Utility Independence of Multiattribute Utility Theory is Equivalent to Standard Sequence Invariance of Conjoint Measurement

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Abstract

Utility independence is a central condition in multiattribute utility theory, where attributes of outcomes are aggregated in the context of risk. The aggregation of attributes in the absence of risk is studied in conjoint measurement. In conjoint measurement, standard sequences have been widely used to empirically measure and test utility functions, and to theoretically analyze them. This paper shows that utility independence and standard sequences are closely related: Utility independence is equivalent to a standard sequence invariance condition when applied to risk. This simple relation between two widely used conditions in adjacent fields of research is surprising and useful. It facilitates the testing of utility independence because standard sequences are flexible and can avoid cancellation biases that affect direct tests of utility independence. Extensions of our results to nonexpected utility models can now be provided easily. We discuss applications to the measurement of quality-adjusted life-years (QALY) in the health domain.

Key Words: Utility independence, standard sequences, multiattribute utility, conjoint measurement, nonexpected utility.

1. Introduction

Utility independence is widely used in decision analysis (Keeney & Raiffa, 1976; Guerrero & Herrero, 2005; Engel & Wellman, 2010). In medical decision making, utility independence underlies the health utility index, a widely used method to derive utilities for multiattribute health states (Feeny, Furlong, Torrance, Goldsmith, Zhu, Depauw, Denton, & Boyle, 2002; Feeny, 2006). Analyses of utility independence are usually based on the normatively convincing, but descriptively problematic, expected utility theory for choices between risky prospects (probability distributions over outcomes). Then the condition usually implies that multiattribute utility is additive, multiplicative, or multilinear.

Utility independence concerns situations where the levels of some attributes are fixed deterministically. The condition then requires that preferences between prospects over the remaining attributes should be independent of the fixed deterministic levels. This requirement has often been tested directly (Miyamoto & Eraker, 1988; Bleichrodt & Johannesson, 1997; Bleichrodt & Pinto, 2005; Spencer & Robinson, 2007). One problem with direct tests of utility independence is that they induce subjects to ignore the common fixed values, not because this is their true preference but rather as a heuristic to simplify the task before any consideration of true preference (Kahneman & Tversky, 1979, the *cancellation heuristic*). That such distorting heuristics can sometimes increase consistency, misleadingly suggesting verification of preference conditions, was emphasized by Loomes, Starmer, & Sugden (2003). For direct tests of utility independence the cancellation heuristic will indeed create artificial support for the condition.

A second problem with traditional analyses of utility independence is that they have been based on expected utility maximization. There is, however, much evidence that expected utility is violated empirically (Allais, 1953; Ellsberg, 1961; Kahneman and Tversky 1979; Starmer, 2000). Extensions of utility independence to nonexpected utility models include Bier & Connell (1994), Bleichrodt, Schmidt, & Zank (2009), Bouyssou & Pirlot (2003), Dyckerhoff (1994), and Miyamoto & Wakker (1996).

The aggregation of attributes is also studied in conjoint measurement (Krantz, Luce, Suppes, & Tversky, 1971). Unlike multiattribute utility theory and decision analysis, conjoint measurement does not assume risk to be present. However, one can still use the techniques of conjoint measurement in the presence of risk. This is the approach to multiattribute utility taken in this paper. A common technique underlying many results in conjoint measurement is the construction of standard sequences.[1] These are sequences of attribute levels that are equally spaced in utility units, endogenously derived from preferences without using the utility function. In marketing, standard sequences are used in the saw-tooth method (Fishburn, 1967; Louviere, Hensher, & Swait, 2000). Krantz et al. (1971) explain the importance of standard sequences in great detail. Many preference conditions amount to invariance of particular standard sequences. By imposing such specific invariance conditions, specific functional forms of the multiattribute utility function can be derived.[2]

This paper shows that there exists a surprisingly simple relation between multiattribute utility and conjoint measurement: utility independence is equivalent to a version of standard sequence invariance. This opens new and useful ways to analyze utility independence. Standard sequence techniques are flexible and efficient and they

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[1] See Abdellaoui (2000), Baron (2008, Chs. 10 and 14), Booij & van de Kuilen (2009), Fishburn & Rubinstein (1982, pp. 682-3 and Figure 1), Loewenton & Luce (1966), von Winterfeldt & Edwards (1986, p. 267).

[2] See Bouyssou & Pirlot (2004), Ebert (2004), Fishburn & Edwards (1997, Axiom 8), Gilboa, Schmeidler, & Wakker (2002) Harvey (1986, p. 1126), Casadesus-Masanell, Klibanoff, & Ozdenoren (2000), Krantz et al. (1971), Nau (2006, Axiom 4), Schmidt (2003), Skiadas (1997) , Stigler (1950), Tversky & Kahneman (1992), Tversky, Sattath, & Slovic (1988), Wakker (1984), Wakker (2010), Wakker & Tversky (1993).

can avoid the aforementioned cancellation bias. Further, they give direct quantitative measurements of utility, which is useful in its own right. They do not directly appeal to risk, as does utility independence, but they focus on tradeoffs between attributes, avoiding the complications of risky decisions. Finally, they can easily be extended to nonexpected utility models, offering the possibility to design tests of utility independence that are robust to violations of expected utility.

2. Notation

We start by assuming a simple model on a simple domain (a rank-ordered set of binary prospects) that is present as a substructure in expected utility but also in most nonexpected utility models. In all these models, the theorems that we obtain within the simple model immediately extend to the whole model. Consequently, our main result, Observation 5.2, applies to all these (non)expected utility models. Miyamoto and Wakker (1996) similarly used rank-ordered binary prospects to obtain results for many nonexpected utility theories.

We consider decision under uncertainty with one *event* E. E is uncertain in the sense that the decision maker does not know for sure if it is true ("will happen") or not. An objective probability p of E may (the case of risk) or may not (the case of uncertainty and ambiguity) be given. Our analysis applies to either case. We consider *prospects* $x_E y$ yielding *outcome* x if E is true and outcome y otherwise. If an objective probability p is given for E, then we can also write $x_p y$. X denotes the *outcome set*.

A preference relation $\succcurlyeq$ is given over the outcomes. The domain of prospects is *rank-ordered*: We assume without further mention that always $x \succcurlyeq y$ in prospects $x_E y$.

The resulting rank-ordered[3] set of prospects is denoted $X^2_{\downarrow}$. A preference relation $\succcurlyeq'$ is given on $X^2_{\downarrow}$. *Constant prospects*, $x_E x$, yielding outcome x for sure are identified with that outcome x. The preference relation $\succcurlyeq'$ generated over outcomes is assumed to agree with $\succcurlyeq$. Thus $\succcurlyeq'$ defined over prospects is an extension of $\succcurlyeq$ defined over outcomes. We will therefore write $\succcurlyeq$ instead of $\succcurlyeq'$ henceforth. Strict preference and indifference are defined as usual, and are denoted $\succ$ and ~.

We assume that the outcome set X is a *two-attribute* product set $\mathcal{Q} \times \mathcal{T}$, with generic element x = (Q,T). Q designates the first attribute and T designates the second, and $\mathcal{Q}$ and $\mathcal{T}$ are *attribute sets*. For example, if outcomes are chronic health states then Q designates a health state and T designates a time period (life duration). The extension of our results to cases of more than two attributes will be presented in §5.

We assume throughout that preferences over prospects $(Q_1,T_1)_E(Q_2,T_2)$ can be represented by

$$\pi U(Q_1,T_1) + (1-\pi)U(Q_2,T_2). \qquad (2.1)$$

Here U: $\mathcal{Q}\times\mathcal{T} \to \mathbb{R}$ is the *utility function*, whose particular form is the central topic of multiattribute utility and of this paper. The *decision weight* of event E is $0 < \pi < 1$. Equation (2) includes virtually all decision theories known today. Well-known examples are: (a) Expected utility where $\pi = P(E)$ is the probability of event E, objective in the case of risk and subjective in the case of uncertainty; (b) rank-dependent utility for risk (Quiggin, 1982) where $\pi = w(p)$ with p the objective probability of event E and w a probability weighting function; (c) rank-dependent utility for uncertainty (also called Choquet expected utility) or prospect theory where

[3] Another widely used term in the literature is comonotonic.

$\pi = W(E)$ with W a nonadditive weighting function or capacity (for gains under prospect theory); (d) maxmin expected utility (Gilboa & Schmeidler, 1989). Further details are in the footnote to Observation 5.2, and in Wakker (2010, §§6.11 and 10.6).

3. Utility independence

The second attribute $\mathcal{T}$ is *utility independent* if

$$(Q,T_1)_E(Q,T_2) \succcurlyeq (Q,T_3)_E(Q,T_4)$$

$$\Leftrightarrow$$

$$(Q',T_1)_E(Q',T_2) \succcurlyeq (Q',T_3)_E(Q',T_4) \tag{3.1}$$

for all Q,Q′ and for all T_1,T_2,T_3,T_4. That is, preferences do not depend on the particular deterministic level at which Q is fixed. As throughout, it is implicitly assumed that all prospects are contained in $X_{\downarrow}^{2}$. *Preferential independence* is utility independence restricted to constant prospects:

$$(Q,T_1) \succcurlyeq (Q,T_3)$$

$$\Leftrightarrow$$

$$(Q',T_1) \succcurlyeq (Q',T_3). \tag{3.2}$$

In economic consumer theory, preferential independence is known as separability of $\mathcal{T}$, and in conjoint measurement (Krantz et al., 1971) it is part of joint independence. Preferential independence implies that we can define preferences over the second attribute $\mathcal{T}$ independently from the first attribute. It is naturally satisfied if $\mathcal{T}$ is an interval and monotonicity holds. A convenient implication of preferential independence is that changing Q in Eq. 3.1 does not affect rank-ordering. That is, the upper two prospects in Eq. 3.1 are contained in $X_{\downarrow}^{2}$ if and only if the lower two are.

Utility independence of $\mathcal{T}$ holds if U is *additive* $\big(U(Q,T) = V(Q) + W(T)\big)$ or *multiplicative* $\big(U(Q,T) = V(Q)W(T)\big)$ with all values V(Q) of the same sign, which can then be taken positive. Under additional conditions, utility independence is not only necessary, but also sufficient for U being additive or multiplicative (Miyamoto and Wakker, 1996, Theorem 3). Then, in Eq. 3.3 below, f or g has to be constant. The following theorem extends a well known result from classical setups to our domain $X^2_{\downarrow}$.

THEOREM 3.1. Assume that the image of the function $T \mapsto U(Q,T)$ is an interval for all Q. Then $\mathcal{T}$ is utility independent if and only if

$$U(Q,T) = f(Q)V(T) + g(Q) \qquad (3.3)$$

for some functions f, V, g with f positive. □

4. Standard sequence invariance

A convenient feature of the standard sequence technique introduced next is that it is directly related to the empirical measurement of utility. $T_0, \ldots, T_n$ is a (*Q*-)*standard sequence* if there exist Q^*, T_g, and T_G such that, for $i = 0,\ldots, n-1$,

$$(Q^*,T_g)_E(Q,T_{i+1}) \sim (Q^*,T_G)_E(Q,T_i)\ . \qquad (4.1)$$

(Q^*,T_g) and (Q^*,T_G) are called *gauge outcomes*. They serve as a measuring rod to peg out the standard sequence. For later purposes, it is of interest to note that Q^* and Q can be different. The proof of the following lemma is given in the main text because it may be clarifying.

LEMMA 4.1. Under Eq. 2.1, a Q-standard sequence is equally spaced in utility units ($U(Q,T_{i+1}) - U(Q,T_i)$ is independent of i).

PROOF. By Eq. 2.1, the $(1-\pi)$ weighted differences $U(Q,T_{i+1}) - U(Q,T_i)$ all match exactly the same π weighted difference $U(Q^*,T_G) - U(Q^*,T_g)$. □

We now turn to comparisons of standard sequences for different values of Q. A Q-standard sequence $T_0, T_1, T_2, \ldots$ and a Q´-standard sequence $T_0', T_1', T_2', \ldots$ are *inconsistent* if they satisfy $T_0 = T_0'$ and $T_1 = T_1'$, but, for some $i > 1$, T_i and T_i' are not equivalent in the sense that $(Q, T_i) \not\sim (Q, T_i')$ or $(Q', T_i) \not\sim (Q', T_i')$.[4] Under Eq. 2.1, inconsistencies are possible because equal spacedness for $U(Q,\cdot)$ need not correspond with equal spacedness for $U(Q',\cdot)$. *Standard sequence invariance on* $\mathcal{T}$ means that such inconsistencies are excluded for all $Q, Q' \in \mathcal{Q}$.

THEOREM 4.2. Assume Eq. 2.1, with the image of the function $T \mapsto U(Q,T)$ an interval for each Q. Preferential independence of $\mathcal{T}$ and standard sequence invariance on $\mathcal{T}$ hold if and only if

$$U(Q,T) = f(Q)V(T) + g(Q) \tag{4.2}$$

for some functions f, V, g with f positive. □

The comparison of Theorems 3.1 and 4.2 establishes an interesting connection between conjoint measurement and multiattribute utility because the necessary and sufficient form in Eq. 3.3 is identical to that in Eq. 4.2: Under preferential independence and richness, standard sequence invariance on $\mathcal{T}$ is equivalent to utility independence of $\mathcal{T}$! That is, we can test utility independence by testing standard sequence invariance. We can now for instance reduce the cancellation heuristic by taking different Q and Q* in Eq. 4.1. This way, we can avoid biases that have distorted traditional tests of utility independence. We will state the relations between utility independence and standard sequence invariance formally in the following section.

We next provide an axiomatization of multiplicative utility, useful for QALY measurement in health (§6). We call $T_0 \in \mathcal{T}$ a *null element* if $(R,T_0) \sim (R',T_0)$ for all R and R´.

OBSERVATION 4.3. Assume that Eqs. 2.1 and 4.2 hold. If $\mathcal{T}$ contains a null element then g(Q) is constant and can be taken equal to 0, giving a multiplicative representation

$$U(Q,T) = f(Q)V(T) . \qquad (4.3)$$

□

For similar results, see Miyamoto, Wakker, Bleichrodt, & Peters (1998, Theorem 3.1) and Bleichrodt and Pinto (2005, Theorem 2). A remarkable implication of the above result is that $\mathcal{Q}$ then also is utility independent on the subdomain where V is positive (which excludes the null element).

We have defined standard sequences for outcomes under not-E, that is, outcomes ranked worst and less preferred than the gauge outcomes. Standard sequences can

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[4] It can be seen that Eq. 2.1 implies $Q' \neq Q$.

equally well be defined for outcomes under E, when they are ranked best and are preferred to the gauge outcomes, using the following indifferences:

$$(Q,T_{i+1})_E(Q^*,T_g) \sim (Q,T_i)_E(Q^*,T_G). \qquad (4.4)$$

For representation theorems, the topic of this paper, it is desirable to use weak preference conditions in order to obtain the logically strongest theorems. For empirical investigations it can be interesting to consider more restrictive preference conditions, to obtain more possibilities to falsify a theory or to measure its concepts. Hence, for empirical purposes it may be interesting to also consider standard sequences defined in Eq. 4.4 and to investigate consistency properties between such larger classes of standard sequences. It easily follows that we should also have invariance here under Eq. 4.2.

Remark A.2 will indicate a mathematical generalization of our theorems that we do not present in the main text because it loses the empirically attractive reduction of the cancellation heuristic. An interesting feature of the weaker preference condition used there is that it is a common weakening of utility independence and standard sequence invariance. Thus the two conditions are different strengthenings of a common underlying necessary and sufficient condition. This observation clarifies the mathematical nature of our results.

5. Generalizations and main result

We first extend our results to n-attribute utility. Assume that X is $X_1 \times \cdots \times X_n$ for a natural number $n \geq 2$, with generic element $(x_1, \ldots, x_n)$. Let $I \subset \{1,\ldots,n\}$ and write $\mathcal{T} = \Pi_{i \in I} X_i$ and $\mathcal{Q} = \Pi_{i \notin I} X_i$. We can write $X = \mathcal{Q} \times \mathcal{T}$. *Utility independence* of I is defined as utility independence of $\mathcal{T}$ (Eq. 3.1). That is, if the attribute levels outside of I are kept fixed at deterministic levels, then the preferences generated over prospects over $\mathcal{T}$ are independent of the deterministic levels chosen. We can define standard sequences on $\Pi_{i\in I}X_i$ exactly as in Eq. 4.1, where now T_g, T_{i+1}, T_G, $T_i \in \Pi_{j\in I}X_j$, and Q^*, $Q \in \Pi_{i\notin I}X_i$. Standard sequence invariance on $\Pi_{i\in I}X_i$ requires consistency between standard sequences in $\Pi_{i\in I}X_i$ for all Q and Q′ in $\Pi_{i\notin I}X_i$. The following theorem immediately follows from Theorems 3.1 and 4.2.

THEOREM 5.1. Assume a preference $\succcurlyeq$ on $X^2_{\downarrow}$, with $X = X_1 \times \cdots \times X_n$, and $I \subset \{1,\ldots,n\}$. Let $\mathcal{T} = \Pi_{i\in I}X_i$ and $\mathcal{Q} = \Pi_{i\notin I}X_i$. Preferences are represented by Eq. 2.1 (with $T = (x_i)_{i\in I}$ and $Q = (x_i)_{i\notin I}$). The image of $(x_i)_{i\in I} \mapsto U\big((x_j)_{j\notin I},(x_i)_{i\in I}\big)$ is an interval for each $(x_j)_{j\notin I}$. Then I is utility independent if and only if $\Pi_{i\in I}X_i$ is preferentially independent and standard sequence invariance on $\Pi_{i\in I}X_i$ holds. □

We next consider decision theories defined on general domains of prospects, leading to our main result. Now prospects can be probability distributions over outcomes with more than one probability involved, or mappings from multi-element state spaces to outcomes, and prospects need not all have the same rank-ordering. The definition of utility independence needs no adaptation: On all subproduct domains, preference is independent of the deterministic level at which outside attributes are kept fixed. We define standard sequence invariance by defining standard sequences on all subsets isomorphic to $X^2_{\downarrow}$ (two outcomes and a fixed event or probability, always with the same rank ordering). No inconsistencies should result both within sets $X^2_{\downarrow}$ and across different sets $X^2_{\downarrow}$. In many theories, this definition can be extended. For example, under rank-dependent utility it can be extended to all multi-event sets of prospects that are comonotonic (defined in Wakker 2010, §10.12). For brevity, we do not elaborate on this point.

OBSERVATION 5.2. Let $X = X_1 \times \cdots \times X_n$ be a set of outcomes, and let $\succcurlyeq$ be a preference relation on a set of prospects. Prospects can be probability distributions over X (risk), or functions from a state space S to X (uncertainty). The set of prospects is rich enough to contain a set of the form $X^2_{\downarrow}$. Preferences are represented by a model that implies Eq. 2.1 on $X^2_{\downarrow}$ with the same utility function U as in Eq. 2.1 used throughout the domain. The utility function is an interval scale, i.e. preferences are not affected if a constant is added to utility or if utility is multiplied by a positive constant.[5] If, for a set $I \subset \{1,\ldots,n\}$, the utility image of $\Pi_{i \in I} X_i$ is an interval whenever the attributes outside of I are kept fixed, then utility independence of I is equivalent to preferential independence and standard sequence invariance on $\Pi_{i \in I} X_i$. □

6. An application to health

This section applies the above results to medical decision making. Outcomes (Q,T) are chronic health states, with Q describing the constant health state and T the life duration spent in this health state, followed by death. Unlike in economics or psychology, statistical probabilities of risks are often available in the health domain. We will assume that prospects are probability distributions over chronic health states.

[5] The requirements in our observation hold for most theories that are popular today. These include expected utility for risk (von Neumann & Morgenstern, 1944) and for uncertainty (Savage, 1954), rank-dependent utility for risk (Quiggin, 1982) and for uncertainty (Gilboa, 1987; Schmeidler, 1989), prospect theory if there are only gains (Luce & Fishburn, 1991; Tversky and Kahneman, 1992), disappointment aversion theory (Gul, 1991), maxmin expected utility (Gilboa and Schmeidler 1989; Wald, 1950) and the α-maxmin model (Hurwicz, 1951; Jaffray, 1994), contraction expected utility (Gajdos, Hayashi, Tallon, & Vergnaud, 2008), and binary rank-dependent utility (Luce, 2000, Ch. 3; Ghirardato & Marinacci, 2001; Wakker, 2010, §§6.11, 10.6). Observation 5.2 applies to all these theories.

The utility of life duration T is described by a function V. The commonly found subjective time preferences and discounting imply that V is concave, with future life years contributing less to V than the first life years to come. Since the 1980s it has become customary to correct life duration for quality of life, leading to the QALY model f(Q)V(T), where f designates the correction factor due to the subjective quality of life of health state Q. The QALY model is widely used in health policy.

Preference axiomatizations can serve to justify the use of QALYs as outcome measure (Pliskin, Shepard, & Weinstein, 1980; Miyamoto & Eraker, 1988; Bleichrodt & Quiggin, 1997; Bleichrodt, Wakker, & Johannesson, 1997; Miyamoto et al., 1998; Miyamoto, 1999; Bleichrodt & Miyamoto, 2003; Doctor & Miyamoto, 2003; Doctor, Bleichrodt, Miyamoto, Temkin, & Dikmen, 2004; Bleichrodt and Pinto, 2005). Observation 4.3, combined with Theorem 4.2, provides a new foundation of the QALY model with standard sequence invariance instead of utility independence. Here $T = 0$ life years naturally serves as the null element required by Observation 4.3. Standard sequence invariance entails that tradeoffs between life-years (discounting) are not different under different health states. This condition will sometimes be more intuitive than utility independence, which appeals to risk attitudes for life-years rather than to direct tradeoffs between life-years and intertemporal preferences.

Obviously, if standard sequence invariance is prescriptively objectionable then Observation 4.3 shows that the QALY model is prescriptively objectionable. Standard sequence invariance can also be used to test the descriptive (rather than prescriptive) validity of the QALY model. A tractable way of testing is as follows. First elicit a Q-standard sequence $T_0, T_1, \ldots T_k$ through indifferences

$$(Q^*, T_g)_p(Q, \mathbf{T_{i+1}}) \sim (Q^*, T_G)_p(Q, T_i) .$$

as in Eq. 4.1, where the new value to be elicited in each indifference has been printed bold. Next take a health state $Q' \neq Q$ and a health state Q**, which can be but need not be different from Q*. Then use a "bridge" question

$$(Q^{**},T'_g)_p(Q',T_1) \sim (Q^{**},\mathbf{T_G'})_p(Q',T_0)$$

to find new gauge outcomes (Q^{**},T'_g)[6] and (Q^{**},T_G') that should provide the same standard sequence starting with T_0 and T_1. Then elicit a second standard sequence T'_0, $T'_1,\ldots,T'_k$:

$$(Q^{**},T'_g)_p(Q',\mathbf{T'_{i+1}}) \sim (Q^{**},T_G')_p(Q',T'_i).$$

We can then test whether the two standard sequences agree, as required by standard sequence invariance and the QALY model. A useful spinoff of these measurements is that they directly measure the utility functions (i.e., discounting) for life duration under Q and Q´ (Wakker & Deneffe, 1996). If these are different under Q than under Q´ then the QALY model is violated.

The measurements proposed above are chained, with answers to one question serving as input of next questions. A drawback of chaining is that errors propagate. Our consistency questions indicated that the errors in most responses were modest. Simulation studies for standard sequences have suggested that the problem of error propagation is not very serious (Bleichrodt & Pinto, 2000, p. 1495; Abdellaoui, Vossmann, & Weber, 2005, p. 1394, §5.3 end; Bleichrodt, Cillo, & Diecidue, 2010, p. 164; van de Kuilen & Wakker, 2011; Conte, Hey, & Moffatt, 2011).

7. Conclusion

We have demonstrated that standard sequences, a tool commonly used in conjoint measurement (where no risk is assumed), can also be used in multiattribute utility theory (where risk is assumed). They provide convenient tools to characterize and analyze utility independence, the most widely used preference condition in multiattribute utility theory. In particular, they facilitate the study of the QALY model for health decisions.

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Appendix. Proofs

PROOF OF THEOREM 3.1. That the functional form implies utility independence follows from substitution. Hence we assume utility independence, and derive the functional form.

Fix a Q*. If the corresponding utility interval is one-point, then by utility independence preference is independent of T, V is constant, and everything follows. Hence, assume that the interval is nonpoint. Then with $V(T) = U(Q^*,T)$, this function is an interval scale in the representation $(T_1,T_2) \rightarrow \pi V(T_1) + (1-\pi)V(T_2)$, which means that it is unique up to level and unit. This uniqueness is well known if we have an expected utility representation on the full, nonrank-ordered, product set $\mathcal{T}^2$ (resulting from X^2 by keeping $Q = Q^*$ fixed), which is a special case of an additive conjoint representation with Krantz et al.´s (1971) restricted solvability satisfied.[7] It

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[6] T'_g can but need not be equal to T_g.

[7] Here, and in what follows, we have continuity with respect to the product topology of the order topology generated over $\mathcal{T}$, where the crucial point is that this topology is connected (it is also

is also well known if we have a rank-dependent representation on the full product set $\mathcal{T}^2$ (Wakker, 1991). That it also holds when restricted to the rank-ordered set $\mathcal{T}^2_\downarrow$ (resulting from $X^2_\downarrow$ by keeping $Q = Q^*$ fixed) as in our setup follows from Chateauneuf & Wakker (1993, Theorem 2.2 and Lemma C.4).

By utility independence the same preferences hold over pairs (T_1,T_2) with Q fixed at every other level $Q' \neq Q^*$. By interval scaling, we have $U(Q',T) = f(Q')V(T) + g(Q')$ with $f(Q')$ positive. This way we obtain the functions f and g. □

PROOF OF THEOREM 4.2. If the functional form in the theorem holds, then all Ts are ordered by V, implying preferential independence. Further, then all standard sequences are equally spaced in V units, and they must be consistent. This implies standard sequence invariance on $\mathcal{T}$.

In the rest of this proof we assume standard sequence invariance on $\mathcal{T}$ and preferential independence and derive Eq. 4.2. By preferential independence we can define a preference relation over $\mathcal{T}$ independently of Q, that we will denote $\succcurlyeq$. Thus $T \succcurlyeq T'$ if $(Q,T) \succcurlyeq (Q,T')$ for some Q, which then holds for all Q.

Take some $Q \neq Q^*$. Define $V(T) = U(Q,T)$ and $V^*(T) = U(Q^*,T)$. By preferential independence, V and V* both represent $\succcurlyeq$ over $\mathcal{T}$ and $V^* = \varphi \circ V$ for a strictly increasing φ that is continuous because it maps an interval onto an interval.

Take a T with V(T) in the interior of $V(\mathcal{T})$. Hence, T is not maximal in $\mathcal{T}$. T will be fixed until the last lines in the proof. Define an open interval S around V(T) so small that there is a “dominating” interval D in $V(\mathcal{T})$ above the interval S large

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topologically separable). The result can be seen in more elementary terms if we transform all values T into V(T), giving a weighted additive representation with linear value functions.

enough to imply, for all T_1 and T_0 in $V^{-1}(S)$, existence of T_g and T_G in $V^{-1}(D)$ such that

$$(Q,T_g)_E(Q,T_1) \sim (Q,T_G)_E(Q,T_0). \qquad \text{(A.1)}$$

In words: each $(1-\pi)$ weighted V difference in S can be matched by a π-weighted V difference in D.

We similarly define an open interval S* around V*(T) so small that there is a dominating interval D* in $V^*(\mathcal{T})$ above the interval S* large enough to imply, for all T_1 and T_0 in $V^{*-1}(S^*)$, existence of T_g^* and T_G^* in $V^{*-1}(D^*)$ such that

$$(Q^*,T_g^*)_E(Q^*,T_1) \sim (Q^*,T_G^*)_E(Q^*,T_0). \qquad \text{(A.2)}$$

That is, each $(1-\pi)$ weighted V* difference in S* can be matched by a π-weighted V* difference in D*.

Take a $T^+ \succ T$ so close to T that both $V(T^+) \in S$ and $V^*(T^+) \in S^*$. Similarly, take a $T^- \prec T$ so close to T that both $V(T^-) \in S$ and $V^*(T^-) \in S^*$. We consider the preference interval $\{T' \in \mathcal{T}: T^- \prec T' \prec T^+\}$ around T and two of its elements $T_0 \prec T_2$. We can find T_1 such that T_0, T_1, and T_2 are equally spaced in V units, and T_1^* such that T_0, T_1^* and T_2 are equally spaced in V* units.

LEMMA A.1. $T_1 \sim T_1^*$.

PROOF. (The end of the proof of this lemma will be indicated by *QED*.) For contradiction, assume $T_1 \prec T_1^*$ (the case with $\succ$ is similar and is not discussed). Because the V values of T_0, T_1, and T_2 are contained in $V^{-1}(S)$, there exist T_g and T_G in $V^{-1}(D)$ such that, for $i = 0$:

$$(Q,T_g)_E(Q,T_{i+1}) \sim (Q,T_G)_E(Q,T_i). \qquad \text{(A.3)}$$

Because T_2 and T_1 have the same V difference as T_1 and T_0, Eq. A.3 also holds for $i = 1$. That is, T_0, T_1, T_2 is a Q-standard sequence.

Because $T_1 \prec T_1^*$, we can find $T_2^* \prec T_2$ such that T_0, T_1, T_2^* are equally spaced in V* units.

Similar to Eq. A.3, because the V* values of T_0, T_1, and T_2^* are contained in $V^{*-1}(S^*)$, there exist T_g^* and T_G^* in $V^{*-1}(D^*)$ such that

$$(Q^*,T_g^*)_E(Q^*,T_1) \sim (Q^*,T_G^*)_E(Q^*,T_0) \qquad \text{(A.4)}$$

and

$$(Q^*,T_g^*)_E(Q^*,T_2^*) \sim (Q^*,T_G^*)_E(Q^*,T_1). \qquad \text{(A.5)}$$

Eqs. A.4 and A.5 imply that T_0, T_1, T_2^* is a Q*-standard sequence. Because $T_2^* \prec T_2$, a contradiction results with standard sequence invariance on $\mathcal{T}$. *QED*

Because $T_1 \sim T_1^*$, T_1 (and also T_1^*) is both the V and the V* midpoint of T_0 and T_2. Hence, on $\{T' \in \mathcal{T}: T^- \prec T' \prec T^+\}$, V and V* midpoints are the same. With $V^* = \varphi \circ V$, the continuous function φ satisfies $\varphi((v_1 + v_2)/2) = (\varphi(v_1) + \varphi(v_2))/2$ on the interval $\left(V(T^-), V(T^+)\right)$ around V(T). It must be affine on this interval (Aczel, 1966 §2.1.3) and have second derivative 0 there, including at T.

The continuous and strictly increasing φ has second derivative 0 at all T in the interior of its domain $V(\mathcal{T})$. This implies that it is affine everywhere. Hence V*(T) = U(Q*,T) = f(Q*)V(T) + g(Q*) for a positive f(Q*). This implies Eq. 4.2.

REMARK A.2. In this proof, we only used standard sequences in Eq. 4.1 with Q* = Q. Hence the theorem remains valid if we define standard sequences only for Q* = Q in Eq. 4.1, and impose standard sequence invariance only for those standard sequences. The resulting condition is mathematically interesting because it is a common weakening of utility independence and standard sequence invariance, implying that the resulting modification of Theorem 4.2 is an immediate generalization of the theorems with utility independence in the literature. We chose the stronger version of standard sequence invariance in our main text because it is empirically more useful. □

PROOF OF OBSERVATION 4.3. Substituting the null element in Eq. 4.2 shows that g(Q) must be constant. It can be taken 0 because U is an interval scale. □

PROOF OF OBSERVATION 5.2. Assume utility independence on a set of the form $X^2_{\downarrow}$. This implies Eq. 3.3 for utility. This, in turn, implies utility independence on the whole domain of prospects because changing the deterministic level of some attributes amounts to an interval rescaling of utility, which does not affect preference. Utility independence on the whole domain trivially implies utility independence on the set $X^2_{\downarrow}$. Hence Eq. 3.3 and the two versions of utility independence are equivalent.

Next assume standard sequence invariance on a set of the form $X_{\downarrow}^{2}$. This implies Eq. 4.2 for utility. This, in turn, implies standard sequence invariance on every set isomorphic to a set $X_{\downarrow}^{2}$. Hence Eq. 4.2 and the two versions of standard sequence invariance are equivalent.

REMARK A.3. Although we did not formally define standard sequences on larger domains, it can readily be seen that such versions are easy to obtain. Replacing the deterministic level of some attributes amounts to an interval rescaling of utility, which does not alter equal spacedness of utility on, for instance, comonotonic subsets under rank-dependent utility. □

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