ON NONCOOPERATIVE GAMES AND MINIMAX THEORY

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On noncooperative games and minimax theory

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Abstract

In this note we review some known minimax theorems with applications in game theory and show that these results form an equivalent chain which includes the strong separation result in finite dimensional spaces between two disjoint closed convex sets of which one is compact. By simplifying the proofs we intend to make the results more accessible to researchers not familiar with minimax or noncooperative game theory.

1 Introduction

In a two person noncooperative zero sum game one faces the following problem. Let X be the set of actions of player 1 and Y the set of actions of player 2. If player 1 chooses action $x \in X$ and player 2 chooses action $y \in Y$, then player 2 has to pay to player 1 an amount $f(x, y)$ with $f : X \times Y \to \mathbb{R}$ a given function. This function is called the payoff function of player 1. Since player 1 likes to gain as much profit as possible, but at the moment he does not know how to achieve this, he first decides to compute a lower bound on his profit. To do this, player 1 argues as follows : if he chooses action $x \in X$, then he wins at least $\inf_{y \in Y} f(x, y)$ irrespective of the action of player 2. Therefore a lower bound on the profit for player 1 is given by

$$r_* := \sup_{x \in X} \inf_{y \in Y} f(x, y). \tag{1}$$

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Similarly player 2 likes to minimize his losses. Therefore, he also decides to compute first an upper bound on his losses. If he decides to choose action $y \in Y$ it follows that he loses at most $\sup_{x \in X} f(x,y)$ and this is independent of the action of player 1. Therefore an upper bound on his losses is given by

$$r^* := \inf_{y \in Y} \sup_{x \in X} f(x,y). \tag{2}$$

Since the profit of player 1 is at least r_* and the losses of player 2 is at most r^* and the losses of player 2 are the profits of player 1, it follows that $r_* \leq r^*$. If $r_* = r^*$, then this equality is called a **minimax result**. If additionally inf and sup are attained, an optimal action for both players can then be easily derived. However, in general $r_* < r^*$, as the following example shows.

Example 1.1 *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ given by $f(x,y) = (x-y)^2$. For this function it holds $0 = r_* < r^* = \frac{1}{4}$. For this example it is not obvious which actions should be selected by the two players.*

By extending the sets of actions of each player, it is possible to show under certain conditions that the extended game satisfies a minimax result. In the next definition we introduce the set of **mixed strategies**.

Definition 1.1 *For a nonempty set D of actions and $d \in D$ let ϵ_d denote the one-point probability measure concentrated on the set $\{d\}$ and denote by $\mathcal{F}(D)$ the set of all probability measures on D with a finite support.*

Introducing the unit simplex $\Delta_k := \{\alpha : \sum_{i=1}^k \alpha_i = 1, \alpha_i \geq 0, 1 \leq i \leq k\}$, it follows by Definition 1.1 that λ belongs to the set $\mathcal{F}(D)$ if and only if there exist some $k \in \mathbb{N}$ and a set $\{d_1, ..., d_k\} \subseteq D$ such that

$$\lambda = \sum_{i=1}^k \lambda_i \epsilon_{d_i}, (\lambda_1, ..., \lambda_k) \in \Delta_k \text{ and } \lambda_i \geq 0.$$

A game theoretic interpretation of a mixed strategy $\lambda \in \mathcal{F}(D)$ is now given by the following. If a player with action set D selects the mixed strategy $\lambda = \sum_{i=1}^k \lambda_i \epsilon_{d_i} \in \mathcal{F}(D)$, then with probability $\lambda_i, 1 \leq i \leq k$ this player will use action $d_i \in D$. By the above definition it is clear that the action set D of any player can be identified with the set of one-point probability measures; therefore the set D is often called the set of **pure strategies** for that player. Assume that player 1 uses the set $\mathcal{F}(X)$ of mixed strategies and the same holds for player 2 using the set $\mathcal{F}(Y)$. This means that the payoff function f should be extended to a function $f_e : \mathcal{F}(X) \times \mathcal{F}(Y) \to \mathbb{R}$ given by

$$f_e(\lambda, \mu) := \sum_{i=1}^m \sum_{j=1}^n \lambda_i \mu_j f(x_i, y_j) \tag{3}$$

with $\lambda = \sum_{i=1}^{m} \lambda_i \epsilon_{x_i} \in \mathcal{F}(X)$ and $\mu = \sum_{j=1}^{n} \mu_j \epsilon_{y_j} \in \mathcal{F}(Y)$. This extension represents the expected profit for player 1 or expected loss of player 2.

In [3] the authors showed that several well known minimax theorems form an equivalent chain and this chain includes the strong separation result in finite dimensional spaces between two disjoint convex sets of which one is closed and the other compact. By reducing the number of results in this equivalent chain and by giving more transparent and simpler proofs, we intend to make the results more accessible to researchers not familiar with minimax or noncooperative game theory.

The first minimax result was proved in a famous paper by von Neumann (cf.[6]) in 1928 for X and Y unit simplices in finite dimensional vector spaces and f affine in both variables. Later on, the conditions on the function f were weakened and more general sets X and Y were considered. These results turned out to be useful also in optimization theory (see for instance [2]) and were derived by means of short or long proofs using a version of the Hahn Banach theorem in either finite or infinite dimensional vector spaces. With von Neumann's result as a starting point, we will show that several of these so-called generalizations published in the literature can be derived from each other using only elementary observations. Before introducing this chain of equivalent minimax results we need the following notations. The set $\mathcal{F}_2(X) \subseteq \mathcal{F}(X)$ denotes the set of two-point probability measures on X. This means that λ belongs to $\mathcal{F}_2(X)$ if and only if

$$\lambda = \lambda_1 \epsilon_{x_1} + (1 - \lambda_1)\epsilon_{x_2} \tag{4}$$

with $x_i, 1 \leq i \leq 2$ different elements of X and $0 < \lambda_1 < 1$ arbitrarily chosen. Also, for each $0 < \alpha < 1$ the set $\mathcal{F}_{2,\alpha}(X)$ represents the set of two point probability measures with $\lambda_1 = \alpha$ in relation (4). On the set Y similar spaces of probability measures with finite support are introduced.

2 Equivalent minimax results

To start in a chronological order we first mention the famous von Neumann's minimax result (cf.[6]).

Theorem 2.1 *(von Neumann, 1928). If X and Y are finite sets, then it follows that*

$$\max_{\lambda \in \mathcal{F}(X)} \min_{\mu \in \mathcal{F}(Y)} f_e(\lambda, \mu) = \min_{\mu \in \mathcal{F}(Y)} \max_{\lambda \in \mathcal{F}(X)} f_e(\lambda, \mu).$$

A generalization of Theorem 2.1 due to Wald [7]) and published in 1945 is given by the next result. This result plays a fundamental role in the theory of statistical decision functions. While in case of Theorem 2.1 the action sets of players 1 and 2 are finite, this condition is relaxed in Wald's theorem claiming that only one set should be finite.

Theorem 2.2 *(Wald, 1945). If X is an arbitrary nonempty set and Y is a finite set, then it follows that*

$$\sup_{\lambda\in\mathcal{F}(X)} \min_{\mu\in\mathcal{F}(Y)} f_e(\lambda,\mu) = \min_{\mu\in\mathcal{F}(Y)} \sup_{\lambda\in\mathcal{F}(X)} f_e(\lambda,\mu).$$

In order to prove Wald's theorem by von Neumann's theorem, we first need the following elementary lemma. For its proof, see for instance [3]. Recall a function is lower semicontinuous if all its upper level sets are closed. For every set D let $< D >$ be the set of all finite subsets of D.

Lemma 2.1 *If the set X is compact and the function $h : X \times Y \to \mathbb{R}$ is upper semicontinuous on X for every $y \in Y$, then $\max_{x\in X} \inf_{y\in Y} h(x,y)$ is well defined and*

$$\max_{x\in X} \inf_{y\in Y} h(x,y) = \inf_{Y_0\in<Y>} \max_{x\in X} \min_{y\in Y_0} h(x,y).$$

Since for every $\mu \in \mathcal{F}(Y)$ and $J \subseteq X$ it is easy to see that

$$\sup_{\lambda\in\mathcal{F}(J)} f_e(\lambda,\mu) = \sup_{x\in J} f_e(\epsilon_x,\mu), \tag{5}$$

we are now ready to derive Wald's minimax result from von Neumann's minimax result. Observe Wald (cf.[7]) uses in his paper von Neumann's minimax result and the Lebesgue dominated convergence theorem.

Theorem 2.3 *von Neumann's minimax result $\Rightarrow$ Wald's minimax result.*

Proof: If $\alpha := \sup_{\lambda\in\mathcal{F}(X)} \min_{\mu\in\mathcal{F}(Y)} f_e(\lambda,\mu)$ then clearly

$$\alpha = \sup_{J\in<X>} \max_{\lambda\in\mathcal{F}(J)} \min_{\mu\in\mathcal{F}(Y)} f_e(\lambda,\mu). \tag{6}$$

Since the set Y is finite we may apply von Neumann's minimax result in relation (6) and this implies in combination with relation (5) that

$$\begin{aligned} \alpha &= \sup_{J\in<X>} \min_{\mu\in\mathcal{F}(Y)} \max_{\lambda\in\mathcal{F}(J)} f_e(\lambda,\mu) \qquad (7)\\ &= \sup_{J\in<X>} \min_{\mu\in\mathcal{F}(Y)} \max_{x\in J} f_e(\epsilon_x,\mu)\\ &= -\inf_{J\in<X>} \max_{\mu\in\mathcal{F}(Y)} \min_{x\in J}(-f_e(\epsilon_x,\mu)). \end{aligned}$$

The finiteness of the set Y also implies that the set $\mathcal{F}(Y)$ is compact and the function $\mu \to f_e(\epsilon_x, \mu)$ is continuous on $\mathcal{F}(Y)$ for every $x \in X$. This shows in relation (7) that we may apply Lemma 2.1 with the set X replaced by $\mathcal{F}(Y)$, Y by X and $h(x, y)$ by $-f_e(\epsilon_x, \mu)$ and so it follows that

$$\alpha = \min_{\mu \in \mathcal{F}(Y)} \sup_{x \in X} f_e(\epsilon_x, \mu). \tag{8}$$

Finally by relation (5) with J replaced by X the desired result follows from relation (8). ■

In 1996 Kassay and Kolumbán (cf.[4]) introduced the following class of functions.

Definition 2.1 *The function $f : X \times Y \to \mathbb{R}$ is called weakly concavelike on X if for every I belonging to $< Y >$ it follows that*

$$\sup_{\lambda \in \mathcal{F}(X)} \min_{y \in I} f_e(\lambda, \epsilon_y) \leq \sup_{x \in X} \min_{y \in I} f(x, y).$$

Since ϵ_x belongs to $\mathcal{F}(X)$ it is easy to see that f is weakly concavelike on X if and only if for every $I \in < Y >$ it follows that

$$\sup_{\lambda \in \mathcal{F}(X)} \min_{y \in I} f_e(\lambda, \epsilon_y) = \sup_{x \in X} \min_{y \in I} f(x, y)$$

and this equality also has an obvious interpretation within game theory. The main result of Kassay and Kolumbán is given by the following theorem (cf.[4]).

Theorem 2.4 *(Kassay-Kolumbán, 1996). If X is a compact subset of a topological space and the function $f : X \times Y \to \mathbb{R}$ is weakly concavelike and upper semicontinuous on X for every $y \in Y$, then it follows that*

$$\inf_{\mu \in \mathcal{F}(Y)} \max_{x \in X} f_e(\epsilon_x, \mu) = \max_{x \in X} \inf_{y \in Y} f_e(x, y).$$

At first sight this result might not be recognized as a minimax result. However, it is easy to verify for every $x \in X$ that

$$\inf_{y \in Y} f(x, y) = \inf_{\mu \in \mathcal{F}(Y)} f_e(\epsilon_x, \mu). \tag{9}$$

By relation (9) an equivalent formulation of Theorem 2.4 is now given by

$$\inf_{\mu \in \mathcal{F}(Y)} \max_{x \in X} f_e(\epsilon_x, \mu) = \max_{x \in X} \inf_{\mu \in \mathcal{F}(Y)} f_e(\epsilon_x, \mu),$$

and so the result of Kassay and Kolumban is actually a minimax result.

We now give an elementary proof for Theorem 2.4 using Wald's minimax theorem.

Proof: Let $\alpha = \inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x,\mu)$, $\beta = \max_{x\in X} \inf_{\mu\in\mathcal{F}(Y)} f_e(\epsilon_x,\mu)$ and suppose by contradiction that $\alpha > \beta$. (The inequality $\beta \leq \alpha$ always holds.) Let γ so that $\alpha > \gamma > \beta$. Then by relation (9) and Lemma 2.1 we have

$$\gamma > \beta = \max_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{Y_0\in<Y>} \max_{x\in X} \min_{y\in Y_0} f(x,y).$$

Therefore, there exists a finite subset $Y_0 \in< Y >$ such that

$$\max_{x\in X} \min_{y\in Y_0} f(x,y) < \gamma$$

and this implies by weak concavelikeness that

$$\sup_{\lambda\in\mathcal{F}(X)} \min_{y\in Y_0} f_e(\lambda,\epsilon_y) < \gamma. \tag{10}$$

Similarly to relation (9), it is easy to see that for every $\lambda \in \mathcal{F}(X)$ and every $\mu \in \mathcal{F}(Y)$ the relations

$$\inf_{\mu\in\mathcal{F}(Y_0)} f_e(\lambda,\mu) = \min_{y\in Y_0} f_e(\lambda,\epsilon_y)$$

and

$$\sup_{\lambda\in\mathcal{F}(X)} f_e(\lambda,\mu) = \max_{x\in X} f_e(\epsilon_x,\mu)$$

hold, and these together with (10) and Wald's theorem imply

$$\alpha > \gamma > \sup_{\lambda\in\mathcal{F}(X)} \inf_{\mu\in\mathcal{F}(Y_0)} f_e(\lambda,\mu) = \inf_{\mu\in\mathcal{F}(Y_0)} \sup_{\lambda\in\mathcal{F}(X)} f_e(\lambda,\mu)$$

$$\geq \inf_{\mu\in\mathcal{F}(Y)} \sup_{\lambda\in\mathcal{F}(X)} f_e(\lambda,\mu) = \inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x,\mu) = \alpha,$$

a contradiction. This completes the proof. ■

In 1952 Kneser (cf.[5]) proved a general minimax result useful in game theory. Its proof is ingenious and very elementary and uses only some simple computations and the well-known result that any upper semicontinuous function attains its maximum on a compact set.

Theorem 2.5 *(Kneser, 1952). If X is a nonempty convex compact subset of a topological vector space and Y is a nonempty convex subset of a vector space and the function $f : X \times Y \to \mathbb{R}$ is affine in both variables and upper semicontinuous on X for every $y \in Y$, then it follows that*

$$\max_{x\in X} \inf_{y\in Y} f(x, y) = \inf_{y\in Y} \max_{x\in X} f(x, y). \tag{11}$$

One year later, generalizing the proof and result of Kneser, Ky Fan (cf.[1]) published his celebrated minimax result. To show his result Ky Fan introduced the following class of functions which we call Ky Fan convex (Ky Fan concave) functions.

Definition 2.2 *The function $f : X \times Y \to \mathbb{R}$ is called Ky Fan concave on X if for every $\lambda \in \mathcal{F}_2(X)$ there exists some $x_0 \in X$ satisfying*

$$f_e(\lambda, \epsilon_y) \leq f(x_0, y)$$

for every $y \in Y$. The function $f : X \times Y \to \mathbb{R}$ is called Ky Fan convex on Y if for every $\mu \in \mathcal{F}_2(Y)$ there exists some $y_0 \in Y$ satisfying

$$f_e(\epsilon_x, \mu) \geq f(x, y_0)$$

for every $x \in X$. Finally, the function $f : X \times Y \to \mathbb{R}$ is called Ky Fan concave-convex on $X \times Y$ if f is Ky Fan concave on X and Ky Fan convex on Y.

By induction it is easy to show that one can replace in the above definition $\mathcal{F}_2(X)$ and $\mathcal{F}_2(Y)$ by $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. Although rather technical, the above concept has a clear interpretation in game theory. It means that the payoff function f has the property that any arbitrary mixed strategy is dominated by a pure strategy. Eliminating the linear structure in Kneser's proof Ky Fan (cf.[1]) showed the following result.

Theorem 2.6 *(Ky Fan, 1953). If X is a compact subset of a topological space and the function $f : X \times Y \to \mathbb{R}$ is Ky Fan concave-convex on $X \times Y$ and upper semicontinuous on X for every $y \in Y$, then it follows that*

$$\max_{x\in X} \inf_{y\in Y} f(x, y) = \inf_{y\in Y} \max_{x\in X} f(x, y).$$

In what follows we show that Ky Fan's minimax theorem can easily be proved by Kassay-Kolumbán's result. Indeed, it is easy to see that every Ky Fan concave function on X is also weakly concavelike on X. By Theorem 2.4 and relation (9) it follows that

$$\max_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{\mu\in\mathcal{F}(Y)}\max_{x\in X} f_e(\epsilon_x,\mu). \tag{12}$$

Also, since f is Ky Fan convex on Y, for every $\mu \in \mathcal{F}(Y)$ there exists $y_0 \in Y$ such that $f_e(\epsilon_x,\mu) \geq f(x,y_0)$ for every $x \in X$. Thus,

$$\max_{x\in X} f_e(\epsilon_x,\mu) \geq \max_{x\in X} f(x,y_0) \geq \inf_{y\in Y}\max_{x\in X} f(x,y)$$

implying that

$$\inf_{\mu\in\mathcal{F}(Y)}\max_{x\in X} f_e(\epsilon_x,\mu) \geq \inf_{y\in Y}\max_{x\in X} f(x,y)$$

and this, together with (12) leads to

$$\max_{x\in X}\inf_{y\in Y} f(x,y) \geq \inf_{y\in Y}\max_{x\in X} f(x,y).$$

Since the reverse inequality always holds, we have equality in the last relation and the proof is complete. ■

We show now that the following well-known strong separation result in convex analysis can easily be proved by Kneser's minimax theorem.

Theorem 2.7 *If $X \subseteq \mathbb{R}^n$ is a closed convex set and $Y \subseteq \mathbb{R}^n$ a compact convex set and the intersection of X and Y is empty, then there exists some $s_0 \in \mathbb{R}^n$ satisfying*

$$\sup\{s_0^\top x : x \in X\} < \inf\{s_0^\top y : y \in Y\}.$$

Proof: Since $X \subseteq \mathbb{R}^n$ is a closed convex set and $Y \subseteq \mathbb{R}^n$ is a compact convex set we obtain that $H := X - Y$ is a closed convex set. It is now easy to see that the strong separation result as given in Theorem 2.7 holds if and only if there exists some $s_0 \in \mathbb{R}^n$ satisfying $\sigma_H(s_0) := \sup\{s_0^\top x : x \in H\} < 0$. To verify this, we assume by contradiction that $\sigma_H(s) \geq 0$ for every $s \in \mathbb{R}^n$. This clearly implies $\sigma_H(s) \geq 0$ for every s belonging to the compact Euclidean unit ball E and applying Kneser's minimax result we obtain

$$\sup_{h\in H}\inf_{s\in E} s^\top h = \inf_{s\in E}\sup_{h\in H} s^\top h \geq 0. \tag{13}$$

Since by assumption the intersection of X and Y is nonempty, we obtain that 0 does not belong to $H := X - Y$ and this implies using H is closed that

$\inf_{h\in H}\|h\| > 0$. By this observation we obtain for every $h \in H$ that $-h\|h\|^{-1}$ belongs to E and so for every $h \in H$ it follows that $\inf_{s\in E} s^\top h \leq -\|h\|$. This implies that

$$\sup_{h\in H}\inf_{s\in E} s^\top h \leq \sup_{h\in H} -\|h\| = -\inf_{h\in H}\|h\| < 0$$

and we obtain a contradiction with relation (13). Hence there must exist some $s_0 \in \mathbb{R}^n$ satisfying $\sigma_H(s_0) < 0$ and we are done. ■

Observe that without loss of generality one may suppose that the vector s_0 in Theorem 2.7 belongs to Δ_n (the unit simplex in $\mathbb{R}^n$). An easy consequence of Theorem 2.7 is the following result.

Lemma 2.2 *If $C \subseteq \mathbb{R}^n$ is a convex compact set, then it follows that*

$$\inf_{u\in C}\max_{\alpha\in\Delta_n}\alpha^\top u = \max_{\alpha\in\Delta_n}\inf_{u\in C}\alpha^\top u.$$

Proof: It is obvious that

$$\inf_{u\in C}\max_{\alpha\in\Delta_n}\alpha^\top u \geq \max_{\alpha\in\Delta_n}\inf_{u\in C}\alpha^\top u. \tag{14}$$

To show the reverse inequality, we assume by contradiction that

$$\inf_{u\in C}\max_{\alpha\in\Delta_n}\alpha^\top u > \max_{\alpha\in\Delta_n}\inf_{u\in C}\alpha^\top u := \gamma. \tag{15}$$

Let $\mathbf{e}$ be the vector $(1, ..., 1)$ in $\mathbb{R}^n$ and introduce the mapping $H : C \to \mathbb{R}^n$ given by $H(u) := u - \beta\mathbf{e}$ with β satisfying

$$\inf_{u\in C}\max_{\alpha\in\Delta_n}\alpha^\top u > \beta > \gamma \tag{16}$$

If we assume that $H(C) \cap \mathbb{R}^n_-$ is nonempty, then there exists some $u_0 \in C$ satisfying $u_0 - \beta\mathbf{e} \leq 0$. This implies $\max_{\alpha\in\Delta_n}\alpha^\top u_0 \leq \beta$ and we obtain a contradiction with relation (16). Therefore $H(C)\cap\mathbb{R}^n_-$ is empty. Since $H(C)$ is convex and compact and $\mathbb{R}^n_-$ is closed and convex, we may apply Theorem 2.7. Hence one can find some $\alpha_0 \in \Delta_n$ satisfying $\alpha_0^\top u - \beta \geq 0$ for every $u \in C$ and using also the definition of γ listed in relation (15) this implies that

$$\gamma \geq \inf_{u\in C}\alpha_0^\top u \geq \beta.$$

Hence we obtain a contradiction with relation (16) and the desired result is proved. ■

Finally we show that von Neumann's minimax theorem (Theorem 2.1) is an easy consequence of Lemma 2.2. In this way we close the equivalent chain of results considered in this note. Indeed, let $m := card(X)$ and introduce the mapping $L : \mathcal{F}(Y) \to \mathbb{R}^m$ given by

$$L(\mu) := (f_e(\epsilon_x, \mu))_{x \in X}.$$

It is easy to see that the range $L(\mathcal{F}(Y)) \subseteq \mathbb{R}^m$ is a convex compact set. Applying now Lemma 2.2 yields

$$\begin{aligned}
\inf_{\mu \in \mathcal{F}(Y)} \max_{\lambda \in \mathcal{F}(X)} f_e(\lambda, \mu) &= \inf_{u \in L(\mathcal{F}(Y))} \max_{\alpha \in \Delta_m} \alpha^\top u \\
&= \max_{\alpha \in \Delta_m} \inf_{u \in L(\mathcal{F}(Y))} \alpha^\top u \\
&= \max_{\lambda \in \mathcal{F}(X)} \inf_{\mu \in \mathcal{F}(Y)} f_e(\lambda, \mu),
\end{aligned}$$

which completes the proof. ■

As we have seen, the equivalent minimax results presented here corresponds to different zero-sum games with different action sets. From our technique it follows that finite pure action sets and compact pure action sets are not really "far apart".

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