ÇERAG PINÇE

Advances in Inventory Management

Dynamic Models

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Nieuwe dynamische modellen voor voorraadbeheersing

PROEFSCHRIFT

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Chapter 1

Introduction

1.1. Inventory Management

Inventories play a crucial role in every economy. For example, in the United States total value of business inventories is about 1.3 trillion dollars, representing 9% of the country's current GDP (U.S. Census Bureau, 2010). One of the main reasons for a company to carry inventories is the mismatch between supply and demand due to uncertainty. For a company selling products, it is often very difficult to predict the exact moment the demand occurs. Thus, without inventories, the company would make all of its customers wait until the product is manufactured or replenished from an outside supplier. Furthermore, the quantity and quality of the products manufactured or received from an outside supplier might be variable due to the uncertainties involved in the in-house or the supplier's production process. Replenishment lead-times might be affected from disruptions in the production process and add up to the uncertainty of the supply process. Thus, inventories exist to bridge the gap between demand and supply due to the uncertainties involved in a supply chain. Moreover, they are instrumental in exploiting the economies of scale in production, transportation and purchasing.

In general, inventory management concerns two fundamental questions: when to order and how much to order. Although in practice, an inventory manager might make different decisions depending on the complexity of the inventory system, essentially most of the decisions are centered around these two fundamental questions. The scientific inventory management literature is mainly focused on answering these two questions for various inventory systems with different characteristics.

In this study, we focus on single item, single location inventory systems. Models studying such systems can be seen as the building blocks of the inventory theory. The earliest and probably the best known model on single item, single location inventory system is the economic-order-quantity (EOQ) model by Harris (1913). The widely used EOQ formula is a fundamental result in inventory theory and generally accepted as the beginning of scientific inventory management. In a simple and elegant way the EOQ formula reveals the relation between the order quantity and the costs of ordering and carrying inventory.

Due to rapid developments in technology and information systems, the speed and the nature of the flow of goods in supply chains have changed drastically compared to the times when the EOQ formula was invented. Today, to meet increased customer expectations, companies need to offer larger assortments, shorter delivery times and better quality for lower prices. As more and newer products are developed and introduced to the markets, the average product life cycles got shorter. Obsolescence risk as well as demand uncertainty has increased significantly. The higher dynamism of markets made the costs more volatile and difficult to predict. As a result of these changes in the surrounding environment, inventory systems became more dynamic.

In this study, we develop and analyze models incorporating two dynamic aspects affecting the inventory systems: nonstationarity in demand and unfixed purchasing prices.

In Part I, we consider an inventory system with a nonstationary demand rate. In particular, we focus on an inventory system of a critical service part where the demand rate drops to a lower level at a known future time. Inventory management of such items is notoriously difficult due to their slow moving character, high downtime costs and high obsolescence risk. In practice, there is a need for policies tailored for service parts taking these aspects into account and easy to implement. We propose an obsolescence based control policy and investigate its performance and impact on costs. We find that ignoring obsolescence in the control policy increases costs significantly and early adaptation of base stock levels can lead to important savings.

In Part II, we consider an inventory system with unfixed purchasing prices. In particular, we focus on an inventory system where the supplier offers price discounts at random points in time. We extend the literature by assuming a more general backordering structure while waiting for a deal. That is, when the system is out of stock, an arriving customer either decides to be backlogged with a certain probability or leaves the system and becomes a lost sale. Such behavior is more general compared to the full backordering or lost sales assumptions which are common in the literature. We derive expressions of the operating characteristics and equations to calculate optimal policy parameters. We demonstrate that allowing backorders in face of random deal offerings can result in considerable savings.

In the following sections, we explain our motivation for the problems considered in Part I and Part II in more detail and give brief outline of the chapters.

1.2. After Sales Services & Service Parts Inventories

Many companies hold service parts inventories either to keep their own products running or to support their customers through after sales services. For example, in commercial aviation industry about 60% of the inventories needed for maintenance, repair and overhaul (MRO) operations are carried by the airlines themselves. Since a commercial aircraft is a complex equipment composed of thousands of parts, on average, a legacy airline carries inventory worth of 2 million dollars in book value per aircraft. The book value of inventories might well exceed 100 million dollars for medium to large commercial MRO service providers. Similarly high costs are observed in the defense industry for military aircrafts or weapon systems.

The need for service parts to keep the critical and durable equipment in operating condition creates a lucrative, robust and long-lasting after sales market. For many durables, margins for after sales services are much higher than margins for sale of the original equipment (Cohen and Whang, 1997) and the revenue streams generated by services are less affected by economical downturns compared to original product sales. Moreover, complexity and number of components and parts used in the final products are increased due to the developments in technology. This makes MRO operations more complicated requiring specialized knowledge about the equipment and therefore, creates a natural demand for after sales services. As a result, in many industries, more and more companies prefer to outsource after sales services in order to avoid technical and logistical difficulties associated with management of service parts.

Therefore, it is no surprise that original equipment manufacturers (OEM) steadily invested in increasing their service capabilities to position themselves as service providers rather than sole manufacturers (Oliva and Kallenberg, 2003). According to a benchmarking study by Cohen et al. (1997), in computing, communication and electronic industries 30% of the product sales revenues are generated by after sales services.

As a result of changes in demand for after sales services, the role of service parts became more significant in service-centric supply chains. Service providers are competing for fast and reliable service delivery due to the customers' demand for prompt response. Simultaneously, maintaining high service levels has become more costly since service providers have to manage a large assortment of complex and expensive service parts. Thus, efficient management of service parts inventories becomes necessary to be competitive in the after sales market.

The term service parts can refer to different types of parts or components depending on their function, reparability, cost and demand rate. A service part is called *critical* if the equipment cannot function properly or safely upon failure of the part or the component. For example, a commercial aircraft can still operate safely if a passenger seat is not properly working but the same is not true for a critical component of engine or landing gear. In commercial aviation such critical items are called *no-go items* since the aircraft can be grounded due to safety regulations until the parts are repaired or replaced. This is a highly undesirable situation for an airline since the revenue loss over a grounded airplane might be more than $150,000 per day.

In general, for equipments requiring regular maintenance and parts replacements downtime costs are very high due to the equipments' critical functionality. For instance, one hour downtime of an electronic tester used in semiconductor production can cost as much as $50,000 since availability of the tester is crucial for smooth production (Cohen et al., 1999). Further examples of critical equipment are mainframe computers, medical electronics, military radar systems. In all these examples, prompt replacement of critical service parts is vital in particular for the operation of the equipment and in general for the whole system.

However, many critical parts and components used in a critical equipment are themselves expensive high-tech products. Due to the advanced technology used in their design and production phases, the reliability of such critical parts and components is often very high. Therefore, they have relatively low failure rates and the inventories of these parts are often *slow moving*, i.e., the inventories remain on the shelf for long time periods before being used. Thus, characterized as slow moving and expensive, critical service parts tend to incur significant inventory holding costs. Moreover, due to their specific nature, the procurement lead-times of critical parts are usually long. Consequently, in many cases, companies have to stock critical service parts to maintain the availability of the system. Therefore, for a company or service provider supporting broad installed base of critical equipments, carrying a large and diverse portfolio of expensive and slow moving items results in substantial inventory costs.

The main tradeoff in the management of critical service parts inventories should be more clear now. On the one hand, high downtime costs of the critical equipment and long procurement lead times require to stock critical parts to guarantee availability of the system. On the other hand, maintaining high inventory levels of such slow moving and expensive items results in high holding costs. Hence, striking the right balance between the two ends of the tradeoff brings a significant competitive advantage for a company conducting service-centric operations.

Most of the available models in the inventory control literature assume stationary demand. This is mainly because the assumption of stationarity yields some nice mathematical properties making the analysis of the inventory system relatively easier. The models developed under stationary demand assumption remain fairly practical for items with nonstationary demand as long as the demand rate can be forecasted from historical data and the life-cycle of the item is long enough. That is, the demand rate is re-estimated in periodic intervals and policy parameters are updated accordingly.

However, lack of historical data and nonstationarity of demand is a fundamental problem in inventory management of service parts. Shortened product life-cycles, increased product quality and diversity make service parts demand low and nonstationary (Fortuin and Martin, 1999; Cohen and Lee, 1990). Since demand is slow moving, the data points to be used for forecast updates are scarce. When demand data is scarce and intermittent, standard black-box forecasting methods usually lag behind the changes in demand rate and yield biased estimates (Shale et al., 2006). Consequently, using stationary inventory control models combined with standard forecasting methods based on historical data are usually not very effective for service parts. The option to increase the safety stocks to hedge against poor forecasts is either very costly or infeasible since most of the critical parts are expensive.

In order to cope with these difficulties, companies keep track of their installed base to collect more information about the changes in demand rate. In principle, the information obtained from installed base tracking can be used to identify a certain pattern for the demand rate of a particular service part. For example, the demand for a specific service part is correlated with the number of equipment in the installed base. Similarly, age, operating conditions, intensity of use and geographical location of the equipment in the installed base affect the demand for the service part directly. The idea of installed base tracking is therefore to exploit these dependencies and to identify the underlying demand pattern for the service part.

The advantage of such approach is that the inventory control would become proactive rather than reactive to the changes in demand rate. For example, when a demand rate of a critical service part is expected to change at certain time points, one can device an inventory control policy based on these expectations. Such a policy would build-up or run-down inventories according to the nonstationarity in demand.

For critical service parts, there are not many models in the inventory literature with nonstationary demand. The models incorporating nonstationarity are mostly periodic review models and mainly suitable for items for which historical data is not scarce. Moreover, for critical service parts, in general, continuous review policies are more suitable since they allow tighter monitoring than periodic review policies.

In this study, we try to fill this gap by proposing a continuous review inventory model specifically for critical service parts and incorporating nonstationarity in demand. However, incorporating nonstationary demand in continuous review control models is mathematically challenging. Even the exact analysis of relatively simple models with linear changes in demand rate is very tedious or intractable. Therefore, in developing the model we limit ourselves to a relatively simple form of nonstationarity. That is, we analyze a case with single drop in demand rate which resembles an obsolescence situation in practice.

We define obsolescence as a single drop in demand rate that is not necessarily to zero. In other words, even after obsolescence occurs the demand might not disappear completely. As such, our definition of obsolescence is more general compared to the conventional definition of obsolescence implying that the demand drops to zero and the inventory is not salable anymore. Since service parts are slow moving items, a large drop in demand rate increases the inventory holding costs significantly and virtually creates an obsolescence situation. Hence, we prefer to use a broader definition of obsolescence. In practice, many critical service parts are subject to this type of obsolescence due to contract expirations, generation upgrades or relocations of installed base.

When there is such a large drop in demand rate it results in excess inventories. Since most of the critical service parts are very specific products it is in general difficult to salvage the excess stock of these specific products. Consequently, most of the excess inventories end up as obsolete stocks increasing holding costs and waiting to be scrapped at a great expense. These issues were first brought to our attention during the meetings at a Netherlands based knowledge platform called Service Logistics Forum Research. At those meetings, occasionally, the logistics managers dealing with critical service parts were complaining about obsolescence problem and looking for methods that help to minimize obsolete stocks. Hence, with this research, we also aim to bring guidance to those managers in their decision making.

In this study, we provide a practical policy for slow movers taking into account obsolescence and give insights about the interplay between backordering costs and obsolescence related holding costs. To this end, we incorporate obsolescence into the one-for-one policy and propose a transition control policy allowing early adaptation of stock levels to reduce obsolete stocks while balancing availability.

1.3. Random Deal Offerings

Temporary price cuts offered by suppliers to retailers are prevalent in practice. Such form of trade promotions often referred to as *deals* in marketing literature are characterized as short-term price cuts which can occur with or without prior announcement. After a deal is over the price goes back to its previous level.

Depending on market fluctuations, inventory holding costs, seasonal effects, shifts in consumer taste and many other factors the degree and frequency of deals can change significantly. Therefore, for suppliers, deals are instrumental in liquidating short term inventory, increasing cashflow or market share, or influencing retailers' promotion activities. Blattberg et al. (1981) suggest that one of the main motivations behind price discounts is to transfer inventory holding costs to consumers. Ailawadi et al. (1999) demonstrate that well-designed trade promotions can influence the retailer's selling activity and coordinate the channel. As a result, a supplier might increase its profits to levels which are not achievable by fixed prices.

On the retailers side, deal offerings necessitate reconsideration of their inventory control policy. One reason is the well known side effect of trade promotions called *forward buying*, i.e., retailers stockpile by taking the advantage of discounted price (Blattberg and Levin, 1987). By forward buying retailers basically trading off the inventory carrying costs with savings obtained by purchasing at the reduced price. In order to exploit occasional deal offerings, some retailers even invest in extra stocking space to stock excess inventory bought during the deal periods (Moinzadeh, 1997).

For a retailer decided to practice forward buying, finding the right balance between the extra inventory carrying costs and the savings by the reduced price is crucial. Otherwise, excess inventories build up and that might be detrimental rather than beneficial. The balance is especially more delicate for goods that are perishable, expensive or subject to sudden obsolescence. Thus, a forward buying retailer shall find the right answers to the questions of how much to order in a deal period to take advantage of the low price and how the regular orders should be adjusted.

Deals do not always have to be initiated by the supplier. Silver et al. (1993) provides a practical example where occasional peaks in otherwise stable demand for a certain product urge the company to place *special* orders to its supplier besides the regular orders. As a result, large demands are usually met directly from the supplier via special orders and do not affect the stock. In case, the supplier gives quantity discounts and special orders are large enough to receive a discount then the special order occasions can be used to order extra units from the discounted price. Consequently, how much extra should be ordered and what should be the regular order size are two questions to be answered.

There is a substantial literature on inventory systems with fluctuating prices or occasional deal offerings. The discussion in the studies with periodic review models mostly centered around the optimal policy structure when the prices or in general the *environment* is fluctuating (Golabi, 1985; Özekici and Parlar, 1999). Whereas the focus in continuous review models is mostly on the analysis of the operating characteristics and the computation of the optimal policy parameters concerning stocking and replenishment decisions (Hurter and Kaminsky, 1968; Silver et al., 1993; Moinzadeh, 1997).

Most of the continuous review models with random deal offerings assume that stockouts are not allowed. Since deal offerings already complicate the analysis considerably one can argue that the assumption of no stockouts significantly decreases the mathematical tedium. However, occasional stockouts are prevalent in many real inventory systems. Moreover, for a system with deal offerings it is intuitive to expect a tradeoff between stockout penalties and the savings obtained by postponing the purchasing decision until the next deal arrives.

In general, many inventory models assume two extreme scenarios when the system is out of stock. Either all demand arriving during the stockout period is backordered (full backordering) or all demand arriving during the stockout period is lost (lost sales). However, in many practical situations backorders and lost sales are mixed. In the literature, this type of backordering structure commonly referred to as *partial backordering*, i.e., during a stockout period a fraction of demand can be backordered and the rest is lost.

Moinzadeh (1989) states that partial backorders can be observed in two situations: i) some of the customers are highly loyal compared to others and they wait when the system is out of stock ii) some customers have a higher priority compared to others and they are replenished immediately at an expense (e.g., emergency replenishment) when the system is out of stock. Here the extra cost of replenishing the high priority customers during a stockout can be interpreted as the lost sales cost.

Kim and Park (1985) show that for the systems where backorders and lost sales are mixed, the assumptions of full backordering and lost sales significantly increase the inventory costs. Thus, assuming partial backorders not only leads to a more general model but also leads to significant cost savings in case the underlying system is not close to one of the two extreme scenarios.

However, partial backorders complicates the analysis of the inventory policies. Even for relatively simple policies like $(S-1, S)$ the handling of partial backorders is not straightforward (Moinzadeh, 1989). The main difficulty in handling partial backorders emerges from the lost sales. Under lost sales the simple relation between the inventory position, the net inventory level and demand during lead time does no longer hold. Thus, especially for stochastic demand case, the analysis of the policies and operating characteristics becomes much more difficult.

To our knowledge there is no study in the literature assuming partial backorders for a system where supplier offers price discounts. Feng and Sun (2001) consider full backordering in a continuous review inventory system facing stochastic demand and random deal offerings. However, the focus of their study is on computation of the optimal policy parameters rather than the tradeoff between backordering costs and savings by waiting for deals by allowing backorders.

In this study, we incorporate partial backorders to a continuous review inventory system with random deal offerings. A deal is a discount over the unit purchasing price of the item. Our aim is to provide relatively simple formulas to calculate the optimal policy parameters and give insights about the impact of partial backorders on policy parameters and costs in a system facing random deal offerings. As we discussed above the analysis of the partial backorders already complicates the analysis even for inventory systems without deals offerings. Thus, in this study, we confined ourselves to a relatively simple assumption of constant demand rate. Furthermore, we assume that the deals arrive according to a Poisson process and they are instantaneous (i.e., deal duration is zero). As such, we extend the model offered by Moinzadeh (1997) by allowing partial backorders.

1.4. Outline

In Chapter 2, we analyze a single location inventory system of a service part with Poisson demand where the demand rate drops to a lower level at a known future time. The inventory system is controlled according to a one-for-one replenishment policy with fixed lead time and full backordering. Adaptation to the lower demand rate is achieved by reducing the base stock level in advance and letting the demand take away the excess stocks.

Our aim in this chapter is to provide a first-cut, qualitative understanding of the impact of the timing of the policy change on obsolescence costs. As such, we assume that pre- and post-obsolescence base stock levels are fixed and the only decision parameter is the time to reduce the base stock level from high to low.

We showed that the timing of the base stock level primarily determines the tradeoff between backordering penalties and obsolescence costs. We propose an approximate solution for the optimal time to shift to the lower base stock level minimizing the expected total cost during the transient period. We found that an advance policy change results in significant cost savings and our model yields near optimal expected total costs when the decision parameter is only the timing of the shift.

In Chapter 3, we consider the same inventory system as in Chapter 2 and propose a three parameter policy allowing not only to control the shift time but also the base stock levels. We carried an exact analysis of the system under the three parameter policy. For the special case of identical base stock levels, we show that the optimal base stock level can be calculated from a critical ratio inequality. For different base stock levels, we derive the exact expression for the expected total discounted cost function by partly relying on the Fast Fourier Transform method and suggest a numerical optimization procedure to find the optimal values of the policy parameters. Our results suggest that the policy change option leads to pronounced cost savings especially when obsolescence requires a relatively large adjustment in base stock level. We find that ignoring obsolescence in the control policy increases costs significantly. Moreover, when obsolescence can be foreseen, early adaptation of base stock levels can lead to important savings.

We do not pursue a comparison between the policies considered in Chapter 2 and Chapter 3. The main reason is that the policy discussed in Chapter 2 is a special case of the policy proposed in Chapter 3. Moreover, in Chapter 3, we conduct an extensive numerical experiment to understand the behavior of optimal policy parameters and benchmark the performance of the three parameter policy. In those experiments, we observe that the interaction between the three policy parameters, i.e., base stock levels and the policy change time, are quite complex. Therefore, we do not think that investigating the performance of a single parameter policy vis-á-vis the three parameter policy would yield interesting insights. Furthermore, such comparison is impractical and limited for the cost criteria are different for both policies. In Chapter 2, we use expected total cost criterion whereas in Chapter 3, we use expected total discounted cost criterion. The reason for the difference is that in Chapter 2 we are dealing with a finite horizon problem where using a total cost function provides better tractability. Moreover, in Chapter 2, assuming that the total cost is not discounted throughout the finite horizon seems to be a reasonable assumption for the length of the finite horizon is not too long as it is a transient period.

In Chapter 4, we consider a single item, single location continuous review inventory system with random deal offerings. We extend the model by Moinzadeh (1997) by letting the reorder point for regular list replenishment to be nonpositive. In other words, we allow the net inventory level to drop below zero while waiting for a deal. Moreover, we assume that when the system is out of stock a fraction of demand can be backordered and the rest is lost, i.e., partial backordering. We derive the exact expressions for operating characteristics of the model and the equations to calculate the optimal policy parameters. We provide qualitative results about the relation between the optimal solution and the system parameters, and demonstrate that allowing backorders in a random deal environment might lead to important cost savings.

In Chapter 5, we summarize our findings and conclude.

Part I

Service Parts & Obsolescence

Chapter 2

An Inventory Model for Slow Moving Items Subject to Obsolescence*

Abstract

In this chapter, we consider a continuous review inventory system of a slow moving item for which the demand rate drops to a lower level at a pre-determined time. Inventory system is controlled according to one-for-one replenishment policy with fixed lead time. Adaptation to the lower demand rate is achieved by changing the control policy in advance and letting the demand take away the excess stocks. We show that the timing of the control policy change primarily determines the tradeoff between backordering penalties and obsolescence costs. We propose an approximate solution for the optimal time to shift to the new control policy minimizing the expected total cost during the transient period. We find that the advance policy change results in significant cost savings and our model yields near optimal expected total costs.

2.1. Introduction

In Chapter 1, we discussed about the challenges in managing the inventories of critical service parts and the need for the control policies taking into account the changes in demand rate. Moreover, we briefly introduced our motivation to study the inventory management under obsolescence, a special form of nonstationarity in demand rate, for critical service parts. In this chapter and the next one, we will elaborate on this problem.

*This chapter is based on Pinçe and Dekker (2009)

As we mentioned in Section 1.2, one of the main challenges in the inventory control of critical service parts is to minimize obsolete stocks while balancing availability. Typically, companies try to mitigate their downtime risk by outsourcing parts management, repair and maintenance activities to a service provider. Since availability of the equipment is crucial for companies' operations, usually service contracts include high penalties for the provider in case it fails to fulfill the specified service target. Moreover, in a competitive service market, occasional stockouts can be detrimental to service provider's business due to loss of good will.

On the other hand, service parts are very specific, high-tech products. When there is a sudden drop in demand rate due to the changes in customer's operations or service preferences, the resulting excess parts are difficult to dispose and likely to become obsolete. Although service providers might try to dispose them via service promotions or secondary markets, the inherent specificity of service parts usually results in low disposal rates and high value loss.

Another example comes from a mainframe computer manufacturer providing after sales services for its customers through different type of service contracts. A manager at the company reports that the decision to relocate or remove the excess service parts often comes too late after a service contract expires. Consequently, due to the drop in demand rate, excess inventories increase holding costs, and usually end up as obsolete stocks to be scrapped at a great value loss. One of the main reasons behind the excess parts is disregarding the contract expiration dates in stocking decisions. In other words, the stock control policy used by the company simply ignores the upcoming drop in demand rate due to contract expirations, and therefore, results in delayed stock adjustments and higher chance of obsolescence.

Companies realizing these facts start keeping track of the changes in their own or customers' base of installed products (installed base) to trace customers and operating units more closely and to react to the changes in demand rate as early as possible. A recent study by Jalil et al. (2009) revealed that at IBM, tracking of the installed base for spare parts can lead to savings up to 58% in transportation and inventory holding costs.

When contextual information is combined with installed base tracking, the timing and the size of the shift in demand rate are either known in advance or can be estimated within a reasonable accuracy. In practice, such shifts typically occur when the size of the installed base at certain geographical location changes. For example, when a customer announces that it is going to relocate its production equipments, after sales service provider anticipates a change in demand for parts between the locations. Similarly, when a customer decides to upgrade its machinery, the old generation equipments usually leave the installed base of the service provider or manufacturer as a result of discarding or salvaging.

When a sudden change in demand rate can be foreseen, timely adaptation of the base stock levels is crucial for optimal stock control. In such cases, upward jumps in demand rate can be handled relatively easily by giving advance or emergency replenishment orders to be delivered before the jump occurs. However, adaptation to the drop in demand rate is more difficult since running down of excess stocks depends on the demand process. For example, when a certain proportion of the installed base is relocated, service providers usually suffer from excess inventories remaining at the previous location. When relocation of spare parts with the installed base is not feasible, it becomes much more difficult to get rid of the excess stocks due to the diminished demand. Consequently, in many cases these excess stocks end up as obsolete stocks.

Generation upgrades may result in a similar problem as well. For example, when an airline announces the selling of their old generation aircrafts to the countries outside Europe, service providers of this airline expect a sudden drop in demand for relevant parts at their service locations in Europe. In such cases, if a prior action is not taken to adjust the base stock levels then the excess stocks might become obsolete.

Motivated by these examples, in this chapter, our aim is to provide a practical policy for slow movers incorporating obsolescence and allowing timely adaptation of stock levels to reduce obsolete stocks while balancing availability.

We assume that the time and the size of obsolescence can be predicted with a high accuracy. As discussed above, in practice, such accurate predictions are possible when a customer announces that it will not extend the service contract for another period, relocate its equipment outside the serviceable area or upgrade it to a newer generation. In all of these scenarios, service provider anticipates a drop in demand rate of the service parts at a predictable time point.

Even when the timing and the size of the drop is known exactly, when to change the inventory control policy to minimize obsolete stocks without staking availability remain as a challenging question. If the adaptation is too early before the drop occurs then the risk of backordering increases as a result of lower base stock level. Since availability is crucial for many companies operations, stockouts can be detrimental to their businesses. On the other hand, if the adaptation is too late or postponed after the drop then the costs associated with obsolescence increase.

We address this issue by focusing on a continuous review inventory system of a slow moving item for which the demand rate drops to a lower level at a known future time. We assume that the inventory system is controlled according to a one-for-one replenishment policy with fixed lead time. Adaptation to the lower demand rate is achieved by changing the control policy in advance and letting the demand process take away the excess stocks.

Our goal is to find the optimal time for a policy change and to investigate its impacts on the costs incurred during the transient period. We assume that pre- and post-obsolescence base stock levels are fixed to their steady state levels and the only decision variable is the time to shift the base stock level from high to low. Thus, in this chapter, our aim is to provide a first-cut, qualitative understanding of the impact of policy change time on transient period costs. We demonstrate the interplay between obsolescence costs and stockout penalties during this period. In Chapter 3, we will relax this assumption and consider a more general three parameter policy where not only the policy change time but also pre- and post-obsolescence base stock levels are the decision variables.

Our work is related to the inventory management models considering obsolescence. Hadley and Whitin (1962) were early contributors in this area. They analyzed a finite horizon periodic review inventory system in which the mean demand rate may vary in every period and there is a finite number of possible obsolescence dates. Pierskalla (1969) studied a similar problem with independent and identically distributed demands and zero lead times.

Brown et al. (1964) offered a more general model for obsolescence in which the demand in each period is generated according to an underlying Markov chain and the state probabilities are updated in Bayesian fashion. Song and Zipkin (1996) also employed a similar Markovian submodel to reflect the processes leading to obsolescence by assuming that the current state of the process is completely observable. They found that the obsolescence has substantial effects on inventory costs and these effects cannot be remedied by simple parameter adjustments.

Masters (1991), Joglekar and Lee (1993), David and Mehrez (1995) considered the EOQ model in which the time to obsolescence is exponentially distributed.

Relative to some of the models available in the obsolescence literature, our model makes the simplifying assumption that the time of obsolescence is deterministic. Although this seems to be a reasonable assumption in the context discussed above, nevertheless, it might be violated if there is gradual obsolescence. However, different than those in the obsolescence literature, we propose a simpler, more practical control policy incorporating obsolescence into a one-for-one policy that is commonly used for slow moving inventories.

Another stream of literature that is related to our study consists of the so called excess stock disposal models. In these models the problem is to determine the economic retention quantity or the time period given the excess stock of an inventory item. Earlier works by Simpson (1955), Mohan and Garg (1961) and Hart (1973) investigated the excess inventory disposal problem for deterministic demand case with the possibility of obsolescence. Stulman (1989) considered continuous review inventory system with stochastic demand but without obsolescence. Rosenfield (1989) investigated the similar problem for slow moving items by including perishability or obsolescence but without stockout penalties. In all of these studies it is assumed that the excess stocks are a result of over purchasing or a drop in demand rate in the past. Therefore, the inventory level is found higher than the maximum level at time zero and the excess inventory is reduced by first disposing, and then letting the demand take away the retained quantity. Although some of the models mentioned above deal with natural attrition of stocks and obsolescence, our model fundamentally differs from these studies in its objective to minimize excess stocks before they occur. As such, different than the excess stock disposal models, we let the demand to take away the stocks before the excess occurs.

In another related study, Teunter and Haneveld (2002) consider the inventory control of a service part in the final phase. After time zero, the price of the part increases since its production ends and the Poisson demand rate drops to zero at a deterministic time point. They propose an ordering policy consisting of initial order-up-to level to take the advantage of low price at time zero and a subsequent series of order-up-to levels gradually decreasing as the obsolescence time approaches. Their objective is to minimize the total undiscounted cost incurred during the final phase under the assumptions that all obsolete stock can be disposed of and backorders occurring within one lead time before obsolescence are lost. Our model mainly differs from theirs in that partial obsolescence is allowed, i.e., demand does not necessarily drop to zero, and all backorders are fulfilled regardless of obsolescence time.

The remainder of this chapter is organized as follows: In Section 2.2, we introduce the model and the transition control policy. In Section 2.3, we give the expressions for the operating characteristics of the transient period and the objective function, and discuss their general behaviors. In Section 2.4, we discuss the results of our numerical study. In Section 2.5, we conclude and provide some future research paths. All proofs are provided in the Appendix 2.6.

2.2. Model

We consider a single item, single location continuous review inventory system for slow moving items with nonstationary demand process and fixed lead times. It is assumed that the demand follows a Poisson process with rate λ_0 up to a pre-determined time point T after which the demand rate drops to a lower state λ_1 and stays there (*i.e.* $\lambda_0 > \lambda_1 \geq 0$). The inventory control policy is based on the $(S-1, S)$ policy which is commonly used for high cost low demand items (Hadley and Whitin, 1963). According to this policy whenever a demand occurs a replenishment order is placed.

We denote the steady state optimal base stock levels for demand rates λ_0 and λ_1 with S_0 and S_1, respectively. They are calculated with the standard formulas given in Hadley and Whitin (1963). We assume that the shift in demand rate is downward (*i.e.* $S_0 > S_1 \geq 0$). In order to adapt to the new base stock level, we employ the following transition control policy based on the inventory position (the net inventory level plus the quantity on order):

Policy: Up to time $T - X$ a replenishment order of size one is placed whenever the inventory position drops to the reorder level $S_0 - 1$. After time $T - X$ a replenishment order of size one is placed whenever the inventory position drops to the reorder level $S_1 - 1$.

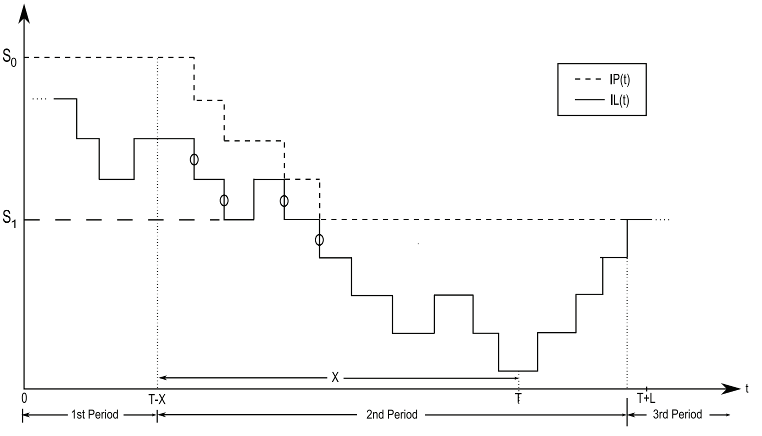
In other words, we use $(S_0 - 1, S_0)$ policy until time $T - X$ and $(S_1 - 1, S_1)$ policy thereafter. Observe that according to this control policy adaptation to the new base stock level is achieved by not giving N $(= S_0 - S_1)$ consecutive orders starting at $X \geq 0$ time units earlier from time T. Hence, we let the demand take away N excess stocks starting from $T - X$. Our goal is to find the optimal time to initiate the excess stock removal process.

The rationale behind the proposed policy is that once the obsolescence date is known with certainty, early adaptation of base stock level should tradeoff the risk of backordering and obsolescence, and decrease the number of excess or obsolete stocks. We do not claim that the transition control policy is optimal. However, as we will demonstrate in our numerical experiments, it indeed leads to significant reduction on obsolescence costs compared to policy without an early adaptation $(X = 0)$.

Figure 2.1 shows a possible realization of the net inventory level process $\{IL(t) : t \geq 0\}$ and the corresponding inventory position process $\{IP(t) : t \geq 0\}$. Note that the trajectories of these processes can be analyzed in three different periods. The first period starts at time zero and ends at time $T - X$. Since a replenishment order is placed upon each demand arrival the inventory position is fixed at S_0 during the first period. We assume that $T - X$ is long enough such that $IL(t)$ is in steady state. This is reasonable since life cycles of many products requiring parts replacements and service support are very long. For example, the average useful life time of a commercial aircraft may last up to 30 years. Thus, the inventory system of a spare part supporting such product has enough time to reach to steady state before the obsolescence occurs.

The second period begins at time $T - X$ and the excess stocks are removed by not giving replenishment orders for N consecutive demands. Hence, the inventory position decreases by one at every demand arrival until it hits the target base stock level S_1. In Figure 2.1, examples of stock removal instances are marked by circles on the inventory level process. If the inventory position process hits S_1 before time T then the replenishment orders are placed again whenever a demand occurs. Thus, the end of the second period

Figure 2.1: Possible realization of $IL(t)$ and $IP(t)$ with stock removals



is the random time point greater or equal to T at which the inventory position is equal to S_1 and all outstanding orders given before time T have arrived (see Figure 2.1).

Note that, the second period is the transient period in which the inventory system adapts itself to the anticipated obsolescence. Since all orders given before time T are replenished before the second period ends, the third period can be seen as a separate inventory system with demand rate $\lambda_1 \geq 0$. If λ_1 is positive then we assume that the net inventory level process during the third period can be described by the stationary process. In many practical situations relocation of installed base or generation upgrades might result in such partial obsolescence situations where the demand is severely diminished but not necessarily vanished. In that case, the third period is similar to the first one but the system is controlled according to (S_1-1, S_1) policy. Clearly, in case of full obsolescence $(\lambda_1 = 0)$ there is no third period.

Our main goal is to find the optimal X minimizing the total expected cost incurred in the second (transient) period. As we will demonstrate in the numerical section, the transient period costs are significant since they include the costs related with obsolescence. Unless a prior action is taken, partial obsolescence $(\lambda_1 > 0)$ results in excess stock situations whereas full obsolescence $(\lambda_1 = 0)$ results in obsolete stocks. As discussed earlier, for many slow movers the costs due to obsolescence are very high under both scenarios.

Hence, in the sequel, we only focus on the analysis of the transient period since savings over obsolescence costs can be achieved only within this period.

Since fixed costs are irrelevant for optimization under one-for-one replenishment policy, we only consider holding and backordering costs incurred per unit per time, denoted by h and π respectively. In addition to that the unit obsolescence/relocation cost c_o is incurred per remaining on hand inventory after T if full obsolescence occurs ($\lambda_1 = 0$).

In the next section, we explain the transient analysis of the net inventory level process, give the expressions for the operating characteristics of the second period and state the optimization problem.

2.3. Operating Characteristics of the Second Period

Our model differs from the standard inventory models due to removal of excess stocks and nonhomogenous demand process. These differences necessitate the transient analysis of the net inventory level process. Unfortunately, outstanding orders before time $T - X$ complicate the analysis considerably. Since the complication results from outstanding orders, conditioning on the net inventory level at time $T - X$ or its expectation does not yield closed form expressions for the operating characteristics. In order to overcome this analytical difficulty and provide approximate formulas for operating characteristics that can be calculated easily, we assume that the net inventory level is equal to S_0 at time $T - X$. We can justify this assumption by appealing to the characteristics of the problem. For slow movers, the base stock levels are usually not very high due to low demand rates and high opportunity costs. On the other hand, due to high backordering penalties the net inventory level process mostly stays in the positive half-plane. Therefore, the average net inventory level at any time is not very far from S_0. Indeed, for all the instances used in our numerical experiments, which are generated to reflect real life scenarios, the average S_0 is found to be 3.24 with maximum of 10. For the same instances, the average difference between S_0 and $E(IL(T - X))$ is found to be 1.1 with maximum of 5. Consequently, we observed that our approximate model performs quite satisfactorily compared to simulation. The effects of our assumptions will be discussed in more detail under Section 2.3.4. Moreover, as we will demonstrate in the numerical section, the optimal X found by using our approximate formulas yields near optimal expected total costs. Hence, we conclude that this assumption does not change the main implications of our study.

The analysis of the net inventory level process is independent of the time axis due to Poisson demand arrivals. Therefore, in the sequel, we shift the beginning of the second period from $T-X$ to 0 for the sake of clarity. Let $\tau_i,\ i=1,\ldots,N$ denote the interarrival times between not replenished demand instances when the arrival rate is λ_0. We refer to $A_k := \sum_{i=1}^{k}\tau_i$ as the arrival time of the kth demand before the drop occurs.

Figure 2.2: Possible realization of $IL(t)$ and $IP(t)$ during 2nd period ($A_N > X$)

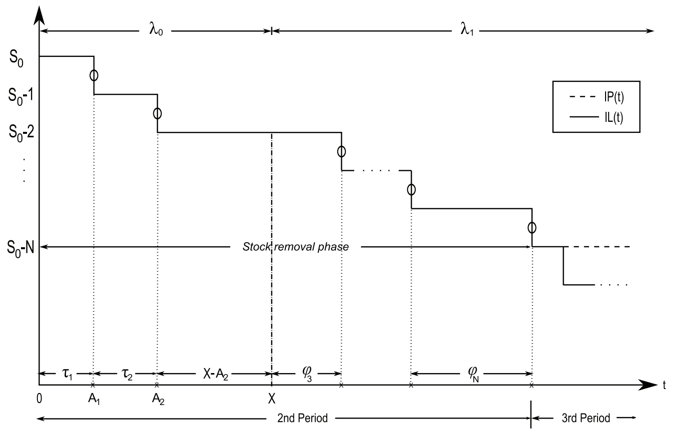
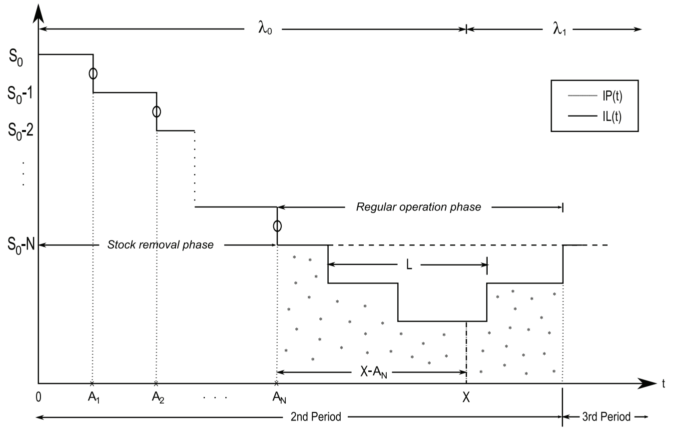


Figure 2.2 shows a realization in which the new base stock level S_1 $(= S_0 - N)$ is hit by the net inventory level process after the drop in demand rate occurs at time X. Observe that, as a result of our assumption about the outstanding orders ($IL(T-X) = S_0$), the net inventory level process is tantamount to the inventory position process until the $N+1$st demand arrives. In the figure, $\varphi_j,\ j=3,\ldots,N$ denote the interarrival times between not replenished demands arriving after time X. Hence, φ_js are exponentially distributed with mean λ_1. Note that, in Figure 2.2, the second period ends immediately after the arrival of the Nth demand since the inventory position is equal to S_1 and there are no outstanding orders before time X. On the other hand, if S_1 is reached before time X then replenishment orders are placed again for every demand arriving thereafter. A

realization of this scenario can be seen in Figure 2.3. Observe that, in Figure 2.3, the second period ends at the moment the last order given between A_N and X is replenished.

Figure 2.3: Possible realization of $IL(t)$ and $IP(t)$ during 2nd period ($A_N \leq X$)



From Figure 2.2 and Figure 2.3, it is clear that the net inventory level process in the second period can be analyzed in two different phases. The first one is the *stock removal phase*. This is the time period in which the excess stocks are taken away by the demand. Thus, the stock removal phase starts at the beginning of the second period and ends when the Nth demand arrives. The second one is the *regular operation phase*. This is the time period in which the replenishment orders are placed again upon every demand arrival since all of the excess stocks are removed before the drop in demand rate occurs. Hence, the regular operation phase starts at A_N and ends when the second period ends (see Figure 2.3). Note that the regular operation phase of the second period exists if and only if the Nth excess stock is removed before time X (*i.e.* $A_N \leq X$).

We begin our analysis with the calculation of the expected total inventory carried during the second period denoted by $E[OH]$. Observe that the random variable OH depends on the arrival time of the kth demand during the stock removal phase, and therefore it can be calculated by conditioning on $A_k,\ k = 1, \ldots, N$. If the arrival time of the first demand A_1 is greater than X then the second period ends at the moment Nth demand arrives. Thus, OH is equal to the inventory carried until time X ($= S_0X$) plus another random variable OH_1' ($= \sum_{i=1}^{N}(S_0 - i + 1)\varphi_i$) representing the inventory carried from time X until the second period ends. Note that if the stock removal phase extends after time X then the trajectory of $IL(t)$ should be analyzed separately for the periods before and after time X due to different demand rates. Hence, the need for an additional random variable $OH_1^{'}$. On the other hand, if A_1 is less than or equal to X then OH is equal to the inventory carried until the first demand arrives ($= S_0\tau_1$) plus another random variable OH_2. Essentially, OH_2 is similar to OH but it depends on A_2 and the new inventory level $S_0 - 1$. Put more formally,

$$OH = \begin{cases} S_0X + OH_1^{'} & \text{if} \quad A_1 > X \\ S_0\tau_1 + OH_2 & \text{if} \quad A_1 \leq X \end{cases} \tag{2.1}$$

If we continue in this fashion for $k = 2, 3, \ldots, N$ when $N \geq 2$ then we come up with the following recursive equations to calculate the total inventory carried during the second period:

$$OH_k = \begin{cases} (S_0 - k + 1)\,(X - A_{k-1}) + OH_k^{'} & \text{if} \quad A_{k-1} \leq X,\ A_k > X \\ (S_0 - k + 1)\tau_k + OH_{k+1} & \text{if} \quad A_k \leq X \\ \quad 0 & \text{o.w.} \end{cases} \tag{2.2}$$

where

$$OH_k^{'} = \sum_{i=k}^{N} (S_0 - i + 1)\varphi_i, \quad k = 1, \ldots, N. \tag{2.3}$$

represents the inventory carried from time X until the end of the stock removal phase when $N - k + 1$ stocks are yet to be removed.

The recursive structure of equations (2.1) and (2.2) gives the positive area under the net inventory level process depending on whether the kth excess stock is removed before time X or not. For example, if all excess stock is not removed before time X (i.e. $A_k > X$ for some k) then equations (2.1) and (2.2) give the area under a similar scenario depicted in Figure 2.2. Otherwise, they give the area similar to the one shown in Figure 2.3.

Note that the equations (2.1)-(2.3) mainly generate the expressions for the total inventory carried during the stock removal phase. Since no orders are given in this phase, the equations are independent of the lead time. The total inventory carried in the regular operation phase is represented implicitly in those equations with the random variable OH_{N+1}. The shaded region in Figure 2.3 shows a possible realization of OH_{N+1}. We will analyze the regular operation phase in detail in the sequel.

Let $p(n;\lambda) = e^{-\lambda}\lambda^n/n!,\ n = 0,1,2,...$ be the pdf of Poisson distribution with parameter $\lambda \geq 0$ and denote its cdf with $P(n;\lambda) = \sum_{k=0}^{n} p(k;\lambda)$. Also, let $\mathbf{1}(\cdot)$ denote the indicator function. Taking expectations of (2.1) and (2.2), and exploiting the recurrence structure, we find $E\left[OH\right]$ as follows:

$$E[OH] \quad = \quad \mathcal{F}(X) + E[OH_{N+1}\mathbf{1}(A_N \leq X)] \tag{2.4}$$

where

$$\mathcal{F}(X) := \lambda_0^{-1} N \left[S_0 - \frac{N-1}{2}\right] + \frac{\lambda_0 - \lambda_1}{\lambda_0 \lambda_1} \sum_{i=0}^{N-1} (S_0 - i) P(i; \lambda_0 X), \quad \lambda_1 > 0 \tag{2.5}$$

In equation (2.4), $\mathcal{F}(X)$ represents the expected inventory carried during the stock removal phase whereas $E[OH_{N+1}\mathbf{1}(A_N \leq X)]$ is the expected inventory carried during the regular operation phase. Note that, OH_{N+1} exists only if the new base stock level is reached before X (*i.e.* $A_N \leq X$).

So far we derived the closed form expressions only for the expected inventory carried during the stock removal phase. In the sequel, we provide an exact transient analysis of the inventory level process during the regular operation phase and derive the expressions for the operating characteristics of this phase.

2.3.1 Analysis of the Regular Operation Phase

We want to compute the expected on hand carried and the expected time weighted backorders incurred in the regular phase which starts at time $A_N (\leq X)$ and lasts until the end of the second period. To compute these operating characteristics we represent the inventory level process in terms of the demand process. Since we are only interested in the time period after A_N, in the sequel, we shift the time axis from A_N to 0 for clarity. Hence, the drop in demand rate occurs at $X - A_N$ time units after the regular operation phase begins (see Figure 2.3). Thus, for $t \geq 0$ the inventory level $IL(t)$ conditional on A_N can be given as:

$$IL(t)|A_N \quad = \quad \begin{cases} S_1 - D(t) & \text{if} \quad t \leq L \\ S_1 - (D(t) - D(t-L)) & \text{if} \quad t > L \end{cases} \tag{2.6}$$

where $\{D(t) : t \geq 0\}$ is a nonhomogenous Poisson process with intensity function $\Lambda(t) : [0, \infty) \to [0, \infty)$ given by

$$\Lambda(t) = \int_0^t \lambda(z)dz \tag{2.7}$$

with arrival rate

$$\lambda(z) = \begin{cases} \lambda_0 & \text{if} \quad z \leq X - A_N \\ \lambda_1 & \text{if} \quad z > X - A_N \end{cases} \tag{2.8}$$

Substituting (2.8) in (2.7) yields

$$\Lambda(t) = \begin{cases} \lambda_0 t & \text{if} \quad t \leq X - A_N \\ (\lambda_0 - \lambda_1)(X - A_N) + \lambda_1 t & \text{if} \quad t > X - A_N \end{cases} \tag{2.9}$$

Equation (2.6) is the representation of the net inventory level at any time point based on the demand up to time t, the lead time demand and inventory position. Recall that the inventory position remains constant at the level S_1 during the regular operation phase since an order is placed each time there is a demand. Therefore, if $t \leq L$ then $IL(t)$ is equal to the inventory position minus the total demand up to time t. Whereas, if $t > L$ then $IL(t)$ is equal to the inventory position minus the lead time demand.

The end of the regular operation phase is a random time point depending on the inventory level at time X. For example, if the net inventory level at time X is equal to $S_0 - N$ then there are no outstanding orders and the regular operation phase ends. Otherwise, it ends when all outstanding orders given between $X - L$ and X are replenished up to time $X + L$. However, dealing with the random end time complicates the analysis beyond tractability. Hence, we assume that the regular operation phase always ends at time $X + L$. This approximation simply results in the overestimation of the expected total cost due to extended calculation period but does not change the optimal X drastically since the shift in the expected total cost is mainly upwards.

We start with the computation of $E[OH_{N+1}\mathbf{1}(A_N \leq X)]$ by conditioning on A_N such that,

$$E[OH_{N+1}\mathbf{1}(A_N \leq X)] = \int_0^X E[OH_{N+1}\mathbf{1}(A_N \leq X)|A_N = s]f_{A_N}(s)\, ds \tag{2.10}$$

where

$$f_{A_N}(s) = \frac{\lambda_0 e^{-\lambda_0 s}(\lambda_0 s)^{N-1}}{(N-1)!}, \qquad s \geq 0.$$

is the pdf of the Erlang distribution with parameters N and λ_0.

We are interested in expected on hand carried from time A_N until $X+L$. Since the time axis is shifted, the expected inventory carried during this period is the positive area under the expected trajectory of the net inventory level process from 0 to $X - A_N + L$. Thus, for a given A_N this area can be computed as follows:

$$E[OH_{N+1}\mathbf{1}(A_N \leq X)|A_N = s] = \int_0^{X-s+L} E\left[(IL(t))^+|A_N = s\right] dt \tag{2.11}$$

From (2.6),

$$\begin{aligned} E\left[(IL(t))^+|A_N = s\right] &= \sum_{n=0}^{S_1-1}(S_1-n)P(D(t)=n)\mathbf{1}(t \leq L) \\ &+ \sum_{n=0}^{S_1-1}(S_1-n)P(D(t)-D(t-L)=n)\mathbf{1}(t > L) \end{aligned} \tag{2.12}$$

Substituting (2.12) in (2.11) yields

$$\begin{aligned} E[OH_{N+1}\mathbf{1}(A_N \leq X)|A_N = s] &= \sum_{n=0}^{S_1-1}(S_1-n)\left[\int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!}\,dt\right. \\ &+ \left.\int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t)-\Lambda(t-L))^n}{n!}\,dt\right] \end{aligned} \tag{2.13}$$

and using the result in (2.10) gives that,

$$\begin{aligned} E[OH_{N+1}\mathbf{1}(A_N \leq X)] &= \sum_{n=0}^{S_1-1}(S_1-n)\int_0^X \left[\int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!}\,dt\right. \\ &+ \left.\int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t)-\Lambda(t-L))^n}{n!}\,dt\right] f_{A_N}(s)\,ds \end{aligned} \tag{2.14}$$

We define the following functions,

$$b_N(r;n,\rho) := \binom{r+n-1}{n-1}\rho^n(1-\rho)^r$$

and

$$\xi(r,n) := np(r+n;\lambda_0 X)\binom{r+n}{n}\sum_{k=0}^{r}\binom{r}{k}\frac{(-1)^k}{n+k}\left(\frac{X-L}{X}\right)^{n+k}$$

where $r \in \{0,1,2,...\}$, $n \in \{1,2,...\}$ and $\rho \in \mathbb{R}$. Moreover, we let $\bar{P}(n;\lambda) := 1-P(n-1;\lambda)$ denote the complementary cdf of Poisson distribution.

The integrals in equation (2.14) can be calculated with respect to the relationship between X and L. Thus, for $\lambda_1 > 0$, the expected on-hand inventory carried during the regular operation phase is found as follows:

$$E[OH_{N+1}\mathbf{1}(A_N \leq X)] = \begin{cases} \sum_{n=0}^{S_1-1}(S_1-n)\left[f(n)+g_1(n)\right] & \text{if} \quad L \leq X \\ \sum_{n=0}^{S_1-1}(S_1-n)\left[f(n)-g_2(n)\right] & \text{if} \quad L > X \end{cases} \tag{2.15}$$

where

$$f(n) = \left[\frac{1}{\lambda_0}+\frac{P(n;\lambda_1 L)}{\lambda_0-\lambda_1}\right]\bar{P}(N;\lambda_0 X)+\frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1}\left[P(N+n;\lambda_0 X)-P(N-1;\lambda_0 X)\right] \tag{2.16}$$

$$\begin{aligned} g_1(n) &= p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L))-\lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right] \\ &- \frac{2\lambda_0-\lambda_1}{\lambda_0(\lambda_0-\lambda_1)}P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L))-\frac{(\lambda_0-\lambda_1)}{\lambda_0\lambda_1}\sum_{i=0}^{n}\xi(i,N) \\ &- \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\sum_{k=0}^{i}p\left(i-k;\lambda_1 L+(\lambda_0-\lambda_1)X\right)b_N\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\delta(k) \end{aligned} \tag{2.17}$$

with $\delta(k) = P(N+k-1;\lambda_1(X-L))-P(N+k-1;\lambda_1 X)$.

$$g_2(n) = \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\sum_{k=0}^{i}p(i-k;\lambda_1 L+(\lambda_0-\lambda_1)X)b_N\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\bar{P}(N+k;\lambda_1 X) \tag{2.18}$$

The expected time weighted backorders incurred during the regular operation phase can be calculated essentially the same way as described above. Hence, we skip the analysis for brevity and directly give the result:

$$E[BO] = \begin{cases} \sum_{n=S_1}^{\infty}(n-S_1)\left[f(n)+g_1(n)\right] & \text{if} \quad L \leq X \\ \sum_{n=S_1}^{\infty}(n-S_1)\left[f(n)-g_2(n)\right] & \text{if} \quad L > X \end{cases} \tag{2.19}$$

2.3.2 Objective Function

We can now obtain the expected total cost incurred in the second period by using the operating characteristics derived above. The general structure of the expected total cost incurred in the second period can be given as follows:

$$TC(X) = hE\left[OH\right]+\pi E\left[BO\right] \tag{2.20}$$

Using equations (2.4), (2.15) and (2.19) in (2.20) and defining,

$$c(x) := \begin{cases} hx & \text{if} \quad x > 0 \\ -\pi x & \text{if} \quad x \leq 0 \end{cases} \tag{2.21}$$

we obtain that,

$$TC(X) \quad = \quad h\mathcal{F}(X) + \begin{cases} \sum_{n=0}^{\infty} c(S_1 - n)\,[f(n) + g_1(n)] & \text{if} \quad L \leq X \\ \sum_{n=0}^{\infty} c(S_1 - n)\,[f(n) - g_2(n)] & \text{if} \quad L > X \end{cases} \tag{2.22}$$

Our goal is to find the optimal time for policy change that minimizes the expected total cost incurred during the second period. That is, we want to minimize $TC(X)$ where $X \geq 0$.

Despite the complicated appearance of equation (2.22) the optimal solution of the problem can be found easily. This is because the equations (2.16)-(2.18) are mainly composed of elementary probability functions and some combinatorial expressions. For the dimensions that we are interested in all of the functions can be calculated easily with a general purpose programming language. Besides, as we will discuss in more detail in Section 2.3.4, $TC(X)$ is observed to be unimodal in X. Hence, X^* can be searched very efficiently with standard nonlinear optimization methods.

2.3.3 Full Obsolescence Case ($\lambda_1 = 0$)

So far we have considered an inventory system facing obsolescence in which the demand drops to a lower level but does not vanish ($\lambda_1 > 0$). However, in some practical cases the demand might disappear after a certain time point and the remaining stocks are either sold in secondary markets or sent to locations where the demand is still healthy. Although the analysis of the net inventory level process for full obsolescence case is essentially the same as described in the previous section, the operating characteristics and the objective function have to be slightly modified.

When $\lambda_1 = 0$ the number of excess stocks to be removed is equal to S_0, and therefore the inventory is carried only during the stock removal phase. Hence, the term $E[OH_{N+1}\mathbf{1}(A_N \leq X)]$ drops from the equation (2.4). Similarly, in equation (2.5) the term representing the expected inventory carried after the drop ($=\lambda_1^{-1}\sum_{i=0}^{N-1}(S_0 - i)P(i;\lambda_0 X)$) becomes irrelevant since under full obsolescence the stock removal can only be possible before time X. Thus, the expected total inventory carried during the second period can be given as:

$$E[OH] = \lambda_0^{-1} \left[\frac{S_0(S_0+1)}{2} - \sum_{i=0}^{S_0-1} (S_0 - i) P(i; \lambda_0 X) \right] \tag{2.23}$$

If full obsolescence occurs before all of the excess stocks are removed then the remaining on hand inventory is usually salvaged (disposed) or relocated. In that case the obsolescence cost c_o is incurred per unit of remaining inventory at the end of the second period. In case of salvaging c_o can be interpreted as the overage cost of the well known newsboy problem. Otherwise, it can be seen as the cost of transporting per unit of remaining inventory to a location where the demand is healthier. Since S_0 items should be removed before time X the expected number of remaining stock at the end of the second period can be given by the following expression:

$$E[RS] = \sum_{i=0}^{S_0-1} (S_0 - i) p(i; \lambda_0 X) \tag{2.24}$$

where $p(i; \lambda_0 X)$ is the probability that i items are demanded from the beginning of the second period until the obsolescence occurs. Note that $E[RS]$ is not affected by our assumption that there are no outstanding orders at the beginning of the second period since the number of stocks removed before time X only depends on the demand arrival process but not the net inventory level process. Moreover, it can be easily shown that $E[RS]$ is convex in X.

The analysis of the regular operation phase is similar to the one with positive λ_1. However, under full obsolescence there are no inventory carried during the regular operation phase since the base stock level is zero. Thus, the expression for the expected time weighted backorders incurred during this phase is found as:

$$E[BO] = \begin{cases} \sum_{n=0}^{\infty} n\,[f(n) + g(n)] & \text{if} \quad L \leq X \\ \sum_{n=0}^{\infty} n f(n) & \text{if} \quad L > X \end{cases} \tag{2.25}$$

where

$$\begin{aligned} f(n) &= \lambda_0^{-1}\left[2\bar{P}(N+n+1;\lambda_0 X) + Np(N+n+1;\lambda_0 X)\right] \\ &- (X-L)p(N+n;\lambda_0 X) \end{aligned} \tag{2.26}$$

$$\begin{aligned} g(n) &= p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L)) - \lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right] \\ &- 2\lambda_0^{-1}\left[P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L)) - \sum_{i=0}^{n}\xi(i,N)\right] \\ &+ (X-L)\xi(n,N) - \lambda_0^{-1}N\xi(n,N+1) \end{aligned} \tag{2.27}$$

Therefore, the expected total cost incurred during the second period under full obsolescence can be given as,

$$TC(X) = hE\left[OH\right] + c_o E\left[RS\right] + \pi E\left[BO\right] \tag{2.28}$$

In our numerical experiments, we observed that equation (2.28) is unimodal in X. Hence, the optimal solution of $TC(X)$ can be found very easily for the full obsolescence case as well.

2.3.4 General Behavior of Objective Function and Operating Characteristics

In this section, we investigate the general behavior of the objective function and the operating characteristics. For comparison purposes we conducted 5000 simulations of the demand arrival process for given λ_0 and λ_1 pair. Then for a given X value, the operating characteristics and the objective function are found by averaging the values calculated at each of the simulated trajectories. In the sequel, we use subscript 's' to denote the simulated values for clarity. Figures 2.4 - 2.5 illustrate the general behavior of the objective function and the operating characteristics. In the figures simulated values are given along with their 95% confidence intervals.

Throughout our numerical study we observed that the expected total cost function is unimodal in X (see Figure 2.4). The intuition behind this behavior can be explained as follows: If X is too short then the inventory system does not have enough time to remove all of the excess stocks (N) before the drop in demand rate occurs. Therefore, the remaining excess stocks either increase the holding costs since the natural attrition of these stocks takes longer due to diminished demand or they result in obsolescence cost -in case of full obsolescence- due to disposal or relocation. In both cases the system incurs extra holding cost or obsolescence cost for not removing all of the excess stocks before the drop. Hence, we observe a decrease in expected total cost function as X diverges from zero.

On the other hand, if X is too long then all of the excess stocks are removed too early and the inventory system returns to its regular operation mode before the drop in demand rate occurs. Consequently, the system operates under a lower base stock level S_1 in order to satisfy the demand until the drop occurs and incurs more backordering costs. Figure 2.5b shows how the expected backorders increase in X. Therefore, there exists an optimal X value balancing the obsolescence related costs (extra holding cost, obsolescence/relocation cost) with the cost of backordering.

Figure 2.4: Behavior of Objective Functions ($\lambda_0 = 10,\ \lambda_1 = 2,\ L = 0.15,\ \pi = 10,\ h = 1,\ N = 2$)

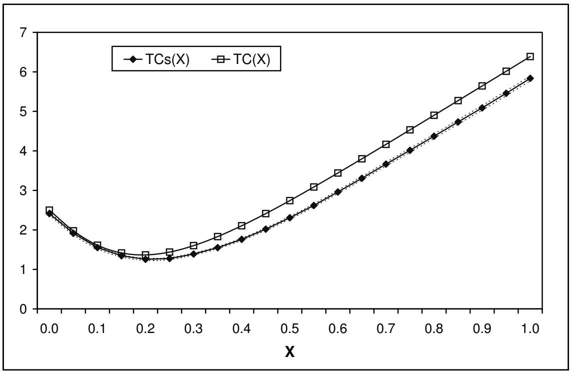
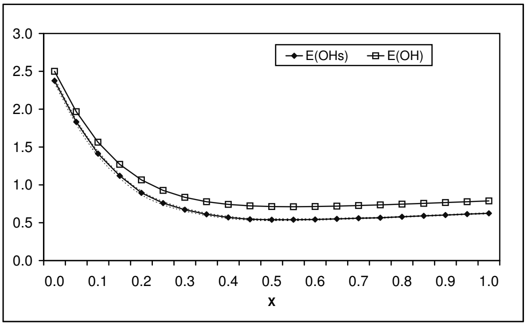


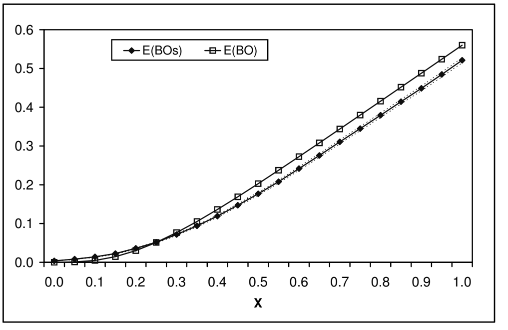
Figure 2.5a presents an example of the rapid decrease in the expected on hand as X diverges from zero when λ_1 is positive. For the full obsolescence case, however, the

Figure 2.5: Behavior of Operating Characteristics ($\lambda_0 = 10,\ \lambda_1 = 2,\ L = 0.15,\ \pi = 10,\ h = 1,\ N = 2$)

(a) Expected On-Hand Carried



(b) Expected Backorder



behavior of the expected on hand is different. The inventory is carried only in the stock removal phase and for small X values it usually ends before all of the excess stocks are removed. Therefore, when $\lambda_1 = 0$ the expected on hand is generally increasing in X until it converges to a constant (the expected positive area under the net inventory level process when the stock removal ends before the drop occurs). Although the inventory system tends to carry less stock as X decreases, the expected total cost increases due to the increase in the expected number of remaining stocks.

Comparison with Simulation

Our two assumptions about the initial inventory level and the end of the second period result in different sample paths of the net inventory level process for our model and simulation in periods $[T - X, T - X + L]$ and $[T, T + L]$. When λ_1 is positive $E[OH]$ always overestimates $E[OH_s]$ due to higher on hand inventory level between $T - X$ and $T - X + L$, and the extended calculation period. This can be observed in Figure 2.5a. On the other hand, when $\lambda_1 = 0$, $E[OH_s]$ is larger for X values near zero since the outstanding orders at time $T - X$ are likely to arrive after time T and therefore, in simulations the second period is likely to be longer compared to our model.

When λ_1 is positive expected backorders are underestimated by $E[BO]$ as long as the initial inventory level S_0 is high enough to cover the demand before time T. However, as X gets larger, the system returns to its regular operation mode earlier and $E[BO]$ begins to overestimate $E[BO_s]$ due to the extended calculation period. For example, in Figure 2.5b, we observe that $E[BO]$ starts to overestimate $E[BO_s]$ for the X values greater than 0.25. Furthermore, we found that the performance of $E[BO]$ is much better for the full obsolescence case. Because when the stock removal phase ends before T, the sample path differences between simulation and our model are only from $T-X$ until $T-X+L$.

For positive λ_1 and X values small enough we observe that the percent difference between $TC(X)$ and $TC_s(X)$ is relatively low since the overestimation of $E[OH_s]$ is compensated by the underestimation of $E[BO_s]$. Moreover, $TC(X)$ underestimates $TC_s(X)$ as long as the real backordering cost is larger than the overestimated quantity in holding cost (*i.e.* $\pi E[BO_s] > h[E[OH]-E[OH_s]]$). Otherwise, $TC(X)$ is larger than $TC_s(X)$ as a result of overestimation in holding costs. Similar intuitive results are observed for the full obsolescence case as well. Finally, for 256 experiment instances, we found that the average absolute percent difference between $TC(X)$ and $TC_s(X)$ is approximately 11% for positive λ_1 while it is only 1.25% when $\lambda_1 = 0$ as a result of increased accuracy in $E[BO]$ and the exact calculation of $E[RS]$.

2.4. Numerical Study

In this section, we first investigate the changes in optimal policy parameter and expected total cost function under different parameter sets. Then, we identify the performance of our model and its impact on expected total costs by comparing it with simulation optimization. Finally, we close the section with a discussion about the value of advance policy change. In the sequel, we use '$*$' to indicate optimality and denote the optimal X value found by simulation optimization with X_s^*.

Throughout the numerical study we assume that simulation is representative of underlying real world model. Thus, we compare $TC_s(X^*)$ with $TC_s^*(X_s^*)$ to measure the impact of operating under X^*. As a simulation optimization technique, we employ response surface methodology as described in Myers and Montgomery (1995).

The experiment instances used in our numerical study is generated with the following parameter set: $\lambda_0 \in \{0.5, 0.7, 1, 5, 7, 10\}$ per year, $\lambda_1 \in \{0, 0.2, 2\}$ per year, $h = 1$ per unit per year, $\pi \in \{5, 15, 25, 50, 75, 150, 300\}$ per unit per year, $c_o \in \{5, 10\}$ per unit,

$L \in \{0.15, 0.25, 0.50, 0.75, 1\}$ years. In total, we generate 281 instances for which the average number of excess stocks to be removed is approximately 3 units. Some of the results from the numerical study are tabulated in Table 2.1-2.2.

Table 2.1: Performance of X^* and Value of Advance Policy Change When $\lambda_1 > 0$

$h = 1$

λ_0	λ_1	π	L	S_0	N	X_s^*	X^*	$\Delta_x\%$	$TC_s^*(X_s^*)$	$TC_s(X^*)$	$\Delta_c\%$	$TC_s(0)$	$\Delta_o\%$
0.5	0.2	50	0.50	2	1	1.80	1.64	-8.94	7.66	7.73	0.88	10.17	31.61
			0.75	2	1	1.29	1.11	-13.77	8.35	8.51	1.89	10.19	19.73
			1.00	2	1	0.96	0.80	-16.60	9.06	9.25	2.07	10.33	11.69
		300	0.50	2	0	-	-	-	-	-	-	-	-
			0.75	3	1	1.65	1.45	-11.88	11.73	11.80	0.65	15.22	28.96
			1.00	3	1	1.24	1.05	-15.43	12.52	12.62	0.81	15.28	21.10
1	0.2	50	0.50	2	1	1.02	0.87	-15.02	6.88	6.88	0.01	10.07	46.34
			0.75	3	2	1.71	1.58	-7.24	14.47	14.53	0.40	24.88	71.21
			1.00	3	2	1.40	1.31	-6.07	16.22	16.41	1.12	24.94	52.03
		300	0.50	3	1	1.16	1.06	-8.32	9.20	9.24	0.42	15.08	63.23
			0.75	4	2	1.74	1.65	-4.78	19.86	20.12	1.28	34.84	73.17
			1.00	5	3	2.26	2.19	-2.98	32.32	32.77	1.40	59.46	81.45
5	2	5	0.05	1	1	0.18	0.15	-13.36	0.39	0.40	1.60	0.50	24.38
			0.15	2	1	0.24	0.19	-20.66	0.70	0.71	0.74	0.96	35.62
			0.25	2	1	0.16	0.12	-23.52	0.84	0.86	2.11	0.98	13.95
		50	0.05	2	1	0.19	0.16	-13.33	0.78	0.78	1.25	0.99	26.33
			0.15	3	1	0.18	0.14	-23.67	1.15	1.19	3.42	1.48	23.79
			0.25	4	2	0.29	0.25	-14.17	2.51	2.55	1.75	3.39	32.75
10	2	5	0.05	1	1	0.12	0.09	-21.22	0.34	0.34	0.19	0.51	47.96
			0.15	3	2	0.29	0.25	-14.01	1.02	1.04	1.30	2.42	133.47
			0.25	4	3	0.35	0.30	-13.07	1.81	1.82	0.27	4.28	135.56
		50	0.05	2	1	0.11	0.09	-20.35	0.68	0.69	1.69	1.02	46.98
			0.15	4	2	0.21	0.18	-12.23	1.88	1.89	0.51	3.44	82.05
			0.25	6	4	0.33	0.31	-6.55	4.27	4.33	1.22	8.79	103.16

2.4.1 General Behavior of Optimal Policy Parameters and Total Cost Functions

As can be seen from Table 2.1, when λ_1 is positive, we do not always observe a monotonic behavior in optimal X values and expected total costs due to discrete jumps in S_0 or S_1 as L or π increases. However, when all other parameters are constant if an increase in L or π does not effect S_0 and S_1 then the optimal X decreases to reduce the risk of backordering.

For the full obsolescence case, we observe a similar non-monotonic behavior in the optimal values with respect to the changes in L or π. However, optimal X and $TC_s(\cdot)$

Table 2.2: Performance of X^* and Value of Advance Policy Change When $\lambda_1 = 0$

$L = 0.25,\ h = 1$

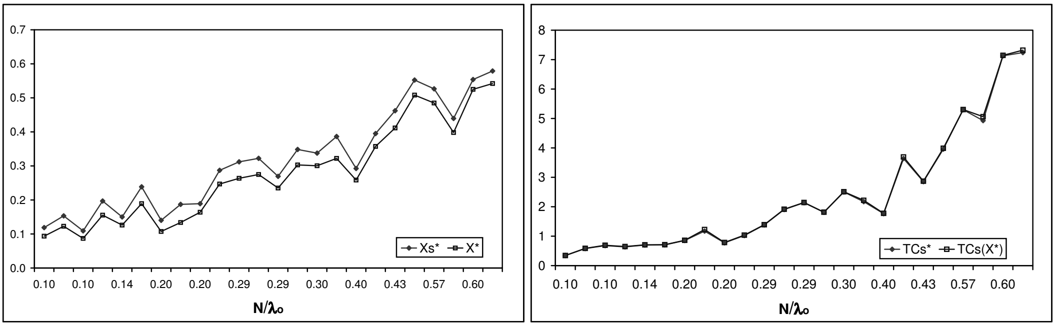
λ_0	π	c_o	N	X_s^*	X^*	$\Delta_x\%$	$TC_s^*(X_s^*)$	$TC_s(X^*)$	$\Delta_c\%$	$TC_s(0)$	$\Delta_o\%$
0.5	50	5	1	0.44	0.43	-2.34	4.73	4.74	0.16	5.03	6.17
		10	1	1.00	0.99	-1.25	8.22	8.22	0.02	10.03	21.99
	300	5	2	0.36	0.40	9.40	9.88	9.89	0.11	10.02	1.35
		10	2	0.93	0.94	1.27	18.11	18.14	0.16	20.02	10.36
1	50	5	2	0.86	0.87	1.19	8.26	8.28	0.28	10.03	21.16
		10	2	1.40	1.40	-0.51	13.31	13.31	0.04	20.03	50.50
	300	5	2	0.29	0.32	11.00	9.44	9.48	0.44	10.06	6.17
		10	2	0.50	0.53	6.05	17.40	17.40	0.03	20.06	15.30
5	5	5	2	0.59	0.59	-0.78	3.37	3.38	0.45	10.14	199.65
		10	2	0.75	0.75	-0.65	4.32	4.34	0.40	20.14	363.86
	50	5	4	0.52	0.52	0.33	11.77	11.82	0.47	20.30	71.73
		10	4	0.67	0.67	0.07	18.33	18.35	0.12	40.30	119.65
10	5	5	4	0.56	0.55	-1.99	4.74	4.74	0.15	20.38	329.45
		10	4	0.66	0.65	-1.79	5.88	5.90	0.36	40.38	584.32
	50	5	6	0.43	0.44	1.09	14.74	14.76	0.12	30.68	107.88
		10	6	0.53	0.53	-0.14	22.75	22.79	0.19	60.68	166.20

are monotonically increasing in c_o since N is independent of the obsolescence cost. Thus, as c_o increases optimal X values also increase to reduce the number of remaining stocks and the expected total costs increase as a result of higher obsolescence penalty (see Table 2.2).

An important indicator for the behavior of the optimal X is the ratio N/λ_0, average time needed to remove Nth excess stock before the drop occurs. In general, we observe that optimal X values and corresponding expected total costs are increasing in N/λ_0 (Figure 2.6). This is to be expected since as the ratio increases more time is needed to complete the stock removal process before the drop occurs. Hence, the system adjusts itself accordingly. On the other hand, the increase in optimal values is not monotonic. This is because the ratio is only a measure of the stock removal process but not the regular operation phase. In other words, the inventory system might incur backordering cost once the stock removal is completed. Hence, the optimal values are not monotonically increasing in N/λ_0.

Figure 2.6: Change in Optimal Values wrt N/λ_0

(a) Optimal X Values (b) Optimal Expected Total Costs



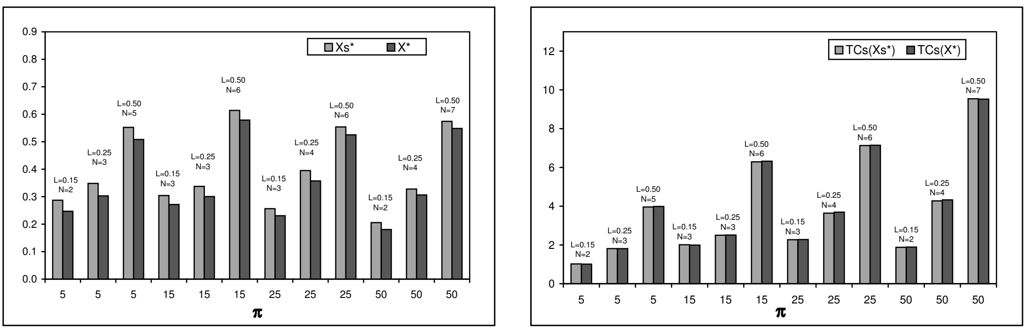
Note. In the figures above, λ_0 is varying from 5 to 10 and N is varying from 1 to 6

2.4.2 Overall Performance of X^*

Next, we compare the performance of X^* *vis à vis* X_s^*. For comparison purposes we use percent error which gives the percentage deviation from the optimal values found by simulation optimization. Hence, we define $\Delta_x\% = \frac{X^* - X_s^*}{X_s^*} \times 100$ as the percent deviation from X_s^* whereas $\Delta_c\% = \frac{TC_s(X^*) - TC_s^*(X_s^*)}{TC_s^*(X_s^*)} \times 100$ is defined as the percent deviation from the optimal expected total cost $TC_s^*(X_s^*)$ as a result of using X^* instead of X_s^*. Figure 2.7 illustrates a comparison of optimal X values and corresponding expected total costs.

Figure 2.7: Performance of X^* ($\lambda_0 = 10,\ \lambda_1 = 2,\ \overline{N} = 4$)

(a) $\overline{X_s^*} = 0.40,\ |\overline{\Delta_x}|\% = 9.21\%$ (b) $\overline{TC_s^*} = 3.86,\ \overline{\Delta_c}\% = 0.67\%$



We observed that the expected total cost is quite robust to the changes in X_s^*. For example, for the instances considered in Figure 2.7 we found that X^* underestimates X_s^*

on average by 9.21%. For the same instances, however, the average deviation from the optimal expected total cost is only 0.67%. The robust behavior of the expected total cost function can also be observed in Figure 2.6b. Moreover, we found that X^* might underestimate or overestimate X_s^* depending on the interplay between the extra costs resulting from our two main assumptions and the costs related with obsolescence. This can be best observed in Table 2.2.

For all instances with positive λ_1 (89 instances out of 281), the mean absolute deviation from X_s^* is found to be 11.87%. For the same instances we found that using X^* instead of X_s^* results in a deviation from the optimal expected total cost on average 1.03% and maximum 4.42%. For the full obsolescence case, we found that the mean absolute deviation from X_s^* is 5.42% and the average deviation from the optimal expected total costs is 0.56% with a maximum of 3.44%. Thus, we conclude that X^* performs satisfactorily and it gives near optimal results for expected total costs. For more detailed results we refer the reader to Tables 2.1-2.2.

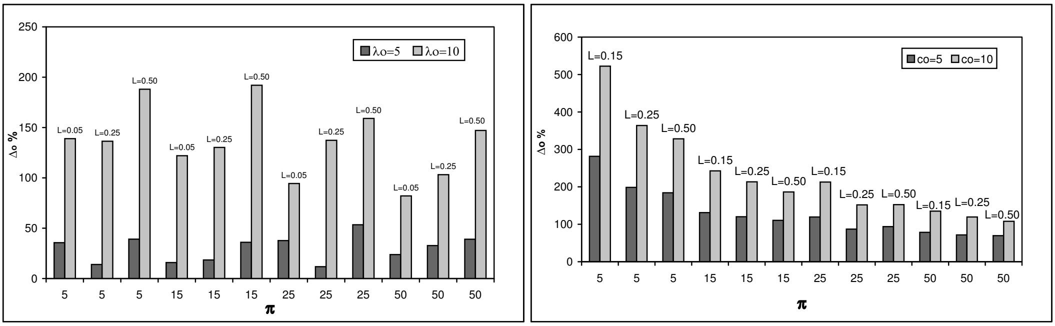
2.4.3 Value of Advance Policy Change

Next, we discuss the value of changing the control policy to initiate the stock removal process before the drop in demand rate occurs. To this end, we compare the expected total cost incurred by changing the policy X^* time units earlier before the drop occurs with the expected total cost incurred by changing it immediately after the drop occurs ($X = 0$). We use the benchmark case where it is possible to change the policy in advance, that is, the decision maker uses X^* and incurs the cost $TC_s(X^*)$. Alternatively, she can postpone the policy change to obsolescence time. As such, the increase in total cost by postponing the policy change reflects the value that the decision maker gets by changing the policy in advance. For comparison purposes we use percent deviation in expected total cost functions defined as $\Delta_o\% = \frac{TC_s(0)-TC_s(X^*)}{TC_s(X^*)} \times 100$. Figure 2.8 illustrates the changes in $\Delta_o\%$ for different λ_0 and λ_1 values.

We found that the impact of advance policy change on costs is significant. For example, in Figure 2.8a when $\lambda_0 = 5$ the average $TC_s(X^*)$ is found to be 2.11. For these instances, waiting until the drop occurs increases the expected total costs on average by 30%. The increase in total cost is due to the increase in holding costs since the natural attrition of

Figure 2.8: Value of Advance Policy Change ($\Delta_o\%$)

(a) $\lambda_0 = 5, 10,\ \lambda_1 = 2$ (b) $\lambda_0 = 5,\ \lambda_1 = 0$



the remaining excess stocks takes longer once the drop occurs. Moreover, we found that when all other parameters are constant, the cost of postponing the policy change increases very rapidly in λ_0 (see Table 2.1). This can be seen clearly from Figure 2.8a; when λ_0 increases from 5 to 10 the average percent deviation due to postponement increases from 30% to 136%.

Our observations for the full obsolescence case are similar. However, when $\lambda_1 = 0$ the increase in total cost is mainly due to the obsolescence/relocation cost charged per remaining excess stock. For the instances given in Figure 2.8b, we found that when $\lambda_0 = 5$, the average $\Delta_o\%$ increases from 129.07% to 228.12% as c_o doubles. Moreover, we observed that $\Delta_o\%$ is decreasing in π. This is because as π increases the cost of obsolescence becomes relatively cheaper compared to backordering and the system tends to postpone the policy change towards the obsolescence time. These behaviors can be seen in more detail in Table 2.2.

We close our discussion about the value of advance policy change by giving the summaries about $\Delta_o\%$. Over all the numerical experiments conducted, we found that for positive λ_1, changing the control policy after the drop occurs increases the expected total costs on average by 60%. In case of full obsolescence, we found that the expected total costs are on average more than doubled as a result of not taking an early action ($\overline{\Delta_o}\% = 133.04\%$). These findings show us that employing an advance policy change when obsolescence is expected, results in important savings.

2.5. Conclusion

In this chapter, we considered a continuous review inventory system of a slow moving item in which the demand rate drops to a lower level at a known time in the future. Adaptation to the new demand rate is achieved by changing the control policy before the drop occurs, and therefore letting the demand process to take away the excess stocks. We focused on the behavior of the net inventory level process during the transient period and proposed an approximate solution for the optimal time to shift to the new control policy minimizing the expected total cost incurred during this period. We found that the advance policy change results in significant cost savings and our model yields near optimal solutions for the expected total costs.

The key contribution of this chapter lies in the analysis of a continuous review inventory system facing nonstationary stochastic demands in the context of obsolescence problem for slow moving, expensive items. Earlier works on obsolescence were focused mostly on periodic review models. The main insights from these works were that the obsolescence has a substantial impact on optimal policies and it should be incorporated into inventory control models explicitly. In our study, we extended these findings for a continuous review system and show that advance policy change in case of known obsolescence time results in significant cost savings. Our numerical experiments revealed that for slow movers the timing of the control policy change primarily determines the tradeoff between backordering penalties and obsolescence costs.

The practical importance of our model comes from its consideration of expensive, slow moving items with high downtime costs for which continuous review policies are preferred over periodic review ones due to lower safety stock requirements. For this class of items, efficient management of inventories is notoriously difficult. Not surprisingly, inventory managers of many companies in after sales service industry are recurrently facing the problem of obsolete or excess inventories of such items. Knowing when to change the control policy is the key to reduce obsolete inventories while balancing the availability. If the change is too early then the risk of backordering is too high and the stockouts can be detrimental to companies' operations. On the other hand, if the change is too late then the risk of obsolescence is too high and obsolete stocks lay as dead weight on the books which in return reduces the competitiveness of companies. Our model can be used to study the impact of the timing of policy change on operational costs and to identify the optimum time that balances the tradeoff between the risk of obsolescence and backordering.

While developing our model, we employed a couple of assumptions to keep the analysis in the boundaries of mathematical tractability. Although some of these assumptions limit the generality of the model, the analysis offers an increased understanding of the transient behavior of inventory systems and the impacts of advance policy change on operational costs. Given the scarcity of research on continuous review systems facing obsolescence, we consider that our model bears a reasonable balance between realism and tractability for the insights obtained. Therefore, it can stand as a building block for more complicated and realistic models.

There are a couple of directions for future research. It would be useful to extend the model with demand rate decreasing by time. Such a model would be more suitable for the products at the the end of their life cycles. Another possibility is to incorporate the uncertainty into the timing of the obsolescence or into the size of the drop in demand rate. These extensions would yield interesting insights about the timing of a policy change. Extending the model for a general class of continuous review control policies seems particularly worthwhile because for many products, the prospect of obsolescence has increased drastically due to the rapid changes in consumer taste and technological innovations.

2.6. Appendix

Proof of Equation (2.4). Taking the expectation of equation (2.2) yields that for $k = 2, \ldots, N$ and $N \geq 2$,

$$\begin{aligned} E[OH_k] &= \left[(S_0 - k + 1)X + E[OH_k']\right] P\left(A_{k-1} \leq X,\ A_k > X\right) \\ &+ (S_0 - k + 1)[E\left[\tau_k \mathbf{1}(A_k \leq X)\right] - E\left[A_{k-1}\mathbf{1}(A_{k-1} \leq X,\ A_k > X)\right]] \\ &+ E[OH_{k+1}\mathbf{1}(A_k \leq X)] \end{aligned} \tag{2.29}$$

Observe that the event $\{A_{k-1} \leq X,\ A_k > X\}$ implies that there are exactly $k-1$ demands during the period of length X. Since the total demand during this period is Poisson distributed with rate λ_0 we obtain that,

$$P\left(A_{k-1} \leq X,\ A_k > X\right) = p(k-1; \lambda_0 X), \quad k = 2, 3, \ldots \tag{2.30}$$

Denote,

$$\varepsilon_k' := E\left[\tau_k\, \mathbf{1}(A_k \leq X)\right], \quad k = 2, 3, \ldots$$

and observe that $A_k = A_{k-1} + \tau_k$. Hence, by conditioning on A_{k-1} and after some algebra we get,

$$\varepsilon_k' = \int_0^X \int_0^{X-s} t f_{\tau_k}(t)\, f_{A_{k-1}}(s)\, dt\, ds = \lambda_0^{-1}[1 - P(k-1; \lambda_0 X)] \tag{2.31}$$

Denote,

$$\varepsilon_k'' := E\left[A_{k-1}\mathbf{1}(A_{k-1} \leq X,\ A_k > X)\right], \quad k = 2, 3, \ldots$$

Similarly, conditioning on A_{k-1} yields,

$$\varepsilon_k'' = \int_0^X sP(\tau_k > X - s)\, f_{A_{k-1}}(s) ds = \lambda_0^{-1}(k-1)p(k; \lambda_0 X) \tag{2.32}$$

Thus, the difference between ε_k' and ε_k'' is found as follows:

$$\varepsilon_k' - \varepsilon_k'' = \lambda_0^{-1}\left[1 - P(k-1; \lambda_0 X)\right] - Xp(k-1; \lambda_0 X), \quad k = 2, 3, \ldots \tag{2.33}$$

Also, note that

$$E[OH_k'] = \lambda_1^{-1} \sum_{i=k}^{N} (S_0 - i + 1), \quad k = 1, \ldots, N. \tag{2.34}$$

Therefore, substituting (2.30), (2.33) and (2.34) in (2.29), and making necessary simplifications yields that for $k = 2, \ldots, N$ and $N \geq 2$,

$$\begin{aligned} E[OH_k] &= \lambda_1^{-1} p(k-1; \lambda_0 X) \sum_{i=k}^{N} (S_0 - i + 1) \\ &+ \lambda_0^{-1}(S_0 - k + 1)\left[1 - P(k-1; \lambda_0 X)\right] \\ &+ E[OH_{k+1}\mathbf{1}(A_k \leq X)] \end{aligned} \tag{2.35}$$

Observe that $E[OH_k] = E[OH_k \mathbf{1}(A_{k-1} \leq X)]$ since $E[OH_k \mathbf{1}(A_{k-1} > X)] = 0$. Thus, by exploiting the recursive structure of (2.35) and after some algebra we obtain that for $N \geq 1$,

$$\begin{aligned} E[OH_2 \mathbf{1}(A_1 \leq X)] &= \lambda_1^{-1} \sum_{i=1}^{N-1} (S_0 - i) \sum_{j=1}^{i} p(j; \lambda_0 X) + \lambda_0^{-1}(N-1) \left[S_0 - \frac{N}{2} \right] \\ &- \lambda_0^{-1} \sum_{i=1}^{N-1} (S_0 - i) P(i; \lambda_0 X) + E[OH_{N+1} \mathbf{1}(A_N \leq X)] \end{aligned} \tag{2.36}$$

with the convention that $\sum_{i=k}^{N}() = 0$ for $N < k$. Now, taking the expectation of equation (2.1) gives,

$$E[OH] = \left[S_0 X + E[OH_1'] \right] P(A_1 > X) + S_0 E\left[A_1 \mathbf{1}(A_1 \leq X) \right] + E[OH_2 \mathbf{1}(A_1 \leq X)] \tag{2.37}$$

Thus, using (2.34) and (2.36) in (2.37), and rearranging the terms yield the expected on-hand as given by equation (2.4). □

Before starting the analysis of $E[OH_{N+1} \mathbf{1}(A_N \leq X)]$ we need the following lemma which is important for our derivations.

Lemma 2.1. *Let $f_E(t)$ be the pdf of Erlang distribution with parameters $\alpha \in \{1, 2, ...\}$, $\beta > 0$ and define*

$$I = \int_a^b p(r; \lambda t + \gamma) f_E(t) dt. \tag{2.38}$$

(i) If $\lambda \neq -\beta$ then

$$I = \sum_{k=0}^{r} p(r-k; \gamma) b_N\left(k; \alpha, \frac{\beta}{\lambda + \beta}\right) \delta(k) \tag{2.39}$$

where

$$\delta(k) = P(\alpha + k - 1; (\lambda + \beta)a) - P(\alpha + k - 1; (\lambda + \beta)b)$$

(ii) If $\lambda = -\beta$ then

$$I = \frac{\beta^\alpha}{(\alpha - 1)!} \sum_{k=0}^{r} p(r-k; \gamma) \frac{(-\beta)^k \left(b^{\alpha+k} - a^{\alpha+k}\right)}{k!(\alpha + k)} \tag{2.40}$$

Proof of Lemma 2.1.

(i) From (2.38) we have,

$$I = \frac{e^{-\gamma}\beta^{\alpha}}{r!(\alpha-1)!}\int_a^b (\lambda t+\gamma)^r t^{\alpha-1} e^{-(\lambda+\beta)t}dt$$

Using the binomial theorem and after some algebra,

$$\begin{aligned} I &= \frac{e^{-\gamma}\beta^{\alpha}}{r!(\alpha-1)!}\int_a^b \sum_{k=0}^{r}\binom{r}{k}\gamma^{r-k}(\lambda t)^k t^{\alpha-1}e^{-(\lambda+\beta)t}dt \\ &= \sum_{k=0}^{r}\frac{e^{-\gamma}\gamma^{r-k}}{(r-k)!}\binom{k+\alpha-1}{\alpha-1}\frac{\beta^{\alpha}}{(\lambda+\beta)^{\alpha-1}}\left(\frac{\lambda}{\lambda+\beta}\right)^k\int_a^b p(\alpha+k-1;(\lambda+\beta)t)dt \end{aligned} \tag{2.41}$$

It can be easily shown that for any $\lambda \neq 0$ the following holds,

$$\int_a^b p(n;\lambda t+\gamma)dt = \frac{1}{\lambda}[P(n;\lambda a+\gamma) - P(n;\lambda b+\gamma)] \tag{2.42}$$

Hence, applying (2.42) to the integral on the right-hand side of (2.41) yields,

$$\begin{aligned} I &= \sum_{k=0}^{r} p(r-k;\gamma)\binom{k+\alpha-1}{\alpha-1}\left(\frac{\beta}{\lambda+\beta}\right)^{\alpha}\left(\frac{\lambda}{\lambda+\beta}\right)^{k}[P(\alpha+k-1;(\lambda+\beta)a) \\ &- P(\alpha+k-1;(\lambda+\beta)b)] \\ &= \sum_{k=0}^{r} p(r-k;\gamma)b_N\left(k;\alpha,\frac{\beta}{\lambda+\beta}\right)[P(\alpha+k-1;(\lambda+\beta)a) - P(\alpha+k-1;(\lambda+\beta)b)]. \end{aligned}$$

(ii) When $\lambda = -\beta$ (2.38) can be simplified as,

$$I = \frac{e^{-\gamma}\beta^{\alpha}}{r!(\alpha-1)!}\int_a^b (-\beta t+\gamma)^r t^{\alpha-1}dt$$

and the result follows from the binomial theorem:

$$\begin{aligned} I &= \frac{e^{-\gamma}\beta^{\alpha}}{r!(\alpha-1)!}\sum_{k=0}^{r}\binom{r}{k}\gamma^{r-k}(-\beta)^k\int_a^b t^{\alpha+k-1}dt \\ &= \frac{\beta^{\alpha}}{(\alpha-1)!}\sum_{k=0}^{r} p(r-k;\gamma)\frac{(-\beta)^k\left(b^{\alpha+k}-a^{\alpha+k}\right)}{k!(\alpha+k)} \end{aligned}$$

Proof of Equation (2.15).

Denote,

$$E[OH_{N+1}\mathbf{1}(A_N \leq X)] = \sum_{n=0}^{S_1-1}(S_1-n)\,[K_1+K_2]$$

with

$$K_1 := \int_0^X \int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!} f_{A_N}(s)\,dt\,ds \tag{2.43}$$

$$K_2 := \int_0^X \int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t)-\Lambda(t-L))^n}{n!} f_{A_N}(s)\,dt\,ds \tag{2.44}$$

(i) If $L \leq X$ then using the definition of $\Lambda(t)$ we can partition the integrals in (2.43) and (2.44) as follows:

$$\begin{aligned} K_1 &= \int_0^{X-L}\int_0^L p(n;\lambda_0 t)\,dt f_{A_N}(s)\,ds \\ &+ \int_{X-L}^X \left[\int_0^{X-s} p(n;\lambda_0 t)\,dt + \int_{X-s}^L p(n;\eta_1(t,s))\,dt\right] f_{A_N}(s)\,ds \end{aligned} \tag{2.45}$$

$$\begin{aligned} K_2 &= \int_0^{X-L}\left[\int_L^{X-s} p(n;\lambda_0 L)\,dt + \int_{X-s}^{X-s+L} p(n;\eta_2(t,s))\,dt\right] f_{A_N}(s)\,ds \\ &+ \int_{X-L}^X \int_L^{X-s+L} p(n;\eta_2(t,s))\,dt f_{A_N}(s)\,ds \end{aligned} \tag{2.46}$$

where $\eta_1(t,s) := \lambda_1 t + (\lambda_0-\lambda_1)(X-s)$ and $\eta_2(t,s) := \eta_1(t,s) - \lambda_0(t-L)$. From identity (2.42) we obtain that,

$$\begin{aligned} K_1 &= \int_0^{X-L} \frac{1}{\lambda_0}[1-P(n;\lambda_0 L)]f_{A_N}(s)\,ds + \int_{X-L}^X \left[\frac{1}{\lambda_0}[1-P(n;\lambda_0(X-s))]\right. \\ &+ \left.\frac{1}{\lambda_1}[P(n;\lambda_0(X-s)) - P(n;\upsilon(s))]\right] f_{A_N}(s)\,ds \end{aligned} \tag{2.47}$$

$$\begin{aligned} K_2 &= \int_0^{X-L}\left[p(n;\lambda_0 L)(X-s-L) - \frac{1}{\lambda_0-\lambda_1}[P(n;\lambda_0 L) - P(n;\lambda_1 L)]\right] f_{A_N}(s)\,ds \\ &+ \int_{X-L}^X \frac{1}{\lambda_0-\lambda_1}[P(n;\lambda_1 L) - P(n;\upsilon(s))]f_{A_N}(s)\,ds \end{aligned} \tag{2.48}$$

with $\upsilon(s) := \lambda_1 L + (\lambda_0 - \lambda_1)(X - s)$. Summing K_1 and K_2 and rearranging the terms yields,

$$\begin{aligned} K_1 + K_2 &= \left[\frac{1}{\lambda_0} - \frac{2\lambda_0 - \lambda_1}{\lambda_0(\lambda_0 - \lambda_1)} P(n;\lambda_0 L) + p(n;\lambda_0 L)(X - L)\right. \\ &+ \left.\frac{P(n;\lambda_1 L)}{\lambda_0 - \lambda_1}\right] \int_0^{X-L} f_{A_N}(s)\,ds - p(n;\lambda_0 L) \int_0^{X-L} s f_{A_N}(s)\,ds \\ &+ \left[\frac{1}{\lambda_0} + \frac{P(n;\lambda_1 L)}{\lambda_0 - \lambda_1}\right] \int_{X-L}^{X} f_{A_N}(s)\,ds \\ &+ \frac{\lambda_0 - \lambda_1}{\lambda_0 \lambda_1} \int_{X-L}^{X} P(n;\lambda_0(X - s)) f_{A_N}(s)\,ds \\ &- \frac{\lambda_0}{\lambda_1(\lambda_0 - \lambda_1)} \int_{X-L}^{X} P(n;\upsilon(s)) f_{A_N}(s)\,ds \end{aligned} \tag{2.49}$$

Note that,

$$\int_{X-L}^{X} P(n;\lambda_0(X - s)) f_{A_N}(s)\,ds = \sum_{i=0}^{n} \int_{X-L}^{X} p(i;\lambda_0(X - s)) f_{A_N}(s)\,ds \tag{2.50}$$

$$\int_{X-L}^{X} P(n;\upsilon(s)) f_{A_N}(s)\,ds = \sum_{i=0}^{n} \int_{X-L}^{X} p(i;\upsilon(s)) f_{A_N}(s)\,ds \tag{2.51}$$

and denote,

$$I_1 := \int_{X-L}^{X} p(i;\lambda_0(X - s)) f_{A_N}(s)\,ds \tag{2.52}$$

$$I_2 := \int_{X-L}^{X} p(i;\upsilon(s)) f_{A_N}(s)\,ds \tag{2.53}$$

Using part (ii) of Lemma 2.1 and after some algebraic manipulations we obtain that,

$$\begin{aligned} I_1 &= \frac{{\lambda_0}^N}{(N-1)!} \sum_{k=0}^{i} p(i-k;\lambda_0 X) \frac{(-\lambda_0)^k \left(X^{N+k} - (X-L)^{N+k}\right)}{k!(N+k)} \\ &= p(N+i;\lambda_0 X) \left[1 - N\binom{N+i}{N} \sum_{k=0}^{i} \binom{i}{k} \frac{(-1)^k}{N+k} \left(\frac{X-L}{X}\right)^{N+k}\right] \\ &= p(N+i;\lambda_0 X) - \xi(i,N) \end{aligned} \tag{2.54}$$

Thus, substituting (2.54) in (2.50) yields,

$$\int_{X-L}^{X} P(n;\lambda_0(X-s)) f_{A_N}(s)\, ds \;=\; P(N+n;\lambda_0 X) - P(N-1;\lambda_0 X) - \sum_{i=0}^{n} \xi(i,N) \quad (2.55)$$

Similarly, from part (i) of Lemma 2.1 we found that

$$I_2 \;=\; \sum_{k=0}^{i} p(i-k;\lambda_1 L + (\lambda_0-\lambda_1)X) b_N\left(k; N, \frac{\lambda_0}{\lambda_1}\right) \delta(k) \quad (2.56)$$

with

$$\delta(k) = P(N+k-1;\lambda_1(X-L)) - P(N+k-1;\lambda_1 X)$$

Substituting (2.56) in (2.51) gives,

$$\int_{X-L}^{X} P(n;\upsilon(s)) f_{A_N}(s)\, ds \;=\; \sum_{i=0}^{n}\sum_{k=0}^{i} p(i-k;\lambda_1 L + (\lambda_0-\lambda_1)X) b_N\left(k; N, \frac{\lambda_0}{\lambda_1}\right) \delta(k) \quad (2.57)$$

Therefore, employing equations (2.55) and (2.57) in (2.49), and using the following identities

$$\int_0^x f_{A_N}(s)\, ds \;=\; \bar{P}(N;\lambda_0 x) \quad (2.58)$$

$$\int_0^x s f_{A_N}(s)\, ds \;=\; \lambda_0^{-1} N \bar{P}(N+1;\lambda_0 x) \quad (2.59)$$

yield that,

$$E[OH_{N+1}\mathbf{1}(A_N \le X)] \;=\; \sum_{n=0}^{S_1-1} (S_1-n)\left[f(n) + g_1(n)\right].$$

with

$$f(n) \;=\; \left[\frac{1}{\lambda_0} + \frac{P(n;\lambda_1 L)}{\lambda_0-\lambda_1}\right] \bar{P}(N;\lambda_0 X) + \frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1}\left[P(N+n;\lambda_0 X) - P(N-1;\lambda_0 X)\right]$$

$$\begin{aligned} g_1(n) \;=\; & p(n;\lambda_0 L)\left[(X-L)\bar{P}(N;\lambda_0(X-L)) - \lambda_0^{-1} N \bar{P}(N+1;\lambda_0(X-L))\right] \\ & - \frac{2\lambda_0-\lambda_1}{\lambda_0(\lambda_0-\lambda_1)} P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L)) - \frac{(\lambda_0-\lambda_1)}{\lambda_0\lambda_1}\sum_{i=0}^{n}\xi(i,N) \\ & - \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)} \sum_{i=0}^{n}\sum_{k=0}^{i} p\left(i-k;\lambda_1 L + (\lambda_0-\lambda_1)X\right) b_N\left(k; N, \frac{\lambda_0}{\lambda_1}\right) \delta(k) \end{aligned}$$

(ii) If $L > X$ then by the definition of $\Lambda(t)$ the integrals in (2.43) and (2.44) can be partitioned as follows:

$$K_1 = \int_0^X \left[\int_0^{X-s} p(n;\lambda_0 t)\,dt + \int_{X-s}^{L} p(n;\eta_1(t,s))\,dt\right] f_{A_N}(s)\,ds \tag{2.60}$$

$$K_2 = \int_0^X \int_L^{X-s+L} p(n;\eta_2(t,s))\,dt f_{A_N}(s)\,ds \tag{2.61}$$

Using the identity (2.42) in K_1 and K_2, and summing the results yield that,

$$\begin{aligned} K_1 + K_2 &= \left[\frac{1}{\lambda_0} + \frac{P(n;\lambda_1 L)}{\lambda_0 - \lambda_1}\right] \bar{P}(N;\lambda_0 X) + \frac{\lambda_0 - \lambda_1}{\lambda_0\lambda_1}\sum_{i=0}^{n}\int_0^X p(i;\lambda_0(X-s)) f_{A_N}(s)\,ds \\ &- \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\int_0^X p(i;\upsilon(s)) f_{A_N}(s)\,ds \end{aligned} \tag{2.62}$$

Denote,

$$I_3 := \int_0^X p(i;\lambda_0(X-s)) f_{A_N}(s)\,ds$$

$$I_4 := \int_0^X p(i;\upsilon(s)) f_{A_N}(s)\,ds$$

By employing part (ii) of Lemma 2.1 in I_3 we obtain that,

$$I_3 = \frac{{\lambda_0}^N}{(N-1)!}\sum_{k=0}^{i} p(i-k;\lambda_0 X)\frac{(-\lambda_0)^k X^{N+k}}{k!(N+k)} = p(N+i;\lambda_0 X) \tag{2.63}$$

Similarly, from part (i) of Lemma 2.1 we have,

$$I_4 = \sum_{k=0}^{i} p(i-k;\lambda_1 L + (\lambda_0-\lambda_1)X) b_N\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\bar{P}(N+k;\lambda_1 X) \tag{2.64}$$

Therefore, employing (2.63) and (2.64) in (2.62) yields that,

$$E[OH_{N+1}\mathbf{1}(A_N \le X)] = \sum_{n=0}^{S_1-1}(S_1-n)\,[f(n) - g_2(n)]$$

with

$$g_2(n) = \frac{\lambda_0}{\lambda_1(\lambda_0-\lambda_1)}\sum_{i=0}^{n}\sum_{k=0}^{i} p(i-k;\lambda_1 L + (\lambda_0-\lambda_1)X) b_N\left(k;N,\frac{\lambda_0}{\lambda_1}\right)\bar{P}(N+k;\lambda_1 X)$$

Proof of Equation (2.25).

Note that the integral expression for the expected time weighted backorders can be found similar to the expected on hand carried during the regular operation phase as described in section 2.3.1. Thus, for $\lambda_1 = 0$ we found that,

$$
\begin{aligned}
E[BO] &= \sum_{n=0}^{\infty} n \int_0^X \left[\int_0^L \frac{e^{-\Lambda(t)}(\Lambda(t))^n}{n!}\, dt \right. \\
&+ \left. \int_L^{X-s+L} \frac{e^{-[\Lambda(t)-\Lambda(t-L)]}\,(\Lambda(t)-\Lambda(t-L))^n}{n!}\, dt \right] f_{A_N}(s)\, ds \\
&= \sum_{n=0}^{\infty} n\, [K_1 + K_2]
\end{aligned}
$$

(i) If $L \leq X$ then from the definition of $\Lambda(t)$ the integral expressions K_1 and K_2 can be partitioned as in equations (2.45) and (2.46) with $\lambda_1 = 0$. Using the identity (2.42) in K_1 and K_2 and summing the results yield that,

$$
\begin{aligned}
K_1 + K_2 &= \left[2\lambda_0^{-1}[1 - P(n;\lambda_0 L)] + p(n;\lambda_0 L)(X-L) \right] \int_0^{X-L} f_{A_N}(s)\, ds \\
&- p(n;\lambda_0 L) \int_0^{X-L} s f_{A_N}(s)\, ds + 2\lambda_0^{-1} \int_{X-L}^{X} [1 - P(n;\lambda_0(X-s))] f_{A_N}(s)\, ds \\
&- (X-L) \int_{X-L}^{X} p(n;\lambda_0(X-s)) f_{A_N}(s)\, ds + \int_{X-L}^{X} p(n;\lambda_0(X-s)) s f_{A_N}(s)\, ds \quad (2.65)
\end{aligned}
$$

Using the identities (2.58) and (2.59) in (2.65), and simplifying gives that,

$$
\begin{aligned}
K_1 + K_2 &= 2\lambda_0^{-1} \left[\bar{P}(N;\lambda_0 X) - P(n;\lambda_0 L)\bar{P}(N;\lambda_0(X-L)) \right] \\
&- p(n;\lambda_0 L) \left[(X-L)\bar{P}(N;\lambda_0(X-L)) - \lambda_0^{-1} N \bar{P}(N+1;\lambda_0(X-L)) \right] \\
&- 2\lambda_0^{-1} \int_{X-L}^{X} P(n;\lambda_0(X-s)) f_{A_N}(s)\, ds - (X-L) \int_{X-L}^{X} p(n;\lambda_0(X-s)) f_{A_N}(s)\, ds \\
&+ \int_{X-L}^{X} p(n;\lambda_0(X-s)) s f_{A_N}(s)\, ds \quad (2.66)
\end{aligned}
$$

Observe that,

$$\int_{X-L}^{X} p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}N\int_{X-L}^{X} p(n;\lambda_0(X-s))f_{A_{N+1}}(s)\,ds \tag{2.67}$$

where $f_{A_{N+1}}$ is the pdf of Erlang distribution with parameters λ_0 and $N+1$. Thus, by applying part (ii) of Lemma 2.1 to the right-hand side of (2.67) we obtain that,

$$\int_{X-L}^{X} p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}N\left[p(N+1+n;\lambda_0X)-\xi(n,N+1)\right] \tag{2.68}$$

Therefore, employing the results (2.54), (2.55) and (2.68) in (2.66), and making necessary simplifications yield the expected backorder as follows:

$$E[BO] = \sum_{n=0}^{\infty} n\left[f(n)+g(n)\right]$$

with

$$f(n) = \lambda_0^{-1}\left[2\bar{P}(N+n+1;\lambda_0X)+Np(N+n+1;\lambda_0X)\right]-(X-L)p(N+n;\lambda_0X) \tag{2.69}$$

$$\begin{aligned} g(n) &= p(n;\lambda_0L)\left[(X-L)\bar{P}(N;\lambda_0(X-L))-\lambda_0^{-1}N\bar{P}(N+1;\lambda_0(X-L))\right] \\ &- 2\lambda_0^{-1}\left[P(n;\lambda_0L)\bar{P}(N;\lambda_0(X-L))-\sum_{i=0}^{n}\xi(i,N)\right] \\ &+ (X-L)\xi(n,N)-\lambda_0^{-1}N\xi(n,N+1) \end{aligned}$$

(ii) If $L>X$ then from the definition of $\Lambda(t)$ the integral expressions K_1 and K_2 can be partitioned as in equations (2.47) and (2.48) with $\lambda_1=0$. Using the identity (2.42) in K_1 and K_2, and summing the results yield that,

$$\begin{aligned} K_1+K_2 &= 2\lambda_0^{-1}\left[\bar{P}(N;\lambda_0X)-\sum_{i=0}^{n}\int_0^X p(i;\lambda_0(X-s))f_{A_N}(s)\,ds\right] \\ &- (X-L)\int_0^X p(n;\lambda_0(X-s))f_{A_N}(s)\,ds+\int_0^X p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds \end{aligned} \tag{2.70}$$

Observe that

$$\int_0^X p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}N\int_0^X p(n;\lambda_0(X-s))f_{A_{N+1}}(s)\,ds \tag{2.71}$$

Thus, using part (ii) of Lemma 2.1 in (2.71) yields that,

$$\int_0^X p(n;\lambda_0(X-s))sf_{A_N}(s)\,ds = \lambda_0^{-1}Np(N+1+n;\lambda_0 X) \tag{2.72}$$

Finally, using (2.63) and (2.72) in (2.70), and after some simplifications we obtain the expected backorder as follows:

$$E[BO] \;=\; \sum_{n=0}^{\infty} nf(n)$$

with $f(n)$ as given in (2.69). $\square$

Chapter 3

Service Parts Inventory Control under Obsolescence *

Abstract

In this chapter, we consider a single location inventory system of a slow moving item with Poisson demand where the demand rate drops to a lower level at a known future time. Under the assumptions of full backordering and fixed lead times, we incorporate obsolescence into a one-for-one policy with the option to reduce the base stock level in advance. We propose a three parameter policy to control the timing of the shift from high base stock level to the low one. For the special case of identical base stock levels, we show that the optimal base stock level can be calculated from a critical ratio inequality. For different base stock levels, we derive the exact expression for the expected total discounted cost function by partly relying on Fast Fourier Transform method and suggest a numerical optimization procedure to find the optimal values of the policy parameters. Our results suggest that the policy change option leads to pronounced cost savings especially when obsolescence requires a relatively large adjustment in base stock level. We find that ignoring obsolescence in control policy increases costs significantly. Moreover, when obsolescence can be foreseen, early adaptation of base stock levels can lead to important savings.

3.1. Introduction

In this chapter, we analyze the inventory system introduced in Chapter 2 under a more general policy. In the previous chapter, the only decision variable was the policy change

*This chapter is based on Pinçe, Frenk, and Dekker (2009) time. There, we analyzed the system in an approximate fashion to derive heuristic formulas for the operating characteristics and provide qualitative results about the impact of policy change time on transient period costs. In this chapter, we propose a more general three parameter policy including pre- and post-obsolescence base stock levels besides the policy change time. Moreover, in this chapter, the analysis of the model is exact and the optimization involves the infinite horizon discounted total cost rather than the transient period total cost.

As in the previous chapter, we consider a continuous review inventory system of a slow moving item for which the Poisson demand rate drops to a lower level at a known future time. We assume a one-for-one replenishment policy with full backordering and a fixed lead time. We propose a transition control policy with advance policy change option. That is, adaptation to the lower demand rate is achieved by reducing the base stock level before the obsolescence occurs and letting the demand process take away the difference. The objective is to find the optimal base stock levels and the optimal policy change time minimizing the expected total discounted cost over infinite horizon. We show that when base stock levels are identical, the optimal base stock level can be calculated from a critical ratio inequality. For different base stock levels, we compute the total discounted cost function by partly relying on Fast Fourier Transform method and suggest a numerical optimization procedure to find the optimal values of the policy parameters.

The rest of the chapter is organized as follows: Section 3.2 introduces the model and states the three parameter transition control policy. Section 3.3 presents the analysis of the three parameter policy and its special case. Section 3.4 presents the optimization method. Section 3.5 presents the numerical study and discusses the results. Section 3.6 concludes and outlines future research directions.

3.2. The Model

We consider a single item, single location continuous review inventory system of a slow moving item where demand rate drops to a lower level at a known future time T. It is assumed that demand follows a non-homogenous Poisson process $\mathbf{N}_\Lambda := \{\mathbf{N}_\Lambda(t) : t \geq 0\}$

with time transformation

$$\Lambda(t) = \int_0^t \lambda(s)ds \tag{3.1}$$

and arrival rate function

$$\lambda(s) = \begin{cases} \lambda_0 & \text{if } 0 \leq s \leq T \\ \lambda_1 & \text{if } s > T \end{cases}$$

where λ_0 denotes the initial (healthy) demand rate and λ_1 denotes the demand rate after obsolescence, and $\lambda_0 > \lambda_1 \geq 0$.

The inventory control policy is based on one-for-one policy which is commonly used for expensive, slow moving items (see Hadley and Whitin, 1963). According to the one-for-one policy a replenishment order is placed whenever a demand occurs. We assume that all unfilled demand is backordered and there is a fixed replenishment lead time L.

The objective is to minimize the expected total discounted cost over an infinite horizon. The total cost function is composed of holding cost h and backordering penalty π both incurred per unit per time. Since unit and fixed ordering costs are independent of the order quantity under one-for-one policy, they are irrelevant for optimization. Therefore, we exclude the ordering costs from total cost calculations.

Moreover, we assume that the inventory system starts to operate only after the base stock quantity is installed. Thus, we ignore the replenishment lead time and the acquisition cost of the base stock quantity.

As we will show in section 3.3.2, by using $\mathbf{N}_\Lambda$, it is possible to incorporate obsolescence into the one-for-one policy and calculate the optimal base stock level such that the drop in demand rate is taken into account. However, using a single base stock level for pre- and post-obsolescence periods may not be efficient under many obsolescence scenarios. For example, when the drop in demand rate is large and not expected to occur at a very near future then balancing backordering cost with obsolescence cost by a single base stock level might be very difficult.

Alternatively, two different base stock levels can be used to bridge the gap between the demand rates. In that case, a transition rule is needed to control the shift from the high base stock level S_0 to the low base stock level S_1. Thus, we propose the following continuous review, three parameter transition control policy based on the inventory position process:

Policy: Up to time x a replenishment order of size one is placed whenever the inventory position drops to the reorder level $S_0 - 1$. After time x a replenishment order of size one is placed whenever the inventory position drops to the reorder level $S_1 - 1$.

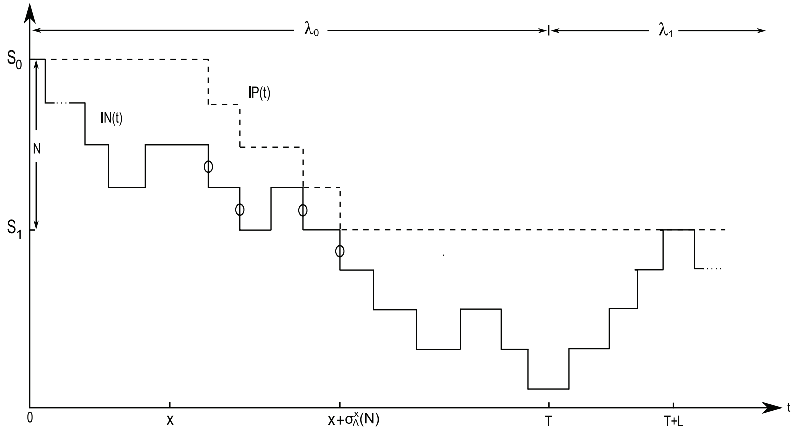
In other words, we use $(S_0 - 1, S_0)$ policy until time x and $(S_1 - 1, S_1)$ policy thereafter. We refer to this policy as the (x, S_0, S_1) policy, where x is the decision variable for policy change, and S_0 and S_1 are the decision variables for stock control. According to the (x, S_0, S_1) policy adaptation to the lower base stock level is achieved by letting the demand take away the *removal quantity* $N(= S_0 - S_1)$ starting from time x. Our goal is to find the optimal base stock levels S_0 and S_1, and the optimal policy change time x minimizing the expected total discounted cost.

It is intuitive that when $S_0 = S_1$, the parameter x becomes redundant and the (x, S_0, S_1) policy boils down to the aforementioned obsolescence based one-for-one policy with single base stock level S_0. In section 3.3.1, we analyze this special case separately and provide a critical ratio inequality to calculate the optimal base stock level.

We let $0 \leq x \leq T$ since postponing the policy change after T would be suboptimal unless the base stock levels are identical.

The rationale behind the (x, S_0, S_1) policy is intuitive: when the time and the size of the drop in demand rate can be foreseen, early adaptation of base stock levels tradeoffs backordering costs with holding costs due to obsolescence, and reduces the number of obsolete stocks while balancing the availability. We do not claim that the (x, S_0, S_1) policy is optimal. Optimal policy structure would probably involve gradual decrease of the base stock level rather than a single adjustment. However, a more sophisticated policy puts the analysis of operating characteristics and the total cost function beyond tractability. Moreover, a policy with multiple base stock adjustments may not be always practical due to increased complexity of execution. Our numerical experiments indicate that the (x, S_0, S_1) policy is a significant improvement over one-for-one policy ignoring obsolescence and a policy without early adaptation ($x = T$). Hence, we claim that the proposed policy brings a good balance between marginal increase in improvement and execution efforts.

Figure 3.1: Possible Realization of $IN(t)$ and $IP(t)$ with Stock Removals



We define the net inventory level process $\mathbf{IN} := \{\mathbf{IN}(t) : t \geq 0\}$ and the inventory position process $\mathbf{IP} := \{\mathbf{IP}(t) : t \geq 0\}$. To better understand the (x, S_0, S_1) policy, we first look at the sample paths of **IN** and **IP** depicted in Figure 3.1. At time zero, the base stock level S_0 is already installed and the inventory system is ready to serve the first demand. Until time x, the system is governed according to $(S_0 - 1, S_0)$ policy, hence, every demand generates a replenishment order. At time x, the ordering policy is changed by reducing the base stock level from S_0 to S_1, and the removal quantity is taken away by N consecutive demand instances. Thus, the inventory position decreases by one at every demand arrival until it hits S_1. In Figure 1, stock removal instances are marked by circles on the sample path of the net inventory level process. In order to determine the time point at which the stock removal process ends, we introduce the hitting time,

$$\sigma_\Lambda^x(N) = \inf\{t \geq 0 : \mathbf{N}_\Lambda(t + x) - \mathbf{N}_\Lambda(x) \geq N\} \tag{3.2}$$

The probability density function of $\sigma_\Lambda^x(N)$ can be given as,

$$f_{\sigma_\Lambda^x}(t) = \frac{d\Lambda(t+x)}{dt} e^{-(\Lambda(t+x)-\Lambda(x))} \frac{(\Lambda(t+x) - \Lambda(x))^{N-1}}{(N-1)!} \tag{3.3}$$

Observe that $x + \sigma_\Lambda^x(N)$ is the random time point at which the inventory position process hits S_1 for the first time, that is, adaptation to the lower base stock level is completed. After $x + \sigma_\Lambda^x(N)$, the replenishment orders are placed again at every demand arrival.

After T, the net inventory level tends to increase due to the drop in demand rate and the arrival of orders given before T. If the net inventory level coincides with the inventory position and the adaptation to the lower base stock level S_1 is not completed yet then both processes slowly decrease due to the diminished demand until they hit S_1. The increase followed by a slow decrease in the net inventory level observed after T is called *inventory hump* (Song and Zipkin, 1996). Long lead times and large removal quantities result in bigger inventory hump, and therefore, higher holding costs. Since the inventory hump is the result of obsolescence, the holding costs incurred during the inventory hump can be considered as a good proxy for obsolescence costs. With the option to change the base stock level before T, the (x, S_0, S_1) policy mainly reduces the inventory hump while balancing backorders.

3.3. Operating Characteristics

3.3.1 Analysis of the (x, S_0, S_1) Policy

To analyze the net inventory level process, we use the inventory position process which provides an easier and more transparent sample path representation. By definition,

$$\mathbf{IP}(t) = \mathbf{IN}(t) + \mathbf{O}(t), \quad t \geq 0$$

where $\mathbf{O}(t)$ is the number of outstanding orders at time $t \geq 0$. By the structure of the considered policy we have,

$$\mathbf{IP}(t) \quad = \quad \begin{cases} S_0 & \text{if} \quad 0 \leq t < x \\ S_0 - (\mathbf{N}_\Lambda(t) - \mathbf{N}_\Lambda(x)) & \text{if} \quad x \leq t < x + \sigma_\Lambda^x(N) \\ S_0 - N & \text{if} \quad t \geq x + \sigma_\Lambda^x(N) \end{cases} \tag{3.4}$$

By the definition of the net inventory level process, the inventory position process and the complete backordering assumption, we obtain that,

$$\mathbf{IN}(t) \quad = \quad \begin{cases} S_0 - \mathbf{N}_\Lambda(t) & \text{if} \quad 0 \leq t < L \\ \mathbf{IP}(t-L) - (\mathbf{N}_\Lambda(t) - \mathbf{N}_\Lambda(t-L)) & \text{if} \quad t \geq L \end{cases} \tag{3.5}$$

Since we let $\mathbf{IN}(0) = \mathbf{IP}(0) = S_0$, from time zero to L, the net inventory level only decreases from S_0 as demand arrives. After time L, the net inventory level can be described by the well known equation relating the net inventory level at time t to the inventory position at time $t-L$ and the lead time demand. Moreover, by the independent increments of the nonstationary Poisson process, we know that $\mathbf{IP}(t-L)$ is independent of the lead time demand $\mathbf{N}_\Lambda(t) - \mathbf{N}_\Lambda(t-L)$ for any $t \geq L$.

The expected total discounted cost of operating under the (x, S_0, S_1) policy is calculated with the following function,

$$TC(x, S_0, S_1) := E\left(\int_0^\infty e^{-\alpha t} c(\mathbf{IN}(t))dt\right) \tag{3.6}$$

where $\alpha > 0$ is the fixed discount rate and $c : \mathbb{R} \to \mathbb{R}$ is the convex cost rate function defined as

$$c(y) := \begin{cases} hy & \text{if} \quad y > 0 \\ -\pi y & \text{if} \quad y \leq 0 \end{cases}$$

Then, the optimization problem can be stated as

$$\begin{aligned} \min \quad & TC(x, S_0, S_1) \\ \text{s.t.} \quad & 0 \leq x \leq T, \\ & 0 \leq S_1 \leq S_0, \\ & S_0, S_1 \in \mathbb{Z}_+. \end{aligned} \tag{3.7}$$

Let $\mathbf{Y}$ be an exponentially distributed random variable with parameter $\alpha > 0$ and let $f(y; \alpha)$ denote the probability density function of $\mathbf{Y}$. Observe that $\mathbf{Y}$ is independent of the process $\mathbf{IN}$ and we have

$$\begin{aligned} E\left(c(\mathbf{IN}(\mathbf{Y}))\right) &= \int_0^\infty E\left(c(\mathbf{IN}(\mathbf{Y}))|\mathbf{Y} = t\right) f(t; \alpha) dt \\ &= \int_0^\infty E\left(c(\mathbf{IN}(t))\right) \alpha\, e^{-\alpha t}\, dt \\ &= \alpha E\left(\int_0^\infty e^{-\alpha t} c(\mathbf{IN}(t))dt\right) \end{aligned} \tag{3.8}$$

where the last step follows from the Fubini's theorem.

Thus, (3.6) can be written in an alternative way as follows

$$TC(x, S_0, S_1) = \alpha^{-1} E\left(c(\mathbf{IN}(\mathbf{Y}))\right) \tag{3.9}$$

Observe that from the definition of $c(y)$ and (3.8) we obtain,

$$E\left(c(\mathbf{IN}(\mathbf{Y}))\right) \;=\; (h+\pi)E\left(\mathbf{IN}(\mathbf{Y})\right)^{+} - \pi E\left(\mathbf{IN}(\mathbf{Y})\right) \tag{3.10}$$

Let $F\left(y;\alpha\right)$ and $\overline{F}\left(y;\alpha\right)$ denote the cumulative distribution function and the tail function of $\mathbf{Y}$. Moreover, let $p(n;\lambda) = e^{-\lambda}\lambda^n/n!,\ n = 0, 1, \ldots$ with parameter $\lambda \in \mathbb{C}$ and $P\left(n;\lambda\right) = \sum_{k=0}^{n} p(k;\lambda)$. Define the functions $g\left(n;\rho\right) = \rho(1-\rho)^n,\ n = 0, 1, 2, \ldots$ with $\rho \in \mathbb{C}$, $b(T, h) = (\lambda_0 - \lambda_1)T + \lambda_1 h$ with $T, h \geq 0$ and $\eta_z = \alpha - (\lambda_0 - \lambda_1)(1 - z)$ with $z \in \mathbb{C}$.

It can easily be shown by (3.5) and $E\left(\mathbf{N}_\Lambda(t)\right) = \Lambda(t)$ that

$$E\left(\mathbf{IN}(\mathbf{Y})\right) = F\left(L;\alpha\right)(S_0 - E\left(\Lambda(\mathbf{Y})\right)) + \overline{F}\left(L;\alpha\right)E\left(\mathbf{IP}(\mathbf{Y})\right) \tag{3.11}$$

where

$$E\left(\Lambda(\mathbf{Y})\right) = \alpha^{-1}\left((\lambda_0 - \lambda_1)F\left(T;\alpha\right) + \lambda_1\right) \tag{3.12}$$

and

$$E\left(\mathbf{IP}(\mathbf{Y})\right) = S_0 - \overline{F}\left(x;\alpha\right)\left(\sum_{k=0}^{N-1} k(\mu_{k+1} - \mu_k) + N(1 - \mu_N)\right) \tag{3.13}$$

with $\mu_0 = 0$ and for $k = 1, 2, \ldots, N$

$$\begin{aligned} \mu_k \;&=\; \sum_{n=0}^{k-1}\sum_{\ell=0}^{n} p(n-\ell;(\lambda_0-\lambda_1)(T-x))\,g\left(\ell;\tfrac{\alpha}{\lambda_1+\alpha}\right)P\left(\ell;(\lambda_1+\alpha)(T-x)\right) \\ &+\; \sum_{n=0}^{k-1} g\left(n;\tfrac{\alpha}{\lambda_0+\alpha}\right)[1 - P\left(n;(\lambda_0+\alpha)(T-x)\right)] \end{aligned} \tag{3.14}$$

The proofs of (3.12) and (3.13) can be found in the Appendix 3.7.

Observe from (3.4) that when $S_0 = S_1$, $\mathbf{IP}(t)$ is equal to S_0 for all $t \geq 0$ since $\sigma_\Lambda^x(0) = 0$. Under fixed inventory position, the analysis of the $E\left(\mathbf{IN}(\mathbf{Y})\right)^{+}$ becomes relatively simple as we will show in Section 3.3.1. However, when $S_0 > S_1$, a closed form expression of $E\left(\mathbf{IN}(\mathbf{Y})\right)^{+}$ is not available. The difficulty arises from the random hitting time $\sigma_\Lambda^x(N)$ which appears in (3.4). By definition, $\sigma_\Lambda^x(N)$ depends on the nonhomogenous Poisson process after time x. Therefore, the computation of $E\left(\mathbf{IN}(\mathbf{Y})\right)^{+}$ requires the calculation of integrals where the dependent random variables $\sigma_{\Lambda}^{x}(N)$ and $\mathbf{N}_{\Lambda}(t)$ for $t \geq x$ are entangled through the maximum function. Unfortunately, analytical computation of such integrals increases the mathematical tedium drastically. To overcome this difficulty, we rely on numerical inversion of probability generating functions as explained below.

Computation of $E\left(\mathbf{IN}(\mathbf{Y})\right)^{+}$ for $S_1 < S_0$

Introduce the integer valued nonnegative random variable

$$\mathbf{D} := S_0 - \mathbf{IN}(\mathbf{Y}) \tag{3.15}$$

representing the deviation from the maximum value of the **IN** process. By relation (3.15) it follows that

$$E\left(\mathbf{IN}(\mathbf{Y})\right)^{+} \quad = \quad E\left(S_0 - \mathbf{D}\right)^{+} = \sum_{k=0}^{S_0-1}(S_0 - k)p_k \tag{3.16}$$

where

$$p_k = P\left(\mathbf{D} = k\right), \quad k = 0, 1, ...$$

Since direct computation of p_k is difficult due to the reasons discussed above, we derive $E\left(z^{\mathbf{D}}\right)$, the probability generating function of $\mathbf{D}$, analytically and use a numerical inversion technique, e.g., Fast Fourier Transform, to recover the original p_k sequence. We refer the reader to Abate and Whitt (1992) for a detailed discussion on recovering the sequence p_k by using the probability generating function.

By relation (3.15), for every $z \in \mathbb{C}$ satisfying $|z| \leq 1$, we have

$$E\left(z^{\mathbf{D}}\right) = z^{S_0} E\left(z^{-\mathbf{IN}(\mathbf{Y})}\right) \tag{3.17}$$

and by (3.5), we obtain that

$$E\left(z^{-\mathbf{IN}(\mathbf{Y})}\right) = \alpha(z^{-S_0}V(\alpha) + e^{-\alpha L}W(\alpha)) \tag{3.18}$$

with

$$V(\alpha) := \int_0^L e^{-\alpha t} E\left(z^{\mathbf{N}_{\Lambda}(t)}\right) dt \tag{3.19}$$

and

$$W(\alpha) := \int_0^\infty e^{-\alpha t} E\left(z^{-\mathbf{IP}(t) + \mathbf{N}_{\Lambda}(t+L) - \mathbf{N}_{\Lambda}(t)}\right) dt \tag{3.20}$$

Observe by relation (3.1) that the time transformation of $\mathbf{N}_\Lambda$ can be written explicitly as,

$$\Lambda(t) = \begin{cases} \lambda_0 t & \text{if} \quad 0 \le t \le T \\ (\lambda_0 - \lambda_1)T + \lambda_1 t & \text{if} \quad t > T \end{cases} \tag{3.21}$$

Since $\mathbf{N}_\Lambda(t)$ is Poisson distributed with parameter $\Lambda(t)$, using (3.21) in (3.19) yields,

$$V(\alpha) = \phi_0^{-1}\left(1 - e^{-\phi_0 \min(T,L)}\right) + \phi_1^{-1} e^{-(\lambda_0-\lambda_1)T(1-z)}\left(e^{-\phi_1 \min(T,L)} - e^{-\phi_1 L}\right) \tag{3.22}$$

where $\phi_i := \alpha + \lambda_i(1-z)$ for $i \in \{0,1\}$.

To analyze $W(\alpha)$, first observe that $\mathbf{IP}(t)$ and $\mathbf{N}_\Lambda(t+L) - \mathbf{N}_\Lambda(t)$ are independent. Second, that $\mathbf{N}_\Lambda(t+L) - \mathbf{N}_\Lambda(t)$ is Poisson distributed with parameter $\Lambda(t+L) - \Lambda(t)$. Thus, we can rewrite (3.20) as

$$W(\alpha) = \int_0^\infty e^{-(\alpha t + (\Lambda(t+L) - \Lambda(t))(1-z))} E\left(z^{-\mathbf{IP}(t)}\right) dt \tag{3.23}$$

Using the definitions of $\mathbf{IP}(t)$ and $\Lambda(t)$, (3.23) can be divided into separate integrals for which the analytical expressions can be derived. Thus, the explicit form of $W(\alpha)$ can be given as follows,

$$W(\alpha) = W_1(\alpha) + W_2(\alpha) + W_3(\alpha) \tag{3.24}$$

where

$$\begin{aligned} W_1(\alpha) &= z^{-S_0}\Big[\alpha^{-1} e^{-\lambda_0 L(1-z)} F\left([\min(T-L,x)]^+;\alpha\right) \\ &+ \eta_z^{-1} e^{-b(T,L)(1-z)}\left(e^{-\eta_z[\min(T-L,x)]^+} - e^{-\eta_z x}\right)\Big] \end{aligned} \tag{3.25}$$

$$W_2(\alpha) = z^{-S_0} e^{-\alpha x} \sum_{k=0}^{N-1} z^k (\underline{\xi}_{k+1} - \underline{\xi}_k) \tag{3.26}$$

with

$$
\begin{aligned}
\underline{\xi}_k &= f_1 + f_2 + f_3 \\
f_1 &:= e^{-b(T-x,L)(1-z)} \begin{cases} \eta_z^{-1} \sum_{n=0}^{k-1} g\left(n; \frac{\eta_z}{\lambda_0+\eta_z}\right) \Delta P_n & \text{if } \lambda_0 \neq -\eta_z \\ \lambda_0^{-1} \sum_{n=0}^{k-1} \Delta p_n & \text{if } \lambda_0 = -\eta_z \end{cases} \\
\Delta P_n &:= P\left(n; (\lambda_0+\eta_z)(T-x-L)^+\right) - P\left(n; (\lambda_0+\eta_z)(T-x)\right) \\
\Delta p_n &:= e^{\lambda_0(T-x)} p(n+1; \lambda_0(T-x)) - e^{\lambda_0(T-x-L)^+} p(n+1; \lambda_0(T-x-L)^+)
\end{aligned}
$$

$$
\begin{aligned}
f_2 &:= \alpha^{-1} e^{-\lambda_1 L(1-z)} \sum_{n=0}^{k-1} \sum_{\ell=0}^{n} p(n-\ell; (\lambda_0-\lambda_1)(T-x)) g\left(\ell; \frac{\alpha}{\lambda_1+\alpha}\right) P\left(\ell; (\lambda_1+\alpha)(T-x)\right) \\
f_3 &:= \alpha^{-1} e^{-\lambda_0 L(1-z)} \sum_{n=0}^{k-1} g\left(n; \frac{\alpha}{\lambda_0+\alpha}\right) \left[1 - P\left(n; (\lambda_0+\alpha)(T-x-L)^+\right)\right]
\end{aligned}
$$

and

$$
W_3(\alpha) = z^{-S_1} e^{-\alpha x} \overline{\xi}_N \tag{3.27}
$$

with

$$
\begin{aligned}
\overline{\xi}_N &= g_1 + g_2 + g_3 - \underline{\xi}_N \\
g_1 &:= \alpha^{-1} e^{-\lambda_1 L(1-z)} \overline{F}(T-x; \alpha) \\
g_2 &:= \eta_z^{-1} e^{-b(T-x,L)(1-z)} \left(e^{-\eta_z(T-x-L)^+} - e^{-\eta_z(T-x)}\right) \\
g_3 &:= \alpha^{-1} e^{-\lambda_0 L(1-z)} F\left((T-x-L)^+; \alpha\right)
\end{aligned}
$$

The proofs of equations (3.24)–(3.27) can be found in the Appendix 3.7.

Thus, p_k sequence can be recovered by a standard inversion algorithm where the probability generating function of $\mathbf{D}$ is calculated by using equations (3.18), (3.22) and (3.24). Although equations (3.25)–(3.27) appear to be somewhat tedious, they can be computed easily since they are composed of well-behaved elementary functions.

In our numerical experiments, we used the inversion algorithm proposed by Abate and Whitt (1992) and found that $TC(x, S_0, S_1)$ can be computed accurately within a couple of seconds for most of the parameter sets considered. Computation time is increasing in S_0 yet it remains feasible for optimization purposes since base stock levels are never too high for expensive, slow moving items.

3.3.2 Analysis of the Special Case: $S_0 = S_1$

When $S_0 = S_1$, no policy change takes place and the (x, S_0, S_1) policy boils down to one-for-one policy with fixed base stock level. We refer to this special case of the (x, S_0, S_1) policy shortly as the *fixed policy*. Note that with the fixed policy, the obsolescence is still taken into account via the nonstationary demand process $\mathbf{N}_\Lambda$ although the base stock level is not changed. In the sequel, we suppress the subscript in S_0 and denote base stock level of the fixed policy with S, for brevity.

When the inventory position process is equal to S for all $t \geq 0$, the net inventory level process given by relation (3.5) becomes,

$$\mathbf{IN}(t) \quad = \quad \begin{cases} S - \mathbf{N}_\Lambda(t) & \text{if} \quad 0 \leq t < L \\ S - (\mathbf{N}_\Lambda(t) - \mathbf{N}_\Lambda(t-L)) & \text{if} \quad t \geq L \end{cases} \tag{3.28}$$

From (3.13), we have $E(\mathbf{IP}(\mathbf{Y})) = S$ since $N = 0$. Thus, (3.11) becomes,

$$E(\mathbf{IN}(\mathbf{Y})) = S - F(L; \alpha) E(\Lambda(\mathbf{Y})) \tag{3.29}$$

When there is no policy change, we can directly calculate the expression $E(\mathbf{IN}(\mathbf{Y}))^+$. Therefore, using relation (3.28) and after some algebra we get

$$E(\mathbf{IN}(\mathbf{Y}))^+ = \sum_{n=0}^{S-1} (S-n) \left[A_1(n) + e^{-\alpha L} A_2(n) \right] \tag{3.30}$$

where

$$\begin{aligned} A_1(n) &= \sum_{\ell=0}^{n} p(n-\ell;(\lambda_0-\lambda_1)T) g\left(\ell;\tfrac{\alpha}{\lambda_1+\alpha}\right) \left[P\left(\ell;(\lambda_1+\alpha)\min(T,L)\right) - P\left(\ell;(\lambda_1+\alpha)L\right)\right] \\ &+ g\left(n;\tfrac{\alpha}{\lambda_0+\alpha}\right)\left[1 - P\left(n;(\lambda_0+\alpha)\min(T,L)\right)\right] \\ A_2(n) &= p(n;\lambda_0 L) F\left((T-L)^+;\alpha\right) + p(n;\lambda_1 L)\overline{F}\left(T;\alpha\right) + f_A(n) \end{aligned}$$

with

$$f_A(n) = \begin{cases} \sum_{\ell=0}^{n} p(n-\ell; b(T,L))\; g\left(\ell;\alpha/\eta_0\right)\; \left[P\left(\ell;\eta_0(T-L)^+\right) - P\left(\ell;\eta_0 T\right)\right] & \text{if } \eta_0 \neq 0 \\ \sum_{\ell=0}^{n} p(n-\ell; b(T,L))\; \frac{(-\alpha)^{\ell+1}}{(\ell+1)!}\; \left[((T-L)^+)^{\ell+1} - T^{\ell+1}\right] & \text{if } \eta_0 = 0 \end{cases} \tag{3.31}$$

and $\eta_0 := \alpha - (\lambda_0 - \lambda_1)$.

The proof of equation (3.30) can be found in the Appendix 3.7.

For notational convenience, we delete the unnecessary policy parameters in $TC(x, S_0, S_1)$ and let $TC(S)$ denote the expected total discounted cost of the fixed policy for the base stock level S. Thus, $TC(S)$ can be calculated from (3.9) by using (3.29) and (3.30).

Observe that for $S_0 = S_1$, the optimization problem given by (3.7) simplifies to

$$\begin{aligned} \min \quad & TC(S) \\ \text{s.t.} \quad & S \geq 0 \ \text{ and } \ S \in \mathbb{Z}_+. \end{aligned} \tag{3.32}$$

It can be easily shown that for any random variable $\mathbf{K}$ taking values in $\mathbb{R}$, $E(c(S-\mathbf{K}))$ is convex in S. Thus, from (3.28) it follows that $E(c(\mathbf{IN}(t)))$ is convex in S for all $t \geq 0$. Since $e^{-\alpha t}$ is positive for all $t \geq 0$ and nonnegative weighted sum of convex functions is also convex, it follows from (3.6) that $TC(S)$ is convex in S. Therefore, the optimal base stock level of the fixed policy S_f^* is the minimum integer S (≥ 0) satisfying $TC(S+1) - TC(S) \geq 0$. That is the smallest S satisfying

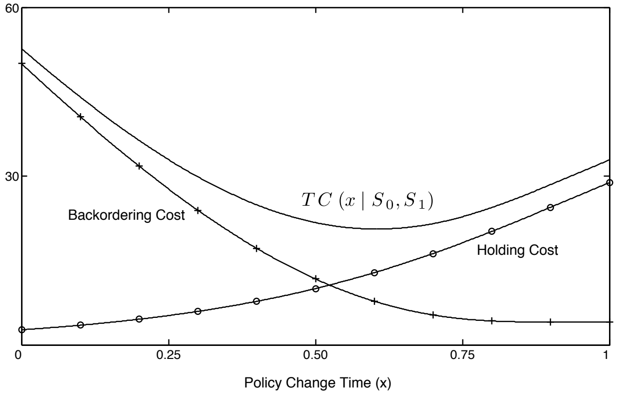
$$\sum_{n=0}^{S} \left[A_1(n) + e^{-\alpha L} A_2(n)\right] \geq \frac{\pi}{\pi + h} \tag{3.33}$$

Therefore, the optimal solution for the original problem (3.7) can be obtained by solving it only for $S_1 < S_0$ and then choosing the optimal policy, i.e., (x^*, S_0^*, S_1^*) or S_f^*, yielding the minimum total cost. In the next section, we will explain this procedure in detail.

3.4. Optimization of $TC(x, S_0, S_1)$

In our numerical experiments, we observed that $TC(x, S_0, S_1)$ is quasi-convex in x. The intuition behind this observation can be explained as follows: For a given S_0 and S_1 such that $S_0 > S_1$, if the base stock policy is changed too late, i.e., if x is near T, then there is not enough time for the demand to take away all of the removal quantity before the drop occurs. Thus, the remaining stocks after T increase the holding costs due to obsolescence. On the other hand, if the policy is changed too early then the system might adapt to the lower base stock level too soon before the obsolescence occurs. Consequently, backordering costs increase due to the suboptimal base stock level used from the time the adaptation is completed until the drop in demand rate occurs. Therefore, there should be an optimal x value that balances the holding costs due to obsolescence with the backordering costs. Figure 3.2 illustrates the behavior of the total discounted cost function in x.

Figure 3.2 Behavior of $TC(x|S_0, S_1)$



Note. $\lambda_0 = 5,\ \rho = 1,\ T = 1,\ \pi = 100,\ L = 0.25,\ \alpha = 0.1,\ S_0 = 3,\ S_1 = 0$

Since optimal S_0 is never too high for slow moving, expensive items, once a reasonably high upper bound $\overline{S}$ is established for the optimal S_0, the minimizer of $TC(x, S_0, S_1)$

can be found by enumerating over the base stock levels and optimizing over the policy change time. More precisely, if we let $TC(x|S_0, S_1)$ denote the total discounted cost as a function of x given (S_0, S_1) then $TC(x|S_0, S_1)$ is minimized over the continuous variable $x \in [0, T]$ for every (S_0, S_1) in $\mathcal{S} = \{(S_0, S_1) : 0 \le S_1 < S_0 \le \overline{S};\ S_0, S_1 \in \mathbb{Z}_+\}$. If we let x^* denote the optimal x given $(S_0^*, S_1^*) \in \mathcal{S}$ and $TC(x^*|S_0^*, S_1^*)$ denote the smallest of the total discounted cost functions minimized over x then the tuple (x^*, S_0^*, S_1^*) is the optimal solution to (3.7) given that $S_1 < S_0$. For $S_0 = S_1$, the optimal solution is S_f^* and it can be calculated by (3.33). Therefore, if $TC(x^*, S_0^*, S_1^*)$ is smaller than $TC(S_f^*)$ then the optimal solution to (3.7) is the three parameter policy with (x^*, S_0^*, S_1^*). Otherwise, the optimal solution to (3.7) is the fixed policy with base stock level S_f^*. We denote the minimum total cost by $TC^* := \min\left\{TC(x^*, S_0^*, S_1^*), TC(S_f^*)\right\}$.

For a given (S_0, S_1), the optimal x can be searched very efficiently with standard nonlinear optimization methods. Moreover, optimal base stock level of a slow moving item being very large is rather unlikely in practice. Thus, we conclude that the computational demands of the optimization procedure described above is reasonable for most of the realistic problems.

Next, we derive the optimal base stock level of the one-for-one policy when there is discounting but no obsolescence. In our numerical experiments, we use the optimal base stock level of the no obsolescence scenario as an heuristic upper bound on optimal S_0.

3.4.1 Heuristic upper bound for S_0^*

Let $TC_\infty(S)$ denote the expected total discounted cost of the one-for-one policy when there is no obsolescence, i.e., demand rate is constant at λ_0. By definition,

$$TC_\infty(S) = \lim_{T\to\infty} TC(S) \tag{3.34}$$

Thus, letting $T \to \infty$ in equation (3.29) yields

$$\lim_{T\to\infty} E\left(\mathbf{IN}(\mathbf{Y})\right) = S - \alpha^{-1}\lambda_0 F\left(L; \alpha\right) \tag{3.35}$$

Similarly, as T goes to infinity (3.31) converges to zero and the equation (3.30) becomes,

$$\lim_{T\to\infty} E\left(\mathbf{IN}(\mathbf{Y})\right)^+ = \sum_{n=0}^{S-1}(S-n)\left[g\left(n; \tfrac{\alpha}{\lambda_0+\alpha}\right)\left[1 - P\left(n; (\lambda_0+\alpha)L\right)\right] + e^{-\alpha L}p(n; \lambda_0 L)\right] \tag{3.36}$$

Therefore,

$$TC_\infty(S) = \alpha^{-1}(h+\pi)\sum_{n=0}^{S-1}(S-n)H(n) - \alpha^{-1}\pi(S-\alpha^{-1}\lambda_0 F(L;\alpha)) \tag{3.37}$$

with

$$H(n) = g\left(n; \tfrac{\alpha}{\lambda_0+\alpha}\right)[1-P(n;(\lambda_0+\alpha)L)] + e^{-\alpha L}p(n;\lambda_0 L)$$

As expected, as α goes to zero $\alpha TC_\infty(S)$ converges to the well known average-cost function of the one-for-one policy (Hadley and Whitin, 1963).

Since $TC(S)$ is a convex function of S, $TC_\infty(S)$ is also convex by (3.34). Thus, the optimal solution for (3.37), denoted by S^*_∞, is the minimum integer S (≥ 0) satisfying $TC_\infty(S+1) - TC_\infty(S) \geq 0$. That is the smallest S satisfying,

$$\sum_{n=0}^{S} H(n) \geq \frac{\pi}{\pi+h} \tag{3.38}$$

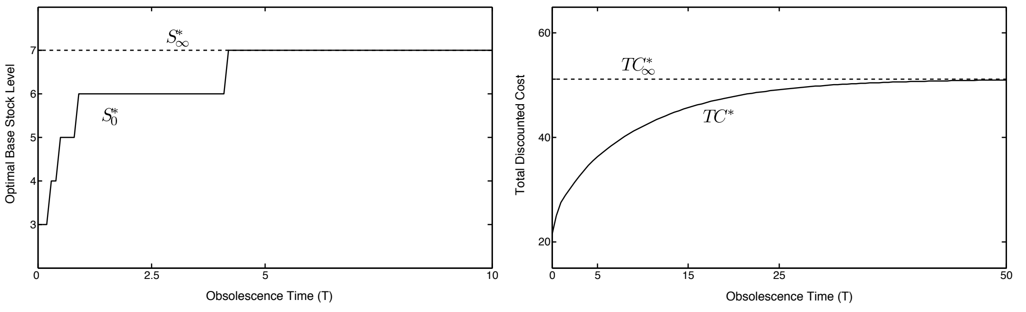
It makes intuitive sense that when there is obsolescence risk, the inventory system would operate under a lower optimal base stock level than it would when there is no obsolescence in order to hedge against obsolete stocks. Therefore, S^*_0 should not be greater than S^*_∞. Song and Zipkin (1996) gave a similar upper bound on the optimal base stock level of the world-dependent base stock policy when the obsolescence time is exponentially distributed and the optimal base stock level is changed immediately after the obsolescence occurs. Unfortunately, we were not able to prove our conjecture about the upper bound due to analytical intractability. However, our numerical experiments indicate that as T goes to infinity, S^*_0 increases to S^*_∞ and TC^* converges to TC^*_∞ (see Figure 3.3). Thus, in our numerical studies, we use S^*_∞ (≥ 0) as an heuristic upper bound on S^*_0.

3.5. Numerical Study

The structure of our numerical study can be summarized as follows: First, we analyze the sensitivity of optimal parameters and optimal total cost of the (x, S_0, S_1) policy due to the changes in system parameters. Second, we investigate the value of the policy change option of the (x, S_0, S_1) policy by comparing the performance of the best (x, S_0, S_1) policy

Figure 3.3 Convergence of S_0^* and TC^* in T

(a) Convergence of S_0^* to S_∞^* (b) Convergence of TC^* to TC_∞^*



Note. The following parameter set is used in both figures: $\lambda_0 = 5,\ \rho = 0.9,\ \pi = 100,\ L = 0.5,\ \alpha = 0.1$

and its special case, the fixed policy. Third, we assess the cost of ignoring obsolescence by comparing the performance of the best (x, S_0, S_1) policy and the best one-for-one policy without obsolescence. Fourth, we identify the value of advance policy change option by comparing the performance of the best (x, S_0, S_1) policy when policy change is optimal (i.e., $S_0^* \neq S_1^*$) and the best (T, S_0, S_1).

The instances used in our numerical experiments are generated from the following input parameter set: $\lambda_0 \in \{0.5, 1, 5, 10\}$ units per year and $\lambda_1 = (1-\rho)\lambda_0$ where $\rho \in \{0.50, 0.75, 0.90, 1\}$ is the percentage drop in λ_0. $T \in \{0.10, 0.50, 1, 2.5, 5\}$ years, $L \in \{0.05, 0.15, 0.25, 0.50\}$ years. Holding cost of an item is normalized to 1 per year and $\pi = \{10, 50, 100, 500\}$ per unit per year. $\alpha = \{0.05, 0.10\}$ per year. In total, we generated 2560 instances from all possible combinations of this parameter set.

In the sequel, we use the notation $\overline{X}$ and $\widehat{X}$ to denote the mean and maximum of X, respectively. The summary of the results of our numerical study is tabulated in Table 3.1. The meaning of Δ symbols in Table 3.1 will be explained throughout the Sections 3.5.2 – 3.5.4.

Table 3.1: Summary of Numerical Experiments

		$\overline{x^*}$	$\overline{S_0^*}$	$\overline{S_1^*}$	$\overline{N^*}$	$\overline{TC^*}$	$\overline{S_f^*}$	$\overline{\Delta}\%$	$\widehat{\Delta}\%$	$\overline{S_\infty^*}$	$\overline{\Delta_o}\%$	$\overline{\Delta_a}\%$
T	0.1	0.02	1.39	1.13	0.25	19.6	1.21	2.5	56.6	3.16	585.7	12.8
	0.5	0.23	2.22	1.14	1.08	22.3	1.54	14.3	127.3	3.16	167.1	29.3
	1	0.61	2.57	1.15	1.42	23.7	1.79	22.9	177.5	3.16	113.1	31.4
	2.5	1.88	2.88	1.17	1.71	25.8	2.20	31.6	260.6	3.16	72.6	28.4
	5	4.16	3.03	1.18	1.85	28.1	2.50	31.0	234.2	3.16	49.9	24.0
λ_0	0.5	1.85	0.74	0.47	0.27	12.4	0.64	3.6	51.5	1.25	250.3	13.2
	1	1.90	1.10	0.68	0.42	15.9	0.93	7.8	96.9	1.63	193.9	19.0
	5	1.80	3.03	1.43	1.60	29.0	2.24	27.2	166.1	3.88	177.7	27.1
	10	1.67	4.81	2.05	2.76	38.3	3.59	43.3	260.6	5.88	168.8	34.3
ρ	0.50	1.97	2.80	2.13	0.68	32.0	2.34	3.4	29.9	3.16	21.3	0.7
	0.75	1.84	2.59	1.44	1.15	25.9	1.94	12.8	77.4	3.16	56.5	2.7
	0.9	1.74	2.41	0.98	1.43	21.0	1.71	25.3	137.1	3.16	111.6	9.4
	1	1.63	1.88	0.09	1.79	16.7	1.41	40.3	260.6	3.16	601.3	80.9
π	10	1.83	1.46	0.60	0.86	12.6	0.84	19.0	260.6	2.06	343.4	53.7
	50	1.82	2.27	1.08	1.20	21.8	1.69	22.1	177.5	3.00	230.4	28.3
	100	1.76	2.63	1.22	1.41	25.9	2.03	21.8	155.2	3.44	146.0	20.8
	500	1.70	3.32	1.73	1.59	35.4	2.83	19.0	123.7	4.13	70.9	12.9
L	0.05	1.91	1.11	0.58	0.53	13.5	0.82	14.2	177.5	1.44	224.7	41.7
	0.15	1.80	1.98	0.94	1.04	20.9	1.46	18.8	228.9	2.50	180.7	23.7
	0.25	1.76	2.65	1.28	1.37	26.0	1.99	21.7	243.9	3.44	213.4	22.6
	0.5	1.66	3.94	1.83	2.11	35.2	3.13	27.2	260.6	5.25	172.0	23.9
Overall		1.77	2.42	1.16	1.26	23.9	1.85	20.5	260.6	3.16	197.7	26.8

3.5.1 Sensitivity Analysis

As T increases, the contribution of obsolete stocks to the total costs starts to decrease compared to the contribution of backorders, and therefore, x^* and S_0^* both increase to balance backordering costs. In general, S_1^* is not significantly affected by the increase in T since it is a policy parameter mainly related with the post-obsolescence period. Furthermore, TC^* is increasing in T due to higher S_0^*.

As the drop in demand rate ρ increases, S_1^* decreases due to lower λ_1. Although slower than S_1^*, we observed that S_0^* is also decreasing in ρ to balance the removal quantity. Consequently, as ρ increases, N^* either increases or stays the same. If N^* increases with ρ, x^* decreases since the average time for the natural attrition of N^* stocks before the obsolescence occurs (N^*/λ_0) increases. On the other hand, if N^* remains the same then x^* tends to increase in ρ to balance backorders. Moreover, TC^* is decreasing in ρ due to lower S_0^* and S_1^*. This result indicates that under the (x, S_0, S_1) policy the inventory system is able to carry less stocks without significantly increasing backorders as a result of the policy change option.

As λ_0 increases, S_0^* increases to satisfy demand. S_1^* is slowly increasing in λ_0 to hedge against backorders in case the adaptation to S_1^* is completed before T. The behavior of x^* is nonmonotic in λ_0. It can decrease or increase depending on the changes in N^* and N^*/λ_0. Moreover, TC^* is increasing in λ_0 as a result of higher base stock levels.

For a given λ_0, if the optimal removal quantity N^* increases with L then x^* decreases to initiate the stock removal process earlier to prevent high holding costs due to obsolescence. If N^* does not increase with L then x^* increases to balance backorders by postponing the policy change. Similarly, as π increases, the system adjusts itself by increasing x^* until the holding costs become significant again. Moreover, as π or L increases, S_0^* and S_1^* both increase to balance the backordering costs, and TC^* increases due to higher base stock levels.

We find that TC^* is decreasing in α. For some instances, x^* is slightly increasing in α since a higher discount rate mainly reduces the effect of obsolescence related costs. In general, S_0^* and S_1^* remain unaffected as α increases.

3.5.2 Value of Policy Change Option

In this section, our aim is to investigate the value of policy change option of the (x, S_0, S_1) policy and identify the parametric regions where the three parameter policy can be substituted with the simpler fixed policy. Although the fixed policy is the special case of the (x, S_0, S_1) policy, when three parameter policy (i.e., policy change) is optimal, the deviation from the optimal cost by imposing a suboptimal fixed policy is not immediately evident. Since in both policies obsolescence is taken into account in base stock calculations, if the deviation from the optimal cost is not too high then the fixed policy may be preferred over the three parameter policy due to its simplicity. Therefore, by comparing the two obsolescence based policies, we are able to measure the added value of policy change option and distinguish the scenarios where policy change makes most sense.

For comparison purposes, we use the percent difference between the total cost of the best fixed policy $TC(S_f^*)$ and the optimal cost TC^*, $\Delta\% = \frac{TC(S_f^*)-TC^*}{TC^*} \times 100$. A positive $\Delta\%$ value indicates the percent increase in total costs by imposing the fixed policy when three parameter policy is optimal. Whereas, the fixed policy is optimal if $\Delta\%$ is equal to zero.

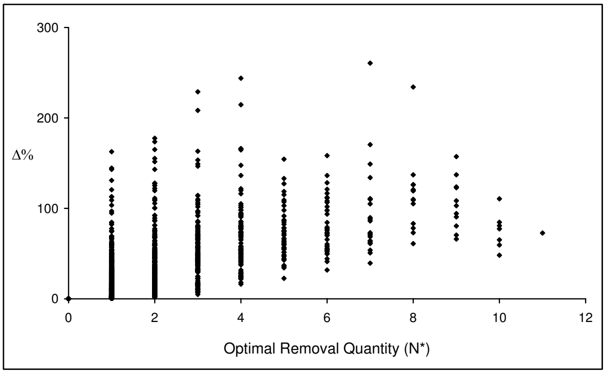
Over all the experiment instances considered, we find that $\Delta\%$ has an average of 20.5% and a maximum of 260.6%. We conclude that the policy change option of the (x, S_0, S_1) policy on average leads to important savings. The savings are largest when the optimal removal quantity N^* is large. The savings are diminishing as obsolescence time T, initial demand rate λ_0 and drop in demand rate ρ decrease. When N^* is low, the suboptimal fixed policy might be used instead of the optimal three parameter policy depending on the system parameters.

As N^* gets larger, the impact of policy change option on costs tends to increase (Figure 3.4). This is because a larger N^* indicates a bigger tradeoff between obsolescence and backordering and it becomes more difficult to balance these costs with a single base stock level. With the option to change the base stock level, the inventory system can operate under a higher base stock level for the pre-obsolescence period to avoid backordering and then reduce it to a lower level to minimize obsolete stocks. Since the fixed policy does not have this flexibility, the performance of the fixed policy decreases in N^*. For example, $\overline{\Delta}\%$ increases from 17.2% to 70% as N^* increases from 1 to 4.

This indicates that the optimal three parameter policy might be substituted with the suboptimal fixed policy only if the optimal removal quantity is low. For the instances with positive $\Delta\%$ (1497 out of 2560 instances), we find that $\Delta\%$ is less than or equal to 5% for 282 instances. Fixed policy substitution might be reasonable for those instances due to low $\Delta\%$. In 261 out of 282 instances, N^* is found to be one. The maximum N^* is found to be three (2 out of 282 instances). However, we shall emphasize that low N^* does not necessarily implies low $\Delta\%$. As can be seen from Figure 3.4, even for low N^* values, the deviation from the optimal costs can be very high depending on the system parameters.

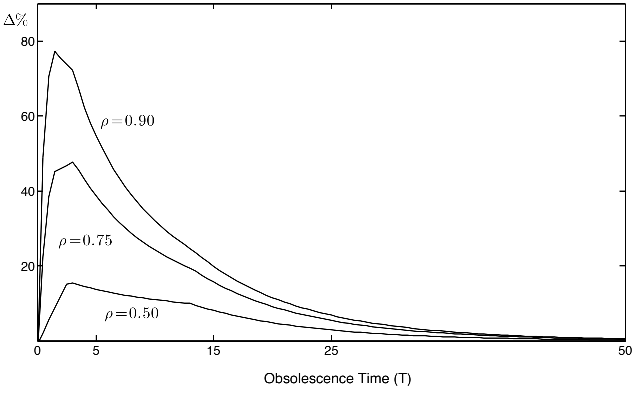
As T increases, $\Delta\%$ first increases and then decreases until it converges to zero. Examples of this behavior for different drop rates can be seen in Figure 3.5. When T is

Figure 3.4 Distribution of Δ% in Optimal Removal Quantity



near zero, the inventory system does not have enough time for a policy change due to the immediate risk of obsolescence. Therefore, for small T values, either the fixed policy is optimal or the improvement by policy change is usually low (e.g., $\overline{\Delta}\% = 2.5\%$ for $T = 0.1$). As T diverges from zero, however, backordering cost increases and the impact of policy change option on costs becomes more pronounced since it delivers a better balance between holding costs and backordering costs. As T approaches to infinity, obsolescence becomes less significant and the total costs of both policies converge to TC^*_∞, and thus, $\Delta\%$ approaches zero.

Figure 3.5 Behavior of Δ% in Obsolescence Time for Different Drop Rates (ρ)



Note. $\lambda_0 = 5,\ \pi = 100,\ L = 0.5,\ \alpha = 0.1$

The average $\Delta\%$ increases as the demand rate and the percentage drop increases due to the increase in $\overline{N}^*$ (see Table 3.2). Especially for high λ_0 and ρ values, the policy change option has pronounced effects on costs as a result of relatively large removal quantities. For instance, in Table 3.2, when $\lambda_0 = 10$ and $\rho = 1$, the optimal removal quantity is 3.84 and $\overline{\Delta}\%$ is equal to 84.95%. On the other hand, for low λ_0 and ρ values, the average contribution of the policy change option is small due to low removal quantities.

Table 3.2: Average Behavior of $\Delta\%$ and N^* in λ_0 and ρ

	$\overline{\Delta}\%$				$\overline{N}^*$			
	ρ				ρ			
λ_0	0.50	0.75	0.90	1	0.50	0.75	0.90	1
0.5	0.58	2.19	4.25	7.34	0.16	0.26	0.27	0.39
1	0.72	4.90	9.97	15.61	0.14	0.40	0.49	0.67
5	4.94	16.72	33.72	53.29	0.93	1.42	1.79	2.26
10	7.45	27.20	53.43	84.95	1.48	2.53	3.19	3.84

As the unit backordering penalty π increases, $\overline{\Delta}\%$ first increases and then decreases depending on the interplay between obsolete stocks and backorders (see Table 3.1). Initially, $\overline{\Delta}\%$ increases in π since it becomes more difficult to balance obsolescence and backordering costs with a single base stock level. Essentially, the three parameter policy reacts to the increase in π by postponing the policy change towards time T to reduce backorders. Since average N^* increases with π, the postponement tends to increase the number of obsolete stocks while trying to balance backorders. Therefore, after π gets large enough, $\overline{\Delta}\%$ starts to decrease due to the relative increase in obsolescence costs as a result of postponement.

$\overline{\Delta}\%$ is increasing in lead time since longer lead times require larger stocks and result in higher removal quantities.

3.5.3 Cost of Ignoring Obsolescence

In this section, our aim is to assess the cost of ignoring obsolescence by using a one-for-one policy which does not take into account the drop in demand rate. We assume that if obsolescence is not taken into account then the system is governed with $(S^*_\infty - 1, S^*_\infty)$ policy. That is the decision maker blindly assumes that the demand rate is constant at level λ_0 and uses the "optimal" base stock level S^*_∞ to satisfy demand where in fact obsolescence occurs at time T. Thus, the expected total discounted cost incurred under this scenario is equal to $TC(S^*_\infty)$ since the system is effectively controlled by the fixed policy with the base stock level S^*_∞.

In order to measure the cost of ignoring obsolescence, we compare $TC(S^*_\infty)$ to the total cost of the best (x, S_0, S_1) policy, i.e., the fixed policy or the three parameter policy. The percentage deviation from the cost of best (x, S_0, S_1) policy is given by $\Delta_o\% = \frac{TC(S^*_\infty)-TC^*}{TC^*} \times 100$. A positive value of $\Delta_o\%$ shows a percentage increase in total costs as a result of ignoring obsolescence in policy choice or, equivalently, the savings obtained by using the obsolescence based (x, S_0, S_1) policy. The average $\Delta_o\%$s with respect to different parameters can be seen from Table 3.1.

We conclude that ignoring obsolescence might lead to substantial cost increases. For all the experiment instances considered, we find that $\Delta_o\%$ has the mean 197.7%. The savings obtained by (x, S_0, S_1) policy are largest when obsolescence occurs at a near future and the drop in demand rate is sharp. As the initial demand rate and unit backordering penalty increases, average savings decrease due to the increase in backordering costs and obsolete stocks.

As T decreases or ρ increases, S^*_0 and S^*_1 decrease to balance obsolete stocks while S^*_∞ stays the same (see Table 3.1). Thus, the holding costs of the $(S^*_\infty - 1, S^*_\infty)$ policy increase as a result of the increase in obsolete stocks and, therefore, $\overline{\Delta}\%$ increases.

The (x, S_0, S_1) policy yields the largest savings when obsolescence is expected at a near future since the increase in the holding costs is most pronounced when discounting period is short (see Table 3.1). On the other hand, the average savings remain significant even when the obsolescence is expected in the mid- or the long-term. For example, in Table 3.1, when $T = 5$, $\overline{\Delta}_o\%$ is 49.9%.

$\overline{\Delta}_o\%$ is increasing in ρ since it takes more time for the demand to take away the obsolete stocks under lower λ_1. The average $\Delta_o\%$ increases nonlinearly in ρ and there is a large jump in its value for $\rho = 1$ (see Table 3.1). For instance, $\overline{\Delta}_o\%$ is 63% when $\rho < 1$ while it is almost ten times bigger when $\rho = 1$. This is because, in our model, the only removal option is the natural attrition of removal quantity by demand. Thus, the holding cost increases sharply if the natural attrition of stocks is not possible after obsolescence occurs.

Moreover, we observe that when a large drop is expected within a relatively short time horizon, the cost of ignoring obsolescence increases steeply due to the compounding effect of sharp drop in demand rate; e.g., when $T = 0.1$ and $\rho = 0.9$, the average $\Delta_o\%$ is equal to 165%, while the same figure drops to 26% if $\rho = 0.5$.

The relative importance of obsolescence decreases as the weight of backordering increases in total costs. Therefore, as π increases, the average savings by taking obsolescence into account tends to diminish. For example, in Table 3.1, $\overline{\Delta}_o\%$ decreases from 343.4% to 70.9% as π increases from 10 to 500.

3.5.4 Value of Advance Policy Change

Next, we investigate the impact of the timing of policy change on total costs and identify the value of changing the control policy before the obsolescence occurs. In order to measure the value of advance policy change, we compute the percentage cost difference between the best (x, S_0, S_1) policy where S_0^* is not equal to S_1^* and the best (T, S_0, S_1) policy. We denote the expected total discounted cost of the best (T, S_0, S_1) policy by TC_T^*. The percent deviation from TC^* is defined as $\Delta_a\% = \frac{TC_T^* - TC^*}{TC^*}$. Note that a positive $\Delta_a\%$ value indicates the percent increase in total cost as a result of postponing the policy change to T. Average $\Delta_a\%$s for various system parameters are reported in Table 3.1.

We conclude that advance policy change leads to significant cost savings. Over all the experiment instances where policy change is optimal (1497 out of 2560 instances), we find that $\Delta_a\%$ has the mean 26.84%. The policy change takes place on average 6 months before the obsolescence occurs and it can be as early as 4.5 years. Savings are most pronounced for the instances with moderate demand rate and full obsolescence. Savings are diminishing as λ_0 decreases, ρ decreases or π increases.

As T diverges from zero, $\overline{N}^*$ increases due to the increase in S_0 and the system starts to carry more stocks. Therefore, early initiation of stock removal becomes necessary and the average $\Delta_a\%$ increases in T. However, as obsolescence time goes to infinity, the impact of obsolescence on costs decreases and the average $\Delta_a\%$ decreases. Hence, $\overline{\Delta}_a\%$ is first increasing and then decreasing in T (see Table 3.1).

$\overline{\Delta}_a\%$ increases as the drop in demand rate gets larger. This is because the natural attrition of removal quantity takes more time under a lower λ_1, and therefore, the obsolescence becomes more costly as the policy change is postponed to time T. $\overline{\Delta}_a\%$ increases sharply for the full obsolescence case since the removal of obsolete stocks is not possible after T.

Postponing the policy change has significant effects on total costs for all λ_0 values. Even for relatively low demand rates, the average $\Delta_a\%$ is found to be high due to the simultaneous effect of large drops (e.g., when $\lambda_0 = 0.5$, $\overline{\Delta}_a\%$ is equal to 13.2%). Moreover, as λ_0 increases, the average optimal removal quantity $\overline{N}^*$ increases and the system becomes more susceptible to sharp drops in demand rate. As a result, advance policy change yields the largest savings when both λ_0 and ρ are high. For instance, when $\lambda_0 = 10$, the average $\Delta_a\%$ is equal to 118% for $\rho = 1$ while the same figure drops to 2.5% for $\rho = 0.75$.

As unit backordering cost π increases, the (x^*, S_0^*, S_1^*) policy tends to postpone the policy change towards the obsolescence time to decrease backordering costs. Thus, the average $\Delta_a\%$ is decreasing in π (see Table 3.1).

3.6. Conclusion

In this chapter, we consider a single location inventory system of a slow moving item with Poisson demand where the demand rate drops to a lower level at a known future time. We assume a one-for-one policy with a fixed lead time, full backordering and two base stock levels. We propose a continuous review control policy to determine the optimal base stock levels and the optimal time to reduce the base stock level, if necessary. That is, if the base stock levels are different, adaptation to lower demand rate is achieved by reducing the base stock level in advance and letting the demand take away the excess stocks. We derive the exact expression for the expected total discounted cost function by partly relying on the Fast Fourier Transform method and obtain the optimal values of the policy parameters by numerical optimization. Moreover, for the special case of identical base stock levels, we derive a critical ratio inequality to calculate the optimal base stock level.

The key insight from our study is that when obsolescence can be foreseen, early adaptation of base stock level can lead to important savings. Our numerical study shows that when obsolescence necessitates a reduction in base stock level, using the proposed three parameter policy decreases the post-obsolescence inventory build-up while balancing the availability. Moreover, we find that ignoring obsolescence in control policy leads to significant cost increases. Important savings over the one-for-one policy without obsolescence indicates that for slow movers, there is a great incentive to incorporate the obsolescence information in stocking decisions.

Our work can be extended in a couple of directions. A model with multiple drops in demand rate and multiple base stock adjustments would be more suitable to capture gradual obsolescence. Incorporating fixed ordering cost into our model would be particularly useful for more general class of products subject to obsolescence.

3.7. Appendix

Before going into the detailed derivations of the equations given in Section 3.3 first observe that if we shift the time axis from x to 0 the shifted process $\mathbf{N}_\Lambda^x := \{\mathbf{N}_\Lambda^x(t) : t \geq 0\}$ with

$$\mathbf{N}_\Lambda^x(t) = \mathbf{N}_\Lambda(t+x) - \mathbf{N}_\Lambda(x) \tag{3.39}$$

is independent of $\{\mathbf{N}_\Lambda(t) : t \leq x\}$ and this shifted process is a nonstationary Poisson process with time transformation $\Lambda_x : [0, \infty) \to \mathbb{R}$ given by

$$\Lambda_x(t) := \Lambda(t+x) - \Lambda(x) \tag{3.40}$$

Therefore, from relation (3.39) we see that

$$\mathbf{N}_\Lambda^x \stackrel{d}{=} \mathbf{N}_{\Lambda_x} \tag{3.41}$$

This implies by relation (3.2) that

$$\sigma_\Lambda^x(N) \stackrel{d}{=} \sigma_{\Lambda_x}(N) \tag{3.42}$$

and $\sigma_\Lambda^x(N)$ is independent of $\{\mathbf{N}_\Lambda(t) : t \leq x\}$.

These observations show that our analysis is independent of the time axis. Thus, in the sequel we use the shifted process when necessary for the sake of clarity.

Lemma 3.1. *Let $f(t) = \beta e^{-\beta t}$, $t \geq 0$ with parameter $\beta \in \mathbb{C}$ and for $\lambda, \gamma \in \mathbb{R}$ define,*

$$I = \int_a^b p(r; \lambda t + \gamma) f(t) dt. \tag{3.43}$$

(i) If $\lambda \neq -\beta$ *then*

$$I = \sum_{k=0}^{r} p(r-k;\gamma) g\left(k; \tfrac{\beta}{\lambda+\beta}\right) \left[P\left(k; (\lambda+\beta)a\right) - P\left(k; (\lambda+\beta)b\right)\right] \tag{3.44}$$

(ii) If $\lambda = -\beta$ *then*

$$I = \sum_{k=0}^{r} p(r-k;\gamma) \frac{\lambda^{k+1}(a^{k+1} - b^{k+1})}{(k+1)!} \tag{3.45}$$

Proof of Lemma 3.1. (i) Observe that (3.43) can be rewritten as follows,

$$I = \frac{e^{-\gamma}\beta}{r!} \int_a^b (\lambda t + \gamma)^r e^{-(\lambda+\beta)t} dt$$

From the binomial theorem and after some algebra we obtain that

$$\begin{aligned} I &= \frac{e^{-\gamma}\beta}{r!} \int_a^b \sum_{k=0}^{r} \binom{r}{k} \gamma^{r-k} (\lambda t)^k e^{-(\lambda+\beta)t} dt \\ &= \sum_{k=0}^{r} \frac{e^{-\gamma}\gamma^{r-k}}{(r-k)!} \beta \left(\frac{\lambda}{\lambda+\beta}\right)^k \int_a^b p(k; (\lambda+\beta)t) dt \end{aligned} \tag{3.46}$$

It can be easily shown that for any $\lambda \neq 0$ and $\gamma \in \mathbb{R}$ the following holds,

$$\int_a^b p(n; \lambda t + \gamma) dt = \frac{1}{\lambda}[P(n; \lambda a + \gamma) - P(n; \lambda b + \gamma)] \tag{3.47}$$

Since $\lambda \neq -\beta$, using the relation (3.47) in (3.46) yields the result as follows,

$$\begin{aligned} I &= \sum_{k=0}^{r} p(r-k;\gamma) \left(\frac{\beta}{\lambda+\beta}\right) \left(\frac{\lambda}{\lambda+\beta}\right)^k [P(k; (\lambda+\beta)a) - P(n; (\lambda+\beta)b)] \\ &= \sum_{k=0}^{r} p(r-k;\gamma) g\left(k; \tfrac{\beta}{\lambda+\beta}\right) [P(k; (\lambda+\beta)a) - P(n; (\lambda+\beta)b)] \end{aligned}$$

(ii) When $\lambda = -\beta$, the integral equation (3.43) can be written as

$$I = \frac{e^{-\gamma}\beta}{r!} \int_a^b (\lambda t + \gamma)^r dt$$

Applying the binomial theorem and after some algebra, the result can be obtained as follows,

$$\begin{aligned} I &= \frac{e^{-\gamma}\beta}{r!} \sum_{k=0}^{r} \binom{r}{k} \gamma^{r-k} \lambda^k \int_a^b t^k dt \\ &= \sum_{k=0}^{r} p(r-k;\gamma) \frac{\lambda^{k+1}(a^{k+1} - b^{k+1})}{(k+1)!} \quad \square \end{aligned}$$

Lemma 3.2. *Let $\{\mathbf{N}_\Lambda(t) : t \geq 0\}$ be a non-homogeneous Poisson process with time transformation*

$$\Lambda(t) \quad = \quad \begin{cases} \lambda_0 t & \text{if} \quad 0 \leq t \leq T \\ (\lambda_0 - \lambda_1)T + \lambda_1 t & \text{if} \quad t > T \end{cases}$$

and introduce the hitting time

$$\sigma_\Lambda(k) = \min\{t \geq 0 : \mathbf{N}_\Lambda(t) \geq k\}, \quad k = 1, 2, ...$$

Define,

$$\underline{\xi}_k(T,h) := E\left(\int_0^{\sigma_\Lambda(k)} e^{-a(t,\,h;\,\Lambda)} dt\right) \quad \text{and} \quad \overline{\xi}_k(T,h) := E\left(\int_{\sigma_\Lambda(k)}^{\infty} e^{-a(t,\,h;\,\Lambda)} dt\right)$$

with

$$a(t,h;\Lambda) := \alpha t + (\Lambda(t+h) - \Lambda(t))(1-z), \quad \forall t, h \geq 0$$

Then,

$$\underline{\xi}_k(T,h) \quad = \quad f_1 + f_2 + f_3$$

where

$$f_1 \quad := \quad e^{-b(T,\,h)(1-z)} \begin{cases} \eta_z^{-1} \sum_{n=0}^{k-1} g\left(n; \frac{\eta_z}{\lambda_0+\eta_z}\right) \Delta P_n & \text{if } \lambda_0 \neq -\eta_z \\ \\ \lambda_0^{-1} \sum_{n=0}^{k-1} \Delta p_n & \text{if } \lambda_0 = -\eta_z \end{cases}$$

$$\Delta P_n \quad := \quad P\left(n; (\lambda_0 + \eta_z)(T-h)^+\right) - P\left(n; (\lambda_0+\eta_z)T\right)$$

$$\Delta p_n \quad := \quad e^{\lambda_0 T} p(n+1; \lambda_0 T) - e^{\lambda_0 (T-h)^+} p(n+1; \lambda_0 (T-h)^+)$$

$$f_2 \quad := \quad \alpha^{-1} e^{-\lambda_1 h(1-z)} \sum_{n=0}^{k-1} \sum_{\ell=0}^{n} p(n-\ell; (\lambda_0-\lambda_1)T) g\left(\ell; \tfrac{\alpha}{\lambda_1+\alpha}\right) P\left(\ell; (\lambda_1+\alpha)T\right)$$

$$f_3 \quad := \quad \alpha^{-1} e^{-\lambda_0 h(1-z)} \sum_{n=0}^{k-1} g\left(n; \tfrac{\alpha}{\lambda_0+\alpha}\right) [1 - P\left(n; (\lambda_0+\alpha)(T-h)^+\right)]$$

and

$$\overline{\xi}_k(T,h) \quad = \quad g_1 + g_2 + g_3 - \underline{\xi}_k(T,h)$$

where

$$g_1 := \alpha^{-1} e^{-\lambda_1 h(1-z)} \overline{F}(T;\alpha)$$

$$g_2 := \eta_z^{-1} e^{-b(T,h)(1-z)} \left(e^{-\eta_z (T-h)^+} - e^{-\eta_z T} \right)$$

$$g_3 := \alpha^{-1} e^{-\lambda_0 h(1-z)} F((T-h)^+;\alpha)$$

Proof of Lemma 3.2. Observe that $\underline{\xi}_k(T,h)$ can be rewritten as follows

$$\underline{\xi}_k(T,h) = E\left(\int_0^\infty e^{-a(t,h;\Lambda)}\ \mathbf{1}_{\{t<\sigma_\Lambda(k)\}}\ dt\right)$$

Using relation

$$\sigma_\Lambda(k) > t \iff \mathbf{N}_\Lambda(t) < k$$

we obtain

$$\begin{aligned} \underline{\xi}_k(T,h) &= E\left(\int_0^\infty e^{-a(t,h;\Lambda)}\ \mathbf{1}_{\{\mathbf{N}_\Lambda(t)<k\}}\ dt\right) \\ &= \int_0^\infty e^{-a(t,h;\Lambda)} P(\mathbf{N}_\Lambda(t) < k) dt \\ &= \sum_{n=0}^{k-1} \int_0^\infty e^{-a(t,h;\Lambda)} p(n;\Lambda(t)) dt \end{aligned} \tag{3.48}$$

Moreover let,

$$\xi(T,h) := \int_0^\infty e^{-a(t,h;\Lambda)} dt \tag{3.49}$$

and observe that $\xi(T,h) = \underline{\xi}_k(T,h) + \overline{\xi}_k(T,h)$ for any $k \in \{1,2,...\}$. Therefore, $\overline{\xi}_k(T,h)$ can be given as

$$\overline{\xi}_k(T,h) = \xi(T,h) - \underline{\xi}_k(T,h) \tag{3.50}$$

(i) If $T < h$ then by the definition of $\Lambda(t)$ we obtain that

$$\Lambda(t+h) - \Lambda(t) = \begin{cases} (\lambda_0 - \lambda_1)T + \lambda_1 h - (\lambda_0 - \lambda_1)t & \text{if} \quad 0 \le t \le T \\ \lambda_1 h & \text{if} \quad t > T \end{cases} \tag{3.51}$$

From the definition of $\Lambda(t)$ and (3.51) the integral in (3.48) can be written as follows

$$\begin{aligned}
\int_0^\infty e^{-a(t,h;\Lambda)} p(n;\Lambda(t))dt &= e^{-((\lambda_0-\lambda_1)T+\lambda_1 h)(1-z)} \int_0^T e^{-(\alpha-(\lambda_0-\lambda_1)(1-z))t} p(n;\lambda_0 t)dt \\
&+ e^{-\lambda_1 h(1-z)} \int_T^\infty e^{-\alpha t} p(n;(\lambda_0-\lambda_1)T+\lambda_1 t)dt \\
&= e^{-b(T,h)(1-z)} \int_0^T e^{-\eta_z t} p(n;\lambda_0 t)dt \\
&+ e^{-\lambda_1 h(1-z)} \int_T^\infty e^{-\alpha t} p(n;b(T,t))dt \qquad (3.52)
\end{aligned}$$

From Lemma 3.1, the first integral in (3.52) can be obtained as
If $\lambda_0 \neq -\eta_z$ then

$$\begin{aligned}
\int_0^T e^{-\eta_z t} p(n;\lambda_0 t)dt &= \eta_z^{-1} \sum_{\ell=0}^{n} p(n-\ell;0) g\left(\ell; \tfrac{\eta_z}{\lambda_0+\eta_z}\right) \left[P(\ell;0) - P(\ell;(\lambda_0+\eta_z)T)\right] \\
&= \eta_z^{-1} g\left(n; \tfrac{\eta_z}{\lambda_0+\eta_z}\right) \left[1 - P(n;(\lambda_0+\eta_z)T)\right] \qquad (3.53)
\end{aligned}$$

If $\lambda_0 = -\eta_z$ then

$$\begin{aligned}
\int_0^T e^{-\eta_z t} p(n;\lambda_0 t)dt &= -\eta_z^{-1} \sum_{\ell=0}^{n} p(n-\ell;0) e^{\lambda_0 T} p(\ell+1;\lambda_0 T) \\
&= \lambda_0^{-1} e^{\lambda_0 T} p(n+1;\lambda_0 T) \qquad (3.54)
\end{aligned}$$

Since $\alpha > 0$, $\lambda_1 \neq -\alpha$ always holds, and from Lemma 3.1, the second integral in (3.52) is found as,

$$\int_T^\infty e^{-\alpha t} p(n;b(T,t))dt = \alpha^{-1} \sum_{\ell=0}^{n} p(n-\ell;(\lambda_0-\lambda_1)T) g\left(\ell; \tfrac{\alpha}{\lambda_1+\alpha}\right) P(\ell;(\lambda_1+\alpha)T) \quad (3.55)$$

Hence, using (3.52), (3.53),(3.54) and (3.55) in (3.48) yields $\underline{\xi}_k(T,h)$ as follows:

$$\begin{aligned}
\underline{\xi}_k(T,h) &= e^{-b(T,h)(1-z)} \begin{cases} \eta_z^{-1} \sum_{n=0}^{k-1} g\left(n; \tfrac{\eta_z}{\lambda_0+\eta_z}\right) \left[1 - P(n;(\lambda_0+\eta_z)T)\right] & \text{if } \lambda_0 \neq -\eta_z \\ \lambda_0^{-1} \sum_{n=0}^{k-1} e^{\lambda_0 T} p(n+1;\lambda_0 T) & \text{if } \lambda_0 = -\eta_z \end{cases} \\
&+ \alpha^{-1} e^{-\lambda_1 h(1-z)} \sum_{n=0}^{k-1} \sum_{\ell=0}^{n} p(n-\ell;(\lambda_0-\lambda_1)T) g\left(\ell; \tfrac{\alpha}{\lambda_1+\alpha}\right) P(\ell;(\lambda_1+\alpha)T) \quad (3.56)
\end{aligned}$$

Similarly, $\xi(T,h)$ can be found by using the definition of $\Lambda(t)$ and (3.51) as follows:

$$\begin{aligned}\xi(T,h) &= e^{-b(T,h)(1-z)}\int_0^T e^{-\eta_z t}dt + e^{-\lambda_1 h(1-z)}\int_T^\infty e^{-\alpha t}dt \\ &= \eta_z^{-1}e^{-b(T,h)(1-z)}(1-e^{-\eta_z T}) + \alpha^{-1}e^{-\lambda_1 h(1-z)}\overline{F}\,(T;\alpha) \qquad (3.57)\end{aligned}$$

Thus, $\overline{\xi}_k(T,h)$ can be easily obtained by substituting (3.56) and (3.57) in (3.50).

(ii) If $T \geq h$ then by the definition of $\Lambda(t)$ we obtain that

$$\Lambda(t+h)-\Lambda(t) = \begin{cases} \lambda_0 h & \text{if} \quad 0 \leq t \leq T-h \\ (\lambda_0-\lambda_1)T + \lambda_1 h - (\lambda_0-\lambda_1)t & \text{if} \quad T-h < t \leq T \\ \lambda_1 h & \text{if} \quad t > T \end{cases} \qquad (3.58)$$

From the definition of $\Lambda(t)$ and (3.58), the integral in (3.48) can be given as follows

$$\begin{aligned}\int_0^\infty e^{-a(t,h;\Lambda)}p(n;\Lambda(t))dt &= e^{-\lambda_0 h(1-z)}\int_0^{T-h} e^{-\alpha t}p(n;\lambda_0 t)dt \\ &+ e^{-b(T,h)(1-z)}\int_{T-h}^T e^{-\eta_z t}p(n;\lambda_0 t)dt \\ &+ e^{-\lambda_1 h(1-z)}\int_T^\infty e^{-\alpha t}p(n;b(T,t))dt \qquad (3.59)\end{aligned}$$

Note that the last integral in (3.59) is already given by equation (3.55). Then, from Lemma 3.1, the first two integrals are found as:

$$\int_0^{T-h} e^{-\alpha t}p(n;\lambda_0 t)dt = \alpha^{-1}g\left(n;\tfrac{\alpha}{\lambda_0+\alpha}\right)\left[1-P\left(n;(\lambda_0+\alpha)(T-h)\right)\right] \qquad (3.60)$$

$$\int_{T-h}^T e^{-\eta_z t}p(n;\lambda_0 t)dt = \begin{cases} \eta_z^{-1}g\left(n;\tfrac{\eta_z}{\lambda_0+\eta_z}\right)\Delta P_n & \text{if } \lambda_0 \neq -\eta_z \\ \lambda_0^{-1}\Delta p_n & \text{if } \lambda_0 = -\eta_z \end{cases} \qquad (3.61)$$

Using (3.55), (3.59), (3.60) and (3.61) in (3.48) yields $\underline{\xi}_k(T,h)$ as follows:

$$\begin{aligned}
\underline{\xi}_k(T,h) &= \alpha^{-1}e^{-\lambda_0 h(1-z)}\sum_{n=0}^{k-1} g\left(n;\tfrac{\alpha}{\lambda_0+\alpha}\right)\left[1-P\left(n;(\lambda_0+\alpha)(T-h)\right)\right] \\
&+ e^{-b(T,h)(1-z)}\begin{cases} \eta_z^{-1}\sum_{n=0}^{k-1} g\left(n;\tfrac{\eta_z}{\lambda_0+\eta_z}\right)\Delta P_n & \text{if } \lambda_0 \neq -\eta_z \\ \lambda_0^{-1}\sum_{n=0}^{k-1}\Delta p_n & \text{if } \lambda_0 = -\eta_z \end{cases} \\
&+ \alpha^{-1}e^{-\lambda_1 h(1-z)}\sum_{n=0}^{k-1}\sum_{\ell=0}^{n} p(n-\ell;(\lambda_0-\lambda_1)T)g\left(\ell;\tfrac{\alpha}{\lambda_1+\alpha}\right)P\left(\ell;(\lambda_1+\alpha)T\right) \quad (3.62)
\end{aligned}$$

$\xi(T,h)$ can be found by using the definition of $\Lambda(t)$ and (3.58) as follows:

$$\begin{aligned}
\xi(T,h) &= e^{-\lambda_0 h(1-z)}\int_0^{T-h} e^{-\alpha t}dt + e^{-b(T,h)(1-z)}\int_{T-h}^{T} e^{-\eta_z t}dt + e^{-\lambda_1 h(1-z)}\int_T^{\infty} e^{-\alpha t}dt \\
&= \alpha^{-1}e^{-\lambda_0 h(1-z)}F\left(T-h;\alpha\right) + \eta_z^{-1}e^{-b(T,h)(1-z)}\left(e^{-\eta_z(T-h)} - e^{-\eta_z T}\right) \\
&+ \alpha^{-1}e^{-\lambda_1 h(1-z)}\overline{F}\left(T;\alpha\right) \quad (3.63)
\end{aligned}$$

Thus, $\overline{\xi}_k(T,h)$ can be easily obtained by substituting (3.62) and (3.63) in (3.50). □

Proof of Equation (3.12). By conditioning on $\mathbf{Y}$ and from the definition of $\Lambda(t)$, we obtain that

$$\begin{aligned}
E\left(\Lambda(\mathbf{Y})\right) &= \int_0^{\infty} \alpha e^{-\alpha t}\Lambda(t)dt \\
&= \int_0^{T} \alpha e^{-\alpha t}\lambda_0 t dt + \int_T^{\infty} \alpha e^{-\alpha t}((\lambda_0-\lambda_1)T+\lambda_1 t)dt \\
&= (\lambda_0-\lambda_1)\left(\int_0^T t\alpha e^{-\alpha t}dt + T\overline{F}\left(T;\alpha\right)\right) + \alpha^{-1}\lambda_1 \quad (3.64)
\end{aligned}$$

Observe that,

$$\int_0^T t\alpha e^{-\alpha t}dt = \alpha^{-1}F\left(T;\alpha\right) - T\overline{F}\left(T;\alpha\right) \quad (3.65)$$

Hence, using (3.65) in (3.64) yields the result. □

Proof of Equation (3.13). By conditioning on $\mathbf{Y}$ and from (3.4), we obtain that

$$E\left(\mathbf{IP}(\mathbf{Y})\right) = E\left(\int_0^\infty \alpha e^{-\alpha t}\mathbf{IP}(t)dt\right) = U_1(\alpha) + U_2(\alpha) + U_3(\alpha) \tag{3.66}$$

with

$$U_1(\alpha) := S_0 \int_0^x \alpha e^{-\alpha t} dt = S_0(1 - e^{-\alpha x}) \tag{3.67}$$

$$U_2(\alpha) := E\left(\int_x^{x+\sigma_\Lambda^x(N)} \alpha e^{-\alpha t}(S_0 - (\mathbf{N}_\Lambda(t) - \mathbf{N}_\Lambda(x)))dt\right) \tag{3.68}$$

$$U_3(\alpha) := (S_0 - N)E\left(\int_{x+\sigma_\Lambda^x(N)}^\infty \alpha e^{-\alpha t}dt\right) \tag{3.69}$$

By shifting the time axis from $[x, x + \sigma_\Lambda^x(N)]$ to $[0, \sigma_\Lambda^x(N)]$ and using the relations (3.39), (3.40) and (3.42) it follows that:

$$\begin{aligned} U_2(\alpha) &= E\left(\int_0^{\sigma_\Lambda^x(N)} \alpha e^{-\alpha(t+x)}(S_0 - (\mathbf{N}_\Lambda(t+x) - \mathbf{N}_\Lambda(x)))dt\right) \\ &= e^{-\alpha x} E\left(\int_0^{\sigma_{\Lambda_x}(N)} \alpha e^{-\alpha t}(S_0 - \mathbf{N}_{\Lambda_x}(t))dt\right) \\ &= e^{-\alpha x}\left[S_0 E\left(\int_0^{\sigma_{\Lambda_x}(N)} \alpha e^{-\alpha t}dt\right) - E\left(\int_0^{\sigma_{\Lambda_x}(N)} \alpha e^{-\alpha t}\mathbf{N}_{\Lambda_x}(t)dt\right)\right] \end{aligned} \tag{3.70}$$

Let, for $k = 0, 1, 2, ...$

$$\mu_k := E\left(\int_0^{\sigma_{\Lambda_x}(k)} \alpha e^{-\alpha t}dt\right) \tag{3.71}$$

and observe that $\mu_0 = 0$ and for $k = 1, 2, ...$

$$\mu_k = \alpha \underline{\xi}_k(T - x, 0) \tag{3.72}$$

Hence, from Lemma 3.2 it follows that for $k = 1, 2, ...$

$$\begin{aligned} \mu_k &= \sum_{n=0}^{k-1}\sum_{\ell=0}^{n} p(n-\ell; (\lambda_0 - \lambda_1)(T-x))\, g\left(\ell; \tfrac{\alpha}{\lambda_1+\alpha}\right) P\left(\ell; (\lambda_1+\alpha)(T-x)\right) \\ &+ \sum_{n=0}^{k-1} g\left(n; \tfrac{\alpha}{\lambda_0+\alpha}\right)[1 - P\left(n; (\lambda_0+\alpha)(T-x)\right)] \end{aligned}$$

Let,

$$U_4(\alpha) := E\left(\int_0^{\sigma_{\Lambda_x}(N)} \alpha e^{-\alpha t}\mathbf{N}_{\Lambda_x}(t)dt\right)$$

Observe that $U_4(\alpha)$ can be computed by dividing the interval $[0, \sigma_{\Lambda_x}(N))$ into non-overlapping parts such that

$$[0, \sigma_{\Lambda_x}(N)) = \bigcup_{k=0}^{N-1} [\sigma_{\Lambda_x}(k), \sigma_{\Lambda_x}(k+1))$$

with $\sigma_{\Lambda_x}(0) = 0$. Moreover, for any $k = 0, 1, ..., N-1$ it is clear that

$$\mathbf{N}_{\Lambda_x}(t) = k \text{ for } t \in [\sigma_{\Lambda_x}(k), \sigma_{\Lambda_x}(k+1)).$$

Hence, using these observations and (3.71), $U_4(\alpha)$ can be given as

$$\begin{aligned} U_4(\alpha) &= \sum_{k=0}^{N-1} kE\left(\int_{\sigma_{\Lambda_x}(k)}^{\sigma_{\Lambda_x}(k+1)} \alpha e^{-\alpha t}dt\right) \\ &= \sum_{k=0}^{N-1} k(\mu_{k+1} - \mu_k) \end{aligned} \tag{3.73}$$

Moreover, substituting (3.71) and (3.73) in (3.70) yields,

$$U_2(\alpha) = S_0 e^{-\alpha x}\mu_N - e^{-\alpha x}\sum_{k=0}^{N-1} k(\mu_{k+1} - \mu_k) \tag{3.74}$$

Similarly, by shifting the time axis from $[x, x + \sigma_\Lambda^x(N)]$ to $[0, \sigma_\Lambda^x(N)]$ and using the relations (3.42), (3.72) and the definition of $\overline{\xi}_k(T, h)$, we can obtain $U_3(\alpha)$ as follows:

$$\begin{aligned} U_3(\alpha) &= (S_0 - N)e^{-\alpha x}E\left(\int_{\sigma_{\Lambda_x}(N)}^{\infty} \alpha e^{-\alpha t}dt\right) \\ &= (S_0 - N)e^{-\alpha x}\alpha\overline{\xi}_N(T - x, 0) \\ &= (S_0 - N)e^{-\alpha x}(1 - \mu_N) \end{aligned} \tag{3.75}$$

Therefore, substituting (3.67), (3.74) and (3.75) in (3.66) and simplifying yields the result. □

Proof of $W(\alpha)$. From (3.4) it follows that $W(\alpha)$ can be written as a summation of the three integrals such that

$$W(\alpha) = W_1(\alpha) + W_2(\alpha) + W_3(\alpha)$$

with

$$W_1(\alpha) := z^{-S_0} \int_0^x e^{-a(t,L;\Lambda)} dt \tag{3.76}$$

$$W_2(\alpha) := z^{-S_0} E\left(\int_x^{x+\sigma_\Lambda^x(N)} e^{-a(t,L;\Lambda)} z^{\mathbf{N}_\Lambda(t)-\mathbf{N}_\Lambda(x)} dt \right) \tag{3.77}$$

$$W_3(\alpha) := z^{-S_0+N} E\left(\int_{x+\sigma_\Lambda^x(N)}^{\infty} e^{-a(t,L;\Lambda)} dt \right) \tag{3.78}$$

The analysis of $W_1(\alpha)$ depends on the relationship between T and L. Therefore, we divide the analysis into two main cases:
(i) If $T < L$ then from the definition of $\Lambda(t)$, for $0 \le t \le T$, we have

$$\Lambda(t+L) - \Lambda(t) = (\lambda_0 - \lambda_1)T + \lambda_1 L - (\lambda_0 - \lambda_1)t \tag{3.79}$$

Thus, using (3.79) in (3.76) yields

$$\begin{aligned} W_1(\alpha) &= z^{-S_0} e^{-((\lambda_0-\lambda_1)T+\lambda_1 L)(1-z)} \int_0^x e^{-(\alpha-(\lambda_0-\lambda_1)(1-z))t} dt \\ &= z^{-S_0} \eta_z^{-1} e^{-b(T,L)(1-z)} (1 - e^{-\eta_z x}) \end{aligned} \tag{3.80}$$

(ii) If $T \ge L$ then the analysis depends on the relation between $x + L$ and T. Hence, we have two subcases:
(a) If $x + L \le T$ then from the definition of $\Lambda(t)$, for $0 \le t \le T$, we obtain that

$$\Lambda(t+L) - \Lambda(t) = \lambda_0 L \tag{3.81}$$

Using (3.81) in (3.76) yields

$$W_1(\alpha) = z^{-S_0} \alpha^{-1} e^{-\lambda_0 L(1-z)} F(x;\alpha) \tag{3.82}$$

(b) If $x+L>T$ then by the definition of $\Lambda(t)$ we obtain that

$$\Lambda(t+L)-\Lambda(t) \;=\; \begin{cases} \lambda_0 L & \text{if} \quad 0\le t\le T-L \\ (\lambda_0-\lambda_1)T+\lambda_1 L-(\lambda_0-\lambda_1)t & \text{if} \quad T-L<t\le x \end{cases} \tag{3.83}$$

Using (3.83) in (3.76) yields

$$\begin{aligned} W_1(\alpha) &= z^{-S_0}\left[e^{-\lambda_0 L(1-z)}\int_0^{T-L} e^{-\alpha t}dt\right. \\ &+ \left. e^{-((\lambda_0-\lambda_1)T+\lambda_1 L)(1-z)}\int_{T-L}^{x} e^{-(\alpha-(\lambda_0-\lambda_1)(1-z))t}dt\right] \\ &= z^{-S_0}\left[\alpha^{-1}e^{-\lambda_0 L(1-z)}F(T-L;\alpha)+\eta_z^{-1}e^{-b(T,L)(1-z)}(e^{-\eta_z(T-L)}-e^{-\eta_z x})\right] \end{aligned} \tag{3.84}$$

Therefore, combining (3.80), (3.82) and (3.84) into a single equation yields $W_1(\alpha)$ as given by (3.25).

By shifting the time axis from $[x, x+\sigma_\Lambda^x(N)]$ to $[0, \sigma_\Lambda^x(N)]$ and using the relations (3.39), (3.40) and (3.42), we can rewrite $W_2(\alpha)$ as follows:

$$\begin{aligned} W_2(\alpha) &= z^{-S_0}E\left(\int_0^{\sigma_\Lambda^x(N)} e^{-a(t+x,L;\Lambda)}z^{\mathbf{N}_\Lambda(t+x)-\mathbf{N}_\Lambda(x)}dt\right) \\ &= z^{-S_0}e^{-\alpha x}E\left(\int_0^{\sigma_{\Lambda_x}(N)} e^{-a(t,L;\Lambda_x)}z^{\mathbf{N}_{\Lambda_x}(t)}dt\right) \end{aligned} \tag{3.85}$$

Let,

$$W_4(\alpha) := E\left(\int_0^{\sigma_{\Lambda_x}(N)} e^{-a(t,L;\Lambda_x)}z^{\mathbf{N}_{\Lambda_x}(t)}dt\right)$$

Observe that $W_4(\alpha)$ can be computed by dividing the interval $[0, \sigma_{\Lambda_x}(N))$ into non-overlapping parts such that

$$[0,\sigma_{\Lambda_x}(N)) = \bigcup_{k=0}^{N-1}[\sigma_{\Lambda_x}(k), \sigma_{\Lambda_x}(k+1))$$

with $\sigma_{\Lambda_x}(0)=0$. Moreover, $\mathbf{N}_{\Lambda_x}(t)=k$ for any $k=0,1,...,N-1$ and $t\in[\sigma_{\Lambda_x}(k), \sigma_{\Lambda_x}(k+1))$. Thus, it follows that

$$W_4(\alpha) \;=\; \sum_{k=0}^{N-1} z^k E\left(\int_{\sigma_{\Lambda_x}(k)}^{\sigma_{\Lambda_x}(k+1)} e^{-a(t,L;\Lambda_x)}dt\right) \tag{3.86}$$

Let, for $k = 1, 2, ...$

$$\underline{\xi}_k := \underline{\xi}_k(T - x, L) := E\left(\int_0^{\sigma_{\Lambda_x}(k)} e^{-a(t,\, L;\, \Lambda_x)} dt\right) \tag{3.87}$$

and note that $\underline{\xi}_k$ can be computed from Lemma 3.2. Therefore, $W_4(\alpha)$ can be written as the summation of the differences of $\underline{\xi}_k$s as follows

$$W_4(\alpha) = \sum_{k=0}^{N-1} z^k (\underline{\xi}_{k+1} - \underline{\xi}_k) \tag{3.88}$$

with $\underline{\xi}_0 = 0$. Hence, substituting (3.88) in (3.89) yields $W_2(\alpha)$ as given in (3.26).

Similarly, by shifting the time axis from $[x, x + \sigma_\Lambda^x(N)]$ to $[0, \sigma_\Lambda^x(N)]$ and using the relations (3.39)-(3.42), $W_3(\alpha)$ can be obtained as follows:

$$\begin{aligned} W_3(\alpha) &= z^{-S_0+N} e^{-\alpha x} E\left(\int_{\sigma_\Lambda^x(N)}^{\infty} e^{-a(t+x,L;\Lambda)} dt\right) \\ &= z^{-S_1} e^{-\alpha x} E\left(\int_{\sigma_{\Lambda_x}(N)}^{\infty} e^{-a(t,\, L;\, \Lambda_x)} dt\right) \\ &= z^{-S_1} e^{-\alpha x} \overline{\xi}_N \end{aligned} \tag{3.89}$$

where $\overline{\xi}_N := \overline{\xi}_N(T - x, L)$ can be computed by Lemma 3.2. □

Proof of Equation (3.30). By conditioning on $\mathbf{Y}$ and from the relation (3.28), we obtain that

$$E\left(\mathbf{IN}(\mathbf{Y})\right)^+ = M_1(\alpha) + e^{-\alpha L} M_2(\alpha) \tag{3.90}$$

with

$$\begin{aligned} M_1(\alpha) &:= \int_0^L \alpha e^{-\alpha t} E(S - \mathbf{N}_\Lambda(t))^+ dt \\ M_2(\alpha) &:= \int_0^\infty \alpha e^{-\alpha t} E(S - (\mathbf{N}_\Lambda(t + L) - \mathbf{N}_\Lambda(t)))^+ dt \end{aligned}$$

Recall that $\mathbf{N}_\Lambda(t)$ and $\mathbf{N}_\Lambda(t+L) - \mathbf{N}_\Lambda(t)$ are Poisson distributed with parameter $\Lambda(t)$ and $\Lambda(t+L) - \Lambda(t)$, resp. Hence,

$$\begin{aligned} M_1(\alpha) &:= \sum_{n=0}^{S_f-1} (S-n) \int_0^L \alpha e^{-\alpha t} p(n; \Lambda(t)) dt \\ M_2(\alpha) &:= \sum_{n=0}^{S_f-1} (S-n) \int_0^\infty \alpha e^{-\alpha t} p(n; \Lambda(t+L) - \Lambda(t)) dt \end{aligned}$$

and

$$E\left(\mathbf{IN}(\mathbf{Y})\right)^+ = \sum_{n=0}^{S_f-1} (S-n) \left[A_1(n) + e^{-\alpha L} A_2(n)\right] \tag{3.91}$$

with

$$\begin{aligned} A_1(n) &:= \int_0^L \alpha e^{-\alpha t} p(n; \Lambda(t)) dt \\ A_2(n) &:= \int_0^\infty \alpha e^{-\alpha t} p(n; \Lambda(t+L) - \Lambda(t)) dt \end{aligned}$$

The integral equations $A_1(n)$ and $A_2(n)$ can be analyzed by conditioning on the relation between T and L as follows:

(i) If $T < L$ then by the definition of $\Lambda(t)$ we have

$$A_1(n) = \int_0^T \alpha e^{-\alpha t} p(n; \lambda_0 t) dt + \int_T^L \alpha e^{-\alpha t} p(n; \lambda_1 t + (\lambda_0 - \lambda_1) T) dt \tag{3.92}$$

Applying Lemma 3.1 to the integrals in (3.92) gives,

$$\begin{aligned} A_1(n) &= g\left(n; \tfrac{\alpha}{\lambda_0+\alpha}\right) \left[1 - P\left(n; (\lambda_0+\alpha)T\right)\right] \\ &+ \sum_{\ell=0}^{n} p(n-\ell; (\lambda_0-\lambda_1)T) g\left(\ell; \tfrac{\alpha}{\lambda_1+\alpha}\right) \left[P\left(\ell; (\lambda_1+\alpha)T\right) - P\left(\ell; (\lambda_1+\alpha)L\right)\right] \end{aligned}$$

From (3.51), the integral equation $A_2(n)$ can be written as,

$$A_2(n) = \int_0^T \alpha e^{-\alpha t} p(n; -(\lambda_0-\lambda_1)t + b(T,L)) dt + \int_T^\infty \alpha e^{-\alpha t} p(n; \lambda_1 L) dt \tag{3.93}$$

Let, $\eta_0 := \alpha - (\lambda_0 - \lambda_1)$ and

$$f_A(n) := \int_0^T \alpha e^{-\alpha t} p(n; -(\lambda_0-\lambda_1)t + b(T,L)) dt$$

Then,

$$A_2(n) = f_A(n) + p(n; \lambda_1 L)\overline{F}\,(T; \alpha)$$

and $f_A(n)$ is found by Lemma 3.1 as follows

$$f_A(n) = \begin{cases} \sum_{\ell=0}^{n} p(n-\ell; b(T,L))\; g\left(\ell; \alpha/\eta_0\right)\; \left[1 - P\left(\ell; \eta_0 T\right)\right] & \text{if } \eta_0 \neq 0 \\ \sum_{\ell=0}^{n} p(n-\ell; b(T,L))\; \frac{(-\alpha)^{\ell+1}}{(\ell+1)!}\; \left(-T^{\ell+1}\right) & \text{if } \eta_0 = 0 \end{cases}$$

(ii) If $T \geq L$ then from the definition of $\Lambda(t)$ and Lemma 1, we obtain that

$$\begin{aligned} A_1(n) &= \int_0^L \alpha e^{-\alpha t} p(n; \lambda_0 t)dt \\ &= g\left(n; \tfrac{\alpha}{\lambda_0+\alpha}\right)\left[1 - P\left(n; (\lambda_0+\alpha)L\right)\right] \end{aligned}$$

From relation (3.58), the integral equation $A_2(n)$ can be written as,

$$\begin{aligned} A_2(n) &= \int_0^{T-L} \alpha e^{-\alpha t} p(n; \lambda_0 L)dt + \int_{T-L}^{T} \alpha e^{-\alpha t} p(n; -(\lambda_0-\lambda_1)t + b(T,L))dt \\ &+ \int_T^{\infty} \alpha e^{-\alpha t} p(n; \lambda_1 L)dt \end{aligned} \tag{3.94}$$

Let,

$$f_A(n) := \int_{T-L}^{T} \alpha e^{-\alpha t} p(n; -(\lambda_0-\lambda_1)t + b(T,L))dt$$

then

$$A_2(n) = p(n; \lambda_0 L)F\,(T-L; \alpha) + f_A(n) + p(n; \lambda_1 L)\overline{F}\,(T; \alpha)$$

and from Lemma 3.1, we obtain that

$$f_A(n) = \begin{cases} \sum_{\ell=0}^{n} p(n-\ell; b(T,L))\; g\left(\ell; \alpha/\eta_0\right)\; \left[P\left(\ell; \eta_0(T-L)\right) - P\left(\ell; \eta_0 T\right)\right] & \text{if } \eta_0 \neq 0 \\ \sum_{\ell=0}^{n} p(n-\ell; b(T,L))\; \frac{(-\alpha)^{\ell+1}}{(\ell+1)!}\; \left[(T-L)^{\ell+1} - T^{\ell+1}\right] & \text{if } \eta_0 = 0 \end{cases}$$

Therefore, combining the expressions of $A_1(n)$, $A_2(n)$ and $f_A(n)$ for the cases $T < L$ and $T \geq L$ yields the result as given in (3.30). $\square$

Part II

Random Deal Offerings

Chapter 4

An Inventory Model for Systems with Random Deal Offerings and Partial Backordering*

Abstract

In this chapter, we consider a single item, single location, continuous review inventory system where the supplier offers price discounts at random points in time. The demand is assumed to be deterministic and lead times are negligible. When the system is out of stock a fraction of demand can be backordered and the rest is lost, i.e., partial backordering. The replenishment and stocking decisions at deal and list prices are controlled by a four parameter control policy. We derive exact expressions for operating characteristics and equations to compute the optimal policy parameters. We provide qualitative results about the optimal solution and demonstrate that allowing backorders in a random deal environment might lead to important cost savings.

4.1. Introduction

In this chapter, our aim is to investigate the impact of price discounts given by a supplier on replenishment and stocking decisions of a firm. We consider a single item, single location, continuous review inventory system where a supplier offers *deals* (price discounts) at random points in time. We assume that deals arrive according to the Poisson process and the deal price is known and fixed. Moreover, we assume that demand is deterministic

*This chapter is based on Pinçe and Berk (2006) and lead times are negligible. If an arriving customer finds the inventory system out of stock then she may either choose to be backlogged or leave the system and become a lost sale. In other words, when the system is out of stock, a certain fraction of demand can be backordered and the rest is lost. We assume that the fraction is known and fixed. In the literature, this type of backordering structure is generally referred to as *partial backordering*.

There are two types of replenishments: at the deal price and at the regular list price. We propose a four parameter control policy to decide the timing (in terms of inventory) and the size for both types of replenishment. That is, the deals can be taken only if the inventory level is below a *can-order* level and the regular list replenishments are given whenever the inventory level hits the reorder point. Both types of replenishment are assumed to be instantaneous. As such, we extend the policy offered by Moinzadeh (1997) by including reorder point and allowing partial backordering. The objective is to determine the optimal reorder point, the deal threshold (can-order level) and the replenishment quantities minimizing the expected total cost rate. We derive the exact expression for the expected total cost rate function and use analytical optimization to determine the optimal policy parameters. We find that allowing backorders while waiting for a deal can lead to considerable savings. Especially, when the expected discount is deep, the benefit of making a *good deal* replenishment offsets the stockout costs on average.

The literature of the inventory systems with price fluctuations can be categorized into three different classes. The papers in the first class consider the systems with deterministic demand rate where a single future price change is announced by supplier or known by experience. Taylor and Bradley (1985) consider an inventory system where the supplier announces a price increase that will occur in a future time and develop optimal ordering strategies for the cases where the price increase does not coincide with the end of an EOQ cycle. Ardalan (1988) investigates the effects of special orders when a supplier reduces the price of a product temporarily and develops EOQ type optimal ordering policies according to the relation between the duration of sale period, replenishment time and on-hand inventory at the replenishment time. Lev et al. (1981) consider EOQ systems in which any or all system parameters may change at some future time known by announcement or prior experience. The demand rate is assumed to be affected by the changes in cost parameters.

They develop myopic and optimal ordering policies depending on the inventory level at the time of the cost change and present a simple method for computing the optimal policy. Lev and Weiss (1990) consider the same model for finite and infinite horizon cases and present more specific results on the structure of the optimal policy. For the infinite horizon case, they show that the model offered is the generalization of the model proposed by Taylor and Bradley (1985) and demonstrate that the results by Taylor and Bradley (1985) are not optimal.

The studies in the second class are mostly periodic review models and their main focus is the optimal policy structure. One of the earliest models in this class belongs to Kalymon (1971). He studies a single item, periodic review inventory system where the prices of future periods are governed by a Markov process and the distribution of period demand is determined by the current price. For finite and infinite horizon cases the optimality of the price dependent (s, S) policy is shown and bounds on the optimal policies are provided. Özekici and Parlar (1999) consider a similar Markov modulated model where the randomly changing environment affects not only demand but also supply and cost parameters. They show that the environment-dependent-basestock policy is optimal when there is no fixed ordering cost and the environment-dependent (S, s) policy is optimal when the fixed ordering cost is positive.

Golabi (1985) examines a periodic review system where random unit prices are determined according to a known distribution function at the beginning of each period. Under deterministic demand assumption, he shows that in any period, the optimal strategy is to order up to a level that satisfies the demands of next n periods when the random unit price is between the critical price levels. Gurnani (1996) presents an optimal ordering policy for a periodic review inventory system with random demands and random deal offerings. He considers two models according to the firm's decision on sharing the cost savings information with customers and presents the optimal policy structure. Recently, Wang (2001) studies a periodic review inventory system where the purchasing prices of successive periods decrease according to a stochastic process. It is assumed that random demands and purchasing prices are independent and the fixed ordering cost is equal to zero. The conditions for cost parameters are derived and the optimal myopic order-up-to levels for each period are determined. Assunção and Meyer (1993) extend the periodic review inventory control systems such as those proposed by Golabi (1985) and Kalymon (1971) by modeling consumer's rate of consumption as a decision variable and the future prices as a first-order stochastic process.

The studies in the third class are composed of continuous review models. Hurter and Kaminsky (1968) analyze a system where the demand arrival process is Poisson and the unit price of an item fluctuates between low and high periods according to another, independent Poisson process. All replenishments are instantaneous and stockouts are not allowed. They propose a three parameter control policy consisting of a deal replenishment threshold and order-up-to points for regular and deal replenishments. According to the policy the regular order-up-to point is used when the price is high and the inventory level is depleted. A deal is taken only if the inventory level is lower than the deal threshold. They suggest 3-dimensional discrete search method to determine the optimal policy parameters. Silver et al. (1993) offer a simple, analytical approximate solution procedure for the model considered by Hurter and Kaminsky (1968). They showed that the procedure of graphical lookup and a single one-dimensional search yields usually optimal values of the three policy parameters. Moreover, they provide a closed form expression for the optimal policy parameters when the deal order-up-to level is equal to the deal threshold.

Moinzadeh (1997) considers the same model with deterministic demand. Under the same three parameter control policy, the exact expressions of the optimal policy parameters are derived and the properties resembling EOQ based intuitions are stated. Furthermore, a heuristic solution is offered for the calculation of the optimal policy parameters and a numerical analysis is provided to investigate the performance of this solution.

Feng and Sun (2001) extend the model considered by Hurter and Kaminsky (1968) and Silver et al. (1993) to allow backorders. They propose a four parameter policy with the three parameters being the same as before and the reorder point for regular replenishments. They suggest a bisection algorithm to obtain the optimal policy parameters.

Abad (2003) considers a model where price discounts offered by supplier lasts a finite time interval. The demand is deterministic and sensitive to reseller's selling price. He formulates the objective function of reseller as a profit maximizing problem and derives the optimal selling price and optimal lotsize expressions for the cases with or without forward-buying possibility. Arcelus et al. (2003) study a similar problem where the promotions offered by supplier lasts for a fixed minimum guaranteed duration but of uncertain total discount period. Such trade promotions are commonly called as “while supplies last” promotion.

The rest of the chapter is organized as follows. Section 4.2 states the assumptions of the model and the control policy. Section 4.3 provides the preliminary results used for the derivation of the operating characteristics. Section 4.4 presents the expressions for the operating characteristics and the equations to calculate optimal policy parameters. Section 4.7 presents the results of the numerical study. Section 4.8 concludes.

4.2. The Model

We consider a single item, single location, continuous review inventory system where the supplier offers price discounts at random points in time. That is, the item can be replenished at the list price c_L at any time and at the discount price c_D when a deal opportunity occurs. Although the deal price can be random in general, in our analysis, we assume that it is fixed. The demand is constant at rate D and the deals arrive according to the Poisson process with rate μ. We assume that the deal duration is negligible and the delivery lead times for both types of replenishment are zero.

There are fixed ordering costs for list and deal replenishments denoted by A_L and A_D, resp. Inventory carrying cost h is incurred per unit per time. When there is no inventory on hand, a certain fraction of demand can be backordered. We denote the backordering fraction with $\hat{p} \in (0, 1)$. Backordering cost is composed of a fixed component π incurred per unit and $\hat{\pi}$ incurred per unit per unit of time. The fraction of demand not backordered is lost against the lost sales cost θ incurred per unit.

To control the timing and the size of the replenishment decisions, we use the following four parameter policy based on the net inventory level (on-hand minus backorders):

Policy: If a deal is offered and the net inventory level is below the deal replenishment threshold s then a replenishment order from the deal price is placed to increase the net inventory level up to $s + Q$. Otherwise, a replenishment order from the list price is placed to increase the net inventory level up to $\tilde{R}$ whenever the net inventory level hits the reorder point $-r$.

We refer to this policy as the (r, R, s, Q) control policy. We let $r, R, s, Q \geq 0$ and $\tilde{R} \in \mathbb{R}$. That is, according to the particular policy choice, the sign of R can vary and therefore, $\tilde{R}$ is either equal to R or $-R$. In the sequel, we identify three different cases according to the sign of R and the relation between s and R. Moreover, we only consider the cases where the deal threshold s cannot be negative. This is because a negative s implies somewhat exotic scenarios where the deals can be accepted only if there are enough stockouts. Although those cases can be analyzed for the sake of completeness, in the same line with the literature, in this study, we limit our attention to the more realistic cases where $s \geq 0$.

Our objective is to find the optimal policy parameters minimizing the expected total cost rate. Depending on the relation between R and s and the sign of R, we can analyze the expected total cost rate function for three different cases:

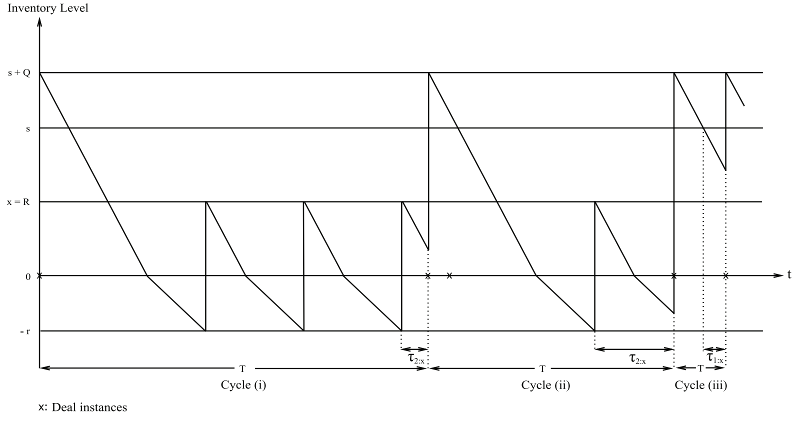
Case 1: $-r \leq 0 \leq R \leq s$

Case 2: $-r \leq 0 \leq s \leq R \leq s + Q$

Case 3: $-r \leq -R \leq 0 \leq s$

Note that for Case 1 and Case 2, $\tilde{R} = R$ and for Case 3, $\tilde{R} = -R$. Figures 4.1–4.3 show realizations of the net inventory level for the three cases.

Figure 4.1: Possible realization of the net inventory level for Case 1



As in Moinzadeh (1997), the cycle is defined as the duration between two consecutive deal replenishments. Observe from Figures 4.1–4.3 that under the (r, R, s, Q) policy, the length of a cycle is a random variable depending not only the deal arrival process but also the net inventory level. If a deal arrives when the inventory level is larger than the threshold s then a deal is not taken. For example, in Figure 4.1, we see three different cycle types according to their end points. Type (i) cycle ends when the net inventory level is between 0 and R, type (ii) cycle ends when the net inventory level is between $-r$ and 0 and type (iii) cycle ends when the net inventory level is between s and R. In the sequel, we will analyze different cycle types for each case to calculate the expected number of list replenishments and the expected cycle length.

Moreover, observe from the figures that the slope of the net inventory level decreases below zero due to partial backordering. In Figure 4.1 and Figure 4.2, list replenishments raise the net inventory up to a nonnegative level R. While in Figure 4.3, the net inventory level might remain nonpositive until a deal arrives since list replenishments do not increase it to a positive level. Intuitively, such a policy might be optimal when the stockout costs are relatively low and deals are very attractive compared to regular list replenishments. Therefore, the system might be better off by incurring stockout costs while waiting for a *good deal*.

Figure 4.2: Possible realization of the net inventory level for Case 2

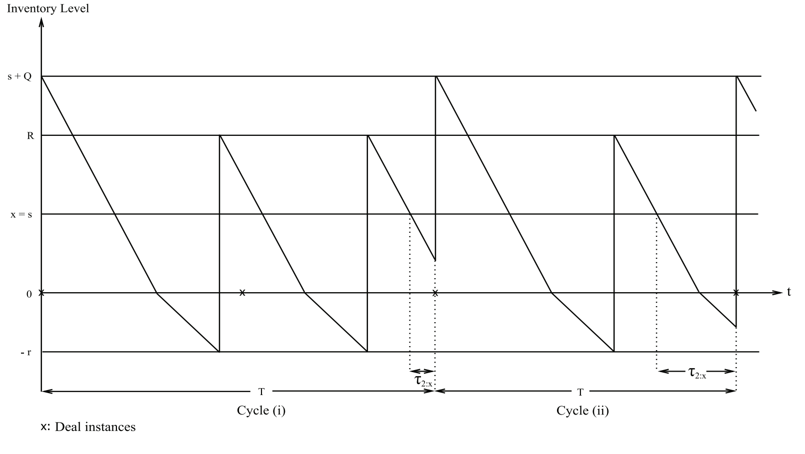
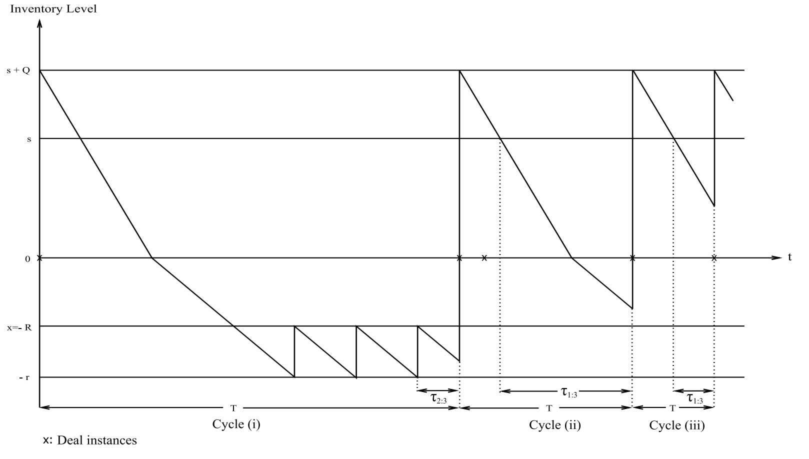


Figure 4.3: Possible realization of the net inventory level for Case 3



In the next section, we provide some preliminary results to be used in the derivation of operating characteristics.

4.3. Preliminaries

Let Y be an exponential random variable with parameter μ representing the duration of the time between two consecutive deal offers.

Preliminaries for Case 1 and Case 2:

Let $x = min(s, R)$ and β_x be the probability that a deal arrives when the net inventory level is between s and x. Example of this scenario can be seen in cycle (iii) in Figure 4.1. In order to calculate β_x, we only need to consider the time interval for the net inventory level to decrease from s to x, i.e., the interval of length $(s-x)/D$, due to the memoryless property of the interarrival time Y. Thus, we have

$$\beta_x = P\left(Y \leq \frac{s-x}{D}\right) = 1 - e^{-\mu(s-x)/D} \tag{4.1}$$

Let α_x be the probability that a deal arrives when the net inventory level is at or below x (see cycles (i) and (ii) in Figure 4.1 and Figure 4.2). Similarly, by only considering the time interval for the net inventory level to decrease from x to $-r$, we obtain that

$$\alpha_x = P\left(Y \leq \frac{x}{D} + \frac{r}{\hat{p}D}\right) = 1 - e^{-\mu(\hat{p}x+r)/\hat{p}D} \tag{4.2}$$

Next, we shall consider the two random time intervals $\tau_{1:x}$ and $\tau_{2:x}$ which are associated with the events described above.

For $x = R$, given that a deal arrives when the net inventory level is between s and R, $\tau_{1:x}$ is the time interval between the first time the net inventory level hits s until the deal arrives (see cycle (iii) in Figure 4.1). Thus, $\tau_{1:x}$ has an exponential distribution truncated from the right at $(s-R)/D$ and its pdf can be given as

$$g_{\tau_{1:x}}(t) = \frac{f_Y(t)}{P(Y \leq \frac{s-R}{D})} = \frac{\mu e^{-\mu t}}{1 - e^{-\mu(s-R)/D}} \qquad \text{for} \quad 0 \leq t \leq \frac{s-R}{D}$$

Similarly, given that a deal arrives when the net inventory level is at or below R, $\tau_{2:x}$ is the time interval between the last time the net inventory level hits R until a deal arrives (see cycles (i) and (ii) in Figure 4.1). Therefore, $\tau_{2:x}$ has an exponential distribution truncated from the right at $R/D + r/\hat{p}D$ and its pdf is found as

$$g_{\tau_{2:x}}(t) = \frac{f_Y(t)}{P(Y \leq \frac{R}{D} + \frac{r}{\hat{p}D})} = \frac{\mu e^{-\mu t}}{1 - e^{-\mu(\hat{p}R+r)/\hat{p}D}} \qquad \text{for} \quad 0 \leq t \leq \frac{R}{D} + \frac{r}{\hat{p}D}$$

For $x = s$, $\tau_{1:x}$ is always zero and $\tau_{2:x}$ has an exponential distribution truncated from right at $s/D + r/\hat{p}D$ (see Figure 4.2).

Observe that depending on the value of x, we can give a general expression for the pdfs of $\tau_{1:x}$ and $\tau_{2:x}$ as follows:

$$g_{\tau_{1:x}}(t) = \frac{\mu e^{-\mu t}}{\beta_x} \qquad \text{for} \quad 0 \leq t \leq \frac{s-x}{D} \tag{4.3}$$

$$g_{\tau_{2:x}}(t) = \frac{\mu e^{-\mu t}}{\alpha_x} \qquad \text{for} \quad 0 \leq t \leq \frac{\hat{p}x+r}{\hat{p}D} \tag{4.4}$$

From (4.3) and (4.4), the first moments of $\tau_{1:x}$ and $\tau_{2:x}$ can be found as,

$$E[\tau_{1:x}] = \frac{1}{\mu} - \frac{(s-x)(1-\beta_x)}{D\beta_x} \tag{4.5}$$

$$E[\tau_{2:x}] = \frac{1}{\mu} - \frac{(\hat{p}x+r)(1-\alpha_x)}{\hat{p}D\alpha_x} \tag{4.6}$$

and the second moment of $\tau_{1:x}$ is found as,

$$E[\tau_{1:x}{}^2] \;=\; \frac{2}{\mu^2} - \frac{(s-x)(1-\beta_x)}{D\beta_x}\left[\frac{2}{\mu} + \frac{s-x}{D}\right] \tag{4.7}$$

Preliminaries for Case 3:

Let β_3 denotes probability that a deal is offered when the net inventory level is between s and $-R$. Moreover, let α_3 denotes the probability that a deal is offered when the net inventory level is at or below $-R$. Thus, from the memoryless property of Y, we have

$$\beta_3 \;=\; P\left(Y \leq \frac{s}{D} + \frac{R}{\hat{p}D}\right) = 1 - e^{-\mu(\hat{p}s+R)/\hat{p}D} \tag{4.8}$$

$$\alpha_3 \;=\; P\left(Y \leq \frac{r-R}{\hat{p}D}\right) = 1 - e^{-\mu(r-R)/\hat{p}D} \tag{4.9}$$

Given that a deal is offered when the net inventory level is between s and $-R$, $\tau_{1:3}$ is the time interval between the first time the inventory net level hits s until a deal is offered (see cycles (ii) and (iii) in Figure 4.3). Thus, it has a truncated exponential distribution such that

$$g_{\tau_{1:3}}(t) = \frac{\mu e^{-\mu t}}{\beta_3} \qquad \text{for} \quad 0 \leq t \leq \frac{\hat{p}s + R}{\hat{p}D} \tag{4.10}$$

Similarly, given that a deal is offered when the net inventory level is at or below $-R$, $\tau_{2:3}$ is the elapsed time from the last time the net inventory level hits $-R$ until a deal is offered (see cycle (i) in Figure 4.3). Thus, the pdf of $\tau_{2:3}$ can be given as

$$g_{\tau_{2:3}}(t) = \frac{\mu e^{-\mu t}}{\alpha_3} \qquad \text{for} \quad 0 \leq t \leq \frac{r-R}{\hat{p}D} \tag{4.11}$$

From (4.10) and (4.11), the first moments of $\tau_{1:3}$ and $\tau_{2:3}$ are found as

$$E[\tau_{1:3}] = \frac{1}{\mu} - \frac{(\hat{p}s + R)(1-\beta_3)}{\hat{p}D\beta_3} \tag{4.12}$$

$$E[\tau_{2:3}] = \frac{1}{\mu} - \frac{(r-R)(1-\alpha_3)}{\hat{p}D\alpha_3} \tag{4.13}$$

and the second moment of $\tau_{2:3}$ can be given as

$$E[\tau_{2:3}{}^2] = \frac{2}{\mu^2} - \frac{(r-R)(1-\alpha_3)}{\hat{p}D\alpha_3}\left[\frac{2}{\mu} + \frac{r-R}{\hat{p}D}\right] \tag{4.14}$$

4.4. Operating Characteristics

We want to determine the optimal policy parameters by using the average cost criterion.

Let $CC(t)$ be the total cycle cost incurred by time t. Therefore, according to the renewal reward theorem (Ross, 1983) we can write the expected total cost rate function as follows:

$$TC(r, R, s, Q) = \lim_{t\to\infty} \frac{CC(t)}{t} = \frac{E[CC]}{E[T]} \tag{4.15}$$

where $E[CC]$ denotes the expected cycle cost and $E[T]$ denotes the expected cycle length.

The expected cycle cost under the (r, R, s, Q) policy can be given as,

$$E[CC] = E[RC] + hE[OH] + \hat{\pi}E[BO_1] + \pi E[BO_2] + \theta E[LS] \tag{4.16}$$

where $E[OH]$ is the expected on-hand inventory carried per cycle, $E[RC]$ is the expected replenishment cost per cycle, $E[N_L]$ is the expected number of list replenishments given in a cycle, $E[BO_1]$ is the expected time-weighted backorders per cycle, $E[BO_2]$ is the expected number of units backordered per cycle, and $E[LS]$ expected number of lost sales per cycle.

Our objective is to find the policy parameters minimizing (4.15). However, depending on the relation between policy parameters given by three different cases above, both the form of (4.15) and the constraint set of the optimization problem are changing. Therefore, we calculate and optimize (4.15) for each case separately and then select the best policy parameter values yielding the minimum average cost.

The operating characteristics for the three cases can be given with the following proposition. We refer the reader to Appendix 4.9 for the proof.

Proposition 4.1. *Under the (r, R, s, Q) policy the operating characteristics can be given as follows,*

1. *For Case 1 and Case 2,*

$$E[N_L] = \frac{(1-\beta_x)(1-\alpha_x)}{\alpha_x} \tag{4.17}$$

$$E[T] = \frac{Q}{D} + \frac{1}{\mu} + \frac{(R-x)}{D}E[N_L] \tag{4.18}$$

$$\begin{aligned} E[RC] &= A_D + c_D D E[T] + \left[A_L + (c_L - c_D)(r+R) + \frac{c_D D(1-\hat{p})}{\mu}\right] E[N_L] \\ &- \frac{c_D D(1-\beta_x)(1-\hat{p})e^{-\mu x/D}}{\mu\alpha_x} \end{aligned} \tag{4.19}$$

$$\begin{aligned} E[OH] &= \frac{Q(Q+2s)}{2D} + \frac{s}{\mu} - \frac{D}{\mu^2}\left[\beta_x + \frac{(1-e^{-\mu x/D})(1-\beta_x)}{\alpha_x}\right] \\ &+ \left[R^2 - x^2 + \frac{2Dx}{\mu}\right]\frac{E[N_L]}{2D} \end{aligned} \tag{4.20}$$

$$E[BO_1] = \frac{\hat{p}D(1-\beta_x)}{\mu^2}\left[1 - \frac{(1-e^{-\mu x/D})}{\alpha_x}\right] - \frac{rE[N_L]}{\mu} \tag{4.21}$$

$$E[BO_2] = \frac{\hat{p}D(1-\beta_x)}{\mu}\left[1 - \frac{(1-e^{-\mu x/D})}{\alpha_x}\right] \tag{4.22}$$

$$E[LS] = \frac{(1-\hat{p})D(1-\beta_x)}{\mu}\left[1 - \frac{(1-e^{-\mu x/D})}{\alpha_x}\right] \tag{4.23}$$

2. For Case 3,

$$E[N_L] = \frac{(1-\beta_3)(1-\alpha_3)}{\alpha_3} \tag{4.24}$$

$$\begin{aligned} E[T] &= \frac{Q}{D} + \frac{1}{\mu} \\ E[RC] &= A_D + c_D D E[T] + [A_L + (c_L - c_D)(r-R)]\, E[N_L] \\ &- \frac{c_D D e^{-\mu s/D}(1-\hat{p})}{\mu} \end{aligned} \tag{4.25}$$

$$E[OH] = \frac{Q(Q+2s)}{2D} + \frac{s}{\mu} - \frac{D(1-e^{-\mu s/D})}{\mu^2} \tag{4.26}$$

$$E[BO_1] = \frac{\hat{p}De^{-\mu s/D}}{\mu^2} - \frac{(r-R)E[N_L]}{\mu} \tag{4.27}$$

$$E[BO_2] = \frac{\hat{p}De^{-\mu s/D}}{\mu} \tag{4.28}$$

$$E[LS] = \frac{(1-\hat{p})De^{-\mu s/D}}{\mu} \tag{4.29}$$

4.5. Optimization of $TC(r, R, s, Q)$

Next, we will state the optimal policy parameters that optimize (4.15) for the three different cases. As can be seen from Proposition 4.1, the form of $TC(r, R, s, Q)$ changes according to different cases. Thus, in order to optimize $TC(r, R, s, Q)$, we need to solve three different optimization problems for different cases.

Define the sets corresponding to the feasible region of the objective function for three cases as follows,

$$\mathbf{\Omega_1} := \{(r, R, s, Q) : 0 \leq R \leq s;\ r, Q \geq 0\}$$

$$\mathbf{\Omega_2} := \{(r, R, s, Q) : 0 \leq s \leq R \leq s + Q;\ r, Q \geq 0\}$$

$$\mathbf{\Omega_3} := \{(r, R, s, Q) : 0 \leq R \leq r;\ s, Q \geq 0\}$$

Thus, the optimization problem for each case is to minimize $TC(r, R, s, Q)$ subject to the constraint set $\mathbf{\Omega_i}$ with $i = 1, 2, 3$.

For each case, by using differential calculus, we identify a set of *candidates* for the optimal solution. The candidates are found either at critical (stationary) points or at the boundaries of the feasible region of the corresponding case. In the sequel, we state rules to eliminate some of the candidates due to the relation between certain system parameters and provide equation systems from which the candidates can be computed easily. Once the candidates are identified for each case, the global optimal solution can be found by picking the candidate yielding the smallest expected total cost rate.

We need the following structural results in order to analyze the objective function and identify the candidates for different cases.

4.5.1 Structural Results

We start with the following lemmas which are useful in the analysis of the derivatives of the objective function with respect to the policy parameters.

Lemma 4.1. *Define* $f : [0, \infty) \to \mathbb{R}$ *by*

$$f(x) = A(1 - ae^{-x} - x) + B$$

with $0 < a \leq 1$, $A \in \mathbb{R} \setminus \{0\}$ *and* $B \in \mathbb{R}$. *Then,*

1. *If* $A > 0$ *then* $f(x)$ *is concave decreasing function and*
   (a) *If* $A(1-a) + B = 0$ *then* $f(x) < 0$ *for* $x > 0$ *and has a unique root at* $x_0 = 0$
   (b) *If* $A(1-a) + B < 0$ *then* $f(x) < 0$ *for* $x \geq 0$
   (c) *If* $A(1-a)+B > 0$ *then* $f(x)$ *has a unique positive root* $x_0 \in \left(1 + \frac{B}{A} - a, 1 + \frac{B}{A}\right)$
2. *If* $A < 0$ *then* $f(x)$ *is convex increasing function and*
   (a) *If* $A(1-a) + B = 0$ *then* $f(x) > 0$ *for* $x > 0$ *and has a unique root at* $x_0 = 0$
   (b) *If* $A(1-a) + B > 0$ *then* $f(x) > 0$ *for* $x \geq 0$
   (c) *If* $A(1-a)+B < 0$ *then* $f(x)$ *has a unique positive root* $x_0 \in (1+\frac{B}{A}-a, 1+\frac{B}{A})$

Proof of Lemma 4.1. If $A > 0$ then $f'(x) = A(ae^{-x} - 1) \leq 0$ and $f''(x) = -Aae^{-x} \leq 0$ since $0 < a \leq 1$ and $x \geq 0$. Thus, $f(x)$ is concave decreasing in x. Since $f(0) = A(1-a) + B$, if $A(1-a) + B \leq 0$ then $f(x) \leq 0$ for $x \geq 0$. On the other hand, if $A(1-a) + B > 0$ then $1 + B/A > a > 0$ and

$$f\left(1 + \frac{B}{A} - a\right) = Aa\left(1 - e^{-\left(1+\frac{B}{A}-a\right)}\right) > 0$$

and

$$f\left(1 + \frac{B}{A}\right) = -Aae^{-\left(1+\frac{B}{A}\right)} < 0$$

Thus, $f(x)$ has a unique root in the interval $\left(1 + \frac{B}{A} - a, 1 + \frac{B}{A}\right)$.

If $A < 0$ then $f'(x) \geq 0$ and $f''(x) \geq 0$ and therefore, $f(x)$ is convex increasing in x. Moreover, if $A(1-a) + B \geq 0$ then $f(x) \geq 0$ for $x \geq 0$. On the other hand, if $A(1-a)+B < 0$ then $1+B/A > a > 0$ and $f\left(1 + \frac{B}{A} - a\right) < 0$ and $f\left(1 + \frac{B}{A}\right) > 0$. Thus, $f(x)$ has a unique root in the interval $\left(1 + \frac{B}{A} - a, 1 + \frac{B}{A}\right)$ $\square$

Lemma 4.2. *Define $f : \mathbb{R}_+ \to \mathbb{R}$ by*

$$f(x) = a + bx + d^{-1}ce^{-dx}$$

with $b, d > 0$ and $a, c \in \mathbb{R}$, and let x^* denotes the minimizer of $f(x)$. Therefore,

1. *If $b \geq c$ then $x^* = 0$*
2. *If $b < c$ then $x^* = -d^{-1} \ln \left(\frac{b}{c}\right)$*

Proof of Lemma 4.2. Taking the first and second derivatives of f yield,

$$\frac{df(x)}{dx} = b - ce^{-dx} \tag{4.30}$$

and

$$\frac{d^2 f(x)}{dx^2} = dce^{-dx} \tag{4.31}$$

If $c < 0$ then $df(x)/dx > 0$ and $d^2 f(x)/dx^2 < 0$. Thus, $f(x)$ is increasing and convex function and it has a minimum at $x^* = 0$.

If $c \geq 0$ then $d^2 f(x)/dx^2 \geq 0$. Thus, $f(x)$ is convex but the sign of (4.30) depends on the relation between b and c such that:

if $b \geq c$ then $df(x)/dx \geq 0$ since $0 < e^{-dx} \leq 1$ for $x \geq 0$. Thus, $f(x)$ is increasing and convex function and $x^* = 0$.

if $b < c$ then $df(x)/dx$ has a unique positive root. Thus, setting (4.30) to zero yields x^* as given above. □

As it will be clear in the sequel, for some cases, optimal (r, R) can be determined independently from (s, Q). In other words, optimization problem can be decomposed. Thus, once the optimal (r, R) is determined, the objective function becomes a function of s and Q which can be represented in a special form. We use the following lemma to find the optimal (s, Q) when decomposition is possible.

Lemma 4.3. *Define $\widetilde{f} : \mathbb{R}^2_+ \to \mathbb{R}_+$ by*

$$\begin{aligned} \widetilde{f}(s, Q) &= A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + h\left(\frac{Q}{D} + \frac{1}{\mu}\right) s \\ &+ \frac{D}{\mu}\left(\frac{h}{\mu} + U\right) e^{-\mu s/D} \end{aligned} \tag{4.32}$$

with $U \in \mathbb{R}$ and let

$$\widetilde{TC}(s,Q) = \frac{\widetilde{f}(s,Q)}{\frac{Q}{D} + \frac{1}{\mu}} \tag{4.33}$$

Then the optimal (s^*, Q^*) minimizing $\widetilde{TC}(s,Q)$ can be given as

1. *If $A_D/D + U/\mu \leq 0$ then $s^* = Q^* = 0$.*
2. *If $A_D/D + U/\mu > 0$ then*

   (a) *if $\sqrt{2A_D Dh} > DU$ then $s^* = 0$ and*

$$Q^* = \frac{D}{\mu}\left(-1 + \sqrt{1 + \frac{2\mu^2}{h}\left(\frac{A_D}{D} + \frac{U}{\mu}\right)}\right)$$

   (b) *If $\sqrt{2A_D Dh} \leq DU$ then*

$$s^* = \frac{-D}{\mu} \ln\left(\frac{\frac{h}{\mu} + \sqrt{\frac{2A_D h}{D}}}{\frac{h}{\mu} + U}\right)$$

   and

$$Q^* = \sqrt{\frac{2A_D D}{h}}$$

Proof of Lemma 4.3. For any given $Q \geq 0$, $\widetilde{f}(s,Q)$ can be represented in the form of $f(s)$ given by Lemma 4.2 since h,μ, D and $\left(\frac{Q}{D} + \frac{1}{\mu}\right)$ are all positive. Therefore, depending on the relation between

$$h\left(\frac{Q}{D} + \frac{1}{\mu}\right) \quad \text{and} \quad \frac{h}{\mu} + U$$

the optimal s minimizing $\widetilde{f}(s,Q)$ and $\widetilde{TC}(s,Q)$ can be found from Lemma 4.2 as follows:

(a) if $hQ/D \geq U$ then $s^* = 0$

(b) if $hQ/D < U$ then

$$s^* = \frac{-D}{\mu} \ln\left(\frac{\frac{h}{\mu} + \frac{hQ}{D}}{\frac{h}{\mu} + U}\right) \tag{4.34}$$

Next, we minimize $\widetilde{TC}(s,Q)$ in Q. We found that for any given Q, $\widetilde{TC}(s,Q)$ is minimized at either $s^* = 0$ or s^* given by (4.34). Therefore, we consider these cases separately in order to minimize $\widetilde{TC}(s^*,Q)$ according to Q.

Taking the first derivative of $\widetilde{TC}(s, Q)$ with respect to Q yields

$$\frac{\partial \widetilde{TC}(s,Q)}{\partial Q} = \frac{F(Q)}{\left(\frac{Q}{D} + \frac{1}{\mu}\right)^2} \tag{4.35}$$

where

$$F(Q) := \frac{\partial \widetilde{f}(s,Q)}{\partial Q}\left(\frac{Q}{D} + \frac{1}{\mu}\right) - \frac{\widetilde{f}(s,Q)}{D} \tag{4.36}$$

It is immediately evident from (4.35) and (4.36) that the behavior of $\widetilde{TC}(s, Q)$ in Q is determined by the sign of $F(Q)$. Thus, in the sequel, we analyze $F(Q)$ for the two cases of s^* mentioned above.

If $s^* = 0$ then from (4.32) we have that

$$f(s^*, Q) = A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + \frac{D}{\mu}\left(\frac{h}{\mu} + U\right) \tag{4.37}$$

and

$$\frac{\partial f(s^*, Q)}{\partial Q} = \frac{hQ}{D} + c_D \tag{4.38}$$

Substituting (4.37) and (4.38) in (4.36) and making necessary simplifications yields

$$F(Q) = \frac{hQ^2}{2D^2} + \frac{hQ}{\mu D} - \left(\frac{A_D}{D} + \frac{U}{\mu}\right) \tag{4.39}$$

Observe from (4.39) that $F(Q)$ is a second-order convex polynomial since $h, \mu, D > 0$ and it is increasing for $Q \geq 0$. Therefore,

(a) if $A_D/D + U/\mu \leq 0$ then $F(Q) \geq 0$ for all $Q \geq 0$. Thus, from (4.35), $\widetilde{TC}(s^*, Q)$ is nondecreasing in Q and minimized at $Q^* = 0$.

Since $s^* = 0$ holds only for $Q \geq DU/h$, optimal Q shall satisfy

$$Q^* \geq DU/h \tag{4.40}$$

Since $A_D/D + U/\mu \leq 0$ this implies that $U \leq 0$ and the above condition is satisfied. Thus, $s^* = Q^* = 0$.

(b) if $A_D/D + U/\mu > 0$ then $F(Q)$ has a unique positive root Q^* and $\widetilde{TC}(s^*, Q)$ is minimized at Q^*. Moreover, it can be showed that $\widetilde{TC}(s^*, Q)$ is convex in Q. Thus, applying quadratic formula for (4.39) and simplifying yields Q^* as follows

$$Q^* = \frac{D}{\mu}\left(-1+\sqrt{1+\frac{2\mu^2}{h}\left(\frac{A_D}{D}+\frac{U}{\mu}\right)}\right) \tag{4.41}$$

Since $s^* = 0$ is true only if the condition (4.40) is satisfied, substituting (4.41) in (4.40) gives,

$$\sqrt{1+\frac{2\mu^2}{h}\left(\frac{A_D}{D}+\frac{U}{\mu}\right)} \geq 1+\frac{\mu U}{h}$$

Squaring both sides of inequality and making necessary simplifications yields the condition

$$\sqrt{2A_D D h} \geq DU \tag{4.42}$$

Note that when $\sqrt{2A_D D h} = DU$, (4.41) simplifies to

$$Q^* = \sqrt{\frac{2A_D D}{h}}$$

If s^* is given by (4.34) then from (4.32) we have

$$\begin{aligned} \widetilde{f}(s^*, Q) &= A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q \\ &+ \left(\frac{hQ}{\mu}+\frac{hD}{\mu^2}\right)\left(1-\ln\left(\frac{\frac{h}{\mu}+\frac{hQ}{D}}{\frac{h}{\mu}+U}\right)\right) \end{aligned} \tag{4.43}$$

and

$$\frac{\partial \widetilde{f}(s^*, Q)}{\partial Q} = \frac{hQ}{D} + c_D - \frac{h}{\mu}\ln\left(\frac{\frac{h}{\mu}+\frac{hQ}{D}}{\frac{h}{\mu}+U}\right) \tag{4.44}$$

Substituting (4.43) and (4.44) in (4.36) and after some algebra we obtain that

$$F(Q) = \frac{hQ^2}{2D^2} - \frac{A_D}{D} \tag{4.45}$$

Since $h, D > 0$ and $A_D \geq 0$, $F(Q)$ is increasing convex function for $Q \geq 0$ and has a unique nonnegative root at

$$Q^* = \sqrt{\frac{2A_D D}{h}} \tag{4.46}$$

Thus, $\widetilde{TC}(s^*, Q)$ is minimized at (4.46) and it can be shown that $\widetilde{TC}(s^*, Q)$ is convex in Q. Substituting (4.46) in (4.34) yields

$$s^* = \frac{-D}{\mu}\ln\left(\frac{\frac{h}{\mu}+\sqrt{\frac{2A_D h}{D}}}{\frac{h}{\mu}+U}\right) \tag{4.47}$$

Since (4.34) holds only for $Q < DU/h$, (4.46) and (4.47) are valid if

$$\sqrt{\frac{2A_D D}{h}} < DU/h$$

That is,

$$\sqrt{2A_D Dh} < DU \tag{4.48}$$

Observe that if $A_D/D + U/\mu \leq 0$ then $U \leq 0$. This violates the condition (4.48) and therefore, (4.46) and (4.47) cannot be valid for this case.

On the other hand, when $A_D/D + U/\mu > 0$ the optimal (s, Q) is uniquely determined by the conditions (4.42) and (4.48). $\square$

4.5.2 Candidate Points for the Optimal Solution

Define,

$$\begin{aligned}
\gamma &:= c_L - c_D - \frac{\hat{\pi}}{\mu} \\
\eta &:= c_L - c_D + \frac{h}{\mu} \\
\Pi &:= \frac{\hat{\pi}}{\mu} + \pi + \frac{\theta(1-\hat{p})}{\hat{p}} \\
\Gamma &:= \frac{h}{\mu} + \hat{p}\,\Pi - (1-\hat{p})c_D \\
\mathcal{K} &:= A_D - A_L - \frac{D(c_L - c_D)^2}{2h} + \frac{D}{\mu}\left(\hat{p}\pi + (1-\hat{p})(\theta - c_L)\right)
\end{aligned}$$

The following theorems identify possible candidate points for the optimal solution.

Theorem 4.1. *For Case 1, the optimal solution can be found at one of the following candidate points,*

1. *If $c_L - c_D \geq \hat{\pi}/\mu$ then there are two candidates*

   (a) *$r^* = 0$ and (R^*, s^*, Q^*) are found as given by Moinzadeh (1997).*

(b) $r^* \to \infty$ *and* R *becomes an irrelevant policy parameter. Let* $U = \hat{p}\,\Pi - (1-\hat{p})c_D$ *then* (s^*, Q^*) *can be given as*

i. *If* $A_D/D + U/\mu \le 0$ *then* $s^* = Q^* = 0$.

ii. *If* $A_D/D + U/\mu > 0$ *then*

A. *if* $\sqrt{2A_D h/D} > U$ *then* $s^* = 0$ *and*

$$Q^* = \frac{D}{\mu}\left(-1 + \sqrt{1 + \frac{2\mu^2}{h}\left(\frac{A_D}{D} + \frac{U}{\mu}\right)}\right)$$

B. *if* $\sqrt{2A_D h/D} \le U$ *then*

$$s^* = \frac{-D}{\mu}\ln\left(\frac{\frac{h}{\mu} + \sqrt{\frac{2A_D h}{D}}}{\frac{h}{\mu} + U}\right)$$

and

$$Q^* = \sqrt{\frac{2A_D D}{h}}$$

2. *If* $c_L - c_D < \hat{\pi}/\mu$ *then there are two candidates*

(a) $r^* = 0$ *and* (R^*, s^*, Q^*) *are found as given by Moinzadeh (1997).*

(b) R^* *is the solution of*

$$A_L - \frac{D}{\mu}\left(\hat{p}\gamma - \eta + \Gamma\right) + \eta R^* - \frac{\hat{p}D\gamma}{\mu}\ln\left(\frac{\eta\, e^{\mu R^*/D} - \Gamma}{\hat{p}\gamma}\right) = 0 \tag{4.49}$$

and r^* *can be found from*

$$r^* = -\frac{\hat{p}D}{\mu}\ln\left(\frac{\eta\, e^{\mu R^*/D} - \Gamma}{\hat{p}\gamma}\right) \tag{4.50}$$

Let $U = c_L - c_D$ *then* $A_D/D + U/\mu > 0$ *and* (s^*, Q^*) *can be given as*

i. *If* $\sqrt{2A_D h/D} > U$ *then* $s^* = R^*$ *and*

$$Q^* = \frac{D}{\mu}\left(-1 + \sqrt{1 + \frac{2\mu^2}{h}\left(\frac{A_D}{D} + \frac{U}{\mu}\right)}\right) \tag{4.51}$$

ii. If $\sqrt{2A_D h/D} \leq U$ *then*

$$s^* = R^* - \frac{D}{\mu} \ln \left(\frac{\frac{h}{\mu} + \sqrt{\frac{2A_D h}{D}}}{\frac{h}{\mu} + U} \right) \tag{4.52}$$

and

$$Q^* = \sqrt{\frac{2A_D D}{h}} \tag{4.53}$$

Proof of Theorem 4.1:

First, we shall rewrite the expected cycle cost function in a form more convenient for optimization purposes:

$$E[CC] \quad = \quad f(s, Q) + \left[A_L + \gamma r + \eta R + \frac{D\Gamma}{\mu} \left(e^{\mu r/\hat{p}D} - 1 \right) \right] \frac{e^{-\mu s/D}}{e^{\mu r/\hat{p}D} - e^{-\mu R/D}} \tag{4.54}$$

with

$$f(s, Q) \quad := \quad A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + h \left(\frac{Q}{D} + \frac{1}{\mu} \right) s \tag{4.55}$$

and the total cost rate function can be given as

$$TC(r, R, s, Q) = \frac{E[CC]}{\frac{Q}{D} + \frac{1}{\mu}} \tag{4.56}$$

Observe that as R, s or Q goes to infinity both $E[CC]$ and $TC(r, R, s, Q)$ go to infinity. This implies that for a given r if the objective function has a minimum then the minimum is found at a finite point (R^*, s^*, Q^*). However, for a given (R, s, Q) the objective function does not necessarily attains its minimum at a finite r. Observe that as r goes to infinity,

$$\lim_{r \to \infty} \frac{r}{e^{\mu r/\hat{p}D} - e^{-\mu R/\hat{p}D}} = 0$$

$$\lim_{r \to \infty} \frac{e^{\mu r/\hat{p}D} - 1}{e^{\mu r/\hat{p}D} - e^{-\mu R/\hat{p}D}} = 1$$

and the expected cycle cost becomes

$$E[CC]|_{r \to \infty} = f(s, Q) + \frac{D\Gamma}{\mu} e^{-\mu s/D} \tag{4.57}$$

Thus, the objective function converges to a positive value as $r \to \infty$. In the sequel, we identify the parametric regions where the objective function attains its minimum as $r \to \infty$.

Since $E[T]$ is independent of r, we begin the analysis of the objective function by investigating the behavior of $E[CC]$ in r.

First Derivative of $E[CC]$ w.r.t. r:

Taking the derivative of (4.54) with respect to r yields,

$$\begin{aligned}\frac{\partial E[CC]}{\partial r} &= \frac{\left(\gamma + \frac{\Gamma}{\hat{p}} e^{\mu r/\hat{p}D}\right) e^{-\mu s/D}}{e^{\mu r/\hat{p}D} - e^{-\mu R/D}} \\ &- \frac{\left[A_L + \gamma r + \eta R + \frac{D\Gamma}{\mu}\left(e^{\mu r/\hat{p}D} - 1\right)\right] e^{-\mu s/D} \frac{\mu}{\hat{p}D} e^{\mu r/\hat{p}D}}{(e^{\mu r/\hat{p}D} - e^{-\mu R/D})^2} \\ &= \frac{e^{-\mu s/D} e^{\mu r/\hat{p}D} f(\frac{\mu r}{\hat{p}D})}{(e^{\mu r/\hat{p}D} - e^{-\mu R/D})^2} \end{aligned} \tag{4.58}$$

with

$$f\left(\frac{\mu r}{\hat{p}D}\right) := \gamma\left(1 - e^{-\mu r/\hat{p}D} e^{-\mu R/D} - \frac{\mu r}{\hat{p}D}\right) - \frac{\mu}{\hat{p}D}\left(A_L + \eta R - \frac{D\Gamma}{\mu}(1 - e^{-\mu R/D})\right) \tag{4.59}$$

For any given $0 \le s < \infty$, let r^* be a critical point satisfying the first order condition $\partial E[CC]/\partial r = 0$. It is clear from (4.58) that the first order condition is satisfied when $f(\mu r^*/\hat{p}D) = 0$

Second Derivative of $E[CC]$ w.r.t. r:

The second derivative of $E[CC]$ with respect to r can be given as,

$$\frac{\partial^2 E[CC]}{\partial r^2} = \frac{e^{-\mu s/D} e^{\mu r/\hat{p}D}}{(e^{\mu r/\hat{p}D} - e^{-\mu R/D})^2}\left[\frac{\mu}{\hat{p}D} f + \frac{\partial f}{\partial r} - \frac{2\mu e^{\mu r/\hat{p}D} f}{\hat{p}D(e^{\mu r/\hat{p}D} - e^{-\mu R/D})}\right] \tag{4.60}$$

Observe from (4.59) that

$$\frac{\partial f}{\partial r} = \frac{\gamma\mu}{\hat{p}D}\left(e^{-\mu r/\hat{p}D} e^{-\mu R/D} - 1\right) \tag{4.61}$$

Thus, evaluating (4.60) at $r = r^*$ and using the identity $f(\mu r^*/\hat{p}D) = 0$ and (4.61) yield

$$\frac{\partial^2 E[CC]}{\partial r^2}|_{r=r^*} = -\frac{\mu e^{-\mu s/D} e^{\mu r^*/\hat{p}D}}{\hat{p}D(e^{\mu r^*/\hat{p}D} - e^{-\mu R/D})}\gamma \tag{4.62}$$

Observe from (4.62) that for any $0 \le s < \infty$,

$$\frac{\mu e^{-\mu s/D} e^{\mu r^*/\hat{p}D}}{\hat{p}D(e^{\mu r^*/\hat{p}D} - e^{-\mu R/D})} > 0$$

and therefore the sign of (4.62) depends only on the the sign of γ. Therefore, for any given $0 \leq s < \infty$,

1. If $\gamma \geq 0$ then TC is quasiconcave in r since
$$\frac{\partial^2 E[CC]}{\partial r^2}|_{r=r^*} \leq 0$$
Thus, TC is minimized either at $r = 0$ or as $r \rightarrow \infty$ depending on the relation between $E[CC]|_{r=0}$ and $E[CC]|_{r\rightarrow\infty}$.

2. If $\gamma < 0$ then TC is strictly quasiconvex in r since
$$\frac{\partial^2 E[CC]}{\partial r^2}|_{r=r^*} > 0$$
Thus, r^* is unique solution of (4.59) $f(\mu r^*/\hat{p}D) = 0$ and it is the global minimum of TC. Therefore, if $r^* < 0$ then TC is minimized at $r = 0$ otherwise r^* is the unique minimizer.

We established that for $\gamma < 0$, TC is minimized either at $r = 0$ or at the critical point $r^* \geq 0$ found by solving $f(\mu r^*/\hat{p}D) = 0$. Thus, in the sequel, we analyze the case where $\gamma < 0$ by assuming that there exists a feasible point $0 \leq r^* < \infty$ satisfying the first order condition $f(\mu r^*/\hat{p}D) = 0$.

First Derivative of $E[CC]$ w.r.t. R:

Taking the derivative of (4.54) with respect to R yields,

$$\begin{aligned}\frac{\partial E[CC]}{\partial R} &= \frac{\eta e^{-\mu s/D}}{e^{\mu r/\hat{p}D} - e^{-\mu R/D}} \\ &- \frac{\left[A_L + \gamma r + \eta R + \frac{D\Gamma}{\mu}\left(e^{\mu r/\hat{p}D} - 1\right)\right] e^{-\mu s/D}\ \frac{\mu}{D} e^{-\mu R/D}}{(e^{\mu r/\hat{p}D} - e^{-\mu R/D})^2} \\ &= \frac{e^{-\mu s/D}\ e^{-\mu R/D}\ h(R)}{(e^{\mu r/\hat{p}D} - e^{-\mu R/D})^2} \end{aligned} \tag{4.63}$$

with

$$h(R) := \eta\left(e^{\mu r/\hat{p}D}\ e^{\mu R/D} - 1 - \frac{\mu R}{D}\right) - \frac{\mu}{D}\left(A_L + \gamma r + \frac{D\Gamma}{\mu}(e^{\mu r/\hat{p}D} - 1)\right) \tag{4.64}$$

For any given $0 \leq r, s < \infty$, let R^* be a critical point satisfying the first order condition $\partial E[CC]/\partial R = 0$. It is clear from (4.63) that the first order condition is satisfied when $h(R^*) = 0$.

Second Derivative of $E[CC]$ w.r.t. R:

The second derivative of $E[CC]$ with respect to R can be given as,

$$\frac{\partial^2 E[CC]}{\partial R^2} = \frac{e^{-\mu s/D}\ e^{-\mu R/D}}{(e^{\mu r/\hat{p}D} - e^{-\mu R/D})^2}\left[-\frac{\mu}{D}h + \frac{\partial h}{\partial R} - \frac{2\mu e^{-\mu R/D}\ h}{D(e^{\mu r/\hat{p}D} - e^{-\mu R/D})}\right] \tag{4.65}$$

Observe from (4.64) that

$$\frac{\partial h}{\partial R} = \frac{\eta\mu}{D}e^{\mu R/D}\left(e^{\mu r/\hat{p}D} - e^{-\mu R/D}\right) \tag{4.66}$$

Thus, evaluating (4.65) at $R = R^*$ and using (4.66) and the identity $h(R^*) = 0$ yield

$$\frac{\partial^2 E[CC]}{\partial R^2}|_{R=R^*} = \frac{\eta\mu e^{-\mu s/D}}{D(e^{\mu r^*/\hat{p}D} - e^{-\mu R^*/D})} \tag{4.67}$$

Note that for any given $0 \leq r, s < \infty$, if (4.64) has a feasible solution $R^* \geq 0$ then

$$\frac{\partial^2 E[CC]}{\partial R^2}|_{R=R^*} > 0$$

and R^* is the local minimum of TC.

Computation of the Critical Point (r^*, R^*):

For any given (s, Q), let (r^*, R^*) be a critical point satisfying $f\left(\mu r^*/\hat{p}D\right) = 0$ and $h(R^*) = 0$. That is,

$$\gamma\left(1 - e^{-\mu r^*/\hat{p}D}\ e^{-\mu R^*/D} - \frac{\mu r^*}{\hat{p}D}\right) - \frac{\mu}{\hat{p}D}\left(A_L + \eta R^* - \frac{D\Gamma}{\mu}(1 - e^{-\mu R^*/D})\right) = 0 \tag{4.68}$$

$$\eta\left(e^{\mu r^*/\hat{p}D}\ e^{\mu R^*/D} - 1 - \frac{\mu R^*}{D}\right) - \frac{\mu}{D}\left(A_L + \gamma r^* + \frac{D\Gamma}{\mu}(e^{\mu r^*/\hat{p}D} - 1)\right) = 0 \tag{4.69}$$

Reorganizing the terms in (4.68) yields

$$\hat{p}\gamma\left(1 - e^{-\mu r^*/\hat{p}D}\ e^{-\mu R^*/D} - \frac{\mu r^*}{\hat{p}D}\right) = \frac{\mu}{D}\left(A_L + \eta R^* - \frac{D\Gamma}{\mu}\right) + \Gamma e^{-\mu R^*/D} \tag{4.70}$$

and reorganizing (4.69) gives

$$\eta\left(e^{\mu r^*/\hat{p}D}\ e^{\mu R^*/D} - 1\right) - \frac{\mu}{D}\left(\gamma r^* + \frac{D\Gamma}{\mu}e^{\mu r^*/\hat{p}D}\right) = \frac{\mu}{D}\left(A_L + \eta R^* - \frac{D\Gamma}{\mu}\right) \tag{4.71}$$

Substituting (4.71) in (4.70) and after some algebra, we obtain that

$$
\begin{aligned}
\hat{p}\gamma\left(1 - e^{-\mu r^*/\hat{p}D}\; e^{-\mu R^*/D}\right) &= \eta\left(e^{\mu r^*/\hat{p}D}\; e^{\mu R^*/D} - 1\right) - \Gamma(e^{\mu r^*/\hat{p}D} - e^{-\mu R^*/D}) \\
\hat{p}\gamma e^{-\mu r^*/\hat{p}D} &= \eta e^{\mu R^*/D} - \Gamma \qquad (4.72)
\end{aligned}
$$

Using (4.72) in (4.70) yields,

$$
\hat{p}\gamma\left(1 - \frac{\eta}{\hat{p}\gamma} + \frac{\Gamma}{\hat{p}\gamma}e^{-\mu R^*/D} - \frac{\mu r^*}{\hat{p}D}\right) = \frac{\mu}{D}\left(A_L + \eta R^* - \frac{D\Gamma}{\mu}\right) + \Gamma e^{-\mu R^*/D}
$$

making necessary simplification gives

$$
\begin{aligned}
\hat{p}\gamma - \eta - \frac{\gamma\mu r^*}{D} &= \frac{\mu}{D}\left(A_L + \eta R^* - \frac{D\Gamma}{\mu}\right) \\
\frac{D}{\mu}\left(\hat{p}\gamma - \eta + \Gamma\right) - \gamma r^* &= A_L + \eta R^*
\end{aligned}
$$

and reorganizing the terms yields that

$$
A_L - \frac{D}{\mu}\left(\hat{p}\gamma - \eta + \Gamma\right) + \eta R^* + \gamma r^* = 0 \qquad (4.73)
$$

Observe from (4.72) that r^* can be given as a function of R^* as follows

$$
r^* = -\frac{\hat{p}D}{\mu}\ln\left(\frac{\eta\, e^{\mu R^*/D} - \Gamma}{\hat{p}\gamma}\right) \qquad (4.74)
$$

Thus, substituting (4.74) in (4.73) yields

$$
A_L - \frac{D}{\mu}\left(\hat{p}\gamma - \eta + \Gamma\right) + \eta R^* - \frac{\hat{p}D\gamma}{\mu}\ln\left(\frac{\eta\, e^{\mu R^*/D} - \Gamma}{\hat{p}\gamma}\right) = 0 \qquad (4.75)
$$

Therefore, the critical point (r^*, R^*) satisfying (4.68) and (4.69) can be found by solving (4.75) for R^* and then calculating r^* from (4.74).

Computation of (s^*, Q^*) **Given** (r^*, R^*)**:**

Define the function

$$
g(r, R) := \frac{A_L + \gamma r + \eta R + \frac{D\Gamma}{\mu}\left(e^{\mu r/\hat{p}D} - 1\right)}{e^{\mu r/\hat{p}D} - e^{-\mu R/D}} \qquad (4.76)
$$

and observe that (4.54) can be expressed as

$$
E[CC] = f(s, Q) + g(r, R)e^{-\mu s/D} \qquad (4.77)
$$

For any given $0 \leq s, Q < \infty$, let (r^*, R^*) be a feasible critical point, that is, $0 \leq R^* \leq s < \infty$. Note that we only need to check R^* since we assume that r^* is always in the feasible region. Then using (4.73) in (4.76), we obtain that

$$g(r^*, R^*) = \frac{D\left(\hat{p}\gamma - \eta + \Gamma e^{\mu r^*/\hat{p}D}\right)}{\mu(e^{\mu r^*/\hat{p}D} - e^{-\mu R^*/D})} \tag{4.78}$$

Substituting Γ from (4.72) in (4.78) and making necessary simplifications yields that

$$\begin{aligned} g(r^*, R^*) &= \frac{D\left(\hat{p}\gamma - \eta + (\eta e^{\mu R^*/D} - \hat{p}\gamma e^{-\mu r^*/\hat{p}D})e^{\mu r^*/\hat{p}D}\right)}{\mu(e^{\mu r^*/\hat{p}D} - e^{-\mu R^*/D})} \\ &= \frac{D\eta\left(e^{\mu R^*/D}e^{\mu r^*/\hat{p}D} - 1\right)}{\mu(e^{\mu r^*/\hat{p}D} - e^{-\mu R^*/D})} \\ &= \frac{D}{\mu}\,\eta\, e^{\mu R^*/D} \end{aligned} \tag{4.79}$$

Thus, from (4.77) and (4.79), the expected cycle cost can evaluated at (r^*, R^*) can be given as

$$E[CC]|_{r^*,R^*} = f(s,Q) + \frac{D}{\mu}\,\eta\, e^{\mu R^*/D}e^{-\mu s/D} \tag{4.80}$$

Next, we optimize (4.80) with respect to s and Q given (r^*, R^*). First, rewrite (4.80) as follows,

$$E[CC]|_{r^*,R^*} = f(s,Q) + \frac{D}{\mu}\,\eta\, e^{\mu R^*/D}e^{-\mu s/D} - h\left(\frac{Q}{D} + \frac{1}{\mu}\right)R^* + h\left(\frac{Q}{D} + \frac{1}{\mu}\right)R^* \tag{4.81}$$

Define $\widetilde{s} := s - R^*$ and reorganize (4.81) as follows

$$E[CC]|_{r^*,R^*} = \widetilde{f}(\widetilde{s}, Q) + h\left(\frac{Q}{D} + \frac{1}{\mu}\right)R^* \tag{4.82}$$

with

$$\begin{aligned} \widetilde{f}(\widetilde{s}, Q) &:= A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + h\left(\frac{Q}{D} + \frac{1}{\mu}\right)\widetilde{s} \\ &+ \frac{D}{\mu}\left(\frac{h}{\mu} + c_L - c_D\right)e^{-\mu\widetilde{s}/D} \end{aligned} \tag{4.83}$$

Note that, for clarity, we substituted η by its definition $c_L - c_D + \frac{h}{\mu}$ in (4.83).

Then, the total cost rate function becomes,

$$TC(r^*, R^*, s, Q) = \frac{\widetilde{f}(\widetilde{s}, Q)}{\frac{Q}{D} + \frac{1}{\mu}} + R^* \tag{4.84}$$

Observe that for a given R^*, optimizing (4.84) according to (s, Q) is equivalent to optimizing

$$\widetilde{TC}(\widetilde{s}, Q) = \frac{\widetilde{f}(\widetilde{s}, Q)}{\frac{Q}{D} + \frac{1}{\mu}} \tag{4.85}$$

according to $(\widetilde{s}, Q)$. Therefore, the optimal $(\widetilde{s}, Q)$ can be found from Lemma 4.3 by substituting $U = c_L - c_D$, and the optimal s follows from $s^* = R^* + \widetilde{s}^{\,*}$. □

Theorem 4.2. *For Case 2, the optimal solution can be found at one of the following candidate points,*

1. *If* $c_L - c_D > \hat{\pi}/\mu$ *then*
   - (a) $r^* = 0$ *and* (R^*, s^*, Q^*) *are found as given by Moinzadeh (1997).*
   - (b) $r^* \to \infty$ *and* R *becomes an irrelevant policy parameter. Let* $U = \hat{p}\,\Pi - (1 - \hat{p})c_D$ *then* (s^*, Q^*) *can be given as*
     - i. *If* $A_D/D + U/\mu \leq 0$ *then* $s^* = Q^* = 0$.
     - ii. *If* $A_D/D + U/\mu > 0$ *then*
       - A. *if* $\sqrt{2A_D h/D} > U$ *then* $s^* = 0$ *and*

$$Q^* = \frac{D}{\mu}\left(-1 + \sqrt{1 + \frac{2\mu^2}{h}\left(\frac{A_D}{D} + \frac{U}{\mu}\right)}\right)$$

       - B. *if* $\sqrt{2A_D h/D} \leq U$ *then*

$$s^* = \frac{-D}{\mu}\ln\left(\frac{\frac{h}{\mu} + \sqrt{\frac{2A_D h}{D}}}{\frac{h}{\mu} + U}\right)$$

and

$$Q^* = \sqrt{\frac{2A_D D}{h}}$$

2. *If* $c_L - c_D < \hat{\pi}/\mu$ *then*

   (a) $r^* = 0$ *and* (R^*, s^*, Q^*) *are found as given by Moinzadeh (1997).*

   (b)
$$R^* = s^* + \sqrt{\frac{2A_D D}{h}} - \frac{D(c_L - c_D)}{h} \tag{4.86}$$

$$r^* = \frac{\mathcal{K} - \eta R^*}{\gamma} \tag{4.87}$$

and define

$$U(s^*) := \hat{p}\left(\Pi + \gamma e^{-\mu r^*/\hat{p}D}\right) - (1-\hat{p})c_D$$

then s^* *can be found by solving*

$$s^* = \frac{D}{\mu} \ln \left(\frac{\frac{h}{\mu} + U(s^*)}{\frac{h}{\mu} + \sqrt{\frac{2A_D h}{D}}} \right) \tag{4.88}$$

and

$$Q^* = \sqrt{\frac{2A_D D}{h}} \tag{4.89}$$

   (c) $s^* = 0$ *and*

$$R^* = Q^* - \frac{D(c_L - c_D)}{h} \tag{4.90}$$

$$r^* = \frac{\mathcal{K} - \eta R^*}{\gamma} \tag{4.91}$$

and Q^* *can be found by solving*

$$\left(\frac{hQ^{*2}}{2D} + \frac{hQ^*}{\mu} - A_D - \frac{D}{\mu}\left(\hat{p}\,\Pi - (1-\hat{p})c_D\right)\right) e^{\mu r^*/\hat{p}D} - \frac{\hat{p}D\gamma}{\mu} = 0 \tag{4.92}$$

   (d) $R^* = s^*$ *and let*

$$\begin{aligned} g(r^*, s^*) &:= A_L + \gamma r^* + \eta s^* + \frac{D\Gamma}{\mu}\left(e^{\mu r^*/\hat{p}D} - 1\right) \\ u(r^*, s^*) &:= \frac{e^{-\mu s^*/D}}{e^{\mu r^*/\hat{p}D} - e^{-\mu s^*/D}} \end{aligned} \tag{4.93}$$

then (r^*, s^*, Q^*) *can be found by solving the following equations simultaneously*

$$\hat{p}\gamma + \Gamma\, e^{\mu r^*/\hat{p}D} \;=\; \frac{\mu}{D}(1 + u(r^*, s^*))g(r^*, s^*) \tag{4.94}$$

$$r^* = -\frac{\hat{p}D}{\mu}\ln\left(\frac{h\left(\frac{Q^*}{D} + \frac{1}{\mu}\right)e^{\mu s^*/D} - \Gamma}{h\left(\frac{Q^*}{D} + \frac{1}{\mu}\right) - \eta + \hat{p}\gamma}\right) \tag{4.95}$$

$$Q^* = \frac{D}{\mu}\left(-1 + \sqrt{1 + \frac{2\mu^2}{h}\left(\frac{A_D}{D} - \frac{h}{\mu^2} + \frac{g(r^*, s^*)\,u(r^*, s^*)}{D}\right)}\right) \tag{4.96}$$

(e) $R^* = s^*$, $Q^* = 0$ *and* (r^*, s^*) *can be found by solving* (4.94) *and* (4.95) *simultaneously.*

Proof of Theorem 4.2:

The expected cycle cost for Case 2 can be rewritten as follows,

$$E[CC] \;=\; f(s,Q) + g(r,R,s)\,u(r,s) \tag{4.97}$$

with

$$f(s,Q) \;:=\; A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + h\left(\frac{Q}{D} + \frac{1}{\mu}\right)s \tag{4.98}$$

$$g(r,R,s) \;:=\; A_L + \gamma r + \eta s + c_L(R - s) + \frac{h}{2D}(R^2 - s^2) + \frac{D\Gamma}{\mu}\left(e^{\mu r/\hat{p}D} - 1\right) \tag{4.99}$$

$$u(r,s) \;:=\; \frac{e^{-\mu s/D}}{e^{\mu r/\hat{p}D} - e^{-\mu s/D}} \tag{4.100}$$

and the total cost rate function can be given as

$$TC(r,R,s,Q) = \frac{E[CC]}{E[T]} \tag{4.101}$$

with

$$E[T] = \frac{Q}{D} + \frac{1}{\mu} + \left(\frac{R-s}{D}\right)u(r,s) \tag{4.102}$$

For the sake of clarity, in the sequel, we suppress the arguments of the functions of the variables r,R,s,Q.

First Derivative of TC w.r.t. r:

Taking the derivative of (4.101) with respect to r yields

$$\frac{\partial TC}{\partial r} = \frac{\frac{\partial E[CC]}{\partial r} E[T] - \frac{\partial E[T]}{\partial r} E[CC]}{E[T]^2} \tag{4.103}$$

The derivative of expected cycle cost with respect to r can be found as

$$\begin{aligned} \frac{\partial E[CC]}{\partial r} &= \frac{\partial g}{\partial r} u + \frac{\partial u}{\partial r} g \\ &= \left(\gamma + \frac{\Gamma}{\hat{p}}\, e^{\mu r/\hat{p}D}\right) u - \frac{\mu}{\hat{p}D} u(1+u) g \end{aligned} \tag{4.104}$$

By adding and subtracting the term $(\mu u/\hat{p}D)f$ on the right-hand side of (4.104), we obtain that

$$\frac{\partial E[CC]}{\partial r} = \frac{\mu u}{\hat{p}D}\left(\frac{\hat{p}D\gamma}{\mu} + \frac{D}{\mu}\Gamma e^{\mu r/\hat{p}D} - g + f\right) - \frac{\mu u}{\hat{p}D}(f + g\,u) \tag{4.105}$$

From (4.97) and after some algebra (4.105) boils down to

$$\frac{\partial E[CC]}{\partial r} = \frac{\mu u}{\hat{p}D}\left(v - E[CC]\right) \tag{4.106}$$

with

$$\begin{aligned} v = v(r,R,s,Q) \quad &:= \quad A_D - A_L - \gamma r + c_D(Q+s) - c_L R - \frac{h}{2D}(R^2 - s^2) \\ &+ \quad \frac{hQ^2}{2D} + \frac{hQs}{D} + \frac{D}{\mu}\left(\hat{p}(c_L + \pi) + (1-\hat{p})\theta\right) \end{aligned} \tag{4.107}$$

The derivative of (4.102) with respect to r can be found as

$$\frac{\partial E[T]}{\partial r} = -\frac{\mu}{\hat{p}D} u(1+u)\left(\frac{R-s}{D}\right) \tag{4.108}$$

By adding and subtracting the term $(\mu u/\hat{p}D)(Q/D + 1/\mu)$ on the right-hand side of (4.108) and reorganizing the terms yield

$$\frac{\partial E[T]}{\partial r} = \frac{\mu u}{\hat{p}D}\left(\frac{Q+s-R}{D} + \frac{1}{\mu} - E[T]\right) \tag{4.109}$$

Using (4.109) and (4.106) in (4.103) yield

$$\begin{aligned}\frac{\partial TC}{\partial r} &= \frac{\mu u}{\hat{p}DE[T]^2}\left[(v - E[CC])\,E[T] - \left(\frac{Q+s-R}{D} + \frac{1}{\mu} - E[T]\right)E[CC]\right] \\ &= \frac{\mu u}{\hat{p}DE[T]}\left[v - \left(\frac{Q+s-R}{D} + \frac{1}{\mu}\right)TC\right] \qquad (4.110)\end{aligned}$$

First Order Condition of TC w.r.t. r:

For any given (R, s, Q), let r^* be a critical point satisfying the first order condition $\partial TC/\partial r = 0$. Since $\mu u/\hat{p}DE[T] > 0$ for $0 \le Q < \infty$ and $0 \le s \le R \le s + Q$, it is clear from (4.110) that the first order condition is satisfied when

$$v(r^*, R, s, Q) = \left(\frac{Q+s-R}{D} + \frac{1}{\mu}\right)TC(r^*, R, s, Q) \qquad (4.111)$$

Second Derivative of TC w.r.t. r:

Define the function

$$z := z(r, R, s, Q) := v(r, R, s, Q) - \left(\frac{Q+s-R}{D} + \frac{1}{\mu}\right)TC(r, R, s, Q)$$

Thus, (4.110) becomes,

$$\frac{\partial TC}{\partial r} = \frac{\mu u z}{\hat{p}DE[T]}$$

and the second derivative of TC with respect to r can be given as,

$$\frac{\partial^2 TC}{\partial r^2} = \frac{\mu}{\hat{p}DE[T]^2}\left[\left(\frac{\partial u}{\partial r}z + u\frac{\partial z}{\partial r}\right)E[T] - uz\frac{\partial E[T]}{\partial r}\right] \qquad (4.112)$$

Observe that

$$\frac{\partial z}{\partial r} = -\gamma - \left(\frac{Q+s-R}{D} + \frac{1}{\mu}\right)\frac{\partial TC}{\partial r} \qquad (4.113)$$

and

$$\frac{\partial z}{\partial r}\Big|_{r=r^*} = -\gamma \qquad (4.114)$$

Moreover, from (4.111) it is clear that $z(r^*, R, s, Q) = 0$. Therefore, evaluating (4.112) at $r = r^*$ yields

$$\frac{\partial^2 TC}{\partial r^2}\Big|_{r=r^*} = -\frac{\mu u \gamma}{\hat{p}DE[T]} \qquad (4.115)$$

Observe that for any $0 \le s \le R < \infty$ and $0 \le Q < \infty$, $\mu u/\hat{p}DE[T] > 0$ and therefore the sign of (4.115) depends only on the the sign of γ. Therefore, for any given $0 \le s \le R < \infty$ and $0 \le Q < \infty$,

1. If $\gamma \geq 0$ then TC is quasiconcave in r since

$$\frac{\partial^2 TC}{\partial r^2}|_{r=r^*} \leq 0$$

   Thus, TC is minimized either at $r = 0$ or as $r \to \infty$.

2. If $\gamma < 0$ then TC is strictly quasiconvex in r since

$$\frac{\partial^2 TC}{\partial r^2}|_{r=r^*} > 0$$

   Thus, r^* is unique solution of (4.111) and it is the global minimum of TC. Therefore, if $r^* < 0$ then TC is minimized at $r = 0$ otherwise r^* is the unique minimizer.

We established that for $\gamma \geq 0$, there are two possibilities for the optimal r. It is either equal to zero and the system boils down to the one analyzed by Moinzadeh (1997) or, the optimal r goes to infinity, R becomes an irrelevant policy parameter and the optimal (s, Q) follows from Lemma 4.3. Similarly, for $\gamma < 0$ the optimal r is either at zero or at the critical point $r^* \geq 0$.

Next, we analyze the case where $\gamma < 0$ by assuming that there exists a critical point $r^* \geq 0$ satisfying the first order condition (4.111).

First Derivative of TC w.r.t. R:

Taking the derivative of (4.101) with respect to R yields,

$$\frac{\partial TC}{\partial R} = \frac{\frac{\partial E[CC]}{\partial R} E[T] - \frac{\partial E[T]}{\partial R} E[CC]}{E[T]^2} \tag{4.116}$$

From (4.97) and (4.102), the derivatives of $E[CC]$ and $E[T]$ with respect to R can be given as

$$\frac{\partial E[CC]}{\partial R} = \frac{\partial g}{\partial R} u = \frac{u}{D} \left(c_L D + hR \right) \tag{4.117}$$

and

$$\frac{\partial E[T]}{\partial R} = \frac{u}{D} \tag{4.118}$$

Substituting (4.117) and (4.118) in (4.116) yields,

$$\frac{\partial TC}{\partial R} = \frac{u}{DE[T]} \left[c_L D + hR - TC \right] \tag{4.119}$$

First Order Condition of TC w.r.t. R:

For any given (r, s, Q), let R^* be a critical point satisfying the first order condition $\partial TC/\partial R = 0$. Since at any feasible point $E[T] > 0$ and $u > 0$ for $0 \leq r, s < \infty$, it follows that $u/\hat{p}DE[T] > 0$. Thus, from (4.119) it is evident that the first order condition is satisfied when

$$c_L D + hR^* = TC(r, R^*, s, Q) \tag{4.120}$$

Second Derivative of TC w.r.t. R:

From (4.119), the second derivative of TC with respect to R yields,

$$\frac{\partial^2 TC}{\partial R^2} \quad = \quad \frac{u}{DE[T]^2}\left[\left(h - \frac{\partial TC}{\partial R}\right) E[T] - [c_L D + hR - TC]\,\frac{\partial E[T]}{\partial R}\right] \tag{4.121}$$

Evaluating (4.121) at $R = R^*$ and using (4.120) gives,

$$\frac{\partial^2 TC}{\partial R^2}|_{R=R^*} = \frac{uh}{DE[T]} \tag{4.122}$$

Thus, for any given $0 \leq r, s, Q < \infty$, if (4.120) has a feasible solution $R^* \geq s$ then

$$\frac{\partial^2 TC}{\partial R^2}|_{R=R^*} > 0$$

and R^* is a local minimum of TC.

First Derivative of TC w.r.t. Q:

Taking the derivative of (4.101) with respect to Q yields,

$$\frac{\partial TC}{\partial Q} = \frac{\frac{\partial E[CC]}{\partial Q} E[T] - \frac{\partial E[T]}{\partial Q} E[CC]}{E[T]^2} \tag{4.123}$$

From (4.97) and (4.102), the derivatives of $E[CC]$ and $E[T]$ with respect to Q can be given as

$$\frac{\partial E[CC]}{\partial Q} = \frac{\partial f}{\partial Q} = \frac{1}{D}\left(c_D D + h(Q + s)\right) \tag{4.124}$$

and

$$\frac{\partial E[T]}{\partial Q} = \frac{1}{D} \tag{4.125}$$

Using (4.124) and (4.125) in (4.123) gives,

$$\frac{\partial TC}{\partial Q} = \frac{1}{DE[T]}\left[c_D D + h(Q + s) - TC\right] \tag{4.126}$$

First Order Condition of TC w.r.t. Q:

For any $0 \leq r < \infty$ and $0 \leq s \leq R < \infty$, let Q^* be a critical point satisfying the first order condition $\partial TC/\partial Q = 0$. It is clear from (4.126) that the first order condition is satisfied when

$$c_D D + h(Q^* + s) = TC(r, R, s, Q^*) \tag{4.127}$$

Second Derivative of TC w.r.t. Q:

From (4.126), the second derivative of TC with respect to Q can be given as,

$$\frac{\partial^2 TC}{\partial Q^2} = \frac{1}{DE[T]^2}\left[\left(h - \frac{\partial TC}{\partial Q}\right) E[T] - [c_D D + h(Q+s) - TC]\frac{\partial E[T]}{\partial Q}\right] \tag{4.128}$$

Substituting $Q = Q^*$ in (4.128) and using (4.127) gives,

$$\frac{\partial^2 TC}{\partial Q^2}|_{Q=Q^*} = \frac{h}{DE[T]} \tag{4.129}$$

Therefore, for any $0 \leq r < \infty$ and $0 \leq s \leq R < \infty$, if (4.127) has a feasible solution $Q^* \geq 0$ then

$$\frac{\partial^2 TC}{\partial Q^2}|_{Q=Q^*} > 0$$

and Q^* is a local minimum of TC.

A Relation Between R^*, Q^* and s:

For any given $0 \leq s, r < \infty$, if a critical point (R^*,Q^*) satisfies (4.120) and (4.127), we have the relation

$$R^* = s + Q^* - \frac{(c_L - c_D)D}{h} \tag{4.130}$$

Moreover, from (4.130), it can be easily shown that

$$\frac{hQ^*(Q^* + 2s)}{2D} = \frac{h(R^{*2} - s^2)}{2D} + (c_L - c_D)R^* + \frac{D(c_L - c_D)^2}{2h} \tag{4.131}$$

Therefore, for any given $0 \leq r, s < \infty$, by using the relations (4.130) and (4.131), we can express (4.127) as a function of Q^* as follows,

$$\frac{hQ^{*2}}{2D} + \frac{hQ^*}{\mu} - A_D + \frac{hD}{\mu^2} = \left[A_L + \gamma r + \eta s - \frac{D}{2h}\left(\frac{hQ^*}{D} - (c_L - c_D)\right)^2 + \frac{D\Gamma}{\mu}(e^{\mu r/\hat{p}D} - 1)\right] u$$

After substituting u, it can be rewritten as

$$\left(\frac{hQ^{*2}}{2D} + \frac{hQ^*}{\mu} - A_D + \frac{hD}{\mu^2}\right)(e^{\mu r/\hat{p}D}\ e^{\mu s/D} - 1) \quad = \quad A_L + \gamma r + \eta s - \frac{D}{2h}\left(\frac{hQ^*}{D} - (c_L - c_D)\right)^2$$

$$+ \quad \frac{D\Gamma}{\mu}(e^{\mu r/\hat{p}D} - 1) \tag{4.132}$$

First Derivative of TC w.r.t. s:

Taking the derivative of (4.101) yields,

$$\frac{\partial TC}{\partial s} = \frac{\frac{\partial E[CC]}{\partial s}E[T] - \frac{\partial E[T]}{\partial s}E[CC]}{E[T]^2} \tag{4.133}$$

The derivative of expected cycle cost with respect to s can be found as

$$\begin{aligned} \frac{\partial E[CC]}{\partial s} &= \frac{\partial f}{\partial s} + \frac{\partial g}{\partial s}u + \frac{\partial u}{\partial s}g \\ &= h\left(\frac{Q}{D} + \frac{1}{\mu}\right) + \left(\eta - c_L - \frac{hs}{D}\right)u - \frac{\mu}{D}u(1+u)g \end{aligned} \tag{4.134}$$

By adding and subtracting the term $f\mu(1+u)/D$ on the right hand side of (4.134), we obtain that

$$\begin{aligned} \frac{\partial E[CC]}{\partial s} &= \frac{\mu(1+u)}{D}\left(\frac{hD}{\mu(1+u)}\left(\frac{Q}{D} + \frac{1}{\mu}\right) + \left(\eta - c_L - \frac{hs}{D}\right)\frac{uD}{\mu(1+u)} + f\right) \\ &- \frac{\mu(1+u)}{D}(f + g\,u) \end{aligned} \tag{4.135}$$

Recall from (4.97) that $E[CC] = f + g\,u$. Thus, after some simplifications (4.135) becomes

$$\frac{\partial E[CC]}{\partial s} = \frac{\mu(1+u)}{D}\left(w - E[CC]\right) \tag{4.136}$$

with

$$w = w(r, s, Q) \quad := \quad A_D + c_D Q + \frac{hQ^2}{2D} + \frac{hQs}{D} + \frac{c_D D + h(Q+s)}{\mu(1+u)} \tag{4.137}$$

The derivative of (4.102) with respect to s can be found as

$$\frac{\partial E[T]}{\partial s} = -\frac{u}{D} - \frac{\mu}{D}u(1+u)\left(\frac{R-s}{D}\right) \tag{4.138}$$

By adding and subtracting the term $(\mu(1+u)/D)(Q/D+1/\mu)$ on the right-hand side of (4.138) and reorganizing the terms give

$$\frac{\partial E[T]}{\partial s} = \frac{\mu(1+u)}{D}\left(\frac{Q}{D}+\frac{1}{\mu(1+u)}-E[T]\right) \tag{4.139}$$

Using (4.139) and (4.136) in (4.133) yields

$$\begin{aligned}\frac{\partial TC}{\partial s} &= \frac{\mu(1+u)}{DE[T]^2}\left[(w-E[CC])\,E[T]-\left(\frac{Q}{D}+\frac{1}{\mu(1+u)}-E[T]\right)E[CC]\right] \\ &= \frac{\mu(1+u)}{DE[T]}\left[w-\left(\frac{Q}{D}+\frac{1}{\mu(1+u)}\right)TC\right] \end{aligned}\tag{4.140}$$

First Order Condition of TC w.r.t. s:

For any $0 \leq r, R, Q < \infty$, let s^* be a critical point satisfying the first order condition $\partial TC/\partial s = 0$. Since $\mu(1+u)/DE[T] \neq 0$ for any finite (r, R, s, Q), it is clear from (4.140) that the first order condition is satisfied when

$$w(r,s^*,Q) = \left(\frac{Q}{D}+\frac{1}{\mu(1+u(r,s^*))}\right)TC(r,R,s^*,Q) \tag{4.141}$$

Q^* as Economic Order Quantity:

Observe that for any Q^* and s^* satisfying (4.127) and (4.141), we have

$$w(r,s^*,Q^*) = \left(\frac{Q^*}{D}+\frac{1}{\mu(1+u(r,s^*))}\right)(c_D D+h(Q^*+s^*)) \tag{4.142}$$

Substituting (4.137) in (4.142) and making necessary simplifications yields

$$\begin{aligned}\frac{Q^*}{D}(c_D D+h(Q^*+s^*)) &= A_D + c_D Q^* + \frac{hQ^{*2}}{2D}+\frac{hQ^*s^*}{D} \\ \frac{hQ^{*2}}{D} &= A_D+\frac{hQ^{*2}}{2D} \\ Q^* &= \sqrt{\frac{2A_D D}{h}} \end{aligned}\tag{4.143}$$

Second Derivative of TC w.r.t. s:

Define the function

$$y := y(r,R,s,Q) := w(r,s,Q)-\left(\frac{Q}{D}+\frac{1}{\mu(1+u(r,s))}\right)TC(r,R,s,Q)$$

Thus, (4.140) becomes,

$$\frac{\partial TC}{\partial s} = \frac{\mu(1+u)y}{DE[T]}$$

and the second derivative of TC with respect to s can be given as,

$$\frac{\partial^2 TC}{\partial s^2} = \frac{\mu}{DE[T]^2}\left[\left(\frac{\partial u}{\partial s}y + (1+u)\frac{\partial y}{\partial s}\right)E[T] - (1+u)y\frac{\partial E[T]}{\partial s}\right] \tag{4.144}$$

Observe that

$$\frac{\partial y}{\partial s} = \frac{\partial w}{\partial s} + \frac{\partial u}{\partial s}\frac{TC}{\mu(1+u)^2} - \left(\frac{Q}{D} + \frac{1}{\mu(1+u)}\right)\frac{\partial TC}{\partial s} \tag{4.145}$$

and

$$\frac{\partial y}{\partial s}|_{s=s^*} = \frac{hQ}{D} + \frac{1}{\mu(1+u)}\left(h + \frac{u\mu(c_D D + h(Q+s^*))}{D}\right) - \frac{u}{D(1+u)}TC|_{s=s^*} \tag{4.146}$$

Moreover, from (4.141) it is clear that $y(r, R, s^*, Q) = 0$. Therefore, evaluating (4.144) at $s = s^*$ yields

$$\begin{aligned}\frac{\partial^2 TC}{\partial s^2}|_{s=s^*} &= \frac{\mu(1+u)}{DE[T]}\frac{\partial y}{\partial s}|_{s=s^*} \\ &= \frac{\mu}{DE[T]}\left[\frac{hQ(1+u)}{D} + \frac{h}{\mu} + \frac{u}{D}\left(c_D D + h(Q+s^*) - TC|_{s=s^*}\right)\right]\end{aligned} \tag{4.147}$$

Recall that Q^* satisfies (4.127). Thus, for $Q = Q^*$, (4.147) becomes,

$$\frac{\partial^2 TC}{\partial s^2}|_{s=s^*,Q=Q^*} = \frac{\mu}{DE[T]}\left[\frac{hQ^*(1+u)}{D} + \frac{h}{\mu}\right] \tag{4.148}$$

Note that in (4.148), $Q^* = \sqrt{2A_D D/h}$ since (4.127) and (4.141) are both satisfied. Thus, for any $0 \le r, R < \infty$, if (4.127) and (4.141) has a feasible solution (s^*,Q^*) such that $0 \le s^* \le R$ and $Q^* \ge 0$ then

$$\frac{\partial^2 TC}{\partial s^2}|_{s=s^*,Q=Q^*} > 0$$

Computation of Critical Point Candidate:

For any given $0 \le r < \infty$, if a critical point (R^*,s^*,Q^*) satisfies (4.120), (4.127) and (4.141) then $Q^* = \sqrt{2A_D D/h}$ and (4.132) becomes,

$$\begin{aligned}\frac{D}{\mu}\left(\sqrt{\frac{2A_D h}{D}} + \frac{h}{\mu}\right)(e^{\mu r/\hat{p}D}\ e^{\mu s^*/D} - 1) &= A_L + \gamma r + \eta s^* - \frac{D}{2h}\left(\sqrt{\frac{2A_D h}{D}} - (c_L - c_D)\right)^2 \\ &+ \frac{D\Gamma}{\mu}(e^{\mu r/\hat{p}D} - 1)\end{aligned} \tag{4.149}$$

Here, we shall note that for $r = 0$, (4.149) boils down to the equation given by Moinzadeh (1997) to find s^*. Similarly, (4.130) becomes

$$R^* = s^* + \sqrt{\frac{2A_D D}{h}} - \frac{(c_L - c_D)D}{h} \tag{4.150}$$

For any given $0 \le s < \infty$, let (r^*,R^*,Q^*) be a critical point satisfying (4.111), (4.120) and (4.127). Then, using (4.130) and (4.131) in (4.111), we obtain that

$$\begin{aligned}\left[\frac{c_L - c_D}{h} + \frac{1}{\mu}\right] TC(r^*, R^*, s, Q^*) &= A_D - A_L - \gamma r^* + \frac{D(c_L - c_D)c_D}{h} + \frac{D(c_L - c_D)^2}{2h} \\ &+ \frac{D}{\mu}\left(\hat{p}(c_L + \pi) + (1 - \hat{p})\theta\right)\end{aligned} \tag{4.151}$$

Substituting (4.120) in (4.151) and reorganizing the terms yield the following relation between r^* and R^*,

$$\eta R^* + \gamma r^* = \mathcal{K} \tag{4.152}$$

with

$$\mathcal{K} := A_D - A_L - \frac{D(c_L - c_D)^2}{2h} + \frac{D}{\mu}\left(\hat{p}\pi + (1 - \hat{p})(\theta - c_L)\right) \tag{4.153}$$

Note that by using (4.130), (4.152) can also be given as a relation between r^* and Q^* as follows,

$$\gamma r^* = \mathcal{K} - \eta\left(s + Q^* - \frac{D(c_L - c_D)}{h}\right) \tag{4.154}$$

Moreover, if (s^*, Q^*) is a point satisfying (4.127) and (4.141) then $Q^* = \sqrt{2A_D D/h}$ and (4.154) becomes,

$$\begin{aligned}A_L + \gamma r^* + \eta s^* &= \frac{D}{2h}\left(\sqrt{\frac{2A_D h}{D}} - (c_L - c_D)\right)^2 - \frac{D}{\mu}\left(\sqrt{\frac{2A_D h}{D}} - (c_L - c_D)\right) \\ &+ \frac{D}{\mu}\left(\hat{p}(c_L + \pi) + (1 - \hat{p})\theta\right)\end{aligned} \tag{4.155}$$

Thus, using (4.155) in (4.149) and making necessary simplifications yields

$$\left(\sqrt{\frac{2A_D h}{D}} + \frac{h}{\mu}\right) e^{\mu r^*/\hat{p}D}\ e^{\mu s^*/D} = \Gamma e^{\mu r^*/\hat{p}D} + \hat{p}\gamma \tag{4.156}$$

Using the definition of Γ and rewriting (4.156) gives s^* as follows

$$s^* = \frac{D}{\mu} \ln \left(\frac{\frac{h}{\mu} + \hat{p}\,(\Pi + \gamma e^{-\mu r^*/\hat{p}D}\ ) - (1-\hat{p})c_D}{\frac{h}{\mu} + \sqrt{\frac{2A_D h}{D}}} \right) \tag{4.157}$$

Therefore, if (r^*,R^*,s^*,Q^*) is a critical point satisfying the first order conditions (4.111), (4.120), (4.127) and (4.141) then $Q^* = \sqrt{2A_D D/h}$ and r^*, R^* and s^* can be found by solving (4.157) by using (4.150) and (4.152).

Computation of Boundary Point Candidate at s=0 :

If the first order condition (4.141) does not yield a feasible critical point s^* then we need to check the boundary $s = 0$ for the optimal solution. We impose $s = 0$ and let (r^*, R^*, Q^*) be a critical point satisfying (4.111), (4.120) and (4.127). Observe that most of the relations derived above for any given $0 \le s \le \infty$ remains valid. Thus, (4.130) becomes,

$$R^* = Q^* - \frac{(c_L - c_D)D}{h} \tag{4.158}$$

and (4.132) becomes

$$\begin{aligned}\left(\frac{hQ^{*2}}{2D} + \frac{hQ^*}{\mu} - A_D + \frac{hD}{\mu^2}\right)(e^{\mu r/\hat{p}D}\ - 1) \ &= \ A_L + \gamma r - \frac{D}{2h}\left(\frac{hQ^*}{D} - (c_L - c_D)\right)^2 \\ &+ \ \frac{D\Gamma}{\mu}(e^{\mu r/\hat{p}D}\ - 1)\end{aligned} \tag{4.159}$$

Moreover, the relation between R^* and r^* given by (4.152) remains the same and the relation between r^* and Q^* given by (4.154) becomes

$$\gamma r^* = \mathcal{K} - \eta\left(Q^* - \frac{D(c_L - c_D)}{h}\right) \tag{4.160}$$

Substituting (4.160) in (4.159) and after some algebra we obtain

$$\left(\frac{hQ^{*2}}{2D} + \frac{hQ^*}{\mu} - A_D - \frac{D}{\mu}\left(\hat{p}\,\Pi - (1-\hat{p})c_D\right)\right) e^{\mu r^*/\hat{p}D}\ - \frac{\hat{p}D\gamma}{\mu} = 0 \tag{4.161}$$

Thus, (r^*, R^*, Q^*) can be found by using (4.152) and (4.158) in (4.161) and solving for Q^*.

Computation of Boundary Point Candidate at R=s:

If $R = s$ then the expected cycle cost given by (4.97) becomes,

$$E[CC] \;=\; f(s,Q) + g(r,s)\,u(r,s) \tag{4.162}$$

with $f(s,Q)$ and $u(r,s)$ being the same and

$$g(r,s) := A_L + \gamma r + \eta s + \frac{D\Gamma}{\mu}\left(e^{\mu r/\hat{p}D} - 1\right) \tag{4.163}$$

and the total cost rate function can be given as

$$TC(r,s,Q) = \frac{E[CC]}{E[T]} \tag{4.164}$$

with the expected cycle time,

$$E[T] = \frac{Q}{D} + \frac{1}{\mu} \tag{4.165}$$

It is clear from (4.164) that the behavior of $TC(r,s,Q)$ in (r,s) depends only on the behavior of $E[CC]$ in (r,s). Thus, taking the derivative of (4.162) with respect to r yields,

$$\begin{aligned}\frac{\partial E[CC]}{\partial r} &= \frac{\partial g}{\partial r}u + \frac{\partial u}{\partial r}g \\ &= u\left(\gamma + \frac{\Gamma}{\hat{p}}\,e^{\mu r/\hat{p}D} - \frac{\mu}{\hat{p}D}(1+u)g\right)\end{aligned} \tag{4.166}$$

For any given (s,Q), let r^* be a critical point satisfying the first order condition $\partial TC/\partial r = 0$. Since $u > 0$ for $0 \le s < \infty$, we observe from (4.166) that the first order condition is satisfied when

$$\gamma + \frac{\Gamma}{\hat{p}}\,e^{\mu r^*/\hat{p}D} = \frac{\mu}{\hat{p}D}(1 + u(r^*,s))g(r^*,s) \tag{4.167}$$

Similarly, the derivative of (4.162) with respect to s can be found as

$$\begin{aligned}\frac{\partial E[CC]}{\partial s} &= \frac{\partial f}{\partial s} + \frac{\partial g}{\partial s}u + \frac{\partial u}{\partial s}g \\ &= u\left(\frac{h}{u}\left(\frac{Q}{D} + \frac{1}{\mu}\right) + \eta - \frac{\mu}{D}(1+u)g\right)\end{aligned} \tag{4.168}$$

For any given (r, Q), let s^* be a critical point satisfying the first order condition $\partial TC/\partial s = 0$. Since $u > 0$ for $0 \leq r < \infty$, it is evident from (4.168) that the first order condition is satisfied when

$$\frac{h}{u(r, s^*)}\left(\frac{Q}{D} + \frac{1}{\mu}\right) = \frac{\mu}{D}(1 + u(r, s^*))g(r, s^*) - \eta \tag{4.169}$$

Thus, for any given Q, if (r^*, s^*) is a critical point satisfying (4.167) and (4.169) then using (4.167) in (4.169) gives

$$\frac{h}{u(r^*, s^*)}\left(\frac{Q}{D} + \frac{1}{\mu}\right) = \hat{p}\gamma + \Gamma\, e^{\mu r^*/\hat{p}D} \; - \eta \tag{4.170}$$

Using the definition of $u(r, s)$ and reorganizing the terms in (4.170) yields

$$r^* = -\frac{\hat{p}D}{\mu}\ln\left(\frac{h\left(\frac{Q}{D} + \frac{1}{\mu}\right)e^{\mu s^*/D} - \Gamma}{h\left(\frac{Q}{D} + \frac{1}{\mu}\right) - \eta + \hat{p}\gamma}\right) \tag{4.171}$$

Therefore, for any given $0 \leq Q < \infty$, (r^*, s^*) can be found by solving any two of the equations (4.167), (4.169) and (4.171).

Taking the derivative of (4.164) with respect to Q yields,

$$\frac{\partial TC}{\partial Q} = \frac{\frac{\partial E[CC]}{\partial Q}E[T] - \frac{\partial E[T]}{\partial Q}E[CC]}{E[T]^2} \tag{4.172}$$

From (4.162) and (4.165), the derivatives of $E[CC]$ and $E[T]$ with respect to Q can be given as

$$\frac{\partial E[CC]}{\partial Q} = \frac{\partial f}{\partial Q} = \frac{1}{D}\left(c_D D + h(Q + s)\right) \tag{4.173}$$

and

$$\frac{\partial E[T]}{\partial Q} = \frac{1}{D} \tag{4.174}$$

Using (4.173) and (4.174) in (4.172) gives,

$$\frac{\partial TC}{\partial Q} = \frac{1}{DE[T]}\left[c_D D + h(Q + s) - TC\right] \tag{4.175}$$

For any $0 \leq r, s < \infty$, let Q^* be a critical point satisfying the first order condition $\partial TC/\partial Q = 0$. It is clear from (4.175) that the first order condition is satisfied when

$$c_D D + h(Q^* + s) = TC(r, s, Q^*) \tag{4.176}$$

Observe that (4.176) can be rewritten as

$$(c_D D + h(Q^* + s))\left(\frac{Q^*}{D} + \frac{1}{\mu}\right) = f(s, Q^*) + g(r, s)\, u(r, s) \tag{4.177}$$

Using the definition of $f(s, Q)$ and after some algebra (4.177) becomes

$$\frac{hQ^{*2}}{2D^2} + \frac{hQ^*}{\mu D} - A_D + \frac{hD}{\mu^2} - g(r, s)\, u(r, s) = 0 \tag{4.178}$$

Applying the quadratic formula to (4.178) yields

$$Q^* = \frac{D}{\mu}\left(-1 + \sqrt{1 + \frac{2\mu^2}{h}\left(\frac{A_D}{D} - \frac{h}{\mu^2} + \frac{g(r, s)\, u(r, s)}{D}\right)}\right) \tag{4.179}$$

Therefore, a critical point (r^*, s^*, Q^*) of the function $TC(r, s, Q)$ can be found by solving (4.167), (4.171) and (4.179).

For $R = s$, we shall also check the boundary $Q = 0$ for the optimal solution. Thus, we impose $Q = 0$ and let (r^*, s^*) be a critical point satisfying the first order conditions (4.167) and (4.169). Observe that for $Q = 0$, (4.171) becomes

$$r^* = -\frac{\hat{p}D}{\mu} \ln\left(\frac{\frac{h}{\mu}\, e^{\mu s^*/D} - \Gamma}{\hat{p}\gamma - (c_L - c_D)}\right) \tag{4.180}$$

Thus, (r^*, s^*) can be found by solving (4.167) and (4.180) simultaneously.

Boundary Point Candidate at R=s+Q :

If an optimal solution exists at the boundary $R = s + Q$ then it means that the system tries to exploit the economies of scale as much as possible in every list replenishment. Since $c_L > c_D$, the optimal R being at its upper bound implies that the deals leads to a marginal cost reduction compared to the reduction obtained by exploiting economies of scale in list replenishments. Such scenario might occur when price discounts are relatively small and the fixed cost of replenishing from the list price is very high compared to the fixed cost of replenishing from the deal price ($A_L >> A_D$). However, this scenario does not seem to be realistic for systems with deal offerings. Although such parameter sets can be constructed for the sake of completeness, we do not think that investigating such sets would yield interesting insights. Thus, we ignore these exotic scenarios and neglect the candidates at the boundary $R = s + Q$. $\square$

Theorem 4.3. *For Case 3, the optimal solution can be found at one of the following candidate points,*

1. *If $c_L - c_D = \hat{\pi}/\mu$ then*

   (a) *if $A_L = 0$ then any r and R satisfying $0 \le R \le r$ is optimal. Let*

   $$U = \hat{p}\,\Pi - (1-\hat{p})c_D$$

   then (s^*, Q^*) can be given as

   i. *If $A_D/D + U/\mu \le 0$ then $s^* = Q^* = 0$.*

   ii. *If $A_D/D + U/\mu > 0$ then*

   A. *if $\sqrt{2A_Dh/D} > U$ then $s^* = 0$ and*

   $$Q^* = \frac{D}{\mu}\left(-1 + \sqrt{1 + \frac{2\mu^2}{h}\left(\frac{A_D}{D} + \frac{U}{\mu}\right)}\right)$$

   B. *if $\sqrt{2A_Dh/D} \le U$ then*

   $$s^* = \frac{-D}{\mu}\ln\left(\frac{\frac{h}{\mu} + \sqrt{\frac{2A_Dh}{D}}}{\frac{h}{\mu} + U}\right)$$

   and

   $$Q^* = \sqrt{\frac{2A_DD}{h}}$$

   (b) *if $A_L > 0$ then $r^* \to \infty$ and R becomes an irrelevant policy parameter. (s^*, Q^*) can be given as in part 1(a).*

2. *If $c_L - c_D > \hat{\pi}/\mu$ then $r^* \to \infty$ and R becomes an irrelevant policy parameter. (s^*, Q^*) can be given as in part 1(a).*

3. *If $c_L - c_D < \hat{\pi}/\mu$ then $R^* = 0$ and*

   (a) *if $A_L = 0$ then $r^* = 0$. Let*

   $$U = \hat{p}\,(\Pi + \gamma) - (1-\hat{p})c_D \tag{4.181}$$

   then (s^*, Q^*) can be found by the formulas given in part 1(a) by substituting U with (4.181).

(b) if $A_L > 0$ then $-A_L/\gamma < r^* < -A_L/\gamma + \hat{p}D/\mu$ and r^* is the solution of

$$1 - e^{-\mu r^*/\hat{p}D} - \frac{\mu r^*}{\hat{p}D} - \frac{A_L \mu}{\gamma \hat{p}D} = 0 \tag{4.182}$$

Let

$$U = \hat{p}\,(\Pi + \gamma e^{-\mu r^*/\hat{p}D}) - (1-\hat{p})c_D \tag{4.183}$$

then (s^*, Q^*) can be found by the formulas given in part 1(a) by substituting U with (4.183).

Proof of Theorem 4.3:

For the optimization of the total cost function according to parameters r and R, we shall first rewrite the expected cycle cost for Case 3 as follows:

$$E[CC] \;=\; f(s,Q) + [A_L + \gamma(r-R)]\,\frac{e^{-\mu s/D}}{e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D}} \tag{4.184}$$

with

$$f(s,Q) \;:=\; A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + h\left(\frac{Q}{D} + \frac{1}{\mu}\right)s + \frac{D\Gamma}{\mu}\,e^{-\mu s/D} \tag{4.185}$$

Taking the first derivative of (4.184) with respect to r yields

$$\begin{aligned}\frac{\partial E[CC]}{\partial r} &= \frac{\gamma e^{-\mu s/D}\,(e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D}) - [A_L + \gamma(r-R)]\,e^{-\mu s/D}\,\frac{\mu}{\hat{p}D}e^{\mu r/\hat{p}D}}{(e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D})^2} \\ &= \frac{\gamma e^{-\mu s/D}\,e^{\mu r/\hat{p}D}\,f(\frac{\mu(r-R)}{\hat{p}D})}{(e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D})^2}\end{aligned} \tag{4.186}$$

with

$$f\left(\frac{\mu(r-R)}{\hat{p}D}\right) := 1 - e^{-\mu(r-R)/\hat{p}D} - \frac{\mu(r-R)}{\hat{p}D} - \frac{A_L\mu}{\gamma\hat{p}D}$$

Observe that we can analyze the behavior of $E[CC]$ according to the sign of γ such that,

1. If $\gamma = 0$ we have to consider two different cases according to the sign of A_L:

   (a) If $A_L = 0$ then $E[CC] = f(s,Q)$ and, therefore, any r and R satisfying the relation $0 \le R \le r$ is optimal.

(b) If $A_L > 0$ then

$$E[CC] = f(s,Q) + \frac{A_L e^{-\mu s/D}}{e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D}}$$

and it can be easily seen that for any given $0 \le Q < \infty$, $0 \le s < \infty$ and $0 \le R \le r$, $E[CC]$ goes to infinity when $r = R$ and $E[CC]$ is decreasing in r. Hence, for any given $0 \le Q < \infty$, $0 \le s < \infty$ and $0 \le R \le r$, $E[CC]$ attains its minimum as $r \to \infty$, and, therefore, R becomes an irrelevant policy parameter. Moreover, observe from (4.184) that as r goes to infinity $(r - R)/(e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D})$ goes to zero, and $E[CC]$ goes to $f(s, Q)$. Thus, $E[CC]|_{r^* \to \infty} = f(s, Q)$ and the optimal (s, Q) follows from Lemma 4.3.

2. If $\gamma > 0$ then $f(\mu(r - R)/\hat{p}D) \le 0$ and the numerator of (4.186) is less than or equal to zero. Thus, $\partial E[CC]/\partial r \le 0$ and for any given $0 \le Q < \infty$, $0 \le s < \infty$, $0 \le R \le r$, $E[CC]$ attains its minimum value as $r \to \infty$ and R becomes irrelevant.

3. If $\gamma < 0$ then we start by analysing the behavior of $E[CC]$ in R.

   The first derivative of (4.184) with respect to R can be given as,

$$\begin{aligned}\frac{\partial E[CC]}{\partial R} &= \frac{-\gamma e^{-\mu s/D}\,(e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D}) + [A_L + \gamma(r-R)]\,e^{-\mu s/D}\,\frac{\mu}{\hat{p}D} e^{\mu R/\hat{p}D}}{(e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D})^2} \\ &= \frac{\gamma e^{-\mu s/D}\, e^{\mu R/\hat{p}D}\, g(\frac{\mu(r-R)}{\hat{p}D})}{(e^{\mu r/\hat{p}D} - e^{\mu R/\hat{p}D})^2} \qquad (4.187)\end{aligned}$$

   with

$$g\left(\frac{\mu(r-R)}{\hat{p}D}\right) := 1 - e^{\mu(r-R)/\hat{p}D} + \frac{\mu(r-R)}{\hat{p}D} + \frac{A_L \mu}{\gamma \hat{p}D}$$

   It can be easily shown that the function $g(x) := 1 - e^x + x + A$ is less than or equal to zero for $x \ge 0$ and $A \le 0$. Thus, if $\gamma < 0$ then $g(\mu(r - R)/\hat{p}D) \le 0$ and the numerator of (4.187) is greater than or equal to zero. Therefore, $\partial E[CC]/\partial R \ge 0$ and for any given $0 \le Q < \infty$, $0 \le s < \infty$, $r \ge R$, $E[CC]$ attains its minimum at $R^* = 0$.

   Substituting $R^* = 0$ in (4.184) yields,

$$E[CC]|_{R^*=0} = f(s,Q) + [A_L + \gamma r]\,\frac{e^{-\mu s/D}}{e^{\mu r/\hat{p}D} - 1} \qquad (4.188)$$

Taking the first derivative of $E[CC]|_{R^*=0}$ with respect to r gives

$$\frac{\partial E[CC]|_{R^*=0}}{\partial r} = \frac{\gamma e^{-\mu s/D}\ e^{\mu r/\hat{p}D}\ f(\frac{\mu r}{\hat{p}D})}{(e^{\mu r/\hat{p}D}\ -1)^2} \tag{4.189}$$

with

$$f\left(\frac{\mu r}{\hat{p}D}\right) := 1 - e^{-\mu r/\hat{p}D}\ - \frac{\mu r}{\hat{p}D} - \frac{A_L \mu}{\gamma \hat{p}D} \tag{4.190}$$

Observe that for any $s \geq 0$, the sign of (4.189) depends on the sign of the function $f(\frac{\mu r}{\hat{p}D})$. Thus,

(a) If $A_L = 0$ then from Lemma 1, we have $f(\frac{\mu r}{\hat{p}D}) \geq 0$ with a unique root at $r = 0$ since $\mu/\hat{p}D > 0$. Thus, (4.189) is always non-negative and (4.188) attains its minimum as $r \to 0$. That is $\lim_{r\to 0} r/(e^{\mu r/\hat{p}D}\ -1) = \hat{p}D/\mu$ and (4.188) becomes

$$E[CC]|_{r^*=R^*=0} = f(s,Q) + \frac{\gamma e^{-\mu s/D}\ \hat{p}D}{\mu} \tag{4.191}$$

Observe that (4.191) can be rewritten as a function of s and Q as follows,

$$\begin{aligned}\widetilde{f}(s,Q) &= A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + h\left(\frac{Q}{D} + \frac{1}{\mu}\right)s \\ &+ \frac{D}{\mu}\left(\frac{h}{\mu} + \hat{p}\,(\Pi + \gamma) - (1-\hat{p})c_D\right)e^{-\mu s/D}\end{aligned} \tag{4.192}$$

Thus, s^* and Q^* can be calculated from Lemma 4.3 by substituting U with $\hat{p}\,(\Pi + \gamma) - (1-\hat{p})c_D$.

(b) If $A_L > 0$ then from Lemma 1, $f(\frac{\mu r}{\hat{p}D})$ is a concave decreasing function with a unique positive root $\mu r^*/\hat{p}D$ such that $-A_L/\gamma < r^* < -A_L/\gamma + \hat{p}D/\mu$. Thus, it is evident from (4.189) that for any $0 \leq Q < \infty$ and $0 \leq s < \infty$, (4.188) is first decreasing and then increasing in r and it reaches its minimum at r^*. Therefore, r^* satisfies $f(\frac{\mu r^*}{\hat{p}D}) = 0$ and (4.188) becomes

$$E[CC]|_{R^*=0,r^*} = f(s,Q) + \frac{\hat{p}D\gamma e^{-\mu r^*/\hat{p}D}\ e^{-\mu s/D}}{\mu} \tag{4.193}$$

For a given r^*, (4.193) can be rewritten as a function of s and Q as follows,

$$\begin{aligned}\widetilde{f}(s,Q) &= A_D + \frac{c_D D}{\mu} - \frac{hD}{\mu^2} + \frac{hQ^2}{2D} + c_D Q + h\left(\frac{Q}{D} + \frac{1}{\mu}\right)s \\ &+ \frac{D}{\mu}\left(\frac{h}{\mu} + U\right)e^{-\mu s/D}\end{aligned} \tag{4.194}$$

with

$$U = \hat{p}\,(\Pi + \gamma e^{-\mu r^*/\hat{p}D}\ ) - (1-\hat{p})c_D$$

and (s^*, Q^*) follow from Lemma 4.3. □

4.6. Discussion

As can be seen from Theorems (4.1)–(4.3) when $c_L - c_D \geq \hat{\pi}/\mu$ the optimal solution can be characterized relatively easily. That is, either $r^* = 0$ and (R^*, s^*, Q^*) is found as given by Moinzadeh (1997) or $r^* \to \infty$ and the optimal policy is characterized by the two parameters (s^*, Q^*).

On the other hand, for $c_L - c_D < \hat{\pi}/\mu$ it becomes more difficult to identify the optimal policy parameters. For Case 1 and Case 3, we can optimize (r^*, R^*) independent from (s^*, Q^*) thanks to the relatively simple form of the objective function. As a result, for Case 1 and Case 3, we are able to reduce the optimization problem to a limited number of candidate points which can be found by solving simple nonlinear equations in a similar fashion as presented by Moinzadeh (1997). For example, we show that for Case 3, r^* is non-negative and finite and $R^* = 0$. This indicates that the optimal solution for Case 3 lies on the boundary of Case 2.

Unfortunately, for Case 2, the form of the objective function is more complicated than the other cases and decomposition of the optimization problem is not possible. Moreover, the objective function remains feasible at most of the boundaries of the constraint set. These factors increase the number of candidates that can only be found by solving systems of nonlinear equations, and this makes the analytical optimization more difficult.

However, for Case 2, we are able to analyze the first order conditions of the system and give a relatively simple set of nonlinear equations given by (4.86)–(4.89) to find the critical points of the objective function. Moreover, we provide a parametric condition analogous to Moinzadeh (1997) to identify whether the interior of the constraint sets $\mathbf{\Omega_1}$ and $\mathbf{\Omega_2}$ contain the optimal solution or not.

In order to see this result observe from Theorem 4.1 that when $c_L - c_D < \hat{\pi}/\mu$, a critical point in the interior of the constraint set can only be found by solving (4.49), (4.50), (4.52) and (4.51). Since $U := c_L - c_D$ in this case, it is clear from the theorem that if $\sqrt{2A_D h/D} > c_L - c_D$ then the optimal solution for Case 1 cannot be in the interior of the constraint set $\mathbf{\Omega_1}$. Similarly, it is clear from (4.86) that if $\sqrt{2A_D h/D} < c_L - c_D$ then the optimal solution of Case 2 cannot be in the interior of the constraint set $\mathbf{\Omega_2}$.

Since the policy analyzed by Moinzadeh (1997) is a special case of our policy, for that problem, the optimization of the objective function is relatively easy and the above condition is enough to locate the optimal solution, i.e., the optimal solution is at the critical point in $\mathbf{\Omega_1}$ or $\mathbf{\Omega_2}$. However, due to the difficulties discussed above, in our problem, the parametric condition is only helpful to narrow down the number of candidates for the optimal solution.

4.7. Numerical Example

In this section, we provide two numerical examples to illustrate the behavior of the optimal policy parameters and optimal total cost rate due to the changes in backordering costs, and the potential benefits of allowing partial backorders. Table 4.1 shows the effects of the changes in backordering costs on optimal policy parameters and optimal cost rate.

Table 4.1: Behavior of (r^*, R^*, s^*, Q^*) and TC^* in $\hat{\pi}$ and π

$\hat{\pi}$	π	r^*	R^*	s^*	Q^*	TC^*
6	0.2	127.87	0.66	7.72	173.21	1980.92
	0.4	126.93	8.95	16.01	173.21	1989.21
	0.8	121.26	22.70	29.75	173.21	2002.96
8	0.2	89.68	16.85	23.91	173.21	1997.12
	0.4	87.44	23.05	30.11	173.21	2003.31
	0.8	81.49	33.62	40.68	173.21	2013.88
10	0.2	69.83	26.95	34.01	173.21	2007.21
	0.4	67.52	31.91	38.97	173.21	2012.17
	0.8	62.15	40.50	47.56	173.21	2020.77

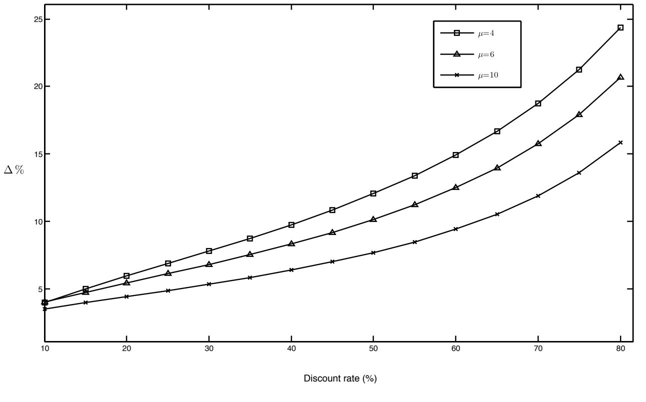
Note. $D = 200, A_L = A_D = 75, c_L = 10, c_D = 9, \theta = 0.4, \hat{p} = 0.9, \mu = 3$

From Table 4.1, we observe that as $\hat{\pi}$ increases $-r^*$ increases to decrease the number of backorders whereas R^* increases to balance the list purchase quantity. We observe that the optimal list order quantity $r^* + R^*$ decreases as $\hat{\pi}$ increases. This is because the time weighted backordering cost has relatively more impact on the reorder level than the order-up-to level. The increase in $\hat{\pi}$ pushes s^* to higher values in order to take advantage of deals earlier and incur less backorders.

Similar behaviors are observed in $-r^*$, R^* and s^* as π increases. Also note that the difference between R^* and s^* remains approximately constant as π increases since s^* moves upward to balance the number of deal orders. Q^* is not effected by the changes in the stockout costs since for all parameter combinations considered the optimal solutions are found in the interior of the constraint sets. Thus, the optimal order quantity is found by the EOQ formula as given in Theorems 4.1–4.3. Moreover, we observe that TC^* increases as the backordering costs increase.

In order to see the benefits of allowing backorders more clearly, we compared the performance of the proposed four parameter policy with the three parameter (R, s, Q) policy offered by Moinzadeh (1997). Figure 4.4 shows savings as the discount rate and the average number of deal offerings (μ) increase. Here, $\Delta\%$ denotes the percentage savings in total cost rate obtained by using the four parameter policy.

Figure 4.4: Comparison with (R, s, Q) policy



Note: $D = 200$, $h = 5$, $c_L = 25$, $A_L = 100$, $A_D = 50$, $\hat{\pi} = 8$, $\pi = 2$, $\theta = 2$, $\hat{p} = 0.95$

As can be seen from Figure 4.4, savings range from 4% up to 25% and increase with the discount rate. This result indicates that allowing backorders may result in considerable savings especially for high discount rates. As the discount rate increases purchasing from the list price becomes relatively expensive compared to backordering. Therefore, waiting for a good deal by allowing backorders becomes more advantageous and using a policy allowing backorders yields cost savings.

We also observe that as the average number of deal offerings (μ) increases, the savings decreases. This is due to the fact that with more frequent deal offers, the number of list replenishments and backorders both decrease and the two models converge to each other. Also note that the decrease in savings is higher for high discount rates due to the bigger tradeoff between backordering and purchasing from the list price.

4.8. Conclusion

In this chapter, we consider an inventory system where price discounts are offered by the supplier at random points in time and a certain fraction of demand is lost during the stockout period. Under the assumptions of deterministic demand, zero replenishment lead time and Poisson deal arrivals, we propose a four parameter continuous review control policy and derive exact expressions for the operating characteristics and the equations to calculate the optimal policy parameters minimizing the expected total cost rate. We provide qualitative results about the location of the optimal solution with respect to the relation between certain system parameters. With an illustrative numerical example, we demonstrate that allowing backorders in a random deal environment indeed leads to cost savings for certain parameter settings. Our model might stand as a stepping stone for the derivation of more complicated and general models with partial backorders which may include random demand, random deal duration and nonzero lead times.

4.9. Appendix

Proof of Proposition 4.1:

Proof of Expected Number of List Replenishments for All Cases:

For all cases, we can write the probability that there exist i list replenishments within a cycle with the following equation:

$$P(N_L = i) = \begin{cases} \beta_j + (1-\beta_j)\alpha_j & \text{if} \quad i = 0 \\ \\ (1-\beta_j)\alpha_j(1-\alpha_j)^i & \text{if} \quad i > 0 \end{cases} \tag{4.195}$$

Thus, taking the expectation of (4.195) yields:

$$\begin{aligned} E[N_L] &= \sum_{i=0}^{\infty} iP(N_L = i) \\ &= (1-\beta_j)\alpha_j \sum_{i=1}^{\infty} i(1-\alpha_j)^i \\ &= \frac{(1-\beta_j)(1-\alpha_j)}{\alpha_j} \quad \square \end{aligned}$$

Proof of Expected Cycle Time for Cases 1 & 2 :

According to possible scenarios given by Figures 4.1 and 4.2 we can write the cycle length as follows:

$$T = \begin{cases} \frac{Q}{D} + \tau_{1:x} & \text{w.p.} \quad \beta_x \\ \\ \frac{Q}{D} + \frac{s-x}{D} + i\left[\frac{R}{D} + \frac{r}{\hat{p}D}\right] + \tau_{2:x} & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i,\ i = 0, 1, 2, ... \end{cases} \tag{4.196}$$

Clearly, from (4.196) the expected cycle time can be written as:

$$\begin{aligned}
E[T] &= \beta_x \left[\frac{Q}{D} + E[\tau_{1:x}]\right] \\
&+ (1-\beta_x)\alpha_x \sum_{i=0}^{\infty} \left[\frac{Q}{D} + \frac{s-x}{D} + i\left[\frac{R}{D} + \frac{r}{\hat{p}D}\right] + E[\tau_{2:x}]\right](1-\alpha_x)^i \\
&= \beta_x \left[\frac{Q}{D} + \frac{1}{\mu} - \frac{(s-x)(1-\beta_x)}{D\beta_x}\right] \\
&+ (1-\beta_x)\alpha_x \sum_{i=0}^{\infty} \left[\frac{Q}{D} + \frac{(s-x)}{D} + i\left[\frac{R}{D} + \frac{r}{\hat{p}D}\right]\right. \\
&+ \left.\frac{1}{\mu} - \frac{(\hat{p}x+r)(1-\alpha_x)}{\hat{p}D\alpha_x}\right](1-\alpha_x)^i \\
&= \frac{\beta_x Q}{D} + \frac{\beta_x}{\mu} - \frac{(s-x)(1-\beta_x)}{D} \\
&+ (1-\beta_x)\alpha_x \left[\frac{Q}{D} + \frac{(s-x)}{D} + \frac{1}{\mu} - \frac{(\hat{p}x+r)(1-\alpha_x)}{\hat{p}D\alpha_x}\right] \sum_{i=0}^{\infty}(1-\alpha_x)^i \\
&+ (1-\beta_x)\alpha_x \left[\frac{R}{D} + \frac{r}{\hat{p}D}\right] \sum_{i=0}^{\infty} i(1-\alpha_x)^i \\
&= \frac{\beta_x Q}{D} + \frac{\beta_x}{\mu} - \frac{(s-x)(1-\beta_x)}{D} \\
&+ (1-\beta_x)\left[\frac{Q}{D} + \frac{(s-x)}{D} + \frac{1}{\mu} - \frac{(\hat{p}x+r)(1-\alpha_x)}{\hat{p}D\alpha_x}\right] \\
&+ \left[\frac{R}{D} + \frac{r}{\hat{p}D}\right]\frac{(1-\beta_x)(1-\alpha_x)}{\alpha_x} \\
&= \frac{Q}{D} + \frac{1}{\mu} + \frac{(R-x)}{D}E(N_L) \quad \square
\end{aligned}$$

Proof of Expected Replenishment Cost for Cases 1 & 2 :

Replenishment cost of Case 1 and Case 2 can be given with the following equation:

$$RC = \begin{cases} A_D + c_D(Q + D\tau_{1:x}) & \text{w.p.} \quad \beta_x \quad \text{and} \quad 0 < \tau_{1:x} \leq \frac{s-x}{D} \\ \\ A_D + c_D(s + Q - x + D\tau_{2:x}) & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \;\;, \;\; i = 0, 1, 2, ... \\ +i\left[A_L + c_L(R + r)\right] & \text{and} \;\; 0 < \tau_{2:x} \leq \frac{x}{D} \\ \\ A_D + c_D(s + Q - \hat{p}x + \hat{p}D\tau_{2:x}) & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \;\;, \;\; i = 0, 1, 2, ... \\ +i\left[A_L + c_L(R + r)\right] & \text{and} \;\; \frac{x}{D} < \tau_{2:x} \leq \frac{\hat{p}x + r}{\hat{p}D} \end{cases} \tag{4.197}$$

In (4.197) let,

$$\begin{aligned} RC_1 &= A_D + c_D(Q + D\tau_{1:x}) \\ \\ RC_2 &= A_D + c_D(s + Q - x + D\tau_{2:x}) + i\left[A_L + c_L(R + r)\right] \\ \\ RC_3 &= A_D + c_D(s + Q - \hat{p}x + \hat{p}D\tau_{2:x}) + i\left[A_L + c_L(R + r)\right] \end{aligned}$$

Therefore, expected replenishment cost can be written as,

$$E[RC] = \beta_x E[RC_1] + (1-\beta_x)\alpha_x \sum_{i=0}^{\infty} \left[E[RC_2] + E[RC_3]\right](1-\alpha_x)^i \tag{4.198}$$

Then

$$\begin{aligned} E[RC_1] &= A_D + c_D\left(Q + DE[\tau_{1:x}]\right) \\ \\ &= A_D + c_D\left[Q + \frac{D}{\mu} - \frac{(s-x)(1-\beta_x)}{\beta_x}\right] \end{aligned} \tag{4.199}$$

Due to limiting conditions on $\tau_{2:x}$, we must derive $E[RC_2]$ as follows

$$
\begin{aligned}
E[RC_2] &= \int_0^{x/D} E[RC_2|\tau_{2:x}=t]g_{\tau_{2:x}}(t)dt \\
&= \{A_D + c_D(s+Q-x) + i\,[A_L + c_L(R+r)]\} \int_0^{x/D} g_{\tau_{2:x}}(t)dt \\
&+ c_D D \int_0^{x/D} t g_{\tau_{2:x}}(t)dt \\
&= -\{A_D + c_D(s+Q-x) + i\,[A_L + c_L(R+r)]\} \frac{e^{-\mu t/D}|_0^{x/D}}{\alpha_x} \\
&+ \frac{c_D D}{\alpha_x}\left[-te^{-\mu t/D}|_0^{x/D} + \int_0^{x/D} e^{-\mu t/D}\right] \\
&= \{A_D + c_D(s+Q-x) + i\,[A_L + c_L(R+r)]\} \frac{(1-e^{-\mu x/D})}{\alpha_x} \\
&- \frac{c_D x}{D}\, e^{-\mu x/D} + \frac{c_D D}{\alpha_x \mu}\,(1-e^{-\mu x/D}) \\
&= \{A_D + c_D(s+Q) + i\,[A_L + c_L(R+r)]\} \frac{(1-e^{-\mu x/D})}{\alpha_x} \\
&- \frac{c_D x}{\alpha_x} + \frac{c_D D}{\alpha_x \mu}\,(1-e^{-\mu x/D}) \qquad (4.200)
\end{aligned}
$$

Similarly,

$$\begin{aligned}
E[RC_3] &= \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} E[RC_3|\tau_{2:x}=t]g_{\tau_{2:x}}(t)dt \\
&= \{A_D + c_D(s+Q-\hat{p}x) + i\,[A_L + c_L(R+r)]\} \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} g_{\tau_{2:x}}(t)dt \\
&+ c_D\hat{p}D \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} t g_{\tau_{2:x}}(t)dt \\
&= -\{A_D + c_D(s+Q-\hat{p}x) + i\,[A_L + c_L(R+r)]\} \frac{e^{-\mu t/D}|_{x/D}^{(\hat{p}x+r)/\hat{p}D}}{\alpha_x} \\
&+ \frac{c_D\hat{p}D}{\alpha_x}\left[-te^{-\mu t/D}|_{x/D}^{(\hat{p}x+r)/\hat{p}D} + \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} e^{-\mu t/D}\right] \\
&= \{A_D + c_D(s+Q-\hat{p}x) + i\,[A_L + c_L(R+r)]\} \frac{\left(e^{-\mu x/D} - e^{-\mu(\hat{p}x+r)/\hat{p}D}\right)}{\alpha_x} \\
&- \frac{c_D(\hat{p}x+r)}{\alpha_x}\, e^{-\mu(\hat{p}x+r)/\hat{p}D} + \frac{c_D\hat{p}x}{\alpha_x}\, e^{-\mu x/D} + \frac{c_D\hat{p}D}{\alpha_x\mu}\left(e^{-\mu x/D} - e^{-\mu(\hat{p}x+r)/\hat{p}D}\right) \\
&= \{A_D + c_D(s+Q) + i\,[A_L + c_L(R+r)]\} \frac{\left(e^{-\mu x/D} - e^{-\mu(\hat{p}x+r)/\hat{p}D}\right)}{\alpha_x} \\
&- \frac{c_D r}{\alpha_x}\, e^{-\mu(\hat{p}x+r)/\hat{p}D} + \frac{c_D\hat{p}D}{\alpha_x\mu}\left(e^{-\mu x/D} - e^{-\mu(\hat{p}x+r)/\hat{p}D}\right) \\
&= \{A_D + c_D(s+Q) + i\,[A_L + c_L(R+r)]\} \frac{\left(e^{-\mu x/D} - (1-\alpha_x)\right)}{\alpha_x} \\
&- \frac{c_D r}{\alpha_x}\,(1-\alpha_x) + \frac{c_D\hat{p}D}{\alpha_x\mu}\left(e^{-\mu x/D} - (1-\alpha_x)\right) \qquad (4.201)
\end{aligned}$$

Hence, from (4.200) and (4.201) we can write,

$$\begin{aligned} E[RC_2] + E[RC_3] &= A_D + c_D(s+Q) + i\,[A_L + c_L(R+r)] \\ &- \frac{c_D x}{\alpha_x} + \frac{c_D D}{\alpha_x \mu}\,(1 - e^{-\mu x/D}) - \frac{c_D r}{\alpha_x}\,(1-\alpha_x) \\ &+ \frac{c_D \hat{p} D}{\alpha_x \mu}\,(e^{-\mu x/D} - (1-\alpha_x)) \end{aligned} \tag{4.202}$$

Let,

$$\begin{aligned} U &= A_D + c_D(s+Q) - \frac{c_D x}{\alpha_x} + \frac{c_D D}{\alpha_x \mu}\,(1 - e^{-\mu x/D}) - \frac{c_D r}{\alpha_x}\,(1-\alpha_x) \\ &+ \frac{c_D \hat{p} D}{\alpha_x \mu}\,(e^{-\mu x/D} - (1-\alpha_x)) \end{aligned} \tag{4.203}$$

Then,

$$E[RC_2] + E[RC_3] = U + i\,[A_L + c_L(R+r)] \tag{4.204}$$

If we substitute (4.199) and (4.204) in (4.198) we obtain,

$$\begin{aligned} E[RC] &= \beta_x \left\{ A_D + c_D \left[Q + \frac{D}{\mu} - \frac{(s-x)(1-\beta_x)}{\beta_x} \right] \right\} \\ &+ (1-\beta_x)\alpha_x \left\{ U \sum_{i=0}^{\infty} (1-\alpha_x)^i + [A_L + c_L(R+r)] \sum_{i=0}^{\infty} i(1-\alpha_x)^i \right\} \\ &= A_D\beta_x + c_D Q \beta_x + \frac{c_D D \beta_x}{\mu} - c_D(s-x)(1-\beta_x) \\ &+ (1-\beta_x)U + [A_L + c_L(R+r)]\,\frac{(1-\alpha_x)(1-\beta_x)}{\alpha_x} \end{aligned}$$

Substituting (4.203) yields,

$$
\begin{aligned}
E[RC] &= A_D\beta_x + c_D Q\beta_x + \frac{c_D D\beta_x}{\mu} - c_D(s-x)(1-\beta_x) \\
&+ (1-\beta_x)\left\{A_D + c_D(s+Q) - \frac{c_D x}{\alpha_x} + \frac{c_D D}{\alpha_x \mu}\left(1-e^{-\mu x/D}\right) - \frac{c_D r}{\alpha_x}\left(1-\alpha_x\right)\right. \\
&+ \left.\frac{c_D \hat{p} D}{\alpha_x \mu}\left(e^{-\mu x/D} - (1-\alpha_x)\right)\right\} + \left[A_L + c_L(R+r)\right]E[N_L] \\
&= A_D + c_D Q + \frac{c_D D}{\mu}\left[\beta_x + \frac{(1-\beta_x)}{\alpha_x}\right] - \frac{c_D D(1-\beta_x)}{\alpha_x \mu}\, e^{-\mu x/D} - c_D x E[N_L] \\
&- c_D r E[N_L] + \frac{c_D \hat{p} D(1-\beta_x)}{\alpha_x \mu}\, e^{-\mu x/D} - \frac{c_D \hat{p} D}{\mu}\, E[N_L] \\
&+ \left[A_L + c_L(R+r)\right]E[N_L]
\end{aligned}
$$

Adding and subtracting the term $c_D D/\mu$ to the third term yields,

$$
\begin{aligned}
E[RC] &= A_D + c_D Q + \frac{c_D D}{\mu} + \frac{c_D D}{\mu}\, E[N_L] - \frac{c_D D(1-\beta_x)(1-\hat{p})}{\alpha_x \mu}\, e^{-\mu x/D} - c_D x E[N_L] \\
&+ (c_L - c_D) r E[N_L] + \frac{c_D \hat{p} D(1-\beta_x)}{\alpha_x \mu}\, e^{-\mu x/D} - \frac{c_D \hat{p} D}{\mu}\, E[N_L] \\
&+ \left[A_L + c_L R\right]E[N_L] \\
&= A_D + c_D Q + \frac{c_D D}{\mu} - c_D x E[N_L] + \left[A_L + c_L R\right]E[N_L] \\
&+ \frac{c_D D(1-\hat{p})}{\mu}\, E[N_L] + (c_L - c_D) r E[N_L] - \frac{c_D D(1-\beta_x)(1-\hat{p})}{\alpha_x \mu}\, e^{-\mu x/D}
\end{aligned}
$$

Adding and subtracting the term $c_D DRE[N_L]$ yields the result,

$$
\begin{aligned}
E[RC] &= A_D + c_D D\left[\frac{Q}{D} + \frac{1}{\mu} + \frac{(R-x)}{D}\ E[N_L]\right] + \left[A_L + (c_L - c_D)R\right] E[N_L] \\
&+ \left[\frac{c_D D(1-\hat{p})}{\mu} + (c_L - c_D)r\right]\ E[N_L] - \frac{c_D D(1-\beta_x)(1-\hat{p})e^{-\mu x/D}}{\mu\alpha_x} \\
&= A_D + c_D DE[T] + \left[A_L + (c_L - c_D)(r+R) + \frac{c_D D(1-\hat{p})}{\mu}\right] E[N_L] \\
&- \frac{c_D D(1-\beta_x)(1-\hat{p})e^{-\mu x/D}}{\mu\alpha_x} \quad \square
\end{aligned}
$$

Proof of Expected On-Hand Inventory for Cases 1 & 2 :

According to the scenarios given by Figures 4.1 and 4.2 the on-hand inventory held in a cycle can be given as follows:

$$
OH = \begin{cases}
\frac{(2s+Q-D\tau_{1:x})}{2}\left[\frac{Q}{D} + \tau_{1:x}\right] & \text{w.p.} \quad \beta_x \ \text{ and } \ 0 < \tau_{1:x} \le \frac{s-x}{D} \\
\frac{(s+Q)^2 - x^2}{2D} + \frac{(2x - D\tau_{2:x})\tau_{2:x}}{2} & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \ , \ i = 0,1,2,... \\
+\frac{iR^2}{2D} & \text{and } \ 0 < \tau_{2:x} \le \frac{x}{D} \\
\frac{(s+Q)^2}{2D} + \frac{iR^2}{2D} & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \ , \ i = 0,1,2,... \\
& \text{and } \ \frac{x}{D} < \tau_{2:x} \le \frac{(\hat{p}x + r)}{\hat{p}D}
\end{cases} \tag{4.205}
$$

In (4.205) let,

$$
\begin{aligned}
OH_1 &= \frac{(2s+Q-D\tau_{1:x})}{2}\left[\frac{Q}{D} + \tau_{1:x}\right] \\
OH_2 &= \frac{(s+Q)^2 - x^2}{2D} + \frac{(2x - D\tau_{2:x})\tau_{2:x}}{2} + \frac{iR^2}{2D} \\
OH_3 &= \frac{(s+Q)^2}{2D} + \frac{iR^2}{2D}
\end{aligned}
$$

Therefore, the expected inventory carried in a cycle can be written as follows:

$$E[OH] = \beta_x E[OH_1] + (1-\beta_x)\alpha_x \sum_{i=0}^{\infty} [E[OH_2] + E[OH_3]]\,(1-\alpha_x)^i \quad (4.206)$$

Next, we will derive the components of (4.206). Thus,

$$E[OH_1] = \frac{Q(Q+2s)}{2D} + sE[\tau_{1:x}] - \frac{D}{2}E[{\tau_{1:x}}^2] \quad (4.207)$$

Substituting (4.5) and (4.7) in (4.207) yields:

$$E[OH_1] = \frac{Q(Q+2s)}{2D} + \frac{s}{\mu} + \frac{(s-x)(1-\beta_x)}{\beta_x}\left[-\frac{s}{D} + \frac{(s-x)}{2D} + \frac{1}{\mu}\right] - \frac{D}{\mu^2} \quad (4.208)$$

$$\begin{aligned}
E[OH_2] &= \int_0^{x/D} E[OH_2|\tau_{2:x}=t]g_{\tau_{2:x}}(t)dt \\
&= \left[\frac{(s+Q)^2-x^2}{2D} + \frac{iR^2}{2D}\right]\int_0^{x/D} g_{\tau_{2:x}}(t)dt \\
&+ x\int_0^{x/D} t g_{\tau_{2:x}}(t)dt - \frac{D}{2}\int_0^{x/D} t^2 g_{\tau_{2:x}}(t)dt \\
&= \frac{1}{\alpha_x}\left[\left[\frac{(s+Q)^2-x^2}{2D} + \frac{iR^2}{2D}\right](1-e^{-\mu x/D}) + x\left[\frac{(1-e^{-\mu x/D})}{\mu} - \frac{xe^{-\mu x/D}}{D}\right]\right. \\
&- \left.\frac{D}{2}\left[-\frac{x^2e^{-\mu x/D}}{D^2} + 2\left[-\frac{xe^{-\mu x/D}}{\mu D} + \frac{(1-e^{-\mu x/D})}{\mu^2}\right]\right]\right] \\
&= \frac{1}{\alpha_x}\left[\left[\frac{(s+Q)^2-x^2}{2D} + \frac{iR^2}{2D} - \frac{D}{\mu^2}\right](1-e^{-\mu x/D}) - \frac{x^2}{2D} + \frac{x}{\mu}\right] \quad (4.209)
\end{aligned}$$

Similarly,

$$\begin{aligned}
E[OH_3] &= \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} E[OH_3|\tau_{2:x}=t]g_{\tau_{2:x}}(t)dt \\
&= \left[\frac{(s+Q)^2}{2D} + \frac{iR^2}{2D}\right]\int_{x/D}^{(\hat{p}x+r)/\hat{p}D} g_{\tau_{2:x}}(t)dt \\
&= \left[\frac{(s+Q)^2}{2D} + \frac{iR^2}{2D}\right]\frac{\left(e^{-\mu x/D} - (1-\alpha_x)\right)}{\alpha_x} \quad (4.210)
\end{aligned}$$

Thence, from (4.209) and (4.210) we have:

$$\begin{aligned} E[OH_2]+E[OH_3] &= \frac{(s+Q)^2}{2D}+\frac{iR^2}{2D} \\ &- \frac{1}{\alpha_x}\left[\frac{x^2}{2D}-\frac{x}{\mu}+\frac{D(1-e^{-\mu x/D})}{\mu^2}\right] \end{aligned} \tag{4.211}$$

Substituting (4.208) and (4.211) in (4.206) gives:

$$\begin{aligned} E[OH] &= \beta_x\left[\frac{Q(Q+2s)}{2D}+\frac{s}{\mu}+\frac{(s-x)(1-\beta_x)}{\beta_x}\left[-\frac{s}{D}+\frac{(s-x)}{2D}+\frac{1}{\mu}\right]-\frac{D}{\mu^2}\right] \\ &+ (1-\beta_x)\alpha_x\left\{\left[\frac{(s+Q)^2}{2D}-\frac{1}{\alpha_x}\left[\frac{x^2}{2D}-\frac{x}{\mu}+\frac{D(1-e^{-\mu x/D})}{\mu^2}\right]\right]\sum_{i=0}^{\infty}(1-\alpha_x)^i\right. \\ &+ \left.\frac{R^2}{2D}\sum_{i=0}^{\infty}i(1-\alpha_x)^i\right\} \\ &= \beta_x\left[\frac{Q(Q+2s)}{2D}+\frac{s}{\mu}+\frac{(s-x)(1-\beta_x)}{\beta_x}\left[-\frac{s}{D}+\frac{(s-x)}{2D}+\frac{1}{\mu}\right]-\frac{D}{\mu^2}\right] \\ &+ (1-\beta_x)\left[\frac{(s+Q)^2}{2D}-\frac{1}{\alpha_x}\left[\frac{x^2}{2D}-\frac{x}{\mu}+\frac{D(1-e^{-\mu x/D})}{\mu^2}\right]\right] \\ &+ \frac{R^2(1-\beta_x)(1-\alpha_x)}{2D\alpha_x} \\ &= \frac{Q(Q+2s)}{2D}+\frac{s}{\mu}-\frac{D}{\mu^2}\left[\beta_x+\frac{(1-e^{-\mu x/D})(1-\beta_x)}{\alpha_x}\right] \\ &+ \left[R^2-x^2+\frac{2Dx}{\mu}\right]\frac{E[N_L]}{2D} \quad \square \end{aligned}$$

Proof of Expected Backorder per Time for Cases 1 & 2 :

Time weighted backorder within a cycle for Case 1 and Case 2 can be stated as follows due to the realizations given by Figures 4.1 and 4.2:

$$BO_1 = \begin{cases} \frac{ir^2}{2\hat{p}D} & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \ \ , \ i = 0,1,2,... \\ & \text{and} \ \ 0 < \tau_{2:x} \leq \frac{x}{D} \\ \\ \frac{\left(\tau_{2:x} - \frac{x}{D}\right)^2 \hat{p}D}{2} + \frac{ir^2}{2\hat{p}D} & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \ \ , \ i = 0,1,2,... \\ & \text{and} \ \ \frac{x}{D} < \tau_{2:x} \leq \frac{(\hat{p}x+r)}{\hat{p}D} \end{cases} \tag{4.212}$$

In (4.212) let,

$$\begin{aligned} BO_{11} &= \frac{ir^2}{2\hat{p}D} \\ BO_{12} &= \frac{\left(\tau_{2:x} - \frac{x}{D}\right)^2 \hat{p}D}{2} + \frac{ir^2}{2\hat{p}D} \end{aligned}$$

Hence, the expected time weighted backorder in a cycle can be written as follows:

$$E[BO_1] = (1-\beta_x)\alpha_x \sum_{i=0}^{\infty} \left[E[BO_{11}] + E[BO_{12}]\right](1-\alpha_x)^i \tag{4.213}$$

The expected BO_{11} and BO_{12}can be found by conditioning on $\tau_{2:x}$. Therefore,

$$\begin{aligned} E[BO_{11}] &= \int_0^{x/D} E\left[BO_{11} | \tau_{2:x} = t\right] g_{\tau_{2:x}}(t)dt \\ &= \frac{ir^2}{2\hat{p}D} \int_0^{x/D} g_{\tau_{2:x}}(t)dt \\ &= \frac{ir^2(1 - e^{-\mu x/D} \ )}{2\hat{p}D\alpha_x} \end{aligned} \tag{4.214}$$

Similarly,

$$
\begin{aligned}
E[BO_{12}] &= \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} E[BO_{12}|\tau_{2:x}=t]g_{\tau_{2:x}}(t)dt \\
&= \frac{\hat{p}D}{2}\int_{x/D}^{(\hat{p}x+r)/\hat{p}D} t^2 g_{\tau_{2:x}}(t)dt - \hat{p}x\int_{x/D}^{(\hat{p}x+r)/\hat{p}D} t g_{\tau_{2:x}}(t)dt \\
&+ \left[\frac{\hat{p}x^2}{2D}+\frac{ir^2}{2\hat{p}D}\right]\int_{x/D}^{(\hat{p}x+r)/\hat{p}D} g_{\tau_{2:x}}(t)dt \\
&= \frac{1}{\alpha_x}\left\{\frac{\hat{p}D}{2}\left[-\left(\frac{\hat{p}x+r}{\hat{p}D}\right)^2 e^{-\mu(\hat{p}x+r)/\hat{p}D} + \left(\frac{x}{D}\right)^2 e^{-\mu x/D}\right.\right. \\
&+ 2\left\{-\frac{(\hat{p}x+r)e^{-\mu(\hat{p}x+r)/\hat{p}D}}{\hat{p}D\mu}+\frac{xe^{-\mu x/D}}{D\mu}\right. \\
&- \left.\left.\frac{\left[(1-\alpha_x)-e^{-\mu x/D}\right]}{\mu^2}\right\}\right] - \hat{p}x\left[-\frac{(\hat{p}x+r)(1-\alpha_x)}{\hat{p}D}\right. \\
&+ \left.\frac{xe^{-\mu x/D}}{D}-\frac{\left[(1-\alpha_x)-e^{-\mu x/D}\right]}{\mu}\right] \\
&+ \left.\left[\frac{\hat{p}x^2}{2D}+\frac{ir^2}{2\hat{p}D}\right]\left[e^{-\mu x/D}-(1-\alpha_x)\right]\right\} \\
&= -\frac{r^2(1-\alpha_x)}{2\hat{p}D\alpha_x}-\frac{r(1-\alpha_x)}{\mu\alpha_x}+\frac{\hat{p}D(1-\beta_x)}{\mu^2}\left[1-\frac{(1-e^{-\mu x/D})}{\alpha_x}\right] \\
&+ \frac{ir^2}{2\hat{p}D}\left[1-\frac{(1-e^{-\mu x/D})}{\alpha_x}\right] \qquad (4.215)
\end{aligned}
$$

Clearly, adding (4.214) and (4.215) yields:

$$
\begin{aligned}
E[BO_{11}]+E[BO_{12}] &= -\frac{r^2(1-\alpha_x)}{2\hat{p}D\alpha_x}-\frac{r(1-\alpha_x)}{\mu\alpha_x} \\
&+ \frac{\hat{p}D}{\mu^2}\left[1-\frac{(1-e^{-\mu x/D})}{\alpha_x}\right]+\frac{ir^2}{2\hat{p}D} \qquad (4.216)
\end{aligned}
$$

Substituting (4.216) in (4.213) gives:

$$
\begin{aligned}
E[BO_1] &= (1-\beta_x)\alpha_x \left\{ \left[-\frac{r^2(1-\alpha_x)}{2\hat{p}D\alpha_x} - \frac{r(1-\alpha_x)}{\mu\alpha_x} \right.\right. \\
&+ \left.\left. \frac{\hat{p}D}{\mu^2}\left[1-\frac{(1-e^{-\mu x/D})}{\alpha_x}\right]\right] \sum_{i=0}^{\infty}(1-\alpha_x)^i + \frac{r^2}{2\hat{p}D}\sum_{i=0}^{\infty} i(1-\alpha_x)^i \right\} \\
&= (1-\beta_x)\left[-\frac{r^2(1-\alpha_x)}{2\hat{p}D\alpha_x} - \frac{r(1-\alpha_x)}{\mu\alpha_x} + \frac{\hat{p}D}{\mu^2}\left[1-\frac{(1-e^{-\mu x/D})}{\alpha_x}\right]\right] \\
&+ \frac{r^2(1-\beta_x)(1-\alpha_x)}{2\hat{p}D\alpha_x} \\
&= \frac{\hat{p}D(1-\beta_x)}{\mu^2}\left[1-\frac{(1-e^{-\mu x/D})}{\alpha_x}\right] - \frac{rE[N_L]}{\mu} \quad \square
\end{aligned}
$$

Proof of Expected Backorder per Unit for Cases 1 & 2 :

Similarly, the number of backordered units within a cycle can be given as:

$$
BO_2 = \begin{cases} ir & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \ , \ i=0,1,2,\ldots \\ & \text{and} \ \ 0 < \tau_{2:x} \le \frac{x}{D} \\ \\ \left(\tau_{2:x} - \frac{x}{D}\right)\hat{p}D + ir & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \ , \ i=0,1,2,\ldots \\ & \text{and} \ \ \frac{x}{D} < \tau_{2:x} \le \frac{(\hat{p}x+r)}{\hat{p}D} \end{cases} \tag{4.217}
$$

In (4.217) let,

$$
\begin{aligned}
BO_{21} &= ir \\
BO_{22} &= \left(\tau_{2:x} - \frac{x}{D}\right)\hat{p}D + ir
\end{aligned}
$$

Hence, the expectation of (4.217) can be written as follows:

$$
E[BO_2] = (1-\beta_x)\alpha_x \sum_{i=0}^{\infty}\left[E[BO_{21}] + E[BO_{22}]\right](1-\alpha_x)^i \tag{4.218}
$$

where,

$$\begin{aligned} E[BO_{21}] &= \int_0^{x/D} E\left[BO_{21}|\tau_{2:x}=t\right] g_{\tau_{2:x}}(t)dt \\ &= \frac{ir(1-e^{-\mu x/D}\ )}{\alpha_x} \end{aligned} \tag{4.219}$$

and

$$\begin{aligned} E[BO_{22}] &= \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} E[BO_{22}|\tau_{2:x}=t]g_{\tau_{2:x}}(t)dt \\ &= \hat{p}D\int_{x/D}^{(\hat{p}x+r)/\hat{p}D} tg_{\tau_{2:x}}(t)dt + (ir-\hat{p}x)\int_{x/D}^{(\hat{p}x+r)/\hat{p}D} g_{\tau_{2:x}}(t)dt \\ &= \frac{1}{\alpha_x}\left\{\hat{p}D\left[-\frac{(\hat{p}x+r)(1-\alpha_x)}{\hat{p}D} + \frac{xe^{-\mu x/D}}{D} + \frac{\left[e^{-\mu x/D}\ -(1-\alpha_x)\right]}{\mu}\right]\right. \\ &+ \left.(ir-\hat{p}x)\left[e^{-\mu x/D}\ -(1-\alpha_x)\right]\right\} \\ &= -\frac{r(1-\alpha_x)}{\alpha_x} + \left[\frac{\hat{p}D}{\mu}+ir\right]\frac{\left[e^{-\mu x/D}\ -(1-\alpha_x)\right]}{\alpha_x} \end{aligned} \tag{4.220}$$

Thus,

$$E[BO_{21}]+E[BO_{22}] = -\frac{r(1-\alpha_x)}{\alpha_x} + \frac{\hat{p}D\left[e^{-\mu x/D}\ -(1-\alpha_x)\right]}{\mu\alpha_x} + ir \tag{4.221}$$

Substituting (4.221) in (4.218) gives the result as follows:

$$\begin{aligned} E[BO_2] &= (1-\beta_x)\alpha_x\left\{\left[-\frac{r(1-\alpha_x)}{\alpha_x} + \frac{\hat{p}D\left[e^{-\mu x/D}\ -(1-\alpha_x)\right]}{\mu\alpha_x}\right]\sum_{i=0}^{\infty}(1-\alpha_x)^i\right. \\ &+ \left. r\sum_{i=0}^{\infty} i(1-\alpha_x)^i\right\} \\ &= (1-\beta_x)\left[-\frac{r(1-\alpha_x)}{\alpha_x} + \frac{\hat{p}D\left[e^{-\mu x/D}\ -(1-\alpha_x)\right]}{\mu\alpha_x}\right] + \frac{r(1-\beta_x)(1-\alpha_x)}{\alpha_x} \\ &= \frac{\hat{p}D(1-\beta_x)}{\mu}\left[1-\frac{(1-e^{-\mu x/D}\ )}{\alpha_x}\right] \quad \square \end{aligned}$$

Proof of Expected Number of Lost Sales for Cases 1 & 2 :

From the scenarios depicted in Figures 4.1 and 4.2, the number of lost sales within a cycle is as:

$$LS = \begin{cases} \frac{ir(1-\hat{p})}{\hat{p}} & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \;\; , \;\; i = 0,1,2,... \\ & \text{and} \;\; 0 < \tau_{2:x} \leq \frac{x}{D} \\ \\ \left(\tau_{2:x} - \frac{x}{D}\right)(1-\hat{p})D + \frac{ir(1-\hat{p})}{\hat{p}} & \text{w.p.} \quad (1-\beta_x)\alpha_x(1-\alpha_x)^i \;\; , \;\; i = 0,1,2,... \\ & \text{and} \;\; \frac{x}{D} < \tau_{2:x} \leq \frac{(\hat{p}x+r)}{\hat{p}D} \end{cases} \tag{4.222}$$

In (4.222) let,

$$\begin{aligned} LS_1 &= \frac{ir(1-\hat{p})}{\hat{p}} \\ LS_2 &= \left(\tau_{2:x} - \frac{x}{D}\right)(1-\hat{p})D + \frac{ir(1-\hat{p})}{\hat{p}} \end{aligned}$$

Similarly, we can write the expected number of lost sales as follows:

$$E[LS] = (1-\beta_x)\alpha_x \sum_{i=0}^{\infty} \left[E[LS_1] + E[LS_2]\right](1-\alpha_x)^i \tag{4.223}$$

where,

$$\begin{aligned} E[LS_1] &= \int_0^{x/D} E\left[LS_1 | \tau_{2:x} = t\right] g_{\tau_{2:x}}(t)dt \\ &= \frac{ir(1-\hat{p})(1-e^{-\mu x/D}\ )}{\hat{p}\alpha_x} \end{aligned} \tag{4.224}$$

and

$$
\begin{aligned}
E[LS_2] &= \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} E[LS2|\tau_{2:x}=t] g_{\tau_{2:x}}(t)dt \\
&= (1-\hat{p})D \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} t g_{\tau_{2:x}}(t)dt + (1-\hat{p})\left[\frac{ir}{\hat{p}} - x\right] \int_{x/D}^{(\hat{p}x+r)/\hat{p}D} g_{\tau_{2:x}}(t)dt \\
&= \frac{1}{\alpha_x}\left\{(1-\hat{p})D\left[-\frac{(\hat{p}x+r)(1-\alpha_x)}{\hat{p}D} + \frac{xe^{-\mu x/D}}{D} + \frac{\left[e^{-\mu x/D} - (1-\alpha_x)\right]}{\mu}\right]\right. \\
&+ \left.(1-\hat{p})\left[\frac{ir}{\hat{p}} - x\right]\left[e^{-\mu x/D} - (1-\alpha_x)\right]\right\} \\
&= -\frac{r(1-\hat{p})(1-\alpha_x)}{\hat{p}\alpha_x} + \left[\frac{D}{\mu} + \frac{ir}{\hat{p}}\right]\frac{(1-\hat{p})\left[e^{-\mu x/D} - (1-\alpha_x)\right]}{\hat{p}\alpha_x} \qquad (4.225)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[LS_1] + E[LS_2] &= -\frac{r(1-\hat{p})(1-\alpha_x)}{\hat{p}\alpha_x} + \frac{(1-\hat{p})D\left[e^{-\mu x/D} - (1-\alpha_x)\right]}{\mu\alpha_x} \\
&+ \frac{ir(1-\hat{p})}{\hat{p}} \qquad (4.226)
\end{aligned}
$$

Substituting (4.226) in (4.223) yields:

$$
\begin{aligned}
E[LS] &= (1-\beta_x)\alpha_x\left\{\left[-\frac{r(1-\hat{p})(1-\alpha_x)}{\hat{p}\alpha_x} + \frac{(1-\hat{p})D\left[e^{-\mu x/D} - (1-\alpha_x)\right]}{\mu\alpha_x}\right]\sum_{i=0}^{\infty}(1-\alpha_x)^i\right. \\
&+ \left.\frac{r(1-\hat{p})}{\hat{p}}\sum_{i=0}^{\infty} i(1-\alpha_x)^i\right\} \\
&= (1-\beta_x)\left[-\frac{r(1-\hat{p})(1-\alpha_x)}{\hat{p}\alpha_x} + \frac{(1-\hat{p})D\left[e^{-\mu x/D} - (1-\alpha_x)\right]}{\mu\alpha_x}\right] \\
&+ \frac{r(1-\hat{p})(1-\beta_x)(1-\alpha_x)}{\hat{p}\alpha_x} \\
&= \frac{(1-\hat{p})D(1-\beta_x)}{\mu}\left[1 - \frac{(1-e^{-\mu x/D})}{\alpha_x}\right] \quad \square
\end{aligned}
$$

Proof of Expected Cycle Time for Case 3:

Due to the realizations depicted in Figure 4.3 the cycle time for Case 3 can be given as follows,

$$T = \begin{cases} \frac{Q}{D} + \tau_{1:3} & \text{w.p.} \quad \beta_3 \\ \frac{Q+s}{D} + \frac{R}{\hat{p}D} + \frac{i(r-R)}{\hat{p}D} + \tau_{2:3} & \text{w.p.} \quad (1-\beta_3)\alpha_3(1-\alpha_3)^i, \quad i = 0, 1, 2, ... \end{cases} \tag{4.227}$$

taking the expectation of (4.227) yields

$$\begin{aligned} E[T] &= \beta_3 \left[\frac{Q}{D} + E[\tau_{1:3}] \right] + (1-\beta_3)\alpha_3 \sum_{i=0}^{\infty} \left[\frac{(Q+s)}{D} + \frac{R}{\hat{p}D} + i\frac{(r-R)}{\hat{p}D} + E[\tau_{2:3}] \right] (1-\alpha_3)^i \\ &= \beta_3 \left[\frac{Q}{D} + \frac{1}{\mu} - \frac{(\hat{p}s + R)(1-\beta_3)}{\hat{p}D\beta_3} \right] \\ &+ (1-\beta_3)\alpha_3 \left[\frac{(Q+s)}{D} + \frac{R}{\hat{p}D} + \frac{1}{\mu} - \frac{(r-R)(1-\alpha_3)}{\hat{p}D\alpha_3} \right] \sum_{i=0}^{\infty} (1-\alpha_3)^i \\ &+ \frac{(1-\beta_3)\alpha_3(r-R)}{\hat{p}D} \sum_{i=0}^{\infty} i(1-\alpha_3)^i \\ &= \frac{\beta_3 Q}{D} + \frac{\beta_3}{\mu} - \frac{(\hat{p}s + R)(1-\beta_3)}{\hat{p}D} \\ &+ (1-\beta_3) \left[\frac{(Q+s)}{D} + \frac{R}{\hat{p}D} + \frac{1}{\mu} - \frac{(r-R)(1-\alpha_3)}{\hat{p}D\alpha_3} \right] \\ &+ \frac{(r-R)(1-\beta_3)(1-\alpha_3)}{\hat{p}D\alpha_3} \\ &= \frac{Q}{D} + \frac{1}{\mu} \quad \square \end{aligned}$$

Proof of Expected Replenishment Cost for Case 3:

Expected replenishment cost of Case 3 can be derived using the following equation,

$$RC = \begin{cases} A_D + c_D(Q + D\tau_{1:3}) & \text{w.p.} \quad \beta_3 \ \text{ and } \ 0 < \tau_{1:3} \leq \frac{s}{D} \\ A_D + c_D\left[Q + s(1-\hat{p}) + \hat{p}D\tau_{1:3}\right] & \text{w.p.} \quad \beta_3 \ \text{ and } \ \frac{s}{D} < \tau_{1:3} \leq \frac{(\hat{p}s+R)}{\hat{p}D} \\ A_D + c_D(Q + s + R + \hat{p}D\tau_{2:3}) & \text{w.p.} \quad (1-\beta_3)\alpha_3(1-\alpha_3)^i, \ \ i = 0, 1, 2, ... \\ +i\left[A_L + c_L(r-R)\right] & \text{and} \quad 0 < \tau_{2:3} \leq \frac{(r-R)}{\hat{p}D} \end{cases} \tag{4.228}$$

In (4.228) let,

$$RC_1 = A_D + c_D(Q + D\tau_{1:3})$$

$$RC_2 = A_D + c_D\left[Q + s(1-\hat{p}) + \hat{p}D\tau_{1:3}\right]$$

$$RC_3 = A_D + c_D(Q + s + R + \hat{p}D\tau_{2:3}) + i\left[A_L + c_L(r-R)\right]$$

Thus, expected replenishment cost can be stated as:

$$E[RC] = \beta_3\left[E[RC_1] + E[RC_2]\right] + (1-\beta_3)\alpha_3\sum_{i=0}^{\infty} E[RC_3](1-\alpha_3)^i \tag{4.229}$$

Expected RC_1 and RC_2 can be found by conditioning on $\tau_{1:3}$ as follows:

$$\begin{aligned} E[RC_1] &= \int_0^{s/D} E[RC_1|\tau_{1:3} = t]g_{\tau_{1:3}}(t)dt \\ &= (A_D + c_DQ)\int_0^{s/D} g_{\tau_{1:3}}(t)dt + c_DD\int_0^{s/D} tg_{\tau_{1:3}}(t)dt \\ &= \frac{1}{\beta_3}\left\{\left[A_D + c_D\left(Q + \frac{D}{\mu}\right)\right](1 - e^{-\mu s/D}\ ) - c_Dse^{-\mu s/D}\right\} \end{aligned} \tag{4.230}$$

Similarly,

$$\begin{aligned}
E[RC_2] &= \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} E[RC_2|\tau_{1:3}=t] g_{\tau_{1:3}}(t)dt \\
&= \left[A_D + c_D\left[Q+(1-\hat{p})s\right]\right] \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} g_{\tau_{1:3}}(t)dt + c_D\hat{p}D \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} t g_{\tau_{1:3}}(t)dt \\
&= \frac{1}{\beta_3}\left\{\left[A_D + c_D\left[Q+(1-\hat{p})s+\frac{\hat{p}D}{\mu}\right]\right]\left(e^{-\mu s/D} - e^{-\mu(\hat{p}s+R)/\hat{p}D}\right)\right. \\
&+ \left. c_D\left[-(\hat{p}s+R)e^{-\mu(\hat{p}s+R)/\hat{p}D} + \hat{p}se^{-\mu s/D}\right]\right\} \\
&= \frac{1}{\beta_3}\left\{\left[A_D + c_D\left[Q+s+\frac{\hat{p}D}{\mu}\right]\right]\left[e^{-\mu s/D} - (1-\beta_3)\right] - c_D R(1-\beta_3)\right\}
\end{aligned} \tag{4.231}$$

Thus,

$$\begin{aligned}
E[RC_1] + E[RC_2] &= A_D + c_D(Q+s) + \frac{c_D}{\beta_3}\left[\frac{D(1-e^{-\mu s/D})}{\mu} - s - R(1-\beta_3)\right. \\
&+ \left.\frac{\hat{p}D\left[e^{-\mu s/D} - (1-\beta_3)\right]}{\mu}\right]
\end{aligned} \tag{4.232}$$

Employing result (4.13), we can write the expected value of RC_3 as follows:

$$\begin{aligned}
E[RC_3] &= A_D + c_D\left[Q+s+R+\hat{p}DE[\tau_{2:3}]\right] + i\left[A_L + c_L(r-R)\right] \\
&= A_D + c_D\left[Q+s+R+\frac{\hat{p}D}{\mu} - \frac{(r-R)(1-\alpha_3)}{\alpha_3}\right] \\
&+ i\left[A_L + c_L(r-R)\right]
\end{aligned} \tag{4.233}$$

Therefore, substituting (4.232) and (4.233) in (4.229) and making necessary simplifications yields the expected replenishment cost as follows:

$$\begin{aligned}
E[RC] &= \beta_3 \left\{ A_D + c_D(Q+s) + \frac{c_D}{\beta_3} \left[\frac{D(1-e^{-\mu s/D})}{\mu} - s - R(1-\beta_3) \right. \right. \\
&+ \left. \left. \frac{\hat{p}D\left[e^{-\mu s/D} - (1-\beta_3)\right]}{\mu} \right] \right\} + (1-\beta_3)\left\{ A_D + c_D \left[Q + s + R + \frac{\hat{p}D}{\mu} \right. \right. \\
&+ \left. \left. - \frac{(r-R)(1-\alpha_3)}{\alpha_3} + \left[A_L + c_L(r-R)\right] E[N_L] \right\} \right. \\
&= A_D + c_D D E[T] + \left[A_L + (c_L - c_D)(r-R)\right] E[N_L] \\
&- \frac{c_D D e^{-\mu s/D}(1-\hat{p})}{\mu} \quad \square
\end{aligned}$$

Proof of Expected On-Hand for Case 3:

For Case 3, the total inventory carried within a cycle is given with the following equation,

$$OH = \begin{cases} \frac{Q(Q+2s)}{2D} + s\tau_{1:3} - \frac{D{\tau_{1:3}}^2}{2} & \text{w.p.} \quad \beta_3 \text{ and } 0 < \tau_{1:3} \leq \frac{s}{D} \\ \frac{(s+Q)^2}{2D} & \text{w.p.} \quad \beta_3 \text{ and } \frac{s}{D} < \tau_{1:3} \leq \frac{(\hat{p}s+R)}{\hat{p}D} \\ \frac{(s+Q)^2}{2D} & \text{w.p.} \quad (1-\beta_3)\alpha_3(1-\alpha_3)^i, \; i = 0,1,2,... \\ & \text{and} \quad 0 < \tau_{2:3} \leq \frac{(r-R)}{\hat{p}D} \end{cases} \tag{4.234}$$

In (4.234) let,

$$\begin{aligned}
OH_1 &= \frac{Q(Q+2s)}{2D} + s\tau_{1:3} - \frac{D{\tau_{1:3}}^2}{2} \\
OH_2 &= \frac{(s+Q)^2}{2D}
\end{aligned}$$

Hence, the expected on-hand inventory carried within a cycle can be stated as:

$$E[OH] = \beta_3\left[E[OH_1] + E[OH_2]\right] + (1-\beta_3)\alpha_3 OH_2 \sum_{i=0}^{\infty}(1-\alpha_3)^i \tag{4.235}$$

where,

$$
\begin{aligned}
E[OH_1] &= \int_0^{s/D} E[RC_1|\tau_{1:3}=t] g_{\tau_{1:3}}(t)dt \\
&= \frac{Q(Q+2s)}{2D}\int_0^{s/D} g_{\tau_{1:3}}(t) + s\int_0^{s/D} t g_{\tau_{1:3}}(t) - \frac{D}{2}\int_0^{s/D} t^2 g_{\tau_{1:3}}(t) \\
&= \frac{1}{\beta_3}\left\{\frac{Q(Q+2s)(1-e^{-\mu s/D}\ )}{2D} + s\left[\frac{(1-e^{-\mu s/D}\ )}{\mu} - \frac{s}{D}\right]\right. \\
&- \left.\frac{D}{2}\left[-\left(\frac{s}{D}\right)^2 e^{-\mu s/D}\ + 2\left[-\frac{se^{-\mu s/D}}{\mu D} + \frac{(1-e^{-\mu s/D}\ )}{\mu^2}\right]\right]\right\} \\
&= \frac{1}{\beta_3}\left\{\left[\frac{Q(Q+2s)}{2D} - \frac{D}{\mu^2}\right](1-e^{-\mu s/D}\ ) - \frac{s^2 e^{-\mu s/D}}{2D} + \frac{s}{\mu}\right\} \qquad (4.236)
\end{aligned}
$$

and

$$
\begin{aligned}
E[OH_2] &= \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} E[RC_2|\tau_{1:3}=t] g_{\tau_{1:3}}(t)dt \\
&= \frac{(s+Q)^2}{2D}\int_{s/D}^{(\hat{p}s+R)/\hat{p}D} g_{\tau_{1:3}}(t) \\
&= \frac{(s+Q)^2\left[e^{-\mu s/D}\ - e^{-\mu(\hat{p}s+R)/\hat{p}D}\ \right]}{2D\beta_3} \\
&= \frac{(s+Q)^2\left[(1-e^{-\mu s/D}\ ) - \beta_3\right]}{2D\beta_3} \qquad (4.237)
\end{aligned}
$$

Therefore, from (4.236) and (4.237) we have:

$$
E[OH_1] + E[OH_2] = \frac{1}{\beta_3}\left\{\frac{s}{\mu} - \frac{D(1-e^{-\mu s/D}\ )}{\mu^2} - \frac{s^2}{2D} + \frac{\beta_3(s+Q)^2}{2D}\right\} \qquad (4.238)
$$

Employing (4.238) in (4.235) and making necessary simplifications yields the expected on-hand inventory carried in a cycle as follows:

$$
\begin{aligned}
E[OH] &= \frac{s}{\mu} - \frac{D(1-e^{-\mu s/D}\ )}{\mu^2} - \frac{s^2}{2D} + \frac{\beta_3(s+Q)^2}{2D} + \frac{(1-\beta_3)(s+Q)^2}{2D} \\
&= \frac{Q(Q+2s)}{2D} + \frac{s}{\mu} - \frac{D(1-e^{-\mu s/D}\ )}{\mu^2} \quad \square
\end{aligned}
$$

Proof of Expected Backorder per Time for Case 3:

According to the possible realizations depicted by Figure 4.3, the time weighted backorders within a cycle can be given as follows:

$$BO_1 = \begin{cases} \left(\tau_{1:3} - \frac{s}{D}\right)^2 \frac{\hat{p}D}{2} & \text{w.p.} \quad \beta_3 \ \text{ and } \ \frac{s}{D} < \tau_{1:3} \leq \frac{(\hat{p}s+R)}{\hat{p}D} \\ \frac{R^2}{2\hat{p}D} + \frac{i(r^2-R^2)}{2\hat{p}D} + R\tau_{2:3} + \frac{\hat{p}D{\tau_{2:3}}^2}{2} & \text{w.p.} \quad (1-\beta_3)\alpha_3(1-\alpha_3)^i, \ i = 0, 1, 2, ... \\ & \text{and} \quad 0 < \tau_{2:3} \leq \frac{(r-R)}{\hat{p}D} \end{cases} \tag{4.239}$$

In (4.239) let,

$$BO_{11} = \left(\tau_{1:3} - \frac{s}{D}\right)^2 \frac{\hat{p}D}{2}$$

$$BO_{12} = \frac{R^2}{2\hat{p}D} + \frac{i(r^2 - R^2)}{2\hat{p}D} + R\tau_{2:3} + \frac{\hat{p}D{\tau_{2:3}}^2}{2}$$

Thus, we can state the expected backorder per time as follows:

$$E[BO_1] = \beta_3 E[BO_{11}] + (1-\beta_3)\alpha_3 \sum_{i=0}^{\infty} E[BO_{12}](1-\alpha_3)^i \tag{4.240}$$

where,

$$\begin{aligned} E[BO_{11}] &= \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} E[BO_{11}|\tau_{1:3} = t] g_{\tau_{1:3}}(t)dt \\ &= \frac{\hat{p}D}{2} \left\{ \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} t^2 g_{\tau_{1:3}}(t)dt - \frac{2s}{D} \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} t g_{\tau_{1:3}}(t)dt \right. \\ &+ \left. \frac{s^2}{D^2} \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} g_{\tau_{1:3}}(t)dt \right\} \\ &= \frac{\hat{p}D}{2\beta_3} \left\{ \left[\frac{s}{D} + \frac{1}{\mu}\right]^2 e^{-\mu s/D} - \left[\frac{(\hat{p}s+R)}{\hat{p}D} + \frac{1}{\mu}\right]^2 (1-\beta_3) \right. \\ &+ \frac{\left[e^{-\mu s/D} - (1-\beta_3)\right]}{\mu^2} - \frac{2s}{D} \left\{ \left[\frac{s}{D} + \frac{1}{\mu}\right] e^{-\mu s/D} \right. \\ &- \left. \left. \left[\frac{(\hat{p}s+R)}{\hat{p}D} + \frac{1}{\mu}\right](1-\beta_3) \right\} + \frac{s^2 \left[e^{-\mu s/D} - (1-\beta_3)\right]}{D^2} \right\} \\ &= \frac{1}{\beta_3} \left\{ \frac{\hat{p}De^{-\mu s/D}}{\mu^2} - \left[\frac{\hat{p}D}{\mu^2} + \frac{R^2}{2\hat{p}D} + \frac{R}{\mu}\right](1-\beta_3) \right\} \end{aligned} \tag{4.241}$$

and the expected value of BO_{12} can be found by employing results (4.13) and (4.14) as follows:

$$
\begin{aligned}
E[BO_{12}] &= \frac{R^2}{2\hat{p}D} + \frac{i(r^2-R^2)}{2\hat{p}D} + RE[\tau_{2:3}] + \frac{\hat{p}DE[\tau_{2:3}{}^2]}{2} \\
&= \frac{R^2}{2\hat{p}D} + \frac{i(r^2-R^2)}{2\hat{p}D} + \frac{R}{\mu} - \frac{R(r-R)(1-\alpha_3)}{\hat{p}D\alpha_3} + \frac{\hat{p}D}{\mu^2} \\
&- \frac{(r-R)(1-\alpha_3)}{2\alpha_3}\left[\frac{2}{\mu} + \frac{(r-R)}{\hat{p}D}\right] \\
&= \frac{R^2}{2\hat{p}D} + \frac{i(r^2-R^2)}{2\hat{p}D} + \frac{R}{\mu} + \frac{\hat{p}D}{\mu^2} \\
&- \frac{(1-\alpha_3)}{\alpha_3}\left[\frac{(r^2-R^2)}{2\hat{p}D} + \frac{(r-R)}{\mu}\right]
\end{aligned} \tag{4.242}
$$

Substituting the results (4.241) and (4.242) in (4.240), we obtain the expected backorder per time as follows:

$$
\begin{aligned}
E[BO_1] &= \frac{\hat{p}De^{-\mu s/D}}{\mu^2} - \left[\frac{\hat{p}D}{\mu^2} + \frac{R^2}{2\hat{p}D} + \frac{R}{\mu}\right](1-\beta_3) \\
&+ (1-\beta_3)\alpha_3\left\{\left[\frac{R^2}{2\hat{p}D} + \frac{R}{\mu} + \frac{\hat{p}D}{\mu^2}\right.\right. \\
&- \left.\left.\frac{(1-\alpha_3)}{\alpha_3}\left[\frac{(r^2-R^2)}{2\hat{p}D} + \frac{(r-R)}{\mu}\right]\right]\sum_{i=0}^{\infty}(1-\alpha_3)^i\right. \\
&+ \left.\frac{(r^2-R^2)}{2\hat{p}D}\sum_{i=0}^{\infty} i(1-\alpha_3)^i\right\} \\
&= \frac{\hat{p}De^{-\mu s/D}}{\mu^2} - \frac{(r-R)E[N_L]}{\mu} \quad \square
\end{aligned}
$$

Proof of Expected Backorder per Unit for Case 3:

Similarly, the number of units backordered within a cycle can be found as:

$$BO_2 = \begin{cases} \left(\tau_{1:3} - \frac{s}{D}\right)\hat{p}D & \text{w.p.} \quad \beta_3 \quad \text{and} \quad \frac{s}{D} < \tau_{1:3} \leq \frac{(\hat{p}s+R)}{\hat{p}D} \\ R + i(r-R) + \hat{p}D\tau_{2:3} & \text{w.p.} \quad (1-\beta_3)\alpha_3(1-\alpha_3)^i, \;\; i = 0,1,2,... \\ & \text{and} \quad 0 < \tau_{2:3} \leq \frac{(r-R)}{\hat{p}D} \end{cases} \tag{4.243}$$

In (4.243) let,

$$\begin{aligned} BO_{21} &= \left(\tau_{1:3} - \frac{s}{D}\right)\hat{p}D \\ BO_{22} &= R + i(r-R) + \hat{p}D\tau_{2:3} \end{aligned}$$

Similarly, we can write the expected backorder per unit as follows:

$$E[BO_2] = \beta_3 E[BO_{21}] + (1-\beta_3)\alpha_3 \sum_{i=0}^{\infty} E[BO_{22}](1-\alpha_3)^i \tag{4.244}$$

where,

$$\begin{aligned} E[BO_{21}] &= \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} E[BO_{21}|\tau_{1:3} = t] g_{\tau_{1:3}}(t)dt \\ &= \hat{p}D \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} t g_{\tau_{1:3}}(t)dt - \hat{p}s \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} g_{\tau_{1:3}}(t)dt \\ &= \frac{1}{\beta_3}\left\{\frac{\hat{p}De^{-\mu s/D}}{\mu} - \left[\frac{\hat{p}D}{\mu} + R\right](1-\beta_3)\right\} \end{aligned} \tag{4.245}$$

and

$$\begin{aligned} E[BO_{22}] &= R + i(r-R) + \hat{p}DE[\tau_{2:3}] \\ &= R + i(r-R) + \frac{\hat{p}D}{\mu} - \frac{(r-R)(1-\alpha_3)}{\alpha_3} \end{aligned} \tag{4.246}$$

Therefore, substituting (4.245) and (4.246) in (4.244) yields the expected backorder per unit as follows:

$$\begin{aligned} E[BO_2] &= \frac{\hat{p}De^{-\mu s/D}}{\mu} - \left[\frac{\hat{p}D}{\mu} + R\right](1-\beta_3) \\ &+ (1-\beta_3)\alpha_3\left\{\left[R + \frac{\hat{p}D}{\mu} - \frac{(r-R)(1-\alpha_3)}{\alpha_3}\right]\sum_{i=0}^{\infty}(1-\alpha_3)^i \right. \\ &+ \left. (r-R)\sum_{i=0}^{\infty} i(1-\alpha_3)^i\right\} \\ &= \frac{\hat{p}De^{-\mu s/D}}{\mu} \quad \square \end{aligned}$$

Proof of Expected Number of Lost Sales for Case 3:

For Case 3 the number of lost sales within a cycle can be given by the following equation:

$$LS = \begin{cases} \left(\tau_{1:3} - \frac{s}{D}\right)(1-\hat{p})D & \text{w.p.} \quad \beta_3 \quad \text{and} \quad \frac{s}{D} < \tau_{1:3} \leq \frac{(\hat{p}s+R)}{\hat{p}D} \\ \frac{R(1-\hat{p})}{\hat{p}} + \frac{i(r-R)(1-\hat{p})}{\hat{p}} + (1-\hat{p})D\tau_{2:3} & \text{w.p.} \quad (1-\beta_3)\alpha_3(1-\alpha_3)^i, \; i = 0,1,2,... \\ & \text{and} \quad 0 < \tau_{2:3} \leq \frac{(r-R)}{\hat{p}D} \end{cases} \tag{4.247}$$

In (4.247) let,

$$\begin{aligned} LS_1 &= \left(\tau_{1:3} - \frac{s}{D}\right)(1-\hat{p})D \\ LS_2 &= \frac{R(1-\hat{p})}{\hat{p}} + \frac{i(r-R)(1-\hat{p})}{\hat{p}} + (1-\hat{p})D\tau_{2:3} \end{aligned} \tag{4.248}$$

Thus, expected number of lost sales within a cycle can be given as

$$E[LS] = \beta_3 E[LS_1] + (1-\beta_3)\alpha_3\sum_{i=0}^{\infty} E[LS_2](1-\alpha_3)^i \tag{4.249}$$

where,

$$
\begin{aligned}
E[LS_1] &= \int_{s/D}^{(\hat{p}s+R)/\hat{p}D} E[LS_1|\tau_{1:3}=t]g_{\tau_{1:3}}(t)dt \\
&= (1-\hat{p})D\int_{s/D}^{(\hat{p}s+R)/\hat{p}D} tg_{\tau_{1:3}}(t)dt - (1-\hat{p})s\int_{s/D}^{(\hat{p}s+R)/\hat{p}D} g_{\tau_{1:3}}(t)dt \\
&= \frac{(1-\hat{p})}{\beta_3}\left\{\frac{De^{-\mu s/D}}{\mu} - \left[\frac{D}{\mu}+\frac{R}{\hat{p}}\right](1-\beta_3)\right\} \qquad (4.250)
\end{aligned}
$$

and

$$
\begin{aligned}
E[LS_2] &= \frac{R(1-\hat{p})}{\hat{p}} + \frac{i(r-R)(1-\hat{p})}{\hat{p}} + (1-\hat{p})DE[\tau_{2:3}] \\
&= \frac{(1-\hat{p})}{\hat{p}}\left\{R + i(r-R) + \frac{\hat{p}D}{\mu} - \frac{(r-R)(1-\alpha_3)}{\alpha_3}\right\} \qquad (4.251)
\end{aligned}
$$

Thus, plugging the results (4.250) and (4.251) in (4.249) gives the expected number of lost sales within a cycle as follows:

$$
\begin{aligned}
E[LS] &= (1-\hat{p})\left\{\frac{De^{-\mu s/D}}{\mu} - \left[\frac{D}{\mu}+\frac{R}{\hat{p}}\right](1-\beta_3)\right\} \\
&+ \frac{(1-\hat{p})(1-\beta_3)\alpha_3}{\hat{p}}\left\{\left[R+\frac{\hat{p}D}{\mu} - \frac{(r-R)(1-\alpha_3)}{\alpha_3}\right]\sum_{i=0}^{\infty}(1-\alpha_3)^i\right. \\
&+ \left.(r-R)\sum_{i=0}^{\infty} i(1-\alpha_3)^i\right\} \\
&= \frac{(1-\hat{p})De^{-\mu s/D}}{\mu} \quad \square
\end{aligned}
$$

Chapter 5

Summary & Conclusion

In this study, we focus on some of the dynamic aspects of single item, single location inventory systems. In three different research projects, we develop models to investigate the impacts of nonstationarity in demand rate and unfixed purchasing prices.

In Chapter 2, we consider a continuous review inventory system of a critical service part in which the demand rate drops to a lower level at a known future time. We assume a one-for-one replenishment policy with full backordering and a fixed lead time. Adaptation to the lower demand rate is achieved by reducing the base stock level before the obsolescence occurs and letting the demand process take away the difference. Under the assumption of fixed base stock levels, we derive approximate expressions for the operating characteristics and propose an approximate solution method for the optimal time to shift to the new control policy by minimizing the expected total cost incurred during the transient period.

There are two main contributions of this chapter:

1. We analyze the obsolescence problem specifically for a slow moving, expensive item in a continuous review setting with nonstationary stochastic demand. Our findings are consistent with earlier works that the obsolescence has significant effects on operating costs and should be taken into account explicitly. We extend these findings by showing that for a continuous review inventory system advance policy change results in significant cost savings. Our numerical experiments revealed that if the control policy is not changed in advance then the transient period costs are on average doubled. Furthermore, we found that the timing of the control policy change

primarily determines the tradeoff between backordering penalties and obsolescence costs.

2. We provide practical heuristic formulas to tradeoff the risk of obsolescence and backordering specifically for expensive, slow moving items with high downtime costs. For this class of items, it is well known that continuous review policies are preferred over periodic review ones since they require lower safety stocks for the same level of availability. Thus, our formulas can be used as a managerial guide in studying the impacts of advance policy change on operational costs and obsolete inventories.

In Chapter 3, we conduct an exact analysis of the system considered in Chapter 2 under a more general policy. We propose a three parameter transition control policy with advance policy change option. As such, the policy parameters consist of pre- and post-obsolescence base stock levels and the time to reduce base stock level from high to low. The objective is to find the optimal base stock levels and the optimal policy change time minimizing the expected total discounted cost over an infinite horizon.

We show that when base stock levels are identical, the optimal base stock level can be calculated from a critical ratio inequality. For different base stock levels, we compute the total discounted cost function by partly relying on the Fast Fourier Transform method and suggest a numerical optimization procedure to find the optimal values of the policy parameters.

The main contributions of this chapter are the following:

1. We find that the policy change option leads to pronounced cost savings especially when obsolescence requires a relatively large adjustment in base stock level. A large adjustment in base stock level indicates a bigger tradeoff between obsolescence and backordering costs. Thus, it becomes more difficult to balance these costs with a fixed base stock level. When there is the option to change the base stock level, the inventory system can operate under a higher base stock level to avoid backordering when demand is healthy and then reduce it to a lower level to minimize obsolete stocks.

2. We measure the cost of ignoring obsolescence and develop insights into the impact of the time and the size of obsolescence on total costs. Our numerical experiments

show that ignoring obsolescence in control policy increases the costs significantly. This is because the inventory system carries more stock than necessary due to the higher base stock level calculated by assuming stationary demand. The increase is largest when obsolescence is expected at a near future and the drop in demand rate is sharp since post-obsolescence holding costs increase steeply when the discounting horizon is short and the post-obsolescence attrition rate is low.

3. We show that when obsolescence can be foreseen, an early adaptation of the base stock level can lead to important savings. When obsolescence requires a reduction in the base stock level, the proposed transition control policy can decrease the post-obsolescence inventory build-up while balancing availability by early adjustment of the base stock level. Savings are most pronounced when a large drop occurs and the demand rate is relatively high since for those instances the potential inventory build-up is highest.

4. Our results indicate that for slow movers, inventory costs can be cut drastically by increasing the efforts to foresee the time and the size of obsolescence.

In Chapter 4, we investigate the impact of price discounts given by a supplier on replenishment and stocking decisions of a firm. We consider a single item, single location, continuous review inventory system where a supplier offers price discounts at random points in time. We assume that deals arrive according to the Poisson process and the deal price is known and fixed. Moreover, we assume that the demand is deterministic and lead times are negligible. We propose a four parameter continuous review control policy and derive exact expressions of the operating characteristics. We provide equations to calculate the optimal policy parameters minimizing the total cost rate function.

The contributions of this chapter can be summarized as follows:

1. We model partial backorders in a system with random deal offerings and constant demand. As such, we generalize the model proposed by Moinzadeh (1997) and extend the available literature.

2. We show that allowing backorders may result in significant savings especially for high discount rates. As supplier offers deeper price discounts, purchasing from the

list price becomes relatively more expensive compared to stockout costs. Therefore, allowing partial backorders while waiting for a good deal becomes more advantageous than replenishing from the list price. Thus, a policy with the option of planned backorders yields lower costs.

3. In case a supplier offers deals more frequently, savings by allowing partial backorders diminishes. This is because with more frequent deal offers, the number of list replenishments and stockouts both decrease. Thus, allowing backorders becomes less beneficial and the (r, R, s, Q) policy converges to the (R, s, Q) policy. Moreover, the decrease in savings is larger for deeper discounts due to the bigger tradeoff between stockout costs and the cost of replenishing from the higher price.

Although the models discussed in Chapters 2 and 3 are concerned with a single location inventory system, it is natural to ask their extension and applicability in a network context. The immediate application of the proposed policies in a network context would be to consider all stocking locations in a certain geographical area as a single, aggregated inventory system supplied from a central warehouse. If the assumptions of the original model about the ordering policy and the demand process remain valid for the aggregated inventory system then one can estimate the average time and the average size of a drop in the aggregated demand rate and use the proposed policies.

Indeed, the results found by aggregation will be approximate due to decoupling of the stocking decisions in the central warehouse from the stocking locations. However, such application might be useful to obtain insights about how the total stocking level of a certain part in a particular geographical region should be adjusted if a drop in demand rate is likely. Currently, we are collaborating with a mainframe computer manufacturer for a similar implementation of the model.

The extension of the three parameter policy proposed in Chapter 3 to the multi-echelon case without aggregation and decoupling would be an interesting and relevant research direction. However, we expect that the exact computation of the optimal policies for this problem would be involved. Yet, METRIC type approximations can be an option.

The models studied in Chapters 2 and 3 require the transient analysis of a stochastic system which increases the complexity of the analysis drastically. Thus, we do not see direct and easy extensions towards a more general changes in demand rate or a general policy including batch ordering. Nevertheless, these extensions might be possible with a different approach such as discretizing the time domain and using dynamic programming. On the other hand, a similar policy to the one proposed in Chapter 3 can be constructed to build-up stocks to adapt to the increase in demand rate. Such policy might be useful, in situations where the replenishment lead times are volatile or there is a limit to the number of items ordered from the supplier at one order. We expect that the analysis of the inventory build-up policy would be relatively easier compared to the inventory run-down policy for the increase in the inventory position process is independent of the demand arrival process. As such, the build-up policy might be extended to incorporate multiple upward jumps in demand rate.

In Chapters 2 and 3, our focus was on managing inventories in face of the changes in demand rate while in Chapter 4, our focus was on managing inventories when there is price uncertainty on the supply side. It would be interesting to combine these two aspects in a model. For example, in Chapters 2 and 3, we assumed that the purchasing price of a part is fixed. Since most of the critical service parts are repairable, this is a reasonable assumption as long as the repairing costs change relatively slowly by time. Moreover, the equipment usually have long life cycles so that the parts are available via suppliers or secondary markets for long time periods. However, when a producer stops producing a certain part then this might create a scarcity in the market and lead to an increase in the purchasing price of the part. Simultaneously, the demand for the part might go down as a result of generation upgrades of the aging equipment. In such a scenario, the base stock level would be affected not only by the drop in demand rate but also the increase in price. Hence, it would be interesting to investigate the behavior of the optimal base stock level and the underlying tradeoffs.

Nederlandse Samenvatting (Summary in Dutch)

In deze studie ontwikkelen en analyzeren we modellen voor een aantal dynamische aspecten van voorraadsystemen. We concentreren ons op twee hoofdthema's die apart geanalyseerd worden: niet-stationariteit van de vraag en wisselende vraagprijzen.

In het eerste deel van de studie bekijken we een voorraadsysteem met een niet-stationaire vraag. We focusseren op kritieke service onderdelen onderhevig aan wegvallende vraag. Voorraadbeheer van zulke onderdelen is moeilijk vanwege de lage omloopsnelheid en de mogelijke hoge kosten bij ontbreken ervan alsook wanneer ze overtollig worden.

In de praktijk is er een behoefte aan strategien voor serviceonderdelen die deze aspecten meenemen en toch makkelijk in te voeren zijn. We stellen een nieuwebesturingsstrategie voor die vraaguitval beschouwt en bepalen de bijbehorende kosten. Het blijkt dat het veronachtzamen van vraaguitval in de voorraadstrategie de kosten behoorlijk doet oplopen en dat een vroege aanpassing van de voorraden de kosten aanzienlijk kan reduceren.

In het tweede deel van de studie bekijken we een voorraadsysteem waar de leverancier prijskortingen aanbiedt op willekeurige tijdstippen. We breiden de wetenschappelijke kennis uit door een meer algemene naleveringstructuur aan te nemen. Dit houdt in dat wanneer er geen voorraad meer is, een klant met een bepaalde kans nageleverd wordt en dat in het andere geval de vraag verloren gaat. We karakteriseren uitdrukkingen voor de parameters van de optimale strategie door een analytische optimalisatie en laten zien dat het toelaten van nalevering in een wisselende prijs omgeving kan leiden tot aanzienlijke kostenbesparingen.

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