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The efficiency of rotating-panel designs in an analysis-of-variance model

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In this paper we consider the relative efficiency of rotating-panel designs in analysis-of-variance models. Throughout we assume that the parameter of interest is a linear combination of period means in the analysis-of-variance model. Results from spectral theory are used to obtain manageable expressions for the variance of the BLUE of this parameter. Relative efficiencies of the BLUE for rotating panels with different rotation periods are presented, e.g., for the period means themselves, of differences, or of averages of means. Moreover we present bounds on the relative efficiency which are valid irrespective of the parameter of interest. The analysis shows that the gains from choosing an optimal rotation design can be quite substantial, even if the cost of a reinterview equals the cost of a first observation. In many cases either the smallest or the highest possible rotation period is optimal. The analysis is illustrated with an empirical example concerning monthly consumer expenditures on food and clothing.

1. Introduction

The collection of micro-economic data, e.g., in consumer surveys, is characterized by its high cost. It is therefore very important to obtain as much information as possible from a given budget by using optimal sample designs. Consider, for example, the choice which a data-collecting agency will have to make in order to monitor average expenditures on some consumption category either to interview the same individuals in several periods or to interview different individuals in different periods. It is well known in the

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literature that the optimal design of the sample will in general depend on the parameter of interest [see, e.g., Raj (1968, p. 152 ff.) or Cochran (1977, p. 345 ff.)]. For example, if one is primarily interested in the total consumption over a number of periods, it is less likely that reinterviews are attractive than if one is primarily interested in period-to-period changes in consumption.

In an earlier paper [Nijman and Verbeek (1990)] we determined the optimal split panel design, i.e., the optimal design if one is allowed to spend part of a given budget on the collection of a series of cross-sections while another part of the budget is spent on a pure panel where all individuals are observed every period. For longer panels it is not attractive however to observe the same individuals in all periods. It has been well documented in the literature that if the number of times respondents have been exposed to a survey gets large, the data may be affected [see, e.g., Binder and Hidiroglou (1988)] and even behavioural changes may be induced. This phenomenon is known as panel-conditioning. Moreover, it is evident that (selective) nonresponse problems will increase if units are interviewed for a larger number of periods. In order to avoid these problems rotating panels, i.e., panels where part of the sample is replaced in each period and every individual is included in the panel for a limited number of periods only, are often used in practice. Little attention however seems to have been paid to the relative efficiency of alternative rotation designs and to the optimal choice of the design of rotating panels. In the early literature, Patterson (1950) and Eckler (1955) paid attention to the estimation of a time-dependent mean from several kinds of rotating samples and to the resulting variances. Rao and Graham (1964) analysed the variance of both the current mean and the change in means in a finite-population context, using a special class of recursive estimators. An excellent survey of the literature in this field is given by Binder and Hidiroglou (1988).

In this paper we derive the design of rotating panels which minimizes the variance of the best linear unbiased estimator (BLUE) of linear combinations $\sum_{j=0}^{J} \xi_j \mu_{\tau-j}$ of the period means μ_t in the analysis-of-variance model,

$$y_{it} = \mu_t + \alpha_i + \varepsilon_{it}, \tag{1}$$

where the α_i and ε_{it} are unobserved i.i.d. random variables with mean zero and variances σ_α^2 and σ_ε^2, respectively, which are mutually independent. Important special cases are of course the determination of the optimal design if the parameter of interest is the period mean μ_τ itself, if the parameter of interest is the change in two subsequent period means $\mu_\tau - \mu_{\tau-1}$, or if the parameter of interest is the average or sum over m subsequent period means $\sum_{k=0}^{m-1} \mu_{\tau-k}$. Throughout this paper we assume for simplicity that the parameters σ_α^2 and σ_ε^2 are known a priori. If these parameters are unknown and replaced by consistent estimates, the same results hold true asymptotically (assuming that the number of individuals in the sample tends to infinity). The constant correlation over time between different observations on the same individual implied by (1) is considered for analytical convenience only. The analysis can easily be extended to more general correlation patterns. Moreover, if no unit is observed for more than two periods, (1) is not restrictive.

We assume that the sample period over which observations on y_{it} are available runs from $t = \tau - T$ to $t = \tau + S$, where τ is the last period of interest. In sections 2 through 5 we restrict ourselves to the estimation of period means not too close to the beginning or end of the sample period, i.e., we will present results for the limiting case when T and S tend to infinity. In section 6 we drop the assumption that an infinite number of future observations is available at the time of estimation, but the assumption that T is large is maintained. As a special case we consider the case in which no future observations are available, i.e., $S = 0$.

A rotating panel with rotation length r is defined in this paper by the property that in every period $100r^{-1}\%$ of the participants is replaced and the assumption that always those units are replaced which participated the largest number of periods. If, e.g., $r = 2$, 50% of the participants in the first wave of the rotating sample will be replaced in the second wave, the other half is replaced in the third wave. New participants in the second wave will be replaced in the fourth wave, etc. Of course a rotating sample with rotation period equal to one is simply a series of cross-sections. We determine the relative efficiency of efficient estimators of linear combinations of the period means μ_t in (1) from a rotating sample with rotation length r and n_r observations in every wave on the one hand and another rotating panel with rotation length s and n_s observations in every wave on the other hand.

The plan of the paper is as follows. In section 2 we will show how results from spectral analysis can be used to obtain manageable expressions for the variance of efficient estimators of linear combinations of the μ_t in (1). In section 3 these results are used to discuss the relative efficiencies of designs for given parameters of interest. Defining $\rho = \sigma_\alpha^2(\sigma_\varepsilon^2 + \sigma_\alpha^2)^{-1}$, we show, e.g., that if the parameter of interest is the period mean μ_t itself, the relative efficiency of a series of cross-sections ($r = 1$) compared to a rotating panel with rotation period $r = 2$ is given by $V\{\hat{\mu}_t^1\}/V\{\hat{\mu}_t^2\} = n_2 n_1^{-1}(1 - \rho^2)^{-1/2}$. In section 4 bounds on the relative efficiency of rotating panels are derived which hold true irrespective of the parameter of interest. There we show, e.g., that the relative efficiency of efficient estimators of any linear combination of the period means based on a series of cross-sections and a panel with rotation period $r = 2$, respectively, will always lie in the interval $(n_2 n_1^{-1}(1 + \rho)^{-1}, n_2 n_1^{-1}(1 - \rho)^{-1})$. In section 5 we consider the choice of the optimal rotation period assuming a simple cost structure. In section 6 the simplifying assumption that the number of periods after the period of interest is large is dropped, and it is shown there that this assumption does not strongly influence most of the results. Finally, section 7 presents an empirical illustration and concludes.

2. Theoretical results on the variances of the parameter estimators

In this section we discuss the main steps in the derivation of manageable expressions for the variance of efficient estimators of (finite) linear combinations of the period means μ_t in (1) from a rotating panel with rotation period r. Details are presented in the appendix.

As stated in the previous section the parameter of interest is assumed to be $\sum_{j=0}^{J}\xi_j\mu_{\tau-j}$. We define $\xi_j = 0$ if $j > J$ or $j < 0$, and we define vectors $\mu = (\mu_{\tau-T}, \dots, \mu_{\tau+S})'$ and $\xi = (\xi_T, \xi_{T-1}, \dots, \xi_{-S})'$ such that $\xi'\mu = \sum_{j=0}^{J}\xi_j\mu_{\tau-j}$. Using the fact that the data in a rotating panel with rotation period r can be divided into r independent subsamples in such a way that each subsample is a time series of independent small panels, we first show in the appendix that the variance of the BLUE of $\xi'\mu$ can be written as

$$V\{\xi'\hat{\mu}^r\} = \sigma_\varepsilon^2\xi'\Psi\xi/n_r, \tag{2}$$

where $\hat{\mu}^r$ is the BLUE of μ from a rotating panel with rotation period r and Ψ is defined by $\Psi = A^{-1}$, where A is a band matrix with elements A_{lk} $(k, l = -T, -T+1, \dots, S-1, S)$ satisfying $A_{lk} = a_{|l-k|}$ if $r - T < l, k < S - r$, and

$$\begin{aligned} a_j &= 1 - \frac{\rho}{1+(r-1)\rho} && \text{if} \quad j = 0, \\ &= -\frac{r-j}{r}\frac{\rho}{1+(r-1)\rho} && \text{if} \quad 0 < j < r, \\ &= 0 && \text{if} \quad j \geq r. \end{aligned} \tag{3}$$

The main problem then is to find expressions for the elements of A^{-1}. Several approaches to this problem are possible. We suggest to use an analogy with a similar problem that has been analysed in the literature on time-series analysis. There the inversion is considered of a matrix Σ^{MA} defined by $\Sigma_{ts}^{\text{MA}} = \mathrm{E}x_t x_s$, with x_t generated by some moving-average process, $x_t = \vartheta(L)e_t$ where $e_t \sim \text{IID}(0, \sigma_e^2)$ and $\vartheta(L) = 1 + \vartheta_1 L + \cdots + \vartheta_{r-1}L^{r-1}$ is a polynomial in the lag operator L. It is well known that the inverse of Σ^{MA} can be approximated by the matrix Σ^{AR} defined by $\Sigma_{ts}^{\text{AR}} = \mathrm{E}z_t z_s$, where z_t is the autoregressive process obtained by inverting the lag polynomial in the moving-average process underlying Σ^{MA}, that is $z_t = \vartheta^{-1}(L)e_t$. More precisely, Shaman (1975) shows that Σ^{MA} and $(\Sigma^{\text{AR}})^{-1}$ are identical except for the $(r-1) \times (r-1)$ submatrices in the upper left and lower right corners.

Now choose σ_e^2 and ϑ_k $(k = 1,\dots, r-1)$ in such a way that $\mathrm{E}x_t x_s = a_{|t-s|}$, which is possible because the a_j $(j = 0,\dots, r-1)$ satisfy the conditions given by Wold (1953, pp. 152–154). If Σ^{MA} and Σ^{AR} are chosen in this way the matrix A which is to be inverted differs from Σ^{MA} only by $(r-1)\times(r-1)$ submatrices in the upper left and lower right corners. Using this fact and the result obtained by Shaman (1975) we show in the appendix that, if $\xi_j = 0$ for $|j| > J$, then

$$\lim_{S,T\to\infty} \xi' A^{-1}\xi = \lim_{S,T\to\infty} \xi'\Sigma^{\mathrm{AR}}\xi. \tag{4}$$

Thus eq. (4) shows how the variance of a linear combination of period means not too close to the beginning or end of the observation period can be approximated.

The simplest way to obtain the elements of the matrix Σ^{AR} (and to obtain the results to be presented in section 4) appears to be to use results from spectral analysis. The spectral density associated with the series of covariances a_j is defined by

$$f_r(\lambda) = \frac{1}{2\pi}\sum_{j=-r+1}^{r-1} a_{|j|}\mathrm{e}^{-i\lambda j}, \qquad -\pi < \lambda < \pi. \tag{5}$$

In the appendix we show that if the a_j are given by (3), the spectral density $f_r(\lambda)$ can be written as

$$f_r(\lambda) = \begin{cases} \dfrac{1}{2\pi}\dfrac{1}{1+(r-1)\rho}\left\{1-\rho+\rho r - \dfrac{\rho}{r}\dfrac{1-\cos(\lambda r)}{1-\cos\lambda}\right\}, & \lambda \neq 0, \\[2ex] \dfrac{1}{2\pi}\dfrac{1}{1+(r-1)\rho}\{1-\rho\}, & \lambda = 0. \end{cases} \tag{6}$$

A direct consequence of standard results in spectral analysis [see, e.g., Fishman (1969), Priestley (1981), or Harvey (1981)] is the fact that the variance of $\sum_{j=0}^{J}\xi_j z_{t-j}$, where $z_t = \vartheta^{-1}(L)e_t$ as before, can be written as

$$\mathrm{V}\left\{\sum_{j=0}^{J}\xi_j z_{t-j}\right\} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} g(\lambda) f_r^{-1}(\lambda)\,\mathrm{d}\lambda, \tag{7}$$

where $f_r^{-1}(\lambda)$ denotes $1/f_r(\lambda)$ and

$$g(\lambda) = \sum_{j=-J}^{J} w_{|j|}\mathrm{e}^{i\lambda j}, \tag{8}$$

with

$$w_k = \sum_{j=0}^{J} \xi_j \xi_{j+k}, \qquad k = 0, \ldots, J. \tag{9}$$

As $V\{\sum_{j=0}^{J} \xi_j z_{t-j}\}$ can also be written as $\xi' \Sigma^{AR} \xi$ we finally obtain the main result of this section from (4) and (7):

$$\lim_{S,T \to \infty} V\{\xi' \hat{\mu}^r\} = \frac{\sigma_\varepsilon^2}{4 n_r \pi^2} \int_{-\pi}^{\pi} g(\lambda) f_r^{-1}(\lambda) \, d\lambda, \tag{10}$$

provided $\xi_j = 0$ if $|j| > J$ for some finite J. For the sake of notation the lim operation will be deleted in the following sections.

3. The relative efficiency of designs for specific parameters of interest

Eq. (10) in the previous section shows how the variance of an efficient estimator will depend on the linear combination of the period means to be estimated, on the choice of the rotation period, and on the number of observations in each wave. In this section we will analyse what this result implies for the relative efficiency of rotating panels if one is interested in some particular linear combination of the period means in (1). In the next section we will use (10) to derive conditions on the relative efficiency of panels which hold true irrespective of the parameter of interest.

An important feature of (10) is that the weights in the linear combination, ξ_j $(j = 0, \ldots, J)$, determine the numerator within the integral while the choice of r affects the denominator only. In fig. 1 we have plotted the reciprocal of the denominator, $f_r^{-1}(\lambda)$, for rotation periods $r = 1, 2, 3, 4, 8, 12$ assuming that $\rho = 0.5$. Similarly the numerator in (10), $g(\lambda)$, is presented in fig. 2 for six important special cases: estimation of the period means themselves ($J = 0$; $\xi_0 = 1$), of differences in means between two successive periods ($J = 1$; $\xi_0 = 1$, $\xi_1 = -1$), and of a k-period sum or average ($J = k - 1$; $\xi_j = 1$, $j = 0, \ldots, k-1$) for $k = 2, 3, 6, 12$. Note that both $f_r(\lambda)$ and $g(\lambda)$ are symmetric in λ and are therefore plotted for nonnegative values of λ only.

It is obvious from these figures and well known in the literature [see, e.g., Cochran (1978, p. 348 ff.)] that the choice of the rotation period which minimizes the variance of the efficient estimator will in general depend on the linear combination of the means to be estimated. If the number of observations per wave does not vary with the choice of the rotation period ($n_r = n$ for all r), a series of cross-sections ($r = 1$) will be optimal if a twelve-period average is to be estimated because it is mainly the behaviour of $f_r^{-1}(\lambda)$ for small values of λ ('low frequency') which is important. If a

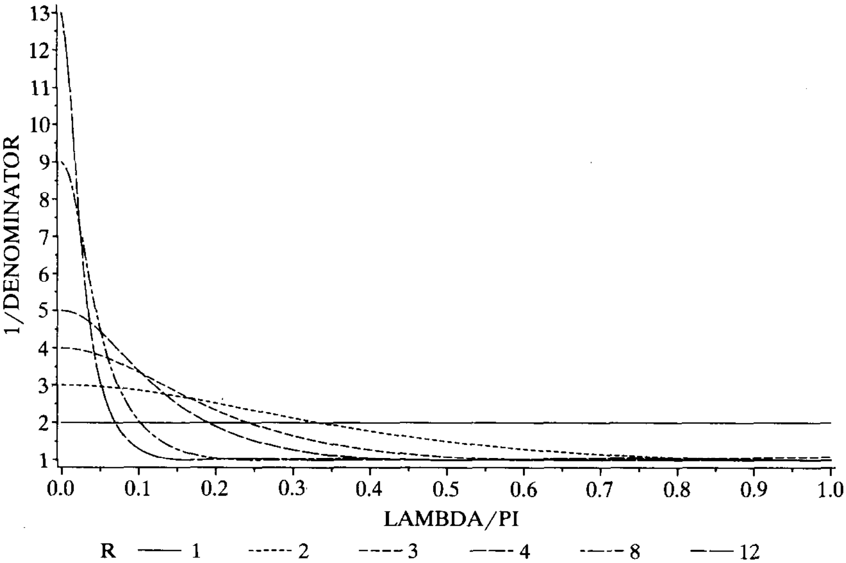

Fig. 1. The inverse of the denominator for several rotation periods with $\rho = 0.5$.

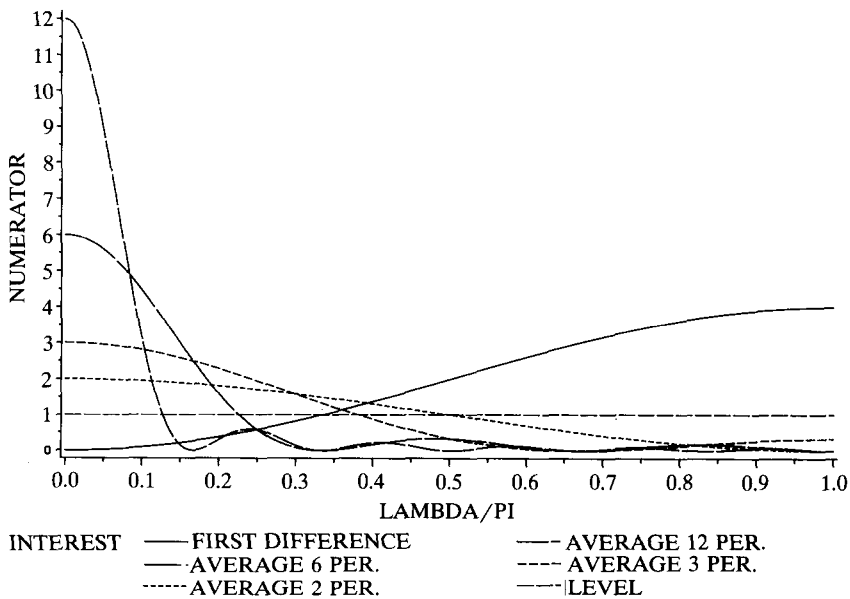

Fig. 2. Values of numerator for several linear combinations of interest.

difference in means is to be estimated, a large value of r is optimal because the 'high frequency' components dominate the variance.

Using (10) it is possible for any given ξ to compute the variance of the efficient estimator of $\xi'\mu$ given r and ρ. The integral in (10) can be computed with the Residue Theorem (see appendix), which yields that

$$\mathrm{V}\{\xi'\hat{\mu}^r\} = \mathrm{Re}\left(\frac{1}{n_r}\sigma_\varepsilon^2 \sum_{j=1}^{r-1} \frac{z_j^{r-2}\left[w_0 + 2\sum_{k=1}^{J} w_k z_j^k \right]}{h'(z_j)} \right), \tag{11}$$

where $\mathrm{Re}(z)$ denotes the real part of z and the z_j are the $r-1$ zeroes within the unit circle of the polynomial $h(z) = z^{r-1}\sum_{j=-r+1}^{r-1} a_{|j|} z^{-j}$, and $h'(z) = \mathrm{d}h(z)/\mathrm{d}z$. The w_k are defined in (9). Note that the z_j are the $r-1$ roots of the lag polynomial of the moving-average process introduced in section 2 and that in (11) it is assumed for simplicity that $h(z) = 0$ has no multiple roots.

In order to compute the variances using (11) one has to determine the zeroes of a polynomial of degree $2r-2$. Although analytical results for $r=3$ and $r=4$ can be obtained, they are not very revealing. Therefore we present analytical results for $r=1$ and $r=2$ only. For $r=1$, the variance of $\xi'\hat{\mu}^r$ is seen to equal

$$\mathrm{V}\{\xi'\hat{\mu}^1\} = \frac{1}{n_1}\sigma^2 w_0, \tag{12}$$

where $\sigma^2 = \sigma_\alpha^2 + \sigma_\varepsilon^2$, and for $r=2$ (11) reduces to

$$\mathrm{V}\{\xi'\hat{\mu}^2\} = \frac{1}{n_2}\sigma^2\sqrt{1-\rho^2}\left(w_0 + 2\sum_{k=1}^{J} w_k \left\{ \frac{1-\sqrt{1-\rho^2}}{\rho} \right\}^k \right). \tag{13}$$

Using (12) and (13) it is straightforward to check by substituting $\xi_0 = 1$ and $J = 0$ that the relative efficiency of the BLUE of the period means μ_τ based on a series of cross-sections and on a rotating panel with rotation period $r = 2$ is given by

$$\frac{\mathrm{V}\{\hat{\mu}_\tau^1\}}{\mathrm{V}\{\hat{\mu}_\tau^2\}} = \frac{n_2}{n_1}\frac{1}{\sqrt{1-\rho^2}}. \tag{14}$$

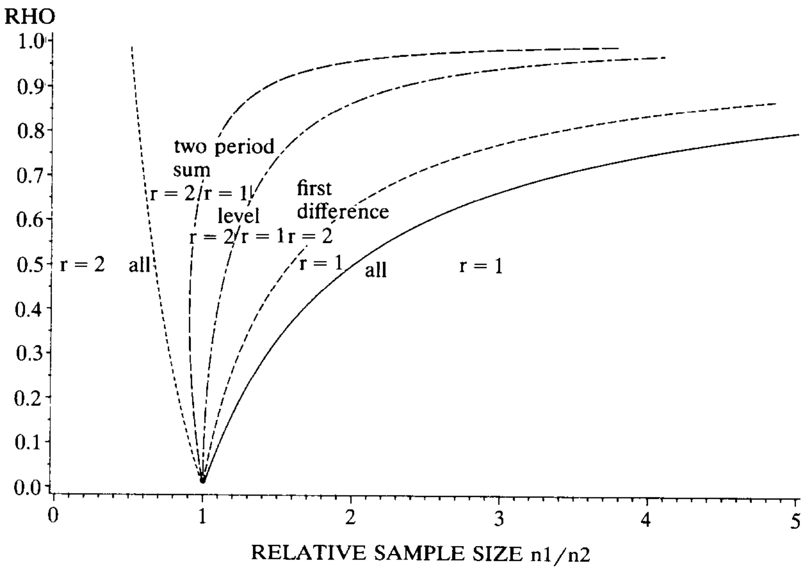

Fig. 3. Comparison of a series of cross-sections ($r = 1$) and a rotating panel with $r = 2$.

Similarly the relative efficiency of the BLUEs of $\mu_\tau - \mu_{\tau-1}$ is

$$\frac{\mathrm{V}\{\hat{\mu}^1_\tau - \hat{\mu}^1_{\tau-1}\}}{\mathrm{V}\{\hat{\mu}^2_\tau - \hat{\mu}^2_{\tau-1}\}} = \frac{n_2}{n_1} \frac{1}{\sqrt{1-\rho^2}} \frac{\rho}{\rho - 1 + \sqrt{1-\rho^2}}, \tag{15}$$

while one obtains

$$\frac{\mathrm{V}\{\hat{\mu}^1_\tau + \hat{\mu}^1_{\tau-1}\}}{\mathrm{V}\{\hat{\mu}^2_\tau + \hat{\mu}^2_{\tau-1}\}} = \frac{n_2}{n_1} \frac{1}{\sqrt{1-\rho^2}} \frac{\rho}{\rho + 1 - \sqrt{1-\rho^2}}, \tag{16}$$

if $\mu_\tau + \mu_{\tau-1}$ is the parameter of interest. The result in (14) implies that a series of cross-sections yields more efficient estimates of the period means than a rotating sample with rotation period $r = 2$ if $n_1/n_2 > (1-\rho^2)^{-1/2}$. If this condition does not hold, the rotating sample is preferable. Similar conditions for the cases where $\mu_\tau - \mu_{\tau-1}$ or $\mu_\tau + \mu_{\tau-1}$ are the parameters of interest are implied by (15) and (16). A graphical illustration of these conditions is given in fig. 3 where it is indicated for which values of the relative sample size $r = 1$ or $r = 2$ is preferable. The lines marked 'all' will be dealt with in the next section. Bounds on n_1/n_2 for other linear combinations of interest can be obtained directly from (12) and (13). It can easily be

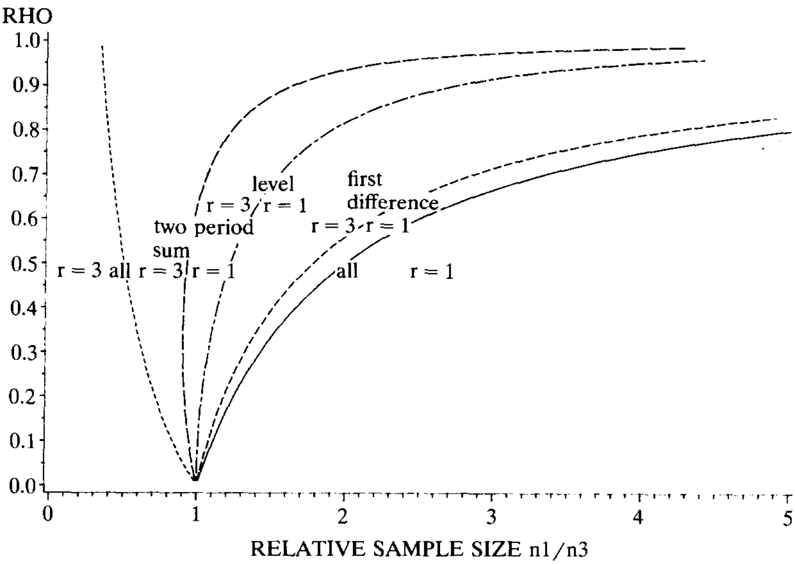

Fig. 4. Comparison of a series of cross-sections ($r = 1$) and a rotating panel with $r = 3$.

verified, e.g., that the condition $n_1/n_2 > (1 - \rho^2)^{-1/2}$ from (14) is sufficient for optimality of a series of cross-sections over $r = 2$ if one is interested in weighted averages of the period means with nonnegative weights only as in that case $w_k \geq 0$ $(k = 0, \ldots, J)$.

Efficiency comparisons similar to (14), (15), and (16) for other combinations of rotation periods can easily be obtained numerically using eq. (11). These efficiency results imply bounds on the relative sample sizes for the one or the other rotation period to yield more efficient estimates than the other. In fig. 4 such bounds are presented for the case where the choice is restricted to $r = 1$ or $r = 3$. Evidently, $r = 1$ is not an attractive choice if the individual effect is dominant unless the number of observations in the cross-sections is much larger than in the rotating panels.

4. The relative efficiency of designs irrespective of the parameter of interest

A problem with the fact that the relative efficiency of panels with different rotation periods generally depends on the aim for which the rotating panel is to be used is that one usually wants to use one panel for the estimation of both levels, differences and averages. In this section we derive bounds on this relative efficiency which hold true irrespective of the linear combination of the means to be estimated.

Evidently, from (10), a panel with rotation period r will yield a more efficient estimator of $\xi'\mu$ than another panel with rotation period s irrespective of the choice of ξ if

$$n_r^{-1} f_r^{-1}(\lambda) < n_s^{-1} f_s^{-1}(\lambda), \qquad -\pi < \lambda \leq \pi. \tag{17}$$

If the value η_{rs} is defined by

$$\eta_{rs} = \max_{\lambda \in [-\pi, \pi]} f_s(\lambda) f_r^{-1}(\lambda), \tag{18}$$

eqs. (10) and (17) imply that the relative efficiency of panels with rotation period r and s respectively will be in the interval $(n_s n_r^{-1} \eta_{sr}^{-1}, n_s n_r^{-1} \eta_{rs})$ irrespective of the parameter of interest. In particular this implies that the panel with rotation period r will yield more efficient estimators of any linear combination of the period means in (1) than a panel with rotation period s if the numbers of observations per wave satisfy $n_r/n_s > \eta_{rs}$.

Let us first of all consider the choice between a series of cross-sections and a rotating panel with rotation period equal to two. Using

$$f_1(\lambda) = \frac{1}{2\pi}(1 - \rho) \tag{19}$$

and

$$f_2(\lambda) = \frac{1}{2\pi}(1 + \rho)^{-1}(1 - \rho \cos \lambda), \tag{20}$$

it is straightforward to verify that $\eta_{12} = (1 - \rho)^{-1}$ and $\eta_{21} = 1 + \rho$. Thus, for any finite linear combination of interest $\xi'\mu$, it holds true that

$$(1 + \rho)^{-1} \frac{n_2}{n_1} < \frac{\mathrm{V}\{\xi'\hat{\mu}^1\}}{\mathrm{V}\{\xi'\hat{\mu}^2\}} < (1 - \rho)^{-1} \frac{n_2}{n_1}. \tag{21}$$

Expression (21) implies that if $n_1 > (1 - \rho)^{-1} n_2$, a series of cross-sections is preferable to a rotating panel with rotation period 2 without ambiguity, while the opposite is true if $n_1 < (1 + \rho)^{-1} n_2$. If neither of these conditions is satisfied, the choice of the optimal design depends on the parameters of interest in the way described in the previous section.

Using (6) it can be shown that $\eta_{1s} = (1 - \rho)^{-1}$ $(s = 2, 3, \ldots)$ and that $\eta_{rs} = \{1 + (r - 1)\rho\}/\{1 + (s - 1)\rho\}$ if $r > s$, so that (21) can be generalized to

$$[1 + (r - 1)\rho]^{-1} \frac{n_r}{n_1} < \frac{\mathrm{V}\{\xi'\hat{\mu}^1\}}{\mathrm{V}\{\xi'\hat{\mu}^r\}} < (1 - \rho)^{-1} \frac{n_r}{n_1}. \tag{22}$$

Table 1

Lower bounds η_{rs} on the quotient of the number of observations per wave n_r/n_s for panel with rotation period r to be unambiguously preferable.

	ρ	$s=1$	$s=2$	$s=3$	$s=4$	$s=8$	$s=12$
$r=1$	0.3	—	1.43	1.43	1.43	1.43	1.43
	0.6	—	2.50	2.50	2.50	2.50	2.50
	0.9	—	10.00	10.00	10.00	10.00	10.00
$r=2$	0.3	1.30	—	1.22	1.38	1.67	1.76
	0.6	1.60	—	1.50	1.92	2.95	3.40
	0.9	1.90	—	1.99	3.06	7.16	10.29
$r=3$	0.3	1.60	1.23	—	1.18	1.67	1.92
	0.6	2.20	1.38	—	1.35	2.61	3.48
	0.9	2.80	1.47	—	1.60	4.54	7.67
$r=4$	0.3	1.90	1.46	1.19	—	1.55	1.92
	0.6	2.80	1.75	1.27	—	2.14	3.15
	0.9	3.70	1.95	1.32	—	3.04	5.55
$r=8$	0.3	3.10	2.38	1.94	1.63	—	1.42
	0.6	5.20	3.25	2.36	1.86	—	1.69
	0.9	7.30	3.84	2.80	1.97	—	1.99
$r=12$	0.3	4.30	3.31	2.69	2.26	1.39	—
	0.6	7.60	4.75	3.45	2.71	1.46	—
	0.9	10.90	5.74	3.89	2.95	1.49	—

In more general cases it does not appear to be possible to obtain simple analytical expressions for η_{rs}, but it is of course straightforward to maximize (18) numerically. Numerical results for three different values of ρ are presented in table 1. If, for example, $\rho = 0.3$, $r = 3$ will be unambiguously preferable to $r = 2$ if $n_3/n_2 > 1.23$, while $r = 2$ will be unambiguously preferable to $r = 3$ if $n_2/n_3 > 1.22$, i.e., if $n_3/n_2 < 0.82$. It is evident from these results that it is relatively simple to choose the optimal rotation period if ρ is small, i.e., if individual effects are relatively not very important in the analysis of variance model (1). Of course the choice of the rotation period is also less important if ρ is small since in that case the obtainable efficiency gain will only be marginal.

In fig. 3, where we restrict ourselves to the choice between $r = 1$ and $r = 2$, we have drawn the bounds η_{12} and η_{21}^{-1} on the relative sample size n_1/n_2. These bounds (marked 'all') determine regions in which $r = 1$ and $r = 2$ respectively are unambiguously preferable to the other. Analogously, bounds for the choice between $r = 1$ and $r = 3$ are drawn in fig. 4. Of course the bounds derived in section 3 for some specific linear combinations of interest will always lie in the region where the choice depends on the parameter of interest.

5. The optimal choice of the rotation period given a budget constraint

In sections 3 and 4 we have shown how the results of section 2 can be used to obtain the relative efficiency of the estimators from two (given) designs and thereby to determine which of these two designs yields a more efficient estimator. In this section we will consider the more general problem of the optimal choice of the rotation period given an assumed cost structure. Assume that a sample still has to be drawn and that one is free to choose the rotation period r as long as it is not larger than some prescribed maximum: $r \leq r^{\max}$. Such a maximum will usually have to be imposed, e.g., to avoid problems concerning panel conditioning and selective nonresponse referred to in the introduction. Let p_1 denote the cost of observing an individual for the first time and p_2 of observing it for a second time. Assume for simplicity that observing it for a third, fourth, etc. time is equally expensive as the second observation. If there is a budget B for each period, the number of observations per wave in case of rotation period r equals

$$n_r = \frac{rB}{p_1 + (r-1)p_2} = \frac{rB^*}{1 + (r-1)\alpha}, \tag{23}$$

where $B^* = B/p_1$ and $\alpha = p_2/p_1$, the relative cost of a repeated observation. Based on experiences from the Panel Study of Income Dynamics, Duncan, Juster, and Morgan (1987) suggest that α is smaller than unity. In particular, they state that the field costs of a cross-section are 30% to 70% higher than for additional waves of the panel, implying values for α between 0.6 and 0.8.

If $r^{\max} = 2$, the choice is again restricted to either $r = 1$ or $r = 2$ and the bound in (16), for example, can easily be rewritten to show that spending the budget on a series of cross-sections will yield more precise estimates of averages over two periods than a rotating panel with rotation period equal to two if

$$\alpha > \frac{2}{\sqrt{1-\rho^2}} \frac{\rho}{\rho + 1 - \sqrt{1-\rho^2}} - 1. \tag{24}$$

Note that (14) and (15) imply that the rotating panel will always be preferable if levels or period-to-period changes are to be estimated as long as $\alpha < 1$ which is likely to be the case.

More general results on the optimal choice of the rotation period can be obtained using (11). The choice of the rotation period if $r^{\max} = 8$ is visualized in figs. 5, 6, 7, and 8 for the case of averages over 2, 3, 6, and 12 periods, respectively. Note that no figures are included for the estimation of a

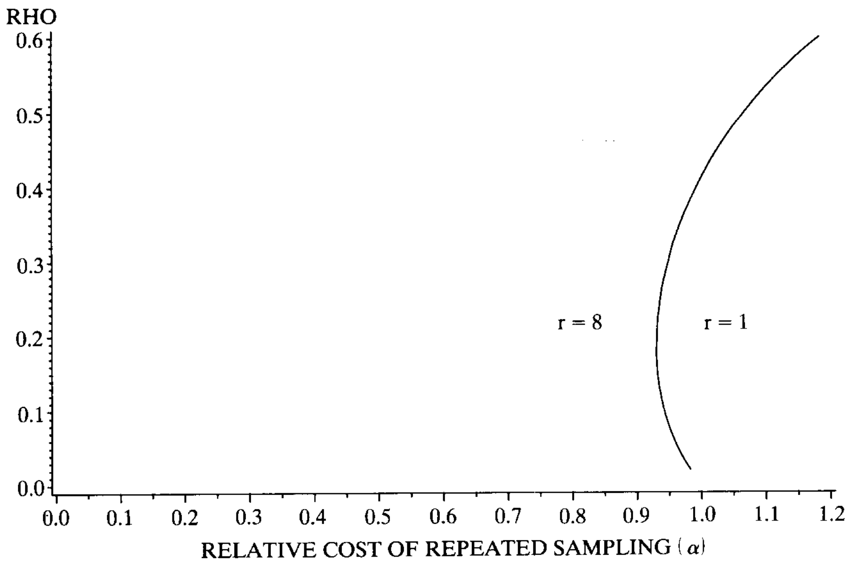

Fig. 5. The optimal rotation period for estimating a two-period average mean.

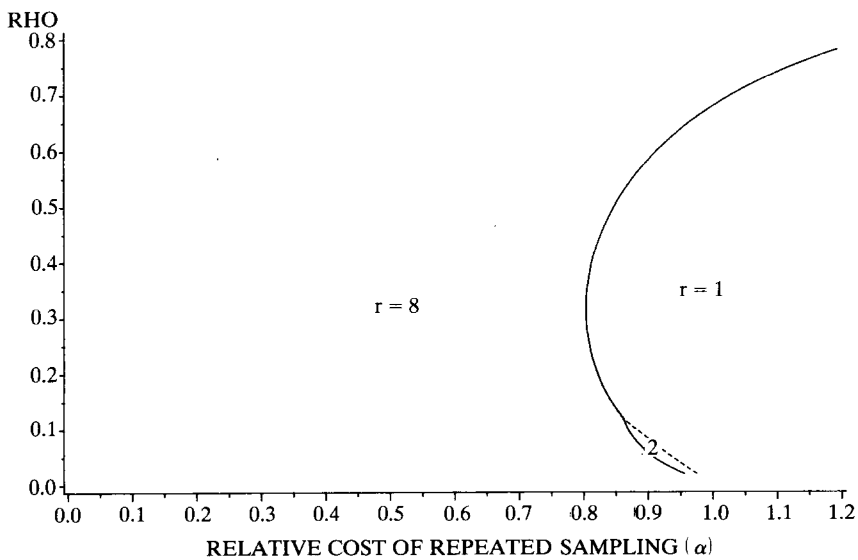

Fig. 6. The optimal rotation period for estimating a three-period average mean.

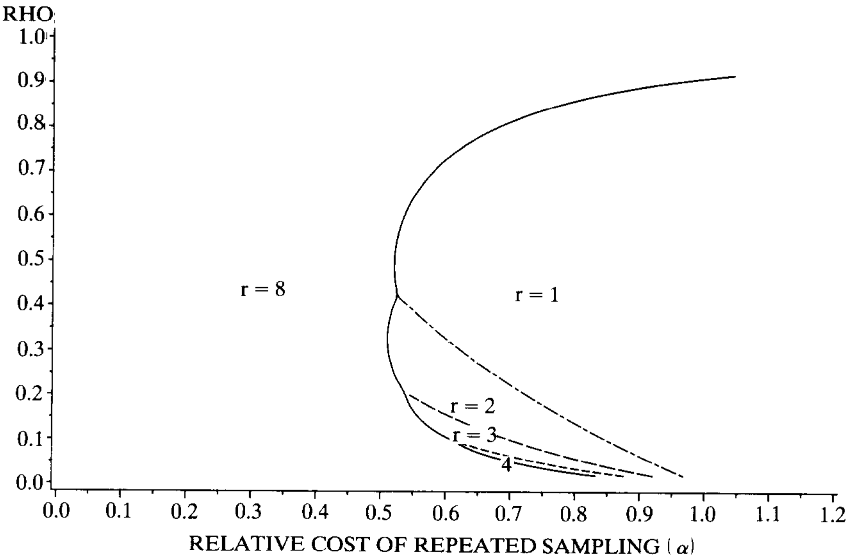

Fig. 7. The optimal rotation period for estimating a six-period average mean.

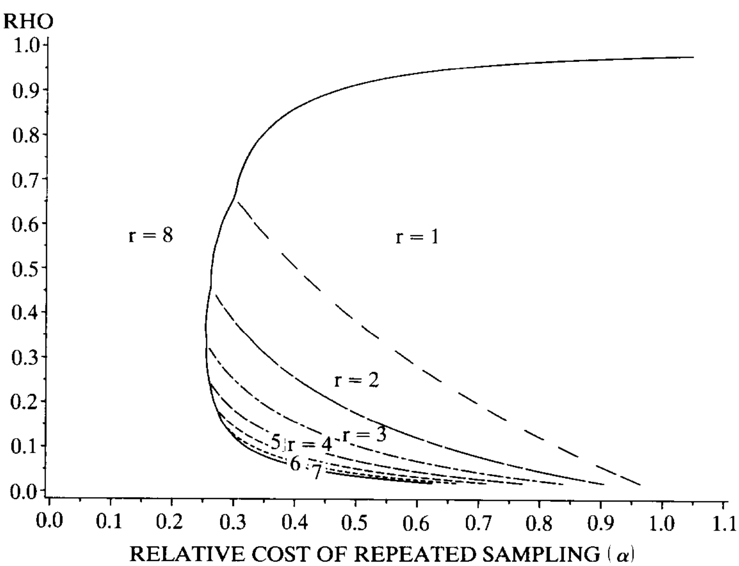

Fig. 8. The optimal rotation period for estimating a twelve-period average mean.

Table 2
Relative efficiency for a panel with rotation period r compared with a series of cross-sections.

	ρ	Levels		First difference		Two-period sum	
		$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$
$r = 2$	0.3	0.71	0.95	0.61	0.81	0.83	1.10
	0.6	0.60	0.80	0.40	0.53	0.80	1.07
	0.9	0.33	0.44	0.12	0.16	0.53	0.71
$r = 3$	0.3	0.62	0.93	0.50	0.76	0.74	1.11
	0.6	0.49	0.74	0.31	0.46	0.68	1.02
	0.9	0.24	0.37	0.08	0.12	0.41	0.61
$r = 4$	0.3	0.57	0.92	0.46	0.73	0.68	1.10
	0.6	0.44	0.71	0.27	0.43	0.61	0.98
	0.9	0.21	0.33	0.07	0.11	0.34	0.55
$r = 8$	0.3	0.49	0.88	0.40	0.71	0.59	1.05
	0.6	0.36	0.64	0.23	0.41	0.49	0.86
	0.9	0.15	0.27	0.06	0.10	0.24	0.43
$r = 12$	0.3	0.46	0.86	0.38	0.70	0.55	1.01
	0.6	0.33	0.60	0.22	0.40	0.43	0.80
	0.9	0.13	0.24	0.05	0.10	0.20	0.38

single period mean or a difference in means as our numerical results suggest that in these cases the largest rotation period will always be optimal. However, one should not be tempted to think that a true panel ($r = \infty$) would yield even more efficient estimates if the preferred choice for the rotation period is $r^{\max}$. In case of equal sample sizes, for example, a true panel will yield estimators of the period means which are as efficient as the ones derived from a series of cross-sections [see, e.g., Cochran (1977, p. 345 ff.)]. In general, figs. 5 to 8 clearly show that intermediary rotation periods ($r = 2,\dots,7$) are optimal in very small parts of the (ρ, α) space only. Usually it will either be the maximal ($r = 8$) or the minimal ($r = 1$) rotation period which is optimal.

It is not only relevant to know how the optimal rotation period can be determined, but also to know how much efficiency will be lost if a suboptimal choice is made. In table 2 we present the relative efficiencies compared with a series of cross-sections ($r = 1$) for several rotation periods, some specific parameters of interest, and values of α and ρ. As an illustration consider the case where $\rho = 0.6$ and $\alpha = 1$ (equal cost). Then it follows from table 2 that the variance of the estimator of a particular μ_t in case $r = 4$ is equal to 71% of the variance for $r = 1$ (a series of cross-sections) and only 43% if one is estimating a first difference. It is clear that the gains in efficiency can be quite substantial if an optimal sample design is chosen. Even in case of equal cost ($\alpha = 1$) gains of more than 50% are not uncommon. Using (23) and the results for $\alpha = 1$, one can easily compute relative efficiency bounds for any α by multiplying the numbers in the table by $\{1 + (r - 1)\alpha\}/r$.

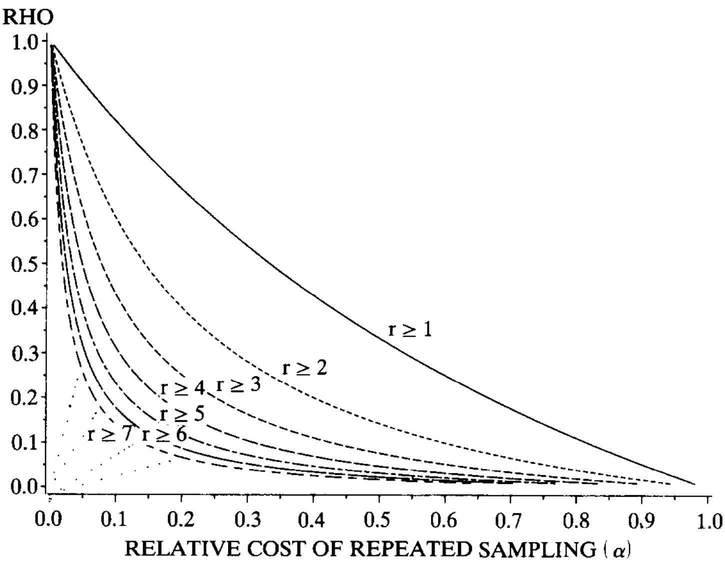

Fig. 9. Regions with restrictions on the optimal rotation period r.

Similar to the approach chosen in section 3, one can check whether it is possible to determine the rotation period which minimizes the variance of the BLUE irrespective of the parameter of interest, assuming that the cost structure (23) holds and imposing $r \leq r^{\max}$. Using (23), each bound η_{rs} can be rewritten as a bound on the relative cost of resampling, α. Pairwise comparisons are used to determine regions of the parameter space where one or more rotation period(s) can never be optimal, whatever the parameter of interest. These regions are presented in fig. 9 where we assumed $r^{\max} = 8$. Note that in most regions there is no unique optimal rotation period since this will depend on the parameter of interest. However, optimality of some rotation periods can be excluded for some values of ρ and α. If, e.g., $\rho < (1-\alpha)/(1+\alpha)$, a series of cross-sections will not be optimal for any choice of the parameter of interest. More general results can be inferred from fig. 9. If, e.g., $\alpha < 0.5$, a series of cross-sections cannot be optimal for any parameter of interest if $\rho < 0.33$, while $r = 2$ and $r = 3$ will always be suboptimal if $\rho < 0.17$ and $\rho < 0.09$, respectively.

6. The optimal design for specific parameters of interest if one is estimating in recent periods

A drawback of the results of the previous sections is that they are only valid if one is estimating period means not too close to the end of the sample period. In those sections we restricted attention to the limiting case where the number of future periods on which data are available, S, tends to infinity. In this section we consider the case of a fixed S, still assuming for convenience that the number of past periods in the sample, T, is infinitely large. The results in this section suggest that, unless ρ is close to unity and S very small, the earlier results are hardly affected.

The main reason for considering the limiting case where S tends to infinity in the previous sections is that in this case a simple expression for the inverse of the matrix A, which arises in the variance–covariance matrix of the efficient estimator, is available. In this section we show how to obtain an expression for the inverse of this matrix if S is fixed. Denote the moving-average process which generates the autocovariances $\mathrm{E}x_t x_s = a_{|t-s|}$ by $x_t = \vartheta(L)e_t$ with $e_t \sim \mathrm{IID}(0, \sigma_e^2)$ as before. Define $z_t = \vartheta^{-1}(L)e_t = \psi(L)e_t$ where $\psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \cdots$. In section 2, where it was assumed that T and S tended to infinity, we have approximated the inverse of the matrix Σ^{MA} defined by $\Sigma_{ts}^{\mathrm{MA}} = \mathrm{E}x_t x_s$ by Σ^{AR} defined as $\Sigma_{ts}^{\mathrm{AR}} = \mathrm{E}z_t z_s$. A valid approximation to $(\Sigma^{\mathrm{MA}})^{-1}$ if S is fixed is to use the matrix of covariances of more-than-S-period-ahead prediction errors of the AR process instead of the matrix of covariances of the variable z_t itself. In the appendix we show that if we define the symmetric matrix $B = (b_{lk})$ by

$$b_{lk} = \sigma_e^{-2} \sum_{j=0}^{S-k} \psi_j \psi_{j+k-l} \qquad k \geq l, \quad -T \leq l, k \leq S, \tag{25}$$

and partition B as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{26}$$

where B_{22} has dimension $(r-1) \times (r-1)$, it holds true that, if $\xi_j = 0$ for $j > J$ for some fixed J,

$$\lim_{T \to \infty} \xi' A^{-1} \xi = \lim_{T \to \infty} \xi' \left\{ B - \begin{pmatrix} B_{12} \\ B_{22} \end{pmatrix} (I + P_{22} B_{22})^{-1} P_{22} (B_{21} B_{22}) \right\} \xi, \tag{27}$$

where S is fixed and P_{22} denotes the lower right $(r-1) \times (r-1)$ block of $A - \Sigma^{\mathrm{MA}}$. Evidently, (27) generalizes (4).

In order to illustrate these results consider again the case where $r = 2$ in which case $\vartheta(L) = 1 - \vartheta L$ with

$$\vartheta = \frac{1 - (1-\rho^2)^{1/2}}{\rho} \quad \text{and} \quad \sigma_e^2 = \frac{1}{2}\frac{\rho}{\vartheta}\frac{1}{1+\rho}. \tag{28}$$

In this case b_{lk} reduces to

$$b_{lk} = \sigma_e^{-2} \sum_{j=0}^{S-k} \vartheta^j \vartheta^{j+k-l} = \sigma_e^{-2} \vartheta^{k-l} \frac{1 - \vartheta^{2(S-k+1)}}{1 - \vartheta^2}, \qquad k \geq l. \tag{29}$$

Using (27) and (29) it is straightforward to verify that the variance of $\xi' \hat{\mu}^r = \sum_{j=0}^{J} \xi_j \hat{\mu}^r_{\tau-j}$ for a rotating panel with rotation period $r = 2$ is given by

$$\mathrm{V}\{\xi'\hat{\mu}^2\} = \frac{1}{n_2}\sigma^2(1-\rho)\left\{ \sum_{k,l=0}^{J} \xi_l \xi_k b_{-k,-l} + \frac{\rho^2}{2(1-\rho\vartheta)(1+\rho)} \left(\sum_{l=0}^{J} \xi_l b_{S,-l} \right)^2 \right\}. \tag{30}$$

Note that (30) is a generalization of (13), while the two expressions coincide if S tends to infinity. For the special case of estimating the period means μ_τ, it is readily verified from (30) that the relative efficiency of a series of cross-sections and a rotating panel with $r = 2$ can be written as

$$\frac{\mathrm{V}\{\hat{\mu}^1_\tau\}}{\mathrm{V}\{\hat{\mu}^2_\tau\}} = \frac{n_2}{n_1} \frac{1}{\sqrt{1-\rho^2}} \{1 + \vartheta^{2S+2}\}^{-1}. \tag{31}$$

Note that this relative efficiency decreases with S and tends to the expression in (14) if S tends to infinity since $|\vartheta| < 1$ if $\rho < 1$. It is not surprising that (31) is always smaller than (14) because the nonavailability of future observations has no impact on the efficient estimator in the cross-section case ($r = 1$), but implies an information loss for the $r = 2$ case.

Table 3

Relative efficiency for a panel with rotation period r compared with a series of cross-sections (in case of equal sample sizes).

	ρ	$S=0$	$S=1$	$S=2$	$S=\infty$
			Level		
$r=2$	0.3	0.98	0.96	0.95	0.95
	0.6	0.89	0.85	0.82	0.80
	0.9	0.61	0.58	0.54	0.44
$r=3$	0.3	0.96	0.95	0.93	0.93
	0.6	0.85	0.79	0.75	0.74
	0.9	0.53	0.47	0.43	0.37
$r=4$	0.3	0.95	0.94	0.93	0.92
	0.6	0.82	0.78	0.74	0.71
	0.9	0.48	0.45	0.42	0.33
			First difference		
$r=2$	0.3	0.82	0.81	0.81	0.81
	0.6	0.55	0.54	0.53	0.53
	0.9	0.17	0.17	0.16	0.16
$r=3$	0.3	0.77	0.76	0.76	0.76
	0.6	0.48	0.46	0.46	0.46
	0.9	0.13	0.13	0.12	0.12
$r=4$	0.3	0.75	0.74	0.74	0.73
	0.6	0.45	0.44	0.44	0.43
	0.9	0.12	0.11	0.11	0.11
			Two-period sum		
$r=2$	0.3	1.12	1.10	1.10	1.10
	0.6	1.15	1.08	1.07	1.07
	0.9	0.94	0.80	0.74	0.71
$r=3$	0.3	1.14	1.12	1.11	1.11
	0.6	1.16	1.08	1.04	1.02
	0.9	0.87	0.78	0.71	0.61
$r=4$	0.3	1.15	1.13	1.11	1.10
	0.6	1.15	1.08	1.03	0.98
	0.9	0.81	0.75	0.70	0.55

For the special case of $S=0$, (31) reduces to

$$\frac{\mathrm{V}\{\hat{\mu}_T^1\}}{\mathrm{V}\{\hat{\mu}_T^2\}}=\frac{n_2}{n_1}\frac{1}{2\sqrt{1-\rho^2}}\frac{\rho^2}{1-\sqrt{1-\rho^2}}. \tag{32}$$

This relative efficiency implies a bound on the relative sample sizes for the series of cross-sections to be preferable to a panel with $r=2$, which is always lower than the one given by Eckler (1955) – in his 'two-level rotation sampling' case – viz.

$$\frac{n_1}{n_2} > \frac{1}{2\sqrt{1-\rho^2}}. \tag{33}$$

This result is not surprising either since for the two-level rotation sampling case, it is assumed that *all* individuals in the sample are observed twice (with one retrospective observation), which implies that the final-period sample size is half of the sample size in the preceding periods.

In table 3 we present values for the lower bounds on the relative sample size n_1/n_r for rotation period r to be preferable to a series of cross-sections for $r = 2, 3, 4$ and three specific parameters of interest. The relative efficiency of two designs is obtained if the entries in the table are multiplied by n_r/n_1. The value of S indicates how many periods of observation are available after period τ. Of course the results for $S = \infty$ coincide with the results of section 3.

Table 3 shows that the relative efficiencies do not strongly depend on S, except possibly for large values of ρ. Moreover, the more observations are available after the estimation period(s), the smaller the difference between the exact bounds and the bounds from section 3 will be. Table 3 therefore clearly suggests that when ρ is known to be moderate, the results of section 3 may be used as an approximation. It is clear from the table that, if the cost structure in (23) is valid and the relative cost of resampling α is smaller than unity, a rotating panel will be preferred to a series of cross-sections, when one is interested in a level as well as a first difference. If α is still smaller (e.g., 0.8), then the rotating panel is also preferable in case of estimation of a two-period sum.

7. Concluding remarks

The collection of data, e.g., in consumer surveys, is characterized by its high cost. Therefore it is important to obtain as much information as possible from a given budget by using optimal sample designs. In this paper we have determined the relative efficiency of rotating-sample designs and have considered the problem of the choice of the rotation period which minimizes the variances of the BLUE of specific linear combinations of the period means or of any linear combination of the period means in an analysis-of-variance model.

The analysis-of-variance model (1) is characterized by an individual effect α_i, implying a constant correlation over time between different observations on the same unit. The results can however easily be extended to more general correlation patterns because the assumptions on the correlation pattern do not affect the structure of the band matrix A to be inverted in order to derive expressions for the variance of efficient parameter estimates.

In a previous paper [Nijman and Verbeek (1990)] where we discussed the choice between a pure panel, a pure series of cross-sections, and a combination of these two data sources, model (1) was used to model monthly consumer expenditures on food and clothing in 1985 in The Netherlands using the so called Expenditure Index panel conducted by INTOMART, a marketing research agency. The assumptions on the error terms appeared to be valid and the maximum-likelihood estimates of ρ in (1) for food and clothing were 0.76 and 0.25 with standard errors 0.005 and 0.002, respectively. If the cost structure introduced in section 5 is valid, these results imply that a series of cross-sections cannot be optimal to monitor expenditures on clothing if the relative cost of resampling is less than 0.60 irrespective of the parameter of interest. The corresponding figure for food where the individual effect is more prominent is 0.14. If one considers one parameter of interest only, these bounds can be sharpened.

Alternatively, the results in this paper can be used to determine the relative efficiency of rotating panels. If one is, e.g., interested in estimates of the average consumer expenditures in the last month of the sample, our results in section 6 imply that the relative efficiency of a panel with rotation period 2 to a series of cross-sections is $0.79n_2n_1^{-1}$ for food and $0.98n_2n_1^{-1}$ for clothing. If the parameter of interest is the period mean in a more distant past, these relative efficiencies drop to $0.65n_2n_1^{-1}$ and $0.97n_2n_1^{-1}$, respectively. Alternatively, the relative efficiencies for a recent change in means are $0.37n_2n_1^{-1}$ and $0.85n_2n_1^{-1}$ for food and clothing, respectively, while these bounds drop to $0.35n_2n_1^{-1}$ and $0.85n_2n_1^{-1}$, respectively, if one is estimating in a more distant past.

In summary, our results show that the gains from choosing an optimal rotation design can be quite substantial, even in the case the cost of a repeated observation equals the cost of a first observation ($\alpha = 1$). Our analysis suggests that in many cases either the smallest ($r = 1$) or the highest possible rotation period is optimal. In the above-mentioned example of food expenditures, a rotating panel with $r = 4$ will yield an efficiency gain of over 70% if one is estimating a difference in subsequent means, compared to a series of independent cross-sections with the same number of observations in every period.

Appendix: Details on some technicalities

In this appendix we will derive the expression for A given by (3) and prove the results in (4), (6), (11), and (27).

A.1. Proof of (3)

In order to derive (3) we split the individuals in the data set into r independent subsamples, each of which containing a time series of independent small panels. If $r = 2$, e.g., a first subsample consists of the units included in the first wave only, of those included in the second and the third wave, of those in the fourth and fifth wave, etc., while the second subsample consists of units observed in even periods and in the preceding period. The ordinary-least-squares estimator is the BLU estimator $\hat{\mu}_j$ of μ in the jth subsample ($j = 1, \ldots, r$). If we define a $k \times k$ matrix Ω_k by

$$\Omega_k = \sigma_\varepsilon^2 I_k + \sigma_\alpha^2 \iota_k \iota_k', \tag{A.1}$$

where I_k is the k-dimensional identity matrix and ι_k is a k-dimensional vector of ones, it can be easily verified that

$$\mathrm{V}\{\hat{\mu}_j\} = \frac{r}{n_r} \Psi_j, \tag{A.2}$$

where Ψ_j is a block-diagonal matrix with upper left block Ω_j, subsequently $[(T-j)/r]$ blocks equal to Ω_r where $[x]$ denotes the integer part of x, and finally a lower right block $\Omega_{T-j-[(T-j)/r]}$, and n_r/r is the number of observations per period in each subsample.

Since the $\hat{\mu}_j$ are independent, the BLUE of μ using all subsamples is

$$\hat{\mu} = \left(\sum_{j=1}^{r} \Psi_j^{-1} \right)^{-1} \sum_{j=1}^{r} \Psi_j^{-1} \hat{\mu}_j \tag{A.3}$$

and the variance of this estimator is

$$\mathrm{V}\{\hat{\mu}\} = \frac{\sigma_\varepsilon^2}{n_r} \Psi \quad \text{where} \quad \Psi = \sigma_\varepsilon^{-2} \left(\frac{1}{r} \sum_{j=1}^{r} \Psi_j^{-1} \right)^{-1}. \tag{A.4}$$

Using the fact that

$$\Omega_k^{-1} = \sigma_\varepsilon^{-2} \left(I_k - \sigma_\alpha^2 \left(\sigma_\varepsilon^2 + k\sigma_\alpha^2 \right)^{-1} \iota_k \iota_k' \right), \tag{A.5}$$

it is easy to check that the elements of $A = \Psi^{-1}$ satisfy eq. (3).

A.2. Proof of (4)

Subsequently, we want to prove eq. (4) which states that

$$\lim_{S,T\to\infty} \Delta = \lim_{S,T\to\infty} \left(\xi' A^{-1}\xi - \xi'\Sigma^{AR}\xi\right) = 0, \tag{A.6}$$

if $\xi_j = 0$ for $|j| > J$ for some finite J.

First define Σ^{MA} and Σ^{AR} as in section 2. Apart from the $(r-1)\times(r-1)$ upper left and lower right corners, A equals Σ^{MA}. Moreover, as stated in the main text, Shaman (1975) shows that Σ^{MA}, apart from the $(r-1)\times(r-1)$ upper left and lower right corners, is equal to $(\Sigma^{AR})^{-1}$. Define the symmetric $(T+S+1)\times(T+S+1)$ matrix W as $W = A - (\Sigma^{AR})^{-1}$. From the results above it is obvious that only the $(r-1)\times(r-1)$ upper left and lower right corners of W contain nonzero elements. Since $(\Sigma^{AR})^{-1}$ is positive definite and W is symmetric, there exists a nonsingular matrix Q such that

$$Q'(\Sigma^{AR})^{-1}Q = I, \tag{A.7}$$

$$Q'WQ = D = \mathrm{Diag}\{\lambda_j\}, \tag{A.8}$$

with D a diagonal matrix containing the eigenvalues λ_j of $\Sigma^{AR}W$ and Q the eigenvectors of $\Sigma^{AR}W$ [see, e.g., Gantmacher (1959, p. 310 ff.)]. Using (A.7) and (A.8), it is easily verified that

$$\Delta = \xi' A^{-1}\xi - \xi'\Sigma^{AR}\xi = -\sum_{j=-T}^{S} \delta_j^2 \frac{\lambda_j}{1+\lambda_j} \quad \text{with} \quad \delta = Q'\xi. \tag{A.9}$$

If the eigenvectors of $\Sigma^{AR}W$ associated with the zero eigenvalues are included in a matrix Q_1 and the remaining $2r-2$ eigenvectors in a matrix Q_2, it is evident that $Q_2'(\Sigma^{AR})^{-1}Q_1 = 0$ and that the first and last $r-1$ rows of Q_1 consist of zero elements only. From (A.9) then follows that

$$\lim_{S,T\to\infty} \Delta = 0 \quad \text{if} \quad \lim_{S,T\to\infty} Q_2'\xi = 0,$$

that is, Δ approaches zero if the first and last r elements of $\Sigma^{AR}\xi$ approach zero. Since $\xi_j = 0$, $|j| > J$, and Σ^{AR} is a covariance matrix of an autoregressive process, this condition is satisfied.

A.3. Proof of (6)

We start with $\lambda \neq 0$. First note that, by using (3),

$$2\pi f(\lambda) = \sum_{j=-r+1}^{r-1} a_{|j|} e^{-i\lambda j}$$

$$= \frac{1}{1+(r-1)\rho}\left(1+(r-2)\rho - \frac{2\rho}{r}\sum_{k=1}^{r-1} k\{e^{i(r-k)\lambda} + e^{-i(r-k)\lambda}\}\right). \quad \text{(A.10)}$$

Furthermore,

$$\sum_{k=1}^{r-1} k\, e^{i(r-k)\lambda} = e^{ir\lambda}\sum_{k=1}^{r-1} k(e^{-i\lambda})^k$$

$$= e^{ir\lambda}\left(\frac{e^{-i\lambda}-e^{-ir\lambda}}{(1-e^{-i\lambda})^2} - \frac{(r-1)e^{-ir\lambda}}{1-e^{-i\lambda}}\right). \quad \text{(A.11)}$$

Using the analogue expression for $\sum_{k=1}^{r-1} k\, e^{-i(r-k)\lambda}$ and substituting $e^{i\lambda k} = \cos(\lambda k) + i\sin(\lambda k)$, it is straightforward to check that

$$2\pi f(\lambda) = \frac{1}{1+(r-1)\rho}\left(1-\rho+\rho r - \frac{\rho}{r}\frac{1-\cos(\lambda r)}{1-\cos\lambda}\right). \quad \text{(A.12)}$$

Secondly, we consider $\lambda = 0$. Since $\cos(k\lambda) = 1$, $\sum_{k=1}^{r-1} k\cos(k\lambda) = \sum_{k=1}^{r-1} k = r(r-1)/2$ proves the second equality in (6).

A.4. Proof of (11)

First note that (8) can be rewritten as

$$g(\lambda) = \operatorname{Re}\left(w_0 + 2\sum_{k=1}^{J} w_k e^{i\lambda k}\right), \quad \text{(A.13)}$$

where $\operatorname{Re}(z)$ denotes the real part of z. With $\tilde{g}(z) = w_0 + 2\sum_{k=1}^{J} w_k z^k$ and

$\tilde{f}_r(z) = (1/(2\pi))\sum_{j=-r+1}^{r-1} a_{|j|} z^{-j}$, the integral in (10) can be written as

$$\int_{-\pi}^{\pi} g(\lambda) f_r^{-1}(\lambda)\,\mathrm{d}\lambda = \mathrm{Re}\left(\int_{-\pi}^{\pi} \tilde{g}(\mathrm{e}^{i\lambda}) \tilde{f}_r^{-1}(\mathrm{e}^{i\lambda})\,\mathrm{d}\lambda\right)$$

$$= \mathrm{Re}\left(\int_{\gamma} \tilde{g}(z) \tilde{f}_r^{-1}(z) \frac{\mathrm{d}z}{iz}\right), \tag{A.14}$$

where γ is the unit circle with positive orientation. With $h(z) = 2\pi z^{r-1}\tilde{f}_r(z)$, it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda) f_r^{-1}(\lambda)\,\mathrm{d}\lambda = \mathrm{Re}\left(-i\int_{\gamma} \frac{z^{r-2}\tilde{g}(z)}{h(z)}\,\mathrm{d}z\right). \tag{A.15}$$

Note that $z^{r-1}\tilde{g}(z)$ is a polynomial of degree $J+r-2$ and $h(z)$ is a polynomial of degree $2r-2$ with $r-1$ roots outside the unit circle and $r-1$ roots within the unit circle. The latter roots are denoted by z_j ($j = 1,\dots,r-1$). We assume that the equation $h(z) = 0$ has no multiple roots. (This assumption is of no importance since the multiple-roots case can be treated as a limit of the no-multiple-roots case). Application of the Residue Theorem [see, e.g., Holland (1980, p. 160)] yields

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} g(\lambda) f_r^{-1}(\lambda)\,\mathrm{d}\lambda = \mathrm{Re}\left(\sum_{j=1}^{r-1} \mathrm{Res}_{z=z_j}\left[\frac{z^{r-2}\tilde{g}(z)}{h(z)}\right]\right), \tag{A.16}$$

where $\mathrm{Res}_{z=z_j}$ denotes the residue at z_j. In the no-multiple-roots case, the z_j's are all single poles and we have

$$\mathrm{Res}_{z=z_j}\left[\frac{z^{r-2}\tilde{g}(z)}{h(z)}\right] = \lim_{z\to z_j}\left\{(z-z_j)\frac{z^{r-2}\tilde{g}(z)}{h(z)}\right\} = \frac{z_j^{r-2}\tilde{g}(z_j)}{h'(z_j)}. \tag{A.17}$$

Thus, we finally obtain from (10)

$$\lim_{S,T\to\infty} \mathrm{V}\{\xi'\hat{\mu}^r\} = \frac{\sigma_\varepsilon^2}{n_r}\mathrm{Re}\left(\sum_{j=1}^{r-1} \frac{z_j^{r-2}\left[w_0 + 2\sum_{k=1}^{J} w_k z_j^k\right]}{h'(z_j)}\right). \tag{A.18}$$

A.5. Proof of (27)

First, we prove that the lower right elements of $(\Sigma^{\mathrm{MA}})^{-1}$ equal the lower right elements of B. It is readily verified that $\Sigma^{\mathrm{MA}} = \sigma_e^2 CC'$ and $B = \sigma_e^{-2} DD'$ with

$$C = \begin{pmatrix} \cdot & & \cdot & & & \\ & \cdot & & \cdot & & \\ & & \cdot & & \vartheta_1 & \vartheta_2 \\ & & & \cdot & 1 & \vartheta_1 \\ & 0 & & & & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \cdot & & & 0 & \\ & \cdot & & & \\ & & \cdot & & \\ & \cdot & \psi_1 & \psi_0 & \\ \cdot & \cdot & \psi_2 & \psi_1 & \psi_0 \end{pmatrix}. \tag{A.19}$$

A sufficient condition for $(\Sigma^{\mathrm{MA}})^{-1}B = I$ is then that $C'D = I$. Elaboration of this equality yields exactly the same conditions as $\vartheta(L)\psi(L) = 1$. To prove (27), use

$$A^{-1} = (\Sigma^{\mathrm{MA}} + P)^{-1} = B(I + PB)^{-1} \quad \text{where} \quad P = \begin{pmatrix} 0 & 0 \\ 0 & P_{22} \end{pmatrix},$$

and standard results on partitioned matrices yield the expression within curved brackets in the right-hand side of (27).

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