



**Manchester  
Metropolitan  
University**

---

[de Freitas, Elizabeth](#) (2017) In: Innovation and technology enhancement in mathematics education. Mathematics Education in the Digital Era . Springer Verlag, pp. 237-257. ISBN 978-3-319-61488-5

---

**Downloaded from:** <http://e-space.mmu.ac.uk/621687/>

**Publisher:** Springer Verlag

**DOI:** <https://doi.org/10.1007/978-3-319-61488-5>

Please cite the published version

<https://e-space.mmu.ac.uk>

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

## **The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes**

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

**Elizabeth de Freitas**  
**[l.de-freitas@mmu.ac.uk](mailto:l.de-freitas@mmu.ac.uk)**  
**Education and Social Research Institute**  
**Manchester Metropolitan University**

### **Abstract**

This chapter focuses on innovative developments of four-dimensional digital mazes, examining how these mazes tap into the ideas of mathematician and fiction writer Charles Hinton (1853-1907) who wrote extensively on perception of a 4<sup>th</sup> *geometric* dimension. Hinton treats mathematical objects as physical and material movements, and draws on non-Euclidean geometry to argue for a virtual dimension to matter. I discuss recent attempts to build digital mazes that develop spatial sense in four dimensions, and show how these are directly linked to Hinton's ideas. I focus on how colour and movement in digital environments are used to develop a distinctive kind of spatial sense. This chapter sheds light on innovative uses of digital software for developing student spatial sense. My aim is to explicate the new materialism of Charles Hinton, contribute to discussions about the nature of spatial sense and spatial reasoning, and to point to possible directions for future research on inventive approaches to geometry.

### **1. Introduction**

Charles Howard Hinton (1853-1907) was a British mathematician and author who published various monographs on the mathematics of higher dimensions. He was particularly interested in conceptualizations of the 4<sup>th</sup> dimension, and was the first to use the term "tesseract" to describe the four-dimensional cube. Hinton argued that people could develop perceptions of higher dimensions if they rid themselves of the conventions of right and left, up and down, through a process of "casting out the self" (De Witt, 2013). In *The fourth dimension*, Hinton (1904) describes how our sensory habits and our capacity to make sense of the world in three dimensions can be altered and opened onto a fourth dimension. Like many others in the late

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

nineteenth century, in response to developments in non-Euclidean geometry and topology, there was a widespread interest in rethinking human spatial sense. Along with Charles Hinton, others working in England, such as Charles Dodgson (Lewis Carroll) and Edwin Abbott, the author of *Flatland* (1884/2007), pursued this burgeoning interest in experiments that contest the limits of human perception.

By examining the experiential qualities in one, two and three dimensions, Charles Hinton attempted to generalize a method or model for how the human mind might come to understand a four-dimensional space. Hinton was motivated to consider the implications of this work for mathematics teaching and learning. Speculating philosophically about these implications led Hinton to propose a “higher” form of existence whereby one might tap into and perceive the fourth dimension. Rather than looking to mystical and religious sources for apprehending this higher being, he turned to mathematics and the physical sciences to study the “physical reality of the fourth dimension” (Hinton, 1902, p. 142). Through developing a more advanced spatial sense, Hinton aspired to what he called the “higher man”.

We must learn to realize the shapes of objects in this world of the higher man; we must become familiar with the movements that objects make in his world, so that we can learn something about his daily experience, his thoughts of material objects, his machinery (Hinton, 1904, p. 121).

In this chapter, I describe Hinton’s methods for developing perception of a 4<sup>th</sup> *geometric* dimension. I show how his approach can be characterized as a kind of materialist approach to the study of mathematics because of the way it fuses mathematics with matter. I then discuss recent attempts to build digital mazes that develop spatial sense in 4 dimensions, and show how these are directly linked to Hinton’s ideas. I focus on how colour and movement in digital environments are used to develop a distinctive kind of spatial sense. Digital environments allow for powerful uses of colour due to the pixel screen and the simulation of depth and movement. This chapter sheds light on innovative uses of digital software for developing student spatial sense, and thereby responds to a need to think more broadly about spatial sense in mathematics education (Kinache, 2012; Wai et al, 2009). I discuss how 4-D digital mazes work and how their use of colour is linked to the philosophical insights of Charles Hinton. My aim is to contribute to discussions about the nature of spatial sense and spatial reasoning, and to point to possible directions for future research on inventive approaches to geometry. In the first sections, I discuss the insights of Hinton, and in the last sections I discuss two examples of 4-D digital mazes, as well as findings regarding participant skill at navigating such mazes.

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

## 2. A new materialist approach to space

Thought experiments run throughout Hinton's work. In these thought experiments, he melds mathematics and physics in creative speculative ways. In *Scientific Romances* (1884) he asks that we imagine a skew line moving through a plane and then states "If the plane were of such a nature as to close up behind the line, if it were of the nature of a fluid, what would be observed would be a moving point." (Hinton, 1884, p.13). This focus on a *fluid* plane on which and through which points, lines, planes and other entities might pass as *physical-mathematical objects*, leaving various traces at varying speeds, allows him to speculate about the nature of both geometry and perception. He carries through with this thought experiment, rigorously pursuing the implications in terms of geometric space, and describes an entire structure of enmeshed lines – a "framework" - cutting through this fluid plane at different angles, and thus producing – for those who live and perceive *on the plane* – points that move across the plane at varying speeds. Indeed, the permanence of any static point on the plane will be the *effect* of one such perpendicular line moving through the plane, and have no other presence than that which is sustained through a particular kind of movement. In this way, Hinton centers movement in the making of mathematics, showing how any individuated bodies (like points) are secondary effects of a particular kind of movement. It's important to note that this secondary effect is not a Platonic image or reflection of a hidden world, because the movement that engenders the form does not operate according to resemblance or copy. Hinton does not hesitate to move back and forth between mathematical entities and material ones as he pursues these ideas:

Let us now assume that instead of lines, very thin threads were attached to the framework: they on passing through the fluid plane would give rise to very small spots. Let us call the spots atoms, and regard them as constituting a material system in the plane (Hinton, 1884, p. 14).

The threads are woven together and form connected shapes that pass through the fluid plane, creating an effect that lasts for some period of time: "These moving figures in the plane are but the traces of the shapes of threads as those shapes pass on. These moving figures may be conceived to have a life and a consciousness of their own." (Hinton, 1884, p. 15). Like many other philosophers in the 1890s, Hinton is exploring the mixture of matter and mind, foreshadowing the convictions of contemporary theorists who describe themselves as new materialist (Coole & Frost, 2010). In *A New Era of Thought* (1888), Hinton says that he prefers to use the term "higher matter" rather than "higher space" because it does not make sense to split concrete matter into extension and impenetrability (Hinton, 1888, p. 106). He hopes that "the horizon of thought is altered" (Hinton, 1888, p.107), not away from matter

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

and towards spiritual existences, but towards the enhanced apprehension of the material. He will suggest that apprehending such “higher matter” demands an attention to detail – a proliferation of detail – so that awareness comes from saturation of detail rather than from generalizing from cases. In other words, He is seeking a way of developing spatial sense that is more immersive, and less based on ideas of abstraction whereby a concept transcends the details and the examples in which it is said to be instantiated. This is an important point as we turn to questions of learning *how to* perceive in four geometric dimensions. Immersive experiences are detail-oriented methods of learning, in which the environment is saturated with pedagogic potential. And although he seeks to study our experiences in three dimensions to learn about the fourth (which might be deemed a kind of generalization), his method of apprehension will ultimately depend on maximizing the apprehension of details.

These highly speculative claims point to Hinton’s interest in mapping the virtual dimension of space. He links consciousness to space in ways that resonate with the historian of mathematics Gilles Chatelet’s work on the virtual dimension of matter (2000), asking “Can our consciousness be supposed to deal with a *spatial profile of some higher actuality?*” (Hinton, 1884, p. 16. my italics). Moreover, the movement that we perceive is but the trace of a “higher” movement, a movement that produces images that *do not resemble* the threaded structures exactly, but are in direct contiguous and haptic relation (an indexical relation) to those structures. What makes this approach highly important for experiments today regarding perception and four dimensions is Hinton’s attempt to think four dimensions according to these kinds of thought experiments where selected invariants – like movement and force – are carried over to the other dimension, although modulated due to the ways that sensation always involves more than geometric relations. Hinton, like others working today on digital 4-D mazes, stresses that our experience in three dimensions can be examined and used to develop a perception in four dimensions.

Again, it sometimes appears to be thought that the fourth dimension is in some way different from the three which we know. But there is nothing mysterious at all about it. It is just an ordinary dimension tilted up in some way, which with our bodily organs we cannot point to.” (Hinton, 1884, p. 46).

We hear in this citation how the fourth dimension is just “tilted up” in some way that we are unable to perceive. But he moves on to make two very important claims about how to build awareness of the fourth dimension. Two everyday spatial experiences are pivotal in his expanding to the fourth dimension. The first is the common feeling of being surrounded or bounded by a space of higher dimension,

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

which implies that all three-dimensional objects touch and are contiguous with this higher dimensional space. The second is the feeling of the continuum, and the fact that any space must be composed of an *infinite number* of objects of lesser dimension (i.e. an infinite number of planes compose a solid). This implies that any four-dimensional space must be composed of an infinite number of three-dimensional objects.

### **3. Continuity and the infinitely smooth texture of matter**

A being existed in four dimensions must then be thought to be as completely bounded in all four directions as we are in three. All that we can say in regard to the possibility of such beings is, that we have no experience of motion in four dimensions. The powers of such beings and their experience would be ampler, but there would be no fundamental difference in the laws of force and motion. (Hinton, 1884, p. 17)

And yet we will see when we discuss digital four dimensional mazes that forces such as gravity will *by necessity* act differently when perceived in four dimensions. But Hinton's point is that in order to become conscious of how these differences will be lived in the higher dimension, we must attend to the "infinitely minute" in matter – "the ultimate particles of matter" - because only then might we be able to compare magnitudes in all four dimensions (Hinton, 1884, p. 21). When we can perceive the infinitely minute, we can then begin to operate according to proportions between such magnitudes in some pragmatic sense (Hinton, 1884, p. 21). It is interesting to note that Hinton seems to be advocating for a certain *infinitesimal* relationship that might help us connect with the fourth dimension. He describes the "thin" dimension of entities, lines for instance, much like Evariste Torricelli might have described the inflatable width of a line, when he broke with the Euclidean definition of line in order to develop the infinitesimal calculus. For Hinton, the same materiality and plasticity of an unperceived *geometric* dimension can be imagined as "thin", if we accept that matter quivers and vibrates with potentiality:

The direction in which it is thin is in a direction which we do not know, in which we cannot move. But although we cannot make any movements which we can observe with our eyes in this direction, still the thin film – thin though infinitely extended in any way which we can measure – this thin film vibrates and quivers in this new direction, and the effects of its trembling and quivering are visible in the results of molecular motion. It only affects matter by its movement in directions at right angles to any paths which we can point to or observe, *and these movements are minute*: but still they are incessant, all-pervading, and the cause of movements of

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

matter. *It is smooth – so smooth that it hinders not at all the gliding of our earth in its onward path*” (Hinton, 1884, p. 52. My italics).

This reference to the fourth direction as being infinitely smooth points again to the infinitesimal as the “smallest interval” or miniscule thread by which the dimensions are stitched together, where the stitch is so fast and the weave so tight, space becomes so smooth and so intense that we are able to slip across dimensions. This is linked to Riemann’s influential essay “The hypotheses which lie at the foundation of geometry” whose ideas Hinton seems to echo at times. And yet the smoothness of Riemann’s continuous manifold was – at the infinitesimal scale – a Euclidean smoothness, whereby the manifold is glued together or composed of patches of Euclidean flat planes (Plotnitsky, 2012). In the 1970s, this use of the infinitesimal developed into synthetic differential geometry and smooth infinitesimal analysis (Bell, 2014). For instance, the mathematician F.W. Lawvere used the infinitesimal to develop this kind of mathematics, assuming the continuum as an autonomous notion, not requiring the notion of the discrete, substituting the idea of the limit with the idea of the nilpotent infinitesimal, a quantity so small that some power of it vanishes. I draw these historical links between Hinton’s ideas and developments in mathematics to show that his musings are not mere quackery. I also want to draw this link to the infinitesimal because it has played such a significant role in fueling the kind of thought experiments that Hinton pursues so well.

The infinitely small plays a pivotal role in how Hinton imagines we are capable of cognition in four dimensions. Hinton speculates that movement at the *quantum level* and the “minute portions of matter” may “go through four-dimensional movements and form four-dimensional structures” (Hinton, 1888, p.109). Like Henri Bergson during the same era, along with many other continental and English philosophers, Hinton was inspired by developments in physics regarding the odd behavior of quantum particles. He suspected that these particles were capable of moving through a geometric fourth dimension. Thus our power to perceive in four dimensions may in fact involve our learning the movements of the very small particles of matter. He then suggests that *thought itself* might be considered the infinitely small, and *therefore the movement of thought is a movement through the fourth dimension*.

The goal of apprehending in four dimensions corresponds to the goal of grasping thought in all its mobility – “by observing, not what we can see, but what we can think.” (Hinton, 1888, p. 110). Hinton makes a Spinoza-like attempt to relink thought *with* matter, and to ascribe to thought a particular kind of movement that is *not* representational of three-dimensional extension, *but accords to four dimensional movement*. Thought, suggests Hinton, and its “small molecules in the brain ... might

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

go through four-dimensional movements and form four-dimensional structures (Hinton, 1888, p. 110). Despite how odd such claims may sound, his approach to consciousness links up with current interest in quantum computing, where the temporal dynamics of 'cognition' disobey the usual space-time rules. For Hinton, we are indeed four-dimensional creatures, ill-equipped to perceive beyond three dimensions, but nonetheless thought and imagination, at their freest, plug into the 4th dimension. Hinton's insights into how we might come to perceive the 4<sup>th</sup> dimension are thus linked to related projects, at the turn of the twentieth century, of developing an "intuitive method" that might tap into the virtual mobility of matter (Bergson, 1903). These projects looked for a virtual dimension buried in matter. More recently, this kind of work supports attempts to study learning in terms of *the movement of thought*, and not simply the movement of already individuated bodies (de Freitas & Ferrara, 2015).

#### **4. Movement and spatial sense**

Hinton was influenced by the work of Lobatchewsky and Bolyai (1830s) on alternative axiomatic foundations for geometry, as well as the work of Riemann (1850s) who had developed analytic tools for mapping the distinctive spatial characteristics of n-dimensional manifolds, and had introduced a new way of thinking about relationships between geometry and space. This latter work, in particular, had led to all sorts of new ways of defining dimension. In 1912, Poincaré (1854–1912) used an inductive strategy for defining dimension that was based on the concept of boundary and border. A space is  $n+1$  dimensional, suggested Poincaré, if its border is  $n$  dimensional. For instance, a 3 dimensional object has 2 dimensional faces (figures), and a 4 dimensional cube (a hypercube) has 3 dimensional faces (cubes). Another way of conceiving dimension is through movement. For instance, one might drag or move an object to enter a new dimension: by dragging a 0-dimensional object in some direction, one obtains a 1-dimensional object. By dragging a 1-dimensional object in a *new direction*, one obtains a 2-dimensional object. These diverse ways of thinking about the concepts of dimension and orientation were debated throughout the nineteenth century.

Like Poincaré, Hinton described the process of moving to higher dimensions in terms of boundaries and movement. He noted that a line is divided in two by a boundary point, that a plane is divided into two distinct planes by a boundary line, and a volume likewise by a boundary plane. In each of these cases, the next dimensional space is generated through the *new movement* of the lesser dimensional entity.



De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

Thus, going on, we may say that space is that which limits two portions of higher space from each other, and that our space will generate the higher space by moving in a direction not contained in itself (Hinton, 1904, p. 122)

What is exciting is the acknowledgement of a movement “in a direction not contained in itself” which underscores two important ideas: (1) this movement is *unrecognizable as movement in the original space*, and (2) such movement points to a virtual dimension or potentiality of space. Because the movement is unrecognizable as movement in the  $n-1$  dimensional space, work on perception in higher dimensions helps us imagine how there might be a movement *in this world* that is beyond our perception.

Hinton (1904) uses examples of physical movement to develop his argument. He describes the movement of a line and a spiral through a film (figure #1), where the intersection of the spiral and film would be a point moving in a circle (see dotted line in Figure 1).

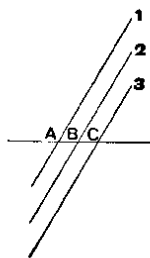


FIGURE 25

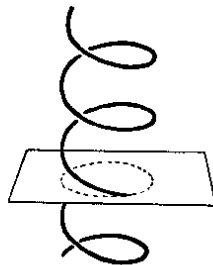


FIGURE 26

Figure 1: Hinton's line and spiral moving through a material film

He then introduces a second movement, that of the film itself moving “vertical” or perpendicular to the lateral space of the film. In the first case, any dweller on the film would simply perceive the circle being drawn on the film, but in the second case, the film dweller would develop an awareness of the film's trajectory through time. This introduces a reflexive perception, that is, a perception of one's environment moving, and thus also introduces, according to Hinton, consciousness. This consciousness is distributed across both the film and the spiral, by way of their intersection:

In the film the permanent existence of the spiral is experienced as a time series – the record of traversing the spiral is a point moving in a circle. *If now we suppose a consciousness connected with the film* in such a way that

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

the intersection of the spiral with the film gives rise to a conscious experience, we see that we shall have in the film a point moving in a circle, conscious of its motion, knowing nothing of that real spiral the record of the successive intersections of which by the film is the motion of the point (my italics, Hinton, 1904, p.125).

Through this combination of movements and diverse points of view, Hinton describes the “apparent” motion within the film, felt by the “plane of consciousness” or “space of consciousness” as a record of the movement – “each atom at every moment is not what it was, but a new part of that endless line which is itself.” (Hinton, 1904, p. 125). These atoms are “acting, living” and partake always of at least two motions, one associated with their interactions within the film, reflecting spatial-temporal relationships immanent to that plane, and the other associated with the film itself moving in an altogether different spatio-temporal dimension. Thus the atoms can only “read off” this motion “in the film” as the trace of an altogether different kind of movement, a movement that is in some *strong sense* inconceivable within the film. This kind of leaping into another dimension where the movement is literally inconceivable in the original space is exactly what is needed to begin operating in four dimensions. Hinton’s approach to the materiality of the spiral intersecting the plane lends itself to a new materialist reading of mathematics (de Freitas & Sinclair, 2014). He isn’t shy to treat circles and other mathematical entities as material or physical objects, and does so in order to think differently about the way that mathematics *is in* the material world. His work is filled with these kinds of thought experiments.

Let us now make the supposition that film after film traverses these higher structures, *that the life of the real being is read off again and again in successive waves of consciousness*. There would be a succession of lives in the different advancing planes of consciousness each differing from the preceding, and *differing in virtue of that will and activity* which in the preceding had not been devoted to the greater and apparently most significant things in life, but the minute and apparently unimportant (Hinton, 1904, p. 126. my italics)

Apprehending this motion that is inherent to life and matter, that is intrinsic to being and becoming, is the task of the “higher man”. This task requires that ‘man’ has “a consciousness of motion which is not as the motion he can see with the eyes of the body” (Hinton, 1904, p. 128). Most notably, this requires becoming conscious of a motion that is not discernable or perceivable in our current perceptual organization. If we are to perceive the fourth dimension, we must achieve this decentering of our current perceptual apparatus. Hinton suggests that this approach

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

to the fourth dimension reflects a certain Eastern philosophy of nature and matter. I would also argue that his approach resonates strongly with that of the French philosopher Henri Bergson who was highly influential in the same period in which Hinton was formulating and writing his ideas. Bergson (1896) argues that there is a difference between *sensori-motor movement*, which is movement trapped by the measure conventions of the perceptible world, and the movement of the whole of duration, a movement that taps the virtual potentiality of matter.

Hinton uses the Pythagorean theorem to elaborate his approach to space. He considers an alternative world where the Pythagorean theorem states that a *sheared* square on the hypotenuse is the difference between the two squares on the other sides of the right triangle. Shearing retains the area of a square, but changes the shape. In this alternative world, we imagine that the inhabitants 'see' the square and its shear image as the same (or as equivalent) if there is a "shear rotation" movement that generates one from the other. As long as such a motion can be identified, two figures will be considered equal. In this alternative world, we emphasize that our own rules no longer apply, and the Pythagorean theorem takes on a different import. Notably, Hinton is experimenting here with how perception and geometry are entwined, and how different geometries entail different topological relationships. He is imagining a different geometry so that he might problematize the very notion of distance or metric, and at the same time link it more directly to matter: "Hence distance independent of position is inconceivable, or practically, distance is solely a property of matter." (Hinton, 1904, p. 136). This reference to "matter" marks his empirical approach, and his realization that only observation and experiment can help us decide whether *our* Pythagorean theorem or this other version of the Pythagorean theorem is appropriate: "There is nothing to connect the definition of distance with our ideas rather than with his, except the behavior of an actual piece of matter." (Hinton, 1904, p. 136).

For Hinton, alternative non-Euclidean geometries should inspire us to reconsider the nature of perception and also the limits of material agency. According to Hinton, the discoveries of non-Euclidean geometry have significance for all sorts of reasons, one of them being how they force us to consider both materiality and sensation in new ways:

By immersing the conception of distance in matter to which it properly belongs, it promises to be of the greatest aid in analysis; for the effective distance of any two particles is the product of complex material conditions and cannot be measured by hard and fast rules. Its ultimate significance is altogether unknown. It is a cutting loose from the bonds of sense, not

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

coincident with the recognition of a higher dimensionality, but indirectly contributory thereto (Hinton, 1904, p. 140).

## 5. Depth of field and colourism

Before turning to a discussion of how Hinton's ideas are linked to current digital four dimensional mazes, it's important to discuss some of the visual cues that are used when humans navigate through space. Visual perception in multiple dimensions entails the notion of *depth of field*. Cutting (1995, 1997) proposes that we think of space in three ways - personal space, action space, and vista space - each defined in terms of their proximity to the body. Perceiving in each of these spaces draws on different kinds of perceptual habits. He shows how we decode various depth cues when making sense of images, identifying various key aspects of unconscious perception. Such cues are crucial as we move through spaces and integrate paths, and these are used extensively by digital game designers, as they develop complex spatial mazes and puzzles. Based on Cutting (1995, 1997), I list here key cues for depth perception, as they play a crucial role in digital maze navigation:

- (1) Occlusion - where part of an object is occluded behind another - is a standard way of rendering depth or dimension, although it does not convey any measure to the depth and thus offers limited information.
- (2) Height in the visual field renders depth through the measures of relations among the bases of objects in the image.
- (3) Relative size [linear perspective] whereby perspective (linear or otherwise) renders depth perception. This engenders a ratio between objects' positions rather than any objective measure of depth between them.
- (4) Relative density refers to the projected number of similar objects or textures per solid visual angle. According to Cutting (1997) it was not until the 15<sup>th</sup> century that artists began rigorously using all four of these in conjunction. He also points out that digital media coordinate all four strategies in generating an image from a single point of view.
- (5) Binocular disparity is the difference in relative position of an object as projected on the retinas of the two eyes. When disparities are small, we perceive solid space. When disparities are greater - often when an object is very near - we suffer double vision.
- (6) Motion perspective where depth is rendered through a moving observer. This technique is very good for judging absolute depths, rather than just determining which objects are in front and which behind.
- (7) Texture gradients.

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

(8) Brightness and shading.

(9) Kinetic depth concerns depth derived from the movement of parts of the image. This often entails (dis)occlusion revealed through motion. With respect to what sort of geometry applies in each of these spaces, Cutting (1997) finds that personal space is perceived as “Euclidean” but that action space is perceived as affine, although movement through such space can reconfigure it as Euclidean.

Most of these points, and especially the last regarding “kinetic depth” perception, are actively deployed in making sense of moving images. Deleuze (1989) suggests that Orson Welles is the master of depth of field in the moving image. In each of his films, the camera plunges through space, away from the viewer, deep into a *beyond* that unfolds through optical barriers, and into nested rooms and distant corners. It feels as though the camera is travelling into space, penetrating a volume of space, poking through what might have been a two-dimensional image and carving out a new dimension. In the history of cinema, depth of field is a hallmark of the first films that documented movement (Lumière and the 1895 “Exiting of the factory”), but Deleuze argues that these films capture depth *in* film or depth *in* image, but not depth *of* field or a depth *of* image. Depth of field achieves a different effect. This distinction can be found, he continues, in tracking changes in Western perspective drawing and painting. Prior to the 17<sup>th</sup> century, depth was conjured through carefully layered planes of vision, where each plane had its occupants and objects, each visually autonomous. In the 17<sup>th</sup> century we see paintings where an element of one plane refers directly to an element of another plane, where characters address each other across planes, and where the foreground comes into immediate contact with the background. This latter effect is depth *of* field. This contrast between a depth that is achieved through juxtaposition of differently sized images and characters, and a depth that is achieved through movement and engagement, can be seen in contrasting the cinema of Welles to that of Griffith’s *Intolerance* where depth was produced by “a simple juxtaposition of independent shots (plans), a succession of parallel planes (plans) in the image” (Deleuze, 1989, p. 107).

Deleuze argues that this new depth of field is principally temporal, and indeed offers a direct image of time:

In this freeing of depth which now subordinates all other dimensions we should see not only the conquest of a continuum but the temporal nature of

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

this continuum: it is a continuity of duration which means that the unbridled depth is of time and no longer of space. (Deleuze, 1989, p. 108)<sup>1</sup>

The temporal dimension of depth is thus extremely important for humans watching a moving image of a particular spatial arrangement. This discussion of the power of depth of field in the moving image helps us appreciate the nature of screen watching when navigating a 4-dimensional maze. How might navigating a 4-dimensional digital maze, ultimately a moving image that unfolds with kinetic depth, tap into this direct relationship with time? Especially as the focus on geometric relationships within such mazes tends to keep one focused on the measurement of space rather than more experiential immersive ways of apprehending? This is where Hinton and current software designers turn to colour.

In *A new era of thought* (1888), Hinton describes working with a system of colour cubes with students, developing their perceptions of the tesseract using different colours for different vertices, edges, and faces. In Hinton's system, colour is used as a way to visualize dimension, and each dimension is assigned a primary colour (Figure 2). Surfaces that stretch into other dimensions have appropriately blended colours. For example, if a cube has edges coloured yellow, red and blue, then each of its faces would be orange (blend of red and yellow) or brown (blend of red and blue) or green (blend of blue and yellow). In order to extend this to the fourth dimension, introduce a new colour – say white – and use the blending to colour the various three-dimensional cross sections of the hypercube. In the fourth dimension, each cross-section is itself a cube, with its faces colour-coded appropriately, but now our vision must grapple with the blending of colours not simply on a surface but in three dimensions.

---

<sup>1</sup> He cites Claudel who said of Rembrandt that depth was “an invitation to recall”. He also refers to how Bergson and Merleau-Ponty showed how depth was principally a temporal dimension.

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

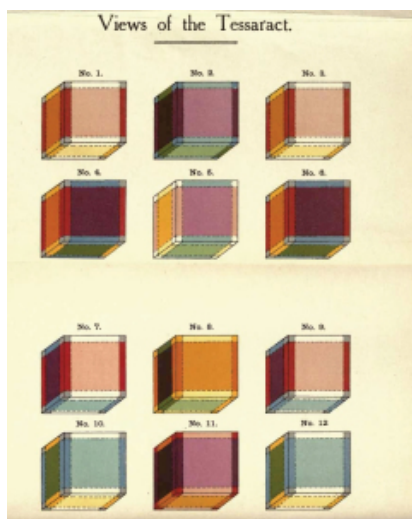


Figure 2: Hinton's (1988) colour cubes

Hinton's ideas about colour and higher dimensions are fascinating for many reasons, particularly because colour is not typically studied as part of spatial reasoning and depth perception. As an under-examined quality of depth perception and orientation skills, colour blending offers important insights into the concept of dimension. Attempts to draw on the modulation of colour intensity in an image are of course familiar to artists. Deleuze (2003) suggests that Francis Bacon's skill at "colouring sensation" is what allows us to get at the distributed nature of consciousness. Bacon's eerie "scrambling and smudging" of the human figure gets to the heart of the movement of thought and what Deleuze calls the "time-image" (Deleuze, 2003, p. 127). Deleuze (2003) states "The formula for the colourists is: if you push colour to its pure internal relations (hot-cold, expansion-contraction), then you have everything" (Deleuze, 2003, p.112). Colourism aims to show how colour itself is the variable relation or differential relation on which individuation depends. The technique of the colourist is "the production of light and even time through the unlimited activity of colour" (Deleuze, 2003, p. 112).

In pushing colour to function in this onto-generative way, Bacon and other colourists, force us to encounter the image differently. In attending to the modulated intensity of colour sensation, the eye is no longer the usual optic device, looking for resemblance, *looking for the line*, but becomes haptic and sculptural. In other words, the eye touches the image, and the sense of sight behaves like the sense of touch. In terms of developing spatial sense, this means that modulations of colour might play an important role. There needs to be more research on how modulating colour is a part of spatial reasoning. As we see below, colour is used for both representing different dimensions, but also for capturing the intensity of a fourth dimension that

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

is folded into the first three in digital environments. In the examples I discuss, colour is used to also refer to the varying 'temperature' (hot/cold) of the fourth dimension, so as to evoke for the participants that sense of intensive quantity that infuses the material world.

## 6. Four-dimensional digital mazes

During the last decade, perception scientists have used experiments with digital mazes to study the potential of developing human perception in 4 dimensions. In this section, I explore the way that innovative colour digital mazes draw on Hinton's earlier ideas to develop student spatial sense. Various software developers use colour to simulate the experience of moving through a fourth dimension. For instance, the mathematician Jeff Weeks (2016) has developed a game that involves moving an object (white ball) along a series of coloured paths to reach a target, such that one must navigate four dimensions (Figure 3).<sup>2</sup> One can train oneself on these mazes, becoming familiar with how the colour coding and the particular corner rainbow transformations embody movement into a fourth dimension with reference to the movements one has already made. In this software, you occupy a 'bird's eye view' on the maze, and can rotate the maze as you try and decide where to move. By rotating the maze, the occluded links between pathways become visible.

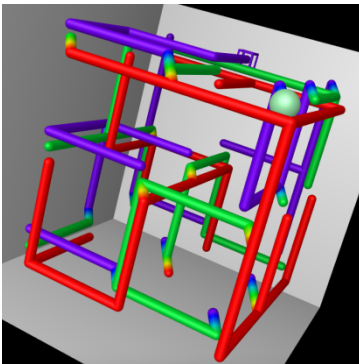


Figure 3: Certain corners are rainbow transitions into the fourth dimension

One can also modify the image with a 4-D shear, so as to see where the linked tubes do not actually connect. This innovation helps one realize how the 3-D version of the maze is also a kind of occlusion, blocking the entire fourth dimension from view (see Figure 4 to see maze without shear on left, and the same maze with shear on right). In other words, colour is used to introduce another fold in the maze, one that would be invisible to anyone operating only in 3-D space.

---

<sup>2</sup> Available at <http://www.geometrygames.org/Maze4D/index.html>



De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

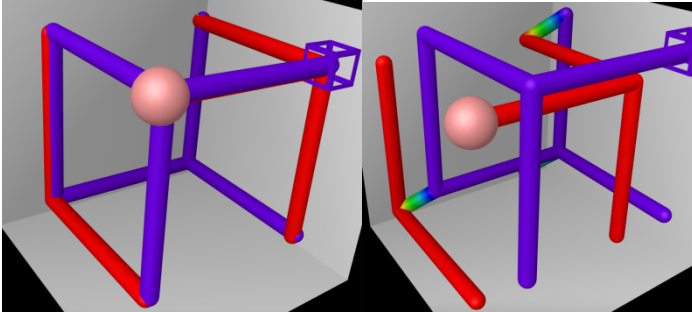


Figure 4: The same maze, showing the 4-D shear on right.

Recent attempts using digital colour mazes to develop students' perceptions of the 4<sup>th</sup> dimension show how colour can enhance "path integration" and successful spatial navigation through 4 dimensional mazes (Aflalo & Graziano, 2008). Gamers using computer simulations claim to develop competence in 4-D navigation using such visual cues (Seyranian, Colantoni, & D'Zmura, 1999). This work points to how students may be able to develop 4-D spatial maps and operate through visualizations of 4-D environments.

Aflalo and Graziano (2008) designed an innovative computer-generated maze that moved through four spatial dimensions, but was different from Weeks' maze in that there was no bird's eye view. The participants were able to develop skills in perceiving in the fourth dimension, based on their ability to perform path integration, which is a standard test in such experiments for measuring spatial sense. After moving along a winding corridor, the participants then had to 'point' back at the occluded starting point of their movement. As the authors note, these experiments indicate the potentiality of the body: "One interpretation is that the brain substrate for spatial navigation is not a built-in map of the 3-dimensional world. Instead it may be better described as a set of general rules for manipulating spatial information that can be applied with practice to a diversity of spatial frameworks" (Aflalo & Graziano, 2008, p. 1066). Much of the work on path integration shows that humans are often better when they do not rely on visual cues (Easton & Sholl, 1995; Farrell & Thomson, 1998; May & Klatzky, 2000; Presson & Montello, 1994). We often have that typical experience of getting lost in a building, and in order to proceed we look up or close one's eyes, as we try to recall how our body moved through the building. Phenomenologists have theorized this in terms of the body and its inherent capacities to navigate through space. Rush (2008) suggests that the body is thus "something that I move with, not something I move, i.e., it has the characteristic of direct motility – I do not have to place my body in order to move it." (Rush,

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

2008, p. 18). This observation is important as developers try to innovate with digital maze technology. Our sense of our own body's movement needs to be considered as we develop innovative technologies. Clearly path integration is only part of the story.

Research into spatial cognition in humans and other animals often uses this skill of path integration, the "short cut" test, to study spatial skills (Biegler, 2000; Newcombe & Huttenlochner, 2000; Wehner, Michel, & Antonsen, 1996). Path integration is the skill of keeping track of the various movements you've taken, summing distances and turns, so that you know where you are in relation to your starting point. Desert ants, for instance, are renowned for wandering around while scavenging, and then able to return to their starting point along a more direct route. Whether this is pheromonal or geometric knowledge (or a combination of both) is unknown.

Four-dimensional digital mazes *without* birds-eye view are built so that the fourth dimension is simulated through engagement with the digital environment. Typically, this entails building a maze so that a fourth dimension is orthogonal to the three dimensions of 3-space. In other words, the movement through the virtual maze entails a movement that is orthogonal to all the movements in 3-space. The virtual environment simulates the experience of moving in this new dimension. Aflolo and Graziano (2008) for instance, created a program which first displayed a menu of selections of either 2-D, 3-D or 4-D mazes. The program contained 100 possible examples of mazes of each type, and would automatically store the time in each maze and the angular accuracy of the subject's response at the end of the maze (the response to the path integration test. See figure #5 E-D). The number of turns and the length of the corridors in each maze varied randomly, with given constraints, such as each corridor was between 3 to 6 units, and could not intersect itself. The view in each maze was of a virtual corridor along which the subject was moving, displaying that which was in front of the participant (Figure 5A).

Using different commands, the subject could simulate travelling and looking in different directions within the virtual world of the maze, translating forward or backward along the corridor and rotating left or right. Texture on the walls was used to enhance perspective cues, but depth was depicted using standard perspective, motion parallax and occlusion. In the 2-D mazes, the bends in the corridor were either to the left or right only. In the 3-D mazes, there were turns that were orthogonal to the horizontal plane. Colour was used to help them identify both the start and the end of each maze – the start cube had silver walls and the end of a corridor cube had five golden walls. Subjects never saw an outside view, and at the end of their journey, were asked to point in the direction of their starting cube,

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

hidden behind the occluding corridor walls. A score indicating their accuracy (angular deviation from direct line joining start to finish) was revealed to the participant after each guess.

In the 4-D mazes, the cubes composing the corridors became hypercubes, and the corridor bends were right-left, up-down bend, and *hot-cold*. The terms hot and cold were used to designate directions in the mathematically defined fourth dimension, but depended on colour perception and sense of orientation. As in the case of representing 3-D space on a flat computer screen, the display of a 4-D maze relies on projection, as well as visual cues. In the virtual computer world, virtual 4-D objects are projected onto the flat display. The subject must reconstruct the 4-D object using perspective, shading, motion parallax, and occlusion – no easy task! As in the maze software of Weeks discussed above, colour was used as an additional perspective cue to aid participants. A greater degree of red hue indicated that the object (in this case a wall or edge) was more in the ‘hot’ direction in relation to the viewer, while a greater degree of blue hue indicated that the object was more in the cold direction in relation to the viewer. If an object was purple, then it was at the same temperature as the viewer – that is, purple indicated that it was on the same plane as the viewer in the four-dimensional space.

Visual cues and sense of orientation are strongly linked in the experience of the maze. In other words, dimensions are always determined in relation to the orientation of the viewer. Thus the viewer has a particular orientation – a front, back, top, down, hot and cold – and as the viewer rotates 90°, these relational terms (top/down) are altered *in relation to* the surrounding environment. For instance, imagine you start the maze and are looking ahead into a corridor, and then you rotate on the horizontal plane, so that the corridor is now to your left (along the left-right axis). Next, you can rotate orthogonal to that horizontal plane, so that the corridor is now above you. In other words, as you move and change your orientation, the environment occupies different dimensions (what was cold is now far away). In the 4-D topology of the space, the participant gains additional rotational degrees of freedom, labeled R4, R5 and R6. These are modeled on conventional rotational movements in 3-D space, but these vary according to the hot-cold dimension. If the viewer comes to a wall in the corridor, she now has additional rotational moves that re-orient her in relation to the four dimensions she faces. The point here is that all dimensions become rotationally swapped as the participant moves through the space. Each rotation redefines what is up/down or right/left or hot/cold. The participants only ever translation movement is in the forward/backward direction, while all other dimensions are experienced through rotation. The dimensions are thus strongly entangled or mixed through rotation, creating an unusual spatial experience. When we move up in an elevator, we do not

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

remix or exchange our relation to the dimensions of the ground floor where we were standing before getting into the elevator. Hence, this 4-D maze invites a radically different way of orienting oneself, compared to the bird's eye view mazes discussed above. Navigating in this space entails understanding the rotational interactions of the dimensions, and seems to entail what Hinton advocated for – the need for “casting out the self” in a process of productive disorientation.

But what is this hot-cold dimension? Aflolo and Graziano (2008) describe it as a kind of additional material quality of the walls in the maze. Each wall has width, length and temperature. Generalizing from two and three dimensions, where the addition of another dimension allows escape from the previous space. For instance, an ant trapped inside a square on a plane, can escape when it is allowed to travel in three dimensions. Trying to keep the ant trapped involves layering squares on top of squares (surrounding him) and giving the initial square height so that the ant cannot crawl out (assuming its movement is on the plane). If we imagine someone trapped inside a cube, they might be also able to escape by moving in a higher-dimensional space. Suppose this dimension is temperature, so that the cube has a particular cold temperature. They can move into a higher temperature and, in theory, escape the particular trappings of the initial cube. In order to keep them trapped, one would have to add additional cubes of higher temperature, so that their movement was still contained by the stacked hypercube environment, an environment of cubes with infinite varying temperature.

Temperature is a useful term because it refers to an intensive quality that applies to all material, and brings a material perspective to the colour coding. In terms of visualizing the 4-D maze, we need to imagine the *barriers* of the corridor no longer are walls, as we imagine them, but cubes. Each cube extends top-down, right-left, hot-cold. These barriers hem the participant in, so that movement within the maze environment only ever happens in ‘one’ direction. In Figure 5F, there are 6 cubes that hem the movement of the participant, acting as barriers. These stop the movement in the left-right direction, the up-down direction, and the hot-cold direction, leaving only the forward-backward direction open for movement. Cube number 1, for instance, is the left hand barrier, with one cold face, one hot face, top and bottom faces, and one near face and one further face. The other barrier cubes (right, and top and bottom) have the same construction. The red cube and the blue cube are the barriers in the temperature direction. Their absence would mean that one could move in that direction (purple). Their presence is thus a barrier to moving – just as in the two-dimensional world of the ant on the plane, a barrier is only a barrier because it occupies that plane. The actual view of the maze navigator is shown in Figure # 5A-B. To simulate the depth of the maze, the structure of 1-D is repeated into the distant corridor, shrinking in size – using basic perspectival

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

technique. In addition, the blue-red temperature dimension is shown to vary, and in the distance (depth of field) the difference between blue and red diminishes. When the participant rotates, the view becomes radically more complicated (see Figure #5B), as does the task of managing all this visual information.

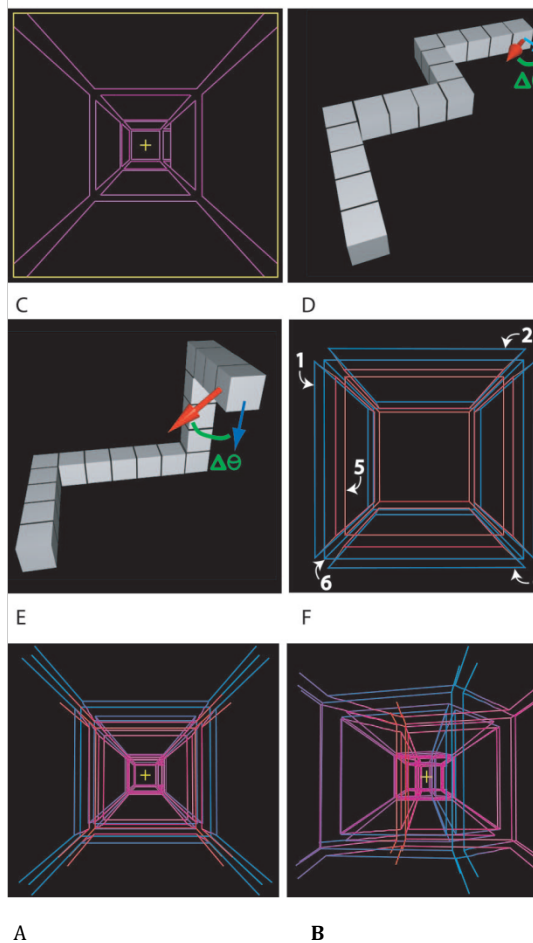


Figure 5: Aflolo and Graziano (2008) p. 1065

One of the fascinating Hinton-like insights of this digital maze experiment, is that if one navigated in this maze using only three dimensions, one has a “better than even-chance” odds of guessing the correct location of the original start when asked to point in that direction. This is simply because if the participant is ignoring one of four dimensions, and if they are expert at three, then their chances are pretty good. The authors used a simulated participant with perfect 3-D path integration skills, and found that such a participant pointed with an angular error of 0 Degrees and a mean accuracy of 28 Degrees. The human participants in this study, however, did

De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.

seem to develop better accuracy, which implies definite improvement of skill at 4-dimensional path integration, rather than simply using 3-D skills and ignoring the fourth dimensional cues.

### **Concluding comments**

In this chapter I have explored the details of Charles Hinton's proposal for how perception in four dimensions is possible. I've shown how his ideas were linked to other ideas and techniques of observation (the moving image) that emerged during the same historical period. In particular, Hinton's ideas are linked to philosophical interest in developing an "intuitive method" as articulated by Henri Bergson, and also linked to developments in non-Euclidean geometry and topological thinking, all of which inform the theoretical turn to new materialism more recently. Although developments in non-Euclidean geometry are often presented as more abstracted from the material world, I have showed how Hinton focused on the specific ways in which geometric concepts are inherently material, and engendered through depth of field, movement, colour, and intensity (temperature).

Hinton's revolutionary approach to spatial sense is now being actualized in innovative uses of digital technology. This chapter shows how two very different attempts to build a digital 4-dimensional maze deploy many of Hinton's ideas about spatial sense, colour, movement and intensity. Moreover, these experiments in virtual navigation raise important research questions about how we can use technology to expand our ways of perceiving and being in the world, suggesting that body syntonicity is radically different in four dimensions. More research is needed on digital technology's potential for advancing our perceptual skills, and on experiments for developing our grasp of what Hinton called a "higher matter". Not only would such research serve perception studies, it would also contribute to our understanding of the nature of the relationship between geometry and the material world, and enhance students' engagement with geometric concepts.

### **References**

- Aflolo, T.N. & Graziano, M.S.A. (2008). Four dimensional spatial reasoning in humans. *Journal of experimental psychology: Human perception and performance*, 34(5), 1066-1077.
- Bell, J.L. (2014). Continuity and infinitesimals. Stanford Encyclopedia of Philosophy. Available at <http://plato.stanford.edu/entries/continuity/>
- Bergson, H. (1903/1999). *Introduction a` la me`taphysique*. (T. E. Hulme: An Introduction to Metaphysics, Trans.). Indianapolis, IN: Hackett Publishing Company, 1999.

- De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.
- Bergson, H. (1896/1988). *Matière et mémoire*. (N. M. Paul & W. S. Palmer: Matter and Memory, Trans.). New York, NY: Zone Books, 1988.
- Biegler, R. (2000). Possible use of path integration in animal navigation. *Animal Learning and Behavior*, 28, 257-277.
- Cha^telet, G. (1993/2000). *Les enjeux du mobile*. (R. Shore & M. Zagha: Figuring space: Philosophy, mathematics and physics, Trans.). Dordrecht, The Netherlands: Kluwer Academy Press, 2000.
- Coole & Frost (2010). *New materialism: Ontology, agency and politics*. Durham, NC: Duke University Press.
- Cutting, J.E. (1997). How the eye measures reality and virtual reality. *Behavior research methods, instruments and computers*, 29(1), 27-36.
- Cutting, J.E., Vishton, P.M., Braren, P.A. (1995). How we avoid collisions with stationary and moving obstacles. *Psychological Review*, 102(4), 627-651.
- de Freitas, E. & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.
- de Freitas, E. & Ferrara, F. (2015). Movement, memory, and mathematics: Henri Bergson and the ontology of learning. *Studies in Philosophy of Education* 33(6), 565-585.
- Deleuze, G. (1989) *Cinema 2: The Time Image*, trans. Hugh Tomlinson and Robert Galeta, Minneapolis, MN: The Athlone Press.
- Deleuze, G. (2003). *Francis Bacon: The logic of sensation*. (Trans. Daniel Smith). Minneapolis: University of Minnesota Press.
- De Witt, A. (2013) *Moral Authority, Men of Science, and the Victorian Novel*, page 173, [Cambridge University Press ISBN 1107036178](#)
- Easton, R. D., & Sholl, M. J. (1995). Object-array structure, frames of reference, and retrieval of spatial knowledge. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 21, 483-500.
- Farrell, M. J., & Thomson, J. A. (1998). Automatic spatial updating during locomotion without vision. *Quarterly Journal of Experimental Psychology: Human Experimental Psychology*, 51(A), 637-654.
- Hinton, C. H. (1902). The recognition of the fourth dimension. Bulletin of the philosophical society of Washington, Vol. #14. Reprinted in *Speculations on the fourth dimension: Selected writings of Charles H. Hinton*. Ed. R.v.B. Rucker. (1980). New York: Dover Publications. 142-162.
- Hinton, C.H. (1904). The fourth dimension. Reprinted in *Speculations on the fourth dimension: Selected writings of Charles H. Hinton*. Ed. R.v.B. Rucker. (1980). New York: Dover Publications. 120-141.
- Hinton, C.H. (1888). A new era of thought. Reprinted in *Speculations on the fourth dimension: Selected writings of Charles H. Hinton*. Ed. R.v.B. Rucker. (1980). New York: Dover Publications. 106-119.

- De Freitas, E. (2017). The new materialism of Charles Hinton: Spatial reasoning in 4D digital mazes. In Faggiano, E., Ferrera, F. & Montone, A. (Eds). *Innovation and technology Enhancing mathematics education*. Springer Verlag.
- Hinton, C.H. (1884). *Scientific Romances*, vol.1. Reprinted in *Speculations on the fourth dimension: Selected writings of Charles H. Hinton*. Ed. R.v.B. Rucker. (1980). New York: Dover Publications. 1-66.
- Kinache, B.M. (2012). Fostering spatial sense vs. metric understanding in geometry. *The mathematics teacher*, 105 (7), 534-540.
- May, M., & Klatzky, R. L. (2000). Path integration while ignoring irrelevant movement. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 26, 169–186.
- Newcombe, N. S., & Huttenlochner, J. (2000). *Making space: The development of spatial representations and reasoning*. Cambridge, MA: MIT Press.
- Plotnitsky, A. (2012). Adventures of the Diagonal: Non-Euclidean Mathematics and Narrative. In *Circles Disturbed: The Interplay of Mathematics and Narrative*, eds. Apostolos Doxiadis and Barry Mazur (Princeton, New Jersey: Princeton University Press, 2012), 407–446.
- Presson, C. C., & Montello, D. R. (1994). Updating after rotational and translational body movements: Coordinate structure of perspective space. *Perception*, 23, 1447–1455.
- Rush, F. (2009). *On Architecture*. New York: Routledge.
- Seyranian, G., Colantoni, P., & D’Zmura, M. (1999). Navigation in environments with four spatial dimensions. *Perception*, 28 (Suppl. 7).
- Wai, J., Lubinski, D. & Benbow, C.P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, 101 (4), 817-835.
- Weeks, J. (2016). 4D mazes. Available at <http://www.geometrygames.org/Maze4D/index.html>
- Wehner, R., Michel, B., & Antonsen, P. (1996). Visual navigation in insects: Coupling of egocentric and geocentric information. *Journal of Experimental Biology*, 199, 129–140.