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Testing for periodic integration[☆]

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Abstract

A periodic autoregressive time-series model assumes that the autoregressive parameters vary with the season. This model can also be represented by a multivariate model for the annual vector containing the seasonal observations. When this multivariate model contains one unit root, a time-series is said to be periodically integrated of order 1. In this paper we propose tests for such a single unit root. These tests for periodic integration are applied to a periodic model for the quarterly German consumption series.

Keywords: Seasonal time series; Integration; Periodic models

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1. Introduction

A class of models that can provide useful descriptions of seasonally observed economic time-series is the class of periodic autoregressions (PAR), i.e. AR models with seasonally varying parameters. Recently, there has been a growing interest in these PAR models since they may reflect the behavior of economic agents with, for example, seasonally varying preferences, see Osborn (1988). Periodic autoregressions can be represented by a multivariate model for the annual series containing the observations per season, see Gladyshev (1961). This model facilitates an investigation into the non-stationarity aspects of a PAR. When the multivariate model contains a single unit root, the univariate series is called a periodically integrated autoregression (PIAR).

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In this paper we propose tests for a single unit root in a multivariate model for the annual vector containing the seasonal observations. First, in Section 2 we discuss notational and representational issues. In Section 3 we analyse the simple first-order autoregressive process. Using Monte Carlo simulations we find that the empirical distributions of the test statistics correspond to the asymptotic distributions. In Section 4 we apply tests for periodic integration to a periodic AR(1) model for German non-durable consumption for 1962.1–1987.4. This particular example is considered since the theoretical model in Osborn (1988) implies that consumption should be described by a PIAR process. Hence, our test procedure can also be useful for evaluating economic theories. In Section 5 we conclude with some remarks.

2. Notation and representation

Consider a time-series y_t, which is quarterly observed during N years, where t runs from 1 to $n = 4N$, and which can be described by a periodic autoregression of order p, PAR(p), or

$$y_t = \sum_{i=1}^{p} \sum_{s=1}^{4} \phi_{is} D_{st} y_{t-i} + \sum_{s=1}^{4} \mu_s D_{st} + \varepsilon_t \,, \tag{1}$$

where ϕ_{is} and μ_s are periodically varying parameters with $s = 1, 2, 3, 4$, ε_t denotes a zero-mean uncorrelated process with constant variance σ^2, and D_{st} represent seasonal dummies. Note that the μ_s parameters do not necessarily indicate that the underlying mean is periodic. This is most easily seen by considering the simple first-order model:

$$y_t - \delta = \phi_{1s}(y_{t-1} - \delta) + \varepsilon_t \,, \tag{2}$$

which can be rewritten as

$$y_t = \phi_{1s} y_{t-1} + \mu_s + \varepsilon_t \,, \tag{3}$$

where μ_s is a function of δ and ϕ_{1s}. In practice, however, one estimates models like (1) unrestrictedly, and we will consider these in the sequel. Note that the lag order is not necessarily p for all seasons, i.e. the orders may be p_s in season s, where $p = \max(p_s)$. The ε_t in (1) can be replaced by an ε_{st} process with seasonally varying variances σ_s^2. Unit root inference is, however, not affected by this extension, and hence we do not consider it any further. See Pagano (1978) and Troutman (1979) for more extensive discussions of periodic autoregressions.

Strictly speaking, the model in (1) is non-stationary, since the variance, the autocovariances and hence the autocorrelations are not constant over the seasons. A more convenient representation of (1) for investigating unit root issues is obtained by stacking the quarterly y_t series into a (4×1) vector Y_T of annual series, where $Y_T = (Y_{1T}, Y_{2T}, Y_{3T}, Y_{4T})'$, with Y_{sT} as the observation in season s in year T, see, for example, Gladyshev (1961), Tiao and Grupe (1980), Osborn (1991) and Franses (1994). The model in (1) can then be represented by

$$A_0 Y_T = A_1 Y_{T-1} + \cdots + A_m Y_{T-m} + \mu + \varepsilon_T \,, \tag{4}$$

where A_i, $i = 0, 1, \ldots, m$, with $m \leq 4p$, are (4×4) parameter matrices, μ is a (4×1) vector of parameters, and ε_T is a vector white noise process with mean zero and covariance matrix $\sigma^2 I_4$. In Eq. (7) below, the A_0 and A_1 matrix for the first-order model as in (2) are given. Note that the model in (4) contains constant parameters.

The vector series Y_T is stationary when the roots of the characteristic equation,

$$|A_0 - A_1 z - \cdots - A_m z^m| = 0 , \tag{5}$$

are outside the unit circle. When one root is on the unit disk one says that the y_t process in (1) is periodically integrated of order 1. In this paper, the focus is on a test for a single unit root.

3. A periodic autoregression of order one

We start by considering the simple periodic first-order autoregressive model

$$y_t = \phi_{1s} y_{t-1} + \varepsilon_t , \tag{6}$$

where $\phi_{1s} \neq \phi_1$ for all s, which can be written in vector notation as

$$A_0 Y_T = A_1 Y_{T-1} + \varepsilon_T , \tag{7}$$

with

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\phi_{12} & 1 & 0 & 0 \\ 0 & -\phi_{13} & 1 & 0 \\ 0 & 0 & -\phi_{14} & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & \phi_{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Define the (4×1) parameter vector $\phi = (\phi_{11}, \phi_{12}, \phi_{13}, \phi_{14})$. The vector process Y_T is stationary if the root of the characteristic equation,

$$|A_0 - A_1 z| = (1 - (\phi_{11}\phi_{12}\phi_{13}\phi_{14})z) = 0 , \tag{8}$$

is outside the unit circle, i.e. if $|g(\phi)| < 1$, where $g(\phi) = \phi_{11}\phi_{12}\phi_{13}\phi_{14}$. Note that the values of some ϕ_{1s} are allowed to exceed unity. The process Y_T is integrated if (8) has a unit root, so that

$$H_0\colon g(\phi) = \prod_{s=1}^{4} \phi_{1s} = 1 , \tag{9}$$

holds. Our subject is to test the null hypothesis (9) against the alternative that $|g(\phi)| < 1$. In the case of (9), the process is said to be periodically integrated of order 1. Otherwise, the process y_t is periodically stationary. Note that the maximum number of unit roots for the Y_T process in (7) is one.

Under the assumption that the errors ε_t in (6) are normally distributed, the maximum likelihood (ML) estimators of ϕ_{1s} are given by the ordinary least squares (OLS) estimators in the regression

$$y_t = \sum_{s=1}^{4} \phi_{1s} D_{st} y_{t-1} + \varepsilon_t \,. \tag{10}$$

Because of the orthogonality of the regressions in (10), we have

$$\hat{\phi}_{1s} = \left[\sum_{t=1}^{n} D_{st} y_{t-1}^2 \right]^{-1} \sum_{t=1}^{n} D_{st} y_{t-1} y_t \,, \tag{11}$$

for $s = 1, 2, 3, 4$. Imposing the null hypothesis leads to the restricted regression

$$y_t = \phi_{11} D_{1t} y_{t-1} + \phi_{12} D_{2t} y_{t-1} + \phi_{13} D_{3t} y_{t-1} + (\phi_{11}\phi_{12}\phi_{13})^{-1} D_{4t} y_{t-1} + \varepsilon_t \,, \tag{12}$$

which can be estimated by non-linear least squares (NLS). A likelihood ratio test statistic may be constructed as

$$LR = n \cdot \ln(RSS_0 / RSS_1) \,, \tag{13}$$

where RSS_0 and RSS_1 denote the residual sums of squares from (12) and (10), respectively. A one-sided test can be constructed as

$$LR_\tau = [\text{sign}(g(\hat{\phi}) - 1) \cdot LR^{1/2} \,. \tag{14}$$

Alternatively, a Wald test can be based on the t-type statistic,

$$\tau = (\hat{V}[g(\hat{\phi})])^{-1/2} (g(\hat{\phi}) - 1) \,, \tag{15}$$

where

$$\hat{V}[g(\hat{\phi})] = (\partial g(\hat{\phi}) / \partial \phi') \hat{V}[\hat{\phi}] (\partial g(\hat{\phi}) / \partial \phi) \,, \tag{16}$$

and $\hat{V}[\hat{\phi}]$ is the usual covariance matrix estimator, which is diagonal because of the orthogonality of the regressors in (10).

Theorem 1. Under the H_0 in (9), and as $n \to \infty$

$$N(g(\hat{\phi}) - 1) \Rightarrow \left[\int_0^1 B(r)^2 \, \mathrm{d}r \right]^{-1} \int_0^1 B(r) \, \mathrm{d}B(r) \,, \tag{17}$$

$$LR_\tau, \tau \Rightarrow \left[\int_0^1 B(r)^2 \, \mathrm{d}r \right]^{-1/2} \int_0^1 B(r) \, \mathrm{d}B(r) \,, \tag{18}$$

where $B(r)$ is a standard Brownian motion process.

Proof. See Boswijk and Franses (1994).

The asymptotic distributions in Theorem 1 are the same as tabulated in Fuller (1976, Tables

8.5.1 and 8.5.2, respectively) for the non-periodic AR model. We see that $N(g(\hat{\phi})-1)$ already has an asymptotic distribution under the null hypothesis that does not depend upon nuisance parameters. So, it can be used as a test statistic alternative to τ, just like the $n(\hat{\rho}-1)$ statistic in the non-periodic case. Observe that $g(\hat{\phi})-1$ should be scaled by N, the number of years in the sample, if it is to be compared with the critical values in Fuller (1976).

Extensions to autoregressive models that include a constant and a trend are straightforward and similar to the standard Dickey–Fuller case, provided that one considers models like

$$y_t = \sum_{s=1}^{4} [\mu_s D_{st} + \delta_s D_{st} T_t + \phi_{1s} D_{st} y_{t-1}] + \varepsilon_t , \tag{19}$$

where T_t represents a deterministic annual trend variable. The test statistics can be calculated along similar lines as above, and the corresponding distributions can be found in the tables in Fuller (1976), see Boswijk and Franses (1994) for details.

In case the order of p in (1) exceeds one, it may be more convenient to consider the LR test statistic in (13), where ϕ becomes a $(4p \times 1)$ parameter vector. To illustrate its use, consider the periodic AR(2) process:

$$y_t = \phi_{1s} y_{t-1} + \phi_{2s} y_{t-2} + \varepsilon_t , \tag{20}$$

which can be written as

$$y_t - \alpha_s y_{t-1} = \beta_s (y_{t-1} - \alpha_{s-1} y_{t-2}) + \varepsilon_t , \tag{21}$$

where $\alpha_{-k} = \alpha_{4-k}$, for $k = 0, 1, 2$. It is easy to recognize that for this model

$$|A_0 - A_1 z| = (1 - \alpha_1\alpha_2\alpha_3\alpha_4 z)(1 - \beta_1\beta_2\beta_3\beta_4 z) = 0 .$$

A non-linear least squares routine can be applied to (21) and the hypothesis that $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ can be tested with the LR test in (13). See Boswijk and Franses (1994) for more details.

To verify whether the asymptotic distributions for models as in (19) provide reasonable approximations, we conduct a small Monte Carlo experiment. The data generating process is a periodic first-order autoregressive process for a mean zero time-series without trends like (6). The test statistics used are the τ test in (15), the LR_τ test in (14) and the $N(g(\hat{\phi})-1)$ test in (17). In Table 1 we report on the empirical size of the tests for two generating processes with the property that $\phi_{11}\phi_{12}\phi_{13}\phi_{14}$ is equal to 1. The maintained regression models contain (no) trends and (no) constants. From the rejection frequencies one can observe that the empirical size is usually too high for the τ test, too low for the $N(g(\hat{\phi})-1)$ test, and approximately adequate for the LR_τ test. The power of the three tests is reported in Table 2. The regression model now contains no trends and no constants. The product of the four AR(1) parameters is about 0.6 and 0.8. From Table 2, it can be concluded that the power of the τ test is usually the highest, while that of the $N(g(\hat{\phi})-1)$ test is the lowest. Note that Table 2 does not report on size-adjusted power, and hence the latter results are likely to be caused by the incorrect size of the $N(g(\hat{\phi})-1)$ test. The power of the LR_τ test is quite reasonable, especially at a nominal size level of 10%. In summary, we recommend the use of the LR_τ test in practice.

Table 1
The empirical size of the tests for periodic integration in a first-order autoregression. The data generating process is $y_t = \phi_{1s} y_{t-1} + \varepsilon_t$, under the restriction $\phi_{1s}\phi_{2s}\phi_{3s}\phi_{4s} = 1$, where ε_t is drawn from a standard normal distribution. Sample size is 100, and the cells contain rejection percentages of the null hypothesis. The number of replications is 1000

Parameters in DGP			Nominal size	Test statistic		
ϕ_{11}	ϕ_{12}	ϕ_{13}		τ	LR_τ	$N(g(\hat{\phi}) - 1)$
The regression model contains no constants and no trends						
0.5	0.90	1.50	5.0	6.0	4.7	2.0
			10.0	11.1	9.5	5.9
1.1	0.91	1.05	5.0	5.5	4.3	3.3
			10.0	11.2	9.3	6.9
The regression model contains constants and no trends						
0.5	0.90	1.50	5.0	8.6	3.9	2.4
			10.0	13.6	7.7	5.1
1.1	0.91	1.05	5.0	9.4	5.0	2.5
			10.0	16.1	9.1	7.0
The regression model contains constants and trends						
0.5	0.90	1.50	5.0	13.6	3.7	0.6
			10.0	19.3	7.4	2.8
1.1	0.91	1.05	5.0	17.0	4.3	0.6
			10.0	21.2	9.5	3.2

Table 2
The empirical power of the tests for periodic integration in a first-order autoregression. The data generating process is $y_t = \phi_{1s} y_{t-1} + \varepsilon_t$, where ε_t is drawn from a standard normal distribution. Sample size is 100, and the cells contain rejection percentages of the null hypothesis. The number of replications is 1000. The regression model contains no constants and no trends

Parameters in DGP				Nominal size	Test statistic		
ϕ_{11}	ϕ_{12}	ϕ_{13}	ϕ_{14}		τ	LR_τ	$N(g(\hat{\phi}) - 1)$
The product $\phi_{11}\phi_{12}\phi_{13}\phi_{14}$ is about 0.6							
0.4	1.2	0.8	1.6	0.05	93.1	86.5	78.4
				0.10	98.6	97.7	94.0
0.6	0.8	1.0	1.2	0.05	96.6	91.7	90.1
				0.10	98.4	97.9	98.3
The product $\phi_{11}\phi_{12}\phi_{13}\phi_{14}$ is about 0.8							
1.3	0.30	1.5	1.3	0.05	56.9	48.1	39.8
				0.10	74.5	70.9	63.8
0.7	0.95	1.2	1.0	0.05	49.0	38.4	29.9
				0.10	69.8	63.3	54.6

4. German consumption, 1962.1–1987.4

In this section, the tests for periodic integration will be applied to the West German consumption data, which cover the period 1962.1–1987.4 as they are given in Appendix E in Lütkepohl (1991). The observations previous to 1962 are used as starting-values.

The pursued model selection strategy amounts to estimating periodic autoregressive models as in (1) of order p, where p is initially set equal to 4, and p is decreased when diagnostic tests indicate no obvious signs of misspecification. It emerges that four trends can be deleted from each of the models. The diagnostic checks used are LM tests for first- and fourth-order residual autocorrelation, F_{AR1} and F_{AR4}, LM tests for first- and fourth-order ARCH effects, F_{ARCH1} and F_{ARCH4}, a $\chi^2(2)$ test for normality of the residuals, and an LM test for first-order periodic autocorrelation, F_{PAR1}.

The specification search yields that there is an additive outlier in the second quarter in 1979, and hence this observation is deleted by including a dummy variable for this observation and for its p next data points. The final model turns out to be

$$c_t = \hat{\mu}_s + \hat{\phi}_{1s} c_{t-1} + \hat{\varepsilon}_t , \tag{22}$$

with

$$\underset{(0.077)}{\hat{\mu}_1 = -0.937} , \quad \underset{(0.069)}{\hat{\mu}_2 = 0.674} , \quad \underset{(0.076)}{\hat{\mu}_3 = 0.053} , \quad \underset{(0.075)}{\hat{\mu}_4 = 0.408} ,$$

$$\underset{(0.010)}{\hat{\phi}_{11} = 1.100} , \quad \underset{(0.009)}{\hat{\phi}_{12} = 0.922} , \quad \underset{(0.010)}{\hat{\phi}_{13} = 0.994} , \quad \underset{(0.009)}{\hat{\phi}_{14} = 0.961} ,$$

where the standard errors are given in parentheses. The diagnostic test values are $F_{AR1} = 0.880$, $F_{AR4} = 1.344$, $F_{ARCH1} = 0.603$, $F_{ARCH4} = 1.571$, $F_{PAR1} = 0.654$ and $\chi^2(2)$ is 1.782. Note that this model is estimated for 104 observations and that the observations for 1979.2 and 1979.3 have been removed to ensure white noise residuals. The point estimates in (22) do not change much when we do not include dummy variables for 1979.2 and 1979.3 in the model.

The F-test for the null hypothesis that the ϕ_{1s} parameters are indeed periodic, i.e. for the hypothesis that $\phi_{1s} = \phi_1$ for all s, obtains a value of 66.636. This suggests that a suitable model for consumption is a periodic AR model of order 1. In Osborn (1988) it is derived from a life-cycle model, which allows for seasonally varying preferences, that a model like (22) is predicted from a standard life-cycle consumption theory. A further implication of that theory is that the PAR(1) model is periodically integrated. This can be verified with the tests given in Section 2. The τ test obtains a value of -1.682, the LR_τ a value of -1.753 and the $N(g(\hat{\phi}) - 1)$ test a value of -0.811. The results in Table 1 indicated that the τ and the $N(g(\hat{\phi}) - 1)$ tests can have incorrect empirical sizes, but that the LR_τ test seems to perform well. Taking these outcomes into account, we conclude that German consumption is periodically integrated, and thus that the life-cycle theory with seasonally varying preferences cannot be rejected for Germany.

5. Concluding remarks

In this paper we have proposed three tests for a unit root in autoregressions, which assume that the autoregressive parameters are periodic, i.e. vary with the seasons. The tests follow standard Fuller (1976) distributions. Monte Carlo evidence indicates that one of the tests is to be preferred. This test can be used as part of an overall strategy to investigate the seasonal and non-stationary properties of univariate time-series, see also Boswijk and Franses (1994). This strategy may consist of estimating a general periodic AR process, and of selecting the most appropriate model via a sequence of tests for parameter restrictions. Future research may be directed to investigate the empirical properties of such a strategy.

In the case when simple periodic time-series models cannot be rejected by the data, one can also use the proposed tests in the present paper to check the predictions from an economic theorem based on the life-cycle hypothesis. In our empirical section we showed that the consumption in Germany obeys some of the restrictions implied by that theory, see Osborn (1988).

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