Adding all the (S, m)-inequalities to formulation (1) and eliminating the inventory variables we get the following mixed integer programming formulations of the economic lot-sizing problem;

(2)   minimize $\sum_{i=1}^{n} c_i x_i + f_i y_i$

     subject to $\sum_{i=1}^{n} x_i \leqslant d_{1n},$ \hfill (2.7)

$$\sum_{i \in S} x_i + \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im}y_i \geqslant d_{1m}, \quad m = 1, \ldots, n, \ S \subseteq \{1, \ldots, m\}, \tag{2.8}$$

$$x_i \geqslant 0, \quad i = 1, \ldots, n,$$

$$y_i \in \{0, 1\}, \quad i = 1, \ldots, n,$$

where $c_i = p_i + \sum_{k=i}^{n} h_k$, $i = 1, \ldots, n$.

Constraint (2.5) is satisfied by combining (2.7) with the $(\{1, \ldots, n\}, n)$-inequality. Constraint (2.6) is equivalent to the $(\{1, \ldots, t\}, t)$-inequality.

In the following section we will construct a feasible solution to the dual of the LP-relaxation (P) of formulation (2). In Section 4 we will construct an integer feasible solution of (P) which together with the dual feasible solution satisfies the complementary slackness relations for (P), thereby proving optimality of both solutions. The combination of both construction methods provides a dual algorithm for solving the economic lot-sizing problem.

3. Construction of the dual solution

In this section we describe a construction method for a feasible solution to te dual problem of the LP-relaxation of the economic lot-sizing formulation (2). The primal linear programming problem is given by

$$(P) \qquad \text{minimize} \quad \sum_{i=1}^{n} c_i x_i + f_i y_i$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i \leqslant d_{1n}, \tag{3.1}$$

$$y_i \leqslant 1, \quad i = 1, \ldots, n, \tag{3.2}$$

$$\sum_{i \in S} x_i + \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im} y_i \geqslant d_{1m}, \quad m = 1, \ldots, n, \quad S \subseteq \{1, \ldots, m\}, \tag{3.3}$$

$$x_i, \ y_i \geqslant 0, \quad i = 1, \ldots, n.$$

If we let the dual variable μ corresponds to (3.1), λ_i to (3.2) and $w(m, S)$ to (3.3), then the dual problem (D) of (P) is given y

$$(D) \qquad \text{maximize} \quad d_{1n}\mu + \sum_{i=1}^{n} \lambda_i + \sum_{m=1}^{n} \sum_{S \subseteq \{1, \ldots, m\}} d_{1m} w(m, S)$$

$$\text{subject to} \quad \mu + \sum_{m=i}^{n} \sum_{S \subseteq \{1, \ldots, m\}, \, i \in S} w(m, S) \leqslant c_i, \quad i = 1, \ldots, n, \tag{3.4}$$

$$\lambda_i + \sum_{m=i}^{n} \sum_{S \subseteq \{1, \ldots, m\}, \, i \notin S} d_{im} w(m, S) \leqslant f_i, \quad i = 1, \ldots, n, \tag{3.5}$$

$$\lambda_i \leqslant 0, \quad i = 1, \ldots, n,$$

$$\mu \leqslant 0,$$

$$w(m, S) \geqslant 0, \quad m = 1, \ldots, n, \quad S \subseteq \{1, \ldots, m\}.$$

The algorithm we will describe for (D) is of the greedy type. We start with $w(m, S) = 0$ for all m and S. During the algorithm, c_i^* and f_i^* will denote the slack in inequality i of (3.4) and (3.5) respectively. In the first step μ and λ_i, $i = 1, \ldots, n$, take on the largest value such that all slacks are nonnegative, i.e.

$$\mu = \min\left\{0, \min_{i=1, \ldots, n} c_i\right\},$$

$$\lambda_i = \min\{0, f_i\}, \quad i = 1, \ldots, n,$$

and the slacks are defined by $c_i^* = c_i - \mu$, $f_i^* = f_i - \lambda_i$, $i = 1, \ldots, n$.

A greedy algorithm with respect to the $w(m, S)$-variables will calculate them in decreasing order of objective coefficient. Note that $d_{1m} > d_{1k}$ if and only if $m > k$.

To motivate the greedy algorithm suppose that for a given m we want to know if there exists a subset $S \subseteq \{1, \ldots, m\}$ for which $w(m, S)$ can take on a positive value. According to (3.4) and (3.5) $w(m, S)$ has to satisfy the constraints

$$w(m, S) \leqslant c_i^*, \quad i \in S, \tag{3.6}$$

and

$$d_{im} w(m, S) \leqslant f_i^*, \quad i \in \{1, \ldots, m\} \setminus S. \tag{3.7}$$

Note that for a given i, $i \leqslant m$, $w(m, S)$ occurs in exactly one of the constraints (3.6) and (3.7) depending on whether $i \in S$ or $i \in \{1, \ldots, m\} \setminus S$.

It follows from (3.6) that $w(m, S) = 0$ if $c_i^* = 0$ and $i \in S$. Therefore, there exists a subset S for which $w(m, S)$ can be made positive only if $S \subseteq S^*$, where $S^* = \{i \mid c_i^* > 0, \ i = 1, \ldots, m\}$.

If we restrict S to subsets of S^*, then it follows from (3.7) that $w(m, S)$ can be made positive if and only if $w(m, S^*)$ can be made positive. The reason is that $w(m, S)$ with $S \subseteq S^*$ has to satisfy all the constraints of the form (3.7) $w(m, S^*)$ has to satisfy.

The value $w(m, S^*)$ can be made positive if and only if there is no $i \in \{1, \ldots, m\} \setminus S^*$ for which $f_i^* = 0$, i.e. there is no i, $i \leqslant m$, for which $c_i^* = f_i^* = 0$.

If $w(m, S^*)$ can be made positive, then the greedy algorithm will give it the largest possible value. So the subset S for which $w(m, S)$ will be made positive is always equal to the subset of $\{1, \ldots, m\}$ corresponding to positive slacks in (3.4). This motivates the definition of S in Step 2 of the algorithm. The largest possible value according to (3.6) is given by

$$\min_{i \in S^*} c_i^*.$$

The largest possible value according to (3.7) is given by

$$\min_{i \in \{1, \ldots, m\} \setminus S^*} f_i^* / d_{im}.$$

The greedy algorithm gives $w(m, S^*)$ the value

$$w(m, S^*) = \min\left\{ \min_{i \in S^*} c_i^*, \ \min_{i \in \{i, \ldots, m\} \setminus S^*} f_i^* / d_{im} \right\}.$$

After the calculation of $w(m, S^*)$ the slacks c_i^* and f_i^* and the set S^* are updated and the procedure is repeated until there no longer exists a subset S for which $w(m, S)$ can be made positive. Let us consider this case more carefully. Assume there exists an i, $i \leqslant m$, for which $c_i^* = f_i^* = 0$. We have shown that in this case for m there does not exist a subset S for which $w(m, S)$ can be made positive. Since slacks are nonincreasing the same holds for all values m', $i \leqslant m' \leqslant m$, i.e. there does not exist a subset S for which $w(m', S)$ can be made positive. As a consequence we can immediately decrease the value of m to the largest value m' for which a subset exists for which $w(m', S)$ can be made positive, i.e.

$$m' = \min_{i: \ i \leqslant m, \ f_i^* = c_i^* = 0} (i - 1).$$

This motivates the update of m in Step 2 of Algorithm 1. The complete construction is summarized in Algorithm 1.

Algorithm 1. *Construction dual solution*
Step 1. $\mu := \min\{0, \min_{i=1, \ldots, n} c_i\}$;
$\quad \lambda_i := \min\{0, f_i\}, \ i = 1, \ldots, n$;
$\quad c_i^* := c_i - \mu, \ i = 1, \ldots, n$;
$\quad f_i^* := f_i - \lambda_i, \ i = 1, \ldots, n$;
$\quad w(m, S) := 0, \ m = 1, \ldots, n, \ S \subseteq \{1, \ldots, m\}$;
$\quad m := n$;
$\quad m := \min\{m, \min_{i: \ i \leqslant m, \ c_i^* = 0, \ f_i^* = 0}(i - 1)\}$.
Step 2. WHILE $m > 0$ DO
$\quad S := \{i \mid i \leqslant m, \ c_i^* > 0\}$;
$\quad w(m, S) = \min\{\min_{i \in S} c_i^*, \ \min_{i \in \{1, \ldots, m\} \setminus S} f_i^* / d_{im}\}$;
$\quad c_i^* := c_i^* - w(m, S) \quad$ for all $i \in S$;
$\quad f_i^* := f_i^* - d_{im} w(m, S)$ for all $i \in \{1, \ldots, m\} \setminus S$;
$\quad m := \min\{m, \min_{i: \ i \leqslant m, \ c_i^* = 0, \ f_i^* = 0}(i - 1)\}$.

Note that each time Step 2 is executed at least one of the slacks becomes zero. Therefore the number of iterations is bounded by $2n$ since we stop whenever $c_1^* = f_1^* = 0$, i.e. $m = 0$. The total complexity of the

construction method is $O(n^2)$ which is the same complexity as the dynamic programming algorithm to solve the economic lot-sizing problem.

By construction the dual solution is a feasible solution of (D). The following lemmas state some properties of the dual solution that we shall use in the next section.

Lemma 3.1. *Let* $0 = i_1 < i_2 < \cdots < i_k = n$ *be the different m-values occuring during the execution of Algorithm* 1 *and define* $I = \{i_s + 1 \mid s = 1, \ldots, k - 1\}$. *Then for every* $p \in I$ *algorithm* 1 *has terminated with* $c_p^* = f_p^* = 0$.

Proof. Take $p = i_s + 1$, $1 \leqslant s \leqslant k - 1$. Consider the iteration in which the m-value was decreased from i_{s+1} to i_s. We know that

$$ i_s = \min_{i: \, i \leqslant i_{s+1}, \, c_i^* = f_i^* = 0} (i - 1). $$

So by this definition and since $i_s + 1 = p$ we have $c_p^* = f_p^* = 0$. Since the slack variables are nonnegative and nonincreasing during the execution of the algorithm the result follows. $\square$

Let us define for every i for which $c_i^* = 0$ at the end of the execution of Algorithm 1 the value $m(i)$ as follows. If c_i^* was zero after Step 1, then $m(i) = n$. If c_i^* has become zero in Step 2, then $m(i)$ is the value of m at the beginning of the iteration in which c_i^* became zero.

Lemma 3.2. *If* $w(m^*, S) > 0$ *as a result of Algorithm* 1, *then* $m(i) < m^*$ *implies that* $i \in S$, *and* $m(i) > m^*$ *implies that* $i \notin S$.

Proof. From the definition of $m(i)$ it follows that during the iterations of Step 2 of Algorithm 1 in which $m = m^*$ we have $c_i^* > 0$ whenever $m(i) > m^*$ and $c_i^* = 0$ whenever $m(i) > m^*$. In the iteration in which $w(m^*, S)$ is determined S equals the set of indices for which the slack variables have a positive value. Combining these results proves the lemma. $\square$

The optimality of the dual solution is shown in the next section where we turn our attention to the construction of an optimal solution of the economic lot-sizing problem.

4. Construction of an optimal primal solution to the economic lot-sizing problem

In this section we show that it is always easy to construct a feasible solution to the economic lot-sizing problem which has the same value as the optimum value of (P). This provides an algorithmic proof of the fact that the feasible region of this LP-relaxation equals the convex hull of the set of feasible solutions of the economic lot-sizing problem; a result for which a rather elaborate proof was already given by Barany et al. (1984). To prove the optimality of a feasible solution to the economic lot-sizing problem, it is sufficient to show that this solution and the dual solution constructed in the previous section satisfy the complementary slackness relations of problems (P) and (D) which are given by

$$ x_i \left(c_i - \mu - \sum_{m=i}^{n} \sum_{S \subseteq \{1, \ldots, m\}, \, i \in S} w(m, S) \right) = 0, \quad i = 1, \ldots, n, \tag{4.1} $$

$$ y_i \left(f_i - \lambda_i - \sum_{m=i}^{n} \sum_{S \subseteq \{1, \ldots, m\}, \, i \notin S} d_{im} w(m, S) \right) = 0, \quad i = 1, \ldots, n, \tag{4.2} $$

$$ \mu \left(\sum_{i=1}^{n} x_i - d_{1n} \right) = 0, \tag{4.3} $$

$$\lambda_i(y_i - 1) = 0, \quad i = 1, \ldots, n, \tag{4.4}$$

$$w(m, S)\left(\sum_{i \in S} x_i + \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im} y_i - d_{1m} \right) = 0, \quad m = 1, \ldots, n, \quad S \subseteq \{1, \ldots, m\}. \tag{4.5}$$

We will show the existence of such a feasible solution to the economic lot-sizing problem with the so called *zero-inventory property*, i.e. a solution such that for all production periods the inventory at the end of the preceding time period is equal to zero. In other words, the production in a given production period equals exactly the sum of the demands from this period until the next production period. In the sequel we shall restrict ourselves to feasible solutions with the zero-inventory property. Furthermore for the moment we shall also confine ourselves to the case that all set-up costs are nonnegative. In this case we can assume without loss of generally that (4.6) holds:

$$y_i = 1 \quad \text{if and only if} \quad x_i > 0, \quad i = 1, \ldots, n. \tag{4.6}$$

Given Lemma 3.1 we see that the complementary slackness relations (4.1) and (4.2) are satisfied if we can find a solution for which the production periods form a subset of I defined in Lemma 3.1. To make a choice for the production periods such that the relations (4.5) are not violated we shall use the following lemma.

Lemma 4.1. *A solution to the economic lot-sizing problem satisfying* (4.6) *and having the zero inventory property satisfies*

$$\sum_{i \in S} x_i + \sum_{i \in \{1, \ldots, m\} \setminus S} d_{im} y_i = d_{1m} \tag{4.7}$$

for given m and S with $w(m, S) > 0$, if and only if the following conditions hold:

(i) *all production periods before the last production period in $\{1, \ldots, m\}$ are in S,*

(ii) *if $m < n$ and $m + 1$ is not a production period, then the last production period in $\{1, \ldots, m\}$ is not in S.*

Proof. Let (4.7) hold. Assume that condition (i) does not hold, and let p be the first production period in $\{1, \ldots, m\}$ which does not belong to S. If q is the last production period in $\{1, \ldots, m\}$, then it follows from our assumption that $p < q$. By the zero-inventory property we know that the total contribution to the left-hand side of (4.7) of production periods previous to p equals $d_{1, p-1}$. The contribution of period p equals $d_{pm} y_p = d_{pm}$. Therefore the total contribution of the first p periods equals d_{1m}. Since period q also makes a positive contribution independent of whether q belongs to S or not, the left-hand side of (4.7) exceeds d_{1m}, which leads to a contradiction. We conclude that condition (i) holds. Assume that condition (ii) does not hold, and let p be the last production period belonging to $\{1, \ldots, m\}$ and $q > m + 1$ the first production period not belonging to $\{1, \ldots, m\}$. If $p \in S$, then according to condition (i) all production periods in $\{1, \ldots, m\}$ belong to S and from the zero-inventory property it follows that the contribution of the left-hand side of (4.7) equals $d_{1, q-1} > d_{1m}$, which leads to contradiction. We conclude that condition (ii) holds.

Let us assume that conditions (i) and (ii) hold. Let p be the last production period in $\{1, \ldots, m\}$ and q the first production period not belonging to $\{1, \ldots, m\}$. By condition (i) the contribution of production periods previous to p equals $d_{1, p-1}$. If $q = m + 1$, then the contribution of production period p equals $x_p = d_{p, q-1} = d_{pm}$ whenever $p \in S$ or $d_{pm} y_p = d_{pm}$ whenever $p \notin S$. Hence (4.7) holds. If $q > m + 1$, then by condition (ii) $p \notin S$ and the contribution of production period p equals $d_{pm} y_p = d_{pm}$. Hence (4.7) holds. $\square$

The following lemma provides sufficient conditions to satisfy the conditions of Lemma 4.1. It uses the $m(i)$-variables defined in relationship with Lemma 3.2.

Lemma 4.2. *Consider a solution with the zero-inventory property to the economic lot-sizing problem in which* $x_i = d_{i,m(i)}$ *for every production period i. Then this solution satisfies conditions* (i) *and* (ii) *of Lemma* 4.1 *for every m and S with $w(m, S) > 0$.*

Proof. From Lemma 3.2 it follows that

$$\text{if } m(i) < m, \text{ then } i \in S \tag{4.8}$$

and

$$\text{if } m(i) > m, \text{ then } i \notin S. \tag{4.9}$$

If i_1 and i_2 are two consecutive production periods belonging to $\{1, \ldots, m\}$, then by the zero-inventory property and $x_i = d_{i,m(i)}$ we have $i_2 = m(i_1) + 1$. It follows that if p, $p \leqslant m$, is the last production period in $\{1, \ldots, m\}$, then $m(i) \leqslant p - 1 < m$ for all production periods i, $i < p$. Combining this with (4.8) proves that condition (i) holds. If q, $q > m + 1$, is the first production period not belonging to $\{1, \ldots, m\}$, then $m(p) = q - 1 > m$. Combining this with (4.9) proves that condition (ii) holds.   □

Lemma 4.2 suggests the following procedure to define the set I^* of production periods:

Algorithm 2. *Construction primal solution*
Step 1. $I^* := \{1\}$; $i := 1$.
Step 2. WHILE $m(i) < n$ DO
        BEGIN
        $i := m(i) + 1$;
        $I^* := I^* \cup \{i\}$
        END.

Note that by definition of $m(i)$ if $m(i) < n$ then $m(i) + 1 \in I$ where I is defined in Lemma 3.1, $i = 1, \ldots, n$. Therefore $I^* \subseteq I$ and the solution obtained satisfies the complementary slackness relations (4.1), (4.2) by Lemma 3.1; (4.3) by construction; and (4.5) according to Lemma 4.1 and Lemma 4.2. Under the assumption that $f_i \geqslant 0$ for all $i = 1, \ldots, m$, also (4.4) is satisfied, because then $\lambda_i = 0$, $i = 1, \ldots, m$, in the dual solution.

It remains to consider the case that there is an i with $f_i < 0$ and thus $\lambda_i < 0$. For such an i we take $y_i = 1$, thereby satisfying (4.4) and also (4.2) because $f_i^* = 0$ after Step 1 of Algorithm 1. This choice of y_i will not affect the relations (4.5), since $w(m, S)$ can not be positive whenever $i \in \{1, \ldots, m\} \setminus S$. To see this consider the iteration in which $w(m, S)$ would have been assigned a positive value. We know that $f_i^* = 0$ holds and $i \in \{1, \ldots, m\} \setminus S$ means $c_i^* = 0$. Therefore m is assigned a value less than or equal to $i - 1$ at the end of the previous iteration.

We have now proven:

Theorem 4.1. *For every objective function of* (P), *there exists an optimal solution which is a feasible (and hence optimal) solution that possesses the zero-inventory property to the corresponding economic lot-sizing problem.*   □

Corollary. *The constraints of the linear program* (P) *describe the convex hull of the set of feasible solutions of the economic lot-sizing problem.*   □

We now summarize our algorithm to solve the economic lot-sizing problem. The complexity of the algorithm is dominated by Step 2 and therefore the running time is $O(n^2)$.

Algorithm (Input: n; c, f, $d \in R^n$; Output: $x \in R^n$, $y \in \{0, 1\}^n$)
Step 1. $\mu := \min\{0, \min_{i=1,\ldots,n} c_i\}$;
    $\lambda_i := \min\{0, f_i\}$, $i = 1, \ldots, n$;

$c_i^* := c_i - \mu$; IF $c_i^* = 0$ THEN $m(i) = n$, $i = 1, 2, \ldots, n$;

$f_i^* := f_i - \lambda_i$, $i = 1, \ldots, n$;

$w(m, S) := 0$, $m = 1, \ldots, n$, $S \subseteq \{1, \ldots, m\}$;

$m := n$;

$m := \min\{m, \min_{i: \, i \leqslant m, \, c_i^* = 0, \, f_i^* = 0}(i - 1)\}$.

Step 2. WHILE $m > 0$ DO

$S := \{i \mid i \leqslant m, \, c_i^* > 0\}$;

$w(m, S) := \min\{\min_{i \in S} c_i^*, \, \min_{i \in \{1, \ldots, m\} \setminus S}(f_i^*/d_{im})\}$;

$c_i^* := c_i^* - w(m, S)$; IF $c_i^* = 0$ THEN $m(i) = m$, $i \in S$;

$f_i^* := f_i^* - d_{im}w(m, S)$, $i \in \{1, \ldots, m\} \setminus S$;

$m := \min\{m, \min_{i: \, i \leqslant m, \, c_i^* = 0, \, f_i^* = 0}(i - 1)\}$.

Step 3. IF $\lambda_i < 0$ THEN $y_i = 1$, $i = 1, \ldots, n$.

Step 4. $x_1 = d_{1,m(1)}$; $y_1 = 1$; $i := 1$;

WHILE $m(i) < n$ DO

$i := m(i) + 1$; $y_i = 1$; $x_i = d_{i, \, m(i)}$.

5. Concluding remarks

We have developed a new algorithm to solve the economic lot-sizing problem. The complexity of this dual algorithm is equivalent to the well-known dynamic programming algorithm. The dual algorithm also provided a new proof of the fact that the linear programming formulation is a complete linear description of the convex hull of feasible solutions for the economic lot-sizing problem. We tried to develope a similar algorithm for the economic lot-sizing problem with start-up costs (Wolsey, 1988) for which a complete linear description of the convex hull of feasible solutions is conjectured, thereby proving this conjecture. However in trying to apply this approach we found a counter example to the conjecture. An extended formulation has been found and its completeness is proven (van Hoesel et al., 1990). The dual algorithm has also been used in the sensitivity analysis for the economic lot-sizing problem, the results can be found in Wagelmans and van Hoesel (1990).

References

Afentakis, P., Gavish, B., and Karmarkar, U. (1984), "Computationally efficient optimal solutions to the lot-sizing problem in multistage assembly systems", *Management Science* 30 (2), 222–239.

Afentakis, P., and Gavish, B. (1986), "Optimal lot-sizing algorithm for complex product structures", *Operations Research* 34, 237–249.

Barany, I., Van Roy, T.J. and Wolsey, L.A. (1984), "Uncapacitated lot-sizing: The convex hull of solutions", *Mathematical Programming Study* 22, 32–43.

Crowston, W.B., and Wagner, M.H. (1973), "Dynamic lot size models for multi-stage assembly systems", *Management Science* 20 (1), 14–21.

Crowston, W.B., Wagner, M.H., and Williams, J.F. (1973), "Economic lot size determinazation in multi-stage assembly systems", *Management Science* 19 (5), 517–527.

Erickson, R.E., Monma, C.L., and Veinott, A.F. (1985), "Minimum concave cost network flows", Stanford University, Stanford, CA.

Florian, M., and Klein, M. (1971), "Deterministic production planning with concave costs and capacity constraints", *Management Science* 26 (7), 12–20.

Florian, M., Lenstra, J.K., and Rinnoy Kan, A.H.G., (1980), "Deterministic production planning: Algorithms and complexity", *Management Science* 26 (7), 669–679.

Hoesel, S. van, Wagelmans, A., and Wolsey, L. (1990), "Economic lot-sizing with start-up costs: The convex hull", Research Report Econometric Institute Erasmus University.

Love, S.F. (1972), "A facilities in series inventory model with nested schedules", *Management Science* 18 (5), 327–338.

Maxwell, W.L., and Muckstadt, J.A. (1985), "Establishing consistent and realistic reorder intervals in production distribution systems", *Operations Research* 33 (6), 1316–1341.

Pochet, Y., (1987), "Lot-sizing problems reformulations and cutting plane algorithms", Ph.D. Louvain-La-Neuve, CORE.

Wagelmans, A., and Hoesel, S. van (1990), "Sensitivity analysis for the economic lot-sizing problem", Research Report Econometric Institute Erasmus University.

Wagelmans, A., Hoesel, S. van, and Kolen, A. (1989), "Economic lot-sizing: An $O(n \log n)$-algorithm that runs in linear time in the Wagner–Whitin case", Research Report RM 89037, Rijksuniversiteit Limburg.

Wagner, H.M., and Whitin, T.M. (1959), "Dynamic version of the economic lot size model", *Management Science* 5 (1), 89–96.

Wolsey, L.A. (1988), "Uncapacitated lot-sizing problems with start-up costs", CORE discussion paper 8801, Louvain-La-Neuve.

Zangwill, W.I. (1966), "A deterministic multiproduct, multifacility production and inventory model", *Operations Research* 14 (3), 486–508.

Zangwill, W.I. (1969), "A backlogging model and a multi-echelon model of a dynamic economic lot size production system – A network approach", *Management Science* 15(9), 506–527.