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by Fractional Programming Techniques

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Abstract. In this paper we adapt the well-known parametric approach from fractional programming to solve a class of fractional programs with a noncompact feasible region. Such fractional problems belong to an important class of single component preventive maintenance models. Moreover, for a special but important subclass we show that the subproblems occurring in this parametric approach are easy solvable. To solve the problem directly we also propose for a related subclass a specialized version of the bisection method. Finally, we present some computational results for these two methods applied to an inspection model and a minimal repair model having both a unimodal failure rate.

Key words: Fractional programming, parametric approach, single component maintenance.

1. Introduction

In [2] a framework in the form of an optimization problem is presented which incorporates almost all of the single component preventive replacement models in the literature where age is the only decision variable. Since in most preventive replacement models in practice this is the only decision variable it is useful for unification purposes to have at hand a general framework describing all these apparently different models. Classical examples of the above models are given by age-replacement, block-replacement, inspection and minimal repair [4, 27]. Having this general framework it is important to present efficient algorithms to solve the corresponding optimization problem. Unfortunately this issue is highly neglected in the maintenance literature which is primarily focused on the construction of models and so the purpose of this paper will be to present two algorithms which solve the optimization problem of [2]. Since the objective function in the general framework is a cost/time ratio with a noncompact feasible region it seems natural to apply fractional programming techniques. This shows another example of a practical problem to which these techniques can be applied and as stressed by [26] in his recent survey on fractional programming this issue is of importance.

We begin by presenting in Section 2 the parametric approach for a special class of fractional programs with a noncompact feasible region. Next we introduce in Section 3 the framework of [2], which can be solved by the described parametric procedure. Since in most cases of practical interest the objective function of the maintenance framework has some nice additional properties we are able to further simplify the parametric approach. In particular, we will show in Section 3.1 for an important subclass that the parametric problem is easy to solve and simultaneously derive localization results for global minima. In Section 3.2 we will derive for a related class an alternative algorithm which can be seen as a specialization of the classical bisection method. Finally, in Section 4 we present and compare the computational results obtained when applying these two procedures to the inspection and the minimal repair model having both a unimodal failure rate function.

To conclude this introduction we like to observe that [1] discuss in very general terms the parametric approach for a more general framework involving also condition based replacement policies. However, their paper is mainly focused on the probabilistic properties of the underlying model and without being aware of related results in fractional programming rediscovers the validity of the parametric approach. Due to the generality of their model they did not give a solution procedure for the parametric problem, which stopping rule to use or derive rate of convergence results. Observe all these issues are addressed in this paper for a slightly less general framework which has important applications in practice [2].

2. Fractional Programming and the Parametric Approach

In this section we consider the fractional programming problem

$$\inf \{g(x) : 0 < x < \infty\} \qquad (P)$$

with $g : (0, \infty) \longrightarrow I\!R$ given by

$$g(x) = \frac{N(x)}{D(x)}.$$

The following conditions are now imposed on the functions N and D.

- The function $N : [0, \infty) \longrightarrow I\!R$ is continuous and strictly positive on $(0, \infty)$ with $N(0) := \lim_{x \downarrow 0} N(x) > 0$ and $N(\infty) := \lim_{x \uparrow \infty} N(x) \leq \infty$.
- The function $D : [0, \infty) \longrightarrow I\!R$ is continuous, strictly increasing and positive on $(0, \infty)$, with $D(0) := \lim_{x \downarrow 0} D(x)$ and $D(\infty) := \lim_{x \uparrow \infty} D(x) \leq \infty$.
- The limit $\infty \geq g(\infty) := \lim_{x \uparrow \infty} g(x)$ exists if $N(\infty) = D(\infty) = \infty$.

Observe, if $D(0) > 0$ we extend the domain of g to $[0, \infty)$ by setting $g(0) = \frac{N(0)}{D(0)}$, and in this case the feasible region of (P) is given by $[0, \infty)$.

By the above assumptions it is clear that g is continuous on $(0, \infty)$ and $\lambda_\star := \inf_{0 < x < \infty} g(x)$ satisfies $0 \leq \lambda_\star < \infty$.

If $N(\infty)$ is finite and $D(\infty) = \infty$ it follows that $g(\infty) = 0$ and so $\lambda_\star$ equals zero. Also in this case $g(x) > 0$ for every feasible x and so the optimization problem (P) has no optimal feasible solution. Since we are interested in optimization problems (P) which have an optimal feasible solution we will only consider the remaining cases.

(i) Both $N(\infty)$ and $D(\infty)$ finite;
(ii) $N(\infty)$ infinite and $D(\infty)$ finite;
(iii) Both $N(\infty)$ and $D(\infty)$ infinite.

Fractional programming problems with noncompact feasible regions yield, beside the existence or not of an optimal feasible solution, problems when applying usual fractional programming solution procedures, as discussed by [22] and [9]. In particular, while for linear fractional programming with a compact feasible region the methods of [20], [21] and [10] are equivalent [28], for the noncompact case this equivalence fails. Moreover, as shown by [22], the methods of [20] and [21] may fail for the noncompact case to recover an optimal feasible solution. Recently [8] and [9] have proposed modifications of respectively the methods of [21] and [10] for linear fractional programs with a noncompact feasible region. Basically, the difficulties created by a noncompact region are whether an optimal feasible solution exists and if so, whether it is possible to apply Dinkelbach's algorithm [14] to recover such a solution. Observe, since Dinkelbach's algorithm uses iteratively an optimal feasible solution of the so-called parametric problem for appropriate values of $\lambda > \lambda_\star$ one presumes that these solutions indeed exist. This clearly holds for compact feasible regions but it is not clear in advance whether it also holds for noncompact feasible regions. To return to the first question and to decide for the class of optimization problems (P) whether an optimal feasible solution exists we note that this always holds for case (ii). For the remaining cases the following necessary and sufficient condition is easy to derive.

LEMMA 1. *For cases (i) and (iii) the optimization problem (P) has an optimal feasible solution if and only if there exists a feasible x_0 satisfying $g(x_0) \leq g(\infty)$.*

Proof. If (P) has an optimal feasible solution then there exists a feasible x_0 satisfying $g(x_0) = \lambda_\star \leq g(\infty)$ and so the if part is verified. Moreover, if there exists some feasible x_0 satisfying $g(x_0) \leq g(\infty)$ and additionally $g(\infty) = \lambda_\star$ then clearly x_0 is an optimal feasible solution. If $\lambda_\star < g(\infty)$ then for $D(0) > 0$ the result follows from the continuity of g on $[0, \infty)$ and Weierstrass theorem [5]. A similar proof applies for $D(0) = 0$ or equivalently $g(0) = \infty$.

We will assume in the remainder that (P) has an optimal feasible solution. This implies that inf in (P) can be replaced by min and the set $\mathcal{X}^\star$ of optimal feasible solutions of (P) is nonempty.

Consider now the following parametric problem associated with (P) given by

$$\inf \{g_\lambda(x) : 0 \leq x < \infty\} \qquad (P_\lambda)$$

with $g_\lambda : [0, \infty) \longrightarrow \mathbb{R}$ defined by

$$g_\lambda(x) := N(x) - \lambda D(x),$$

for $\lambda \in \mathbb{R}$. Let also $p : \mathbb{R} \longrightarrow [-\infty, \infty)$ be the associated parametric function

$$p(\lambda) := \inf \{g_\lambda(x) : 0 \leq x < \infty\}$$

and $\mathcal{X}^\star(\lambda)$ the (possibly empty) set of optimal feasible solutions of (P_λ). For $D(0) = 0$ the feasible region of (P_λ) differs from the feasible region of the "classical" parametric problem with a compact feasible region as introduced by [14] or with an arbitrary feasible region as discussed by [11]. This modification is made in order not to exclude zero beforehand to become an optimal feasible solution of (P_λ). However, for the optimization problem (P) under some conditions the point zero will never be an optimal feasible solution of (P_λ) for $\lambda \geq \lambda_\star$ and so taking the closure of the original feasible region will not influence the structure of the parametric problem. Similarly as in [11] one can show the following result.

LEMMA 2.
(a) $p(\lambda) < +\infty$ *for every* $\lambda \in \mathbb{R}$*, and the function* p *is decreasing.*
(b) $p(\lambda) < 0$ *if and only if* $\lambda > \lambda_\star$*. Moreover,* $p(\lambda_\star) \geq 0$*.*
(c) If (P*) has an optimal feasible solution then* $p(\lambda_\star) = 0$*.*

Another useful result relating (P) and (P_λ) is given by the next lemma. Observe the first part is also discussed by [11]) but due to our modification of the feasible region for $D(0) = 0$ we also need to show its validity for this case.

LEMMA 3. *If* $p(\lambda_\star) = 0$*, then (*P*) and (*$P_{\lambda_\star}$*) have the same set of optimal feasible solutions (which may be empty).*

Proof. If $D(0) > 0$ then the proof is given by (d) of Proposition 2.1 of [11]. Let now $D(0) = 0$ and suppose that $x_\star$ is an optimal feasible solution of (P). Then clearly $\frac{N(x)}{D(x)} \geq \frac{N(x_\star)}{D(x_\star)} = \lambda_\star$ for every $x > 0$ and this implies that $N(x) - \lambda_\star D(x) \geq 0 = N(x_\star) - \lambda_\star D(x_\star)$. Since the functions N and D are continuous this yields that $x_\star$ is an optimal feasible solution of ($P_{\lambda_\star}$). Moreover, if $p(\lambda_\star) = 0$ and $x_\star$ is an optimal feasible solution of ($P_{\lambda_\star}$) then $x_\star$ must be positive due to $N(0) > 0$ and $D(0) = 0$. Since for every $x > 0$ we have that $N(x) - \lambda_\star D(x) \geq N(x_\star) - \lambda_\star D(x_\star) = 0$ this implies that $x_\star$ is an optimal feasible solution of (P). Finally, if either the optimal feasible solution set of (P) or of ($P_{\lambda_\star}$) is empty then by contradiction and the previous observation one can show that the other set is also empty.

It is well known that Dinkelbach's algorithm is nothing else than the Newton–Raphson root finding procedure applied to the function p. Therefore we must know at least for which values of λ the function p is finite-valued. Among other results this issue is discussed in the next lemma.

LEMMA 4. *If* $N(\infty)$ *and* $D(\infty)$ *are finite, or* $N(\infty)$ *is infinite and* $D(\infty)$ *is finite then the function* p *is finite-valued, concave and continuous on* $(-\infty, \infty)$.

If both $N(\infty)$ *and* $D(\infty)$ *are infinite then the function* p *is finite-valued, continuous and concave on* $(-\infty, g(\infty))$.

Proof. Clearly for the case that $N(\infty)$ and $D(\infty)$ are finite or $N(\infty)$ is infinite but $D(\infty)$ is finite, the function p is finite-valued and concave. Hence, by a standard result in convex analysis [18] the function p must be continuous on $\mathbb{R}$.

Finally, if both $N(\infty)$ and $D(\infty)$ are infinite and $g(\infty)$ exists we obtain easily that $g_\lambda(\infty) = \infty$ if $\lambda < g(\infty)$ and $g_\lambda(\infty) = -\infty$ if $\lambda > g(\infty)$. This implies that the function p is finite-valued on $(-\infty, g(\infty))$ and since p is concave it must be continuous on $(-\infty, g(\infty))$ [18].

The next example illustrates that for $N(\infty)$ and $D(\infty)$ infinite it might occur that $p(g(\infty)) = -\infty$ for $g(\infty) < \infty$.

EXAMPLE 5. Let the functions N and D be given by

$$N(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ x - \ln x & \text{if } x > 1 \end{cases},$$

and $D(x) = x$. For these functions it follows that the optimization problem (P) satisfies $\lambda_\star = 1 - \exp(-1)$ and the optimal feasible solution is given by $x_\star = \exp(1)$. Moreover, $g(\infty) = 1$ and $p(1) = -\infty$.

By the concavity of the function p the next result is easy to verify. Remember the scalar $a \in \mathbb{R}$ is called a supergradient of the function p at the point λ_0 if

$$p(\lambda) \leq p(\lambda_0) + a\,(\lambda - \lambda_0)$$

for every $\lambda \in \mathbb{R}$. Observe the next result holds without any monotonicity conditions on D.

LEMMA 6. *If (P_λ) has an optimal feasible solution x_λ then $-D(x_\lambda)$ is a supergradient of p at the point λ. Moreover, for x_{λ_1} and x_{λ_2} optimal feasible solutions of (P_{λ_1}), respectively (P_{λ_2}) with $\lambda_2 < \lambda_1$ we have $D(x_{\lambda_2}) \leq D(x_{\lambda_1})$.*

Since for the optimization problem (P) it is assumed that the function D is strictly increasing the following result is an easy consequence of Lemma 6.

LEMMA 7. *If x_{λ_1} and x_{λ_2} are optimal feasible solutions of (P_{λ_1}), respectively (P_{λ_2}) with $\lambda_2 < \lambda_1$ then $x_{\lambda_2} \leq x_{\lambda_1}$. Moreover, if the nonempty optimal feasible solution set $\mathcal{X}^\star$ of (P) satisfies $\mathcal{X}^\star \subseteq (0, \infty)$ then 0 is no optimal feasible solution of (P_λ) for every $\lambda \geq \lambda_\star$.*

Proof. The first part is an immediate consequence of Lemma 6 and D strictly increasing. Clearly the second part holds for $\mathcal{X}^\star(\lambda)$ empty. Therefore assume $\mathcal{X}^\star(\lambda)$ nonempty and by the first part it is sufficient to show the result for $\lambda = \lambda_\star$. Observe now by (c) of Lemma 2 and Lemma 3 that $\mathcal{X}^\star(\lambda_\star) = \mathcal{X}^\star \subseteq (0, \infty)$ and this proves the result.

As already observed applying Dinkelbach's algorithm might not be possible if we only know that (P) has an optimal feasible solution. Additionally, we also need to identify some interval containing $\lambda_\star$ on which the parametric problem has an optimal feasible solution. Unfortunately we cannot use Lemma 4 to identify such an interval since it might happen for $p(\lambda)$ finite that the corresponding problem (P_λ) has no optimal feasible solution as shown by the next example.

EXAMPLE 8. Let the functions N and D be given by

$$N(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases}, \quad \text{and } D(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ x - \frac{1}{x} + 1 & \text{if } x > 1 \end{cases}.$$

For these functions it follows that the optimization problem (P) satisfies $\lambda_\star = \frac{4}{5}$ and its optimal feasible solution is given by $x_\star = 2$. Moreover, $g(\infty) = 1$ and $p(1) = -1$. However the parametric problem does not have an optimal feasible solution for $\lambda = 1$.

The next results relate for the three cases (i), (ii) and (iii) the conditions that (P) has an optimal feasible solution to the conditions that (P_λ) has an optimal feasible solution for appropriate values of λ. At the same time it identifies an interval on which the parametric problem has an optimal feasible solution.

LEMMA 9. *If $N(\infty)$ and $D(\infty)$ are finite the optimization problem (P) has an optimal feasible solution if and only if (P_{λ_0}) with $\lambda_0 := g(\infty)$ has an optimal feasible solution. Moreover, if (P) has an optimal feasible solution then (P_λ) has an optimal feasible solution for all $\lambda \leq g(\infty)$, and $p(\lambda) = 0$ implies $\lambda = \lambda_\star$.*

Proof. If (P_{λ_0}) has an optimal feasible solution then there exists some feasible x_0 satisfying

$$p(\lambda_0) = g_{\lambda_0}(x_0) \leq g_{\lambda_0}(\infty) = 0$$

and so for $D(0) = 0$ the point x_0 cannot be zero due to $N(0) - \lambda_0 D(0) = N(0) > 0$. Since $g_{\lambda_0}(x_0) \leq 0$ this implies $g(x_0) \leq \lambda_0$ and applying Lemma 1 yields that (P) has an optimal feasible solution.

To show the reverse implication we observe since (P) has an optimal feasible solution that by Lemma 1 there exists a feasible x_0 satisfying $g_{\lambda_0}(x_0) \leq 0 = g_{\lambda_0}(\infty)$. Hence for every $\lambda \leq \lambda_0$ we obtain by the monotonicity of the function D that

$$\begin{aligned} 0 &\leq g_{\lambda_0}(\infty) - g_{\lambda_0}(x_0) = (N(\infty) - N(x_0)) - \lambda_0 (D(\infty) - D(x_0)) \\ &\leq (N(\infty) - N(x_0)) - \lambda (D(\infty) - D(x_0)) = g_\lambda(\infty) - g_\lambda(x_0) \end{aligned}$$

and this shows that $p(\lambda) \leq g_\lambda(x_0) \leq g_\lambda(\infty)$ for every $\lambda \leq \lambda_0$. Since the functions D and N are continuous we obtain by a similar argument as used in Lemma 1 that the parametric problem (P_λ) has an optimal feasible solution for $\lambda \leq \lambda_0$.

To prove the last part we observe by (b) of Lemma 2 that $\lambda \leq \lambda_\star \leq \lambda_0$ if $p(\lambda) = 0$. Moreover, since (P) has an optimal feasible solution this implies using the just proved result that (P_λ) has an optimal feasible solution. Hence, there exists some feasible x_λ such that

$$0 = p(\lambda) = N(x_\lambda) - \lambda D(x_\lambda) \leq N(x) - \lambda D(x)$$

for every $x \geq 0$. Clearly for $D(0) = 0$ it follows that x_λ cannot be zero and so $g(x) \geq g(x_\lambda) = \lambda$ for every feasible x proving the last result.

Using the same arguments as in Lemma 9 one can show the following result, for case (ii), i.e. $N(\infty)$ infinite and $D(\infty)$ finite.

LEMMA 10. *If $N(\infty)$ is infinite and $D(\infty)$ is finite the optimization problems (P) and (P_λ) for all $\lambda \in I\!R$, have an optimal feasible solution. Moreover, if $p(\lambda) = 0$ then $\lambda = \lambda_\star$.*

Finally we will consider case (iii).

LEMMA 11. *If both $N(\infty)$ and $D(\infty)$ are infinite then the parametric problem (P_λ) has an optimal feasible solution for every $\lambda < g(\infty)$. Moreover, if (P) has an optimal feasible solution and $p(\lambda) = 0$ then $\lambda = \lambda_\star$.*

Proof. The first part is easy to verify and so we omit its proof.

To verify the second part observe since $p(\lambda) = 0$ that $g_\lambda(x) \geq 0$ for every $x > 0$. This implies that $g(x) \geq \lambda$ for every $x > 0$ and hence $\lambda_\star \geq \lambda$. If $\lambda < \lambda_\star \leq g(\infty)$ then by the first part the problem (P_λ) has an optimal feasible solution and so there exists some $x_\lambda \geq 0$ for $D(0) > 0$ or $x_\lambda > 0$ for $D(0) = 0$ satisfying $g_\lambda(x_\lambda) = 0$ and hence $g(x_\lambda) = \lambda < \lambda_\star$. This yields a contradiction with the definition of $\lambda_\star$ and so $\lambda = \lambda_\star$.

As shown by Example 8 it is in general not true that also (P_{λ_0}) has an optimal feasible solution and so the above result is the best possible.

The following important theorem relates, for all the three cases, the optimization problem (P) to its parametric problem (P_λ).

THEOREM 12. *If (P) has an optimal feasible solution then $p(\lambda) = 0$ if and only if $\lambda = \lambda_\star$. Also, if (P) has an optimal feasible solution then the optimal feasible solution set of (P) equals the optimal feasible solution set of (P_λ) for $\lambda = \lambda_\star$.*

Proof. The above result is an immediate consequence of Lemma 2 and respectively Lemma 9 for case (i), Lemma 10 for case (ii) and Lemma 11 for case (iii).

By Lemmas 9, 10 and 11 and Theorem 12 it is clear that with a proper chosen starting point λ_1 we can apply Dinkelbach's algorithm to find an optimal feasible solution of (P). Observe that such a value can be found by simply taking $\lambda_1 := g(x_0) \leq g(\infty)$, with x_0 a feasible point. Clearly, if $N(\infty)$ and $D(\infty)$ are infinite and $\lambda_\star < g(\infty) < \infty$ we could have used some feasible x_0 satisfying $g(x_0) = g(\infty)$, but if this holds it may happen that (P_{λ_0}) does not have an optimal feasible solution. However, due to $\lambda_\star < g(\infty)$ there must exist some feasible x_0 satisfying $g(x_0) < g(\infty)$ and hence, by using this value we can still properly apply Dinkelbach's algorithm. Notice, by Lemma 1 we know that there exists some feasible x_0 satisfying $g(x_0) \leq g(\infty)$ or in case $N(\infty)$ and $D(\infty)$ infinite and $\lambda_\star < g(\infty)$ some x_0 satisfying $g(x_0) < g(\infty)$.

Dinkelbach's algorithm can now be summarized as follows:

ALGORITHM 1 (Dinkelbach).

Step 1.
Let $\lambda_1 := g(x_0)$, $k := 1$ and `Goto` **Step** 2.
Step 2.
Determine $x_{\lambda_k} := \text{argmin}_{0 \leq x < \infty} g_{\lambda_k}(x)$ and `GoTo` **Step** 3.
Step 3.
Let $\lambda_{k+1} := g(x_{\lambda_k})$.
`If` $\lambda_{k+1} = \lambda_k$
  `Then` `Stop`.
  `Else` Let $k := k + 1$ and `GoTo` **Step** 2.

By the previous lemmas it follows that the parametric problem (P_{λ_k}) always has an optimal feasible solution and by a similar reasoning as in [14] one can show that the sequence $\{\lambda_k : k \geq 1\}$ is decreasing with $\lim_{k \uparrow \infty} \lambda_k = \lambda_\star$. Moreover, if $\lambda_k = \lambda_{k+1}$ for some finite k then $\lambda_k = \lambda_\star$ and the algorithm stops with the optimal feasible solution x_{λ_k} (Theorem 12). As shown by [25] the sequence $\{\lambda_k : k > 1\}$ converges superlinear or sometimes quadratic to the value $\lambda_\star$ if the algorithm does not stop after a finite number of iterations.

Observe also that the stopping rule $p(\lambda_k) \geq -\epsilon$ with $\epsilon > 0$ some given constant can be used. If at iteration k this stopping rule holds it follows by the supergradient inequality applied at λ_k and using Lemma 6 and Theorem 12 that

$$-\epsilon \leq p(\lambda_k) \leq p(\lambda_\star) - (\lambda_k - \lambda_\star)\, D(x_{\lambda_k}) = -\,(\lambda_k - \lambda_\star)\, D(x_{\lambda_k})$$

with x_{λ_k} belonging to $\mathcal{X}^\star(\lambda_k)$. This inequality implies that $\lambda_k < \lambda_\star + \epsilon D(x_{\lambda_k})^{-1}$ and since the sequence λ_k, $k > 1$ decreases to $\lambda_\star$ we obtain

$$\lambda_\star \leq g(x_{\lambda_k}) = \lambda_{k+1} \leq \lambda_k \leq \lambda_\star + \epsilon \left(D(x_{\lambda_k})\right)^{-1}.$$

Finally, observe that the efficiency of Dinkelbach's algorithm strongly relays on the ease to solve the parametric problem (P_λ), and that this depends on the structure of the original problem (P). As we will see in the next section, there are several classes of maintenance problems for which the parametric problem (P_λ) is easy to solve. Moreover, in that section we will also specialize the above results to a general single-component preventive maintenance model.

3. A General Single-Component Maintenance Model

[2] argue that the objective function of any preventive maintenance model, where age is the only decision variable to start preventive maintenance, necessarily has the form of the optimization problem (P) with

$$N(x) := c + \int_0^x m(z)\, h(z)\, \mathrm{d}z$$

and

$$D(x) := d + \int_0^x h(z)\, \mathrm{d}z$$

with $c > 0, d \geq 0$ and $m : [0, \infty) \longrightarrow I\!R$ a continuous and positive function, while the function $h : [0, \infty) \longrightarrow I\!R$ is continuous and strictly positive on $(0, \infty)$. A proof of their result can be found in [15]. Well-known examples are the average cost and expected total discounted cost versions of the age replacement and block replacement model [6], the minimal repair and standard inspection model [12] and models where preventive maintenance is only possible at opportunities [13, 2]. Other less known models which fit into the above framework are discussed in [7, 12] and [2].

Observe that this problem falls into the fractional programming class discussed in the previous section. Introducing

$$\infty \geq \beta_0 := \lim_{x \uparrow \infty} \int_0^x m(z)\, h(z)\, \mathrm{d}z \geq 0$$

and

$$\infty \geq \beta_1 := \lim_{x \uparrow \infty} \int_0^x h(z)\, \mathrm{d}z > 0$$

we have that $N(\infty) := c + \beta_0$ and $D(\infty) := d + \beta_1$. Moreover, to describe maintenance activities the only realistic situations correspond also to cases (i), (ii) and (iii). Also, if the optimization problem (P) has an optimal feasible solution and $d > 0$ it is only realistic from a maintenance point of view to assume that any optimal feasible solution is strictly positive. An example of (i) is given by the age replacement model, while the block replacement and inspection model satisfy the assumptions of (iii). To guarantee that $g(\infty)$ also exists for case (iii) we introduce the additional assumption that $m(\infty) := \lim_{x \uparrow \infty} m(x)$ exists, implying $g(\infty) = m(\infty)$. For this special problem we can specialize Lemma 1 as follows:

LEMMA 13. *If β_0 and β_1 are finite then the optimization problem (P) has an optimal feasible solution if and only if there exists a feasible x_0 satisfying*

$$\frac{\int_{x_0}^{\infty} m(z)\, h(z)\, \mathrm{d}z}{\int_{x_0}^{\infty} h(z)\, \mathrm{d}z} \geq \frac{c + \beta_0}{d + \beta_1}.$$

Moreover, if β_0 and β_1 are infinite and $m(\infty) < \infty$ then the optimization problem (P) has an optimal feasible solution if and only if there exists some feasible x_0 satisfying

$$\int_0^{x_0} (m(\infty) - m(z))\, h(z)\, \mathrm{d}z \geq c - d\, m(\infty).$$

The above result does not exclude for $d > 0$ that 0 is an optimal feasible solution of (P). However, in the remainder we will always assume that for $d > 0$ the point 0 is not an optimal feasible solution and so the nonempty set $\mathcal{X}^\star$ of optimal feasible solutions satisfies $\mathcal{X}^\star \subseteq (0, \infty)$. Observe that, for β_0 and β_1 finite it is easy to verify, if $m(\infty)$ exists and $m(\infty) > g(\infty) = \frac{c+\beta_0}{d+\beta_1}$, that the sufficiency condition stated in Lemma 13 holds. Also for this case, if we have found some x_0 satisfying $m(x) > g(\infty)$ for every $x \geq x_0$, then $g(x_0) < g(\infty)$ and so $\lambda_1 = g(x_0)$ can be used to start Dinkelbach's algorithm.

3.1. Applying the Parametric Approach

Before specializing the results derived in Section 2 notice that if (P) has an optimal feasible solution and $d > 0$ it can be easily shown that $\mathcal{X}^\star \subseteq (0, \infty)$ if $m(0) < g(0)$. Using now Lemma 7 it is relatively easy to prove the validity of the following lower bound on the optimal feasible solution of (P_λ) for λ appropriately chosen.

LEMMA 14. *If the nonempty optimal feasible solution set $\mathcal{X}^\star$ of (P) satisfies $\mathcal{X}^\star \subseteq (0, \infty)$ and the continuous function $m : [0, \infty) \longrightarrow \mathbb{R}$ is decreasing on $[0, b]$ for some $b \geq 0$ then $x_\lambda > b$ for any optimal feasible solution x_λ of (P_λ) with $\lambda_\star \leq \lambda$. Moreover, the same result holds for any optimal feasible solution of (P).*

Proof. Clearly, for $b = 0$ the result follows by Lemma 7 and so we assume that $b > 0$. Using Lemma 6 and the function D strictly increasing it is enough to prove these results for $\lambda_\star$. Hence, suppose $x_{\lambda_\star}$ is an optimal feasible solution of ($P_{\lambda_\star}$). By Lemma 7 it follows that $x_{\lambda_\star} > 0$ and so by the first order necessary conditions for a global minimum and h strictly positive on $(0, \infty)$ we obtain that $m(x_{\lambda_\star}) - \lambda_\star = 0$. If $x_{\lambda_\star} \leq b$ this implies since m is decreasing on $[0, b]$ and h is strictly positive on $(0, \infty)$ that

$$g'_{\lambda_\star}(x) = h(x)(m(x) - \lambda_\star) \geq h(x)\left(m(x_{\lambda_\star}) - \lambda_\star\right) = 0$$

for every $0 < x \leq x_{\lambda_\star}$. Hence by the mean value theorem [24] this yields $g_{\lambda_\star}(0) \leq g_{\lambda_\star}(x_{\lambda_\star})$ and since $x_{\lambda_\star}$ is an optimal feasible solution it must follow that 0 is also an optimal feasible solution contradicting Lemma 7. This proves the first part and by the first part and Theorem 12 the second part follows immediately.

Before discussing a class of functions for which the parametric problem is easy to solve we observe if (P) has an optimal feasible solution that for any scalar λ_k

generated by Dinkelbach's algorithm with starting value $\lambda_1 := g(x_0)$ it follows by Lemmas 9, 10 and 11 that the set $\mathcal{X}^\star(\lambda_k)$ of optimal feasible solutions of (P_{λ_k}) is a nonempty set. Moreover, if additionally 0 is no optimal feasible solution of (P) (only needed for $d > 0$) and m is decreasing on $[0, b]$ with $b \geq 0$ we know by Lemma 14 that $x_{\lambda_k} > b$ for every x_{λ_k} belonging to $\mathcal{X}^\star(\lambda_k)$. By these observations it follows that the constrained optimization problem (P_{λ_k}) is actually unconstrained and so by the Karush–Kuhn–Tucker conditions for an optimum and the strict positivity of h on $(0, \infty)$ we obtain that

$$\mathcal{X}^\star(\lambda_k) \subseteq \{x \in I\!R : m(x) - \lambda_k = 0, b < x\} . \tag{1}$$

Hence to find an optimal feasible solution of the parametric problem (P_λ) we have to generate in the general case all stationary points $x > b$.

Another way to solve the global optimization problem (P_λ) is given by one of the algorithms discussed by [16] if we know additionally an upper bound u on the set of optimal feasible solutions of (P_λ) and an upper bound on the values $\max\{m(x) : b \leq x \leq u\}$ and $\max\{h(x) : b \leq x \leq u\}$. In this case the optimization problem (P_λ) reduces to the optimization of an univariate Lipschitz continuous function over a compact interval. Moreover, by Lemma 7 it follows for $\lambda_{k+1} \leq \lambda_k$ that

$$x_{\lambda_{k+1}} \leq x_{\lambda_k}$$

and hence we have an upper bound on the set $\mathcal{X}^\star(\lambda_{k+1})$

$$\mathcal{X}^\star(\lambda_{k+1}) \subseteq \{x \in I\!R : m(x) - \lambda_{k+1} = 0, b < x \leq x_{\lambda_k}\} .$$

At the same time it shows using Theorem 12 that $\lim_{k\uparrow\infty} x_{\lambda_k} = x_b$ with x_b the biggest optimal feasible solution of (P). By the previous observation it is now easy to introduce a class of functions for which the parametric problem (P_λ) has a nice structure and is therefore almost trivially solvable.

DEFINITION 15. [3] A function $f : [0, \infty)$ is called unimodal if there exists some $0 \leq z_0 < \infty$ such that f is decreasing on $[0, z_0]$ and increasing on (z_0, ∞).

It is clear for any unimodal function f that $\infty \leq f(\infty) := \lim_{z\uparrow\infty} f(z)$ exists. Moreover, if additionally f is continuous, one can also find constants $0 \leq b_1 \leq b_2 \leq \infty$ satisfying f attains its minimum on $[b_1, b_2]$. In this case the continuous function f is called unimodal with parameters $0 \leq b_1 \leq b_2 \leq \infty$. The next theorem characterizes the optimal feasible solution set of (P_λ) and (P) and shows that under a unimodality assumption the inclusion in (1) can be replaced by an equality. Observe a weaker result for the optimization problem (P) with $d = 0$ using a completely different approach is given by [12].

THEOREM 16. *If the optimal feasible solution set* $\mathcal{X}^\star$ *of* (P) *satisfies* $\mathcal{X}^\star \subseteq (0, \infty)$, *the continuous function* $m : [0, \infty) \longrightarrow I\!R$ *is unimodal with parameters*

$0 \leq b_1 \leq b_2 < \infty$ and for $\lambda \geq \lambda_\star$ the parametric problem (P_λ) has an optimal feasible solution then the optimal feasible solution set $\mathcal{X}^\star(\lambda)$ of (P_λ) equals the closed interval

$$\{x \in I\!R : m(x) - \lambda = 0,\, b_2 < x\}\,.$$

Moreover, under the same conditions the optimal feasible solution set of (P) is given by the closed interval

$$\{x \in I\!R : m(x) - \lambda_\star = 0,\, b_2 < x\}\,.$$

Proof. Clearly by (1) we have that

$$\mathcal{X}^\star(\lambda) \subseteq \{x \in I\!R : m(x) - \lambda = 0,\, b_2 < x\}\,.$$

Consider therefore an arbitrary point $x > b_2$ satisfying $m(x) - \lambda = 0$ and let x_λ denote an optimal feasible solution of (P_λ). Since by Lemma 14 the inequality $x_\lambda > b_2$ holds and m is increasing on $[b_2, \infty)$ we obtain due to $m(x_\lambda) = \lambda$ that $m(y) = \lambda$ for every

$$\min\{x, x_\lambda\} \leq y \leq \max\{x, x_\lambda\}$$

and so $g'_\lambda(y) = 0$. Applying now the mean value theorem yields g_λ is constant on the interval $[\min\{x, x_\lambda\}, \max\{x, x_\lambda\}]$ and hence $g_\lambda(x) = g_\lambda(x_\lambda)$. This shows the first part and the second part follows immediately from this and Theorem 12.

By Theorem 16 it is relatively easy using for example a standard bisection algorithm [5] to compute an optimal feasible solution of (P_λ) for the appropriate λ. Clearly for the subset of unimodal functions m which are strictly increasing on (b_2, ∞) we obtain that (P_λ) for the appropriate λ has a unique optimal feasible solution and this implies by Theorem 16 that the optimization problem (P) also has a unique optimal feasible solution.

Finally, if the continuous function m is unimodal with parameters $0 \leq b_1 \leq b_2 < \infty$ and the function h is decreasing it is also possible for $d = 0$ to solve (P) by using a specialized version of the bisection method. This will be discussed in the next section.

3.2. Specialized Bisection Method

Besides the Newton–Raphson root finding approach, another important method to find the unique root of the univariate equation $p(\lambda) = 0$ is given by the classical bisection method. Basically this method constructs a succession of smaller intervals containing the root. Although at each iteration the diameter of the interval is halved, the convergence rate of this method is only linear in opposition to the Dinkelbach algorithm. [19] proposes a variant of this method, resorting to the bounds produced by the Dinkelbach method, which converges superlinearly. However, the bisection-like method introduced in this section aims at solving directly (P), and fully exploits the very special structure of the maitenance model discussed in this section.

Since the function D is strictly increasing its inverse $D^{\leftarrow} : [d, d+\beta_1) \longrightarrow [0,\infty)$ exists and is also strictly increasing. Let us consider also the function $\upsilon : (\frac{1}{d+\beta_1}, \frac{1}{d}] \longrightarrow [0,\infty)$ given by

$$\upsilon(y) = yN(D^{\leftarrow}(\tfrac{1}{y})).$$

Clearly by the above definition it follows that the objective function g of the optimization problem (P) equals $g(x) = \upsilon(\frac{1}{D(x)})$. It is now possible to prove the following result for the function υ. In the next lemma notice that if both $b = 0$ and $D(0) = d = 0$ then $\frac{1}{D(b)} = \infty$.

LEMMA 17. *The function υ is convex on $(\frac{1}{d+\beta_1}, \frac{1}{D(b)})$ if and only if the continuous function m is increasing on (b,∞).*

Proof. Clearly, the function υ is convex on $(\frac{1}{d+\beta_1}, \frac{1}{D(b)})$ if and only if the function $y \longmapsto yN(D^{\leftarrow}(\frac{1}{y}))$ is convex on $(\frac{1}{d+\beta_1}, \frac{1}{D(b)})$. Adapting slightly the proof of Theorem I.1.1.6 of [18] one can easily show using the criteria of increasing slopes that $y \longmapsto yN(D^{\leftarrow}(\frac{1}{y}))$ is convex on $(\frac{1}{d+\beta_1}, \frac{1}{D(b)})$ if and only if $y \longmapsto N(D^{\leftarrow}(y))$ is convex on $(D(b), d+\beta_1)$. The derivative of the last function equals $m(D^{\leftarrow}(y))$ and since by our assumption the function m is increasing on (b,∞) and $D^{\leftarrow}$ is strictly increasing the equivalence follows.

A direct consequence of the above lemma and the subgradient inequality for convex functions is given by the next result.

LEMMA 18. *If the continuous function m is increasing on (b,∞) and $a > b$ is some arbitrary point then*

$$g(x) \geq m(a) + (g(a) - m(a))\,\frac{D(a)}{D(x)}$$

for every $x > b$.

Proof. By Lemma 17 it follows that the function υ is convex on $(\frac{1}{d+\beta_1}, \frac{1}{D(b)})$. This implies by the subgradient inequality that

$$\begin{aligned} g(x) &= \upsilon(\tfrac{1}{D(x)}) \geq \upsilon(\tfrac{1}{D(a)}) + \upsilon'(\tfrac{1}{D(a)})\left(\tfrac{1}{D(x)} - \tfrac{1}{D(a)}\right) \\ &= g(a) + \upsilon'(\tfrac{1}{D(a)})\left(\tfrac{1}{D(x)} - \tfrac{1}{D(a)}\right). \end{aligned}$$

It is easy to verify that $\upsilon'(y) = \frac{\upsilon(y) - m(D^{\leftarrow}(y^{-1}))}{y}$ and substituting $y = \frac{1}{D(a)}$ yields the desired inequality.

To derive a specialized bisection algorithm we restrict ourselves in the remainder of this section to an optimization problem (P) with $d = 0$ for which the continuous function h is decreasing and the function m is unimodal and continuous with parameters $0 \leq b_1 \leq b_2 < \infty$. Examples of these models are given by age replacement, inspection and minimal repair with a unimodal failure rate function [27].

Under the above conditions it follows that the strictly increasing function D is concave and so its inverse function $D^{\leftarrow}$ is convex. Moreover, since $d = 0$ we have that $D^{\leftarrow}(0) = 0$ and using its convexity we obtain that $D^{\leftarrow}(\alpha x) \leq \alpha D^{\leftarrow}(x)$ for every $0 < \alpha < 1$.

To start the discussion on our specialized bisection algorithm we observe the following. Let $[d_0 - r_0, d_0 + r_0]$ be the initial interval of uncertainty containing an optimal feasible solution of (P). By Theorem 16 we may choose $d_0 - r_0$ equal to b_2 and $d_0 + r_0$ equal to x_{λ_1} with x_{λ_1} obtained in **Step** 2 of Algorithm 1 for $\lambda = \lambda_1$. If d_k is the midpoint of the $(k+1)$th uncertainty interval $[d_k + r_k, d_k - r_k] \subseteq [d_0 - r_0, d_0 + r_0]$ then we know for $l_k := \min\{g(d_i) : i \leq k\}$ that $l_k \geq g(x_\star)$ with $x_\star$ an optimal feasible solution and so by Theorem 16 and Lemma 18 it follows for $x = x_\star$ and $a = d_k$ that

$$l_k \geq m(d_k) + \left(g(d_k) - m(d_k)\right) \frac{D(d_k)}{D(x_\star)}. \tag{2}$$

This implies for $g'(d_k) > 0$ or equivalently $g(d_k) - m(d_k) < 0$ that $l_k - m(d_k) \leq g(d_k) - m(d_k) < 0$ and so the above inequality yields that

$$D(x_\star) \leq \frac{g(d_k) - m(d_k)}{l_k - m(d_k)} D(d_k).$$

Hence by our previous observation about the function D we obtain that

$$x_\star = D^{\leftarrow}(D(x_\star)) \leq D^{\leftarrow}\left(\frac{g(d_k) - m(d_k)}{l_k - m(d_k)} D(d_k)\right) \leq \frac{g(d_k) - m(d_k)}{l_k - m(d_k)} d_k.$$

Moreover, if $g'(d_k) < 0$ or equivalently $g(d_k) - m(d_k) > 0$ it follows by (2) that $l_k > m(d_k)$ and

$$D(x_\star) \geq \frac{g(d_k) - m(d_k)}{l_k - m(d_k)} D(d_k).$$

Again by a similar argument we obtain

$$x_\star \geq \frac{g(d_k) - m(d_k)}{l_k - m(d_k)} d_k.$$

Observe, if $g'(d_k) = 0$ then d_k is the optimal feasible solution and we stop the algorithm. By these observations it follows for

$$\alpha_k := \frac{g(d_k) - m(d_k)}{l_k - m(d_k)}$$

that the midpoint of the new interval of uncertainty is given by

$$d_{k+1} = \begin{cases} \frac{1}{2}(d_k + r_k + \alpha_k d_k) & \text{if } g'(d_k) < 0 \\ \frac{1}{2}(d_k - r_k + \alpha_k d_k) & \text{if } g'(d_k) > 0 \end{cases} .$$

Moreover, the radius of the new interval equals

$$r_{k+1} = \begin{cases} \frac{1}{2}(r_k + (1 - \alpha_k)d_k) & \text{if } g'(d_k) < 0 \\ \frac{1}{2}(r_k + (\alpha_k - 1)d_k) & \text{if } g'(d_k) > 0 \end{cases} .$$

By the above results the interval of uncertainty is reduced by more than a half if $l_k < g(d_k)$ and so this algorithm improves the classical bisection method [5]. We can state this specialized bisection algorithm in the following way:

ALGORITHM 2 (Specialized Bisection).

Step 1.
Let $[b_2, x_{\lambda_1}]$ be the initial interval.
Compute d_1, r_1, let $k := 1$ and `Goto` **Step** 2.
Step 2.
Compute $g'(d_k)$ and `Goto` **Step** 3.
Step 3.
`If` $g'(d_k) = 0$
`Then` `Stop` with d_k the optimal feasible solution.
`Else` Compute $\alpha_k, d_{k+1}, r_{k+1}$, let $k := k + 1$ and `GoTo` **Step** 2.

For this specialized bisection algorithm we can also derive a stopping rule which is comparable to the stopping rule of the Dinkelbach algorithm. Such a stopping rule is needed in order to give a fair comparison in the next section of the specialized bisection algorithm and the Dinkelbach algorithm. First notice the trivial result

$$0 \geq \lambda_\star - g(d_k) = g(x_\star) - g(d_k). \tag{3}$$

Using Lemma 18 and substituting $x = x_\star > b_2$ and $a = d_k > b_2$ we have

$$g(x_\star) \geq m(d_k) + (g(d_k) - m(d_k)) \frac{D(d_k)}{D(x_\star)} \tag{4}$$

and so by (4) and (3) we obtain

$$0 \geq \lambda_\star - g(d_k) \geq (m(d_k) - g(d_k)) \frac{D(x_\star) - D(d_k)}{D(x_\star)}.$$

Since the function D is concave and hence Lipschitz continuous on the interval $[d_k - r_k, d_k + r_k]$ with Lipschitz constant $L_k := D'(d_k - r_k)$ it follows that

$$0 \geq \lambda_\star - g(d_k) \geq -L_k \mid m(d_k) - g(d_k) \mid \frac{r_k}{D(x_\star)}.$$

Hence, the specialized bisection algorithm stops if the stopping rule

$$-L_k \mid m(d_k) - g(d_k) \mid r_k > -\epsilon$$

for some positive $\epsilon > 0$ holds and by the above inequality this guarantees the absolute error $-\frac{\epsilon}{D(x_\star)}$. This stopping rule is comparable with the stopping rule for the Dinkelbach algorithm discussed in Section 2. Observe, if $g'(d_k) = 0$ and hence d_k is the optimal point then $m(d_k) - g(d_k) = 0$ and the stopping rule is automatically satisfied.

In the next section we will present some computational results comparing both methods.

4. Computational Results

In this section we present some computational experiments with the Dinkelbach algorithm and the specialized bisection algorithm applied to the inspection and minimal repair model. For both models we assume that the distribution F of the time to failure of one component has a failure rate function given by

$$r(z) = \begin{cases} \kappa\theta(\kappa z)^{\theta-1} & \text{if } 0 \leq z < \gamma \\ \kappa\theta(\kappa z)^{\theta-1} + \frac{\beta}{\eta}(\frac{z-\gamma}{\eta})^{\beta-1} & \text{if } z \geq \gamma \end{cases}$$

Such a function was proposed by [17] to model a unimodal failure rate. Observe for both models we have $d = 0, c := c_2 > 0$ and $h(z) = 1$ for every $z \geq 0$. Moreover, for the inspection model it follows that $m(z) = c_1 F(z)$, $c_1 > 0$, while for the minimal repair model we have $m(z) = c_1 r(z)$, $c_1 > 0$.

Before discussing the results we first mention how the parameters of the above failure rate function were generated. Since κ is a scale factor this value was set equal to one in all the test problems. The remaining parameters $\theta, \beta, \eta, \gamma$ were generated uniformly on the intervals $[0.5, 3.5]$, $[0, 1]$, $[0, 10]$ and $[0.5, 3.5]$. Also, the parameters c_1 and c_2 are uniformly drawn from $[0, 0.5]$ and $[0, 1.5]$. For both models we generated ten problems. Finally, the stopping rule of both the Dinkelbach algorithm and the specialized bisection algorithm used $\epsilon = 10^{-6}$. By doing so both stopping rules are comparable. Observe, since $r(\infty) = \infty$ that the minimal repair model has a feasible solution. Moreover, since

$$\int_0^x r(z)\,\mathrm{d}z = \begin{cases} (\kappa x)^\theta & \text{if } 0 \leq x < \gamma \\ (\kappa x)^\theta + (\frac{x-\gamma}{\eta})^\beta & \text{if } x > \gamma \end{cases}$$

we do not have to use for the minimal repair model any numerical procedure to evaluate the objective function at any point. However, for the inspection model we need to approximate the objective function. Therefore we used Romberg integration [23] to evaluate this function. Observe, as the computational results in Table II show, that this affects the total execution time. All computations were done on a COMPAQ PROLINEA $4/66$ with mathematical coprocessor.

We will start by discussing the results contained in Table I which refers to the minimal repair model. For this table the entries on row *Dinkel.* report the results obtained using the Dinkelbach algorithm. Similarly, the entries of row *Sp.Bis.* report the results using the specialized bisection method. The column *It* refers to the average number of iterations performed by the corresponding algorithm. The column *Sec* refers to the average execution time in seconds measured by the unit ”TPTIMER” in Turbo Pascal version 5.0. The last column *% Sub* refers to the percentage of the total time needed to solve the subproblems in the Dinkelbach algorithm. To solve the subproblems in **Step** 2 of this algorithm we used the standard bisection method.

Table I. Minimal Repair Model

Method	*It*	*Sec*	*% Sub*
Dinkel.	5.8	0.017	89.65
Sp.Bis.	11.0	0.008	

Although the average number of iterations in the Dinkelbach algorithm is less it takes on average more time to solve the minimal repair model. This is obviously caused by the fact that in **Step** 2 of the Dinkelbach algorithm the subproblems are solved using bisection.

If both algorithms are used to solve the inspection model the integral $\int_0^x F(z)\,\mathrm{d}z$ has to be approximated. Therefore a numerical approximation is needed in every iteration. The results for solving the inspection model using both methods are contained in Table II.

Table II. Inspection Model

Method	*It*	*Sec*	*% Num*	*% Sub*
Dinkel.	4.2	0.770	98.31	1.50
Sp.Bis.	8.9	1.499	96.14	

This table is constructed in a similar way as Table I, but it contains an extra column, *% Num*. This column contains the percentage of the total running time needed for the Romberg integration to approximate the integrals in the model.

Also in this case the average number of iterations in the Dinkelbach algorithm is less than the average number of iterations in the specialized bisection method. Furthermore, the specialized bisection method takes on average considerably more time than the Dinkelbach algorithm. This is due to the fact that in each iteration of both methods we need to compute an integral and this takes more time than solving the subproblems in **Step** 2 of the Dinkelbach algorithm. From the above results, it seems, if the parametric problem is easy solvable, that the Dinkelbach algorithm is preferred to the improved bisection method to solve classical one-component maintenance models whenever we need to call for a numerical procedure to evaluate the objective function.

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References

1. Aven, T. and Bergman, B. (1986), Optimal Replacement Times. A General Setup, *Journal of Applied Probability* **23**, 432–442.
2. Aven, T. and Dekker, R. (1996), A Useful Framework for Optimal Replacement Models. Technical Report TI 96 – 64/9, Tinbergen Institute Rotterdam, to appear in *Reliability Engineering and System Safety*.
3. Avriel, M., Diewert, W.E., Schaible, S., and Zang, I. (1988), *Generalized Concavity*, volume 36 of *Mathematical Concepts and Methods in Science and Engineering*, Plenum Press, New York.
4. Barlow, R.E. and Proschan, F. (1965), *Mathematical Theory of Reliability*, Wiley, New York.
5. Bazaraa, M.S., Sherali, H.D., and Shetty, C.M. (1993), *Nonlinear Programming: Theory and Algorithms*, Wiley, New York. (Second edition).
6. Berg, M. (1980), A Marginal Cost Analysis for Preventive Maintenance Policies, *European Journal of Operational Research* **4**, 136–142.
7. Berg, M. (1995), The Marginal Cost Analysis and Its Application to Repair and Replacement Models, *European Journal of Operational Research* **82**(2), 214–224.
8. Cambini, A. and Martein, L. (1992), Equivalence in Linear Fractional Programming, *Optimization* **23**, 41–51.
9. Cambini, A., Schaible, S., and Sodini, C. (1993), Parametric Linear Fractional Programming for an Unbonded Feasible Region, *Journal of Global Optimization* **3**, 157–169.
10. Charnes, A. and Cooper, W. (1962), Programming with Linear Fractional Functionals, *Naval Research Logistics Quarterly* **9**, 181–186.
11. Crouzeix, J.P., Ferland, J.A., and Schaible, S. (1985), An Algorithm for Generalized Fractional Programs, *Journal of Optimization Theory and Applications* **47**(1), 35–49.
12. Dekker, R. (1995), Integrating Optimization, Priority Setting, Planning and Combining of Maintenance Activities, *European Journal of Operational Research* **82**(2), 225–240.
13. Dekker, R. and Dijkstra, M.C. (1992), Opportunity Based Age Replacement: Exponentially Distributed Times Between Opportunities, *Naval Research Logistics* **39**, 175–192.
14. Dinkelbach, W. (1967), On Nonlinear Fractional Programming, *Management Science* **13**(7), 492–498.
15. Frenk, J.B.G., Dekker, R., and Kleijn, M.J. (1996), A Note on the Marginal Cost Approach in Maintenance, Technical Report TI 96 – 25/9, Tinbergen Institute Rotterdam, to appear in *Journal of Optimization Theory and Applications*, November 1997.
16. Hansen, P. and Jaumard, B. (1995), Lipschitz Optimization. In R. Horst and P. Pardalos, editors, *Handbook of Global Optimization*, volume 2 of *Nonconvex Optimization and Its Applications*, pages 407–493. Kluwer Academic Publishers, Dordrecht.
17. Hastings, N.A.J. and Ang, J.Y.T. (1995), Developments in Computer Based Reliability Analysis, *Journal of Quality in Maintenance Engineering* **1**(1), 69–78.
18. Hiriart-Urruty, J.B. and Lemaréchal, C. (1993), *Convex Analysis and Minimization Algorithms I: Fundamentals*, volume 1. Springer-Verlag, Berlin.
19. Ibaraki, T. (1983), Parametric Approaches to Fractional Programs, *Mathematical Programming* **26**, 345–362.
20. Isbell, J.R. and Marlow, W.H. (1956), Attrition Games, *Naval Research Logistics Quarterly* **3**, 71–93.

21. Martos, B. (1964), Hyperbolic Programming, *Naval Research Logistics Quarterly* **11**, 135–155. English translation from: Publications of Mathematic Institute Hungarian Academy Sciences, 5: 383–406, 1960.
22. Mond, B. (1981), On Algorithmic Equivalence in Linear Fractional Programming, *Mathematics of Computation* **37**(155), 185–187.
23. Press, W.H., Flannery, B.P., Tenkolsky, S.A., and Vetterling, W.T. (1992), *Numerical Recipes in Pascal. The Art of Science Computing*. Cambridge University Press, Cambridge.
24. Rudin, W. (1976), *Principles of Mathematical Analysis*, Mathematical Series. McGraw-Hill, New York.
25. Schaible, S. (1976), Fractional Programming. II, On Dinkelbach's Algorithm, *Management Science* **22**, 868–873.
26. Schaible, S. (1995), Fractional Programming, In R. Horst and P. Pardalos, editors, *Handbook of Global Optimization*, volume 2 of *Nonconvex Optimization and Its Applications*, pages 495–608. Kluwer Academic Publishers, Dordrecht.
27. Tijms, H.C. (1994), *Stochastic Models. An Algorithmic Approach*, John Wiley, Chichester.
28. Wagner, H.M. and Yuan, J.S.C. (1968), Algorithmic Equivalence in Linear Fractional Programming, *Management Science* **14**(5), 301–306.