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INDEPENDENCE OF IRRELEVANT ALTERNATIVES AND REVEALED GROUP PREFERENCES

BY HANS PETERS AND PETER WAKKER

1. INTRODUCTION

IN CONSUMER DEMAND THEORY the concept of revealed preference is based on the assumption that, by choosing from budget sets, a consumer reveals his preferences over the available commodity bundles. Analogously, in bargaining game theory the agreements reached in bargaining games may be thought to reveal the preferences of the bargainers as a group. In this paper we consider, more generally, single-valued choice functions defined on the convex compact subsets of the positive orthant of $\mathbb{R}^n$. These subsets are called choice situations. In bargaining game theory choice functions are called bargaining solutions and choice situations are called bargaining games. In consumer demand theory choice functions are called demand functions and choice situations are called budget sets. Compact convex budget sets may be regarded as "generalized" budget sets where certain commodity bundles from the full simplices (linear budget sets) are not available. An example is the case of piecewise linear budget sets (see Hausman (1985)); our results would remain valid under the restriction to this case as well. Works concerned with revealed preference in consumer demand theory are, e.g., Richter (1971), Varian (1982), and Pollak (1990). The latter discusses generalized budget sets.

One purpose of this paper is to find conditions under which a choice function maximizes a real-valued function. In consumer demand theory such a function is called the consumer's utility function. Another purpose is to provide a thorough study of the consequences of the well-known independence of irrelevant alternatives (IIA) condition. A third purpose is to generalize the Nash bargaining solution.

We will first observe that a choice function maximizes a binary relation if and only if it satisfies IIA. This condition was introduced by Nash in his seminal 1950 paper on the bargaining problem. Next we show that the combination of Pareto optimality and IIA for a choice function in general only excludes cycles of length 1 or 2 in the revealed binary relation. If the dimension is 2, then also cycles of length 3 are excluded, but cycles of length at least 4 may still occur. For the latter case (i.e., $n = 2$), adding a weak form of continuity called Pareto continuity suffices to exclude circularity of the revealed binary relation; in general, however, even "full" continuity does not exclude cycles. For the case of 2-dimensional linear budget sets, related work was done by Samuelson (1948) and Rose (1958).

The main result of the paper is obtained by strengthening Pareto continuity to continuity: this condition together with Pareto optimality, and IIA for $n = 2$ or the (stronger) strong axiom of revealed preference for $n > 2$, is sufficient for the existence of a function representing the revealed binary relation, i.e., of a function which is maximized by the choice function. We finally show that this representing function must be strongly monotonic and strictly quasiconcave and, conversely, that the existence of a representing function with these properties implies the conditions of continuity, Pareto optimality, IIA, and the strong axiom of revealed preference for the choice function.

The organization of the paper is as follows. Section 2 gives elementary definitions and considers the role of IIA. Sections 3 and 4 study the (a)cyclicity of revealed preference without and with continuity conditions, respectively. Section 5 is devoted to the aforementioned main result and briefly discusses an application to bargaining game theory. Section 6 shows that the results can be extended to other domains, and concludes.

2. THE ROLE OF IIA

We denote by X the set of all possible *alternatives* (for a consumer, a group of bargainers,...). In this paper, with the exception of Section 6, $X = \mathbb{R}^n_{++}$, and a *choice situation* is a nonempty convex compact subset of X. The collection of all choice situations is denoted by Σ.

A *choice function* is a map $F: \Sigma \to X$ with $F(S) \in S$ for every $S \in \Sigma$. Note that in this paper a choice function is *single-valued* by definition. From F we derive a binary relation R on X as follows: xRy ("x is *directly revealed preferred to* y") if there is an $S \in \Sigma$ with $x = F(S)$, $y \in S$.

Sometimes choice functions can be derived from binary relations. A binary relation $\succcurlyeq$ on X *represents* a choice function F if for every choice situation S we have

$$\{F(S)\} = \{x \in S : x \succcurlyeq y \text{ for every } y \text{ in } S\}, \tag{2.1}$$

i.e., F uniquely maximizes $\succcurlyeq$ on S.

Obviously not every binary relation represents a choice function, and not every choice function can be represented by a binary relation. The following condition will characterize, within the set-up of this paper, the choice functions which can be represented by a binary relation. It was introduced in Nash (1950) for bargaining game theory, and is central in this paper.

DEFINITION 2.1: The choice function F satisfies *independence of irrelevant alternatives* (IIA) if for all choice situations S and T with $S \subset T$ and $F(T) \in S$ we have $F(S) = F(T)$.

THEOREM 2.2: *The choice function F can be represented by a binary relation $\succcurlyeq$ if and only if F satisfies IIA.*

PROOF: First suppose F is represented by $\succcurlyeq$. Let $S, T \in \Sigma$ with $S \subset T$ and $F(T) \in S$. By definition $\{F(T)\} = \{x \in T : x \succcurlyeq y$ for every $y \in T\}$. So $\{F(T)\} = \{x \in S : x \succcurlyeq y$ for every $y \in S\}$. From this we conclude that F satisfies IIA.

In order to prove the converse, suppose F satisfies IIA. Define $\succcurlyeq := R$. Then, for every $S \in \Sigma$, $F(S) \succcurlyeq y$ for every $y \in S$. We still have to show that $F(S)$ uniquely maximizes $\succcurlyeq$ on S, for every $S \in \Sigma$. Suppose there is an $S \in \Sigma$ with $y \in S$ and $y \succcurlyeq F(S)$, i.e., $yRF(S)$. Then there is a $T \in \Sigma$ with $F(S) \in T$ and $y = F(T)$, so by IIA applied twice, $y = F(T \cap S) = F(S)$. This completes the proof. *Q.E.D.*

In defending the IIA-condition Nash (1950, p. 195) argues that (two) rational individuals, agreeing on a common choice x from T, should find the agreement to choose x from $S \subset T$ "of lesser restrictiveness" than the agreement to choose x from T, and thus should also agree to choose x from S. Theorem 2.2 and Formula (2.1) clarify how the presence of fewer points in S may make it "of lesser restrictiveness" to agree on the choice x from S: in S the players must agree on $[x \succcurlyeq y]$ for fewer points y. Thus Theorem 2.2 clarifies two ideas which may have been underlying Nash's intuition: firstly, that the two players should choose in accordance with a binary "group preference" relation, and, secondly and more basic, the idea that the two players may be considered as one decision unit on which consistency requirements can be imposed.

Let us further note that Theorem 2.2 essentially depends on the restrictive framework in this paper, in which the choice function is single-valued and has a domain which is intersection-closed. Under more general circumstances many other conditions for choice functions have been formulated in the literature which in the context of this paper are equivalent to IIA. We mention the weak axiom of revealed preference (see Samuelson (1938)), property α and property β of Sen (1971), renamed nonincreasing eligibility and nondecreasing eligibility in Wakker (1989a), the independence of/from irrelevant alternatives of Luce (1959) and Kaneko (1980), and the V-axiom of Richter (1971). Most of these properties were studied in the context of consumer demand theory. Arrow (1959) showed that IIA (called C4 there) is necessary and sufficient for the existence of a transitive complete representing binary relation under the restrictive assumption that the domain of the choice function contains all finite subsets of X.

The next two sections deal with the (a)cyclicity of the binary relation in Theorem 2.2. In Section 3 we consider choice functions without the (Pareto) continuity property; in Section 4 we will add Pareto continuity and continuity.

3. (A)CYCLICITY OF REVEALED PREFERENCE WITHOUT CONTINUITY

Let F be a choice function and R the corresponding directly revealed preference relation. We write xPy if there exists an $S \in \Sigma$ with $x = F(S)$ and $y \in S$, $y \neq F(S)$. P is called the *directly revealed strict preference relation*. For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in X$, we write $x \geqslant y$ if $x_i \geqslant y_i$ for $i = 1, 2, \ldots, n$ and $x > y$ if $x_i > y_i$ for $i = 1, 2, \ldots, n$; $x \leqslant y$, $x < y$ are analogous. For $T \subset X$, $\operatorname{conv}(T)$ denotes the convex hull of T and $\operatorname{comv}(T) := \{x \in X : x \leqslant y \text{ for some } y \in \operatorname{conv}(T)\}$ denotes the *comprehensive* convex hull of T. For $S \in \Sigma$, $P(S) := \{x \in S : \text{there is no } y \in S \text{ with } y \geqslant x,\ y \neq x\}$ denotes the *Pareto optimal subset of* S. F satisfies *Pareto optimality* (PO) if $F(S) \in P(S)$ for every $S \in \Sigma$.

LEMMA 3.1: (i) *For every $x \in X$ we have xRx and not xPx.* (ii) *Suppose F satisfies PO and IIA. Let $x, y \in X$ with $x \neq y$. Then the following three statements are equivalent*: (a) *xRy*, (b) *xPy*, (c) *$x = F(S)$ for every $S \in \Sigma$ with $x \in S$ and $S \subset \operatorname{comv}\{x, y\}$.*

PROOF: (i) xRx since $F(\{x\}) = x$. [Not xPx] is obvious. (ii) (b) $\Rightarrow$ (a) by definition. To prove (a) $\Rightarrow$ (c), suppose xRy. Then $x = F(T)$ for some $T \in \Sigma$ with $\operatorname{conv}\{x, y\} \subset T$; so $F(\operatorname{conv}\{x, y\}) = x$ by IIA, hence by PO and IIA, $F(S) = x$ for every $S \in \Sigma$ with $P(S) = \operatorname{conv}\{x, y\}$; so by IIA again $x = F(S)$ for every $S \in \Sigma$ with $x \in S$ and $S \subset \operatorname{comv}\{x, y\}$. We have proved (a) $\Rightarrow$ (c). Suppose (c) is true; then $x = F(\operatorname{conv}\{x, y\})$ so xPy. (c) $\Rightarrow$ (b) follows, which completes the proof. *Q.E.D.*

Lemma 3.1 (and the arguments in its proof) will often be used without explicit mentioning.

DEFINITION 3.2: The choice function F satisfies the *strong axiom of revealed preference* (SARP) if there does not exist a *cycle* $x = x^0 P x^1 P x^2 \cdots x^{k-1} P x^k = x$, where $k > 0$ is the *length* of the cycle.

The condition SARP and the following result (formulated here in a way suited for our context) have been obtained by Ville (1946) and, independently, by Houthakker (1950) for general contexts. Kim (1987) has shown that slight weakenings of the transitivity of the binary relation do not affect the characterizing condition.

THEOREM 3.3: *There exists a transitive binary relation representing F if and only if F satisfies SARP.*

The following question arises: Are there in our context simpler and more appealing conditions which are still strong enough to imply SARP? In view of Theorem 2.2, IIA is a necessary condition. Further, we have the following lemma.

LEMMA 3.4: *Let F satisfy IIA. Then there do not exist cycles of length* 1 *or* 2 *in the revealed preference relation.*

PROOF: Cycles of length 1 are excluded by Lemma 3.1(i), which also excludes cycles of length 2: $xPyPx$ would by IIA imply $x = F(\text{conv}\{x, y\}) = y$ and hence xPx. *Q.E.D.*

The last property in Lemma 3.4, nonexistence of cycles of length 2, is known as the *Weak Axiom of Revealed Preference* (WARP); see e.g. Richter (1971). Further discussion is postponed until the end of the next section.

In the sequel we shall always assume Pareto optimality. In consumer demand theory it is an implicit condition; in bargaining game theory it is fairly standard. In what follows, $l(a, b)$ denotes the straight line through the points $a \neq b$ in X.

LEMMA 3.5: *Let $n = 2$, and let F satisfy PO and IIA. Then there do not exist cycles of length* 3.

PROOF: Assume the following:

(3.1) $a, b, x \in X$ satisfy aPb and $x\not{R}b$.

In view of the reflexivity of R (Lemma 3.1(i)) and the definition of P it follows from (3.1) that $a \neq b$, $a \neq x$, $b \neq x$. We will show that $x\not{R}a$; in some cases the additional requirement bPx will be needed. Nonexistence of cycles of length 3 then follows immediately.

In order to prove $x\not{R}a$, we list the following cases, which essentially exhaust all possible configurations of $\{a, b, x\}$.

(3.1.a) $a \geqslant b$,

(3.1.b) $b_1 < a_1$, $b_2 > a_2$,

(3.1.b.1) $x_1 < b_1$, x on or above $l(a, b)$,

(3.1.b.2) $x_1 < b_1$, $x_2 > b_2$, x strictly below $l(a, b)$,

(3.1.b.3) $x \in \text{comv}\{a, b\}$,

(3.1.b.4) $x_1 > a_1$, x on or below $l(a, b)$,

(3.1.b.5) $x_2 < a_2$, x strictly above $l(a, b)$,

(3.1.b.6) $x_1 \geqslant a_1$, $a_2 \leqslant x_2 < b_2$,

(3.1.b.7) $x_1 < a_1$, $x_2 < b_2$, x strictly above $l(a, b)$.

Note that the case $b \geqslant a$ is excluded by aPb and PO. Also the case $x \geqslant b$ is excluded by $x\not{R}b$ and PO. Further, the cases with $b_1 > a_1$, $b_2 < a_2$ are analogous to (3.1.b.1)–(3.1.b.7) and are therefore omitted. The proof of $x\not{R}a$ is given in two steps.

STEP 1: In the cases (3.1.a), (3.1.b.1), (3.1.b.3), and (3.1.b.4), we have $x \not R a$.

PROOF: (3.1.a): xRa would by Lemma 3.1(ii) imply $x = F(\operatorname{conv}\{x, a, b\})$, contradicting $x \not R b$.

(3.1.b.1): Same proof as for case (3.1.a).

(3.1.b.3): By aPb and Lemma 3.1(ii) we have $a = F(\operatorname{conv}\{a, b, x\})$, so aPx, hence $x \not R a$.

(3.1.b.4): Let $S := \operatorname{conv}\{x, a, b\}$. If $F(S) \in \operatorname{conv}\{a, b\}$ then $F(S) = a$ and hence aPx. If $F(S) \in \operatorname{conv}\{a, x\}$, then $F(S) \neq x$ since otherwise xRb; so by IIA also $F(\operatorname{conv}\{a, x\}) \neq x$, which by Lemma 3.1(ii) implies $x \not R a$. This completes the proof of Step 1.

STEP 2: Suppose bPx. Then $x \not R a$ in the cases (3.1.a), (3.1.b.1)–(3.1.b.4), and (3.1.b.7). The cases (3.1.b.5) and (3.1.b.6) cannot occur.

PROOF: In view of Step 1 we still have to consider the cases (3.1.b.2), (3.1.b.5)–(3.1.b.7).

(3.1.b.2): Let $S := \operatorname{conv}\{x, a, b\}$. If $F(S) \in \operatorname{conv}\{a, b\}$ then $F(S) = a$ by IIA, so aPx. If $F(S) \in \operatorname{conv}\{x, b\}$, then $F(S) = b$ since bPx, which leads to the contradiction bPa.

(3.1.b.5), (3.1.b.6): By Lemma 3.1(ii), bPx, and $a \in \operatorname{comv}\{b, x\}$, we would have bPa, a contradiction.

(3.1.b.7): Let $T := \operatorname{conv}\{a, b, x\}$. If $F(T) \in \operatorname{conv}\{b, x\}$, then $F(T) = b$, which would imply bPa, a contradiction. So $F(T) \in \operatorname{conv}\{x, a\}$ and $F(T) \neq x$ since otherwise xPb. So by IIA, $F(\operatorname{conv}\{x, a\}) \neq x$, hence $x \not R a$ by Lemma 3.1(ii).

This completes the proof of Step 2, and of the lemma. *Q.E.D.*

The following example, which was not easy to construct, shows that for $n = 2$, IIA and PO are not sufficient to exclude cycles of length greater than 3.

EXAMPLE 3.6: We define the following subsets of $X = \mathbb{R}^2_{++}$ (see Figure 1):

$$
\begin{aligned}
G_1 := &\left\{x \in X : x_1 \geqslant 4,\ x_2 \geqslant 4 + 2\sqrt{2}\right\} - \{(4,8)\} \\
&- \left\{x \in X : x_1 \leqslant 4 + 2\sqrt{2},\ x_2 \leqslant 8,\ \left(x_1 - 4 - 2\sqrt{2}\right)^2 \right. \\
&\qquad\qquad \left. + \left(x_2 - 8 - 2\sqrt{2}\right)^2 > 16\right\},
\end{aligned}
$$

$$G_2 := \{x \in X : 2 \leqslant x_1 < 4\} - \left\{x \in X : x_2 \leqslant 10,\ (x_1 - 4)^2 + (x_2 - 10)^2 > 4\right\},$$

$$
\begin{aligned}
G_3 := &\left\{x \in X : 4 \leqslant x_2 < 4 + 2\sqrt{2}\right\} - \{(8,4)\} \\
&- \left\{x \in X : x_1 \leqslant 8,\ \left(x_1 - 8 - 2\sqrt{2}\right)^2 + \left(x_2 - 4 - 2\sqrt{2}\right)^2 > 16\right\},
\end{aligned}
$$

$$G_4 := \{x \in X : 2 \leqslant x_2 < 4\} - \left\{x \in X : x_1 \leqslant 10,\ (x_1 - 10)^2 + (x_2 - 4)^2 > 4\right\},$$

$$G_5 := \{(8,4)\},$$

$$G_6 := \{x \in X : x_2 \geqslant 9\} - G_1 - G_2,$$

$$G_7 := \{x \in X : x_1 > 8\} - G_1 - G_3 - G_4,$$

$$G_8 := \{(4,8)\},$$

$$G_9 := X \setminus (G_1 \cup \cdots \cup G_8).$$

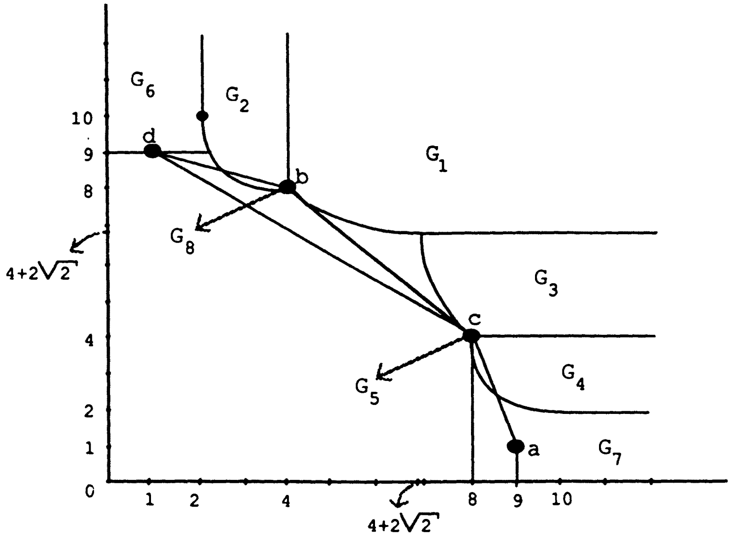

FIGURE 1.—A choice function satisfying IIA and PO, but violating SARP. $G_1 \succ \cdots \succ G_9$. On G_1, G_4, G_5, G_7, the first coordinate is maximized. On G_2, G_3, G_6, G_8, the second coordinate is maximized. On G_9, $x_1 x_2$ is maximized. A cycle *aPbPcPd* results.

Further $a := (9,1)$, $b := (4,8)$, $c := (8,4)$, and $d := (1,9)$. Let the transitive binary relation $\tilde{\succcurlyeq}$ on X be defined as follows:

(i) $x \tilde{\succ}$ for all $i<j$, $x \in G_i$, $y \in G_j$, $i,j \in \{1,2,\ldots,9\}$.

(ii) On G_1, G_4, G_5, G_7, $\tilde{\succcurlyeq}$ is the lexicographic order.

(iii) On G_2, G_3, G_6, G_8, $\tilde{\succcurlyeq}$ is the reversed lexicographic order (first maximizing the second coordinate).

(iv) On G_9, $\tilde{\succcurlyeq}$ maximizes the product $x_1 x_2$.

We define $\tilde{F}$ as the choice function maximizing $\tilde{\succcurlyeq}$. It can be seen that $\tilde{F}$ is well-defined, and satisfies IIA, PO, and SARP. We define $\succcurlyeq$ to be equal to $\tilde{\succcurlyeq}$ with one exception: $b \succcurlyeq c$ instead of $c \tilde{\succcurlyeq} b$. So $\succcurlyeq$ is not transitive. We define F as the choice function maximizing $\succcurlyeq$. Then also F is well-defined and satisfies PO and IIA (by Theorem 2.2), but F does not satisfy SARP: *aPbPcPdPa*, a cycle of length 4.

This section is concluded by an example showing that if $n > 2$, IIA and PO admit cycles of length 3.

EXAMPLE 3.7: Let $n = 3$ and let the choice function $F: \Sigma \to X$ be defined as follows. Let $Y := \{x \in X : x \geqslant (1,1,1)\}$ and let $S \in \Sigma$. If S contains an interior point of Y, then let $F(S)$ be the unique point of $Y \cap S$ where the product $(x_1 - 1)(x_2 - 1)(x_3 - 1)$ is maximized on this set; then $F(S) > (1,1,1)$. If $S \cap Y = \varnothing$, then let $F(S)$ be the unique point of S where the product $x_1 x_2 x_3$ is maximized on S. If $S \cap Y \neq \varnothing$ and x_1 (resp. x_2, x_3) $= 1$ for all $x \in S \cap Y$, then let $F(S)$ be the Pareto optimal point of $S \cap Y$ with maximal third (resp. first, second) coordinate. Then F can be seen to be a well-defined choice function satisfying IIA and PO. The corresponding revealed preference relation contains cycles of length 3, e.g. $(2,1,1)P(1,1,2)P(1,2,1)P(2,1,1)$.

4. (A)CYCLICITY OF REVEALED PREFERENCE WITH CONTINUITY

The following additional condition for a choice function was introduced in Peters (1986).

Definition 4.1: A choice function $F:\Sigma \to X$ satisfies *Pareto continuity* (PC) if for every sequence $S, S_1, S_2, \ldots$ in Σ with $S_k \to S$ and $P(S_k) \to P(S)$ (where the limits are taken with respect to the Hausdorff metric) we have $F(S_k) \to F(S)$.

For $n = 2$ and $S \in \Sigma$, let $D^1(S)$ be the point of $P(S)$ with maximal first coordinate, and let $D^2(S)$ be the point of $P(S)$ with maximal second coordinate. D^1 and D^2 are choice functions satisfying PO, IIA, and Pareto continuity but not continuity (see Def. 4.9). Note that for choice functions F satisfying PO and IIA we have $F(S) = F(T)$ whenever $P(S) = P(T)$: so, for such F, requiring Pareto continuity instead of continuity seems reasonable.

The remainder of this section is devoted, firstly, to proving that the combination of PO, PC, and IIA for a choice function F implies SARP if $n = 2$; secondly, to showing that for $n > 2$ these conditions, even with full continuity instead of PC, do not suffice to exclude cycles. For $x \neq y$, $l_x(x, y)$ denotes the straight closed halfline through x and y with endpoint x.

Lemma 4.2: *Let F satisfy PO, IIA, and PC. Let* $v, w \in X$ *with* $v \neq w$.

(i) *If* wPv *then* wPx *for all* $x \in l_w(w, v) \setminus \{w\}$, *and* wPx' *for all* $x' \leqslant x \in l_w(w, v) \setminus \{w\}$.

(ii) $[xPv$ *or* $xPw]$ *for all* $x \in \operatorname{conv}\{v, w\} \setminus \{v, w\}$, *and* $[x'Pv$ *or* $x'Pw]$ *for all* $x' \geqslant x$ *with* $x \in \operatorname{conv}\{v, w\} \setminus \{v, w\}$.

Proof: (i) Suppose wPv. By convexity of choice situations this immediately implies wPx for all $x \in \operatorname{conv}\{v, w\} \setminus \{w\}$. The case remains where $x \in l_w(w, v)$, x not between v and w. Let $S := \operatorname{conv}\{x, w\}$. If $F(S) \in \operatorname{conv}\{v, w\}$ then, by IIA, $F(S) = F(\operatorname{conv}\{v, w\}) = w$, so wPx. The case remains where $F(S) \notin \operatorname{conv}\{v, w\}$. We will show that this case cannot occur. By PC the function $y \mapsto F(\operatorname{conv}\{y, w\})$ is continuous on $l(v, w)$. Its image must be connected, so there is a $y \in \operatorname{conv}\{x, v\}$ such that $F(\operatorname{conv}\{y, w\}) = v$. This and $F(\operatorname{conv}\{v, w\}) = w$ contradict IIA. So everything concerning x in (i) has been proved. The result concerning x' follows from consideration of comv$\{x, w\}$.

(ii) Let x' be as in (ii) (possibly $x' = x$). If $x' \geqslant v$ or $x' \geqslant w$, then we are done. Otherwise, note that $\operatorname{conv}\{w, x'\} \cup \operatorname{conv}\{x', v\}$ is the Pareto optimal subset of $\operatorname{conv}\{w, x', v\}$. W.l.o.g. suppose $F(\operatorname{conv}\{w, x', v\}) \notin \operatorname{conv}\{w, x'\}$. By PC the function $y \mapsto F(\operatorname{conv}\{y, x', v\})$ from $\operatorname{conv}\{w, x'\}$ to $\operatorname{conv}\{w, x'\} \cup \operatorname{conv}\{x', v\}$ is continuous. Its image must be connected; hence $F(\operatorname{conv}\{y, x', v\}) = x'$ for some $y \in \operatorname{conv}\{x', w\}$. This implies $x'Pv$. *Q.E.D.*

Up to Theorem 4.8 we make the following assumption:

(4.1) $n = 2$ and F satisfies PO, IIA, and PC.

We will show that P has no cycles by induction based on Lemma 3.5, which says that there are no cycles of length 3. Fix a sequence $a, b, \ldots, y, x$ of length at least 4 with $aPbP \cdots PyPx$. We want to show: $x \not R a$. The induction hypothesis is that no cycles of length smaller than the length of $(a, b, \ldots, y, x)$ exist. This implies:

(4.2) For all v and w in this sequence with $vP \cdots Pw$ and not both $v = a$ and $w = x$, we have $w \not R v$. Further, $x \not R a$ if there are v and w in the sequence with w not the immediate successor of v and vPw.

Note that aPb and $x \not R b$. Again (3.1.a)–(3.1.b.7), distinguished in the proof of Lemma 3.5, are essentially all possible cases. Step 1 in the proof of Lemma 3.5 (in which only $x \not R b$ is used) implies the following lemma.

LEMMA 4.3: *In the cases* (3.1.a), (3.1.b.1), (3.1.b.3), *and* (3.1.b.4), *we have* $x\not{R}a$.

The remaining cases (3.1.b.2), (3.1.b.5), (3.1.b.6), and (3.1.b.7), are treated in the following lemmas.

LEMMA 4.4: *In case* (3.1.b.2): $x_1 < b_1 < a_1$, $x_2 > b_2 > a_2$, *x strictly below $l(a,b)$, we have* $x\not{R}a$.

PROOF: From Lemma 4.2(i) with a in the role of w and b in the role of v, it follows that (even) aPx. *Q.E.D.*

LEMMA 4.5: *Case* (3.1.b.6): $x_1 \geqslant a_1 > b_1$, $a_2 \leqslant x_2 < b_2$, *cannot occur.*

PROOF: yPx, $x \geqslant a$, and Lemma 3.1(ii) imply $y = F(\text{conv}\{a, y, x\})$. So yPa in contradiction with (4.2). *Q.E.D.*

LEMMA 4.6: *Case* (3.1.b.7): $b_1 < a_1$, $x_1 < a_1$, $b_2 > a_2$, $b_2 > x_2$, *x strictly above $l(a,b)$, cannot occur.*

PROOF: We consider all possible locations of y. If $y_1 \leqslant a_1$ and y on or below $l(a,b)$, then aPy in view of Lemma 4.2(i), so from (4.2) we obtain $x\not{R}a$. Since by (4.2) also $x\not{R}b$, a contradiction with Lemma 4.2(ii) follows. If $y_1 > a_1$, and y on or below $l(x,a)$, then xPa would by Lemma 4.2(i) imply xPy which is a contradiction. So $x\not{R}a$, but as before that is also impossible. If $y_2 \geqslant b_2$ and y on or above $l(a,b)$, then $b \in \text{comv}\{x, y\}$, so yPb by Lemma 3.1(ii) (since yPx), in contradiction with (4.2). If $y_2 \leqslant a_2$ and y on or above $l(x,a)$, then $a \in \text{comv}\{x, y\}$, so yPa (since yPx), in contradiction with (4.2). Also $y \geqslant a$ would imply the contradiction yPa. The only possibility left is: y strictly above $l(a,b)$, $y_2 < b_2$, $y_1 < a_1$. In that case, yPa or yPb by Lemma 4.2(ii), in contradiction with (4.2). *Q.E.D.*

LEMMA 4.7: *In case* (3.1.b.5): $b_1 < a_1$, $b_2 > a_2 > x_2$, *x strictly above $l(a,b)$, we have* $x\not{R}a$.

PROOF: Suppose xPa. Then $xPaPb \cdots Py$, and yPx. By the previous lemmas, yPx is excluded in all possible configurations except for the configuration described in this lemma, so $a_1 < x_1$, $a_2 > x_2 > y_2$, y strictly above $l(x,a)$. If z is the immediate predecessor of y, then $yPxPaPb \cdots Pz$ and zPy. Again, the only possible configuration for this is: $x_1 < y_1$, $x_2 > y_2 > z_2$, z strictly above $l(y,x)$. Repeating this argument we find for the final step $bPcP \cdots PzPyPxPa$ and aPb: $c_1 < b_1$, $c_2 > b_2 > a_2$, a strictly above $l(b,c)$. In particular, $b_1 > c_1 > \cdots > y_1 > x_1 > a_1 > b_1$, an obvious impossibility. *Q.E.D.*

Lemmas 3.4 and 3.5, and Lemmas 4.3–4.7, imply the following theorem.

THEOREM 4.8: *For $n = 2$, PO, IIA, and PC imply SARP.*

Samuelson (1948) and Rose (1958) essentially showed that PO and WARP suffice to exclude cycles, for a single-valued choice function defined on only 2-dimensional linear choice situations (i.e., budget sets of the form $\text{comv}\{(a,0),(0,b)\}$ where $a, b \in \mathbb{R}_+$). Theorem 4.8 extends this result to choice functions defined on nonlinear 2-dimensional budget sets, while weakening WARP to IIA.

The next question is whether Theorem 4.8 will still hold if $n > 2$. Gale (1960) has provided an example of a continuous demand function defined on 3-dimensional linear budget sets which satisfies PO and WARP but not SARP. In other words, the result of Rose (1958) mentioned before does not have to hold if there are more than 2 commodities. In the Appendix we will show that Gale's example can be extended to 3-dimensional nonlinear budget sets (our choice sets) as well. This can be done even with PC strengthened to full continuity:

DEFINITION 4.9: A choice function $F: \Sigma \to X$ is *continuous* if for every sequence $S, S_1, S_2, \ldots$ in Σ with $S_k \to S$ (where the limit is taken with respect to the Hausdorff metric) we have $F(S_k) \to F(S)$.

The appendix shows that WARP does not imply SARP for dimension $n = 3$, by extending the example of Gale (1960) to nonlinear choice sets. The extension to higher dimensions, also for linear budget sets, will be given in Peters and Wakker (1991). For linear budget sets a theoretical argument has already been given in Kihlstrom, Mas-Colell, and Sonnenschein (1976, first paragraph of page 975).

Another interesting question is whether IIA can be strengthened in an appealing way in order to imply SARP. For instance, for each dimension n, can one find a natural number $k(n)$ such that requiring the exclusion of cycles of length smaller than or equal to $k(n)$, instead of IIA, implies SARP? For linear budget sets the answer is negative, as follows from Shafer (1977). For our case the answer is also negative: this can be shown by extending Shafer's 3-dimensional example to nonlinear budget sets in the same way as is done in the Appendix with Gale's example.

5. REPRESENTATION OF REVEALED PREFERENCE

Let F be a choice function. $x \in X$ is *revealed preferred to* $y \in X$, notation $x\bar{R}y$, if there exists a sequence $x = x^0, x^1, \ldots, x^k = y$ in X with $x^0Rx^1R \cdots Rx^k$. If in this sequence x^iPx^{i+1} for some $i \in \{0, 1, \ldots, k-1\}$, x is *revealed strictly preferred to* y, notation $x\bar{P}y$. By Wakker (1989b, Corollary I.2.12, (vi) and (vii), and Theorem I.2.5, (ii) and (vi)), F satisfies SARP if and only if $\bar{P}$ is the asymmetric part of $\bar{R}$. Note that in our case, by Lemma 3.1(ii), if $x \neq y$ and $x\bar{R}y$, then $x\bar{P}y$.

Although it is not impossible that $\bar{R}$ is *complete* (i.e., $x\bar{R}y$ or $y\bar{R}x$ for all $x, y \in X$; for instance let $n = 2$ and $F = D^1$), this will in general not be the case. For instance, if $n = 2$ and F is the *Nash choice function* N (that is, $N(S)$ is the point of $S \in \Sigma$ where the product x_1x_2 is maximized over S), then neither $(1,2)\bar{R}(2,1)$ nor $(2,1)\bar{R}(1,2)$). Also, $\bar{R}$ does not have to be "representable" by a real-valued function on X; $f: X \to \mathbb{R}$ *represents* the binary relation $\succcurlyeq$ on X if $[x \succcurlyeq y \Rightarrow f(x) \geqslant f(y)]$ and $[x \succ y \Rightarrow f(x) > f(y)]$ for all $x, y \in X$, where $\succ$ is the asymmetric part of $\succcurlyeq$. For instance, if $\bar{R}$ is revealed by D^1 then $\bar{R}$ is the lexicographic order on X which is well-known not to be representable by a real-valued function.

The main purpose of this section is to find sufficient conditions for F such that the corresponding revealed preference relation $\bar{R}$ is representable by a real-valued function f. Such a function will be called a *utility function* (of the consumer, or the group of bargainers). It will be shown that f is strongly monotonic and strictly quasi-concave (see above Lemma 5.4). Up to Theorem 5.3 we assume:

(5.1) F is continuous and satisfies PO and SARP.

The following lemma can be derived from Corollary 1 in Jaffray (1975) applied to the transitive, asymmetric partial order $\bar{P}$.

LEMMA 5.1: *If there exists a countable subset A of X such that for all $x, y \in X$ with $x\bar{P}y$ there is an $a \in A$ with $x\bar{P}a\bar{P}y$, then there exists a function $f: X \to \mathbb{R}$ such that $[x\bar{P}y \Rightarrow f(x) > f(y)]$ for all $x, y \in X$.*

REMARK: Lemma 5.1 is a variation on a result by Debreu (1954, Lemma II); the latter holds for weak orders (transitive complete binary relations). Actually, given an enumeration $A = \{a_1, a_2, \ldots\}$ of the set A, a function f as in Lemma 5.1 is easily defined: $f: x \mapsto \sum_{k: x\bar{P}a_k} 2^{-k-1}$. See Jaffray (1975) for further details.

A set A as in Lemma 5.1 can be obtained as follows:

$$(5.2) \qquad A := \{a \in X: a = F(\text{conv}\{x, y\}) \text{ for some } x, y \in X \cap \mathbb{Q}^n\}.$$

LEMMA 5.2: *Let $x, y \in X$ with $x\bar{P}y$. Then there exists an $a \in A$ with $x\bar{P}a\bar{P}y$.*

PROOF: First assume xPy. Choose sequences $(x^j), (y^j) \subset X \cap \mathbb{Q}^n$ with $x^j \to x$, $y^j \to y$, and with for all $j: x^j < x$, $y^j > y$, and $\frac{1}{2}x^j + \frac{1}{2}y^j \in \text{comv}\{x, y\}$. By the continuity of F we have $F(\text{conv}\{x^j, y^j\}) \to F(\text{conv}\{x, y\}) = x$ which implies: there is some $k \in \mathbb{N}$ such that $a := F(\text{conv}\{x^k, y^k\}) \in \text{comv}\{x, y\}$. So $a \in A$, and xPa in view of Lemma 3.1(ii). Since $y \in \text{comv}\{a, y^k\}$, also aPy. So this point a has the desired properties.

Next assume $x\bar{P}y$. Then $x = x^0Rx^1 \cdots Rx^{j-1}Rx^j \cdots Rx^k = y$ with, say, ${}^{j-1}Px^j$. So by the first part of the proof we have $x^{j-1}PaPx^j$ for some $a \in A$, hence also $xPa\bar{P}y$. *Q.E.D.*

For an arbitrary choice function F and a real-valued function f on X, *F maximizes f* if $f(F(S)) > f(x)$ for every $S \in \Sigma$ and $x \in S$, $x \neq F(S)$.

THEOREM 5.3: *Let F be a Pareto optimal continuous choice function. Then the following two statements are equivalent*:

(a) *F satisfies SARP.*

(b) *F maximizes a real-valued function f on X.*

PROOF: Suppose F satisfies SARP. Then F satisfies condition (5.1), so by Lemmas 5.1 and 5.2 there is an $f: X \to \mathbb{R}$ with $x\bar{P}y \Rightarrow f(x) > f(y)$ for all $x, y \in X$. Since $F(S)Px$ for all $F(S) \neq x \in S$ and $S \in \Sigma$, F maximizes f. The implication (b) $\Rightarrow$ (a) is straightforward. *Q.E.D.*

Consequently, if the consumer's demand function, or the bargainers' solution, is continuous, Pareto optimal, and satisfies SARP, then the consumer chooses as if maximizing a utility function, and the bargainers reach a compromise as if maximizing a group utility function.

Next we will show that the function f in Theorem 5.3 is *strongly monotonic*, i.e., strictly increasing in each coordinate, and *strictly quasiconcave*, i.e., the set $\{y \in X: f(y) \geqslant f(x)\}$ is "strictly convex," for every $x \in X$. A set $T \subset X$ is *strictly convex* if $\alpha x + (1 - \alpha)y$ is an interior point of T whenever $x, y \in T$, $x \neq y$, $0 < \alpha < 1$.

LEMMA 5.4: *Let F be a Pareto optimal continuous choice function which maximizes a real-valued function f on X. Then f is strongly monotonic and strictly quasiconcave.*

PROOF: Let $x, y \in X$ with $x \geqslant y$, $x \neq y$. Then $F(\text{conv}\{x, y\}) = x$ by PO of F, so $f(x) > f(y)$. This proves strong monotonicity of f.

Next, for contradiction let $z \in X$ and $T := \{x \in X : f(x) \geqslant f(z)\}$. For convexity of T, let $x, x' \in T$ with $x \neq x'$ and $y = \alpha x + (1-\alpha)x'$ where $0 < \alpha < 1$. We have to show $f(y) \geqslant f(z)$. By Lemma 4.2(ii), we have yPx or yPx', so $f(y) > f(z)$ and $y \in T$.

Finally, suppose that T is not strictly convex. If $v \neq w \in T$ and $\mathrm{conv}\{v, w\}$ contains an interior point t of T, then by convexity of T all points in $\mathrm{conv}\{v,w\} \setminus \{v,w\}$ are interior. So the assumption that T is not strictly convex implies the existence of $v \neq w \in T$ such that $\mathrm{conv}\{v,w\}$ is a subset of the boundary of T. Let $F(\mathrm{conv}\{v,w\}) = v$ (otherwise continue the proof with $F(\mathrm{conv}\{v,w\})$ in the role of v if $F(\mathrm{conv}\{v,w\}) \neq w$, or with the roles of v and w reversed if $F(\mathrm{conv}\{v,w\}) = w$). Note that $f(v) > f(w) \geqslant f(z)$. Also, $f(x) < f(z)$ for every x in the interior of $\mathrm{comv}\{v,w\}$ since otherwise, by PO, $\mathrm{conv}\{v,w\}$ would contain an interior point of T. Let $v^1, v^2, \ldots \in X$ be a sequence in the interior of $\mathrm{comv}\{v,w\}$ converging to v. Then $F(\mathrm{conv}\{v^k, w\}) = w$ for every $k \in \mathbb{N}$ whereas $F(\mathrm{conv}\{v,w\}) = v$. This contradicts the continuity of F. *Q.E.D.*

LEMMA 5.5: *Let F be a choice function which maximizes a strongly monotonic and strictly quasiconcave real-valued function f on X. Then F is Pareto optimal and continuous.*

PROOF: Pareto optimality of F is an immediate consequence of strong monotonicity of f. Next suppose for contradiction that F is not continuous. Using compactness, subsequences, and IIA, we can arrange sequences $p, p^1, p^2, \ldots$ and $q, q^1, q^2, \ldots$ in X with $p^k \to p$, $q^k \to q$, $F(\mathrm{conv}\{p^k, q^k\}) = p^k$, $F(\mathrm{conv}\{p,q\}) = q$. From $f(q) > f(p)$ and strict quasiconcavity of f it follows that $\frac{1}{2}p + \frac{1}{2}q$ is an interior point of $\{x : f(x) \geqslant f(p)\}$; so $f(\frac{1}{2}p + \frac{1}{2}q) > f(p)$ by monotonicity of f. Similarly $3q/4 + p/4$ is an interior point of $\{x : f(x) \geqslant f(\frac{1}{2}p + \frac{1}{2}q)\}$; so by monotonicity of f there is a $\hat{q} < 3q/4 + p/4$ such that $f(\hat{q}) > f(\frac{1}{2}p + \frac{1}{2}q)$. Further, there is a $\hat{p} > p$ such that $\frac{1}{2}p + \frac{1}{2}q > \hat{w}$ for some $\hat{w} \in \mathrm{conv}\{\hat{p}, \hat{q}\}$. Then $f(\hat{q}) > f(\frac{1}{2}p + \frac{1}{2}q) > f(\hat{w})$. By strict quasiconcavity of $f : f(\hat{w}) \geqslant \min\{f(\hat{p}), f(\hat{q})\}$. We conclude that $f(\hat{p}) < f(\hat{q})$.

So $\hat{q} < 3q/4 + p/4$, $\hat{p} > p$, $f(\hat{p}) < f(\hat{q})$. Take $k \in \mathbb{N}$ so large that $p^k < \hat{p}$ and $\hat{q} \in \mathrm{comv}\{p^k, q^k\}$. Then $f(p^k) < f(\hat{p}) < f(\hat{q})$ whereas $F(\mathrm{conv}\{p^k, q^k\}) = p^k$. Since $\hat{q} \in \mathrm{comv}\{p^k, q^k\}$ this contradicts Lemma 3.1(ii). *Q.E.D.*

Theorem 5.3 and Lemmas 5.4 and 5.5 lead to the following theorem.

THEOREM 5.6: *For a choice function F the following two statements are equivalent*:

(a) *F is continuous and satisfies PO and SARP.*

(b) *F maximizes a strongly monotonic strictly quasiconcave real-valued function f on X.*

For $n = 2$, Theorems 4.8 and 5.3 imply the following corollary, which further illustrates the meaning of Nash's IIA.

COROLLARY 5.7: *Let $n = 2$ and let F be a Pareto optimal continuous choice function. Then the following two statements are equivalent*:

(a) *F satisfies IIA.*

(b) *F maximizes a real-valued function f on X.*

The function f in Theorem 5.6 may fail to be continuous. This can be inferred from the straightforward adaptation of Example 1 and Remark 4 in Hurwicz and Richter (1971) to our context.

We conclude this section with some consequences for bargaining game theory. If $n = 2$, a choice situation S may be interpreted as a 2-person *bargaining game* where bargainer i $(= 1, 2)$ derives utility x_i from a compromise $x \in S$; if no compromise is reached, then the game results in the *status quo alternative* $(0, 0)$. To solve their problem the bargainers may appeal to a choice function called (*bargaining*) *solution* in this context. This is essentially the model proposed by Nash (1950), who showed that the solution N defined before is the unique solution satisfying, besides some other properties, IIA. The results of this paper may contribute to clarifying the role of IIA. Theorem 2.2 and Corollary 5.7 characterize large classes of solutions with the IIA property. These solutions can be interpreted as generalizations of the Nash bargaining solution that allow for interaction between players, and payments in more realistic quantities than von Neumann-Morgenstern utilities. Similarly, Theorem 5.6 characterizes a large class of n-person solutions with the SARP property. Further discussion was given at the end of Section 2.

6. DOMAIN EXTENSIONS AND CONCLUDING REMARKS

The choice of domain $X = \mathbb{R}^n_{++}$ was made for convenience. Examples 3.6 and 3.7 (with any strictly quasiconcave strongly monotonic function instead of $x_1 x_2 x_3$ for case (c) below) can be adapted to the cases below in a straightforward manner. Also the example elaborated in the Appendix can be adapted to these cases (see there). Further we have the following:

(a) If $X = \mathbb{R}^n$, then all theorems and lemmas in this paper remain true.

(b) If $X = \mathbb{R}^n_+$, then all theorems and lemmas before Lemma 5.5 remain true. Lemma 5.5 and Theorem 5.6 are no longer valid. This case is the mathematically most deviating one since the (essential) domain is not open.

Details are as follows. In the first part of the proof of Lemma 5.2 allow x^j and y^j to have zero coordinates whenever the corresponding coordinates of both x and y are zero, and suppress these coordinates. Take any $x' = \mu x + (1 - \mu)y$ $(0 < \mu < 1)$. Then $xPx'Py$ by Lemma 4.2. Proceed with x' instead of x. In the last part of the proof of Lemma 5.4 the interior of $\text{comv}(\{v, w\})$ must be replaced by the interior relative to the subspace where those coordinates are zero that are zero for both v and w. The claim about (nonrelative) interior points of T then remains true. In the proof of Lemma 5.5, the point $\hat{q}$ does not have to exist.

(c) If $X = \mathbb{R}^n_+$, Σ is restricted to the sets T which contain a strictly positive point, $F(T) > 0$ for all T, and the function f and the revealed and representing binary relations are restricted to $\mathbb{R}^n_{++}$, then all theorems and lemmas of this paper remain true. (Note that all intersections required in the proof of Theorem 2.2 are in Σ.)

Case (b) is of interest for consumer demand theory. Case (c) can be used to derive the Nash bargaining solution from our results. One of the conclusions from this paper is that the IIA condition, combined with Pareto optimality and continuity, only has strong implications in the 2-dimensional case. This case is relatively important: bargaining situations often include two parties, and if there are more than two parties intermediate coalitions should usually be allowed, which restricts the importance of n-person pure bargaining games; in consumer demand theory, many situations can be modeled as involving only two goods by considering composite goods. Nevertheless it is unfortunate that, in general, we obtain the n-dimensional analogue only by strengthening IIA to SARP.

We conclude by indicating the relation between our results and those of Lensberg (1987). In a context where the dimension may vary and where a choice function is a (countably infinite) list of prescriptions (one for each dimension), Lensberg shows that a condition called multilateral stability is necessary and sufficient for a Pareto optimal continuous choice function to maximize an additively separable strictly quasiconcave function. Further, if the dimension may vary but has an upper bound of at least 3, then Lensberg shows that this result still holds under the weaker condition of bilateral stability. Additive separability excludes interactions between dimensions. It has been discussed in many contexts; see Wakker (1989b, Section II.5). In consumer theory where dimensions refer to commodities which may have physical interaction, and even more in group decision making where dimensions refer to individuals who may have social interaction, violations of additive separability are of considerable interest. This motivated the general approach of this paper.

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APPENDIX

In this appendix a choice function F is constructed for dimension $n=3$, satisfying continuity, PO, and IIA, but not SARP. F extends a demand function, proposed by Gale (1960) in order to show that WARP does not imply SARP if there are at least three goods. Let A be the matrix

$$\begin{bmatrix} -3 & 4 & 0 \\ 0 & -3 & 4 \\ 4 & 0 & -3 \end{bmatrix}.$$

For all (price) vectors $p, q \in X$ with (demand vectors) $Ap \geqslant 0$, $Aq \geqslant 0$ (cf. Gale, 1960, sect. 3):

(*) If $pAq \leqslant pAp$ and $qAp \leqslant qAq$, then $Ap = Aq$ ("WARP").

Let

$$B := A^{-1} = 1/37 \begin{bmatrix} 9 & 12 & 16 \\ 16 & 9 & 12 \\ 12 & 16 & 9 \end{bmatrix}.$$

Let $S \in \Sigma$ be fixed, and let $M := \{x \in S : \text{there is no } y \in S \text{ with } x_i = y_i \text{ for all } i \neq 1 \text{ and } x_1 < y_1\}$. For every $x \in M$ let $\pi(x) \in R^3_+$ be defined by $\pi(x)_i = x_i$ for all $i \neq 1$, $\pi(x)_1 = 0$, i.e., π is the projection on the hyperplane $x_1 = 0$. Then $\pi : M \to \pi(M)$ is a homeomorphism, and $\pi(M)$ is nonempty, compact, and convex. Further, for every $x > 0$ let $H(x)$ be the supporting hyperplane of S with normal x and such that S is below $H(x)$. Then the correspondence $I : x \mapsto H(x) \cap S = H(x) \cap P(S)$ for every $x > 0$ is upper semicontinuous (as can be shown directly, or as a consequence of the Maximum Theorem).

Finally, let the correspondence $\mu : \pi(M) \to \pi(M)$ be defined by

$$\mu(x) = \pi\big(I\big(B\big(\pi^{-1}(x)\big)\big)\big) \qquad \text{for every } x.$$

Then clearly μ is convex-valued and upper semicontinuous, so by Kakutani's fixed point theorem there exists a fixed point $x^* \in \mu(x^*)$.

Next we show that such a fixed point x^* is unique. Suppose $z^* \in \mu(z^*)$ is another fixed point. Then $\pi^{-1}(x^*) \in I(B(\pi^{-1}(x^*)))$ and $\pi^{-1}(z^*) \in I(B(\pi^{-1}(z^*)))$. So by definition of I:

$$\big(B\pi^{-1}(z^*)\big)\pi^{-1}(x^*) \leqslant \big(B\pi^{-1}(z^*)\big)\pi^{-1}(z^*) \qquad \text{and}$$

$$\big(B\pi^{-1}(x^*)\big)\pi^{-1}(z^*) \leqslant \big(B\pi^{-1}(x^*)\big)\pi^{-1}(x^*).$$

Hence

$$\big(B\pi^{-1}(z^*)\big)A\big(B\pi^{-1}(x^*)\big) \leqslant \big(B\pi^{-1}(z^*)\big)A\big(B\pi^{-1}(z^*)\big) \qquad \text{and}$$

$$\big(B\pi^{-1}(x^*)\big)A\big(B\pi^{-1}(z^*)\big) \leqslant \big(B\pi^{-1}(x^*)\big)A\big(B\pi^{-1}(x^*)\big).$$

From these inequalities and (*), we conclude $\pi^{-1}(x^*) = \pi^{-1}(z^*)$, and so $x^* = z^*$.

Let F assign the point $\pi^{-1}(x^*)$ to every choice situation, with x^* the unique fixed point as above. Then F is a well-defined choice function. PO and IIA of F follow straightforwardly from its definition. Next we prove that F is continuous.

Let $S, S_1, S_2, \ldots \in \Sigma$, $S_i \to S$ in the Hausdorff-metric, and $F(S_i) = y^i \to y \in S$. For every i let $p^i = B(y^i)$; by construction, p^i is a normal of a supporting hyperplane of S_i at y^i. Since $y^i \to y$, we have $B(y^i) \to B(y) := p$, so $p^i \to p$. It is straightforward to show that $\{x : px = py\}$ supports S at y. (Incidentally, it also follows that $p = B(y) > 0$, since all entries of B are positive. Hence $y \in P(S)$.) So $\pi^{-1}(y)$ is the fixed point of μ, and $F(S) = y$ follows.

Finally, a violation of SARP is obtained, adapting the example of Gale (1960, Section 5). The following observation will be used. Let $S \in \Sigma$ be such that $P(S) \subset \{x \in X : px = c\}$ for some vector $p > 0$ and some constant $c > 0$. If the point $c(pAp)^{-1}Ap$ is an element of $P(S)$, then by construction of F it is equal to $F(S)$. We now turn to the example.

Let $x^1 = (1, 0.001, 0.001)$, $x^2 = (0.6, 0.001, 0.3)$, $x^3 = (0.3, 0.001, 0.6)$, $x^4 = (0.001, 0.001, 1)$, and let $p^1 = (9.028, 16.021, 12.025)$, $p^2 = (10.212, 13.209, 9.916)$, $p^3 = (12.312, 12.009, 9.016)$, $p^4 = (16.021, 12.025, 9.028)$. Then each x^i is a multiple of Ap^i. Further, we have:

$$p^1x^1 > p^1x^2, \quad \text{so} \quad F(\operatorname{conv}\{x^1, x^2\}) = x^1,$$

$$p^2x^2 > p^2x^3, \quad \text{so} \quad F(\operatorname{conv}\{x^2, x^3\}) = x^2,$$

$$p^3x^3 > p^3x^4, \quad \text{so} \quad F(\operatorname{conv}\{x^3, x^4\}) = x^3.$$

So x^1 is revealed preferred to x^4, i.e., $(1, 0.001, 0.001)$ is revealed preferred to $(0.001, 0.001, 1)$. By interchanging the appropriate numbers one similarly shows that $(0.001, 0.001, 1)$ is revealed preferred to $(0.001, 1, 0.001)$, and that $(0.001, 1, 0.001)$ is revealed preferred to $(1, 0.001, 0.001)$. So F violates SARP.

The above construction holds for the prevalent case in the paper where all choice situations are strictly positive. We conclude this Appendix by modifying the construction for the cases (a)–(c) in Section 6. We construct G as follows. Let $c > 0$ be a constant such that the set

$$C = \{x \in \mathbb{R}^3 : x > 0,\ x_1x_2x_3 \geqslant c\}$$

contains all the points needed for the construction of the cycle above. Now for $S \in \Sigma$, let $G(S) := F(S \cap C)$ if $S \cap C \neq \varnothing$. If $S \cap C = \varnothing$, then let $\alpha_S > 0$ be the minimal number such that $S \cap (C - \alpha_S(1,1,1)) \neq \varnothing$, and let $G(S)$ be the (unique!) point in this intersection. This G is continuous and satisfies PO and IIA, but not SARP. It can be used in the cases (a) and (b) in Section 6. For the case (c) there, take, instead, the choice function G' with $G'(S) = G(S)$ if $S \cap C \neq \varnothing$, and $G'(S)$ is the unique point of S with maximal product of the coordinates, otherwise.

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