Optimal Zone Boundaries for Two-class-based Compact 3D AS/RS

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ABSTRACT AND KEYWORDS	
Abstract	Compact, multi-deep (3D), Automated Storage and Retrieval Systems (AS/RS) are becoming more common, due to new technologies, lower investment costs, time efficiency and compact size. Decision-making research on these systems is still in its infancy. We study a particular compact system with rotating conveyors for the depth movement and a Storage/Retrieval (S/R) machine for the horizontal and vertical movement of unit loads. We determine the optimal storage zone boundaries for such systems with two product classes: high and low turnover, by minimizing the expected Storage/Retrieval (S/R) machine travel time. We propose a mixed-integer nonlinear programming model to determine the zone boundaries. A decomposition algorithm and a one dimensional search scheme are developed to solve the model. The algorithm is complex, but the results are appealing since most of them are in closed-form and easy to apply to optimally layout the 3D AS/RS rack. The results are compared with those under random storage, and show that a significant reduction of the machine travel time can be obtained. Finally, a practical example is studied to demonstrate the use and validate our findings.
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Optimal zone boundaries for two class-based compact 3D AS/RS

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Abstract

Compact, multi-deep (3D), Automated Storage and Retrieval Systems (AS/RS) are becoming more common, due to new technologies, lower investment costs, time efficiency and compact size. Decision-making research on these systems is still in its infancy. We study a particular compact system with rotating conveyors for the depth movement and a Storage/Retrieval (S/R) machine for the horizontal and vertical movement of unit loads. We determine the optimal storage zone boundaries for such systems with two product classes: high and low turnover, by minimizing the expected Storage/Retrieval (S/R) machine travel time. We propose a mixed-integer nonlinear programming model to determine the zone boundaries. A decomposition algorithm and a one dimensional search scheme are developed to solve the model. The algorithm is complex, but the results are appealing since most of them are in closed-form and easy to apply to optimally layout the 3D AS/RS rack. The results are compared with those under random storage, and show that a significant reduction of the machine travel time can be obtained. Finally, a practical example is studied to demonstrate the use and validate our findings.

Keywords: Order picking; Storage rack design; AS/RS; Travel time model; Class-based storage

1. Introduction

Solutions aiming at improving order picking efficiency and reducing storage space often play an important role in shortening customer response times in supply chains, decreasing costs and improving associated customer service. It is widely recognized that these improvements are obtainable by well selected system types, optimized warehouse layout, and storage policies.

In the past decades, AS/R (Automated Storage and Retrieval) Systems, replacing conventional manual warehouses, have been designed to enhance order picking efficiency by automating the product S/R (Storage/Retrieval) process. Since the breakthrough paper of Hausman et al. (1976) on AS/R systems, different storage policies have been studied in literature and have been implemented in warehouses to shorten the S/R machine travel time for delivering and picking up unit-loads (pallets, containers or totes) in storage racks. The results show that compared to random storage policy, class-based storage can reduce the S/R machine’s travel time by storing high-turnover unit loads closer to the I/O point (Hausman et al. 1976; Eynan and Rosenblatt 1994; Kouvelis and Papanicolaou 1995; Ruben and Jacobs 1999; Park 2006). Unit loads are stored in single deep racks (with a depth of only one unit load). To store or retrieve unit loads, aisles are required between every two racks, wasting much floor space, and requiring a large building. Also, a warehouse with such a layout has relative long travel times to store and retrieve unit loads.

Solutions exist in the form of multi-deep (3D) storage systems, also called *compact* or *super high-density* storage systems, which have recently been introduced to further improve order picking efficiency, and to save floor space (Retrotech 2006; Westfalia 2006). In the most common type of 3D compact ***AS/RS***, every load can be accessed individually, through an automated S/R machine and some automated orthogonal movement mechanisms (De Koster et al. 2006). Storing unit-loads multi-deep saves storage aisle space compared to conventional warehouses or AS/R systems. The total S/R travel time may therefore be shorter than in a traditional system. The full automation of such systems implies they can work around the clock. As a result, the system may save cost from its high productivity, land space saving, and labor reduction.

Compact AS/R systems become increasingly popular (Van den Berg and Gademann 2000; Hu et al. 2005) for storing products. An example is the system of Miele in Gütersloh (D), where a combination of machines and shuttles store and retrieve individual palletized white goods (like washing machines and dish washers), and automatically sequence them for loading trains and trailers. Some examples have been described by Graves et al. (2002), and more examples can be found at websites of system suppliers (e.g., Retrotech 2006; Westfalia 2006). We have studied their applications in dense container stacking at a container yard and at the Distrivaart barge in the Netherlands (Waals 2005), where pallets are transported by barge shipping between several suppliers and supermarket warehouses. This project has actually been implemented and has resulted in a fully automated storage system on a barge.

This paper focuses on optimizing class zone boundaries for compact (3D) AS/R systems containing two storage zones/regions: high turnover and low turnover (2-class-based storage policy). These boundaries include three dimensional lengths for the 1st and 2nd class regions, and optimal zone sizes (e.g., volume ratio between the first class region and the rack). This problem is more complicated than in 2D AS/R systems with class-based storage where the number of decision variables is much less than for a 3D AS/RS as only zone sizes (Rosenblatt and Eynan 1989; Eynan and Rosenblatt 1994; Park et al. 2006) are the variables (from which the optimal boundaries are then obtained). This paper will propose a mathematical model for determining the optimal boundaries for 2-class-based 3D AS/RS by minimizing the S/R machine's travel time for the single command operating mode. In order to do so, we have to formulate the expected S/R machine travel time for different cases. The model is nonlinear mixed-integer, but we can optimally solve it by splitting it into several solvable sub-models and reduce the feasible area of the decision variables without losing the optimal solution. Finally, a one-dimensional search is introduced to determine the optimal ratio between the first class storage volume and the total rack volume. Although the procedure for determining the optimal solutions is complex, the optimal results obtained are simple to apply in practical applications.

This paper, similar to de Koster et al. (2006), studies compact 3D AS/R systems which can be used in at least two variants of the depth movement mechanisms for unit loads: one with gravity conveyors and the other with powered conveyors. The system is sketched in Figure 1 and consists of a 3D storage rack, a depot (or I/O point), an S/R machine (or crane), and orthogonal conveyors operating in pairs responsible for the depth movement. The unit loads enter and leave the system via the I/O point and are stored in the rack. The unit loads can flow to the back and front end of the rack on inbound and outbound conveyors respectively controlled by gravity or power. The S/R machine can drive and lift simultaneously and takes care of the movements in the horizontal and vertical directions. It picks up unit loads from the I/O point to bring them to an inbound conveyor or retrieves them from an outbound conveyor to bring them to the I/O point.

<Insert Figure 1 here>

The system with non-powered (i.e. gravity) conveyors is schematically shown in Figure 2. At the back of the rack a simple inexpensive elevator lifts the unit loads one by one from the inbound conveyor to the neighboring outbound conveyor from which they flow to the front end of the rack. In this way the unit loads on the two conveyors can rotate until a requested one (for example, at position "A") reaches the S/R position and is stopped by a stop switch at the front side of the rack for a retrieval request. The lift drives the rotation of unit loads and, as it is slower than its two conveyers, it determines the effective rotation speed. In order to retrieve a unit load, the two neighboring gravity conveyors should have at least one empty slot (at position "E").

The depth movement mechanisms and the S/R machine also can be used to sequence unit loads according to their turnovers; the S/R machine picks up a disordered or new incoming unit-load, and inserts it into the required relative position when this position flows to storage position “E”, and becomes empty.

The depth movement mechanism with powered conveyors does not use a lift and conveyors are mounted in the rack horizontally. The retrieval operation is identical to gravity conveyors, but for storage there are two differences in operation. First, the empty slots may be at any position on the two conveyors. Second, the storage time of a unit load may be longer than in the case of gravity conveyors because the powered conveyors may need more time to rotate an empty slot to the storage position. In this paper, in order to obtain the expected cycle time for the two system variants, we only consider retrieval travel time. This consideration is motivated by the fact that retrieval time is more critical in operations to reduce the customer response time.

<Insert Figure 2 here>

In the studied 3D AS/RS, the two-class-based storage policy is applied to sequentially assign items on unit loads, sorted by decreasing turnover frequencies to storage positions in the rack. Within the rack, all unit loads are grouped into two classes to which storage regions are assigned. In every class region, the unit loads are randomly stored.

The following Section 2 reviews related literature. In Sections 3 & 4, we propose the general draft and detailed models for determining the optimal class-dimensions of both class regions. Section 5 optimally solves the general model which is further extended. We compare our results with those of random storage in Section 6. Section 7 gives an example to illustrate how to implement and validate our results. Section 8 concludes the paper.

2. Literature review

In the past decades, designing an efficient AS/RS has interested many researchers resulting in numerous papers for 2D AS/R systems, and few papers for 3D AS/R systems. Performance measures used include travel time per S/R operation cycle, throughput, and average cost per S/R operation. Much literature focuses on the travel time per S/R operation cycle which depends on the shape of the storage rack: SIT (square in time), or NSIT (non-SIT), unit load storage policies (random, class-based, or full turnover-based policies), the S/R machine’s operation modes (single, dual, and multiple commands per cycle), dwell point policies (at the middle or corner of the rack), and rack depth (2D or 3D racks). We here only review literature closely related to our research, focusing on travel time calculation with different rack shapes, on storage policies, on optimal class-zone boundaries, and on 3D storage systems.

Travel time calculation. Calculating the travel time based on different rack shapes with Chebyshev travel has received considerable interests since the paper of Hausman et al. (1976). They calculate the one-way travel time for a single command cycle based on a SIT-rack system (the ratio of the horizontal and vertical rack dimensions, measured in travel time, equals 1) with different storage policies: random, turnover, and class-based storage. Bozer and White (1984) obtain the travel time for single and dual command cycles for NSIT rack systems under the random storage policy, and prove that with a constant S/R speed, the SIT rack is the optimal 2D-rack configuration. In practice other rack shapes exist, given the various cost components as well as height and length constraints. Based on Bozer and White's travel time model for NSIT racks, Eynan and Rosenblatt (1994) develop a procedure for dividing a rectangular rack into storage classes and calculate the travel time resulting from class-based storage. Also, Pan and Wang (1996), Park et al. (2003), Hu et al. (2005), Park et al. (2006), and Park (2006) derive travel time to be performance criterion for different shapes of racks and operation modes in AS/R systems. Some of these papers take the travel time as a function of the rack dimensions, and obtain an optimal result by minimizing the travel time (e.g. Bozer and White 1984), or analyze the influence of rack shapes on the travel time (e.g. Park et al. 2003) for 2D AS/R systems.

Storage policies. Under the random storage policy S/R requests are allocated randomly over the available storage locations in a rack. This policy is considered widely in the literature, see Bozer and White (1984), Lee and Elsayed (2005), and de Koster et al. (2006). In many studies, like Hausman et al. (1976) and Lee and Elsayed (2005), it is used as benchmark to measure improvements of other storage policies. The full turnover-based policy was first described by Heskett (1963; 1964) as the Cube-per-Order index (COI) rule without a proof of its optimality. Kallina and Lynn (1976) discuss the implementation of the COI rule in practice. Hausman et al. (1976) assume a Pareto (or ABC)-demand curve and a basic EOQ (Economic Order Quantity)-based reordering policy, in their derivation of an expression for the expected single-command travel time for random and full turnover-based storage. They formulate a universal expression to calculate the one way travel time which can be used for NSIT racks and multi-deep racks as well, because in its derivation only EOQ assumptions and an ABC-demand function are used. It has been used by other researchers in different warehouse settings. For example, Koh et al. (2002)

apply it to estimate the travel time for a warehousing system with a crane in combination with a carousel. For NSIT racks, Park et al. (2003) derive dual command travel times with turnover-based storage. The travel time reduction from full turnover-based storage policy is substantial, but it is not realistic in the sense that the turnover of every pallet stored in the system needs to be known and should be constant over time (Hausman et al. 1976). The class-based storage therefore is more popular in practice by roughly dividing the pallets into high and low turnover pallets, and assigning high turnover pallets closer to the I/O point. The travel time for class-based storage is derived by Hausman et al. (1976), and discussed by many researchers in 2D AS/R systems in two class-based storage settings (Kouvelis and Papanicolaou 1995; Park 2006; Park et al. 2006) or multiple class based settings (Rosenblatt and Eynan 1989; Eynan and Rosenblatt 1994; Thonemann and Brandeau 1998; Ruben and Jacobs 1999). Some of them also study the optimal zone boundaries.

Optimal zone boundaries. The problem of determining optimal sizes for different turnover zones has been first studied by Hausman et al. (1976) with a grid search method by minimizing the S/R one way travel time in a SIT rack face. The rack storage positions are partitioned into two or three zones. Rosenblatt and Eynan (1989) develop a solution procedure which allows determining those boundaries for any desired number of regions. They demonstrate that most of savings of a full turnover policy can be achieved by dividing the warehouse into a relatively small number of regions. Eynan and Rosenblatt (1994) extend the above two papers by determining the optimal zone boundaries (sizes) for multiple classes by dividing a pre-designed rectangular warehouse (NSIT case). The above papers only consider single command cycles. For dual command cycles, due to the complexity of the problem, the optimal zone sizes are only numerically investigated for the SIT case, for 2D systems with two storage classes (Park et al. 2006). Park (2006) further determines the mean and variance of the single and dual command travel times for AS/R systems for the NSIT case with two-class storage, and analyzes the influence of rack factors and skewness parameters on the system throughput. But the optimal zone size is not given.

3D storage systems. Park and Webster (1989b) propose a conceptual model that can help a warehouse planner design certain 3D pallet-storage systems by minimizing the total storage system costs. The costs consist of land, building, handling equipment, storage-rack, labor, maintenance, and operating costs. Park and Webster (1989a) consider a “cubic-in-time” layout to minimize the travel time of selected handling equipment. These two publications study conventional 2D (single deep) storage systems from a three dimensional point of view by considering multiple 2D racks and aisles. The dimensions of the 3D storage systems are given or, in other words, the problem of determining the optimal system dimensions is neglected. Sari et al. (2005) study a 3D flow-rack AS/RS where the pallets are stored and retrieved at different rack sides by two cranes. In order to retrieve a particular pallet, the retrieval crane has to move all pallets in front of it and store these on a special restoring conveyer. They derive the travel time for the random storage policy with given lengths of the three rack dimensions. De Koster et al. (2006) develop a model for compact 3D AS/R systems with built-in circular conveyors and find the optimal design of these systems by deriving the expected travel time of the S/R machine of random S/R requests under random storage policies. They conclude that the optimal ratio of the three dimensions in vertical, horizontal and conveyor directions is $0.72 : 0.72 : 1$ for single-command systems.

In conclusion, the class-based storage policy is one of the most important storage policies for conventional 2D warehousing in science and practice. Its optimal zone boundaries for class-regions are widely studied by many researchers (Rosenblatt and Eynan 1989; Park 2006) after the seminal work done by Hausman et al. (1976). However, literature on 3D compact storage systems is far from redundant with only a few papers. This paper fills this gap as the first paper to study class-based storage for a compact 3D warehouse system with the objective to optimize the rack design and class-zone boundaries.

3. Assumptions and general model

3.1 Assumptions

We study the compact 3D AS/R system sketched in Figure 1. The basic assumptions made throughout this paper, and also commonly used by other AS/RS papers (see also Hausman et al. 1976; Bozer and White 1984; Rosenblatt and Eynan 1989; Eynan and Rosenblatt 1994; Ashayeri et al. 2002; De Koster et al. 2006) are:

- The 3D rack is considered to have a continuous rectangular pick face, where the I/O point (or depot) is located at the lower left-hand corner of the rack (see Figure 1). When the S/R machine is idle, it stops at the I/O point.

- The S/R machine is capable of simultaneously moving in vertical and horizontal directions at constant speeds. Thus, the travel time required to reach any location at the front side of the rack (a storage conveyor pair) is represented by the Chebyshev metric.
- Each conveyor moving mechanism can move loads in an orthogonal depth direction, independent of the S/R machine movement, at a constant speed.
- The S/R machine operates on a single-command basis (multiple stops in the aisle are not allowed).
- Each unit load holds only one item type. All storage locations and unit loads have the same size. Therefore all storage locations can be used for storing any unit load. The items are replenished according to the EOQ model.
- Following Hausman et al. (1976), the pick-up/deposit (P/D) time for the machine to pick up or deposit a unit load is ignored because the P/D time is fairly small compared to the total machine travel time.
- Retrieval requests are generated instantaneously. We therefore do not consider prepositioning of unit load retrieval requests (see also, Hausman et al. 1976; Bozer and White 1984).
- A two class-based storage policy is implemented. The unit loads and rack space are partitioned into Class I and II regions based on the travel distances (single command travel times) and unit-load turnovers. Class I region is used for higher-turnover unit loads which are nearer to the I/O point, while Class II is used for lower-turnover unit loads which are farther from the I/O point. In each given class, unit loads are assigned to locations randomly.
- The rack volume is a known positive constant.

3.2 Notations and general model

The length (*L*), the height (*H*) of the rack, and the perimeter of two conveyors in a pair (with length 2S) form three orthogonal dimensions of the system. The speed of the conveyor moving mechanism and the S/R machine's speed in the horizontal and vertical direction, are denoted by s_c, s_h, and s_v respectively.

For sake of convenience and without loss of generality, we suppose that the travel time to the end of the rack (class II) is no less than the travel time to the highest location in the rack (class II):

$L / s_h \geq H / s_v$ (see also Bozer and White 1984; Eynan and Rosenblatt 1994). We define $t_c = 2S / s_c$ as length (in time) of two conveyors in a pair; $t_h = L / s_h$ as length (in time) of the rack; and $t_v = H / s_v$ as height (in time) of the rack.

Following the assumption that the rack volume is given, $t_h t_v t_c = V$ is then a constant (*V* can be considered as the system storage capacity in cubic time units). By setting $H * L * S = V'$ (volume in cubic meter units), i.e., $(t_v s_v)(t_h s_h)(0.5 t_c s_c) = V'$, the relationship between V' and V can be expressed as:

$$V = \frac{2V'}{s_h s_v s_c} \tag{1}$$

To standardize the system, we normalize $V = 1$ the volume of the rack equal to 1 by setting $t_{h2} = t_h / \sqrt[3]{V}$, $t_{v2} = t_v / \sqrt[3]{V}$, and $t_{c2} = t_c / \sqrt[3]{V}$ to represent the boundaries (in time) of class II (or the rack) on horizontal, vertical, and orthogonal dimensions respectively. Correspondingly, the boundaries (in time) of class I can be set as $t_{h1} = L_1 / \left(s_h \sqrt[3]{V}\right)$, $t_{v1} = H_1 / \left(s_v \sqrt[3]{V}\right)$, and $t_{c1} = 2S_1 / \left(s_c \sqrt[3]{V}\right)$ where L_1, H_1, and S_1 are the boundaries (in meter units) of class I on horizontal, vertical, and orthogonal dimensions respectively. According to the assumptions, $L_1 \leq L$, $H_1 \leq H$, $S_1 \leq S$, and $t_{h1} t_{v1} t_{c1} = V_1' / V' < 1$ where V_1' is denoted as the volume of class I in cubic meter units, and assumed to be a constant. If $t_{hi} = t_{vi}$, $i = 1 \; or \; 2$, we call the rack class i square-in-time (SIT).

Assume that a random retrieval location is represented by (X, Y, Z) where X, Y and Z refer to the coordinates on the rack horizontal, vertical, and the conveyor (deep) directions respectively. The expected S/R machine's retrieval time for single command (denoted as ESC) consists of the following two components:

- Time needed for the S/R machine to go from the depot to an S/R position (as shown in Figure 2) to pick-up an available unit load. The unit load is made available to the S/R position by the conveyor movement mechanism. As a result, because the movements on three dimensions are independent, this travel time, denoted by W, is the maximum of the following three quantities:
  - time needed to travel horizontally from the depot to the S/R position,
  - time needed to travel vertically from the depot to the S/R position,

- Time needed for the conveyor moving mechanism to circulate the requested load from its current position to the S/R position.

- Time needed for the S/R machine to return to the depot from the S/R position (denoted by U).

  That is $W = \max(X, Y, Z)$, and $U = \max(X, Y)$. Hence, the expected S/R travel time ESC can be expressed as follows:

$$ESC = E(W) + E(U), \tag{2}$$

which is a function of the boundary values of classes I and II ($t_{hi}, t_{vi}, t_{ci} \geq 0,\ i = 1 \text{and} 2$). More specifically, similar to Hausman et al. (1976), Rosenblatt and Eynan (1989), and Eynan and Rosenblatt (1994), in compact 3D AS/RS ESC can be expressed as:

$$ESC = \frac{\int_{j \in R_1} \lambda(j)[E(W \mid R_1) + E(U \mid R_1)]dj + \int_{j \in R_2} \lambda(j)[E(W \mid R_2) + E(U \mid R_2)]dj}{\int_{j=0}^{1} \lambda(j)dj} \tag{3}$$

where

R_i: set (region) of storage locations in class i, $i = 1,2$;

$E(A \mid R_i)$: expected value of A (=U or W) where the requested location is in class region i, $i = 1,2$;

$\lambda(j)$: turnover of the $j\text{th}$ unit load in the rack.

According to Hausman et al. (1976), with EOQ policy, $\lambda(j) = (2s / K)^{1/2} j^{(s-1)/(s+1)}$, $0 < j \leq 1$ where K is the ratio of order cost to holding cost and is assumed to be identical for all items. s is determined by the well-known ABC curve representation in Equation (4).

$$A(i) = i^s,\ 0 < s \leq 1, \tag{4}$$

where i is the percentage of inventoried items, $0 < i \leq 1$, s is the skewness of the ABC curve, and $A(i)$ is the cumulative percentage of demand in full unit loads.

In order to derive ESC, we substitute $\lambda(j)$ into Equation (3), and obtain

$$ESC = G_1^{2s/(1+s)}(E(U \mid R_1) + E(W \mid R_1)) + (1 - G_1^{2s/(1+s)})(E(U \mid R_2) + E(W \mid R_2)) \tag{5}$$

where G_1 is V_1' / V' (or V_1 / V).

We can now develop the following general model (denoted by **GM**) to optimize the boundaries of the two-class-based rack:

Model GM:

$$\text{Min}\ \ ESC = G_1^{\frac{2s}{1+s}}(E(U \mid R_1) + E(W \mid R_1)) + (1 - G_1^{\frac{2s}{1+s}})(E(U \mid R_2) + E(W \mid R_2)) \tag{6}$$

Subject to

$$t_{h1} \leq t_{h2} \tag{7}$$

$$t_{v1} \leq t_{v2} \tag{8}$$

$$t_{c1} \leq t_{c2} \tag{9}$$

$$t_{v1}\, t_{h1}\, t_{c1} = G_1 \tag{10}$$

$$t_{v2}\, t_{h2}\, t_{c2} = 1 \tag{11}$$

where decision variables are $t_{hi}, t_{vi}, t_{ci} \geq 0,\ i = 1,2$.

Constraints (7)-(9) state that region boundaries of class I are closer to the I/O point than those of class II. Constraints (10) and (11) show that the volume of class I accounts for 100G_1% of the total rack volume, and the volume of class II accounts for 100(1-G_1)% of the total rack volume where G_1 is given constant, which will be extended as a decision variable later in Subsection 5.4. If the optimal values of t_{hi}^*, t_{vi}^*, and t_{ci}^*, $i = 1, 2$ are determined, we then can obtain the optimal rack length L^* (the boundary dimensions of class II) in Equation (12a) from $t_h = L / s_h$, $t_{h2} = t_h / \sqrt[3]{V}$ and Equation (1). Similarly, the other optimal variable values can be obtained from Equations (12b)-(12f):

$$L^* = t_{h2}^* \sqrt[3]{\frac{2V' s_h^2}{s_v s_c}} \tag{12a}$$

$$H^* = t_{v2}^* \sqrt[3]{\frac{2V' s_v^2}{s_h s_c}} \tag{12b}$$

$$S^* = \frac{t_{c2}^*}{2} \sqrt[3]{\frac{2V' s_c^2}{s_h s_v}} \tag{12c}$$

$$L_1^* = t_{h1}^* \sqrt[3]{\frac{2V' s_h^2}{s_v s_c}} \tag{12d}$$

$$H_1^* = t_{v1}^* \sqrt[3]{\frac{2V' s_v^2}{s_h s_c}} \tag{12e}$$

$$S_1^* = \frac{t_{c1}^*}{2} \sqrt[3]{\frac{2V' s_c^2}{s_h s_v}} \tag{12f}$$

And the minimal objective function value for any given V will be

$$ESC_V^* = \sqrt[3]{V} ESC^* = ESC^* \sqrt[3]{\frac{2V'}{s_h s_v s_c}} \tag{12g}$$

according to the definition of W and U.

In order to find the optimal variable values, we must first derive $E(U \mid R_1)$, $E(W \mid R_1)$, $E(U \mid R_2)$, and $E(W \mid R_2)$. However, because they are highly dependent on the shapes of regions I and II, the complete derivation of them should consider more than 10 cases which makes the model tough to solve, and which may also be not necessary since the S/R machine's movements on horizontal and vertical dimensions are symmetrical. To simplify the model, we therefore assume that the two regions are SIT on the S/R machine's moving face (indicated as the SIT case from now on) and develop a corresponding detailed mathematical model. After that, we will prove that the optimal solution of the model for the SIT case is an optimal solution of Model GM above.

4. Modeling the SIT case

Subsections 4.1-4.3 derive $E(U \mid R_1)$, $E(U \mid R_2)$, $E(W \mid R_1)$, and $E(W \mid R_2)$. The corresponding mathematical model for the SIT case ($t_{h2} = t_{v2}$ and $t_{h1} = t_{v1}$) is then developed in Subsection 4.4.

4.1 Computation of $E(U \mid R_1)$ and $E(W \mid R_1)$

$E(U \mid R_1)$ is the same as the one way S/R machine travel time that proven by Hausman et al. (1976) and Bozer and White (1984) for a 2-dimensional rack with SIT, and can be calculated as:

$$E(U \mid R_1) = 2t_{h_1} / 3. \quad (13)$$

For $E(W \mid R_1)$, from de Koster et al. (2006), we can get its formula for SIT in two cases:

- When the orthogonal (deep) dimension for region I is the longest ($t_{h1} \le t_{c1}$),

$$E(W \mid R_1) = \frac{1}{4}(2t_{c1} + \frac{t_{h1}^2}{t_{c1}}) \quad (14a)$$

- When the orthogonal dimension for region I is the shortest ($t_{c1} \le t_{h1}$),

$$E(W \mid R_1) = \frac{1}{12t_{h1}^2}(t_{c1}^3 + 8t_{h1}^3). \quad (14b)$$

Combining Equations (14a) and (14b), $E(W \mid R_1)$ turns out to be

$$E(W \mid R_1) = \frac{v}{4}(2t_{c1} + \frac{t_{h1}^2}{t_{c1}}) + \frac{(1-v)(t_{c1}^3 + 8t_{h1}^3)}{12t_{h1}^2} \quad (15)$$

where v is a binary variable; v =1 corresponds to $t_{c1} \ge t_{h1}$, and v =0 corresponds to $t_{c1} \le t_{h1}$.

4.2 Computation of $E(U \mid R_2)$

In order to derive $E(U \mid R_2)$, we have to split R_2 into two sub-regions $R_{2,1}$ and $R_{2,2}$ (see Figure 3) to obtain the probability distribution function of $U = \max\{X, Y\}$, where $R_{2,1}$ and $R_{2,2}$ are defined in Figure 3. Since (*X*, *Y*, *Z*) coordinates are independently randomly generated in every class, $E(U \mid R_2)$ can be written as:

$$E(U \mid R_2) = E(U \mid R_{2,1}) \cdot P(R_{2,1}) + E(U \mid R_{2,2}) \cdot P(R_{2,2}) \tag{16}$$

where

$E(U \mid R_{2,i})$: The conditional expected value of U where $(X,Y,Z) \in R_{2,i}$, $i = 1,2$.

$P(R_{2,i})$: The probability of $(X,Y,Z) \in R_{2,i}$, $i = 1,2$ in R_2.

According to Hausman et al. (1976) and Eynan, and Rosenblatt (1994), we have

$$E(U \mid R_{2,1}) = E(U \mid R_1) = \frac{2t_{h1}}{3}, \tag{17a}$$

$$E(U \mid R_{2,2}) = \frac{2}{3}\frac{(t_{h2}^3 - t_{h1}^3)}{(t_{h2}^2 - t_{h1}^2)}. \tag{17b}$$

since the machine return travel time is independent of the storage depth.
In every region, we use randomized storage; the location coordinates are uniformly distributed. The probabilities $P(R_{2,1})$ and $P(R_{2,2})$ then only depend on the volumes of R_2, $R_{2,1}$ and $R_{2,2}$, and can be calculated as:

$$P(R_{2,1}) = \frac{R_{2,1}\text{'s volume}}{R_2\text{'s volume}} = \frac{(t_{c2} - t_{c1})t_{h1}^2}{t_{c2}t_{h2}^2 - t_{c1}t_{h1}^2}, \tag{18a}$$

$$P(R_{2,2}) = \frac{R_{2,2}\text{'s volume}}{R_2\text{'s volume}} = \frac{t_{c2}(t_{h2}^2 - t_{h1}^2)}{t_{c2}t_{h2}^2 - t_{c1}t_{h1}^2}. \tag{18b}$$

Substituting Equations (17a) - (18b) into Equation (16), we have

$$E(U \mid R_2) = \frac{2(t_{c2}t_{h2}^3 - t_{c1}t_{h1}^3)}{3(t_{c2}t_{h2}^2 - t_{c1}t_{h1}^2)}. \tag{19}$$

4.3 Computation of $E(W \mid R_2)$

Similar to the derivation of $E(U \mid R_2)$, $E(W \mid R_2)$ can be developed as:

$$E(W \mid R_2) = E(W \mid R_{2,1}) \cdot P(R_{2,1}) + E(W \mid R_{2,2}) \cdot P(R_{2,2}) \tag{20}$$

The values of $E(W \mid R_{2,1})$ and $E(W \mid R_{2,2})$ therefore should be derived first.

- ***Computation of*** $E(W \mid R_{2,1})$

We first have to derive the probability distribution function of $W = \max\{X, Y, Z\}$. Let $F(w \mid R_{2,1})$ denote the probability distribution that W is less than or equal to w where $(X, Y, Z) \in R_{2,1}$. Because (X, Y, Z) coordinates are independently randomly generated along the x, y and z-axes, and belong to $R_{2,1}$, we have

$$\begin{aligned} F(w \mid R_{2,1}) &= P(W \le w \mid R_{2,1}) = P(\max\{X, Y, Z\} \le w \mid R_{2,1}) \\ &= P(X \le w \mid R_{2,1}).P(Y \le w \mid R_{2,1}).P(Z \le w \mid R_{2,1}) \end{aligned} \tag{21}$$

Furthermore, as we use randomized storage in every class region, the location coordinates are uniformly distributed. Therefore,

$$P(X, Y \le w \mid R_{2,1}) = \begin{cases} w/t_{h1} & \text{if } 0 < w \le t_{h1} \\ 1 & \text{if } t_{h1} < w \end{cases} \tag{22}$$

$$P(Z \le w \mid R_{2,1}) = \begin{cases} 0 & \text{if } 0 < w \le t_{c1} \\ (w - t_{c1})/(t_{c2} - t_{c1}) & \text{if } t_{c1} < w \le t_{c2} \\ 1 & \text{if } t_{c2} < w \end{cases} \tag{23}$$

However, because t_{c1} and t_{h1} are decision variables, and their relative magnitudes (i.e. which one is the longer) are still unknown, the expression for $F(w \mid R_{2,1})$ has to be discussed in three cases as shown in Figure 4.

<Insert Figure 4 here>

Case 1 ($0 < t_{h1} \le t_{c1}$): substituting Equations (22) and (23) into Equation (21), we have

$$F(w \mid R_{2,1,1}) = \begin{cases} 0 & \text{if } 0 < w \le t_{c1} \\ \dfrac{(w - t_{c1})}{(t_{c2} - t_{c1})} & \text{if } t_{c1} < w \le t_{c2} \\ 1 & \text{if } t_{c2} < w \end{cases}$$

$$\Rightarrow f(w \mid R_{2,1,1}) = \begin{cases} \dfrac{1}{(t_{c2} - t_{c1})} & \text{if } t_{c1} < w \le t_{c2} \\ 0 & \text{otherwise} \end{cases},$$

where the subscripts of R represent region index, sub-region index, and case index, respectively. Therefore,

$$E(W \mid R_{2,1,1}) = \int_{t_{c1}}^{t_{c2}} \frac{w}{(t_{c2} - t_{c1})} dw = \frac{t_{c1} + t_{c2}}{2}. \tag{24}$$

Case 2 ($t_{c1} < t_{h1} \le t_{c2}$): substituting Equations (22) and (23) into Equation (21), we have

$$F(w \mid R_{2,1,2}) = \begin{cases} 0 & \text{if } 0 < w \le t_{c1} \\ \dfrac{w^2(w - t_{c1})}{t_{h1}^2(t_{c2} - t_{c1})} & \text{if } t_{c1} < w \le t_{h1} \\ \dfrac{w - t_{c1}}{t_{c2} - t_{c1}} & \text{if } t_{h1} < w \le t_{c2} \\ 1 & \text{if } t_{c2} < w \end{cases},$$

$$\Rightarrow f(w \mid R_{2,1,2}) = \begin{cases} \dfrac{w(3w - 2t_{c1})}{t_{h1}^2(t_{c2} - t_{c1})} & \text{if } t_{c1} < w \le t_{h1} \\ \dfrac{1}{t_{c2} - t_{c1}} & \text{if } t_{h1} < w \le t_{c2} \\ 0 & \text{otherwise} \end{cases}.$$

Therefore,

$$E(W \mid R_{2,1,2}) = \frac{t_{c1}^4 + 8t_{c1}t_{h1}^3 - 3(2t_{c2}^2 t_{h1}^2 + t_{h1}^4)}{12(t_{c1} - t_{c2})t_{h1}^2}. \tag{25}$$

Case 3 ($t_{c2} < t_{h1}$): substituting Equations (22) and (23) into Equation (21), we have

$$F(w \mid R_{2,1,3}) = \begin{cases} 0 & \text{if } 0 < w \le t_{c1} \\ \dfrac{w^2(w - t_{c1})}{t_{h1}^2(t_{c2} - t_{c1})} & \text{if } t_{c1} < w \le t_{c2} \\ \dfrac{w^2}{t_{h1}^2} & \text{if } t_{c2} < w \le t_{h1} \\ 1 & \text{if } t_{h1} < w \end{cases},$$

$$\Rightarrow f(w \mid R_{2,1,3}) = \begin{cases} \dfrac{w(3w - 2t_{c1})}{t_{h1}^2(t_{c2} - t_{c1})} & \text{if } t_{c1} < w \le t_{c2} \\ \dfrac{2w}{t_{c1}^2} & \text{if } t_{c2} < w \le t_{h1} \\ 0 & \text{otherwise} \end{cases}.$$

Therefore,

$$E(W \mid R_{2,1,3}) = \frac{(t_{c1} + t_{c2})(t_{c1}^2 + t_{c2}^2)}{12t_{h1}^2} + \frac{2t_{h1}}{3}. \tag{26}$$

Combining Equations (24), (25), and (26) for the above three cases, we have

$$E(W \mid R_{2,1}) = \sum_{k=1}^{3} u_{2,1,k} \cdot E(W \mid R_{2,1,k}) \tag{27}$$

where $u_{2,1,k}$ $k = 1,2,3$ are binary variables: $u_{2,1,k} = 1$ when case k is selected, and $u_{2,1,k} = 0$ otherwise. Only one of three cases can be selected: $\sum_{k=1}^{3} u_{2,1,k} = 1$.

- ***Computation of*** $E(W \mid R_{2,2})$

Similar to the derivation of $E(W \mid R_{2,1})$, because t_{c2}, t_{h1} and t_{h2} are decision variables, and their relative magnitudes (i.e. which one is the longer) are still unknown, the derivation process of $E(W \mid R_{2,2})$ has to be investigated in three cases as shown in Figure 5.

<Insert Figure 5 here>

The derivation process of $E(W \mid R_{2,2,1})$ corresponding to case 1 is complex and given in Appendix A, from which we have

$$E(W \mid R_{2,2,1}) = \frac{2(t_{h1}^2 + t_{h1}t_{h2} + t_{h2}^2)}{3(t_{h1} + t_{h2})} \tag{28}$$

Cases 2 and 3 are similar to case 1 in sub-region $R_{2,2}$. We give their results here without detailed derivation:

$$E(W \mid R_{2,2,2}) = \frac{t_{c2}^4 - 6t_{c2}^2 t_{h1}^2 - 3t_{h1}^4 + 8t_{c2}t_{h2}^3}{12t_{c2}(t_{h2} - t_{h1})(t_{h1} + t_{h2})} \tag{29}$$

$$E(W \mid R_{2,2,3}) = \frac{2t_{c2}^2 + t_{h1}^2 + t_{h2}^2}{4t_{c2}} \tag{30}$$

Combining Equations (28), (29), and (30) for the above three cases, we have

$$E(W \mid R_{2,2}) = \sum\nolimits_{k=1}^{3} u_{2,2,k} \cdot E(W \mid R_{2,2,k}) \tag{31}$$

where $u_{2,2,k}$ $k = 1, 2, \text{or } 3$ are binary variables: $u_{2,2,k} = 1$ when case k is selected, and $u_{2,2,k} = 0$ otherwise. Only one of three cases can be selected: $\sum\nolimits_{k=1}^{3} u_{2,2,k} = 1$.

Substituting Equations (27) and (31) into (20), and considering constraints on case matching and relative dimension magnitudes among different regions, we have:

$$E(W \mid R_2) = P(R_{2,1}) \cdot \sum_{k=1}^{3} u_{2,1,k} E(W \mid R_{2,1,k}) + P(R_{2,2}) \cdot \sum_{k=1}^{3} u_{2,2,k} E(W \mid R_{2,2,k}) \tag{32}$$

Subject to Constraints (7), (9) and

$$\sum_{k=1}^{3} u_{2,j,k} = 1 \qquad j = 1, 2 \tag{33}$$

$$u_{2,1,3} = u_{2,2,1} \tag{34}$$

where $P(R_{2,1})$ and $P(R_{2,2})$ are from (18a) and (18b) respectively. Constraint (34) shows that case $u_{2,1,3}$ =1 can only match $u_{2,2,1}$ =1.

4.4 Modeling compact AS/RS for the SIT case

Now *ESC* can be obtained by substituting Equations (13), (15), (19), and (32) into Equation (6). We therefore can obtain a detailed model (denoted as **DM**) for the SIT case by optimizing the two-class-based boundaries as follows:

Model DM:

$$Min\ ESC = G_1^{\frac{2s}{1+s}}(E(U \mid R_1) + E(W \mid R_1)) + (1 - G_1^{\frac{2s}{1+s}})(E(U \mid R_2) + E(W \mid R_2)) \qquad (35)$$

Subject to Constraints (7), (9), (33), (34), and

$$t_{h1}^2\, t_{c1} = G_1 \qquad (36)$$

$$t_{h2}^2\, t_{c2} = 1 \qquad (37)$$

where $t_{hi}, t_{ci} \geq 0,\ i = 1 \text{ and } 2$; v, and $u_{2,j,k}$ $j = 1,2$, $k = 1,2,3$ are binary variables. Constraints (36) and (37) correspond to Constraints (10) and (11) respectively in the SIT case.

The above model is nonlinear and mixed-integer which is generally difficult to solve. We aim at getting a closed-form expression for its optimal solution, and then decide to decompose it into several subproblems.

5. Optimizing region boundaries

The outline of our algorithm is depicted in Figure 6. In Steps 1 and 2, we decompose Model DM into two cases: $t_{h1} \leq t_{c1}$ (or $v = 1$) and $t_{c1} \leq t_{h1}$ (or $v = 0$) respectively. The optimal solutions of the two sub-models corresponding to these two cases are then derived in closed form. After that, the optimal solution of Model DM is obtained by comparing the two solutions and selecting the solution with minimum objective value in Step 3. Step 4 is used to prove that the optimal solution of model DM is an optimal solution of Model GM.

<Insert Figure 6 here>

For Steps 1 and 2, solving Model DM for two cases ($t_{h1} \leq t_{c1}$ and $t_{c1} \leq t_{h1}$) can be further decomposed into four sub-steps:

Sub-step 1: *Determine the optimal solution for t_{c1} as a function of t_{h1} (denoted by $t_{c1}^*(t_{h1})$).* From constraints (36), for either of the two cases we have

$$t_{c1}^*(t_{h1}) = G_1 / t_{h1}^2 \, . \qquad (38)$$

Sub-step 2: *Determine the optimal solutions of* t_{h2} *and* t_{c2} *as functions of* t_{h1} *(denoted by* $t_{h2}^*(t_{h1})$ *and* $t_{c2}^*(t_{h1})$ *respectively)*. The solutions should satisfy all constraints of Model DM and Equation (38). The solutions are discussed in Subsection 5.1 for the case $t_{h1} \le t_{c1}$ and in 5.2 for the case $t_{c1} \le t_{h1}$, respectively. We then obtain $t_{h2}^*(t_{h1})$ and $t_{c2}^*(t_{h1})$ by selecting the case that contains the optimal solution t_{h2}^* and t_{c2}^* of Model DM.

Sub-step 3: *Determine the optimal* t_{h1}^*. We do this by substituting $t_{c1}^*(t_{h1})$, $t_{h2}^*(t_{h1})$ and $t_{c2}^*(t_{h1})$ into Model DM, and determining t_{h1}^*.

Sub-step 4: *Determine the optimal solutions of* t_{c1}^*, t_{h2}^* *and* t_{c2}^* *with* t_{h1}^*. Substitute t_{h1}^* into $t_{c1}^*(t_{h1})$, $t_{h2}^*(t_{h1})$ and $t_{c2}^*(t_{h1})$, and obtain $t_{c1}^* = t_{c1}^*(t_{h1}^*)$, $t_{h2}^* = t_{h2}^*(t_{h1}^*)$ and $t_{c2}^* = t_{c2}^*(t_{h1}^*)$. The minimum ESC is then obtained by substituting them into Equation (35) for $t_{h1} \le t_{c1}$ (or $v = 1$) and $t_{c1} \le t_{h1}$ (or $v = 0$).

5.1 Step 1: Optimizing Model DM with $t_{h1} \le t_{c1}$

Sub-step 1 determines $t_{c1}^*(t_{h1})$ (see Equation (38)). For Sub-step 2 (Determine $t_{h2}^*(t_{h1})$ and $t_{c2}^*(t_{h1})$), because $t_{h1} \le t_{c1}$, we have $u_{2,1,1} = 1$ (see Figure 4), which can only match $u_{2,2,3} = 1$ or $u_{2,2,2} = 1$ (see Figure 5). In this situation, we therefore have two possible cases satisfying Constraints (33) and (34) shown in Figure 7.

<Insert Figure 7 here>

Therefore, for Case 1 in Figure 7, we obtain the following subproblem:

Subproblem 1 for Model DM with $t_{h1} \le t_{c1}$:

$$\begin{aligned} &\mathrm{Min}\ E(U \mid R_2) + E(W \mid R_2) \\ &\quad = Min\ E(U \mid R_2) + P(R_{2,1}) \cdot E(W \mid R_{2,1,1}) + P(R_{2,2}) \cdot E(W \mid R_{2,2,3}) \end{aligned} \tag{39}$$

subject to Constraints (7), (9), (37), (38) and

$$t_{h2} \le t_{c2} \tag{40}$$

Decision variables: t_{c2} and t_{h2}. Constraint (40) corresponds to $u_{2,2,3} = 1$.

In objective function (39), $E(U \mid R_1) + E(W \mid R_1)$ are not included because it is not a function of t_{h2} *and* t_{c2}. for any given t_{h1}.

Theorem 1. The optimal variable solution of Subproblem 1 for Model DM with $t_{h1} \le t_{c1}$ is:

$$t_{h2}^*(t_{h1}) = \begin{cases} \sqrt[3]{(\sqrt{10}-1)/3} & \text{if } \sqrt{G_1}\sqrt[3]{(\sqrt{10}-1)/3} < t_{h1} \le \sqrt[3]{(\sqrt{10}-1)/3} \\ t_{h1}/\sqrt{G_1} & \text{if } t_{h1} \le \sqrt{G_1}\sqrt[3]{(\sqrt{10}-1)/3} \\ t_{h1} & \text{if } \sqrt[3]{(\sqrt{10}-1)/3} < t_{h1} \le \sqrt[3]{G_1} \end{cases} \quad (41a)$$

where $\sqrt[3]{(\sqrt{10}-1)/3} < t_{h1} \le \sqrt[3]{G_1}$ only holds when $G_1 > \sqrt{10}-1)/3$;

$$t_{c2}^* = (t_{h2}^*(t_{h1}))^{-2}. \quad (41b)$$

Proof. See Appendix B.

Similarly, corresponding to Case 2 in Figure 7, we have

Subproblem 2 for Model DM with $t_{h1} \le t_{c1}$**:**

$$\text{Minimize}\, E(U \mid R_2) + P(R_{2,1}) \cdot E(W \mid R_{2,1,1}) + P(R_{2,2}) \cdot E(W \mid R_{2,2,2}) \quad (42)$$

Subject to Constraints (7), (9), (37), (38) and

$$t_{c2} \le t_{h2} \quad (43)$$

Decision variables: t_{c2} and t_{h2}. Constraint (43) corresponds to $u_{2,2,2} = 1$.

Theorem 2. The optimal variable solution of Subproblem 2 for Model DM with $t_{h1} \le t_{c1}$ is:

$$t_{h2}^* = t_{c2}^* = 1. \quad (44)$$

Proof. See Appendix C.

From Theorem 2, we find the minimum objective value of function (42) as

$$\frac{17 - 6G_1^2 t_{h1}^{-2} - 8G_1 t_{h1} - 3t_{h1}^4}{12(G_1 - 1)} \quad (45)$$

where $t_{h1} \in (\sqrt{G_1}, \sqrt[3]{G_1})$ according to Constraints $t_{h1} \le t_{c1}$ and (43).

We can check that the value of function (39) is equal to the value of Equation (45) at $t_{h2}^* = t_{c2}^* = 1$ which is only a feasible solution of Subproblem 1 for Model DM with $t_{h1} \le t_{c1}$. Therefore, Subproblem 2 can be omitted and the optimal solution of Model DM with $t_{h1} \le t_{c1}$ must be identical to that of Subproblem 1.

From the analysis above, we conclude:

"*In Model DM with* $t_{h1} \le t_{c1}$, *for any given* t_{h1}, *the optimal variable values* t_{h2}^*, t_{c2}^* *can be determined by Equations (41a) and (41b) respectively*."

Sub-steps 3 and 4: Determine t_{h1}^***.** t_{c1}^***,** t_{h2}^* ***and*** t_{c2}^***.**

By substituting (41a), (41b), and v=1 ($t_{h1} \le t_{c1}$) into Model DM, we get the problem to determine t_{h1}^* given in Equation (46a). t_{c1}^*, t_{h2}^* and t_{c2}^* are then determined by substituting t_{h1}^* into Equations (38), (41a) and (41b). We then can obtain Theorem 3 below.

Theorem 3. The optimal solution (t_{h1}^*. t_{c1}^*, t_{h2}^*, t_{c2}^*.) of Model DM with $t_{h1} \le t_{c1}$ is

$t_{h1}^* = \sqrt[3]{(\sqrt{10}-1)G_1/3}$ (46a) $t_{c1}^* = \sqrt[3]{9G_1/(\sqrt{10}-1)^2}$ (46b)

$t_{h2}^* = \sqrt[3]{(\sqrt{10}-1)/3}$ (46c) $t_{c2}^* = \sqrt[3]{9/(\sqrt{10}-1)^2}$ (46d)

The proof of Theorem 3 is similar that of Theorem 1, and is omitted here.

Correspondingly,

$$ESC^* = \frac{0.035G_1^{4/3}(39.97 - 39.97G_1^{-4/3} + 39.97G_1^{-0.67(2-s)/(1+s)} - 39.97G_1^{-(1-s)/1+s)})}{-1 + G1} \quad (46e)$$

5.2 Step 2: Optimizing Model DM with $t_{c1} \le t_{h1}$

For $t_{c1} \le t_{h1}$, in Sub-step 2, there are three possible cases that satisfy Constraints (33) and (34), and are shown in Figure 8.

<Insert Figure 8 here>

Similar to the methodology in Subsection 5.1, the solutions corresponding to the three cases can be determined, based on which the optimal solution of Model DM with $t_{c1} \le t_{h1}$ can be determined by selecting the best one of them. The optimal solution of Model DM with $t_{c1} \le t_{h1}$ is given directly without proofs by Theorem 4 below:

Theorem 4. The optimal solution of Mode DM with $t_{c1} \le t_{h1}$ is:

When $G_1 \le (\sqrt{10}-1)/3$,

$t_{h1}^* = t_{c1}^* = \sqrt[3]{G_1}$ (47a) $t_{h2}^* = \sqrt[3]{(\sqrt{10}-1)/3}$ (47b)

$t_{c2}^* = \sqrt[3]{9/(\sqrt{10}-1)^2}$ (47c)

$$ESC^* = \frac{0.035(41G_1^2 - 39.97G_1^{2/3} - 41G_1^{1+2s/(1+s)} + 39.97G_1^{2/3+2s/(1+s)})}{(G_1 - 1)G_1^{2/3}} \quad (47d)$$

When $G_1 > (\sqrt{10}-1)/3$,

$t_{h1}^* = t_{c1}^* = t_{h2}^* = \sqrt[3]{G_1}$ (47e) $t_{c2}^* = 1/\sqrt[3]{G_1^2}$ (47f)

$$ESC^* = \frac{0.083(6 + 14G_1 - 6G_1^{2s/(1+s)} + 3G_1^{1+2s/(1+s)})}{G_1^{2/3}}. \quad (47g)$$

5.3 Step 3 & 4: Optimizing boundaries for Model DM and GM

In Steps 1 and 2, the minimum value of ESC^* corresponding to $t_{h1} \le t_{c1}$ and $t_{c1} \le t_{h1}$ have been obtained. Based on these two results, the optimal solution of the class region boundary can be obtained by selecting the solution providing the smaller objective value. We therefore obtain Theorem 5 below.

Theorem 5. The optimal solution of Model DM is determined by (46a)-(46e).

Proof. See Appendix D.

Based on Theorem 5, an optimal solution of Model GM can then be obtained for the general case of NSIT by Theorem 6:

Theorem 6. The optimal class region boundaries for a 3D AS/RS rack (or the optimal solution of Model GM) are SIT ($t_{v1}^* = t_{h1}^*$ and $t_{v2}^* = t_{h2}^*$) on the S/R machine moving face and the optimal variable values t_{c1}^*, t_{h1}^*, t_{h2}^*, and t_{c2}^* are determined by Equations (46a)-(46e).

Proof. See Appendix E.

5.4 Extension

In the above analysis, G_1, the storage size of the class I region, is a predetermined parameter. Since the product turnovers change over time, warehouse managers can only roughly classify their products into high and low turnover classes.

However, if detailed turnover information of stored products is available, it is quite possible to take G_1 as a decision variable to further decrease the S/R machine travel time. In Theorem 6, for any given G_1, the optimal solution $(t_{v1}^*, t_{h1}^*, t_{c1}^*, t_{v2}^*, t_{h2}^*, t_{c2}^*)$ has been determined as a function of G_1. Therefore, to optimize Model GM with relaxing G_1 as a decision variable is equivalent to minimizing Equation (46e) by giving an optimal $G_1 \in (0,1)$. Because Equation (46e) is neither a convex nor a concave function of G_1, the optimal solution is difficult to solve analytically. Similar to the case in Hausman et al. (1976), a grid search method can be used to find the global optimal value G_1^* numerically. Once G_1^* is obtained, the optimal variable values $(t_{v1}, t_{h1}, t_{c1}, t_{v2}, t_{h2}, t_{c2})$ are then determined by substituting G_1^* into Equations (46a)-(46d).

In order to differentiate between the two optimal solutions with a fixed G_1 and with the optimal G_1^*, we use the notations $(t_{v1}^*, t_{h1}^*, t_{c1}^*, t_{v2}^*, t_{h2}^*,\ t_{c2}^*)$ and ESC^* for the optimal solution with fixed G_1, and $(\overline{t}_{v1}^*, \overline{t}_{h1}^*, \overline{t}_{c1}^*, \overline{t}_{v2}^*, \overline{t}_{h2}^*,\ \overline{t}_{c2}^*)$ and $\overline{ESC}^*$ for the optimal solution with optimal G_1^* in the rest of this paper.

6. Comparing the results with those of random storage

In this section, we present numerical results for several ABC-curve skewness parameters, and compare them with single command travel times for random storage policy as derived in De Koster et al. (2006).

Using the results in Subsection 5.3 and 5.4, we can find the optimal solution and its expected travel time for the 3D AS/RS rack system for any given skewness parameter value with or without fixed G_1 corresponding to a particular ABC curve. Using Equation (4), the notation $i\,/\,A(i)$ denotes that a fraction i of the inventoried items represents a fraction $A(i)$ of the total demand. For different $i\,/\,A(i)$ combinations (or given s), Table 1 tabulates the values of the optimal solutions for a normalized rack.

<Insert Table 1 here>

The following observations can be made from analyzing Table 1:

(1) The two values of ESC decrease with the decrease in skewness parameter s. Significant reductions in travel times are obtained based on 2-class-based storage policy compared with randomized storage policy, when the skewness parameter s is small. For example, for s=0.07 (20%/90%) the percentages of travel times saved for a fixed (given) G_1=0.2 and optimal G_1^* are 32.21% and 44.52%, respectively.

(2) When $s = 1$ (20%/20%), our result is the same as that of the random storage policy in De Koster et al. (2006). The problem in their paper is a special case of that in this paper with $s = 1$. This also can be theoretically confirmed by setting $s = 1$ in Theorem 6 leading to $t_{h2}^* = t_{v2}^* = 0.90$, $t_{c2}^* = 1.24$ and ESC^*=1.38.

(3) If the sknewness of the ABC curve increases, the optimal region size G_1 increasingly influences the travel time. When the demand distribution is not skewed ($s \geq 0.22$) class-based storage with optimal G_1^* hardly outperforms class-based storage with given G_1. However, when $s < 0.22$, an optimal G_1 can decrease the expected travel time significantly.

For example, when s =0.07 (20%/90%) the optimal solution with optimal G_1^* reduces the travel time compared with given G_1 =0.2 by (0.94-0.77)/0.94×100%=18.16%.

(4) The more skewed the ABC curve is, the more sensitive the expected travel time is to a change in the skewness s. For example, when the ABC curve changes from 20%/20% to 20%/30%, the ESC with optimal G_1^* decreases $(1.38-1.34)/1.38=2.73\%$. However when it changes from 20%/80% to 20%/90%, the ESC decreases $(0.98-0.77)/0.98=22.13\%$.

(5) Class region I's optimal boundaries change significantly with a change in ABC curves, but the rack (region II) optimal dimensions (boundaries) do not change for different ABC curves. This phenomenon makes it possible to find robust rack dimensions good for various ABC-curves.

Equations (12a) - (12g) show that the optimal rack layout for a 2-class-based storage policy for any given rack storage capacity V' can be obtained from the results for a normalized rack storage capacity, as indicated in Table 1. The table can therefore be used as a reference to design and zone a 3D rack for any required rack storage capacity.

7. An example

Here we demonstrate how to use the optimal solutions given in Table 1 to design and zone a practical discrete AS/RS. It is assumed that we have to design a 3D compact system with data in Table 2, as taken from de Koster et al. (2006). The layout of the system refers to Figure 1, and we assume G_1 a variable.

The five steps below demonstrate the use of the optimal results for a normalized compact rack indicated in Table 1 to find the (near) optimal boundaries/dimensions for this system for different given ABC curves.

<Insert Table 2 here>

Step 1: calculate the rack storage volume in cubic meters and seconds. The rack should have sufficient capacity to store 1000 pallets, which means that the rack should have at least $V'=1.2\times1.2\times2\times1000$ =2880 (m^3) or $V=3600$ ($\mathrm{seconds}^3$) using Equation (1).

Step 2: look up the optimal solution for normalized AS/RS from Table 1: From Table 1**,** 1) for the 20%/20% ABC curve, $G_1^* = \overline{t}_{h1}^*\overline{t}_{v1}^*\overline{t}_{c1}^*$ where $\overline{t}_{h1}^*$, $\overline{t}_{v1}^*$, $\overline{t}_{c1}^*$ can be any real number among $(0,1)$, $\overline{t}_{h2}^* = \overline{t}_{v2}^* = 0.90$, $\overline{t}_{c2}^*=1.24$, and $\overline{ESC}^*$ =1.38; 2) for the 20%/90% ABC curve, $\overline{t}_{h1}^*$ = $\overline{t}_{v1}^*$ =0.23, $\overline{t}_{c1}^*$=0.32, G_1^* =0.02, $\overline{t}_{h2}^* = \overline{t}_{v2}^* = 0.90$, $\overline{t}_{c2}^*=1.24$, and $\overline{ESC}^*$ =0.77.

Step 3: calculate the optimal solution for a continuous rack face. Using Equations (12a)-(12f), 1) for the 20%/20% ABC curve, we have $L^* = 34.35$ meters, $H^* = 10.99$ meters, $S^* = 7.63$ meters, and $\overline{ESC}^*$ =21.18 seconds; 2) for the 20%/90% ABC curve, $L_1^* = 8.81$ meters, H_1^* =2.82 meters, S_1^* =1. 96 meters, $L^* = 34.35$ meters, $H^* = 10.99$ meters, $S^* = 7.63$ meters, and $\overline{ESC}^*$ =11.74 seconds.

Step 4: obtain a near optimal solution for a discrete rack face. In a real-world setting, AS/R systems are discrete and the rack dimensions must be an integral multiple of the pallet dimensions. In our case the rack horizontal dimension must be an even multiple of the pallet's horizontal dimension because the conveyors work in pairs in the orthogonal movement system. Therefore, we choose 'practical near optimal' rack dimensions such that they are as close as possible to the corresponding optimal dimensions found while the system storage capacity is at least 1000 pallets, and the volume ratio between Class I region and the rack is near to G_1^* =0.02. We therefore obtain the following practical near optimal dimensions and expected travel time for both ABC curves: 1) for the 20%/20% ABC curve, $\hat{L}^* = 36$ meters (30 pallets), $\hat{H}^* = 10$ meters (5 pallets), $\hat{S}^* = 8.4$ meters (7 pallets). The practical expected S/R machine travel time $\widehat{ESC}^*$ =20.10 seconds which is calculated by a discrete enumeration method, and a pallet's position/coordinate is defined by its middle point on three dimensions. The real rack capacity is 1050 pallets. 2) For the 20%/90% ABC curve, $\hat{L}_1^* = 7.2$ meters (6 pallets), $\hat{H}_1^*$ =4 meters (pallets), $\hat{S}_1^*$ =2. 4 meters (2 pallets), $\hat{L}^* = 36$ meters (30 pallets), $\hat{H}^* = 10$ meters (5 pallets), $\hat{S}^* = 8.4$ meters (7 pallets), and $\widehat{ESC}^*$ =12.07 (seconds). The real rack capacity is 1050 pallets.

Step 5: evaluate the near optimal solution. From the above results we find that the deviation of the near optimal solutions from the optimal solutions is fairly small: the deviation percentages (i.e. ($\widehat{ESC}^*$ - $\overline{ESC}^*$)/ $\overline{ESC}^*$ $\times 100\%$) for 20%/20% and 20%/90% are 1.28% and 2.71%, respectively.

We conclude that the result obtained in this paper can be easily implemented and help in obtaining a good near optimal solution for discrete racks.

8. Conclusions and further research

This paper aims to determine optimal class boundaries of two storage classes for a compact 3D AS/RS by minimizing the expected travel time of the S/R machine. Some main results helpful for AS/RS designers and warehouse managers can be summarized as follows:

- The optimal rack layout can be obtained for compact 3D AS/R systems with 2-class-based storage, by using Theorem 6 or the results in Subsection 5.4 for any rack storage capacity, and any ABC curve. For some commonly used ABC curves, Table 1 provides the optimal results which can be used to optimally layout the compact 3D AS/R systems for any storage capacity.
- In compact 3D AS/R systems, the 2-class-based storage policy is a good assignment rule for improving the expected travel time of the S/R machine for a single command cycle. The more skewed (smaller s) the ABC curve is, the more expected time is saved compared with the random storage policy. For example, for $s = 0.07$, (a 20%/90% ABC curve), the saved time is 44.52%. The problem with the random storage policy discussed by de Koster et al. (2006) is a special case of our problem with skewness parameter $s = 1$.
- The skewness parameter s of the ABC curve has great impact on the optimal percentile G_1^* of the first class region, then the optimal dimensions of the region. For the 20%/30% ABC curve, G_1^* equals 0.25 while G_1^* only equals 0.02 for a 20%/90% ABC curve.
- The example shows that the optimal results for our continuous 3D AS/RS are helpful to find a near optimal solution for practical discrete settings. The gap between the near minimum expected travel time and the minimum one is very small. In our example, this gap is less than 3%.
- The expected travel times for the 3D compact AS/RS are derived in several cases where the class-based storage policy is implemented. These analytic formulas for the travel times can be used for evaluating the performance of 3D compact AS/R systems.

So far, compact 3D storage systems have received only little attention in academic literature. Our initial results may help in stimulating more research in this direction. Many problems still remain to be addressed. For example, it is interesting to extend 2-class-based storage to *n*-class-based storage since random storage, two-class-based storage, and full turnover-based storage are all special cases of *n*-class-based storage. Another interesting issue is the prepositioning of queued requested unit loads to locations closer to the I/O point, to shorten their retrieval times, assuming the demand information is available ahead of time. Multiple command cycles may also be considered to improve the performance of the 3D AS/RS. For those further researches, the analysis however may become too cumbersome to obtain closed-form analytic results. Optimal numerical results might still be tractable.

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Appendix A. Computation of $E(W \mid R_{2,2,1})$

As shown in Figure 9, we split region $R_{2,2}$ into three sub-spaces: $R_{2,2}^A$, $R_{2,2}^B$, and $R_{2,2}^C$ which are defined in Figure 9. Similar to the derivation of $E(U \mid R_2)$, $E(W \mid R_{2,2,1})$ can be developed with:

$$\begin{aligned} E(W \mid R_{2,2,1}) &= E\left(W \mid R_{2,2}^A \bigcap R_{2,2,1}\right) \cdot P((X,Y,Z) \in R_{2,2}^A) \\ &+ E\left(W \mid R_{2,2}^B \bigcap R_{2,2,1}\right) \cdot P((X,Y,Z) \in R_{2,2}^B) + E\left(W \mid R_{2,2}^C \bigcap R_{2,2,1}\right) \cdot P((X,Y,Z) \in R_{2,2}^C) \end{aligned} \quad (48)$$

in which $E\left(W \mid R_{2,2}^A \bigcap R_{2,2,1}\right)$ represents the expected value of $W \in R_{2,2}$ satisfying Case 1 shown in Figure 5. $P((X,Y,Z) \in R_{2,2}^A)$ represents the probability of $W \in R_{2,2}^A$ under the condition that $W \in R_{2,2}$. Other similar notations are not explained again.

We have to derive the values of every component in Equation (48).

<Insert Figure 9 here>

In sub-space $R_{2,2}^A$, the probability functions of a random storage position on three dimensions are:

$$P(X \le w \mid R_{2,2}^A) = \begin{cases} w/t_{h1} & \text{if } 0 < w \le t_{h1} \\ 1 & \text{if } t_{h1} < w \end{cases} \quad (49)$$

$$P(Y \le w \mid R_{2,2}^A) = \begin{cases} 0 & \text{if } 0 < w \le t_{h1} \\ (w - t_{h1})/(t_{h2} - t_{h1}) & \text{if } t_{h1} < w \le t_{h2} \\ 1 & \text{if } t_{h2} < w \end{cases} \quad (50)$$

$$P(Z \le w \mid R_{2,2}^A) = \begin{cases} w/t_{c2} & \text{if } 0 < w \le t_{c2} \\ 1 & \text{if } t_{c2} < w \end{cases} \quad (51)$$

Considering $0 < t_{c2} \le t_{h1}$ from Figure 5, the probability function of $W = \max\{X,Y,Z\}$ can be found from:

$$F(w \mid R_{2,2}^{A} \bigcap R_{2,2,1}) = F(X \le w \mid R_{2,2}^{A} \bigcap R_{2,2,1}) \cdot F(Y \le w \mid R_{2,2}^{A} \bigcap R_{2,2,1}) \cdot F(Z \le w \mid R_{2,2}^{A} \bigcap R_{2,2,1})$$

$$= \begin{cases} 0 & \text{if } 0 < w \le t_{h1} \\ (w - t_{h1}) \big/ (t_{h2} - t_{h1}) & \text{if } t_{h1} < w \le t_{h2} \\ 1 & \text{if } t_{h2} < w \end{cases}$$

$$\Rightarrow f(w \mid R_{2,2}^{A} \bigcap R_{2,2,1}) = \begin{cases} 1 \big/ (t_{h2} - t_{h1}) & \text{if } t_{h1} < w \le t_{h2} \\ 0 & \text{otherwise} \end{cases} \quad (52)$$

Therefore,

$$E\left(W \mid R_{2,2}^{A} \bigcap R_{2,2,1}\right) = \int_{t_{h1}}^{t_{h2}} w \cdot f(w \mid R_{2,2}^{A} \bigcap R_{2,2,1}) = \frac{t_{h1} + t_{h2}}{2}. \quad (53)$$

Similarly, we have

$$E\left(W \mid R_{2,2}^{B} \bigcap R_{2,2,1}\right) = \frac{t_{h1} + 2t_{h2}}{3}, \quad (54)$$

and

$$E\left(W \mid R_{2,2}^{C} \bigcap R_{2,2,1}\right) = E\left(W \mid R_{2,2}^{A} \bigcap R_{2,2,1}\right) = \frac{t_{h1} + t_{h2}}{2}. \quad (55)$$

Because in class II, the items are randomly stored, $P((X,Y,Z) \in R_{2,2}^{A})$ equals the volume of $R_{2,2}^{A}$ divided by that of $R_{2,2}$, that is,

$$P((X,Y,Z) \in R_{2,2}^{A}) = \frac{2t_{c2}t_{h1}}{2t_{c2}t_{h1} + t_{c2}(t_{h2} - t_{h1})}. \quad (56)$$

Similarly,

$$P((X,Y,Z) \in R_{2,2}^{B}) = \frac{t_{c2}(t_{h2} - t_{h1})}{2t_{c2}t_{h1} + t_{c2}(t_{h2} - t_{h1})}, \quad (57)$$

$$P((X,Y,Z) \in R_{2,2}^{C}) = P((X,Y,Z) \in R_{2,2}^{A}) = \frac{2t_{c2}t_{h1}}{2t_{c2}t_{h1} + t_{c2}(t_{h2} - t_{h1})}. \quad (58)$$

Substituting Equations (53)-(58) into (48), we obtain $E(W \mid R_{2,2,1})$.

Appendix B. Proof of Theorem 1

From Constraints (37) and (38), we have $t_{c2} = 1 / t_{h2}^{2}$ and $t_{c1} = G_1 / t_{h1}^{2}$ respectively. By substituting them into the objective function (39), it becomes a function of t_{h2} as follows:

$$E(U \mid R_2) + E(W \mid R_2) = \frac{6t_{h1}^{2} - 6G_1^{2}t_{h2}^{2} - 8G_1t_{h1}^{3}t_{h2}^{2} - 3t_{h1}^{6}t_{h2}^{2} + 8t_{h1}^{2}t_{h2}^{3} + 3t_{h1}^{2}t_{h2}^{6}}{12(1 - G_1)\ t_{h1}^{2}t_{h2}^{2}} \quad (59)$$

For the domain of variable t_{h2}, from Constraint (9) considering Equations (36) and (37), and, we have $t_{c1} = G_1 / t_{h1}^2 \leq t_{c2} = 1/t_{h2}^2$, and then $$t_{h2} \leq t_{h1} / \sqrt{G_1} \quad (60)$$

From $t_{h1} \leq t_{c1}$ and Constraint (40) considering Constraints (38) and (37), we have

$$t_{h2} \leq 1 \text{ and } t_{h1} \leq G_1^{1/3}. \quad (61)$$

Therefore, a model equivalent to Subproblem for Model DM with $t_{h1} \leq t_{c1}$ is the following constrained-optimization problem (denoted by Subproblem E1):

Subproblem E1:

Minimize function (59)

Subject to $$t_{h1} \leq t_{h2} \leq \min\{1,\ t_{h1} / \sqrt{G_1}\} \quad (62)$$

where $t_{h2} \geq 0$ is a decision variable.

Because

$$\frac{d^2(E(U \mid R_2) + E(W \mid R_2))}{dt_{h2}^2} = \frac{3(1 + t_{h2}^2)(1 - t_{h2}^2 + t_{h2}^4)}{3(1 - G_1)t_{h2}^4} > 0 \quad (63)$$

$E(U \mid R_2) + E(W \mid R_2)$ is a convex function of t_{h2}.

Let $\frac{d(E(U \mid R_2) + E(W \mid R_2))}{dt_{h2}}$=0, only one critical point can be obtained at $t_{h2} = \sqrt[3]{(\sqrt{10} - 1)/3}$.

Therefore:

If $t_{h1} > \sqrt[3]{(\sqrt{10} - 1)/3}$, then $\frac{d(E(U \mid R_2) + E(W \mid R_2))}{dt_{h2}}$<0 and $E(U \mid R_2) + E(W \mid R_2)$ is a increasing function of t_{h2} in a feasible area (satisfying Constraint (62)), and the minimum value of Equation (59) will be obtained at

$$t_{h2}^* = t_{h1} \qquad \text{if } t_{h1} > \sqrt[3]{(\sqrt{10} - 1)/3}. \quad (64a)$$

If $t_{h1} / \sqrt{G_1} \leq \sqrt[3]{(\sqrt{10} - 1)/3}$, that is, $t_{h1} \leq \sqrt{G_1}\sqrt[3]{(\sqrt{10} - 1)/3}$, we have the minimum value of (59) will be obtained at

$$t_{h2}^* = t_{h1} / \sqrt{G_1} \qquad \text{if } t_{h1} \leq \sqrt{G_1}\sqrt[3]{(\sqrt{10} - 1)/3}. \quad (64b)$$

Otherwise, we have

$$t_{h2}^* = \sqrt[3]{(\sqrt{10} - 1)/3} \qquad \text{if } \sqrt{G_1}\sqrt[3]{(\sqrt{10} - 1)/3} < t_{h1} \leq \sqrt[3]{(\sqrt{10} - 1)/3}. \quad (64c)$$

Considering $t_{h1} > \sqrt[3]{(\sqrt{10} - 1)/3}$ in Equation (64b), and $t_{h1} \leq G_1^{1/3}$ in Equation (61), Equation (64 b) holds if and only if $\sqrt[3]{(\sqrt{10} - 1)/3} < G_1^{1/3}$, that is

$$G_1 > (\sqrt{10} - 1)/3. \tag{65}$$

Summarizing Equations (64a)-(64c), and (65), Equation (41a) is proved.

Using Constraint (37), Equation (41b) is proved.

Appendix C. Proof of Theorem 2

From Constraints (37) and (38), we have $t_{c2} = 1/t_{h2}^2$ and $t_{c1} = G_1/t_{h1}^2$ respectively. By substituting them into the objective function (39), it becomes a function of t_{h2} as follows:

$$E(U \mid R_2) + E(W \mid R_2) = \frac{-6G_1^2 t_{h1}^{-2} - 3t_{h1}^4 + t_{h2}^{-8} + 8t_{h2} - 8G_1 t_{h1} + 8t_{h2}}{12(G_1 - 1)} \tag{66}$$

For the domain of variable t_{h2}, from Constraint (9) considering Equations (36) and (37), we have $t_{c1} = G_1/t_{h1}^2 \le t_{c2} = 1/t_{h2}^2$, and then $$t_{h2} \le t_{h1}/\sqrt{G_1} \tag{67}$$

From $t_{h1} \le t_{c1}$ and Constraint (40) considering Constraints (36) and (37), we have

$$t_{h2} > 1 \text{ and } t_{h1} \le G_1^{1/3}. \tag{68}$$

From Equations (7), (67), and (68), t_{h2} satisfies:

$$\max\{1, t_{h1}\} < t_{h2} \le t_{h1}/\sqrt{G_1}. \tag{69}$$

Considering Equation (68) and $G_1 < 1$, then $\max\{t_{h1}, 1\}$ =1.

Therefore, a model equivalent to Model **Subproblem 2 for Model DM** with $t_{h1} \le t_{c1}$ is the following constrained-optimization problem (denoted by Subproblem E2):

Subproblem E2:

Minimize function (66)

subject to $$1 < t_{h2} \le t_{h1}/\sqrt{G_1} \tag{70}$$

where $t_{h2} \ge 0$ is a decision variable.

Because $$\frac{d(E(U \mid R_2) + E(W \mid R_2))}{dt_{h2}} = \frac{2(2t_{h2}^9 - 1)}{3(1 - G1)t_{h2}^9} > 0, \tag{71}$$

considering Constraint (70). $E(U \mid R_2) + E(W \mid R_2)$ therefore is an increasing function of t_{h2}. Theorem 2 is proved.

Appendix D. Proof of Theorem 5

From the optimal solution of Subproblem 2 for Model DM with $t_{c1} \le t_{h1}$ given in Theorem 4 is $t_{h1}^{*}=t_{c1}^{*}$ which is a special case in subproblem 1 for Model DM with $t_{h1} \le t_{c1}$. The solution given in Theorem 4 therefore can be considered as a feasible solution of Subproblem 1 for Model DM under $t_{h1} \le t_{c1}$, but is not an optimal solution of subproblem 1 for Model DM with $t_{h1} \le t_{c1}$ from Theorem 1. Thus the optimal solution of Model DM must be identical that of Model DM with $t_{h1} \le t_{c1}$ given in Theorem 3.

Appendix E. Proof of Theorem 6

In order to prove that the optimal class region boundaries are SIT on the S/R machine moving face, we only need to prove the justification of the assumption:

"under any given storage capacity and the depth dimension lengths of class regions I and II, if a boundary of class region I or II is closer to SIT than the other, its corresponding value ESC of the objective function (5) is smaller."

If it is true, we can start to adjust either of the two class region boundary that is not SIT, then the other. Continue the process until the two class region boundaries are both SIT.

For class region I, without loss of generality, we assume that there are two boundaries (denoted by regions R_1 and $\bar{R}_1$ in Figure 10) where $t_{h1} / t_{v1} > \bar{t}_{h1} / \bar{t}_{v1} \ge 1$ (the region $\bar{R}_1$ is closer than R_1 to SIT), and V_{R_1} (the volume of region R_1) = $V_{\bar{R}_1}$ (the volume of region $\bar{R}_1$). We denote ESC_{R_1} and $ESC_{\bar{R}_1}$ as the ESC for R_1 and $\bar{R}_1$ respectively. If we change R_1 to $\bar{R}_1$, using Equation (5),

$$\triangle ESC_1 = ESC_{\bar{R}_1} - ESC_{R_1} = G_1^{\frac{2s}{1+s}}(\triangle E(U \mid R_1) + \triangle E(W \mid R_1)) + (1 - G_1^{\frac{2s}{1+s}})(\triangle E(U \mid R_2) + \triangle E(W \mid R_2))$$

where it is not difficult to obtain:

$$\triangle E(U \mid R_1) = \frac{V_{R_2^1} \cdot E(U \mid R_2^1)}{G_1} - \frac{V_{R_1^2} \cdot E(U \mid R_1^2)}{G_1} = \frac{V_{R_1^2} \cdot [E(U \mid R_2^1) - E(U \mid R_1^2)]}{G_1},$$

$$\triangle E(W \mid R_1) = \frac{V_{R_1^2} \cdot [E(W \mid R_2^1) - E(W \mid R_1^2)]}{G_1}, \ \triangle E(U \mid R_2) = \frac{V_{R_1^2} \cdot [E(U \mid R_1^2) - E(U \mid R_2^1)]}{1 - G_1}, \text{ and}$$

$$\triangle E(W \mid R_2) = \frac{V_{R_1^2} \cdot [E(W \mid R_1^2) - E(W \mid R_2^1)]}{1 - G_1}.$$ R_2^1 and R_1^2 are defined in Figure 10.

From $t_{h1}/t_{v1} > \overline{t}_{h1}/\overline{t}_{v1}$ and $V_{R_1} = V_{\overline{R}_1}$, we have $t_{v1} < \overline{t}_{v1} < \overline{t}_{h1} < t_{h1}$. And U in R_2^1 equals $\max\{X, Y \mid X, Y \in R_2^1\} \leq \overline{t}_{h1} < \max\{X, Y \mid X, Y \in R_1^2\}$ which equals U in R_1^2. $\triangle E(U \mid R_1)$ is then smaller than 0. Similarly, for any given depth length, $\triangle E(W \mid R_1)$<0.

Moreover, because the turnover per item in class I is bigger than that in class II, we have $(1 - G_1^{\frac{2s}{1+s}})/(1 - G_1) < 1 < G_1^{\frac{2s}{1+s}}/G_1$. Therefore,

$$
\begin{aligned}
\triangle ESC_1 &= G_1^{\frac{2s}{1+s}}\left(\frac{V_{R_1^2} \cdot [E(U \mid R_2^1) - E(U \mid R_1^2)]}{G_1} + \frac{V_{R_1^2} \cdot [E(W \mid R_2^1) - E(W \mid R_1^2)]}{G_1}\right) \\
&\quad + (1 - G_1^{\frac{2s}{1+s}})\left(\frac{V_{R_1^2} \cdot [E(U \mid R_1^2) - E(U \mid R_2^1)]}{1 - G_1} + \frac{V_{R_1^2} \cdot [E(W \mid R_1^2) - E(W \mid R_2^1)]}{1 - G_1}\right) \\
&< G_1^{\frac{s-1}{1+s}}(V_{R_1^2} \cdot [E(U \mid R_2^1) - E(U \mid R_1^2)] + V_{R_1^2} \cdot [E(W \mid R_2^1) - E(W \mid R_1^2)]) \\
&\quad + G_1^{\frac{s-1}{1+s}}(V_{R_1^2} \cdot [E(U \mid R_1^2) - E(U \mid R_2^1)] + V_{R_1^2} \cdot [E(W \mid R_1^2) - E(W \mid R_2^1)]) = 0
\end{aligned}
$$

That is, $\triangle ESC_1$<0. Similarly, we can get the same result in class region II. Therefore Theorem 6 is proved.

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Figure and table list


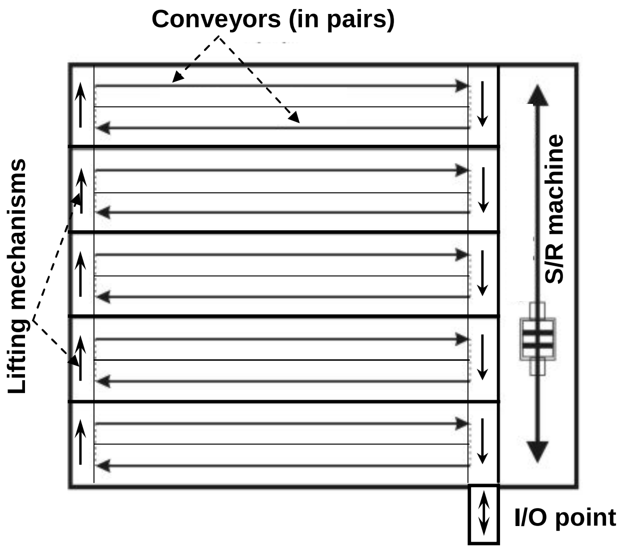

Figure 1: A compact S/RS with gravity conveyors (De Koster et al. 2006)

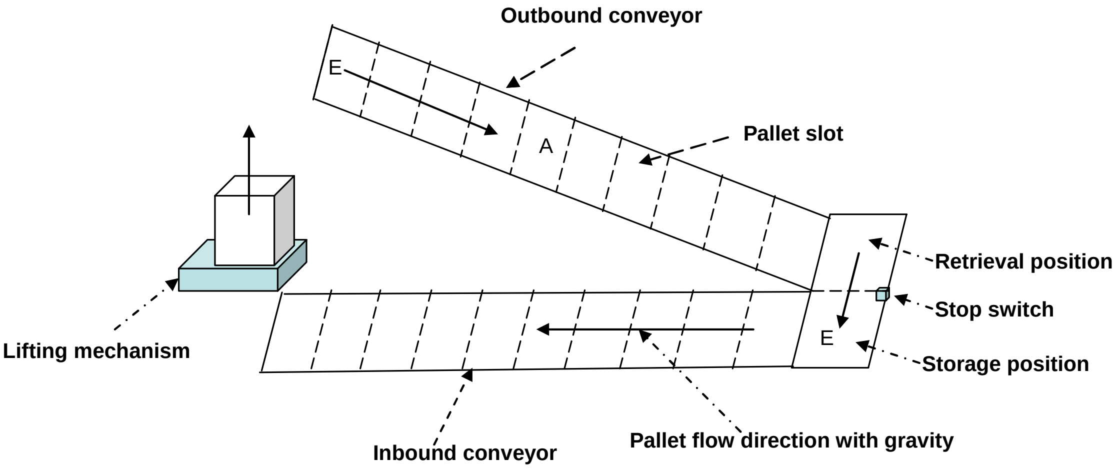

Figure 2: Deep dimension work mechanism with gravity conveyors

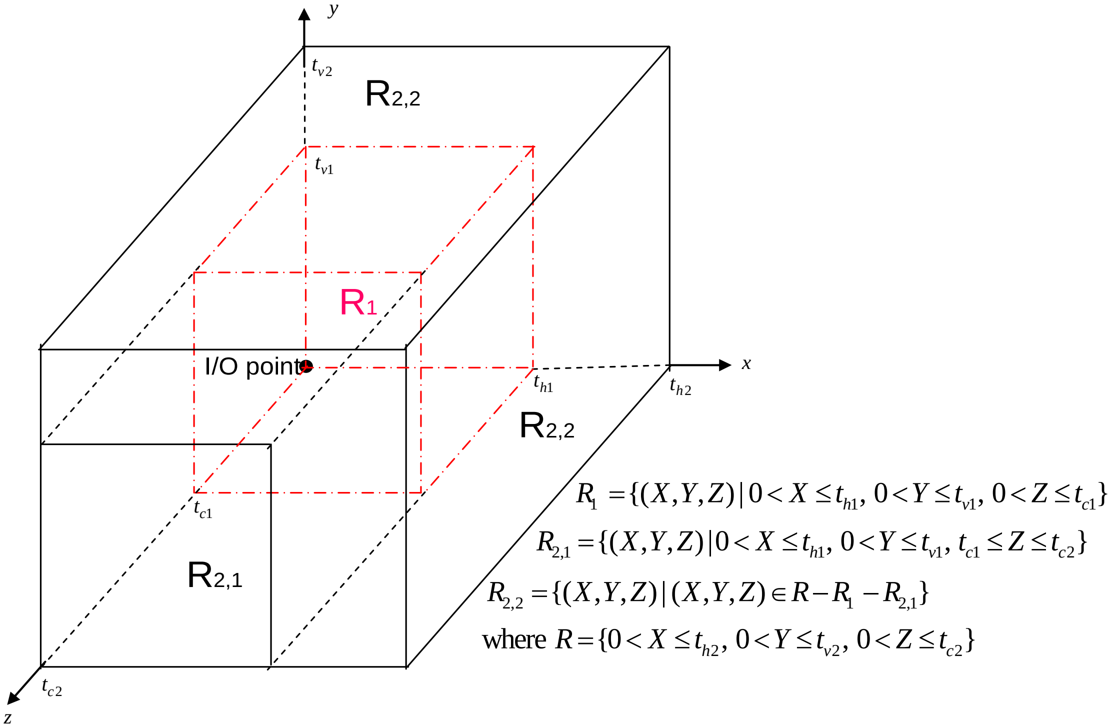

Figure 3: Three storage regions in the 3D rack

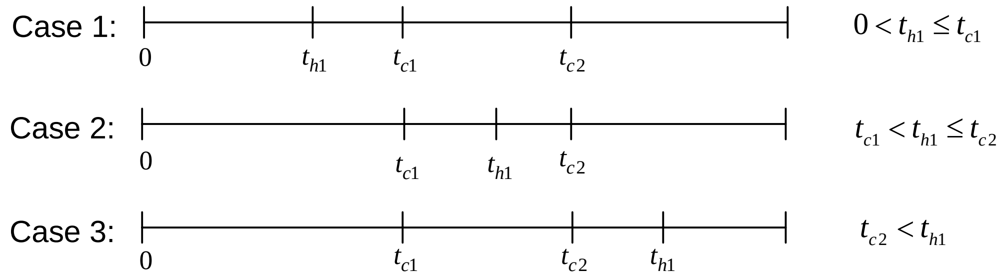

Figure 4: Three cases in sub-region $R_{2,1}$

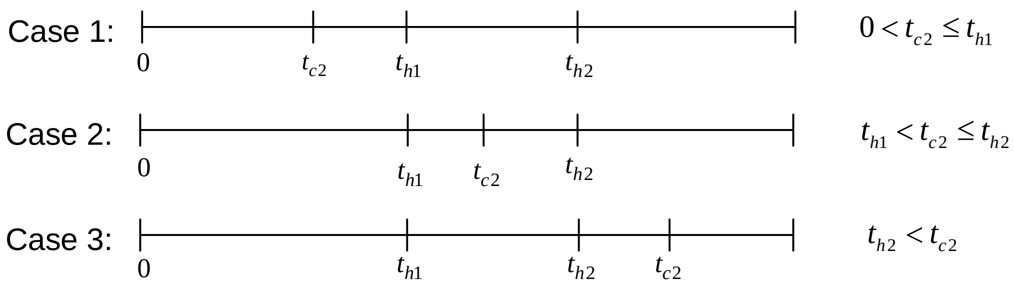

Figure 5: Three cases in sub-region $R_{2,2}$

Step 1: Solving the Model DM under $t_{h1} \le t_{c1}$ Step 2: Solving the Model DM under $t_{c1} \le t_{h1}$ Step 3: Compare the results in Steps 1&2, and get the optimal solution of Model DM.	Optimizing Model DM by Decomposing it into a series of sub-problems
Step 4: Prove the optimal solution of model DM will be an optimal solution of Model GM	Obtaining the optimal solution of Model GM

Figure 6: Algorithm outline for determining class boundaries

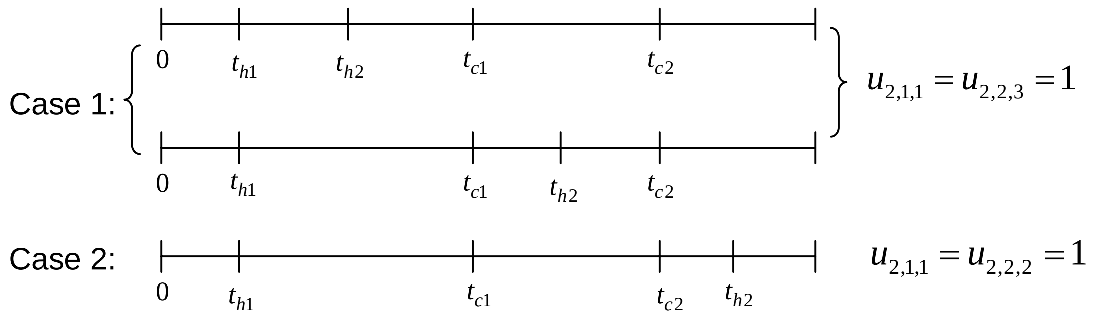

Figure 7: Two cases in Model DM with $t_{h1} \le t_{c1}$

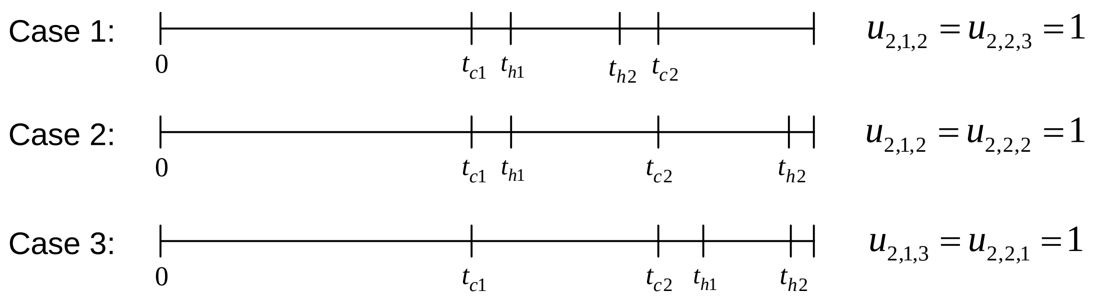

Figure 8: Three cases in Model DM with $t_{c1} \le t_{h1}$

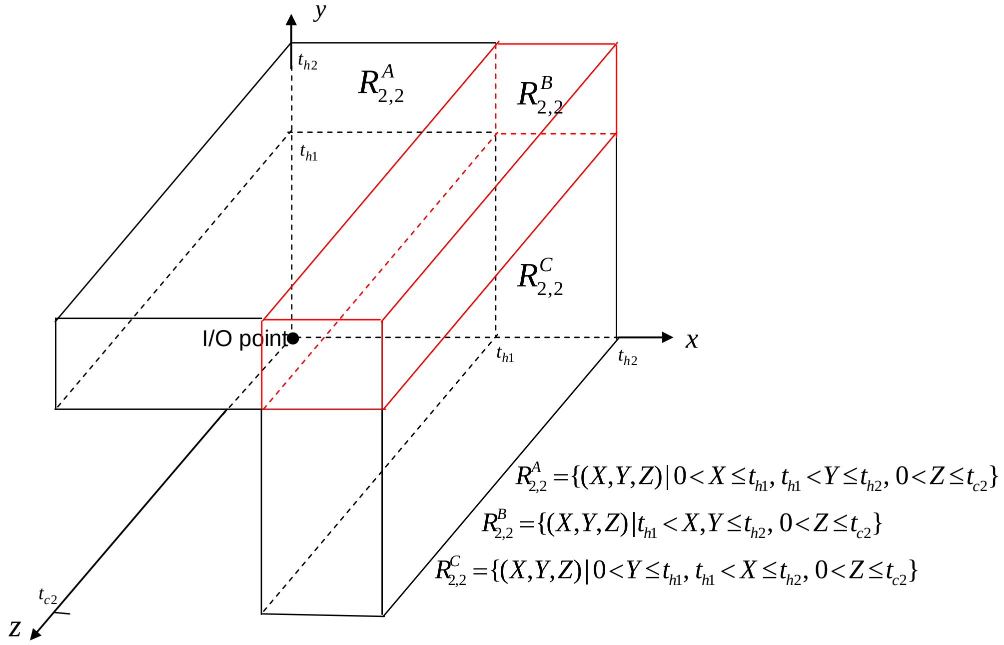

Figure 9: Three partitioned spaces in sub-region $R_{2,2}$

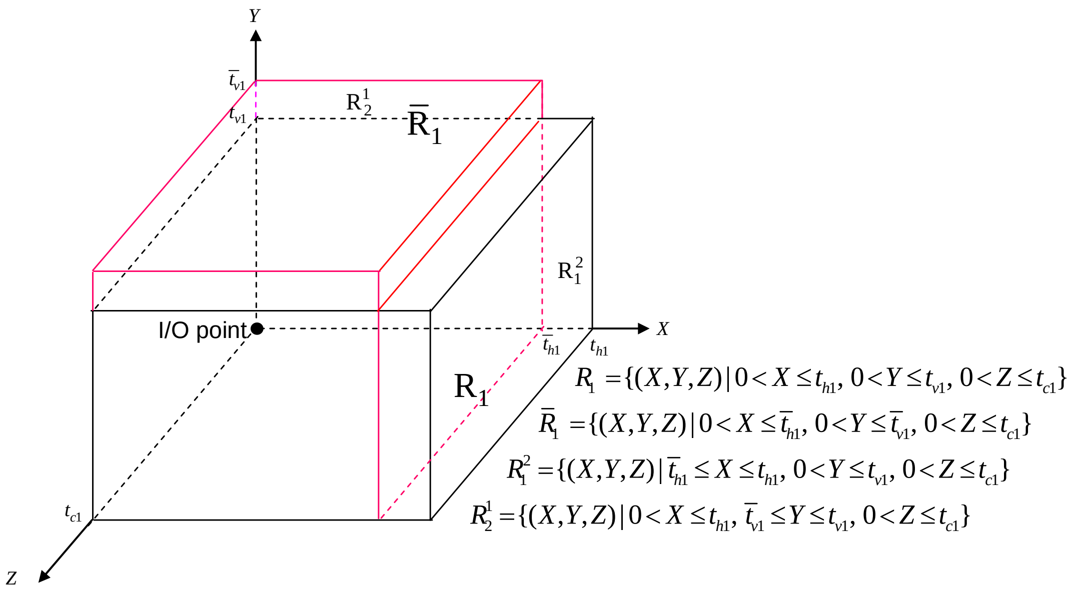

Figure 10: Boundary changes on class region I

Table 1: The optimal solutions for different skewness parameters (ABC curves)

s	ABC Curve	$\overline{t}_{h1}^{*}$	$\overline{t}_{c1}^{*}$	G_1^{*}	$\overline{ESC}^{*}$	ESC^{*}	$\mathrm{Impro}_{G_1^*}$ (%)	Impro (%)
1.00	20%/20%	-	-	/	1.38	1.38	0.00	0.00
0.75	20%/30%	0.55	0.77	0.24	1.34	1.34	2.73	2.71
0.57	20%/40%	0.53	0.73	0.21	1.30	1.30	5.76	5.76
0.43	20%/50%	0.50	0.69	0.17	1.25	1.25	9.36	9.31
0.32	20%/60%	0.46	0.64	0.14	1.19	1.19	13.85	13.52
0.22	20%/70%	0.41	0.57	0.10	1.11	1.12	19.87	18.56
0.14	20%/80%	0.34	0.48	0.06	0.98	1.04	28.75	24.69
0.07	20%/90%	0.23	0.32	0.02	0.77	0.94	44.52	32.21

Note: "-": any value between 0 and 1; "/": $G_1^* = \overline{t}_{h1}^{*2} \cdot \overline{t}_{c1}^{*}$; $\mathrm{Impro}_{G_1^*}$: *ESC* reduction in percentage (%) from two-class based storage and $G_1 = G_1^*$, and measured by $\left(ESC_{RN}^* - \overline{ESC}^*\right)/ESC_{RN}^* \times 100$ where ESC_{RN}^* is ESC for random storage policy from De Koster et al. (2006); Impro : ESC reduction in percentage (%) from two class based storage and given $G_1 = 0.2$, and measured by $\left(ESC_{RN}^* - ESC^*\right)/ESC_{RN}^* \times 100$. The values not given above are the same for different ABC curves with $t_{h1}^* = t_{v1}^*$ =0.52, t_{c1}^* =0.73, $t_{h2}^* = t_{v2}^* = \overline{t}_{h2}^* = \overline{t}_{v2}^*$ =0.90, $\overline{t}_{c2}^* = t_{c2}^*$ =1.24.

Table 2: System parameters

Total system capacity in pallets		1000 pallets
Storage policy		Two class-based storage
Pallet size in meter (width x length x height)	Net	1 x 1 x 1.5
	Gross	1.2 x 1.2 x 2
S/R machine	Operating policy	Single-command cycle
	Vertical speed (s_v)	0.8 (*meter per second*)
	Horizontal speed (s_h)	2.5 (*meter per second*)
Conveyor speed (s_c)		0.8 (*meter per second*)
Cases of ABC curve considered		20%/20% and 20%/90%

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