STRESS TESTING WITH STUDENT'S *T* DEPENDENCE

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Stress testing with Student's t dependence*

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Abstract

In this study we propose the use of the Student's t dependence function to model dependence between asset returns when conducting stress tests. To properly include stress testing in a risk management system, it is important to have accurate information about the (joint) probabilities of extreme outcomes. Consequently, a model for the behavior of risk factors is necessary, specifying the marginal distributions and their dependence. Traditionally, dependence is described by a correlation matrix, implying the use of the dependence function inherent in the multivariate normal (Gaussian) distribution. Recent studies have cast serious doubt on the appropriateness of the Gaussian dependence function to model dependence between extreme negative returns. The Student's t dependence function provides an attractive alternative. In this paper, we introduce four tests to analyze the empirical fit of both dependence functions. The empirical results indicate that probabilities assigned to stress tests are largely influenced by the choice of dependence function. The statistical tests reject the Gaussian dependence function, but do not reject the Student's t dependence function.

Key words: Stress testing, dependence, extreme values, copulas, tail dependence

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1 Introduction

Sound risk management techniques are at the heart of financial management. The recent large declines in stock prices following for instance the Asian crisis, the tumbling of Enron and other corporate scandals serve to illustrate the importance of proper risk management techniques. Risk management should clarify the risks to which portfolios are exposed and provide information on the size of the exposures.

Stress tests form an important and relevant element of current risk management. Stress tests are meant to evaulate the influence of large, unexpected movements in financial markets on portfolio values. Stress tests become even more important, if those movements are accompanied by structural breaks or temporal breakdowns.

A well-known stylized fact of asset returns is their relatively strong tail dependence (see Longin and Solnik, 2001; Hartmann et al., 2001). In this paper we investigate dependence in relation to stress testing. We propose to model stress tests with dependence in asset returns by the dependence function implied by the multivariate Student's t distribution. We compare the results of the Student's t dependence function with the results of the Gaussian dependence function, which is implied by the traditional correlation approach. Our work builds on and is an extension of the work of Kupiec (1998), who discusses stress-test in a multivariate normal context. It also extends Longin (2000), who allows univariately fat-tailed behavior, but aggregates risks in a way related to the variance/covariance method. Furthermore, it adds to the work of Tan and Chan (2003), who include univariate fat tails in the framework set up by Kupiec (1998), but still use the Gaussian dependence function.

In a more general sense our study can be seen as an application of Embrechts et al. (2001), who advocate the use of copulas (or dependence functions) to model dependence. Correlation only describes dependence completely, if the dependence is as implied by the multivariate normal distribution. Several empirical studies, among which Longin and Solnik (2001) and Hartmann et al. (2001) are the most notable, show that the hypothesis of a constant correlation matrix to model dependence should be rejected due to the actual observed dependence in the tails[1]. Instead, a dependence model with either time-varying correlation or a different copula with constant parameters can be considered. Since the first option would oblige the specification of a correlation process, and would still impose a Gaussian dependence structure, we choose the second option and consider the Student's t copula. The Student's t copula differs considerably from the Gaussian in its modelling of dependence between tail observations. Furthermore, Campbell et al. (2003) report empirical evidence supporting a Student's t based model for dependence.

In our empirical work we find strong and significant results favoring the Student's t dependence function over the Gaussian function. The estimation results point at a clear and significant difference. Though the correlation matrices do not show large differences, the degrees of freedom parameters of the Student's t dependence function is fairly low, indicating strong tail dependence. Statistical tests confirm that the Student's t dependence fits the data more accurately. Furthermore, we demonstrate that the selected dependence function has serious consequences for stress testing. The Student's t dependence function assigns a larger probability to stress tests than the Gaussian dependence function. Furthermore, the speed at which the probability of a joint occurrence of extreme returns decreases, if the returns become more extreme, is lower in case of the Student's t dependence function.

The plan of this paper is as follows. In the next section we develop the framework for stress testing for an investor with a well-diversified portfolio. We pay specific attention to the role of dependence models. Therefore, we consider several dependence functions and discuss estimation and testing. We apply the Gaussian and Student's t dependence function to the case of an investor holding stocks, bonds and real estate. We estimate and test the proposed dependence functions for the indices in section 3. In the subsequent section we propose several stress tests and analyze the probabilities that the different dependence functions assign to them. We finish with a discussion and conclusions.

[1]Other studies investigating the constant correlation hypothesis include Forbes and Rigobon (2002), Loretan and English (2000), Campbell et al. (2002) and Ang and Chen (2002), who all consider correlation conditional on the size of the returns, and Ramchand and Susmel (1998), Edwards and Susmel (2001) and Ang and Chen (2002), who take a Markov switching approach. The reported evidence is mixed.

2 Including stress tests in the risk management system

According to Berkowitz (2000) a risk management system consists of two parts. In the first part, the risk manager needs to identify the risk factors that influence the value of the portfolio under consideration. The second part of the system is the model that links the risk factors to the value (or return) of the portfolio. To keep things simple, we consider an asset portfolio containing no derivatives. As risk factors, we identify all constituents of this portfolio. We assume the portfolio consist of n different assets. We concentrate on the return of the portfolio, R_P. Quantifying the risk factors by their returns $R_i,\ i = 1, \dots, n$, the relation between the portfolio return and the risk factors is linear: $R_P = \sum_{i=1}^{n} w_i R_i$, where w_i gives the portfolio weight for asset i.

Kupiec (1998) presents a general applicable setup for stress testing, which main element is a subset of the risk factors taking on extreme values. The other risk factors can remain constant (or equal zero in case of returns), or can take on their expected value (possibly conditional on the value of the stressed risk factors). We assume that all risk factors are stressed. We do not assume that each risk factor takes on a specific value, but allow for a range of values with an upper bound. Since the relation between the portfolio return and the risk factors is linear, this corresponds with a portfolio return that also has an upper bound.

Subsequently, the stress scenario should be assigned a probability. Such an assignment enables a comparison of stress scenarios and hence makes it possible to evaluate the implications of different stress scenarios. In order to determine joint probabilities a multivariate model that describes the behavior of the risk factors is needed. We advocate the copula approach for such a model. Using those dependence functions allows for separately modelling the marginal distribution of each risk factor (i.e. return) and their joint dependence. Consequently, we can apply the sophisticated techniques that have been developed for univariate return series to each factor in isolation, and in a second step apply an appropriate dependence function on the marginal models. This way the simultaneous behavior of returns is completely specified. Now the advantage of using one model for both the center and the extreme parts of the distribution can be used, since no arbitrary judgement has to be made when the model for the center applies and when the model for the tails. Therefore, we first discuss the modelling of univariate return series and turn to dependence functions later. We finish with a treatment of the tests for the fit of the dependence functions.

2.1 Univariate models

It is a stylized fact that asset returns exhibit fat tails. Since risk management focuses mainly on extreme downside returns, a model that correctly incorporates tail behavior is of great value. We consider two models for univariate returns that are able to model fat tails. The first one is the Student's t distribution. This symmetric distribution is characterized by three parameters: location (μ), scale (σ) and shape (ν), which is also known as the degrees of freedom.

The second model we consider uses the semi-parametric approach proposed by Daníelsson and de Vries (2000). They combine the empirical distribution function with extreme value theory. The empirical distribution is used for the center of the distribution, since enough data points are generally available to provide a proper description. Extreme value theory describes the behavior of random variables in the upper or lower part of their domain[2]. Among others, it provides models for the cumulative probability of an event x, given that is extreme: $P(X \leq x | X \leq z) = F^{[z]}(x), x \leq z$, independent of specifying $F(x)$ for the complete domain. For fat-tailed distributions $F^{[z]}(x)$ can be modelled by the Pareto distribution:

$$F^{[z]}(x) = \left(\frac{x - \mu}{z - \mu} \right)^{-\alpha} \qquad x \leq z < \mu, \tag{1}$$

where μ is the mean of X. The shape parameter α is referred to as the tail index[3].

[2]See Reiss and Thomas (1997) for an introduction and Embrechts et al. (1997) for a more rigorous treatment.

[3]The tail index a is the parameter that actually characterizes the tail behavior of fat-tailed distri-

The semi-parametric model is the most flexible of the two. The tail behavior of the left and right tail can be modelled separately, whereas the Student's t model imposes the same behavior for both. Furthermore, the semi-parametric model can handle skewness contrary to the symmetric Student's t distribution. An advantage of the Student's t distribution is the fact that one does not need to specify the beginning of the tail, parameterized by z, making the Student's t model more robust.

Central in the estimation of the models is the estimation of the tail index (i.e. α for the Pareto distribution and ν for the Student's t distribution). We choose to use the modified Hill-estimator developed by Huisman et al. (2001), because of its unbiasedness[4]. Estimates for the means and the shape parameter of the Student's t model can be constructed straightforwardly out of their sample counterparts. The parameter z in (1) is chosen in such a way that $P(X \leq z)$ has a low probability.

2.2 Dependence functions

Dependence of random variables can be modelled uniquely by copulas, also referred to as dependence functions. We will use three different copulas: the empirical, the Gaussian and the Student's t copula. We start with a definition of copulas, first[5]. Then we discuss the different copulas we use in more detail. Finally, we consider estimation.

A copula is a function that links the joint probability of events to the marginal probability of each event. Let X be a random vector of size n, and let $F_{X_i}(x_i)$ denote the marginal probability $P(X_i \leq x_i)$, then the copula C is defined by:

$$P(X \leq x) = C(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)) \tag{2}$$

An important theorem regarding copulas is due to Sklar (1959). It states that each

butions. A distribution F is called fat-tailed if it is regularly varying at negative infinity, defined as $\lim_{t\to\infty} F(-tx)/F(-t) = x^{-\alpha}$.

[4]Other methods for tail index estimation can be found in Daníelsson et al. (2001) and Drees and Kaufmann (1998).

[5]A general discussion of copulas can be found in Joe (1997). For a discussion applied to finance we refer to Bouyé et al. (2000).

multivariate distribution function with continuous marginals has a unique copula representation. As a consequence, we can derive a unique copula from the multivariate normal or Student's t distribution.

2.2.1 The empirical copula

Let $x_t,\ t = 1, 2, \ldots, T$ be a sample of realizations of a $n \times 1$ random vector X . The empirical copula C_E is defined by:

$$C_E\left(\frac{m}{T}\right) = \frac{1}{T}\sum_{t=1}^{T} 1_{\left[x_{1,t} \leq x_1^{[m_1]}, \ldots, x_{n,t} \leq x_n^{[m_n]}\right]}, \quad 0 \leq m_i \leq T \quad i = 1, \ldots, n \tag{3}$$

In this notation, the function $1_{[y \leq z]}$ equals 1 if the subscript part is true and 0 otherwise. $x_i^{[k]}$ denotes the k^{th} order statistic for the subsample $x_{i,t},\ t = 1, 2, \ldots, T$.

The empirical copula has some severe disadvantages. If the number of random variables increases, the copula becomes less informative, since the number of observations per dimension decreases. Secondly, the probability assigned to the minimum and maximum are unsatisfactory. In the univariate model for a random variable Y it is possible to have $P(Y \leq \min_t y_t) = P(Y \geq \max_t y_t) = 1/2T$. Though arbitrary, both events have positive probability. In the case of the empirical copula, $P(X \leq \min_t x_t) = C_E(1/T) = 0$[6], unless the minima are jointly observed. This condition is very restrictive, implying that the event will mostly have zero probability. For maxima, a similar restrictive condition applies as well.

2.2.2 The Gaussian copula

The Gaussian copula is implied by the multivariate normal distribution. It is parameterized by a correlation matrix Ω. Its functional form is given by:

$$C_\Phi(u; \Omega) = \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n); \Omega) \tag{4}$$

where Φ_n denote the standard normal n-variate cumulative distribution function and Φ^{-1} denotes the inverse of the standard normal univariate cumulative distribution function. (4)

[6] Here $\min_t x_t$ is defined as $[\min_t x_{1,t}, \ldots, \min_t x_{n,t}]$.

states that the joint probability of events with marginal probability u_i can be found by first transforming them to standard normally distributed variables $\Phi^{-1}(u_i)$ and subsequently computing their joint probability using the multivariate normal distribution.

The correlation matrix Ω of the Gaussian copula and the empirical correlation matrix coincide if all n marginal distributions are Gaussian as well. If (as in this paper) the marginal distributions deviate from the Gaussian, the two correlation matrices will be different.

2.2.3 The Student's t copula

The third copula we consider is the one that models the dependence inherent in the multivariate Student's t distribution. Its density function is given by:

$$f_{t,n}(x;\nu,\Omega) = \frac{\Gamma\left(\left(\nu+n\right)/2\right)}{\Gamma\left(\nu/2\right)\left(\nu\pi\right)^{n/2}\sqrt{|\Omega|}}\left(1+\frac{x'\Omega x}{\nu}\right)^{-\frac{\nu+n}{2}} \tag{5}$$

where ν is the scalar degrees of freedom, Ω the correlation matrix and Γ refers to the gamma-function. The Student's t copula is defined as:

$$C_t(u;\nu,\Omega) = F_{t,n}(F_t^{-1}(u_1;\nu),\ldots,F_t^{-1}(u_n;\nu);\nu,\Omega) \tag{6}$$

where $F_{t,n}$ denotes the n-variate cumulative Student's t distribution function and F_t^{-1} denotes the inverse of the univariate cumulative Student's t distribution function. The degrees of freedom used in $F_{t,n}$ and F_t^{-1} are similar. (6) can be interpreted similarly as (4). The degrees of freedom in (6) can be varied independently of (the tail behavior of) the actual marginal distributions.

The main difference between the Student's t and Gaussian copula lies in the probability they assign to extreme events. Of course, the joint probability of extreme events decreases if the events become more extreme. However, for a correlation matrix that is equal for both copulas, the probability that the Gaussian copula assigns decreases faster than is the case for the Student's t copula.[7]

[7] Joe (1997) introduces the concept tail dependence to describe the tail behavior of bivariate copulas.

2.3 Estimating copulas

The definition of a copula implies that it is a cumulative distribution function on the domain $[0,1]^n$. The corresponding density function can be found by n times differentiating the copula function C. Based on the density function the parameters can be estimated using general estimation techniques such as maximum likelihood. The likelihood function can be constructed based on the observations, transformed to have uniform marginal distributions. Empirically, two approaches can be followed. The first approach, exact maximum likelihood, jointly estimates the parameters for the marginal models and the copula. Hence, the transformation of the observations is part of the estimation. The second approach is a two-step approach that estimates the parameters for the marginal models first and conditional upon these values, estimates the parameters for the copula in a second step[8]. While the two-step procedure is less efficient than one step maximum likelihood, it computationally more attractive. Moreover, it is possible to use different estimation techniques for the marginal models. Therefore, we use the second method.

In case of the Gaussian copula, estimation of the correlation matrix Ω is straightforward. Using a model for the marginals (i.e. the Student's t distribution or the semi-parametric method), all observations are transformed to the $[0,1]$-interval. Then the observations are transformed to standard normally distributed series, using the inverse function Φ^{-1}. Based on these series, the correlation matrix can be estimated using maximum likelihood. However, estimation of the parameters of the Student's t copula is not as straightforward. The reason is that the transformation of the uniformly distributed marginals to a Student's t distributed series depends on the copula degrees of freedom parameter that has to be

For a bivariate copula $C(U)$ with U having uniform marginals, the tail dependence function $\lambda(u)$ is defined as $\lambda(u) = P(U_1 \leq u | U_2 \leq u) = C(u,u)/u$. If $\lim_{u \downarrow 0} \lambda(u)$ exists and is unequal to 0, the copula exhibits tail dependence. If the limit exists but equals 0, the copula exhibits tail independence. Embrechts et al. (2001) show that the bivariate Gaussian copula exhibits tail independence if the correlation coefficient is smaller than 1. They also show that the bivariate Student's t copula exhibits tail dependence, even if the correlation coefficient equals 0.

[8] Joe (1997) calls this the the inference functions for margins method. The resulting estimators belongs to the general class of sequential estimators (see e.g. Newey, 1984).

estimated. Consequently, the arguments that maximize the likelihood function can not be expressed in closed form. Instead, a numerical optimization is used.

2.4 Testing the fit of copulas

General non-parametric tests for the fit of a distribution to data can be applied to test the fit of a specific copula. Malevergne and Sornette (2001) describe a method to test the fit of a Gaussian copula. Suppose the Gaussian copula with correlation matrix Ω is used to describe the dependence of the n uniform marginals of the vector U. The random variables $X_i = \Phi^{-1}(U_i),\ i = 1, \ldots, n$ follow a multivariate normal distribution with correlation matrix Ω. Consequently, the random variable

$$Z = X'\Omega^{-1}X \tag{7}$$

has a χ^2 distribution with n degrees of freedom. Starting with the empirical marginals on which the Gaussian copula is estimated, they derive an empirical Z-series. Based on this series, they construct the empirical distribution $F_E(z)$. For each observation z_t, they compare $F_E(z_t)$ with the theoretical value $F_{\chi^2_n}(z_t)$. They define the following test-statistics:

$$d_1 = \max_t |F_E(z_t) - F_{\chi^2_n}(z_t)| \tag{8}$$

$$d_2 = \int |F_E(z) - F_{\chi^2_n}(z)| dF_{\chi^2_n}(z) \tag{9}$$

$$d_3 = \max_t \frac{|F_E(z_t) - F_{\chi^2_n}(z_t)|}{\sqrt{F_{\chi^2_n}(z_t)(1 - F_{\chi^2_n}(z_t))}} \tag{10}$$

$$d_4 = \int \frac{|F_E(z) - F_{\chi^2_n}(z)|}{\sqrt{F_{\chi^2_n}(z)(1 - F_{\chi^2_n}(z))}} dF_{\chi^2_n}(z) \tag{11}$$

The distance measure d_1 is known as the Kolmogorov distance; d_2 is its average. The distance measure d_3 is named after Anderson and Darling (1952); d_4 is its average. The Kolmogorov distances are more sensitive to deviations in the center of the distribution, whereas the Anderson-Darling measures are more sensitive to deviations in the tails of the distributions. Malevergne and Sornette (2001) advocate the use of d_2 and d_4 because they are less sensitive to outliers.

It can be investigated whether the calculated statistics differ significantly from their theoretical values under the null hypothesis that the Gaussian copula is appropriate. . Since the distributions of the statistics is not known analytically, they use simulation to derive their distributions. Based on these distributions, they determine p-values for the hypothesis that the statistics equal their theoretical values.

We propose a similar approach to test the Student's t copula. Suppose that the dependence of a random vector U is given by a Student's t copula with parameters Ω and ν. Then the random variables $Y_i = F_t^{-1}(U_i; \nu),\ i = 1, \ldots, n$ have a multivariate Student's t distribution, parameterized by Ω and ν. Consequently, Y can be written as $W/\sqrt{S/\nu}$, with W multivariately normally distributed with correlation matrix Ω, $S \sim \chi^2$ with ν degrees of freedom, and W is independent from S. The variable

$$H = Y'\Omega^{-1}Y/n = (W'\Omega^{-1}W/n)/(S/\nu) \tag{12}$$

follows an F-distribution with n and ν degrees of freedom respectively, since $W'\Omega^{-1}W$ and S have a χ^2 distribution and are independent. Starting with the uniform marginals, we construct a H-series. We use the statistics defined above, replacing the theoretical χ_n^2 distribution by the appropriate $F_{n,\nu}$ distribution. In that way we compare the empirical distribution of the H-series with its theoretical distribution. Again, simulation can be used to derive the distribution of the test statistics under the null hypothesis.

3 Estimating and testing the models

Since we are interested in the effect of the assumed dependence model on stress testing in practice, we consider a portfolio that consist of investments in stocks, bonds and real estate, all in the US. The models we propose for the behavior of the returns on these assets use copulas. As a consequence, we have models for the marginal distributions of returns and models for their dependence. We estimate the models subsequently: first the models for the marginal distributions and then the models for the joint distribution. We start with a presentation of the data. Since our attention lies mainly in the dependence models, we will treat estimation of the marginal models concisely and focus on the estimation of the dependence functions and tests on their fit.

3.1 Data

The data consist of almost 4 years of daily returns on stocks, bonds and real estate, relevant for a domestic US investor. For the stock index, we use the Standard & Poor's 500 Composite index; for bonds the JP Morgan US Government Bond Index and for real estates the NAREIT All Index. All indices are total return indices. The data set runs from January 1, 1999 to December 12, 2002. Excluding non-trading days, this results in 995 daily observations, that we transform to 994 returns (in %). All data are obtained from Datastream.

Table 1 presents the summary statistics for the different indices. The stock and real estate index have an insignificant mean, whereas the mean return for bonds is significant (p-value $< 1\%$). As can be expected, the volatility for stocks is highest, followed by real estate, while bonds have lowest volatility. All three indices exhibit skewness, pointing at an asymmetric distribution. Furthermore, they all exhibit excess kurtosis, indicating the presence of fat tails. This is most pronounced in the real estate returns. The Jarque-Bera test clearly rejects the hypothesis of normality (p-values are all smaller than 0.001%).

3.2 Modelling the marginal distributions

The summary statistics in Table 1 indicate the presence of fat tails. In order to investigate this issue further, we estimate tail indices using the modified Hill-estimator proposed by Huisman et al. (2001). The Hill-estimator yields an estimate for the inverse of the tail index[9]. If the inverse tail index is closer to zero, the distribution is less fat-tailed. Table 2 presents the estimated tail indices. The hypothesis of exponentially declining tails (i.e. $\gamma = 0$) can be rejected. For each index, the estimates for the right and left tail do not

[9]The inverse of the tail index is represented by γ in the Von-Mises representation of extreme value distributions, see Reiss and Thomas (1997).

differ significantly from each other. Furthermore, the estimates for the real estate index point out it has significantly fatter tails than the other two indices.

The graphs in Figure 1 give an illustration of the alternative specification for modelling the left tail of the distribution. The solid lines, that represent the empirical distribution show a rather jagged pattern. Both approaches are close to the empirical tail at the right end of the graph, but deviate more from it, if one moves to the left. For the stock index, the Student's t model seems to attribute considerably more weight to negative extreme returns than the semi-parametric model. For the bond index and the real estate index, the difference is negligible.

The Student's t distribution is less flexible than the empirical distribution. For example, it can not capture skewness, which is present according to Table 1. We test the fit of the Student's t distribution using a χ^2 goodness-of-fit test. We partition the sample into 20 subsamples, having equal probability. Then the test-statistic w can be determined as:

$$w = \sum_{j=1}^{m} \frac{(n_j - T \cdot p)^2}{T \cdot p}, \tag{13}$$

where m is the number of intervals into which the sample is partitioned, n_j is the number of observations in interval j, T is the total number of observations and p is the probability assigned to each interval. The distribution of W is bounded between a χ^2_{m-1} and a χ^2_{m-k-1}, with k equal to the number of estimated parameters (Chernoff and Lehmann, 1954). Mittelhammer (1996) advises to calculate p-values for both distributions. In general, the p-value based on a χ^2_{m-1}-distribution will be biased towards not rejecting the hypothesized distribution, while the p-value based on a χ^2_{m-1}-distribution will be biased towards rejection. The values of the test statistic and the corresponding p-values can be found in Table 3. For bonds and real estate the hypothesis of a Student's t distribution is clearly rejected. For mild significance levels, the hypothesis is rejected for stocks as well.[10]

[10]The χ^2 goodness-of-fit test turns out to be quite sensitive to the number of intervals chosen. We calculated test-statistics and p-values for m ranging from 10 to 25. The average value for p_1 (p_2) for stocks equals 0.128 (0.050). For bonds we find 0.077 (0.026). For real estate both averages are smaller than 0.001. Hence for stocks, rejection of the hypothesis is debatable, whereas the evidence for bonds and real estate

3.3 Estimation of the dependence functions

Using the results of the previous subsection we estimate the parameters for the different dependence functions. Since we use the two-step procedure, the first step is the transformation of the observed returns using a marginal distribution function (i.e. the Student's t distribution or the semi-parametric method). Then the parameters of the dependence functions are determined.

The estimated correlation matrix for the Gaussian copula based on the semi-parametric marginal distribution is presented in Table 4. We observe that bonds on the one hand and stocks or real estate on the other hand are weakly, negatively correlated. The correlation between stocks and real estate is positive and has a moderate value. Using the Student's t distribution for the marginal distributions produces only slightly different estimates.

Next we consider the estimation of the parameters of the Student's t dependence function. We start again by modelling the marginals using the semi-parametric method. The estimates for the correlation matrix and degrees of freedom are found by numerically maximizing the likelihood function. Table 5 shows the estimates and their standard errors. The estimates for the correlation matrix are very similar to those reported in Table 4, though all estimates seem to be slightly lower in absolute sense. Since correlation is driven mainly by observations in the center of the distribution, this similarity is not surprising. The low estimate for the degrees of freedom indicates stronger dependence in the tails of the distribution than implied by the Gaussian copula. If we use the Student's t distribution to model the marginals, we find an even lower estimate for degrees of freedom: 7.18 (standard error: 1.32). The resulting correlation matrix is again only slightly different.

The estimate for the degrees of freedom parameter points at a stronger dependence between tail events. Since the Student's t dependence function has only one degrees of freedom parameter, its estimate can be largely due to the joint tail observations of only two random variables (e.g. stocks and bonds). To see whether this effect is present, we estimate the parameters for bivariate Student's t dependence functions for all combinations.

is less ambiguous.

The bivariate Student's t dependence function is parameterized by a single correlation coefficient and a degrees of freedom parameter. The estimation results can be found in Table 6. A first inspection of the degrees of freedom estimates reveals that the estimate in Table 5 are not driven by the dependence between two tails. Apparently, the estimated degrees of freedom parameter in Table 5 is a mixture of the parameters found for the bivariate dependence functions, which appear not to differ too much from one another. A formal test on the difference of the estimated parameters is complicated. Instead, we construct confidence intervals using the fact that maximum likelihood estimates have a normal distribution, asymptotically. The confidence intervals for stocks and bonds, and stocks and real estate estimates largely overlap. The confidence interval for bonds and real estate completely includes the other two intervals. We conclude that the estimated degrees of freedom parameter of the Student's t dependence function is not driven by joint tail observations of only two asset returns. However, the estimate does not necessarily fall in the confidence interval of estimates based on two tails separately.

3.4 Testing the fit of dependence functions

In order to assess the fit of the dependence functions, we apply the test strategy outlined in subsection 2.4. Thus, using the marginal models we transform the observations to a sample having uniform marginals. To test the fit of the Gaussian dependence function, the sample is transformed again to a sample having a multivariate normal distribution. Then the empirical Z-series is constructed, as given by (7). For Ω we use the estimates discussed in the previous subsection. Based on that series, the four test statistics discussed in subsection 2.4 are constructed. Subsequently, we simulate 10,000 samples whose dependence is modelled by the Gaussian dependence function with correlation matrix Ω. For each simulated sample, the test statistics are constructed as well, resulting in an empirical distribution for the statistics. That distribution is used to determine the p-value of the statistics that belong to the actual observations. The fit of the Student's t dependence function is tested using a similar procedure. Instead of a Z-series, a H-series is constructed, as defined by

(12). We construct 10,000 simulated samples that have a dependence that is modelled by the Student's t dependence function.

The value of the different statistics can be found in Table 7. We test the hypothesis of the Gaussian and Student's t copula being appropriate and compare the values of the test-statistics with their expected value under the null hypothesis. The alternative hypothesis reads that the found values exceed the theoretical values, making the test one-sided. The corresponding p-values are listed in parentheses. The results favor the combination of the semi-parametric method to model the marginal distributions and the Student's t copula to model dependence. All statistics except d_3 are smaller than their theoretical values, indicating a close fit. The combination of Gaussian dependence and semi-parametric marginals provide a considerable worse fit. Three out of four statistics have p-values around 0.10. Among them are both Anderson-Darling measures, which are more sensitive to deviations in the tails. If the Student's t distribution is used to model the marginal distribution, the Gaussian copula is clearly rejected as well, nor does the combination of Student's t marginals with Student's t dependence yield satisfying results. A reason for this poor fit may be the assumption of uniform marginals. The conducted tests assume that the marginal distributions are correctly specified and hence yield uniformly distributed marginals, when the corresponding cumulative density function is used to transform the sample. Tests cast profound doubt on the fit of the univariate Student's t distribution on the marginal distributions, however. The semi-parametric method is by construction less sensitive to this misfit. A comparison of the first with the third, and the second with the fourth column indeed points in this direction, since the values in the first and second column are all considerably smaller than the values reported in the third and fourth column.

The superior performance of the Student's t copula compared with the Gaussian copula is confirmed by the pseudo likelihood ratio test designed by Mashal and Zeevi (2002). The values of both likelihood functions are compared, with the Gaussian copula formulated as a Student's t copula with the degrees of freedom parameter restricted to a high value (e.g. 10,000). Mashal and Zeevi (2002) show that the traditional likelihood ratio has a scaled χ^2 distribution with 1 degree of freedom and scale factor $(1+\gamma)$. The scale factor is due to the uncertainty in the marginal distributions. The authors typically find a scale factor around 1.1. To be prudent they use the scale factor equal to 2 in their calculations. For the models with semi-parametric marginals we find a test statistic equal to 29.92 and for the models with Student's t marginals we find 38.46. With a scale factor equal to 2, the critical value corresponding with the 95% (99%) level equals 7.68 (13.27). Hence, the Gaussian dependence function is clearly rejected.

4 Stress tests and their probabilities

The specification of the risk management system as outlined in section 2 is complete. We identified the risk factors and modelled their behavior. In the previous section we explained the estimation and testing of the proposed models. Furthermore we established the link between the portfolio return and the risk factors. Hence, we are able to perform stress tests and assign probabilities to the tests. By a comparison of those probabilities for different dependence functions, we can analyze the influence of dependence functions. The analysis uses a basic stress test that is made more extreme. This enables us to compare the dependence function on specific stress tests and to analyze the sensitivities of the assigned probabilities to changes in the stress tests.

The basic stress test consists of events that are on themselves not too extreme. We assume that stock decrease at least 1.75%, bonds at least 0.40% and real estate at least 0.75%. The events are chosen to have almost equal marginal probability according to their empirical distribution: stocks 0.104, bonds 0.102 and real estate 0.106 respectively. Under the obviously unrealistic assumption of independence, the joint probability of the events would be 0.00112. Using the empirical distribution, we assign a probability of 0.00302 to this stress test. Based on the Student's t dependence function we find a slightly different probability of 0.00297. In contrast, the probability that is attributed when applying the Gaussian dependence function is almost twice as small: 0.00146. The impact of this difference becomes clearer if we consider waiting times. The reciprocal of the probabilities can be interpreted as an expected waiting time. So, based on the Student's t dependence function, an observation of the stress event is expected to happen once every 337 days. Based on the empirical distribution the waiting time equals 331 days. If the Gaussian dependence function is applied, such an event happens only once every 684 days. Independence would imply an expected waiting time of 889 days. We conclude that already for not too extreme stress events, the model assumed for dependence has a serious influence on the assigned probability. The Gaussian dependence function seems to underestimate the probability of a joint large decrease of stocks, bonds and real estate.

We continue the analysis by reducing one of the maximum returns smaller, while the other two remain the same. In Figure 2 we plot the probability attributed to the stress test as a function of the stock return. To ease the interpretation of the results, we plot the expected waiting time (i.e. the reciprocal of the probability). So Figure 2 shows the expected waiting time for a joint stress event as a function of the stress return on stocks that ranges from -1.75% to -3.75%. The other two stress events are a decrease of at least 0.4% (bonds) and 0.75% (real estate). The joint probabilities are calculated using the empirical, Gaussian and Student's t dependence function. Figure 3 repeats the analysis for bonds and Figure 4 for real estate.

Of course, the expected waiting time rises if the stress return gets more extreme. If the return on stocks is at most -2.5% (with a marginal probability of 0.0294), the expected waiting time based on the Student's t dependence function equals 802 days. According to the Gaussian dependence function, it equals 2281 days, whereas the empirical distribution assigns 994 days. For an even more extreme event in which stocks decrease by more than 3.5% (marginal probability 0.0070), the Student's t dependence function assigns a waiting time of 2375 days, so once every 10 years. The Gaussian dependence function would lead one to expect this event to happen once every 9648 days, almost 40 years! This event falls outside the observations of the last 4 years, so the empirical dependence function can not be used.

Considering the complete graphs confirms these observations. The empirical dependence function can attribute probabilities only to a fixed set of stress returns, which does not cover the complete domain. It is also obvious that the speed at which the waiting times increase is larger for the Gaussian than for the Student's t dependence function. Moreover, the difference between the two can be observed already for stress returns that have a moderate probability ($p > 0.001$). Finally, if we compare the graph for the empirical with the graph for the Student's t dependence function, we note that the Student's t dependence function fits the empirical one quite well in case of stocks and real estate and not too bad in case of bonds.

To investigate the significance of the results, we construct confidence intervals. Based on the estimated covariance matrix of the estimates, we create 200 possible new parameter estimates. Using the new parameter estimates we assign probabilities to the stress events. These in turn are used to determine the bounds of the 95% confidence intervals. The thin lines in the Figure 2 to 4 represent the bounds of the confidence intervals. We observe that the confidence intervals are non-overlapping in all cases. This implies that the probabilities assigned by the Gaussian and Student's t dependence functions are significantly different. Furthermore, the empirical probabilities fall almost completely in the confidence interval of the Student's t dependence function. We interpret this as further support for the Student's t dependence function.

The analysis makes clear that the chosen dependence model has substantial effects. The Gaussian dependence function makes investors more optimistic about the occurrence of joint extreme events. On the contrary, the Student's t dependence function assigns larger probabilities to joint extreme events. The difference is a factor 2 for moderate probability levels (once per 14 months) and increases to 4 for extreme levels (once per 10 years). Consequently, the Student's t dependence function makes investors more prudent. Our analysis shows that this dependence function is more realistic.

5 Conclusions and discussion

A crucial ingredient of a risk management system is understanding the magnitudes of the risks of extreme events and their dependence. This requires knowledge of the joint distribution of the underlying risk factors. While the tail behavior of univariate asset returns can be appropriately modelled by a fat-tailed distribution or by means of a semi-parametric method, a simple correlation matrix often appears inappropriate to model dependence between the different returns. In this paper we advocate the use of the Student's t dependence function and illustrate its use for carrying out stress tests.

Unlike the Gaussian dependence function, which is characterized by a correlation matrix and implies linear dependence, the Student's t dependence function is parameterized by a correlation matrix and a degrees of freedom parameter. This additional flexibility allows for stronger dependence between extreme returns than implied by the Gaussian dependence function. The robust empirical evidence in Longin and Solnik (2001) demonstrates that asset return actually exhibit stronger dependence in the tails than implied by the Gaussian copula. Since stress tests deal with extreme events, an appropriate dependence model is crucial. Moreover, the use of the Student's t dependence function reduces the need to separately model extreme returns and their dependence.

In this paper we considered the implications of the use of the Student's t dependence function for stress tests for a US portfolio consisting of stocks, bonds and real estate. We model the marginal distributions by the semi-parametric method of Daníelsson and de Vries (2000) or a Student's t distribution and use both the normal and Student's t dependence functions. The estimated degrees of freedom parameter of the Student's t dependence function is rather low, indicating stronger tail dependence than implied by a correlation matrix.

To formally test the fit of the dependence functions, we extended the testing procedure developed by Malevergne and Sornette (2001). These tests are based on alternative comparisons of Kolmogorov and Anderson-Darling distances. When the Student's t distribution is used to model the marginal distributions of all returns, both the Gaussian and Student's t dependence function have to be rejected, although the evidence against the latter dependence function is much weaker. However, when the semi-parametric method is used for the marginals, only the Gaussian dependence function is statistically rejected for each of the four distance measures. Replacing the Gaussian by the Student's t dependence function improves the fit substantially. Overall, the tests clearly indicate that on statistical grounds the semi-parametric method combined with the Student's t dependence function is to be preferred to other models.

Finally, we computed probabilities assigned to stress tests. Starting with a stress test that is not too extreme, we found that the Student's t dependence function deems the event twice as likely as the Gaussian dependence function. For more extreme stress tests the differences become even larger. Furthermore, the probabilities implied by the Student's t dependence function are more in line with the empirical data. Confidence intervals show that the difference between the probabilities based on the Gaussian and Student's t dependence functions differ significantly.

Our results contribute to the ongoing discussion on dependence present in financial markets. While stress testing is only a part of the risk management system, it may be interesting to see the consequence of modelling dependence on other elements of the risk management system, such as the calculation of risk measures. Furthermore, using the Student's t dependence function can improve the analysis of the diversification possibilities actually available to investors. This is a topic for further research.

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	stocks	bonds	real estate
mean	-0.0162	0.0275	0.0377
volatility	1.40	0.332	0.736
skewness	0.212	-0.343	0.267
kurtosis	4.17	3.80	8.01
min	-5.83	-1.40	-3.49
max	5.73	1.17	4.68

Table 1: *Summary statistics for the returns (in %) of the S&P 500 Composite index (stocks), the JP Morgan US Government Bond Index (bonds) and the NAREIT All Index (real estate).*

	stocks	bonds	real estate
left tail			
$\gamma = 1/\alpha$	0.194 (0.047)	0.185 (0.044)	0.366 (0.088)
α	5.14	5.42	2.73
right tail			
$\gamma = 1/\alpha$	0.219 (0.053)	0.196 (0.047)	0.313 (0.075)
α	4.57	5.09	3.20
both tails			
$\gamma = 1/\alpha$	0.208 (0.050)	0.165 (0.040)	0.347 (0.083)
α	4.82	6.06	2.88

Table 2: *Tail index estimates based on the modified Hill-estimator from Huisman et al. (2001) applied to the returns of the S&P 500 Composite index (stocks), the JP Morgan US Government Bond Index (bonds) and the NAREIT All Index (real estate). Standard errors are in parentheses.*

	stocks	bonds	real estate
w	28.86	33.12	51.07
p_1	0.068	0.023	< 0.001
p_2	0.025	0.007	< 0.001

Table 3: χ^2 *goodness-of-fit test applied to the Student's* t *distribution, which is used to model the returns of the S&P 500 Composite index(stocks), JP Morgan US Government Bond Index (bonds) and the NAREIT All Index (real estate). The degrees of freedom used are reported in Table 2. The parameters for the mean and variance are determined using their sample counterparts. The value of the* w*-statistic is determined as in* (13) *for 20 equiprobable intervals.* p_1 *(*p_2*)is the p-value based on a* χ^2_{19} *(*χ^2_{16}*) distribution.*

	stocks	bonds	real estate
stocks	1	-0.206 (0.030)	0.463 (0.023)
bonds	-0.206 (0.030)	1	-0.111 (0.031)
real estate	0.463 (0.023)	-0.111 (0.031)	1

Table 4: *Two-step maximum likelihood estimates for the correlation matrix of the Gaussian dependence function, applied to the returns of the S&P 500 Composite index (stocks), the JP Morgan US Government Bond Index (bonds) and the NAREIT All Index (real estate). The semi-parametric method of Daníelsson and de Vries (2000) is used to model the marginal distributions, using the tail index estimates of Table 2. The left (right) tail starts at 0.01 (0.99) probability. Standard errors are in parentheses.*

	stocks	bonds	real estate
stocks	1	-0.189 (0.033)	0.454 (0.026)
bonds	-0.189 (0.033)	1	-0.0912 (0.034)
real estate	0.454 (0.026)	-0.0912 (0.034)	1
degrees of freedom	8.17 (1.66)		

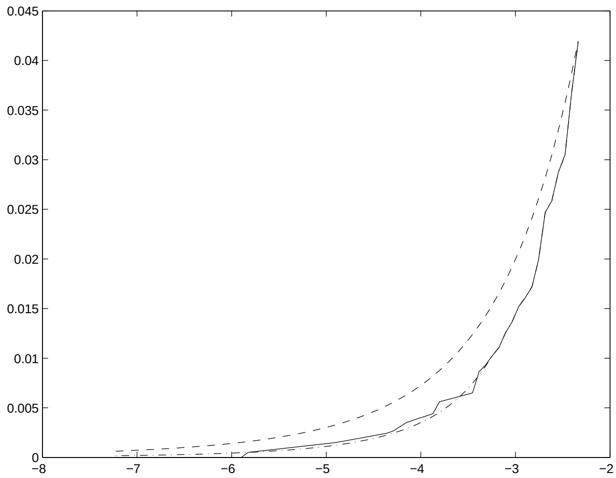
Table 5: *Maximum likelihood estimates for the correlation matrix and degrees of freedom parameter of the Student's t dependence function, applied to the returns of the S&P 500 Composite index (stocks), the JP Morgan US Government Bond Index (bonds) and the NAREIT All Index (real estate). The semi-parametric method of Daníelsson and de Vries (2000) is used to model the marginal distributions, using the tail index estimates of Table 2. The left (right) tail starts at 0.01 (0.99) probability. Standard errors are in parentheses.*

combination	correlation coefficient	degrees of freedom
stocks and bonds	-0.182 (0.035)	5.21 (1.16) [2.94, 7.48]
stocks and real estate	0.458 (0.027)	6.03 (1.44) [3.21, 8.84]
bonds and real estate	-0.0973 (0.034)	11.6 (4.89) [2.01, 21.2]

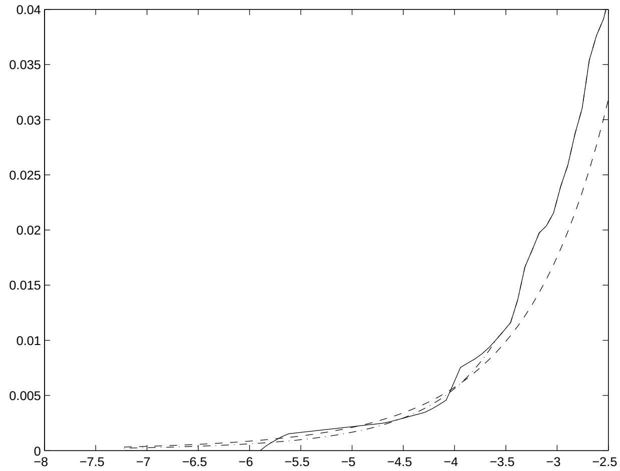
Table 6: *Maximum likelihood estimates for the correlation coefficient and degrees of freedom parameter of the bivariate Student's t copula, applied to all combinations of the returns of the S&P 500 Composite index (stocks), the JP Morgan US Government Bond Index (bonds) and the NAREIT All Index (real estate). The semi-parametric method of Daníelsson and de Vries (2000) is used to model the marginal distributions, using the tail index estimates of Table 2. The left (right) tail starts at 0.01 (0.99) probability. Standard errors are in parentheses. Between brackets a 95%-confidence interval is reported.*

marginal model	semi-parametric		Student's t	
dependence function	Gaussian	Student's t	Gaussian	Student's t
d_1	0.032 (0.227)	0.019 (0.883)	0.066 $(2.6 \cdot 10^{-4})$	0.089 $(1.6 \cdot 10^{-4})$
d_2	0.014 (0.118)	0.004 (0.994)	0.034 $(2.2 \cdot 10^{-4})$	0.049 $(6.8 \cdot 10^{-4})$
d_3	0.108 (0.103)	0.094 (0.371)	0.136 (0.043)	0.182 (0.041)
d_4	0.038 (0.090)	0.011 (0.996)	0.080 $(2.3 \cdot 10^{-4})$	0.102 (0.0011)

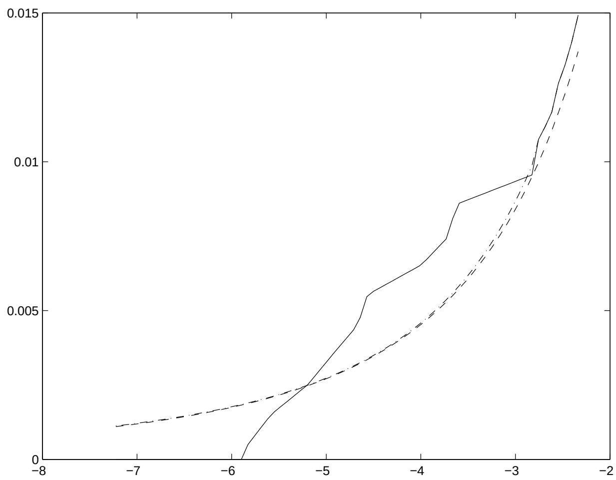
Table 7: *Test statistics for the fit of the Gaussian and Student's t dependence function.* d_1 *is the Kolmogorov distance, defined in* (8)*, and* d_2 *its average as defined in* (9)*.* d_3 *is known as Anderson-Darling distance, given by* (10) *and* d_4 *its average as given by* (11)*. The results are derived using either the semi-parametric method or the Student's t distribution to model the marginal distributions of the returns and maximum likelihood estimation for the parameters of the dependence functions. p-values for the hypothesis that the values of the statistics equal their theoretical values versus the alternative of exceedance are given in parentheses.*



(a) stocks



(b) bonds



(c) real estate

Figure 1: *Different models applied to the left tails of the return series of the S&P 500 Composite index (stocks), the JP Morgan US Government Bond Index (bonds) and the NAREIT All Index (real estate). The solid line gives the empirical cumulative probability function, the dashed line gives the approximating Student's t distribution function and the dotted line gives the semi-parametric approximation with a tail that start at 0.01 probability.*

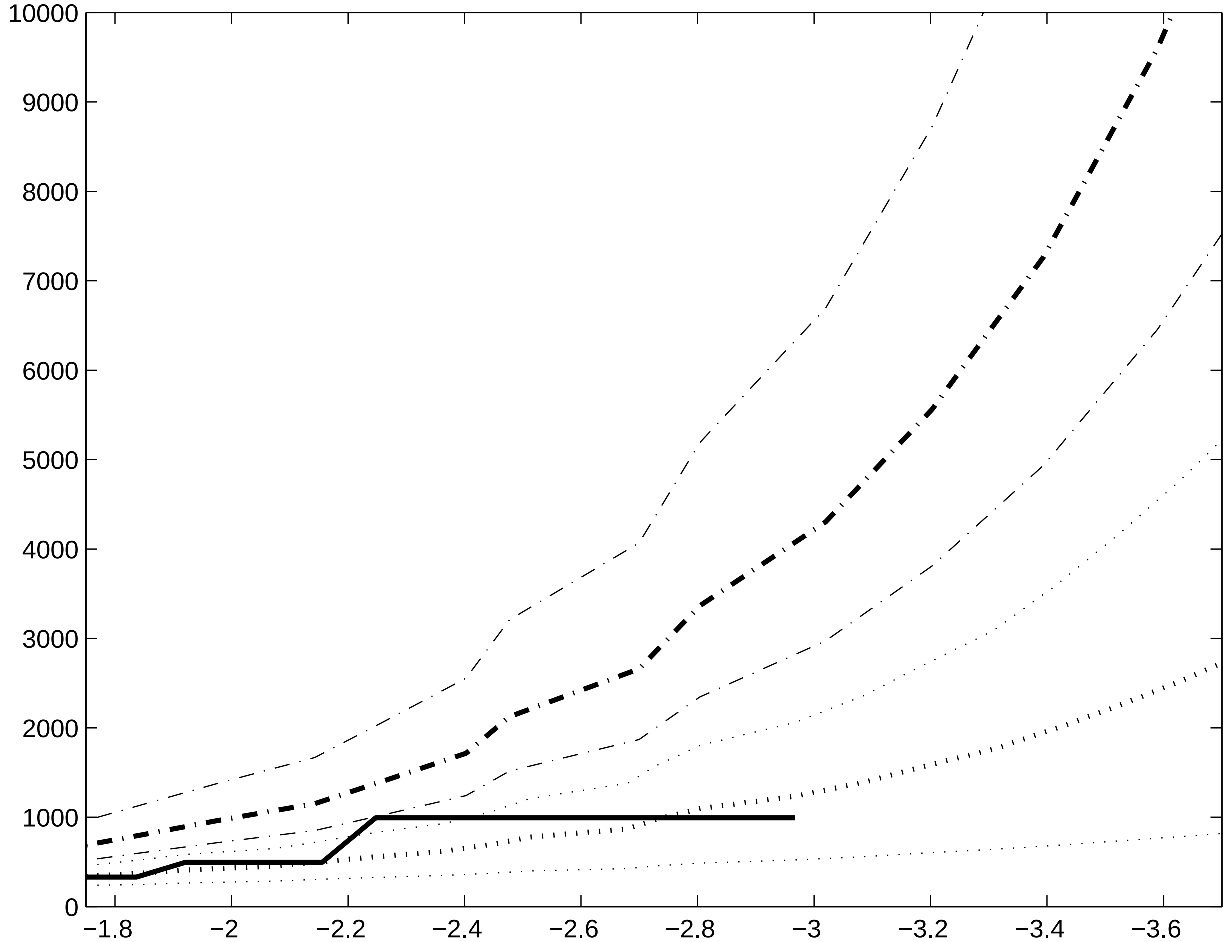


Figure 2: *The joint probability for stress returns as a function of a stress return for stocks. Stress returns are reported as expected waiting time* $1/p$ *in days. For the remaining asset categories fixed stress returns are chosen: bonds -0.40%, real estate -0.70%. The semi-parametric method by Daníelsson and de Vries (2000) is used to model the marginal distribution of each return series (with tails starting at 0.01 and 0.99 probability). Three dependence functions are used to calculate the joint probability: empirical (solid line), Gaussian (dash-dotted) and Student's t (dotted). The thick lines represent the probabilities based on the estimates in Table 5. The thin lines represent the bounds of the 95% confidence interval around the probabilities.*

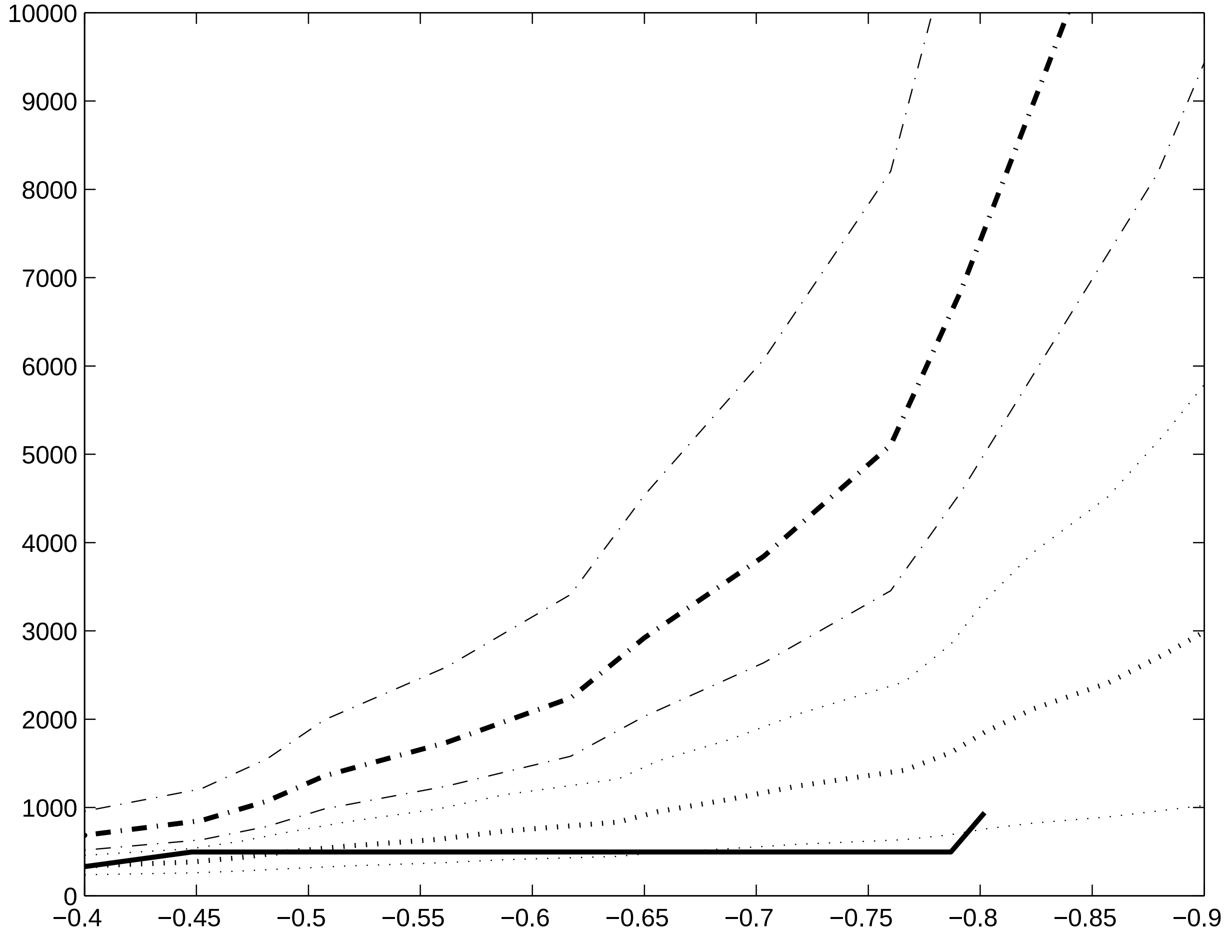


Figure 3: *The joint probability for stress returns as a function of a stress return for bonds. Stress returns are reported as expected waiting time* $1/p$ *in days. For the remaining asset categories fixed stress returns are chosen: stocks -1.75%, real estate -0.70%. The semi-parametric method by Daníelsson and de Vries (2000) is used to model the marginal distribution of each return series (with tails starting at 0.01 and 0.99 probability). Three dependence functions are used to calculate the joint probability: empirical (solid line), Gaussian (dash-dotted) and Student's t (dotted). The thick lines represent the probabilities based on the estimates in Table 5. The thin lines represent the bounds of the 95% confidence interval around the probabilities.*

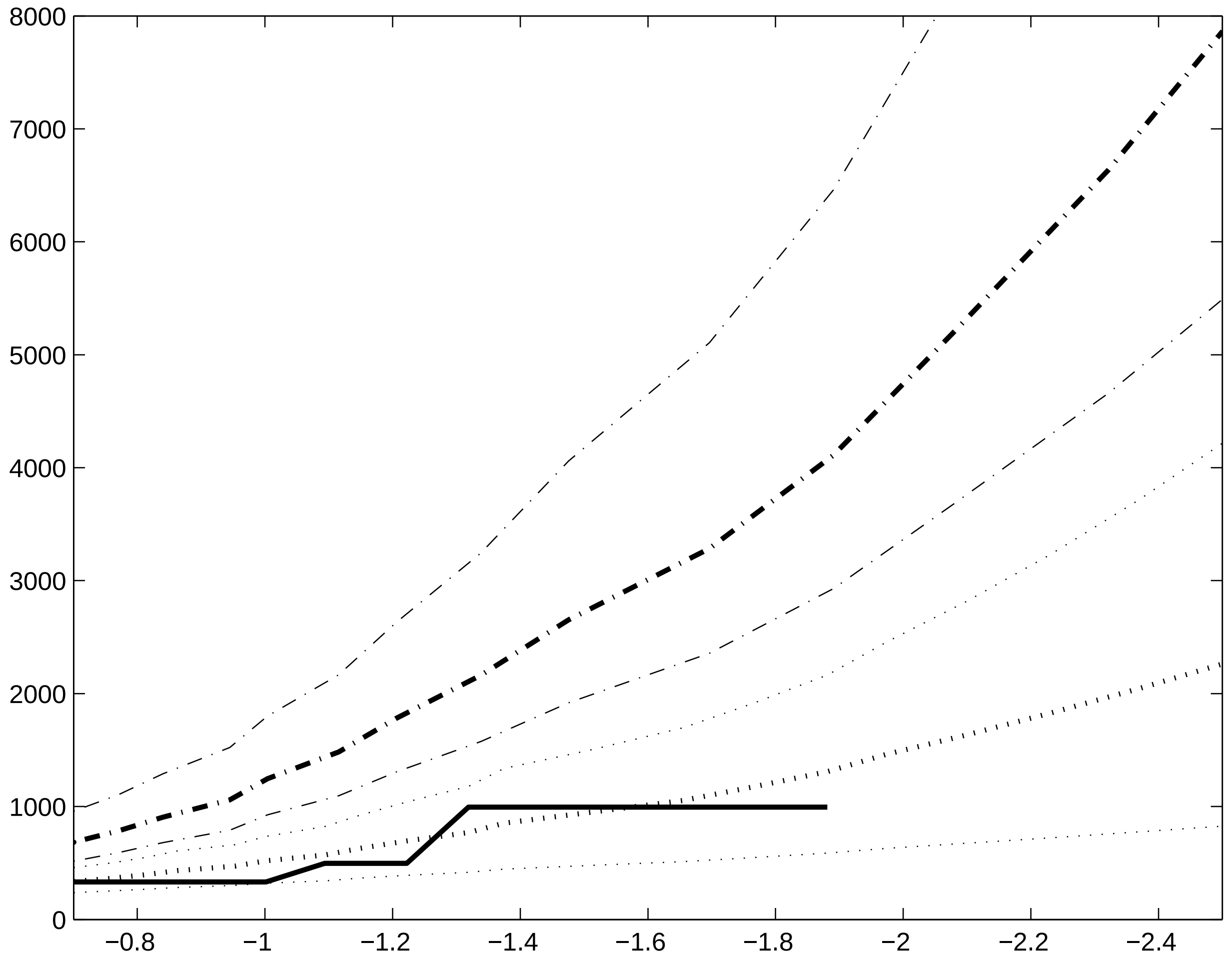


Figure 4: *The joint probability for stress returns as a function of a stress return for real estate. Stress returns are reported as expected waiting time* $1/p$ *in days. For the remaining asset categories fixed stress returns are chosen: stocks -1.75%, bonds -0.40%. The semi-parametric method by Daníelsson and de Vries (2000) is used to model the marginal distribution of each return series (with tails starting at 0.01 and 0.99 probability). Three dependence functions are used to calculate the joint probability: empirical (solid line), Gaussian (dash-dotted) and Student's t (dotted). The thick lines represent the probabilities based on the estimates in Table 5. The thin lines represent the bounds of the 95% confidence interval around the probabilities.*

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