Axiomatic characterization of the absolute median on cube-free median networks

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Abstract

In [15] a characterization of the absolute median of a tree network using three simple axioms is presented. This note extends that result from tree networks to cube-free median networks. A special case of such networks is the grid structure of roads found in cities equipped with the Manhattan metric.

1 Introduction

A problem often encountered in the provision of a service is how best to locate a service facility so as to optimize efficiency and accessibility. This problem has received a great deal of attention, as indicated by the reference lists of [6] and [12]. Typically, the problem is formulated as an optimization problem in which a facility has to be optimally located at some point on a network of roads. Optimality is defined in terms of the utility of clients. Client utility (or more precisely disutility) is modelled as some monotone function of the distance the client has to travel to reach the facility.

One of the best studied objective functions in the literature is the sum of all client-facility distances. A point (there can be more than one) in the network that minimizes this objective function is called an *absolute median*. Although we mention only the absolute median there are numerous other choices for a location. The conceptual issue of why one location should be preferred to another is what prompts this paper.

Following Arrow [1], cf. [5], we take an axiomatic approach to this question. Holzmann [7] was the first to take this approach in the context of location problems. The first axiomatic characterization of the absolute median in a location context was derived in [15]. The absolute median was characterized on tree networks using three simple axioms: Unanimity, Cancellation, and Consistency. The main axiom here is consistency. Loosely speaking, it means that, if client set S_1 agrees on outcome x and client set S_2 agrees on x as well, then the combined client set $S_1 \cup S_2$ should also agree on outcome x. We will make this more precise below. The idea for this type of consistency appears in a paper of H.P. Young [16], in which he presented an axiomatization of Borda's rule for voting procedures.

Tree networks are a very special class of networks and it is natural to ask if this characterization can be extended to more general networks. In [11] an extension was proved but with a restriction to graphs: the same characterization holds for so called cube-free median graphs. These graphs might seem very special, but in [8] a one-to-one correspondence between the class of triangle-free graphs and a restricted subclass of the cube-free median graphs is established. So this latter class is not as restrictive as it seems. In this note, using the relation between median graphs and median networks from [3], we extend the characterization to the class of cube-free median networks. While still restrictive, the class is rich enough to encompass one category of real world network structure: the grid equipped with the Manhattan metric (or city-block norm). However we have to sacrifice something: the client locations are restricted to the vertices (or nodes) of the network.

First we present a characterization of tree networks that shows the difference between networks and graphs in our context. This difference explains why clients must be restricted to lying at vertices only. Next we review some results on median networks from [3], and extend these results so as to suit our purposes. Then we present some key results from [11]. Finally, we combine this material into the main result of this paper: *On cube-free median networks the median function is the unique consensus function that satisfies Anonymity, Betweenness, and Consistency.*

2 The model

A network $N = (V, E, \lambda)$ consists of a finite set of *vertices* V, a set of *arcs* E of unordered pairs of vertices, and a mapping λ that assigns a *length* to each pair uv in E. Multiple arcs are not allowed. Each arc uv is a line segment of length $\lambda(uv)$ with interior points and the vertices u and v as its extremal points. The vertices and interior points will be called the *points* of the network.[1] For any two points p and q on the same arc, the length $\lambda(pq)$ of the subarc between p and q is just the length of the part of the curve between p and q. A *path* R joining two points p on arc uv_1 and q on arc $v_k w$ is either a subarc or a sequence $p \to v_1 \to v_2 \ \ldots \ \to v_k \to q$, where $v_i v_{i+1}$ is a arc, for $i = 1, \ldots, k$, such that each vertex occurs at most in the sequence. Since there is at most one arc between any two vertices, this definition of a path uniquely determines the arcs that are used to get from p to q. The length of the path is

$$\lambda(pv_1) + \lambda(v_1v_2) + \ \ldots \ \lambda(v_{k-1}v_k) + \lambda(v_kq).$$

A shortest path is a path of minimum length. The *distance* $\delta(p, q)$ between two points p and q is the length of a shortest p, q-path. We assume that there are *no redundant arcs*, that is, each arc uv is the unique shortest path between u and v. Note that this implies the absence of multiple arcs (which were excluded anyway by our definition). Moreover, for each arc uv, any other path between u and v has length greater than $\lambda(uv)$. This assumption is a necessary condition for Theorem A and Lemma D to be true. At the end of the paper we explain why this assumption is without from the viewpoint of computing median sets.

The *underlying graph* of the network N is just $G = (V, E)$. When referring to G the elements of E will be called *edges*. In the underlying graph $G = (V, E)$ we define a path in the usual way. The length of a path in G is the number of edges on the path. To distinguish between the network N and its underlying graph G, we call a shortest path in G a *geodesic*, and we denote the graph distance between two vertices u and v by $d(u, v)$. In general there is *no* relation between $\delta(u, v)$ and $d(u, v)$, except that, because of the irredundancy of arcs, we have $\delta(u, v) = \lambda(uv)$ if and only if $d(u, v) = 1$.

We need some additional concepts. Let W be a subset of V. Then $\langle W \rangle_G$ denotes the subgraph of G induced by W, that is, the subgraph with W as its vertex set and all edges with both ends in W as its edge set. Furthermore $\langle W \rangle_N$ denotes the subnetwork of N induced by W, that is, the subnetwork with W as its vertex set and all arcs with both ends in W as its set of arcs. The *segment* between two points p and q in N is the set

$$S(p, q) = \{ \ r \mid \delta(p, r) + \delta(r, q) = \delta(p, q)\}.$$

Since there are no redundant arcs, $S(p, q)$ is the subnetwork of N consisting of all shortest paths between p and q. The *interval* between two vertices u and v in G is the

[1] One can make the definition of network more formal by taking λ to be a mapping that maps the vertices onto distinct points of some euclidian m-space, and that maps arc uv onto a curve of length $\lambda(uv)$ with extremities u and v. We require that the curves do not intersect in interior points.

set

$$I(u,v) = \{\ w \in V \mid d(u,w) + d(w,v) = d(u,v)\}.$$

Note that $I(u,v)$ is a *subset* of V, but we may consider the subgraph $\langle I(u,v)\rangle_G$ of G or the subnetwork $\langle I(u,v)\rangle_N$ of N induced by $I(u,v)$. The subgraph $\langle I(u,v)\rangle_G$ may consist of more than the geodesics between u and v, viz. if there exists some edge between vertices, say x and y, in $I(u,v)$ with $d(u,x) = d(u,y)$ (so that we also have $d(x,v) = d(y,v)$). Such an edge will be called a *horizontal edge* in $I(u,v)$. Note that, if G is a bipartite graph, then such horizontal edges do not exist.

A *m-cycle* in N, with $m \geq 3$, is a closed path with m arcs, or more precisely, a sequence $v_1 \to v_2 \to \ \ldots\ \to v_m \to v_{m+1}$ with $v_1, v_2,\ \ldots\ , v_m$ distinct vertices and $v_1 = v_{m+1}$, such that v_iv_{i+1} is a arc, for $i = 1, 2, \ldots, m$. A *rectangle* in N is a 4-cycle such that non-adjacent (i.e. opposite) arcs have equal length.

A connected graph G is a *median graph* if $|I(u,v) \cap I(v,w) \cap I(w,u)| = 1$, for any three vertices u, v, w in G. The class of median graphs includes trees, hypercubes and grid graphs. A connected network N is a *median network* if

$$|S(u,v) \cap S(v,w) \cap S(w,u)| = 1,$$

for any three vertices u, v, w in N. A characterization of median networks was given in [3]:

Theorem A *A network N is a median network if and only if its underlying graph G is a median graph and all 4-cycles in N are rectangles.*

The usefulness of this theorem lies in the fact that we can make use of the rich structure theory for median graphs, see e.g. [14, 9, 11].

The *cube* is the graph with the eight 0,1-vectors of length 3 as its vertices and two vertices are adjacent if and only if they differ in exactly one coordinate. So a cube graph is just the set of vertices and edges of a solid 3-dimensional cube. The *cube network* is the network with underlying graph the cube graph, in which all 4-cycles are rectangles. Our main result holds for cube-free median networks. The necessity for this restriction is explained later in the paper.

3 Axioms

Let X denote the set of possible client positions, and Y denote the set of possible locations. For our purposes, $X \subseteq Y$.

A *profile* on a set X is a sequence $\pi = x_1, x_2,\ \ldots\ , x_k$ of elements in X, with $|\pi| = k$ the *length* of the profile. In case k is odd, we call π an *odd profile*, and in case k is even, we call π an *even profile*. Denote by X^* the set of all profiles on X. A *consensus function* on a set X is a function $f : X^* \to 2^Y - \emptyset$ that returns a nonempty subset of Y for each profile on X. We use sequences so as to accommodate multiple occurrences in π. This allows more than one client at the same point, or, if the clients are weighted, we can replace a client of weight j by j copies of this client in π.[2] So that the location choice does not depend on the order we impose the axiom of Anonymity: any permutation of the profile π produces the same output. In our approach, this axiom has to be added to the axioms used in [15].

The *median function* M on a connected network N is an example of a consensus function. In the network setting it is usual to take $X = Y$ to be the set of all points. In the graph setting it is usual to take $X = Y = V$. Let $\pi = x_1, x_2, \ldots, x_k$ be a profile on X. For a connected network N and a point p of N, we write $\Delta(p, \pi) = \sum_{i=1}^{k} \delta(p, x_i)$. For a connected graph G and a vertex v of G, we write $D(v, \pi) = \sum_{i=1}^{k} d(v, x_i)$. Then the respective median functions are given by

$$M_N(\pi) = \{ \ x \mid x \text{ a point minimizing } \Delta(x, \pi) \ \},$$

and

$$M_G(\pi) = \{ \ v \mid v \text{ a vertex minimizing } D(v, \pi) \ \}.$$

Since no confusion will arise, we often omit the subscripts $_N$ or $_G$. For our purposes the following axioms for consensus functions f on networks (graphs) are relevant. Where necessary, we assume that $X \subseteq Y$.

Anonymity (A) : For any profile $\pi = x_1, x_2, \ldots, x_k$ on X and any permutation σ of $\{1, 2, \ldots, k\}$, we have $f(\pi) = f(\pi^\sigma)$, where $\pi^\sigma = x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}$.

Betweenness (B) : $f(x, y) = S(x, y)$, for all $x, y \in X$ [Network version].

Betweenness (B) : $f(u, v) = I(u, v)$, for all $u, v \in V$ [Graph version].

Consistency (C) : If $f(\pi) \cap f(\rho) \neq \emptyset$ for profiles π and ρ, then $f(\pi, \rho) = f(\pi) \cap f(\rho)$.

It is easily seen that the median function M satisfies all these axioms in the network case, where $X = Y$ is the set of all points, and we choose the network version of (B). Similarly, M satisfies all these axioms in the graph case, where $X = Y = V$, and we choose the graph version of (B).

The main result on the absolute median in [15] can be rephrased as follows.

Theorem B *Let N be a tree network, and let f be a consensus function on N where $X = Y$ is the set of all points in N. Then $f = M$ if and only if f satisfies (A), (B) and (C).*

In [11] an extension from the tree case is offered but with a restriction to graphs.

[2] In [15] the input for the consensus function was taken as a multiset (to allow for multiple occurrences). Because multisets are a bit tricky to work with, we choose sequences.

Theorem C *Let $G = (V, E)$ be a cube-free median graph, and let f be a consensus function on G where $X = Y = V$. Then $f = M$ if and only if f satisfies (A), (B) and (C).*

Both results raise the question for which networks (for which graphs, respectively) the median function is characterized amongst all consensus functions by (A), (B), and (C). This is an open problem and seems to be very difficult.

In this note we extend Theorem B to the network case. We will prove the following:

Theorem *Let N be a cube-free median network, and let f be a consensus function on N where $X = V$ and Y is the set of all points in N. Then $f = M$ if and only if f satisfies (A), (B) and (C).*

The reader will notice that client positions are restricted to lie at vertices of the network. This is essential as shown below.

4 Main Result

The model used in [15] is a tree network. A tree network is a connected, cycle-free network. Trees have the characterizing property that, for any two points p and q, there exists a unique path connecting them. Trivially, this unique path is the shortest p, q-path. For any three points p, q, r in a tree network, the three paths between the pairs of p, q, r have a unique common point, which is necessarily a vertex (unless one of the three points is an interior point on the path between the other two). From the viewpoint of the problem we discuss in this Note, a striking difference between networks and graphs arises with respect to this property.

Theorem 1 *Let N be a network. Then N is a tree network if and only if*

$$S(p, q) \cap S(q, r) \cap S(r, p) \neq \emptyset,$$

for any three points p, q, r in N.

Proof. Let N be a tree. Then $S(p, q) \cap S(q, r) \cap S(r, p)$ consists of the unique point (vertex) lying simultaneously on the shortest paths between the three pairs of p, q, and r.

Assume that N is not a tree. If N is disconnected, we choose p and q in different components, whence $S(p, q) = \emptyset$, so that, for any r, we have three points for which the corresponding segments have empty intersection. Now suppose that N contains cycles. Let C be a cycle of minimal length t. Minimality of t implies that C is an *isometric* cycle in N, i.e., the distance along the cycle between any two points p and q on C equals their distance $\delta(p, q)$ in N. Now choose two distinct points q and r on C such that $\delta(q, r) < \frac{1}{2}t$. Then $\delta(q, r)$ is the length of the shortest of the two arcs on C between q and r. Let p be the point on the other arc with equal distance to q and

r. Then $\delta(p,q)$ is the length of the shorter arc on C between p and q, and $\delta(p,r)$ is the length of the shorter arc on C between p and r. Now, if $S(p,q) \cap S(q,r)$ contained a point x different from q, then a shortest p,x-path together with a shortest x,r-path and the shorter arc of C between r and p would contain a cycle shorter than C, which contradicts the minimality of C. So $S(p,q) \cap S(q,r) = \{q\}$. Similarly, any two of the segments intersect only in their common endpoint. Since p,q,r are distinct points on C, the intersection of all three segments is empty. This settles the proof of the Theorem. □

Because of the uniqueness of the point on the three paths between the pairs of p,q,r, we may refine the property in Theorem 1 to obtain the following characterization.

Corollary 2 *Let N be a network. Then N is a tree if and only if only if $S(p,q) \cap S(q,r) \cap S(r,p)$ is a unique point, for any three points p,q,r in N.*

If $S(p,q) \cap S(q,r) \cap S(r,p) \neq \emptyset$, for some points p,q,r, then, by (C) and (B), we have $M(p,q,r) = S(p,q) \cap S(q,r) \cap S(r,p)$. If N is a network with cycles, then let C be any cycle of minimal length, and let p,q,r be points as in the proof of Theorem 1. Now $S(p,q) \cap S(q,r) \cap S(r,p) = \emptyset$, whereas $M(p,q,r) = \{q,r\}$. So we cannot determine $M(p,q,r)$ from the three segments using consistency. This fact forces us to restrict ourselves to profiles consisting of vertices only.

In the graph case the situation is quite different: we consider vertices only. We can have $I(u,v) \cap I(v,w) \cap I(w,u) \neq \emptyset$, for all u,v,w, whereas the graph is *not* a tree. For example, consider any hypercube or grid. The intersection consists of a unique vertex, for any three vertices u,v,w in the hypercube (or grid).

In a median network, we have the following property: For any arc vw and any vertex u, either $v \in S(u,w)$ or $w \in S(u,v)$. Just use the median property and the irredundancy of arcs (i.e. $S(v,w) = \{v,w\}$). This property was called the *bottleneck property* in [3]. In [3] the following Lemma was proved.

Lemma D *Let $N = (V,E,\lambda)$ be a network with the bottleneck property. Then*

$$I(u,v) = S(u,v) \cap V,$$

for any two vertices u,v in V.

The proof in [3] contains a minor but repairable gap. An examination of the proof reveals that one can prove more. First we give some notation: $I_1(u,v)$ is the set of all neighbors of u in $I(u,v)$. Clearly, we have the following equation:

$$I(u,v) = \{u\} \cup \ [\bigcup_{x \in I_1(u,v)} I(x,v) \ ].$$

Lemma D can be strengthened to the following lemma. Instead of giving a full proof here by extending the one in [3], we just use Lemma D as a starting-point, and restrict ourselves to completing the proof.

Lemma 3 *Let $N = (V, E, \lambda)$ be a network with the bottleneck property. Then $S(u, v)$ is the subnetwork of N induced by $I(u, v)$, for any two vertices u and v in V.*

Proof. We use induction on the length $n = d(u, v)$ of the intervals $I(u, v)$ in G. If $d(u, v) = 0$, then $u = v$, so that $S(u, u) = \{u\} = I(u, u)$. If $d(u, v) = 1$, then uv is an edge in G and a arc in N, and we have $I(u, v) = \{u, v\}$ and $S(u, v)$ is the arc uv. So assume that $n \geq 2$. First we observe that there are no horizontal edges in $I(u, v)$. For otherwise, suppose xy is a horizontal edge in the interval, so that $d(u, x) = d(u, y) < n$. Then, by induction, $S(u, x) = \langle I(u, x) \rangle_N$ and $S(u, y) = \langle I(u, y) \rangle_N$. So we have $x \notin S(u, y)$ and $y \notin S(u, x)$, which contradicts the bottleneck property. So all edges in $\langle I(u, v) \rangle_G$ are on u, v-geodesics. By Lemma D, any shortest u, v-path in N starts with a arc ux with x in $I(u, v)$. So it starts with an edge ux, where x is a neighbor of u in $I(u, v)$. Hence we have

$$S(u, v) = \bigcup_{x \in I_1(u,v)} [S(u, x) \cup S(x, v)],$$

which, by induction is equal to

$$\bigcup_{x \in I_1(u,v)} [S(u, x) \cup \langle I(x, v) \rangle_N].$$

Since there are no horizontal edges in $I(u, v)$, the assertion now follows. □

Note that we have even proved that, in any network satisfying the bottleneck property, each u, v-geodesic in G can be obtained from a shortest u, v-path in N by ignoring the lengths of the arcs. Conversely, each shortest u, v-path in N can be obtained from the corresponding u, v-geodesic in G by assigning the appropriate lengths to the arcs. This more informal version of Lemma 3 is the one we will use.

An important consequence of Lemma 3 involves the notion of convexity. Let W be a subset of vertices. Then $\langle W \rangle_N$ is *convex* in N if $S(u, v) \subseteq \langle W \rangle_N$, for any two vertices u, v in W. Similarly, $\langle W \rangle_G$ is *convex* in G if $I(u, v) \subseteq W$, for any two vertices u, v in W. In a network N with the bottleneck property we have $\langle W \rangle_N$ is convex in N if and only if $\langle W \rangle_G$ is convex in the underlying graph G. Moreover, $\langle W \rangle_N$ can be obtained from $\langle W \rangle_G$ by assigning the appropriate lengths, and, vice versa, $\langle W \rangle_G$ can be obtained from $\langle W \rangle_N$ by ignoring lengths of arcs. These observations imply that all results for median graphs that can be proved using the concepts of distance, geodesic, and convexity have their counterparts for median networks provided we restrict ourselves to profiles over vertices only. So, in the sequel *the median function on a network $N = (V, E, \lambda)$ is a consensus function with profiles on V and with the set of all points Y as set of possible outcomes*:

$$M_N : V^* \rightarrow 2^Y - \{\emptyset\}.$$

It would lead to far afield to give a full description of the structure of median networks that follows from these observations. So we restrict ourselves to some features of median networks that show that they are a generalization of road networks of Manhattan type, and to some properties of the median function that we need.

Let N be a median network, and let uv be an arbitrary arc in N. By W_u^{uv} we denote the set of vertices that are strictly closer to u than to v. Note that, by Lemma 3, we can use the same notation for the underlying median graph. Clearly W_u^{uv} and W_v^{uv} are disjoint subsets of V. By the bottleneck property, there are no vertices with equal distance to u and v, so $W_u^{uv} \cup W_v^{uv} = V$. We call W_u^{uv} the *u-side* of arc uv, and W_v^{uv} the *v-side*. It follows from the structure theory for median graphs in [14] that the sets W_u^{uv} and W_v^{uv} are convex. Moreover, for any other arc xy between the two sets with, say $x \in W_u^{uv}$ and $y \in W_v^{uv}$, it turns out that $W_x^{xy} = W_u^{uv}$ and $W_y^{xy} = W_v^{uv}$. So arc xy defines the same sides as arc uv. Moreover, for any shortest u, x-path $u \to u_1 \to \ \dots \ \to u_k \to x$ in the u-side there exists a shortest v, y-path $v \to v_1 \to \ \dots \ \to v_k \to y$ in the v-side such that $u_i v_i$ is a arc, for $i = 1, \dots, k$. From the rectangle property we deduce that all these arcs have the same lengths, in particular uv and xy have the same length. This property is the typical Manhattan type structure of the network that we are looking for.

We need some properties of the median sets $M_G(\pi)$ in G, see [11]. Let uv be an edge in G, and π a profile on V. Then π_u is the subprofile of π consisting of all elements of π in W_u^{uv}, that is, those elements that are closer to u than to v. Similarly, π_v is the subprofile consisting of all elements closer to v than to u. We will use the same notation for a arc uv in N. Then we have the following properties of the median function on a median graph G. Let uv be an edge with $|\pi_u| = |\pi_v| = \frac{1}{2}|\pi|$. Then either both u and v are in $M_G(\pi)$ or none is. Let uv be an edge with $u \in M_G(\pi)$ and $v \notin M_G(\pi)$. Then $|\pi_u| > \frac{1}{2}|\pi|$. So $M_G(\pi)$ consists of all vertices u such that a majority (not necessarily strict) of the profile is closer to u than any of its neighbors. Finally, the sets $M_G(\pi)$ are convex in median graphs.

Theorem 4 *Let $N = (V, E, \lambda)$ be a median network, let $G = (V, E)$ be its underlying median graph, and let π be a profile on V. Then*

$$M_N(\pi) = \langle M_G(\pi) \rangle_N.$$

Proof. By Lemma 3, we know that $M_N(\pi) \cap V = M_G(\pi)$. So we only have to check interior points. Let uv be any arc of N, and let p be an interior point of uv. By the definition of W_u^{uv}, and the bottleneck property, the distance from v to any vertex in W_u^{uv} can be measured via u. Hence the distance from p to any vertex in W_u^{uv} can also be measured via u. The same holds if we interchange the roles of u and v. This implies

$$\Delta(p, \pi) = \Delta(p, \pi_u) + \Delta(p, \pi_v) =$$

$$= |\pi_u|\lambda(pu) + \Delta(u, \pi_u) + |\pi_v|\lambda(pv) + \Delta(v, \pi_v).$$

From this equality we deduce the following facts. If uv is a arc with $|\pi_u| = |\pi_v| = \frac{1}{2}|\pi|$, then $\Delta(p, \pi) = \Delta(u, \pi) = \Delta(v, \pi)$. Hence, either the entire arc uv is in $M_N(\pi)$ or none of it is in $M_N(\pi)$. Finally, if $|\pi_u| > |\pi_v|$, then $\Delta(p, \pi) > \Delta(u, \pi)$. Hence p is not in $M_N(\pi)$. From these observations and the facts on median sets in median graphs preceding the theorem we deduce that the proof is complete. □

Finally, we present some of the main theorems from [11] that we need for our axiomatic result on median networks. For a profile π and an element x_i in π, we denote by $\pi - x_i$ the subprofile of π obtained by deleting x_i.

Theorem E *Let $\pi = x_1, x_2, \ldots, x_k$ be an odd profile on the median graph G with $k > 1$. Then*

$$M_G(\pi) = \bigcap_{i=1}^{k} M_G(\pi - x_i).$$

The presence of a cube complicates the structure of the set of absolute medians. If we take a profile $\pi = u, v, x, y$ consisting of four pairwise non-adjacent vertices in the cube, then $M(\pi)$ is the set of all vertices in the cube. Note that we have $I(u, v) \cap I(x, y) = \emptyset$. This example shows the necessity of the restriction to the cube-free case in the next theorem.

Theorem F *Let G be a cube-free median graph, and let π be a profile on V of even length $2m$. Then there exists a permutation $y_1, y_2, \ldots, y_{2m-1}, y_{2m}$, such that*

$$M_G(\pi) = \bigcap_{i=1}^{m} I(y_{2i-1}, y_{2i}).$$

A simple consequence of Theorems E and F is Theorem B.

By Lemma 3, we get an analogue of Theorem F for networks.

Theorem 5 *Let N be a cube-free median network, and let π be a profile on V of even length $2m$. Then there exists a permutation $y_1, y_2, \ldots, y_{2m-1}, y_{2m}$, such that*

$$M_N(\pi) = \bigcap_{i=1}^{m} S(y_{2i-1}, y_{2i}).$$

Now we are ready to prove our main result.

Theorem 6 *Let N be a cube-free median network, and let f be a consensus function on N where $X = V$ and Y is the set of all points in N. Then $f = M$ if and only if f satisfies (A), (B) and (C).*

Proof. Let f be a consensus function on N satisfying (A), (B), and (C). We use induction on the length of the profiles to prove that $f(\pi) = M_N(\pi)$, for all profiles π on V.

By (C) and (B), we have $f(x) = f(x) \cap f(x) = f(x, x) = S(x, x) = \{x\} = M_N(x)$. Now let $\pi = x_1, x_2, \ldots, x_k$ be a profile of length $k > 1$. If k is even, then, by Theorem

5, we can write $\pi = y_1, y_2, \ \dots \ , y_{2m}$ such that $M_N(\pi) = \bigcap_{i=1}^{m} S(y_{2i-1}, y_{2i})$. By (B) and (C), we conclude that $f(\pi) = M_N(\pi)$.

If k is odd, then, by Theorem E, we have $M_N(\pi) = \bigcap_{i=1}^{k} M_N(\pi - x_i)$. Hence, by the induction hypothesis, we have $M_N(\pi) = \bigcap_{i=1}^{k} f(\pi - x_i)$. Since $M_N(\pi) \neq \emptyset$, axiom (A) and repeated use of (C) gives $M_N(\pi) = f(x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_k, \dots, x_k)$ with the i-th element x_i appearing exactly $k-1$ times in f. Using (A), we have $M_N(_\pi) = f(\pi, \dots, \pi)$ with π appearing exactly $k-1$ times in f. Hence, by (C), we deduce that $f(\pi) = M_N(\pi)$. □

At first glance it might be thought that one could just "lift" Theorem B up to median networks. This is true in the case that the range, Y, of the consensus function f is V as well. Then Theorem F would suffice to prove this result. But in Theorem 6 we include interior points in the range of f as well.

5 Redundant arcs

As noted above, we excluded redundant arcs to ensure that the unique shortest path between a pair of adjacent vertices was the arc connecting them. Here we outline why this entails no great loss.

First we give a precise definition of redundant arcs. Let N be a connected network. Consider two vertices u and v. If there exists a shortest u, v-path with more than one arc, then any arc with ends u and v is a *redundant* arc. If there is no such shortest path, then each shortest u, v-path is an arc with ends u and v. Take one such arc, and call this arc the irredundant arc between u and v. All other arcs between u and v are *redundant arcs*. The *reduced network* $\bar{N}$ is the network obtained from N by deleting all redundant arcs. We will argue that $M_{\bar{N}}(\pi) \subseteq M_N(\pi)$.

Observe first that the distance between any pair of points in $\bar{N}$ is the same as their distance in N. Therefore, if $M_N(\pi)$ contains no point interior to a redundant arc, then $M_N(\pi) = M_{\bar{N}}(\pi)$.

Now let p be an interior point of a redundant arc e incident to the vertices u and v, say, with $\delta(u, p) \leq \delta(p, v)$. Take a shortest u, v-path P in $\bar{N}$, and let p' be the point on P with $\delta(u, p) = \delta(u, p')$. Then it is straightforward to check that $\Delta(p, \pi) \geq \Delta(p', \pi)$. This implies that, if p lies in $M_N(\pi)$, then p' lies in $M_{\bar{N}}(\pi)$ as well as $M_N(\pi)$. It is again straightforward to deduce from this fact that $M_{\bar{N}}(\pi) \subseteq M_N(\pi)$.

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