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A NOTE ON A PROFIT MAXIMIZING LOCATION MODEL

Shuzhong Zhang*

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ABSTRACT

In this paper we discuss a locational model with a profit-maximizing objective. The model can be illustrated by the following situation. There is a set of potential customers in a given region. A firm enters the market and wants to sell a certain product to this set of customers. The location and demand of each potential customer are assumed to be known. In order to maximize its total profit, the firm has to decide: 1) where to locate its distribution warehouse to serve the customers; 2) the price for its product. Due to existence of competition, each customer holds a reservation price for the product. This reservation price is a decreasing function in the distance to the warehouse. If the actual price is higher than the reservation price, then the customer will turn to some other supplier and hence is lost from the firm's market. The problem of the firm is to find the best location for its warehouse and the best price for its product at the same time in order to maximize the total profit. We show that this problem can be solved in polynomial time.

Key words: location, pricing, profit maximization.

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* Econometric Institute, Erasmus University Rotterdam, The Netherlands.

1 Introduction

Traditionally the literature on locational models under the microeconomic marketing environment is quite limited. See [1] and [2] for a survey. One reason for this could be that most locational models have an accent of minimizing costs of some type. One of the most celebrated models in location science is certainly the Weber model introduced by A. Weber in 1909 [5]. This model can be briefly described as follows. There is a set of customers to be served. Each customer is known by its location and the quantity of the demand. The problem is to find a location for the distribution warehouse to serve these customers, where the objective is to minimize the total transportation costs. This model has been extensively studied in the literature, because it captures an essence common in many locational problems. At the same time, it receives criticisms as well. One of the criticisms was raised by Lösch in [4]: "Weber's solution for the problem of location proves to be incorrect as soon as not only cost but also sales possibilities are considered. His fundamental error consists in seeking the place of lowest cost. This is as absurd as to consider the point of largest sales as the proper location. Every such one-sided orientation is wrong. Only search for the place of greatest profit is right".

In Chapter 15 of [1] Peeters and Thisse discussed several operational models which combine location problems with profit-maximization objectives. Moreover, it is assumed that there is no competition in the market and that the demand is a decreasing affine linear function in the price, as in many other microeconomic pricing models. We refer the reader to Chapter 4 of [3] for a survey of various pricing models.

The emphasis of the model to be discussed in this paper is different. We shall take competition into account. Consider the following situation. There is a firm which produces a certain product. This product will be brought to a warehouse for distribution. There are n potential cutomers who are interested in this product. The locations of these n customers are known to be a_i with $i = 1, \cdots, n$. Moreover, it is known that the demand quantity of customer i is Q_i for $i = 1, \cdots, n$. For each customer, it is attractive if the warehouse is located in his/her close neighborhood since the customer bares the transportation costs. Due to the existence of competition in the market, the reservation price (the maximum price up to which a customer is willing to pay) of each customer is a strictly decreasing function in the distance to the warehouse. As soon as the actual price goes beyond a customer's reservation price, he/she will turn to some other supplier, and thus is lost completely from the market of the firm. The problem of the firm is to choose the location for its warehouse and the price for the product in order to maximize the total profit.

The goal of this note is to show that in fact there is a simple polynomial time solution method for solving the problem. Certainly, the model is not supposed to be immediately operational due to its simplicity in nature. However, many extensions of the model are possible.

2 The model

Mathematically, the problem discussed at the beginning of the previous section can be formulated as

$$\begin{array}{ll} \text{maximize} & p\sum_{i=1}^{n} Q_i \chi_{\{p \le r_i(\|x-a_i\|)\}} \\ \text{subject to} & p \in \Re^1_+ \text{ and } x \in \Re^2 \end{array}$$

where r_i is the reservation price of the customer i given the location of the warehouse at x. In this expression χ_s stands for the characteristic function of a statement s:

$$\chi_s = \begin{cases} 1, & \text{if } s \text{ is true} \\ 0, & \text{otherwise.} \end{cases}$$

According to our model description, the function r_i is a strictly decreasing function.

This model looks quite messy in the sense that it is neither convex or concave in its decision variables p and x. Even for fixed p, the objective is in general not continuous in x. It is interesting to see, however, that the problem can be solved when p is fixed as a parameter. To illustrate this, let us introduce

$$F_p(x) = p\sum_{i=1}^{n} Q_i \chi_{\{p \le r_i(\|x-a_i\|)\}}.$$

The original model can be rewritten as

$$\max_{p \in \Re^1_+} \max_{x \in \Re^2} F_p(x).$$

The first question we pose is: *Can we efficiently evaluate* $\max_{x\in\Re^2} F_p(x)$ *when* p *is fixed as a parameter?*

To keep analysis simple, we consider the case where the norm is Euclidean. In that case, the above problem is equivalent to the following combinatorial circle covering problem:

(Circle Covering) We are given n circles $\{C_i \mid 1 \leq n \leq n\}$ on the plane. Circle C_i is centered at a_i with radius R_i, $i = 1, \cdots, n$. It is assumed that if a point is in C_i then this point receives a weight w_i. The total weight of a point x is denoted by $w(x)$ and is simply the summation of all weights from the circles that the point is contained in, i.e.

$$w(x) = \sum_{\{i|x \in C_i\}} w_i.$$

The problem is to find a point on the plane with the maximum total weight. Namely, we wish to find $x \in \Re^2$ that maximizes $w(x)$.

In the picture below we consider an example of the circle covering problem.

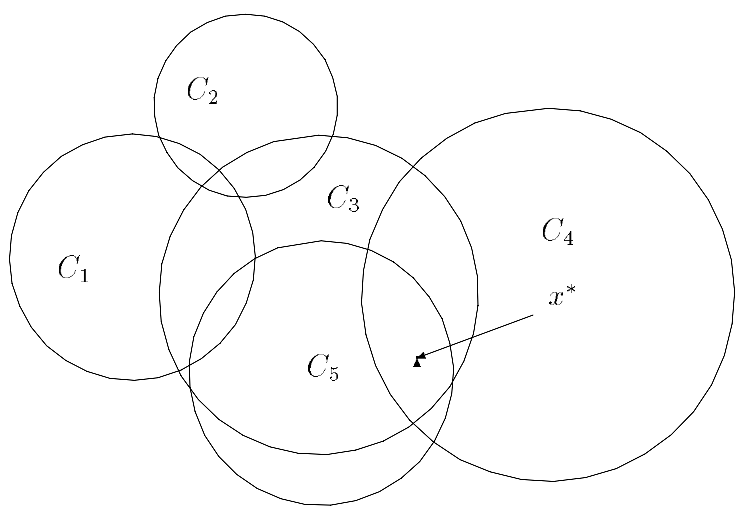

Figure 1. Circle covering.

In this instance, we let $w_1 = 7$, $w_2 = 6$, $w_3 = 10$, $w_4 = 8$ and $w_5 = 9$. Therefore, an optimal solution x^* lies in the intersection of C_3, C_4 and C_5.

This problem is combinatorial in nature. Although the circles may form a partition of the plane in a very complex way, we will see in the rest of this paper that the problem in general allows an easy solution method.

Observe that for each pair of circles C_i and C_j there can be maximally two intersection points. Let them form a set $I_{\{i,j\}}$. To be more precise, we define

$$I_{\{i,j\}} = \begin{cases} \text{The two intersection points;} & \text{if } C_i \text{ and } C_j \text{ intersect,} \\ \text{The touching point;} & \text{if } C_i \text{ and } C_j \text{ tangentially touch at a point,} \\ \text{The center of } C_j; & \text{if } C_i \text{ contains } C_j. \end{cases}$$

Let the set of all intersection points be

$$I = \cup_{1 \leq i < j \leq n} I_{\{i,j\}}.$$

We observe that the set I must contain at least one optimal solution, as the following lemma shows.

Lemma 2.1 *For the circle covering problem, it holds that*

$$\max_{x \in \Re^2} w(x) = \max_{x \in I} w(x).$$

Proof.

No matter how complex the partition of the plane formed by the circles may look like, the function $w(x)$ remains constant within each region given by this partition. Hence it is sufficient to check representatives from each region to find a maximum point for $w(x)$. Clearly, each region of the partition will have a representative in the set I. Therefore, checking all the points in I yields an optimal solution.

Q.E.D.

Note that $|\, I \,| \leq n(n-1)$ and that checking the weight of a given point amounts to $O(n)$ operations. Hence, checking all the intersection points in I yields an optimal solution in $O(n^3)$ operations.

This computational complexity can be further improved by a more careful sorting.

Let the circles be ordered by their natural indices:

$$C_1, C_2, \cdots, C_n.$$

First, take the circle C_1. Sort all of its intersection points with other circles, i.e. sort the points in the set I_1 with

$$I_1 := \cup_{j=2,\ldots,n} I_{\{1,j\}}$$

in the clockwise direction with respect to C_1. For simplicity, one may start sorting with an intersection point between C_1 and C_2.

Take one intersection point in I_1 in that order.

Let d be a n-dimensional vector defined as

$$d_i = \begin{cases} 1, & \text{if the current point is contained in } C_i, \\ 0, & \text{otherwise.} \end{cases}$$

Adding w_i's with $d_i = 1$ gives the total weight for the point under consideration.

Now, take the next point in I_1 according to the ordering described previously. Note that the d vector of the next point differs only in one position from the previous one. Hence, it takes only constant time to update the total weight. Remark that it involves a sorting for the points with respect to their positions in C_1, and so this gives an computational complexity of order $O(n \log n)$.

After we are done with the points in I_1, delete I_1 from I and continue the procedure with C_2. This will be repeated for all circles. In total the procedure will require $O(n^2 \log n)$ basic operations to find a point with maximum circle covering weight.

3 A procedure for solving the pricing/location model

In this section we discuss the original problem in which the price p plays as a part of the decision variable. For ease of exposition, assume that the reservation price of a customer is affine, i.e.

$$r_i(\|x - a_i\|) = \max\{u_i - v_i \|x - a_i\|, 0\}$$

where $u_i > 0$ and $v_i > 0$. This formula can be explained as follows. Suppose that apart from the current firm, the best price customer i gets elsewhere is u_i (including the transportation cost). Moreover, if customer i sticks to the current firm, then the transportation cost is assumed to be $v_i \|x - a_i\|$. Therefore, customer i will stay as long as $p \le r_i(\|x - a_i\|)$.

Let the inverse function of r_i be r_i^-. In particular,

$$r_i^-(t) = u_i/v_i - t/v_i$$

for $0 \le t \le u_i/v_i$.

Now we reformulate the pricing/location model slightly:

$$\max_{p \in \Re^1_+} \max_{x \in \Re^2} F_p(x)$$

with

$$F_p(x) = p \sum_{i=1}^{n} Q_i \chi_{\{\|x-a_i\| \le r_i^-(p)\}}.$$

For a given p value, construct an $n \times n$ matrix $M = (m_{ij})_{n \times n}$ as follows.

Let $m_{ii} = 0$ for all i.

On its (i, j) $(i \neq j)$ position we make a link to two $(n + 1)$-dimensional vectors $w_1^{(i,j)}$ and $w_2^{(i,j)}$ representing two possible intersection points between C_i and C_j.

For each vector $w_l^{(i,j)}$ $(l = 1, 2)$ and $k = 1, 2, ..., n$, let

$$(w_l^{(i,j)})_k = \begin{cases} 1, & \text{if this intersection point is contained in } C_k, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, let $(w_l^{(i,j)})_{(n+1)}$ store the current weight of the intersection point.

Remark that in the case C_i and C_j do not intersect, then we let these two vectors be empty, and when C_i and C_j tangentially touch at a point, or one circle contains the other, then these two vectors reduces to only one vector.

For initialization, we let p be a sufficiently small number.

It takes $O(n^3)$ operations to install all the entries of M and the associated w vectors. As p increase, all the circles will shrink simultaneously at different rates. In particular, the intersection points between C_i and C_j can be computed analytically as

$$I_{\{i,j\}} = \{a_j + r_j^-(p)e\}$$

with $\|e\| = 1$ and

$$\angle(e, a_i - a_j) = \arccos\{[(r_j^-(p))^2 - (r_i^-(p))^2 + \|a_i - a_j\|^2]/[2r_j^-(p)\, \|a_i - a_j\|]\}.$$

At a certain stage, the state of the positions will change, i.e. one of the following situations occurs:

- For certain i with $1 \le i \le n$, $r_i^-(p)$ becomes zero;

- The intersection points of two circles converge to only one point;
- One intersection point of two circles lies on a third or more circles;
- One intersection point of two circles gets off from a third or more circles.

Knowing the analytical formula of all intersection points and the formula for r_i^-, it is easy to detect when (the value p) any of the above situations will occur. If the first situation occurs, i.e. one of $r_i^-(p)$ becomes zero for $i = 1, 2, ..., n$, then we simply need to delete C_i from further consideration and continue the procedure for the rest of the circles. For all the other cases, since r_i^- remains affine, it is easy to detect which circles will first change their relative positions. Should this change takes place, we need to update the matrix M and its associated records. However, this update requires only constant time as we know precisely what has changed, i.e. which circles have changed their positions. Notice that as p increase, the relative position of any two circles can only possibly evolve in the direction:

one contains the other $\rightarrow$ tangentially touch from inside $\rightarrow$ intersect
$\rightarrow$ tangentially touch from outside $\rightarrow$ non-intersect.

This ensures that one needs in total $O(n^3)$ basic operations to keep track with the whole procedure. By simply keeping the highest weight among the intersection points at each update, we select the best one and hence the problem is solved in $O(n^3)$ time. To summarize we have shown the following result:

Theorem 3.1 *The pricing/location model can be solved in at most $O(n^3)$ basic operations.*

4 Discussions

The algorithm is based on a real-number computational model. In fact, one may apply the same arguments if any other explicit form of norms are used, including the L_q-norms or polyhedral gauges. The form of r_i can be quite general as well. It is important, however, that one is able to detect the timing when two circles change their relative positions.

In a more general form we may consider a model in which several firms start competing in a given region for the same set of potential customers. It can be interesting to investigate, for instance, whether or not an equilibrium exists.

Remark that the problem we discuss is a planar one. It remains a topic for further research how to solve the problem in polynomial time as the dimension becomes more than two.

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