Selecting Copulas for Risk Management*

Erik Kole[†a], Kees Koedijk[b,c], and Marno Verbeek[b]

[a] *Econometric Institute, Erasmus School of Economics and Business Economics, Erasmus University Rotterdam, The Netherlands*
[b] *Dept. of Financial Management, RSM Erasmus University, Erasmus University Rotterdam, The Netherlands*
[c] *CEPR, London, UK*

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Abstract

Copulas offer financial risk managers a powerful tool to model the dependence between the different elements of a portfolio and are preferable to the traditional, correlation-based approach. In this paper we show the importance of selecting an accurate copula for risk management. We extend standard goodness-of-fit tests to copulas. Contrary to existing, indirect tests, these tests can be applied to any copula of any dimension and are based on a direct comparison of a given copula with observed data. For a portfolio consisting of stocks, bonds and real estate, these tests provide clear evidence in favor of the Student's t copula, and reject both the correlation-based Gaussian copula and the extreme value-based Gumbel copula. In comparison with the Student's t copula, we find that the Gaussian copula underestimates the probability of joint extreme downward movements, while the Gumbel copula overestimates this risk. Similarly we establish that the Gaussian copula is too optimistic on diversification benefits, while the Gumbel copula is too pessimistic. Moreover, these differences are significant.

Key words: financial dependence, risk management, copulas, distributional tests, tail dependence
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†Corresponding author: P.O.Box 1738, 3000 DR Rotterdam, The Netherlands, Tel. +31 10 408 12 58. E-mail addresses: `kole@few.eur.nl` (E. Kole), `ckoedijk@rsm.nl` (K. Koedijk), `mverbeek@rsm.nl` (M. Verbeek)

1 Introduction

Modelling dependence is of key importance to portfolio construction and risk management. An inappropriate model for dependence can lead to suboptimal portfolios and inaccurate assessments of risk exposures. Traditionally, correlation is used to describe dependence between random variables, but recent studies have ascertained the superiority of copulas to model dependence, as they offer much more flexibility than the correlation approach (see e.g. Embrechts et al., 2002). An important reason to consider other copulas than the correlation-implied Gaussian copula is the failure of the correlation approach to capture dependence between extreme events, as shown by Longin and Solnik (2001), Bae et al. (2003) and Hartmann et al. (2004). However, up to now no consensus has been reached on which copula to use in specific applications or on how to test the accuracy of a specific copula.

In this paper we propose an approach to evaluate copulas and investigate the importance of accurate copula selection. Generally, theory offers little guidance in choosing a copula, making the selection an empirical issue. Since a copula is equivalent to a distribution function, we discuss how traditional goodness-of-fit tests such as the Kolmogorov-Smirnov test and the Anderson-Darling test can be applied. These tests are based on a direct comparison of the dependence implied by the copula with the dependence observed in the data. These direct tests of the fit of a copula have several advantages over alternative approaches proposed in the literature. First, they are applicable to any copula, not only to the Student's t and Gaussian copulas. Second, they can be used for copulas of any dimension, not only for bivariate copulas. Third, they indicate whether a copula captures the observed dependence accurately, and not only whether it can be rejected against another specific copula. Finally, they take the complete dependence into account, contrary to selection procedures that consider only part of the dependence pattern (i.e. dependence of extreme observations).

To determine the importance of selecting the right copula for risk management, we consider a portfolio of stocks, bonds and real estate. As investors are generally averse to downside risk, a copula should capture both the risk of joint downward movements of asset prices, and the diversification opportunities that assets offer. This is particularly relevant in the case of stocks, bonds and real estate, as a proper allocation over these assets should lead to the main risk reduction in investments. Therefore, we test the Gaussian, the Student's t

and the Gumbel copulas to model the dependence of the daily returns on indexes that approximate these three asset classes. The Gaussian copula is the traditional candidate for modelling dependence. The Gumbel copula is directly related to multivariate extensions of extreme value theory, which has gained popularity in risk management over the last decade (see e.g. Longin, 1996). The Student's t copula can be seen as a compromise, because it can capture dependence in the tails without giving up flexibility to model dependence in the center.

In our application, the Student's t copula passes the tests with success, but both the Gaussian and Gumbel copulas are rejected. To stress the economic importance of copulas in this application, we show that the different copulas lead to significantly different assessments both of the risk of downward movements and of diversification benefits. We examine the risk of downward movements by stress tests. The Student's t copula leads to probabilities that do not differ significantly from the empirical copula. On the contrary, the Gaussian copula significantly underestimates the risk of joint downward movements and the Gumbel copula overestimates it. With Value-at-Risk computations we establish that the Gaussian copula is too optimistic on the diversification benefits of the assets, while the Gumbel copula is too pessimistic. The differences are pronounced and stress the importance of copula selection.

The contributions of this article are threefold. First, we show that the impact of copulas on the risk management of asset portfolios is substantial. While the impact of copulas has been studied in relation to option pricing (see e.g. Frey and McNeil, 2003; Mashal et al., 2003; Hamerle and Rösch, 2005), the term structure of interest rates (see Junker et al., 2006) and credit risk (see Giesecke, 2004; Meneguzzo and Vecchiato, 2004), knowledge on the consequences of copulas for portfolios of linear assets is limited. Poon et al. (2004) address this issue, but only consider Gaussian and Gumbel copulas. Moreover, their analysis only considers pairwise dependence. Second, we add to the literature on diversification breakdown (see e.g. Loretan and English, 2000; Campbell et al., 2002; Ang and Chen, 2002), as we find that the traditional correlation approach overstates diversification benefits, most notably for assets with a low or negative correlation. Third, we provide straightforward tests for the fit of copulas, that are easy to implement in comparison with Fermanian (2005) and Chen et al. (2004).

The remainder of this article is structured as follows. Section 2 discusses the tests and their application to the Gaussian, the Student's t and the Gumbel copulas. In Section 3 we tests the different copulas and investigate their consequences for stress testing and gauging diversification benefits. Section 4 concludes.

2 Goodness-of-fit tests for copulas

In this section we explain how the Kolmogorov-Smirnov and Anderson-Darling tests can be implemented for copulas. We start with a short introduction on copulas.[1] In the second subsection we present the tests. The third subsection discusses the implementation.

2.1 Copulas

Dependence between random variables can be modelled by copulas. A copula returns the joint probability of events as a function of the marginal probabilities of each event. This makes copulas attractive, as the univariate marginal behavior of random variables can be modelled separately from their dependence. For a random vector $\boldsymbol{X}$ of size n with marginal cumulative density functions (cdf) F_i, the copula with cdf $C(\cdot)$ gives the cumulative probability for the event $\boldsymbol{x}$:

$$P(\boldsymbol{X} \leq \boldsymbol{x}) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

The applicability of copulas is wide, as Sklar (1959) proves that each multivariate distribution with continuous marginals has a unique copula representation. Moreover, any function $C : [0,1]^n \to [0,1]$ satisfying some regularity restrictions implies a copula.[2]

Tail dependence is an important property of copulas. It describes the behavior of copulas when the value of the marginal cdf F_i reaches its bounds of zero (lower tail dependence) or one (upper tail dependence) and is defined as the limiting probability that a subset of the variables in X has extreme values, given that the complement has extreme values.[3] If the limiting probability equals zero, a copula exhibits tail independence; if the probability exceeds zero it exhibits tail dependence.

[1] A more rigorous treatment of copulas can be found in Joe (1997) and Nelsen (1999). For a discussion applied to finance we refer to Bouyé et al. (2000) and Cherubini et al. (2004).

[2] See Definition 1 in Embrechts et al. (2002).

[3] Joe (1997) Sec. 2.1.10 gives a definition for the bivariate case, which is generalized by Schmidt and Stadmüller (2006) to $n > 2$ dimensions.

The traditional use of correlation to model dependence implies using the Gaussian copula[4] which has cdf:

$$C_n^{\Phi}(\boldsymbol{u};\boldsymbol{\Omega}^{\Phi}) = \Phi_n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n);\boldsymbol{\Omega}^{\Phi}), \tag{2}$$

where $\boldsymbol{u}$ is a vector of marginal probabilities, Φ_n denotes the cdf for the n-variate standard normal distribution with correlation matrix $\boldsymbol{\Omega}^{\Phi}$, and Φ^{-1} is the inverse of the cdf for the univariate standard normal distribution. For imperfectly correlated variables, the Gaussian copula implies tail independence (see Embrechts et al., 2002).

Closely related to the Gaussian copula is the Student's t copula, with cdf:

$$C_n^{\Psi}(\boldsymbol{u};\boldsymbol{\Omega}^{\Psi},\nu) = \Psi_n(\Psi^{-1}(u_1;\nu),\ldots,\Psi^{-1}(u_n;\nu);\boldsymbol{\Omega}^{\Psi},\nu), \tag{3}$$

where Ψ_n denotes the cdf of an n-variate Student's t distribution with correlation matrix $\boldsymbol{\Omega}^{\Psi}$ and degrees of freedom parameter $\nu > 2$, and Ψ^{-1} is the inverse of the cdf for the univariate Student's t distribution with mean zero, dispersion parameter equal to one and degrees of freedom ν. The Gaussian and Student's t copula belong to the class of elliptic copulas. A higher value for ν decreases the probability of tail events. As the Student's t copula converges to the Gaussian copula for $\nu \to \infty$, the Student's t copula assigns more probability to tail events than the Gaussian copula. Moreover, the Student's t copula exhibits tail dependence (even if correlation coefficients equal zero).

The third copula we consider in the paper is the Gumbel copula, which belongs to the class of Archimedean copulas. The Gumbel copula is an extreme value copula.[5] Its standard cdf is given by

$$C_n^{\mathrm{G}}(u;a) = \exp\left(-\left(\sum_{i=1}^{n}(-\log u_i)^a\right)^{1/a}\right), \tag{4}$$

with $a \geq 1$, where $a = 1$ implies independence. Because the standard Gumbel copula implies the same dependence between all combinations of marginal variables u_i, we use the extension proposed by Bouyé (2002). He uses a recursive definition, in which the

[4]Correlation can always be used as a dependence *measure*. However, if correlation is used as a *model*, i.e. a complete characterization, of dependence it implies the Gaussian copula.

[5]Joe (1997) provides a detailed, general discussion of extreme value theory in relation to copulas, while Bouyé (2002) discusses it from a risk management perspective.

dependence of the marginal probability u_{i+1} with the preceding marginal probabilities $u_1, \ldots, u_i$ is characterized by a specific parameter a_i:

$$C_n^{\mathrm{B}}(u_1, \ldots, u_n; a_1, \ldots, a_{n-1}) = \begin{cases} C_2^{\mathrm{G}}(u_1, u_2; a_1) & \text{if } n = 2 \\ C_2^{\mathrm{G}}(C_{n-1}^{\mathrm{B}}(u_1, \ldots, u_{n-1}; a_1, \ldots, a_{n-2}), u_n; a_{n-1}) & \text{if } n > 2, \end{cases} \quad (5)$$

with $a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq 1$. $C_2^{\mathrm{G}}()$ denotes the standard bivariate Gumbel copula as defined in Eq. (4). The restrictions on the a's impose a descending dependence order: the dependence between u_1 and u_2, governed by a_1, is at least as strong as the dependence between u_1 and u_2 on the one hand and u_3 on the other, governed by a_2. The ordering of the variables is therefore important. The Gumbel copula exhibits upper tail dependence but lower tail independence, which can be reversed by using the survival copula.[6]

2.2 Test statistics for the fit of copulas

The tests we propose belong to the large class of goodness-of-fit tests for distributions. Suppose that we want to test whether a specific distribution for a random variable accurately fits the corresponding observations. Under the hypothesis that this is the case, the empirical cumulative distribution of the observations F_{E} will converge to the hypothesized cumulative distribution F_{H} almost surely, as stated by the Glivenko-Cantelli theorem (see Mittelhammer, 1996, p. 313). Therefore, we can use the deviations of the empirical distribution from the hypothesized distribution to test the fit. Let $\boldsymbol{x}_t$ be a realization of the random variable $\boldsymbol{X}$ out of sample of T realizations. We propose the following four

[6]The cumulative joint probability of events u is calculated by the survival copula: $P(\boldsymbol{U} \leq \boldsymbol{u}) = \bar{C}(\boldsymbol{\imath}_n - \boldsymbol{u})$, where $\bar{C}$ denotes the joint survival function. For a random vector $\boldsymbol{X}$ with (multivariate) density function $F(\boldsymbol{x})$ (not necessarily a copula) the joint survival function is defined as $\bar{F}(\boldsymbol{x}) = P(\boldsymbol{X} \geq \boldsymbol{x})$. Joe (1997) (p. 10, item 39) gives the general formula that relates $\bar{F}$ to F (e.g. for the two dimensional case $\bar{F}(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2)$, where F_i denotes a marginal distribution). For an extensive discussion of survival copulas and survival functions we refer to Cherubini et al. (2004), Ch. 2.5 and 4.4.

statistics:

$$D_{\mathrm{KS}}^{\mathrm{m}} = \max_t |F_{\mathrm{E}}(\boldsymbol{x}_t) - F_{\mathrm{H}}(\boldsymbol{x}_t)|; \tag{6}$$

$$D_{\mathrm{KS}}^{\mathrm{a}} = \int_{\boldsymbol{x}} |F_{\mathrm{E}}(\boldsymbol{x}) - F_{\mathrm{H}}(\boldsymbol{x})| \,\mathrm{d}F_{\mathrm{H}}(\boldsymbol{x}); \tag{7}$$

$$D_{\mathrm{AD}}^{\mathrm{m}} = \max_t \frac{|F_{\mathrm{E}}(\boldsymbol{x}_t) - F_{\mathrm{H}}(\boldsymbol{x}_t)|}{\sqrt{F_{\mathrm{H}}(\boldsymbol{x}_t)(1 - F_{\mathrm{H}}(\boldsymbol{x}_t))}}; \tag{8}$$

$$D_{\mathrm{AD}}^{\mathrm{a}} = \int_{\boldsymbol{x}} \frac{|F_{\mathrm{E}}(\boldsymbol{x}) - F_{\mathrm{H}}(\boldsymbol{x})|}{\sqrt{F_{\mathrm{H}}(\boldsymbol{x})(1 - F_{\mathrm{H}}(\boldsymbol{x}))}} \,\mathrm{d}F_{\mathrm{H}}(\boldsymbol{x}). \tag{9}$$

The first distance measure is commonly referred to as the Kolmogorov-Smirnov distance, of which the second is an average. The third distance measure is known as the Anderson-Darling distance after Anderson and Darling (1952), and the fourth is again an average of it. The Kolmogorov-Smirnov distances are more sensitive to deviations in the center of the distribution, whereas the Anderson-Darling distances give more weight to deviations in the tails. Originally, the measures focus on the largest deviation in a sample but to get more complete information on the goodness-of-fit the average can be used as well. To reduce the influence of outliers in the Anderson-Darling distances, we follow Malevergne and Sornette (2003) by replacing the original $(F_{\mathrm{E}}(\boldsymbol{x}_t) - F_{\mathrm{H}}(\boldsymbol{x}_t))^2$ term by $|F_{\mathrm{E}}(\boldsymbol{x}_t) - F_{\mathrm{H}}(\boldsymbol{x}_t)|$. The distributions of the statistics under the null hypothesis are non-standard. Moreover, the parameters for the hypothesized distribution are often estimated on the same data. Therefore, simulations are necessary to evaluate the test statistics.[7]

One way to test the fit of a specific copula is to derive the test statistics directly, by transforming each observation to the corresponding marginal probabilities, based on which the distance measures are then calculated. The hypothesized and empirical copulas take the place of F_{H} and F_{E}, respectively. The empirical copula C_E based on a sample $\mathcal{X}$ gives the joint probability for a vector of marginal probabilities u as follows:

$$C_E(\boldsymbol{u}; \mathcal{X}) = \frac{1}{T} \sum_t^T I(x_{1,t} \leq x_1^{\lfloor u_1 \cdot T \rfloor}) \cdot \ldots \cdot I(x_{n,t} \leq x_n^{\lfloor u_n \cdot T \rfloor}), \tag{10}$$

where $I(\cdot)$ is the indicator function, which equals 1 if the statement in parentheses is true and zero otherwise, and $x_j^{\lfloor u_j \cdot T \rfloor}$ is the k^{th} (ascending) order statistic, k being the largest integer not exceeding $u_j \cdot T$.

[7] If the parameters are not estimated but known, it is possible to derive multivariate goodness-of-fit test statistics that follow a standard distribution (see e.g. Khmaladze, 1993; Cabaña and Cabaña, 1997).

Inspired by Malevergne and Sornette (2003) we propose a slightly different approach for elliptic copulas. As the cumulative distribution functions of elliptic distributions are generally not available in closed form, calculation of the hypothesized probabilities will be computationally demanding if the number of dimensions increases. We use a faster procedure and evaluate the fit of elliptic copulas in terms of the fit of a univariate random variable. This approach is based on the property that the density functions of elliptic distributions are constant on ellipsoids. Each elliptically distributed random variable implies a univariate random variable with a specific distribution that corresponds with the radii of the ellipsoids of constant density (see Fang et al., 1990, for a formal treatment). Instead of considering the observation itself we consider the squared radius of the ellipsoid of constant density that it implies. We compare the empirical distribution of the squared radii with their theoretical distribution, which are standard distributions in case of the Gaussian and Student's t copula.

For a random vector $\boldsymbol{U} = (U_1, \ldots, U_n)'$ with marginal uniform distributions on $[0, 1]$ and dependence given by the Gaussian copula with correlation matrix $\boldsymbol{\Omega}^{\Phi}$, we construct the squared radius as:

$$Z_{\Phi} = \tilde{\boldsymbol{U}}'(\boldsymbol{\Omega}^{\Phi})^{-1}\tilde{\boldsymbol{U}}, \tag{11}$$

where $\tilde{\boldsymbol{U}} = (\Phi^{-1}(U_1), \ldots, \Phi^{-1}(U_n))'$ and $\Phi^{-1}(\cdot)$ is the inverse of the standard normal cdf. The random variable Z_{Φ} has a χ^2_n-distribution. This follows easily upon realizing that $\tilde{\boldsymbol{U}}$ has a normal distribution with correlation matrix $\boldsymbol{\Omega}$, which makes Z_{Φ} the sum of n squared random variables that are independently, standard normally distributed. So, starting with a sample having uniform marginal distributions, we transform each observation $\boldsymbol{u}$ to $z = \tilde{\boldsymbol{u}}'(\Omega^{\Phi})^{-1}\tilde{\boldsymbol{u}}$ and calculate its associated cumulative probability by the cdf of the χ^2_n-distribution.

For the Student's t copula we use a similar transformation. Let $\boldsymbol{V} = (V_1, \ldots, V_n)'$ be a random vector with each V_i being uniformly distributed on $[0, 1]$ and whose dependence is given by the Student's t copula with correlation matrix $\boldsymbol{\Omega}^{\Psi}$ and degrees of freedom ν. Now we construct the squared radius as

$$Z_{\Psi} = \tilde{\boldsymbol{V}}'(\boldsymbol{\Omega}^{\Psi})^{-1}\tilde{\boldsymbol{V}}/n, \tag{12}$$

where $\tilde{\boldsymbol{V}} = (\Psi^{-1}(V_1; \nu), \ldots, \Psi^{-1}(V_n; \nu))'$ and $\Psi^{-1}(V_j; \nu)$ is the inverse function of the standard Student's t distribution with degrees of freedom parameter ν. The variable Z_{Ψ} is distributed according to an F-distribution with degrees of freedom parameters n and ν. Note that the variable $\tilde{\boldsymbol{V}}$ has a Student's t distribution and can therefore be written as $\boldsymbol{W}/\sqrt{S/n}$, with $\boldsymbol{W}$ being an n-dimensional normally distributed random variable with correlation matrix $\boldsymbol{\Omega}^{\Psi}$ and S being a univariate random variable with a χ^2_ν-distribution. Consequently, we can write

$$Z_\Psi = \frac{\boldsymbol{W}'(\Omega^\Psi)^{-1}\boldsymbol{W}/n}{S/\nu},$$

which makes Z_Ψ the ratio of two χ^2-distributed variables divided by their respective degrees of freedom. Therefore, it has a Snedecor's $F_{n,\nu}$ distribution. So when we test the Student's t copula, we start with a sample having uniform marginal distributions, transform each observation $\boldsymbol{v}$ to $\tilde{\boldsymbol{v}}'(\Omega^\Psi)^{-1}\tilde{\boldsymbol{v}}/n$ and calculate the cumulative probability with the cdf of the $F_{n,\nu}$ distribution.

2.3 The procedure

Suppose that we want to use a specific copula with cdf C and parameters $\boldsymbol{\theta}$ to model the dependence of a random variable $\boldsymbol{X}$ for which we have a sample available of size T. The procedure that we propose to evaluate the fit of this copula consist of four steps:

Estimation step We estimate the parameters $\boldsymbol{\theta}$. In general two approaches can be used for estimating copula parameters. For our test procedure we advocate the inference functions for margins method (IFM) (see Joe, 1997, Ch. 10). In this two-step approach the parameters for the marginal models are estimated first. In the second step, the copula parameters are estimated with the marginal distribution parameters treated as given.[8] It is also possible to apply maximum likelihood to jointly estimate the parameters for the marginal models and the copula. The IFM is less efficient than one-step maximum likelihood, but it is computationally more attractive and allows larger flexibility in the estimation techniques for the marginal models.

Evaluation step We evaluate the fit of the copula with the estimated parameters by calculating the four distance measures of the previous subsection. If the copulas belong to the elliptical family, we propose to base the calculation on a transformation

[8]The resulting estimators $\hat{\boldsymbol{\theta}}$ belong to the general class of sequential estimators (see Newey, 1984).

of the uniform marginals. We use $\hat{d}^{\mathrm{m}}_{\mathrm{KS}}$, $\hat{d}^{\mathrm{a}}_{\mathrm{KS}}$, $\hat{d}^{\mathrm{m}}_{\mathrm{AD}}$ and $\hat{d}^{\mathrm{a}}_{\mathrm{AD}}$ to refer to the distance measures for the original sample.

Simulation step To test whether the distance measures provide evidence against the fit of the copula, we need to construct the distribution of the distance measures under the null hypothesis of accurate fit. Given the form of the distance measures and the fact that the parameters of the copula are not known but estimated, simulations have to be used. For each simulation, we generate a random sample of size T from the copula with parameters $\hat{\boldsymbol{\theta}}$.[9] We apply the previously mentioned estimation and evaluation step on this simulated sample (and find a new estimate for $\boldsymbol{\theta}$, which will be close to $\hat{\boldsymbol{\theta}}$ in expectation though not necessarily equal). Each simulation yields new values for the distance measures. Combined, the simulations result in a distribution of random variables corresponding to $\hat{d}^{\mathrm{m}}_{\mathrm{KS}}$, $\hat{d}^{\mathrm{a}}_{\mathrm{KS}}$, $\hat{d}^{\mathrm{m}}_{\mathrm{AD}}$ and $\hat{d}^{\mathrm{a}}_{\mathrm{AD}}$.

Test step Finally, we use the distribution that results from the simulation step to judge the values $\hat{d}^{\mathrm{m}}_{\mathrm{KS}}$, $\hat{d}^{\mathrm{a}}_{\mathrm{KS}}$, $\hat{d}^{\mathrm{m}}_{\mathrm{AD}}$ and $\hat{d}^{\mathrm{a}}_{\mathrm{AD}}$, by determining their p-value. p-values below the commonly used thresholds of 10%, 5% or 1% lead to rejection of the fit of the copula on that sample.

This procedure can be implemented straightforwardly. Note that the estimation step within the simulation step should be applied to the marginal parameters as well. If, for example, the empirical distributions are used to model the marginal distributions of the original sample, they should be used for the simulated sample, too.

3 Applying copulas in risk management

In this section we consider three copulas to model the dependence between the returns on the main asset classes considered for asset allocation, being the returns on stocks, bonds and real estate. The risk of a portfolio is directly related to the dependence between the portfolio's constituents. Consequently the model that is used for dependence is of key importance for portfolio management and portfolio selection. In this section we use the tests of the previous section to choose among the Gaussian, the Student's t, and the

[9]Simulation techniques for copulas can be found in Bouyé et al. (2000) and Cherubini et al. (2004), Ch. 6. General simulation techniques are discussed in Devroye (1986). Aas (2004) discusses a specific simulation technique for Gumbel copulas.

Gumbel copula. We provide empirical evidence of the importance of choosing the right copula by applying these copulas in stress testing and to determine diversification benefits.

Our main motivation to consider these three copulas is investors' sensitivity to downside risk.[10] Investors are particularly averse to extreme negative returns. Consequently, the dependence model should capture the risk entailed by the joint tail behavior of returns, without failing to exploit the diversification possibilities represented by the center of the return distribution. The Gaussian copula, the traditional method to model dependence, mostly reflects dependence in the center of the distribution and implies tail independence. The Gumbel copula mostly reflects tail dependence. Being an extreme value copula, it extends the successful univariate extreme value theory techniques in risk management, as shown by Longin (1996) and Jansen et al. (2000). In their study of dependence of extreme returns Longin and Solnik (2001) and Poon et al. (2004) also use the Gumbel copula. The Student's t copula can capture both dependence in the center and the tails of the distribution, and has been proposed as an alternative to the Gaussian copula by several authors including Glasserman et al. (2002), Campbell et al. (2003), Mashal et al. (2003), Valdez and Chernih (2003) and Meneguzzo and Vecchiato (2004).

The following subsection introduces the data. We briefly discuss how the marginal distributions for each return can be modelled. In the second subsection we test the fit of the Gaussian, the Student's t and the Gumbel copulas. The subsequent subsection analyzes the dependence in the tails for stress testing. Subsection four discusses the implications of the different copulas for diversification benefits.

3.1 Data and marginal models

We use indexes to proxy for the returns on stocks, bonds and real estate: Standard & Poor's 500 Composite Index (stocks), JP Morgan's US Government Bond Index (bonds) and the NAREIT All Index (real estate). We base the analysis on daily returns to ensure that enough observations of the tails of the distributions are available. We collect data from DataStream over the period January 1, 1999 to December 17, 2004. Excluding non-trading days the sample consist of 1499 returns.

[10]See Kahneman and Tversky (1979) and Tversky and Kahneman (1991) for a general discussion. Benartzi and Thaler (1995) and Berkelaar et al. (2004) discuss the implications of downside risk aversion from a finance perspective.

Table 1 presents summary statistics on the returns in the sample. Over our sample period, real estate yielded the highest average return equal to a significant average return of 0.065% per day, or 16.4% per year. Bonds come in second with a significant average return of 0.022% per day. The average return on stocks is insignificant. Measured by volatility, stocks are most risky, followed by real estate and then bonds. All return series exhibit fat tails as indicated by the kurtosis estimates. Kurtosis is highest for real estate, implying that extreme returns on real estate have a relatively high probability of occurrence. The estimates for the tail indices also point at fat tails. A distribution is fat tailed if the hypothesis $1/\alpha = 0$ is rejected in favor of the alternative $1/\alpha > 0$, where α reflects the tail index. The tail index estimates are based on the modified Hill-estimator developed by Huisman et al. (2001) because of its unbiasedness.[11] For each asset class we reject the hypothesis $1/\alpha = 0$, both for the left tail index α_{l} and the right tail index α_{r}. For real estate the left tail is significantly fatter than the right tail. For stocks and bonds we do not find significant differences between the index of the left tail and the index of the right tail.

The semi-parametric method of Daníelsson and de Vries (2000) can be used to model the marginal distributions.[12] They model the center of the distribution by the empirical distribution function and rely on univariate extreme value theory to model the tails. This method enables a combination of the good approximation of the center by the actual distribution that the empirical distribution provides, and the statistical rigor from extreme value theory to model the tails of the distribution. Given the evidence of fat tails, the Pareto distribution should be used to model the tails. We model the tails separately, the left (right) tail applying to cumulative probabilities below 0.01 (above 0.99).

3.2 Selecting a copula

We use the procedure outlined in Section 2.3 to select from the Gaussian, Student's t and Gumbel copulas. The copula parameters are estimated by the IFM method of Joe (1997). The marginal distributions constructed in the first step are based on the semi-parametric

[11] Brooks et al. (2005) conclude that this estimator outperforms other methods for tail index estimation when applied in Value-at-Risk calculations.

[12] This method takes an unconditional approach. While it can be argued that conditional aspects should be taken into account (e.g. ARCH-effects), it is debated whether models for extreme returns benefit from a conditional approach (see also the discussion in Daníelsson and de Vries, 2000).

method of Daníelsson and de Vries (2000). In the second step we apply maximum likelihood estimation. In Bouyé (2002)'s extension of the standard Gumbel copula, the ordering of the variables is important. By definition, the dependence between the first two random variables is stronger than the dependence between the first two and the third random variables, i.e. $a_{1,2} \leq a_{(1,2),3} \leq 1$. The order is determined by putting those two variables first for which the a resulting from estimating a bivariate Gumbel copula is highest.[13] To allow for lower tail dependence which carries the main interest from a downside risk perspective, we use the survival copula (see footnote 6). For each copula we calculate the resulting distance measures and evaluate them by constructing their distributions under the null hypothesis of an accurate fit by means of 10,000 simulated samples.

The outcomes of this analysis are reported in Table 2. All parameter estimates indicate the presence of diversification opportunities. Bonds in particular offer the possibility to diversify risk, as the correlation estimates for bonds and stocks, and bonds and real estate are negative for the Gaussian and the Student's t copula. The $a_{(\mathrm{s},\mathrm{r}),\mathrm{b}}$-estimate in the Gumbel copula implies independence between bonds on the one hand and stocks and real estate on the other. Both the correlation estimates for stocks and real estate and the $a_{\mathrm{s},\mathrm{r}}$-estimate in the Gumbel copula indicate a moderate level of dependence between stocks and real estate. However, the degrees of freedom estimate in the Student's t copula is low, implying that extreme events have a stronger tendency to occur jointly than captured by the Gaussian copula. The $a_{\mathrm{s},\mathrm{r}}$-estimate in the Gumbel copula differs significantly from 1, which implies tail dependence between stocks and real estate. Though the different asset classes may offer ample diversification opportunities, this dependence in the tails renders the diversification of downside risk more difficult. Of course, this is particularly relevant for downside risk averse investors.

The test results in panel (d) of Table 2 indicate that the copula should be selected with care. The Gaussian copula does not provide a good fit, as three out of four distance measures reject this hypothesis with p-values below 5%. Neither does the Gumbel copula match the actual dependence, which is also rejected by three out of four distance measures. However, the Student's t copula is not rejected on any of the four distance measures. Hence, our procedure provides a clear positive advise for selecting the Student's t copula.

[13] To stress the ordering, we deviate from the notation in section 2, and attach it as a subscript to a. So, $a_{(\mathrm{s},\mathrm{r}),\mathrm{b}}$ is the coefficient for the dependence between stocks and real estate on the one hand and bonds on the other hand.

We conclude from the estimates and test results that the Student's t copula has the right qualities to match both the dependence in the center and the dependence in the tails. The Gaussian copula fails to capture tail dependence. The Gumbel copula probably offers too less flexibility to model the complete dependence accurately. In the following subsection we examine the differences between the copulas from an economic point of view. We investigate whether the different copulas lead to significantly different assessments of the risk of joint downward movements. Next, we consider their consequences for gauging diversification benefits.

3.3 Joint downward price movements

The consequences of extreme downward movements in prices are an important ingredient of a risk management system. They fall in the broader category of stress tests, in which the risk manager analyzes the consequences of extreme events for a portfolio of assets (see Kupiec, 1998; Longin, 2000). If a portfolio only contains long positions in linear assets, as in our case, stress tests consist of an examination of the effects of large downward price movements. However, they can also consider other extreme events, such as volatility increases in case of equity option portfolios, or a widespread deterioration of credit ratings in case of portfolios with credit derivatives. Berkowitz (2000) argues that the probability of the extreme events that define the stress test should be included in the analysis of the consequences to retain consistency with other elements of the risk management system. In this subsection, we compare the different copulas by the probabilities that they attach to joint downward movements.

We define extreme downward movements by thresholds below which we regard returns extreme. Our starting point for stress tests are thresholds corresponding with the 10%-quantiles of the empirical marginal return distributions, which equal -1.54% for stocks, -0.40% for bonds and -0.81% for real estate. We can apply the different copulas to calculate the joint probability of returns below these thresholds. The copula choice has a large impact on that joint probability. Under the assumption of independence, the joint probability simply equals $0.10^3 = 0.001$ or one day per 48 months. Using the Gaussian copula, this probability becomes 0.0015 (one day per 30 months), for the Student's t copula it increases to 0.0024 (one day per 20 months), while it equals 0.0042 (one day per 11 months) for the Gumbel copula.

Figure 1 shows how the expected waiting time (calculated as the inverse of the joint probability) of returns below the three thresholds changes, if one threshold is reduced while the others are kept fixed at the 10%-quantile of the marginal distributions. The three subfigures clearly demonstrate the growing difference between the three copulas. For each stress test, the expected waiting time for its occurrence are lowest for the Gumbel copula (long dashed lines), followed by the Student's t copula (solid lines) and highest for the Gaussian copula (dotted lines). If stress tests become more extreme, these differences grow. A reduction of the threshold for the return on stocks to -2.54% leads to expected waiting times of 144 months in case of the Gaussian copula, 65 months for the Student's t copula and 34 months for the Gumbel copula. The empirical copula produces an estimate of 71 months, and is close to the Student's t copula.

Figure 1 also contains 95% confidence intervals for the expected waiting times. The good fit of the Student's t copula, as indicated by the test procedure, is highlighted by the fact that the expected waiting times that result from the empirical copula fall mainly in the 95% confidence interval around the expected waiting times derived from the Student's t copula. Moreover, the confidence intervals corresponding with the different copulas hardly overlap, indicating that the copulas lead to significantly different joint probabilities or expected waiting times.

We conclude that the weight given to a stress test is largely influenced by the copula choice, as a stress test has a considerably different probability of occurrence depending on which copula is used. We find that the Student's t copula provides an accurate estimate of the risk of joint downward movements. On the contrary, the Gaussian copula significantly underestimates this risk, while the Gumbel copula overestimates it.[14]

3.4 Diversification benefits

Stress tests provide information on the portfolio consequences if things go wrong in financial markets. A natural question in that case is how one constituent of a portfolio can counterbalance losses of other elements of a portfolio. The parameter estimates for the Gaussian, Student's t and Gumbel copula point at ample diversification opportunities, but

[14]In this section we concentrate on the risk of downward movements, which is the main concern of investors as they generally hold long portfolios. Investors holding short portfolios would be most concerned by large upward movement. We show in Figure 3 (in Appendix A) that the Student's t copula also performs significantly better than the Gaussian copula or the Gumbel copula to assess the risk of joint price increases.

the presence of tail dependence implies that these benefits may be lower in times when they are needed most. This hypothesis has been put forward by many authors.[15] In this subsection we investigate this hypothesis by comparing the diversification benefits that are implied by the different copulas.

To examine the diversification benefits of investing in stocks, bonds and real estate, we calculate the Value-at-Risk (VaR)[16] of one asset class under the assumption that the investor already incurs a loss on the other two asset classes. In each class \$ 1,000 is invested. The big question then is whether the first asset class can compensate these losses. We assume that the losses are at least as large as the VaR for those two asset classes with a confidence level of 99%. Moreover, we assume that the losses per asset class have an equal marginal cumulative probability.

The results in Table 2 point out that bonds offer the best diversification possibilities, as it is negatively correlated with stocks and real estate, according to the Gaussian and Student's t copula, or independent of these two classes if we rely on the Gumbel copula. As a benchmark, consider the Student's t copula, since it was not rejected by the goodness-of-fit tests. With a confidence level of 99%, a loss on stocks does not exceed \$ 21.50 and a loss on real estate does not exceed \$ 12.20. If we take these losses as given, the probability that bonds do not incur a loss equals 0.69. With a probability of 0.95, a loss on bonds does not exceed \$ 5.00. A similar calculation for the Gaussian copula yields a probability of no losses on bonds, given the losses on stocks and real estate, equal to 0.71, which does not deviate much from the outcome of the Student's t copula. With a probability of 0.95, a loss on bonds does not exceed \$ 3.90. Here the difference is larger, and the Gaussian copula would imply larger diversification possibilities of bonds than the Student's t copula. For the Gumbel copula, the probability of no loss on bonds conditional on the loss on stocks and real estate is much smaller, 0.55, implying less diversification benefits. The maximum loss corresponding with the 0.95-quantile equals \$ 5.40, which is comparable to the result of the Student's t copula. Figure 2(b) provides a graphical representation of this analysis. The curve that reflects the Gaussian copula lies below the other two, indicating diversification benefits are larger according to the Gaussian copula. The Gumbel copula

[15]See for instance Loretan and English (2000); Longin and Solnik (2001); Campbell et al. (2002); Ang and Chen (2002); Ang and Bekaert (2002); Campbell et al. (2003).

[16]The Value-at-Risk s_{VaR} of a portfolio with stochastic future value S with a confidence level p is commonly defined by the equation $\Pr[S \leq -s_{\mathrm{VaR}}] = p$.

produces VaR-estimates that are larger than those produced by the Student's t copula for confidence levels below 97.5%. Consequently, diversification benefits are less. For higher confidence levels, the Gumbel copula indicates larger benefits than the Student's t copula, which reflects the estimated independence between stocks and real estate on the one hand, and bonds on the other hand.

We conduct the same analysis for stocks, given losses on bonds and real estate. From the estimation results in Table 2 we can already conclude that the diversification benefits of stocks for a portfolio of bonds and real estate are less than those of bonds for a stock-real estate portfolio. Indeed, the probabilities we find for no losses are smaller. In case of the Gaussian and Student's t copula this probabilities equals 0.30, and in case of the Gumbel copula 0.17. The loss that corresponds with a 95% confidence level equals \$ 33.10 for the Gumbel copula, \$ 28.60 for the Student's t copula and \$ 24.30 for the Gaussian copula. Figure 2(a) shows the results for the VaR of an investment in stocks, given losses on bonds and real estate. Here we see that the Gumbel copula indicates substantially less diversification benefits than the other two copulas. The Gaussian copula indicates most diversification benefits, but is closer to the Student's t copula than in the case for bonds.

Not surprisingly, we find similar results for the VaR of an investment in real estate, given losses in stocks and bonds. However, Figure 2(c) indicates that the differences are less than for the other two asset classes. The probability of no losses varies from 0.18 (Gumbel copula) to 0.22 (Student's t copula) and 0.23 (Gaussian copula). The loss corresponding with a 95% confidence level equals \$ 21.30 (Gaussian copula), \$ 26.50 (Student's t copula) or \$ 27.50 (Gumbel copula).

Summarizing, we also find that the choice of a copula is crucial to gauge the diversification benefits of asset classes. The Gumbel copula leads to a serious underestimation of diversification effects. On the other hand, the Gaussian copula overestimates the possibilities to diversify risks. It gives a reliable estimate of the probability of no losses in one asset class, given losses in the other two asset classes, but the estimate of (large) losses in that asset class are too optimistic.

4 Conclusions

In this paper we have examined the importance of copulas for risk management. Both recent theoretical and empirical evidence have cast doubt on the accuracy of the Gaussian copula that is implied by using correlations. We discuss how traditional tests for distributional assumptions, being the Kolmogorov-Smirnov and Anderson-Darling tests, can be implemented to determine the accuracy of the Gaussian and alternative copulas, such as the Student's t and Gumbel copula. These tests directly compare the fit of the copula on observed dependence. Moreover, they can be applied more generally, while several existing tests can only be used in bivariate cases or for elliptical copulas. Finally, while the choice of test leaves some flexibility – the Kolmogorov-Smirnov-based tests are more sensitive to fit in the center and the Anderson-Darling-based tests more to fit in the tails – the complete dependence pattern is taken into account, contrary to approaches that focus exclusively on dependence of extreme returns.

By analyzing the consequences of the Gaussian, Student's t and Gumbel copulas for the risk management of a portfolio of stocks, bonds and real estate, we find that the impact of copula selection is large. The tests do not reject the Student's t copula, but do reject the Gaussian and Gumbel copula. Indeed, the Student's t copula attaches probabilities to observing extreme negative returns that do not deviate significantly from the empirical probabilities. On the contrary, the Gaussian copula significantly underestimates these probabilities, while the Gumbel copula leads to overestimation. These difference also show up in the assessment of diversification benefits. The Gaussian copula overstates these benefits and the Gumbel copula understates them. This indicates that optimal portfolios that are based on correlations alone can be suboptimal and carry more risk than calculated.

In more general sense, the preference that we find for the Student's t copula over the Gaussian and Gumbel copula points out that both dependence in the center and dependence in the tails are important. The Gaussian copula ignores tail dependence, while the Gumbel copula, as an extreme value copula, cannot accurately capture dependence for non-extremes, i.e. the center of a distribution. Studies that only compare the Gaussian and Gumbel copula like Longin and Solnik (2001) and Poon et al. (2004) may therefore be too limited. We conclude that correlations are useful to characterize dependence in the center, and that the degrees of freedom parameter of the Student's t copula is a good addition to capture stronger dependence in the tails.

A Joint upward price movements

To investigate the consequences of the different copulas for the assessment of the risk of joint large price increase, we redo the analyses in Section 3.3. In stead of returns below thresholds, we now concentrate on returns above thresholds. The basic thresholds in Section 3.3 correspond with a marginal cumulative probability of 10%. In this section, we use basic thresholds with a marginal cumulative probability of 90%, being 1.49% for stocks, 0.44% for bonds and 1.00% for real estate. Figure 3 draws the expected waiting time of returns jointly above the three thresholds, if one threshold is increased, while the others are kept fixed.

Figure 3 shows again significant differences between the application of the different copulas. This could be expected because of the symmetry of the Gaussian and Student's t copulas. The Gumbel copula is asymmetric. Since the survival copula is used, the Gumbel copula exhibits left tail dependence, but right tail independence. Consequently, Figure 3 reveals that both the Gaussian copula and the Gumbel copula lead to a smaller assessment of the risk of joint upward movements than the Student's t copula. The expected waiting times produced by the empirical copula are generally closest to those produced by the Student's t copula.

Figure 3 also contains 95% confidence intervals. They stress the good fit of the Student's t copula with respect to the empirical copula, and indicate that the differences with the other two copulas are significant. We conclude that also for the risk of joint upward movements the Student's t copula is the best candidate, as was the case for the risk of joint downward movements. In both cases, the Gaussian copula significantly underestimates this risk. Because of its asymmetry, the Gumbel copula overestimates the risk of joint downward movements, but underestimates the risk of joint upward movements.

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Table 1: Summary statistics and tail indices

	stocks	bonds	real estate
mean	0.012	0.022	0.065
volatility	1.26	0.34	0.81
skewness	0.18	−0.34	−0.33
kurtosis	4.60	3.83	7.34
minimum	−5.83	−1.38	−5.19
maximum	5.73	1.08	4.68
α_l	5.31	4.62	2.60
α_r	4.44	4.96	4.23

This table reports summary statistics for the three index return series (in %) in our sample: S&P 500 Composite Index (stocks), JP Morgan Government Bond Index (bonds) and NAREIT All Index (real estate). The series consist of 1499 returns from January 1, 1999 to December 17, 2004. α_l and α_r report the estimates for the left and the right tail indices, respectively. The tail indices are estimated by Huisman et al. (2001)'s modified Hill-estimator, with the maximum number of observations used (κ) equal to 149.

Table 2: Estimation and test results

(a) Gaussian copula

$\rho_{\mathrm{s,b}}$	-0.200	(0.024)
$\rho_{\mathrm{s,r}}$	0.471	(0.018)
$\rho_{\mathrm{b,r}}$	-0.073	(0.026)
$\log L$	218.44	

(b) Student's t copula

$\rho_{\mathrm{s,b}}$	-0.195	(0.026)
$\rho_{\mathrm{s,r}}$	0.471	(0.020)
$\rho_{\mathrm{b,r}}$	-0.074	(0.027)
ν	12.1	(2.76)
$\log L$	230.47	

(c) Gumbel copula

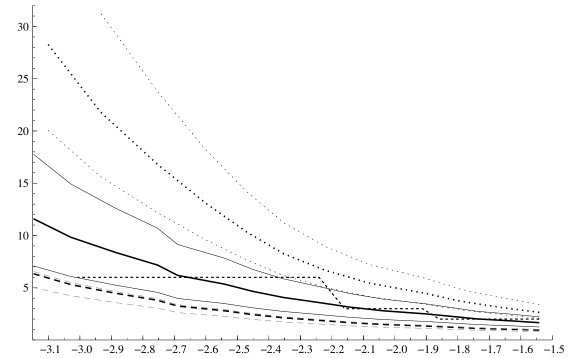
$a_{\mathrm{s,r}}$	1.42	(0.028)
$a_{\mathrm{b,(s,r)}}$	1.000	(0.038)*
$\log L$	183.55	

(d) Test results

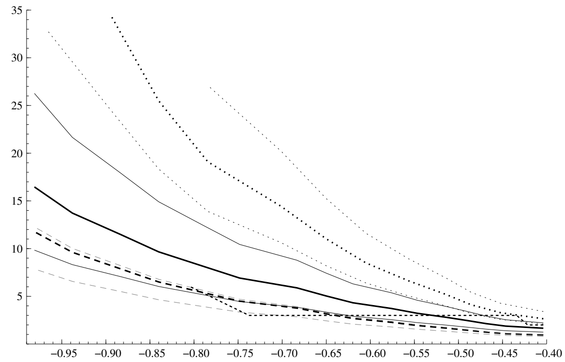
	Gaussian		Student's t		Gumbel	
$\hat{d}^{\mathrm{m}}_{\mathrm{KS}}$	0.026	[0.013]	0.0095	[0.98]	0.035	[$< 0.5 \cdot 10^{-4}$]
$\hat{d}^{\mathrm{a}}_{\mathrm{KS}}$	0.012	[0.0006]	0.0024	[0.9980]	0.0082	[0.0003]
$\hat{d}^{\mathrm{m}}_{\mathrm{AD}}$	0.058	[0.34]	0.044	[0.69]	0.28	[0.33]
$\hat{d}^{\mathrm{a}}_{\mathrm{AD}}$	0.030	[0.00073]	0.0065	[0.9993]	0.026	[0.0016]

Estimation and test results for the Gaussian, Student's t and Gumbel copula. Panels (a) to (c) report the parameter estimates, standard errors and log likelihood values. The copulas are estimated on daily returns from the S&P 500 Composite Index, the JP Morgan Government Index and NAREIT All Index from January 1, 1999 to December 17, 2004 using the IFM method (Joe, 1997). The marginal distributions are constructed by the semi-parametric method of Daníelsson and de Vries (2000), with cut-off probabilities 0.01 and 0.99 for the left and right tail respectively, and tail indices estimated by the modified Hill-estimator of Huisman et al. (2001) (see Table 1). For both the Gaussian and the Student's t copula we report the correlation coefficients for stocks and bonds ($\rho_{\mathrm{s,b}}$), stocks and real estate ($\rho_{\mathrm{s,r}}$), and bonds and real estate ($\rho_{\mathrm{b,r}}$). For the Student's t copula we include the degrees of freedom parameter ν. The parameters for the Gumbel copula refer to Bouyé (2002)'s extension of the standard Gumbel copula, applied to the survival copula. $a_{\mathrm{s,r}}$ refers to the dependence between stocks and real estate; $a_{\mathrm{b,s,r}}$ to the dependence between stocks and real estate on the one hand, and bonds on the other. Standard errors are reported in parentheses. In the estimation $a_{\mathrm{b,s,r}} = 1 + x^2$ is used; the standard error marked with an asterisk corresponds with x. Panel(d) reports the distance measures resulting from the tests. The values for the distance measures result from the evaluation step, applying the transformation in Eq. (11) for the Gaussian copula and in Eq. (12) for the Student's t copula. The p-values, based on 10,000 simulations as described in the simulation step, are reported in brackets.

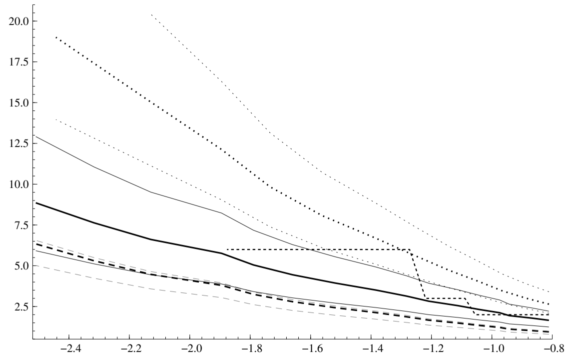
Figure 1: Expected waiting time of extreme negative events



(a) stocks



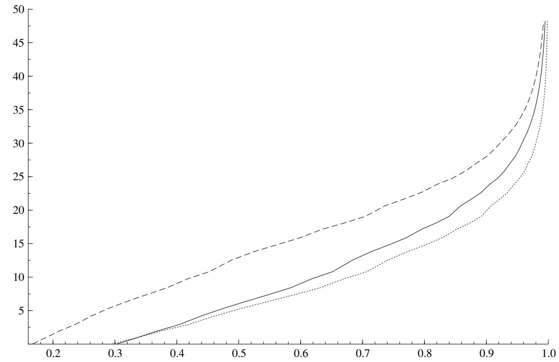
(b) bonds



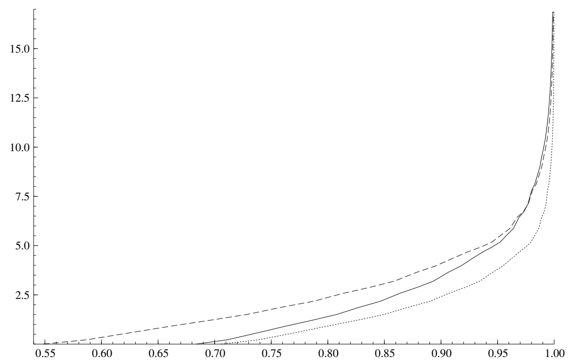
(c) real estate

This figure presents the expected waiting time (in years) for the joint occurrence of returns below thresholds. The expected waiting time is calculated as the inverse of the joint probability. The basic thresholds correspond with the 10%-quantile of the marginal cumulative distribution, which gives -1.54% for stocks, -0.40% for bonds and -0.81% for real estate. The subfigure for a specific asset class shows the expected waiting time for different values of the thresholds for returns on that asset class, while the other thresholds remain at their basic level. We plot waiting times for the Gaussian (dotted), Student's t (solid), Gumbel (long dashed) and empirical (dashed, piecewise linear) copulas. The thick lines show the point estimates, the thin lines show the 95% confidence intervals. The confidence intervals are based on 200 parameters drawings based on the estimated Hessian matrix.

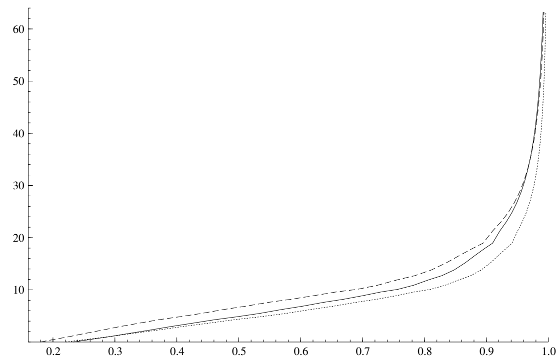
Figure 2: Portfolio Value-at-Risk for an equally weighted portfolio



(a) VaR for stocks only



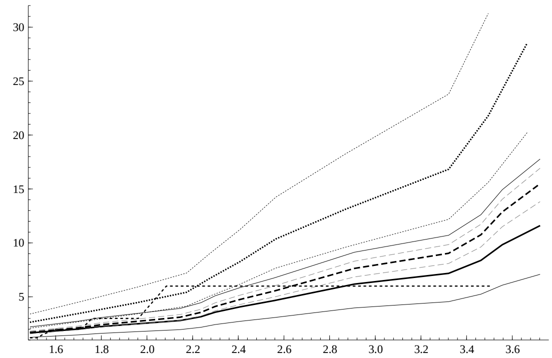
(b) bonds



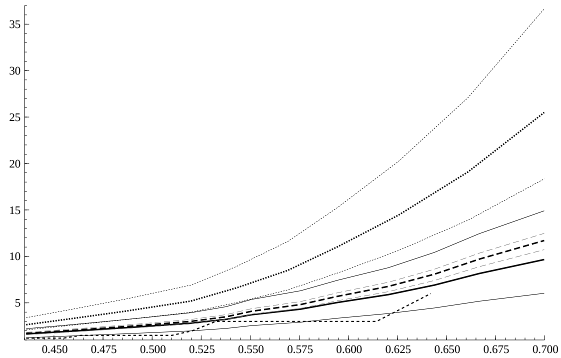
(c) real estate

This figure shows the one-day Value-at-Risk for an equal investments of $ 1,000 in stocks, bonds and real estate. We plot the effect of one asset class on the portfolio VaR for different confidence levels (on the x-axis), conditional on a loss in the other two asset classes corresponding with the VaR with a confidence level of 99%. We choose the minimum losses in those two classes such that their marginal probabilities are equal. Panel a graphs the influence of stocks on the portfolio VaR, conditional on a loss in bonds and real estate. In Panel b the effect of bonds is shown, given a loss on stocks and real estate. Panel c shows how real estate affects VaR, given a loss on stocks and bonds. In each subfigure, the solid, dashed and dotted lines show the VaR resulting from applying the Student's t, the Gaussian and the Gumbel copula, respectively.

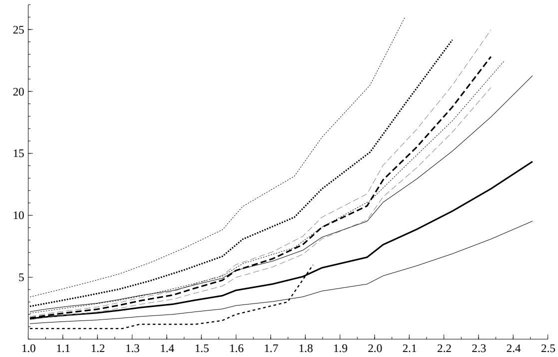
Figure 3: Expected waiting time of extreme positive events



(a) stocks



(b) bonds



(c) real estate

This figure presents the expected waiting time (in years) for the joint occurrence of returns above thresholds. (See also Figure 1 in the original paper.) The expected waiting time is calculated as the inverse of the joint probability. The basic thresholds correspond with the 90%-quantile of the marginal cumulative distribution, which gives 1.49% for stocks, 0.44% for bonds and 1.00% for real estate. The subfigure for a specific asset class shows the expected waiting time for different values of the thresholds for returns on that asset class, while the other thresholds remain at their basic level. We plot waiting times for the Gaussian (dotted), Student's t (solid), Gumbel (long dashed) and empirical (dashed, piecewise linear) copulas. The thick lines show the point estimates, the thin lines show the 95% confidence intervals. The confidence intervals are based on 200 parameters drawings based on the estimated Hessian matrix.