Self-Organization in Communication Networks

Venkatesh Bala* Sanjeev Goyal†

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Abstract

We develop a dynamic model to study the formation of communication networks. In this model, individuals periodically make decisions concerning the continuation of existing information links and the formation of new information links, with their cohorts. These decisions trade off the costs of forming and maintaining links against the potential rewards from doing so. We analyze the long run behavior of this process of link formation and dissolution.

Our results establish that this process always *self-organizes*, i.e., irrespective of the number of agents, and the initial network, the dynamic process converges to a limit social communication network with probability one. Furthermore, we prove that the limiting network is invariably either a wheel network or the empty network.

We show in the (corresponding) static network formation game that, while a variety of architectures can be sustained in equilibrium, the wheel is the unique efficient architecture for the interesting class of parameters. Thus, our results imply that the dynamics have strong equilibrium selection properties.

Key Words: networks, coordination, learning, path-dependence, self-organization.

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*Dept. of Economics, McGill University, 855 Sherbrooke Street West, Montreal, Canada H3A 1A8. E-mail: bala@inbv.musicb.mcgill.ca

†Econometric Institute, Erasmus University, PO Box 1738, 3000 DR, Rotterdam, The Netherlands. E-mail: goyal@few.eur.nl

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1 Introduction

The importance of communication networks has been extensively documented in empirical work and their role in social learning has been highlighted in recent theoretical research.[1] The finding that communication networks matter naturally leads to the question: which network structures are reasonable? This question motivates the formulation of a theory of network formation.

In this paper, we propose an approach to network formation which is inspired by the following story: consider a group of individuals who have to periodically choose an action without being fully informed about the true payoffs from the different options. Each agent has some information concerning the payoffs, which includes the agent's personal experiences as well as information gathered from other individuals. Prior to choosing his next action, an individual has an opportunity to contact[2] a subset of agents to access their information. In deciding with whom to form a link, the individual trades-off the costs and the potential benefits from doing so. His costs and benefits take into account the fact that information has a public good aspect: well-connected people (i.e. those who possess information collected from many sources) generate a positive externality, and the individual has an incentive to contact such people directly rather than all the individual sources accessed by them.

In this setting, we study the dynamic process of social communication. In particular, we ask:

- Does the process of network formation settle down, and if so, what is the architecture of the communication network that emerges?

- Given that information has some of the characteristics of a public good, what is the relationship between socially efficient networks and the networks derived from choices made by self-interested individuals?

[1] See Allen (1982), Coleman (1966), Granovetter (1974) and Rogers and Shoemaker (1971) for early work and Bala and Goyal (1993) for recent work and related references.

[2] The nature of this contact depends on the context. For a scientist who is about to carry out a new project, and would like to know about recent research in related areas, reading a working paper constitutes a contact. For a consumer wishing to find information about different brands of cars, computers or mutual funds, accessing a web page devoted to the evaluation of these products constitutes a contact.

The framework we use to address the above questions has the following structure. We consider a group of n individuals, each of whom has (private) information which is worth $V > 0$. This information can be augmented by forming *pairwise* links with other agents; every link has an associated cost $c > 0$. The public good aspect of network formation is captured by the following assumption: when an agent i forms a link with some other agent j he gets access to all the information possessed by the latter, *including* the information that j has acquired by forming links himself. At regular intervals, an individual gets an opportunity to revise his links with other agents.[3] When faced with this opportunity, an agent chooses a strategy – forms links with a subset of his cohorts – which is a (myopic) best response to the existing set of links of the other agents. If more than one strategy is optimal then he is assumed to randomize over the set of optimal pure strategies.

The above action revision process generates a Markov chain on the state space of all networks. In analyzing the process of network evolution, we are naturally lead to the concept of *self-organization*. We say that the dynamic process exhibits self-organization from an initial network if the Markov chain converges (in finite time) to a limiting network, with probability one.

In our formulation, with n agents the total number of networks is given by $2^{n(n-1)}$, implying that the cardinality of the state space increases very rapidly with the number of agents in the society. Intuitively, because the coordination problem becomes increasingly complicated as the size of the society increases, it would seem that self-organization will at best occur in small societies and perhaps only from certain initial networks. Nevertheless, our main results, Theorems 3.1 and 3.2, yield the following startling conclusion: *irrespective of the number of agents in the society, and the initial network, the learning process invariably exhibits self-organization, i.e. converges with probability one to a limit network.* Our results also characterize precisely the architecture[4] of the limiting networks: if the value of private information with any agent V is more than the cost of forming a link c, then the process almost surely converges to a wheel network, which is the unique efficient

[3]Formally, in every period each agent, with positive probability, gets an option to form or dissolve links with other agents. This probability is assumed to be independent across agents.

[4]By architecture we mean the equivalence class of all networks obtained by permuting the strategies of agents in a given network. See the discussion in Section 2.

architecture in our model.[5] Thus not only is there self-organization, but it is also efficient. When V is less than c the process converges either to the empty network or to a wheel. Thus self-organization also occurs in this case, but efficiency may not be attained. In fact, it is possible to show that the process exhibits *path-dependence*: starting from certain initial networks, there is a positive probability of converging either to a wheel or to the empty network.

The results on self-organization are striking for several reasons. The first reason is related to the convergence of the dynamic process. Our theorems reveal that even when agents pursue self-interested goals in a myopic way, and make no attempt to coordinate their actions with other agents through a social planner or institution, they can nevertheless achieve a stable pattern of communication links in the long run.

Secondly, we note that the static (one-shot) game has many equilibrium networks with widely varying architectural and welfare properties (see Propositions 2.1-2.5).[6] The dynamics thus possess remarkable equilibrium selection properties: while there are a number of architectures which can be supported as Nash equilibria of the static game, the learning process converges to a network having one of only two such architectures – the empty network and the wheel.

The third reason relates to the rate of convergence. The value of Theorems 3.1 and 3.2 would be compromised if self-organization took place very slowly. Our simulations of the learning dynamics suggest that the rate of convergence to a limiting network is extremely rapid both when communication costs are low ($V > c$) and when they are high ($V < c$). In the former case, with $n = 7$ agents there are $2^{n(n-1)} = 2^{42} \approx 4 \times 10^{12}$ possible networks, yet the process converges on average in less than 40 periods to a wheel! When the communication costs are high ($V < c$) convergence is even more rapid. Furthermore, in virtually all cases, the limit network is a wheel rather than the empty network.

[5] The graph of a wheel has agents located on a circle, with each agent accessing his predecessor. We say that a network is efficient if it maximizes the net aggregate benefit, i.e. the total value of information that all the individuals acquire less the cost of all the communication channels that support the network (for formal definitions see Section 2).

[6] See Figure 2 below for some examples of equilibrium networks in a society with $n = 5$ agents. In addition, our computations reveal that if $V > c$ then for $n = 3, 4, 5$ and 6 the number of equilibrium networks in the one-shot game is 5, 58, 1069 and in excess of 20,000, respectively. This corresponds to 2, 5, 16 and more than 30 possible architectures for n = 3,4,5 and 6 respectively.

In order to study the evolution of networks, we have chosen a particularly simple parametric model of social communication. Our choice is motivated by the need for analytical tractability: even in such a basic model, a full characterization of the dynamics is quite difficult. One simplifying assumption we have made in particular is to suppose that information does not "decay" in the process of transmission across agents.[7] To investigate the robustness of our results on self-organization, we briefly discuss an extension of the model where the quality of information decays as it is communicated across agents. This is a very complex problem: however, our preliminary analysis and simulations indicate that in societies of moderate size, self-organization occurs with high probability and that the limiting networks are natural generalizations of a wheel network. Interestingly, in this case, self-organizing networks seem to be constituted of *local neighborhoods*: different subsets of agents form small wheels – the local neighborhoods – while other agents have links across these wheels.[8]

Our paper is a contribution to the theory of network formation. There is a large literature in economics (in addition to work in sociology and computer science) on the subject of networks.[9] Much of this work is concerned with the efficiency aspects of different communication networks within firms and takes a 'planner's problem' approach to characterize optimal networks.[10] By contrast, we are concerned with the self-organization properties of networks and look at social and economic settings where individuals decide independently on their sources of information and these decisions define a social communication network.

In recent years, Jackson and Wolinsky (1996), among others, have studied network formation in a similar spirit.[11] The existing papers have been concerned with the relationship between efficient and sustainable networks, in static settings. This relationship is also one of our concerns, but the present paper departs from the existing work by considering the

[7] Assuming no information decay is analogous to the assumption of "no friction" in physics, "markets with perfect information" in a general equilibrium model and "zero transaction costs" in finance.

[8] See Figure 18 below for some examples of such networks.

[9] For detailed reviews of this literature, see Antonelli (1992), Flament (1963), Harary (1972), Luce (1951), Marshak and Radner (1972), Rogers (1971).

[10] For recent work in this tradition, see Bolton and Dewatripont (1994), Radner (1993), and the papers referred to therein. Hendricks, Piccione and Tan (1995) use a similar approach to characterize the optimal flight network for an unregulated airline which has to serve a given set of cities.

[11] Their paper draws upon the work of Myerson and Aumann, among others, which studies coalition formation in (cooperative) game theoretic models. This work is surveyed in Myerson (1991) and van den Nouweland (1993). See also Dutta, van den Nouweland and Tijs (1995) and Dutta and Muttuswamy (1996).

dynamics of network formation. We believe that the dynamics are important for several reasons. Firstly, networks are observed to change over time and it is natural to study their evolution. Secondly, a dynamic formulation allows us to consider phenomena such as path-dependence which are of central importance and cannot be understood in a static model. Finally, it helps us select between different equilibria: the results in this paper are especially useful in this context.

Our paper also departs from previous work in considering asymmetric link formation: we allow for agent i to form a link with agent j without the converse being true. By contrast, previous literature has concentrated on models of symmetric link formation. This difference in formulation is motivated by a wide range of examples where communication is naturally viewed as asymmetric.[12] In view of these differences, our work should be viewed as complementary to the earlier analyses of networks.

More generally, the present paper should be seen as part of a research program in which the structure of interaction among individuals is explicitly modeled and its aggregate implications are studied. This research problem has received growing attention in the recent work on the evolution of conventions as well as on the diffusion of new technologies.[13] For the most part, this work takes as given the existence of some network structure and proceeds to analyze its implications, see, e.g., the survey paper by Kirman (1993).[14]

We end this discussion by relating our results to the recent research on boundedly rational players learning to play Nash equilibrium of the one-shot game (see e.g., Hurkens (1994) and Sanchirico (1996)). This line of research shows, roughly speaking, that if the learning process satisfies certain properties then the dynamics converge to a minimal 'curb' set of the one-shot game in the long run. The game we analyze in this paper is quite 'large' and the real issue here is: what do the minimal curb sets look like? Our main results, Theorems 3.1 and 3.2, characterize the minimal curb sets as well as establish convergence.

[12] The examples mentioned earlier in the introduction fall in this category.

[13] For studies of interaction structure and evolution of conventions, see Anderlini and Ianni (1996), Ellison (1993), Goyal and Janssen (1993) and Goyal (1996). For work on diffusion of new technologies/products see Allen (1982), An and Kiefer (1992), Bala and Goyal (1993), Besley and Case (1994), Coleman (1966), Ellison and Fudenberg (1993, 1995) and Rogers (1971, 1983).

[14] An exception to this approach is the work by Mailath, Samuelson and Shaked (1992,1996), which explores endogenous interactions within a matching model. They show that when agents may choose whom they wish to interact with, heterogeneous outcomes which partition the society into groups with different payoffs can be evolutionarily stable. Our concern is not about heterogeneity of outcomes, but whether agents can learn to attain a stable pattern of communication when information is a public good.

We also note that since the learning process we study is different, the convergence results are of independent interest.

The plan of the paper is as follows. We introduce the basic model and present the static results in Section 2. The dynamics are analyzed in Section 3. We consider a model with information decay in Section 4, while Section 5 concludes. All the proofs are collected in an appendix at the end of the paper.

2 The Model and Static Results

Let N be a set of agents and let i and j be typical members of this set. The agents are numbered from 1 to n. To avoid trivialities, we shall assume throughout that $n \geq 3$. Each agent has some private information which is commonly valued at $V > 0$. An agent can augment this information by communicating with other people; this communication takes time and effort and is made possible via the setting up of *pair-wise* channels of communications, each of which cost $c > 0$.[15]

A *strategy* of agent $i \in N$ is a (row) vector $g_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n})$ where $g_{i,j} \in \{0,1\}$ for each $j \in N \backslash \{i\}$. The statement '$g_{i,j} = 1$' is interpreted as saying that agent i forms a link with agent j (in other words, has direct access to j's information) while $g_{i,j} = 0$ states that i does not directly communicate with agent j. The set of all strategies of agent i is denoted by $\mathcal{G}_i$. Since agent i has the option of forming or not forming a link with each of the remaining $n-1$ agents, the number of strategies of agent i is clearly $|\mathcal{G}_i| = 2^{n-1}$. The set $\mathcal{G} = \mathcal{G}_1 \times \ldots \times \mathcal{G}_n$ is the strategy space of all the agents. A strategy profile $g = (g_1, \ldots, g_n)$ can be represented as a *directed network*. Figure 1 below provides an example with $n = 3$ agents:

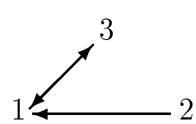

Figure 1

[15]Some of the material in this section is drawn from an earlier paper by Goyal (1993), which was circulated under the title "Sustainable Communication Networks".

Here agent 1 has formed links with agents 2 and 3, agent 3 has a link with agent 1 while agent 2 does not link up with any other agent. Note that $g_{i,j} = 1$ is represented by an *edge* starting at j with the arrowhead pointing at i. Since there is a one-to-one correspondence between the set of all networks with n vertices and the set of strategies $\mathcal{G}$, we shall use the terms 'network' and 'strategy profile' interchangeably.

Our model assumes that even if i does not form a link with j, i may be able to obtain j's information indirectly, say by forming a link with someone who forms a link with j, and so on. To formalize this notion, we require some additional definitions. In the network g, let $N_d(i;g) = \{k \in N \mid g_{i,k} = 1\}$, i.e. $N_d(i;g)$ is the set of agents with whom i maintains a link. We next say that there is a *path* from j to i in the network g if either $g_{i,j} = 1$ or there exist agents $j_1, \ldots, j_m$ distinct from each other and from i and j such that $g_{i,j_1} = g_{j_1,j_2} = \ldots = g_{j_m,j} = 1$. For example in Figure 1 there is a path from agent 2 to agent 3. The notation "$j \xrightarrow{g} i$" indicates that there exists a path in the network g from j to i. Likewise $j \xleftrightarrow{g} i$ indicates that there is a path from i to j and also from j to i. The length of the path $j \xrightarrow{g} i$ is given by the number of intervening links. Thus in Figure 1, the length of the path $2 \xrightarrow{g} 3$ is 2. In more complicated networks, there will typically exist more than one path between any two points. A *geodesic* from j to i is a path from j to i of minimum length. If there is a path from j to i in a network g, then the *distance* from j to i denoted by $d(i,j;g)$ is the length of a geodesic from j to i. We shall adopt the convention that if in a graph g there exists no path between two points j and i then $d(i,j;g) = \infty$. Also we shall assume that a single agent i constitutes a trivial path and that $d(i,i;g) = 0$, for all $g \in \mathcal{G}$.

Furthermore we define $N(i;g) = \{k \in N \mid k \xrightarrow{g} i\} \cup \{i\}$. The set $N(i;g)$ is the set of all agents whose information i accesses either directly or through other agents. We use the convention that i belongs to his own neighborhood, i.e. $i \in N(i;g)$ for all $g \in \mathcal{G}$. Finally, let g_{-i} denote the network obtained when all of agent i's links are removed. Note that the network g_{-i} can be regarded as the strategy profile where i chooses not to link with anyone. The network g can be written as $g = g_{-i} \oplus g_i$ where the '$\oplus$' indicates that g is formed as the union of the links in g_{-i} and g_i. Agent i's payoff from the network $g = g_{-i} \oplus g_i$ is defined as

$$\Pi_i(g) \;=\; |N(i;g)|V - |N_d(i;g)|c. \tag{1}$$

In other words, i's payoff is V times the number of agents accessed in the network less c times the number of agents with whom he forms links. Thus in Figure 1 agent 1's payoff is $3V - 2c$, agent 2's is V and agent 3's is $3V - c$. Given a network g, the strategy g_i is said to be a *best response* of agent i to g_{-i} if

$$\Pi_i(g_{-i} \oplus g_i) \geq \Pi_i(g_{-i} \oplus g_i'), \quad \text{for all } g_i' \in \mathcal{G}_i. \tag{2}$$

The set of all of agent i's *pure strategy* best responses to g is denoted $BR_i(g)$. Lastly, a network $g = (g_1, \ldots, g_n)$ is said to be *sustainable* if agents are playing a Nash equilibrium, i.e. for each $j \in N$ we have $g_j \in BR_j(g)$.

In our framework, the effectiveness with which useful private information gets communicated across the society is of interest. Given the cost-reward structure developed so far, this issue turns on the pattern and number of channels of communication in a network. The main welfare property of networks we consider is *aggregate efficiency*. A communication network is said to be efficient if it maximizes the difference between the aggregate payoffs and the aggregate cost of the channels. Formally, the social welfare level of a network g is given by

$$W(g) = \sum_{i \in N} \Pi_i(g) \;=\; \sum_{i \in N} |N(i; g)|V - \sum_{i \in N} |N_d(i; g)|c. \tag{3}$$

A network g is said to be efficient if $W(g) \geq W(g')$, for all $g' \in \mathcal{G}$. A communication network g is called *connected* if for every pair of agents $i,j \in N$, we have $i \overset{g}{\longleftrightarrow} j$. This is equivalent to saying that $N(i; g) = N$ for all $i \in N$. A network which is not connected is referred to as being disconnected. Furthermore, a network is said to be *empty* if $N(i; g) = \{i\}$ for all $i \in N$. We denote the empty network as g^e. Next, a network g is said to be *minimally connected* if the deletion of any link in g renders the network disconnected, i.e. if for any $i,j \in N$ satisfying $g_{i,j} = 1$, the network created by setting $g_{i,j} = 0$ is disconnected.

Lastly, we make a distinction between a network and a network architecture. A network g is simply an element of $\mathcal{G}$. Two networks $g \in \mathcal{G}$ and $g' \in \mathcal{G}$ are equivalent if g' is obtained as a permutation of the strategies of agents in g. (For example, if g is the network in Figure 1, and g' is the network where agents 1 and 2 are interchanged, then g and g' are equivalent). The equivalence relation partitions $\mathcal{G}$ into classes: each class is referred to as an *architecture*. While the relevant concept for agents in the game is the specific network which is played, our primary interest in analyzing the game lies in the architecture of the network.

2.1 Sustainable and Efficient Communication Networks

We begin by characterizing sustainable and efficient networks. Our first result considers the implications of sustainability for the structure of networks.

Proposition 2.1 *A sustainable network is either connected or empty.*

This proposition highlights a general property of networks in which agents are symmetrically located vis-a-vis information and costs of access: such networks cannot be partially connected. The intuition underlying this property may be understood in terms of the incentives to form links in a network with (say) two distinct components. In a sustainable network, if an agent forms links then she must be getting non-negative payoffs. It follows then that any agent outside a component can always increase his payoff by simply linking up with some member of this component. Thus either all members of a component are getting negative payoffs, in which case the component cannot be part of a sustainable network, or the network will be connected.

The set of connected networks is quite large and we would like to further specify the nature of sustainable networks. This motivates the next two propositions. Recall that a network is minimally connected if the deletion of any link renders it disconnected.

Proposition 2.2 *Suppose $V > c > 0$. Then a network is sustainable if and only if it is minimally connected.*

This proposition characterizes the set of sustainable networks for the case where $V > c$. The following 'monotonicity' result concerns the case of higher cost levels. Given a pair of values for V and c, let $\mathcal{S}(V, c)$ denote the set of sustainable networks.

Proposition 2.3 *(a) Suppose $V < c < c'$. Then $\mathcal{S}(V, c') \subset \mathcal{S}(V, c)$. (b) If instead we have $0 < c < V < c'$ then $\mathcal{S}(V, c') \subset \mathcal{S}(V, c) \cup \{g^e\}$ where g^e is the empty network.*

Figure 2a shows two important sustainable network architectures for $V > c$ when there are $n = 5$ agents. Note that in the first network every agent communicates with agent 4 and vice-versa. We refer to this as a star network g^s. The other network in Figure 2a is termed a wheel network g^w.[16] Many other minimally connected (and hence sustainable for $V > c$) networks may be found. Figure 2b provides some examples.

Taken together, Propositions 2.1-2.3 show that the set of sustainable networks consists of a subset of the set of minimally connected networks and the empty network g^e. In particular we note the following:

Proposition 2.4 *The architecture of sustainable networks is related to the cost levels. (a) If $0 < c < V$ then the wheel and star, among other architectures, are sustainable. (b) If $V < c < (n-1)V$ then the wheel is sustainable, but so is the empty network. The star is not sustainable. (c) If $c > (n-1)V$ then the empty network g^e is the unique sustainable architecture.*

The above results help us to understand the types of networks that will emerge when individuals make link formation decisions, based on rational calculations concerning personal payoffs. Given this characterization, it is natural to ask: are individual incentives for link formation consistent with aggregate (social) welfare maximization? The following result responds to this question by providing a complete characterization of efficient networks.

Proposition 2.5 *The efficient architecture depends on the cost levels. (a) If $0 < c < (n-1)V$ then the wheel is the unique efficient architecture. (b) If $c > (n-1)V$ then the empty network is the unique efficient architecture.*

A comparison between Propositions 2.4 and 2.5 suggests both over-provision as well as under-provision of communication links is possible (relative to the socially efficient level). Related to this is the finding that the efficient network is always sustainable. These results point to a more general feature: there exist multiple equilibria in the (static) communication network game. These equilibria are welfare ranked and correspond to very different architectures.[17] This fact motivates an inquiry into the dynamic stability of different networks, a subject that is studied in the following section.

[16] Note that with n agents there are n possible 'star' networks corresponding to which agent i acts as the 'central coordinator'. All of these come under the equivalence class of the star architecture. Likewise, the wheel architecture is the equivalence class of $(n-1)!$ networks consisting of all permutations of n agents in a circle. The empty architecture coincides with the empty network g^e since there is only one network with no links across agents.

[17] If we suppose $c \in (0, V)$, then the number of sustainable networks equals 5, 58, 1069 and in excess of 20000 for $n =$ 3,4,5 and 6 respectively, with the corresponding number of architectures being 2, 5, 16 and more than 30.

3 The Dynamics of Network Formation

We analyze a simple social learning model, which is based on a modified version of the best response dynamic. The first modification is that agents exhibit 'inertia': i.e. in each period, with a fixed positive probability less than one, agent i maintains the action chosen in the previous period. Furthermore, if the agent does not exhibit inertia, then he chooses a myopic pure strategy best response to the actions of all other agents in the previous period: if there are many such best responses, each of them is assumed to be chosen with positive probability. The last assumption introduces a certain degree of 'mixing' in the dynamic process.[18]

To state these assumptions formally, let $\mathcal{G}_{-i}$ denote the strategy space of all agents except i and for given a set A, let $\Delta(A)$ denote the set of probability distributions on A. We suppose that for each agent i there exists a number $p_i \in (0,1)$ and a function $\phi_i : \mathcal{G} \to \Delta(\mathcal{G}_i)$ where ϕ_i satisfies

$$\phi_i(g) \in \text{Interior } \Delta(BR_i(g_{-i})), \quad \forall g_{-i} \in \mathcal{G}_{-i}. \tag{4}$$

For $\hat{g}_i$ in the support of $\phi_i(g)$, the notation $\phi_i(g)(\hat{g}_i)$ denotes the probability assigned to $\hat{g}_i$ by the probability measure $\phi_i(g)$. If the network at time $t \geq 1$ is $g^t = g^t_{-i} \oplus g^t_i$, the strategy of agent i at time $t+1$ is assumed to be given by:

$$g_i^{t+1} = \begin{cases} \hat{g}_i \in \text{support } \phi_i(g), & \text{with probability } p_i \times \phi_i(g)(\hat{g}_i); \\ g_i^t, & \text{with probability } 1 - p_i. \end{cases} \tag{5}$$

Equation (5) states that with probability $p_i \in (0,1)$, agent i chooses a naive best response to the strategies of the other agents. The function ϕ_i dictates how agent i randomizes between best responses if more than one exists. Furthermore, with probability $1 - p_i$ agent i exhibits 'inertia', i.e. maintains his previous strategy. Thus in the network of Figure 1 (reproduced below for convenience) assuming $V > c$, agent 2 has two best responses; to form a link with either 1 or 3 (but not both). If agents 1 and 3 exhibit inertia, either Figure 1' or Figure 1" can occur with positive probability.

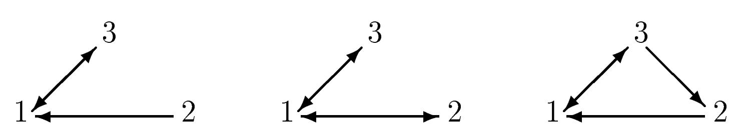

Figure 1 Figure 1' Figure 1"

[18]In recent years, considerable work has been done using the best response dynamic process. For a discussion of the behavioral and informational assumptions implicit in such a dynamic, see the discussion in Mailath (1992).

Supposing in addition that agents randomize independently of each other, our rules concerning agent choices induce a transition matrix T mapping the statespace $\mathcal{G}$ to the set of all probability distributions $\Delta(\mathcal{G})$ on $\mathcal{G}$. Let $\{X_t\}$ be the stationary Markov chain starting from the initial network $g \in \mathcal{G}$ with the above transition matrix. The process $\{X_t\}$ describes the dynamics of network evolution given our assumptions on agent behavior.

To get a first impression of the dynamics we simulate a sample trajectory with $n = 5$ agents, for a total of twelve periods (Figure 3). The initial network (labelled $t = 1$) has been drawn at random from the set of all directed networks with 5 agents.[19] As can be seen, the choices of agents evolve rapidly and settle down by period 11. The resultant communication network is a wheel. Having reached a wheel, the process stays there forever since the wheel is an absorbing state (formally, it is a strict Nash equilibrium and all such equilibria are absorbing states). The simulation naturally suggests the concept of *self-organization* in communication networks, which we now define.

Definition 3.1 *Fix an initial network g. The stochastic process $\{X_t\}$ is said to exhibit self-organization if starting at g the process converges to a limiting network, with probability 1. If convergence occurs with probability less than 1, we say that there is incomplete self-organization.*

The notion of self-organization (when it occurs) is an appealing one because it implies that agents who are myopically pursuing self-interested goals are nevertheless able to attain a stable pattern of communication links in a finite amount of time. Since the number of possible networks increases very rapidly with the number of agents,[20] it would seem that self-organization can only be expected for small values of n, and even then perhaps only from certain initial networks. We can however show the following result:

Theorem 3.1 *Suppose $V > c > 0$. For any n and any initial network g the learning process $\{X_t\}$ exhibits self-organization, i.e. converges with probability one in finite time to a limiting network. The limiting network is always a wheel.*

[19] In period $t \geq 2$, the choices of agents who exhibit inertia have been drawn in uninterrupted lines, while those whose choices are best responses have been drawn in lines interrupted by dots.

[20] Recall that for n agents there are $2^{n(n-1)}$ networks.

The broad strategy behind the proof is to demonstrate that every network different from the wheel g^w is transient. We do this by showing that starting from any network g the learning process converges to a wheel in finitely many steps with positive probability. Since a wheel (being a strict Nash equilibrium network) is an absorbing state, the conclusion then follows from standard results in the theory of Markov chains.

The main ideas of the proof can then be understood in the context of a simple example which we now discuss. Suppose that the initial 'state' of the system is given by the network g depicted in Figure 4. We initially suppose that there are seven agents labelled $E_1, \ldots, E_7$ with links depicted as in the figure. There is also a distinguished agent labelled 'n' in the network upon whom we focus: in the proof, given the initial network g, the distinguished agent may be chosen arbitrarily.

We first characterize the best response of agent n given this network. Lemmas 3.1 to 3.3 in the Appendix show that this can be found as follows: first delete all the links starting from or going to n in the graph g. Next, we can order the agents E_1 to E_7 in the form of a 'tree' as depicted in Figure 5 (ignore the arrows with dots for the moment). Note that agents E_3, E_6 and E_7 represent the 'tree-tops' as no other agents observe these agents. Since $V > c$ it is worthwhile for agent n to have a link with each of these agents; furthermore since the 'tree-top' agents observe every other agent, n need not establish any other link in his best response. Consequently n's best response is as shown by the red arrows in Figure 5. Since each agent independently chooses a best response with probability $p_i \in (0,1)$, with probability of at least $p_n \times \prod_{j \neq n}(1 - p_j)$ the stochastic process will move from g to the network g' in Figure 5. We use this idea repeatedly in the proof: at each stage we pick a suitably chosen agent, allow him to play his best response and suppose that all other agents display inertia. Under the assumptions of our learning process, the new network occurs with positive probability given its predecessor. Before passing to the next step, we remark that the network g' in Figure 5 does not depict links that other agents have with agent n which are unaltered when n chooses his best response. Thus in Figure 4, E_1, E_3, E_5 and E_7 have links with n and these are not shown in g'. The reason for our omission is that given the rules of the process, it does not matter for the proof whether or not these links exist.

In the next step, we pick agent E_1 to play his best response, with all other agents exhibiting inertia. Agent E_1 is a 'bottom' or 'root' agent vis-a-vis agent n since, in g', he does not observe any other agent (apart possibly from n himself). Furthermore we see that agent n obtains information from all the agents in the society in g'. It follows that agent E_1 can obtain all the information in the society by establishing a link with n. Since $V > c$, forming a link with n is in fact a best response for E_1. The graph g'' in Figure 6 shows the new state of the system after E_1 has played this best response (with of course all other agents exhibiting inertia). We see that in Figure 6, E_2 and E_3 continue to be 'bottom' agents vis-a-vis agent n, i.e. they have no links apart (possibly) from n himself. Thus a best response for E_2 is to simply link up with E_1, since in g'', E_1 obtains all the information in the society. Subsequently a best response for E_3 is to have a link with E_2. In Figure 7, we have collapsed these two steps into one to show the resulting network $\hat{g}$.

For the next step we see that E_4 is a 'second level' agent: she has a link with some of the 'bottom' agents of the tree in $\hat{g}$ but with no one else (again, apart possibly from n). Thus a best response of agent E_4 to $\hat{g}$ is simply to form a link with E_3 since E_3 now obtains all the information in the society. The resulting network in shown in Figure 8. It now follows that agent E_5, who is a 'third-rung' agent (i.e. observes no agent higher than 'second-rung' agents) can simply form a link with agent E_4 as his best response. Furthermore, agent E_6 can then link with E_5 as his best response, to yield the network in Figure 9. The penultimate step occurs when agent 7 forms a link with E_6 as a best response. The resulting network is a 'hyperwheel', i.e. a network which contains the wheel as a sub-network, as in Figure 10. If agent n now chooses his best response, a wheel results (with positive probability), as in Figure 11.

The complications which make the actual proof lengthier than the above description suggests arise from the possibility that E_1 to E_7 in the original network g of Figure 4 may not be agents, but in fact groups of agents who all communicate with each other by links entirely within the group. (In the proof such a group is referred to as a component). Lemmas 3.1 to 3.3 show that given any network g, and a distinguished agent n, the remaining agents can be classified into components $E_1, \ldots, E_m$ partially ordered as in Figure 5. Lemma 3.7 in the Appendix shows that when we move from the network like the one in Figure 5 to the one in Figure 6, all the agents within the component E_1 will (with positive probability) arrange themselves in a linear 'chain'. The same is shown at subsequent steps for the components E_2, E_3, etc. (also in Lemma 3.7). The result then follows in the manner of the above example.

We next examine the case where $V < c$. The analysis here is more complicated. To illustrate the main differences we present an example where $n = 3$. Clearly, if $c > 2V$ then the only sustainable network is the empty one. Furthermore, it is easy to see that in this case, starting from any network, the process will converge to the empty network eventually. The interesting case arises when $c \in (V, 2V)$. We show that for this parameter range the process may exhibit *path-dependence*: there exist initial networks g from which there is a positive probability of the process converging either to a wheel or to the empty network. To demonstrate the possibility of path dependency, suppose that the initial network is a star as shown in Figure 12a.

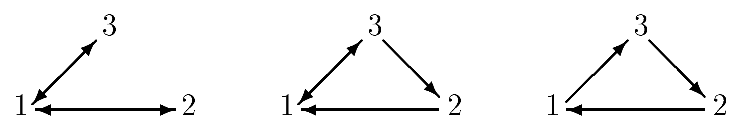

Figure 12a Figure 12b Figure 12c

Starting from the network in Figure 12a, note that a best response for agent 2 is to form a link with agent 3 and disassociate from agent 1. If the other two agents exhibit inertia, there is a positive probability of moving to the network depicted in Figure 12b. Likewise, if agent 1 chooses his best response to the new network with the other two agents maintaining their original links, the process transits to the wheel displayed in Figure 12c where it is absorbed. On the other hand, starting from the same network as in 12a, it is easily established that if the agents choose their best response in the order 1, 3, 2 and 3 then the process will be absorbed into the empty network. Our analysis thus shows that there is a 'phase transition' in the dynamics when c crosses the threshold V.

The above example raises the general question: under what circumstances does the process display self-organization and if so, what are the limiting networks when $V < c$? The following result provides a complete answer to this question.

Theorem 3.2 *Suppose $V < c$. For any n and starting from any initial network g, the (stochastic) network $\{X_t\}$ exhibits self-organization. The limiting network is either a wheel or the empty network.*

As in Theorem 3.1, the first point to note concerns the best response of agent n. When $V < c$ it is not always optimal to connect with all tree-tops. The proof exploits this idea and begins by dividing the set of networks into two subsets. The first subset consists of networks with the property that the best response of every agent involves forming no links. We observe that for this subset of networks the dynamic process converges to the empty network, with probability 1. The second subset comprises those networks in which at least one agent has a best response which involves link formation. The first step here is to note that there is an agent n whose best response involves forming links with *some* of the tree-tops. Consider the new network formed after n has chosen his best response. The second step considers the best response of a bottom agent such as E_1 (as in Figures 4-7). Two cases arise here: first, that n observes all agent who observe him, and second, that n is observed by other agents who are not observed by n. We focus on the first case. We show that it is optimal for the bottom agent E_1 to form a link with agent n only and with no other agent. For the second case, we show that agent E_1 forms a link with some agent $k \neq n$ who is linked with n and is "furthest" away from E_1. The subsequent steps follow along the lines of Theorem 3.1. The proof thus establishes that starting from *any* network in this subset, there is a strictly positive probability of transiting to a wheel (which is an absorbing network). This observation along with the example above establishes that if the initial network lies in the second set then the dynamic process converges with probability 1, and the limiting network is either a wheel or the empty network.

Theorems 3.1 and 3.2 show that the process converges and characterize the limiting networks. The value of these results would be diminished if the rate of convergence was very slow. In our setting a slow rate of convergence is a definite possibility since with as few as $n = 7$ agents there are $2^{42} \approx 4 \times 10^{12}$ possible networks. These considerations motivate an examination of the rate of convergence.

In what follows we report on some simulations of the dynamic process for n varying from 3 to 7. In the simulations we assume that $p_i = p$ for all agents. Furthermore, let $\hat{\phi}$ be such that it assigns equal probability to all best responses of an agent given a network g. We assume that all agents have the same function $\hat{\phi}$. For a fixed value n, the initial network g is chosen at random from the set of all networks with n vertices, and the process is simulated until it converges to a limiting network. When $V > c > 0$, the average convergence times over 2000 simulations for different values of n and p are shown in Figure 13. Note that except for $n = 3$, the average convergence time increases if p is close to zero or one. The intuition for this finding is that when p is small, there is a very high probability that the state of the system does not change very much from one period to the next, which raises the convergence time. When p is very large, there is a high probability that "most" agents move simultaneously. This raises the likelihood of mis-coordination which slows the process. The convergence time is thus lowest for intermediate values of p where these two effects are balanced. Remarkably the minimum convergence time increases only slowly as n increases. Even with over four trillion networks ($n = 7$), the average convergence time for $p = 0.4$ is less than 40 periods! The average convergence times are even lower when the communication cost is higher, as in Figure 14, which displays the results when $c \in (V, 2V)$.[21] Overall, there is a strong tendency towards rapid self-organization in our model.

We finally comment on the role of two assumptions on the dynamic process: the randomizing over best responses and the inertia hypothesis. Recall, that in the above process, if an agent i has two or more best responses to a given network then he randomizes between them according to the function ϕ_i. Given our observations in Section 2, it can be seen that Theorems 3.1-3.2 will not hold in the absence of this condition. This is because in the absence of randomization all Nash equilibria are absorbing and there are a large number of equilibria in the one-shot link formation game. Randomization ensures that the dynamic process moves away from non-strict Nash equilibria eventually and that only the strict Nash equilibria are rest points of the system. This property of the dynamics is thus crucial for equilibrium selection.

Our assumption that an agent i exhibits "inertia" with probability $1 - p_i$ is also crucial for the results. Consider the network depicted in Figure 15a and suppose that $V > c > 0$.

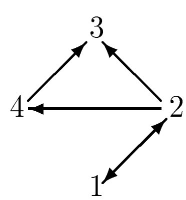

Figure 15a

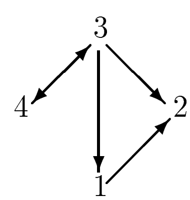

Figure 15b

It is easily verified that in this network each agent has a unique best response. If $p_i = 1$ for all i (every agent chooses his best response with probability one) then the network in Figure 15b results. Likewise, starting from the latter network, the former results if every agent chooses their best response, leading to a two-period cycle. This implies

[21]The faster convergence is not because the higher cost of communication results in a greater tendency to converge to the empty network. In our simulations for $n = 4$, convergence to the empty network occurred only once or twice per 2000 simulations and for $n = 5$,6 and 7 it was never once observed. The faster rate of self-organization seems to be because there are fewer best responses for each agent when $c > V$.

that Theorem 3.1 no longer obtains if agents do not exhibit some inertia. The role of the inertia assumption is thus to eliminate cycles which arise due to miscoordination by agents.

4 Information Decay

While the results in previous section provide a sharp characterization of limit outcomes in the learning process, they require the assumption that information does not "decay", i.e. information obtained through indirect links have the same value as that obtained through direct communication. This assumption can possibly serve as a reasonable approximation if the size of the society is "small", say $n = 3$,4 or 5. For large n, however, it is unlikely that a network such as the wheel will be sustainable because information will be transmitted through a long chain of links, with the attendant possibility that its quality gets degraded in the process.

In general, the process of network formation in the presence of information decay is very complex. In this section, we report some preliminary work which focuses on the issue of how information decay affects our earlier results on self-organization.

We model the possibility of decay by introducing a parameter $\delta \in [0, 1]$. Given a network g, it is now assumed that if an agent i has a link with another agent j, i.e. $g_{i,j} = 1$, then agent i receives information of value δV from j. More generally if the shortest path in the network from j to i is $k \geq 1$ links, then the value of agent j's information to i is $\delta^k V$. Thus decay is assumed to be geometric, and our earlier analysis of "no decay" corresponds to the case $\delta = 1$. The costs of link formation are still taken to be $c > 0$ per link for each agent.

The learning dynamics are taken to be the same as in Section 3. We start by noting that what matters for our analysis are the relative values of c/V and δ. It is easy to show that if $0 < c/V < \delta - \delta^2$ then the dominant strategy for all agents is to form links with every other agent. Hence the dynamic process will self-organize to the full network with probability 1. Likewise, in a society with n agents, if $c/V > \sum_{i=1}^{n-1} \delta^i$ then it is easily seen that the dominant strategy for each agent is not to form links with any other agent. Hence, self-organization to the empty network will occur with probability 1. The interesting range for studying self-organization is therefore as follows:

$$\delta - \delta^2 < \frac{c}{V} < \sum_{i=1}^{n-1} \delta^i. \tag{6}$$

If $n = 3$ or 4, the number of communication networks is relatively small, and it is possible to prove self-organization in a number of situations in the above parameter range. We characterize the outcomes in Figures 16 and 17.

Figure 16 covers the case of $n = 3$. It shows that if $c/V \in (\delta - \delta^2, \delta)$ then the process $\{X_t\}$ converges with probability 1 to a wheel, while if $c/V \in (\delta, \delta + \delta^2)$ then the process converges to a limit network with probability 1. In the latter case the limit network is either a wheel or the empty network.[22]

Figure 17 summarizes the findings for the case of $n = 4$. We see that if $c/V \in (\delta - \delta^3, \delta)$ then $\{X_t\}$ process exhibits self-organization, and the limit networks are either a star or a wheel, while if $c/V \in (\delta, \delta + \delta^2 + \delta^3)$ then self-organization also occurs, either to a wheel or to the empty network.[23] If $c/V \in (\delta - \delta^2, \delta - \delta^3)$ then our simulations reveal that self-organization may be incomplete from certain initial networks, i.e. with strictly positive probability convergence to a limiting network does not occur. This is related to the fact that a wheel is no longer sustainable in this parameter region. Given a choice of getting information third-hand (as will happen in a wheel with four agents) each agent will strictly prefer to form two communication links. Thus agents continually attempt to form a wheel, but are always thwarted by individual incentives to behave otherwise.

For higher values of n, analytical results are difficult to obtain as the number of cases to be considered becomes very large. Our simulations indicate that for each n there are parameter regions defined by complicated polynomial boundaries in which self-organization and incomplete self-organization are seen to occur.[24] By simulating the process at a grid of parameter values[25] in the range $c/V \in (\delta - \delta^2, \sum_{i=1}^{n-1} \delta^i)$ we found that convergence

[22]With $n = 3$ there are 64 possible networks; these networks may be classified into 3 possible types, with each type having at most four subtypes. The above characterization is proved by showing that the network of each subtype is transient and converges to a wheel or (depending upon the parameter values) to a wheel or the empty network. The proof is available from the authors on request.

[23]With 4 agents in the society, the number of possible networks is $2^{12} = 4096$, which complicates the analysis. The proof, which consists of examining about 120 different subtypes of networks, is extremely tedious and as in the case of $n = 3$, omitted.

[24]Based upon our simulations, we believe that these boundaries are defined by polynomials up to degree $n - 1$.

[25]More than 2000 combinations of c/V and δ for each n.

to a limiting network occurred in 100%, 92.6%, 89.1%, 84.0%, 79.7%, 80.1% and 76.2% of the parameter values as n ranged from $n = 3$ to $n = 9$. Since with $n = 9$ agents there are more than 10^{21} networks, the high probabilities of self-organization are remarkable.

The set of limiting networks in the presence of information decay is also of interest. When $c/V < \delta$, limiting networks which are stars, wheels, or combinations of the two types tend to occur. When $c/V > \delta$, the star, or star-like networks are no longer sustainable. In this case, limit networks tend to be wheels, collections of wheels connected to each other or the empty network. The (non-empty) self-organizing networks thus appear to be constituted of local neighborhoods. Figure 18 displays some of the possible limit networks for different n.

To summarize our discussion, in moderately sized societies, self-organization occurs with high probability, and the limiting networks are either the empty network or intuitive generalizations of the wheel. Thus, while our results on network formation in the presence of information decay are not as clear-cut as in the earlier section, the simulations and analysis suggest that Theorems 3.1 and Theorem 3.2 are fairly robust.

5 Conclusion

In this paper we present an approach to the theory of social communication based upon the notion that information networks are created by individual decisions which trade off the cost of forming and maintaining communication links against the potential rewards from doing so.

This approach is developed with the help of a simple dynamic model in which individual agents decide to form or severe links with some subset of their cohorts, at regular intervals. We examine the long run behavior of this process. Our results establish that this process invariably *self-organizes*, i.e., starting from every initial configuration of links it converges to a limit social communication network, with probability one. Moreover, we show that the limit network is either empty or a wheel, thus providing a complete characterization of the set of self-organizing networks.

We show in the (corresponding) static network formation game that, while a variety of architectures can be sustained in equilibrium, the wheel is the unique efficient architecture for the interesting class of parameters. Thus, our results imply that the dynamics have remarkable equilibrium selection properties.

Our analysis is carried out within the framework of a simple parametric model of social communication. This framework is used for reasons of analytical tractability. In order to investigate the robustness of our results, we consider an extension of the model which allows for the quality of information to decay as it is communicated across agents. Preliminary work on this model suggests that in societies of moderate size, self-organization occurs with high probability and that the limiting networks are natural generalizations of a wheel network. Interestingly, in this case, self-organizing networks seem to be constituted of *local neighborhoods*: different subsets of agents form small wheels – the local neighborhoods – while other agents have links across these wheels.

The results we obtain are striking and it seems worthwhile to examine further extensions of the framework. In particular, we assume that all agents have the same amount of valuable private information and the same costs of link formation. We also assume that maintaining an already established link has the same cost as creating a new link. Finally, we suppose that information links are entirely asymmetric, while in reality both asymmetric and symmetric communication are seen to occur. The implications of relaxing these assumptions need to be explored in future research.

6 Appendix

Proof of Proposition 2.1: If $V > c$ then it is obvious that a sustainable network will be connected. We shall assume that $V < c$.[26] Suppose g is a sustainable network which is not empty. We show that it must be connected. The proof is by contradiction. Since g is not the empty network, there exist distinct agents i and i' such that $g_{i,i'} = 1$. As $V < c$ and g is sustainable, there must exist $i'' \notin \{i, i'\}$ such that $g_{i',i''} = 1$, for otherwise i is better off by cutting his link with i'. By the same token there exists $i''' \notin \{i', i''\}$ such that $g_{i'',i'''} = 1$. Since n is finite, we see that eventually there is a cycle of agents $D = \{j_1, \ldots, j_m\}$ such that

$$g_{j_1,j_2} = g_{j_2,j_3} = \cdots = g_{j_{m-1},j_m} = g_{j_m,j_1} = 1. \tag{7}$$

[26]The proof for the case $V = c$ is available from the authors upon request.

Let $\bar{D}$ be defined as

$$\bar{D} \;=\; \Big\{k \in N \mid \exists\, j, j' \in D \text{ such that } j \xrightarrow{g} k \text{ and } k \xrightarrow{g} j'\Big\}. \qquad (8)$$

The set $\bar{D}$ consists of agents who observe some member of D and in turn are observed by some member of D. It is easy to see by virtue of (7) that given $k \in \bar{D}$, we have $k \xrightarrow{g} k'$ and $k' \xrightarrow{g} k$ if $k' \in \bar{D}$, while for $k' \notin \bar{D}$ at least one of these conditions is violated. Now define the set F as consisting of those agents outside of $\bar{D}$ who are accessed by at least one agent in $\bar{D}$, i.e.

$$F \;=\; \Big\{i \in N \backslash \bar{D} \mid \exists\, j \in \bar{D} \text{ such that } i \xrightarrow{g} j\Big\}. \qquad (9)$$

We show that F is empty. Suppose $i \in F$. Clearly there is no path in g from an agent in $\bar{D}$ to i or else i would belong to $\bar{D}$. Let $j \in \bar{D}$ be an agent who is at the shortest distance from i, i.e. $d(j,i;g) = \min_{\{j' \in \bar{D}\}} d(j',i;g)$. Then there exists i_1,i_2,...,i_s not in $\bar{D}$ such that $g_{j,i_1} = \cdots = g_{i_s,i} = 1$. Clearly, $\{i_1, \ldots, i_s\} \subset F$ as well. As $V < c$, there must be i' such that $g_{i,i'} = 1$, for otherwise agent i_s would be strictly better off not forming a link with i. Moreover, we see that $i' \notin \bar{D}$, for otherwise $i \in \bar{D}$ contradicting $i \in F$. Hence $i' \in F$ as well. Arguing the same way, there must be $i'' \in F$ such that $g_{i',i''} = 1$. Since F is a finite set, there must be a cycle of agents $\hat{D} = \{k_1, \ldots, k_r\} \subset F$ such that $g_{k_1,k_2} = \cdots = g_{k_r,k_1} = 1$. Now suppose agent k_r chooses to break the link with k_1 and instead form a link with agent $j \in D$. If the new network is denoted g' then $\Pi_{k_r}(g') - \Pi_{k_r}(g) \geq |\bar{D}|V > 0$ since the agent k_r's cost is the same in both networks, while the agent also obtains the information of all agents in $\bar{D}$ in the network g'. Since k_r can deviate and do better, g is not sustainable, contrary to supposition. Hence F must be empty as required. In turn, this implies that for every $j \in \bar{D}$ we have $N(j;g) = \bar{D}$.

We now demonstrate that if $k \in N \backslash \bar{D}$, then $j' \xrightarrow{g} k$ for some $j' \in \bar{D}$. If not, then by definition, $N(k;g) \cap \bar{D} = \emptyset$, or equivalently $N(k;g) \cap N(j;g) = \emptyset$ for all $j \in \bar{D}$. But if we consider the network g' which is the same as g except that agent k forms an additional link with some $j \in \bar{D}$, then

$$\Pi_k(g') - \Pi_k(g) \;=\; |N(j;g)|V - c \geq |N(j;g)|V - |N_d(j;g)|c \;=\; \Pi_g(j) \geq V > 0. \qquad (10)$$

where the first equality follows from the fact that agent k accesses the cycle $\bar{D}$ in g' and did not do so in g, and the last but one inequality follows because agent j is individually rational and must be getting at least as much payoff in g as by not forming any links. This contradicts the assumption that g is sustainable. Hence for every

$k \in N \backslash \bar{D}$ there is some $j' \in \bar{D}$ such that $j' \xrightarrow{g} k$. Now let j be an agent in $\bar{D}$ at the closest distance to k, i.e. $d(k, j; g) = \min_{\{j' \in D\}} d(k, j'; g)$. By construction of $\bar{D}$ there is some $j^* \in \bar{D}$ with $g_{j^*,j} = 1$. Suppose agent j^* deviates by cutting his link with j and forming one with k instead: if g' denotes the new network, then $\Pi_{j^*}(g') - \Pi_{j^*}(g) \geq V > 0$. This holds because j^*'s costs are the same in the two networks and j^* accesses all other agents in $\bar{D}$ either through his remaining links with agents in $\bar{D}$ or via k in g', and in addition gets k's information. Thus g is not sustainable as supposed. The contradiction establishes that there does not exist k outside of $\bar{D}$, i.e. g is connected. □

Sketch of Proof of Proposition 2.2: If a network g is sustainable and $V > c$ it is clearly connected. Suppose it is not minimally connected. Then there exist agents i and j such that $g_{i,j} = 1$, and the network $\hat{g}$ obtained by setting $g_{i,j} = 0$ is still connected. Since $c > 0$, agent i obtains a higher payoff from $\hat{g}$ than from g, which contradicts the assumption that g is sustainable. In the reverse direction, suppose g is minimally connected. This implies that for any agent $i \in N$, $g_{i,k} = 1$ only if $k \in E$ where $E \in \mathbf{T}$, i.e. agent i forms one and only one link with each of the 'top' maximal components of the graph g'_{-i} and no links with members of other maximal components.[27] The proof now follows from the characterization of an agent's best response which is given in Lemma 3.3 below. □

Proof of Proposition 2.3: For part (a) we show that $g \in \mathcal{S}(V, c') \Rightarrow g \in \mathcal{S}(V, c)$. Proposition 2.1 implies that g is either connected or empty. If g is empty then the claim is clearly true since an empty network is sustainable for all $c > V$. We therefore focus on the case that $g \in \mathcal{S}(V, c')$ is connected. The proof proceeds by contradiction. Suppose $g \notin \mathcal{S}(V, c)$. Then there exists an agent i and a strategy $\hat{g}_i$ such that

$$\Pi_i(g_{-i} \oplus \hat{g}_i \mid c) > \Pi_i(g_{-i} \oplus g_i \mid c). \tag{11}$$

where $\Pi_i(\cdot \mid c)$ is agent i's payoff when the communication cost is c. In other words,

$$|N(i; g_{-i} \oplus \hat{g}_i)|V - |N_d(i; g_{-i} \oplus \hat{g}_i)|c > |N(i; g_{-i} \oplus g_i)|V - |N_d(i; g_{-i} \oplus g_i)|c \tag{12}$$

which is equivalent to

$$\Big\{|N(i; g_{-i} \oplus \hat{g}_i)| - |N(i; g_{-i} \oplus g_i)|\Big\}V > \Big\{|N_d(i; g_{-i} \oplus \hat{g}_i)| - |N_d(i; g_{-i} \oplus g_i)|\Big\}c. \tag{13}$$

[27] This terminology is defined in the proof of Theorem 3.1 which is presented later in this appendix.

The connectedness of g implies that $|N(i; g_{-i} \oplus g_i)| = n$. Since $|N(i; g_{-i} \oplus \hat{g}_i)| \leq n = |N(i; g_{-i} \oplus g_i)|$ the above inequality implies $|N_d(i; g_{-i} \oplus \hat{g}_i)| < |N_d(i; g_{-i} \oplus g_i)|$. As $c' > c$, this implies that

$$\Big\{|N_d(i; g_{-i} \oplus \hat{g}_i)| - |N_d(i; g_{-i} \oplus g_i)|\Big\}c > \Big\{|N_d(i; g_{-i} \oplus \hat{g}_i)| - |N_d(i; g_{-i} \oplus g_i)|\Big\}c'. \qquad (14)$$

From (13) and (14) we obtain, after rearrangement:

$$|N(i; g_{-i} \oplus \hat{g}_i)|V - |N_d(i; g_{-i} \oplus \hat{g}_i)|c' > |N(i; g_{-i} \oplus g_i)|V - |N_d(i; g_{-i} \oplus g_i)|c' \qquad (15)$$

or equivalently

$$\Pi_i(g_{-i} \oplus \hat{g}_i \mid c') > \Pi_i(g_{-i} \oplus g_i \mid c'). \qquad (16)$$

This contradicts the hypothesis that $g \in \mathcal{S}(V, c')$ and (a) follows. The proof for part (b) is now straightforward. If $g \in \mathcal{S}(V, c')$ is the empty network g^e then (b) holds trivially. Otherwise the proof is exactly as in part (a). □

Proof of Proposition 2.4: Consider part (a): since $c < V$ any sustainable network must be connected. Since the wheel and the star are minimally connected it is immediate that no agent can profitably deviate from their specified strategies, in these two cases. It is easy to construct other networks, such as a sequence of stars linked together, which are sustainable. Consider part (b) next: first note that the wheel is sustainable since every agent is forming the minimum number of links, 1, thus incurring a cost c, while getting the maximum amount of benefits, $(n-1)V$. The empty network is sustainable because no agent has an incentive to form a link with an isolated agent, since $c > V$. Similar considerations lead to the conclusion that the star is not sustainable. Finally in part (c) note that since $c > (n-1)V$, the best response to any g_{-i} is the strategy g_i with $g_{i,j} = 0, \forall j \in N \backslash \{i\}$. Thus the only sustainable network is the empty network. □

Proof of Proposition 2.5: First consider part (a). We begin by noting that if a network is connected then it must have at least n links. Furthermore, as proved in the Claim below, if a connected network has exactly n links it must be a wheel. For $V > c$, an efficient network must be connected. Since the minimum number of links needed to connect n agents is n, the above assertion thus directly implies that for these parameter values the wheel is the unique efficient network. Next consider the case where

$V < c < (n-1)V$. Note that the welfare level provided by a wheel network g^w is given by

$$W(g^w) = n^2V - nc. \tag{17}$$

By hypothesis, $c < (n-1)V$ and so it follows that $W(g^w) > nV$. Since the empty network g^e provides welfare $W(g^e) = nV < W(g^w)$, it is not efficient. Let g be a network in which the number of links $L > n$. Then $W(g) \leq n^2V - Lc < n^2V - nc = W(g^w)$ so that g is not efficient. Likewise, if $L = n$ and $g \neq g^w$ then g is not connected and is again dominated by g^w. Consider a network g in which $L < n$, so that g is not connected. Suppose g is efficient. In such a network, at least $n - L$ agents have no links with other agents. Denote this set of agents by D. Since g is efficient, $W(g) \geq W(g^w) > nV$. Thus at least one agent (say) k must be getting a payoff $\Pi_k(g) > V$. In particular, this implies $|N_d(k;g)| \geq 1$. Now construct a network g' which has all the links in g and in addition $g'_{j,k} = 1$ for some $j \in D$. By construction, the payoff in g' of every agent $i \in N\backslash\{j\}$ will be at least as high as in g. In addition the payoff of j is strictly higher since

$$\Pi_j(g') = |N(j;g')|V - c \geq |N(k;g)|V - c \geq \Pi_k(g) > V. \tag{18}$$

Thus $W(g') > W(g)$, contradicting the hypothesis that g is efficient. Hence an efficient network[28] must have at least n links, and we have seen that g^w is the unique architecture which maximizes $W(\cdot)$ among all such networks. Consider part (b) next: note that the minimum cost of link formation is c while the maximum benefits are $(n-1)V$. For $c > (n-1)V$ it is then immediate that the unique efficient network must be the empty network g^e. □

Claim: *A connected network g with n links is a wheel.*

Proof: Recall that to avoid trivialities we have assumed $n \geq 3$. Since g is connected and has n links, for every $i \in N$ there is one and only one $j \in N\backslash\{i\}$ such that $g_{i,j} = 1$. Consider agent 1. Renumbering the agents if necessary, let $g_{1,2} = 1$. If $g_{2,1} = 1$ as well, then there is no path from agent 3 to either 1 or 2 which violates connectedness. Hence $g_{2,1} = 0$ and suppose without loss of generality that $g_{2,3} = 1$. More generally, suppose for some $k < n$ we have $g_{1,2} = g_{2,3} = \cdots = g_{k-1,k} = 1$. If $g_{k,i} = 1$ for some $i \in I \equiv \{1, \ldots, k-1\}$ then there is no path from agent $k+1$ to the agents in I, violating the connectedness of g. Thus $g_{k,j} = 1$ for some $j \in N\backslash I$; renumbering if necessary, let $j = k+1$. Proceeding inductively we see that $g_{1,2} = \cdots = g_{n-1,n} = 1$. If $g_{n,i} = 1$

[28] The existence of an efficient network is guaranteed since the number of feasible networks is finite.

for some $i > 1$ then there is no path from 1 to agent n violating connectedness. The contradiction proves that $g_{n,1} = 1$. Since all the n links are accounted for, g is a wheel. □

Proof of Theorem 3.1

Our first objective is to characterize the best response of an agent to a network $g \in \mathcal{G}$. We start by showing the following property of a best response. Recall that for a network g, the network g_{-i} is obtained by deleting all of i's links, i.e. replacing all edges of the form $g_{i,j} = 1$ with $g_{i,j} = 0$.

Lemma 3.1 *Given a network $g = g_{-i} \oplus g_i$ in $\mathcal{G}$, let g'_{-i} be the network obtained by replacing all edges of the form $g_{j,i} = 1$ with $g_{j,i} = 0$ in g_{-i}. Then $BR_i(g_{-i}) = BR_i(g'_{-i})$.*

Proof: Consider a strategy $\bar{g}_i \in \mathcal{G}_i$. Let $\bar{g} = g_{-i} \oplus \bar{g}_i$. Clearly $N_d(i; \bar{g}) = N_d(i; g'_{-i} \oplus \bar{g}_i)$, so that the cost of strategy $\bar{g}_i$ is the same in both networks. Since $g'_{-i} \oplus \bar{g}_i$ is a sub-network of $\bar{g}$, we have $N(i; g'_{-i} \oplus \bar{g}_i) \subset N(i; \bar{g})$. We show that $N(i; \bar{g}) \subset N(i; g'_{-i} \oplus \bar{g}_i)$ as well. Suppose $k \neq i$ belongs to the former set. Then by definition there is a path $k \xrightarrow{\bar{g}} i$. Since the path does not involve any link of the form $\bar{g}_{j,i} = 1$, and the absence of such links constitutes the only difference between $\bar{g} = g_{-i} \oplus \bar{g}_i$ and $g'_{-i} \oplus \bar{g}_i$, we also have a path from k to i in the network $g'_{-i} \oplus \bar{g}_i$. Hence $k \in N(i; g'_{-i} \oplus \bar{g}_i)$. Thus i's payoff is the same in both $\bar{g}$ and $g'_{-i} \oplus \bar{g}_i$. The result follows. □

Lemma 3.1 implies that to obtain a best response of agent i we can start from a network in which all the links that other agents have with i have been removed. Next, for a network g, given a set $E \subset N$ of agents and two distinct agents $i,j \in E$, we say that there is a path from j to i in E if either $g_{i,j} = 1$ or if there exist agents $j_1, \ldots, j_m \in E$ distinct from each other and i and j such that $g_{i,j_1} = \ldots = g_{j_m,j} = 1$. A path of this type is denoted $j \xrightarrow{g,E} i$. Note that the relation '$\xrightarrow{g,E}$' is transitive. If both $j \xrightarrow{g,E} i$ and $i \xrightarrow{g,E} j$ hold, we write this as $i \xleftrightarrow{g,E} j$. We can now define the useful notion of a component.

Definition 3.1 *A component of a network g is a set of agents $E \subset N$ such that for all $i,j \in E$ with $i \neq j$, we have $j \xrightarrow{g,E} i$. A component E of g is called maximal if there is no strict superset $E' \subset N$ which is also a component of g.*

A single agent in a network g vacuously constitutes a component of g. An agent who either does not have links with other agents or whom nobody has a link with is a maximal component of g. We now have the following technical result:

Lemma 3.2 *Given a network g and $i \in N$, let g'_{-i} be the network defined in Lemma 3.1. Then there exists a unique partition of all agents in $N \backslash \{i\}$ into maximal components E_1, $E_2, \ldots, E_m$ of g'_{-i}. The class of sets in the partition is denoted as $\mathbf{E}$.*

The proof of this result is omitted due to space constraints; a copy of the proof is available from the authors. Lemma 3.2 allows us to characterize the nature of the best response for an agent. Henceforth to distinguish the role of agent i, he will be referred to as agent n. We start by extending the relation '$\leftrightarrow$' to sets of agents: given a network g and disjoint sets E, $E' \subset N$, we write $E \xrightarrow{g} E'$ if for every $j \in E$ and every $j' \in E'$ we have $j \xrightarrow{g} j'$. We now apply this notion to the partition $\mathbf{E}$ obtained in Lemma 3.2 associated with the network $\bar{g} = g'_{-n}$. Given distinct E,$E' \in \mathbf{E}$, it may be verified that $E \xrightarrow{\bar{g}} E'$ if and only if the following seemingly weaker condition holds: there exist $k \in E$ and $k' \in E'$ such that $k \xrightarrow{\bar{g}} k'$. Furthermore, $E \xrightarrow{\bar{g}} E'$ implies $E' \xrightarrow{\bar{g}} E$ *cannot* hold, since the two sets are maximal components. As the relation '$\xrightarrow{\bar{g}}$' on $\mathbf{E}$ is nonreflexive and transitive, it constitutes a strict partial order on $\mathbf{E}$. We define the class $\mathbf{T} \subset \mathbf{E}$ of 'top' maximal components consisting of the largest elements of $\mathbf{E}$, i.e. $E \in \mathbf{T}$ if there does not exist $E' \in \mathbf{E}$ such that $E \xrightarrow{\bar{g}} E'$. It can be seen that the class $\mathbf{T}$ is non-empty. Furthermore note that if $E \in \mathbf{E} \backslash \mathbf{T}$ then there must exist $\hat{E} \in \mathbf{T}$ such that $E \xrightarrow{\bar{g}} \hat{E}$. We can now provide the following characterization of agent n's best response. Let $T = \cup_{E \in \mathbf{T}} E$ be the set of all 'top' agents for agent n.

Lemma 3.3 *Suppose $V > c > 0$. Given the network $g \in \mathcal{G}$, $\hat{g}_n \in BR_n(g_{-n})$ if and only if for $\mathbf{T}$ as defined above for the network g'_{-n}, we have for all $j \notin T$, $\hat{g}_{n,j} = 0$ and for all $E \in \mathbf{T}$, $\hat{g}_{n,j(E)} = 1$ for exactly one agent $j(E) \in E$.*

Proof: By Lemma 3.1, we can consider the agent n's best responses to the network $\bar{g} = g'_{-n}$. Suppose $\hat{g}_n$ is a best response to $\bar{g}$. Let $\mathbf{E}$ and $\mathbf{T}$ be as defined above. Fix $E \in \mathbf{T}$ and let $k \in E$. Since $V > c$ and there is no agent $k' \in N \backslash E$ such that $k \xrightarrow{\bar{g}} k'$ (by definition of $\mathbf{T}$) it must be the case that $\hat{g}_{n,j(E)} = 1$ for some $j(E) \in E$. Furthermore, it can easily be seen that $N(j;\bar{g}) = N(j';\bar{g})$ for all j,$j' \in E$. Hence if $\hat{g}_{n,j} = 1$ for some $j \in E$ other than $j(E)$, the strategy obtained from $\hat{g}_n$ by replacing $\hat{g}_{n,j} = 1$ with $\hat{g}_{n,j} = 0$ will yield a strictly higher payoff, as $c > 0$. The contradiction implies that the agent $j(E)$ must be unique. We now note that if $k \notin T$ then there exists $E \in \mathbf{T}$ such that $k \in N(j(E);\bar{g})$. Thus if $\hat{g}_{n,k} = 1$ the strategy obtained by deleting this link would yield a strictly higher payoff. The contradiction shows that $\hat{g}_{n,k} = 0$

must necessarily hold for all $k \notin T$. To show sufficiency, suppose $\hat{g}_n$ satisfies the conditions of the Lemma. It is clearly the case that $N(n; \bar{g} \oplus \hat{g}_n) = N$. Suppose that $\tilde{g}_n$ is a best response to $\bar{g}$. Then since $V > c$, we must have $N(n; \bar{g} \oplus \tilde{g}_n) = N$ as well. Hence for all $E \in \mathbf{T}$ there must exist $k(E) \in E$ such that $\bar{g}_{n,k(E)} = 1$. It follows that $|N_d(n; \bar{g} \oplus \tilde{g}_n)| \geq |N_d(n; \bar{g} \oplus \hat{g}_n)|$. Thus $\Pi_n(\bar{g} \oplus \hat{g}_n) \geq \Pi_n(\bar{g} \oplus \tilde{g}_n)$, and the result follows. □

The proof of Theorem 3.1 repeatedly invokes the following useful property concerning the best responses of an agent i.

Lemma 3.4 *Let g be a network, and for $i \in N$ suppose $g_i \in BR_i(\bar{g})$ where $\bar{g} = g'_{-i}$. Let K be a non-empty set of agents such that $g_{i,k} = 1$ for all $k \in K$. If $\hat{k}$ is an agent satisfying $k \xrightarrow{\bar{g}} \hat{k}$ for all $k \in K$, then the strategy $\hat{g}_i$ given by*

$$\hat{g}_{i,j} = g_{i,j} \text{ for all } j \notin K \cup \{\hat{k}\}, \ \hat{g}_{i,k} = 0 \text{ for all } k \in K, \text{ and } \hat{g}_{i,\hat{k}} = 1. \tag{19}$$

is also a best response for agent i.

Proof: Note that by (19), $|N_d(i; g_{-i} \oplus \hat{g}_i)| \leq |N_d(i; g_{-i} \oplus g_i)|$, so that agent i's cost for strategy $\hat{g}_i$ is at most that of using g_i. The result is shown if $N(i; g_{-i} \oplus g_i) \subset N(i; g_{-i} \oplus \hat{g}_i)$, which we now demonstrate. Let j belong to the former set. If $j \in K$ then since $\hat{g}_{i,\hat{k}} = 1$ and $j \xrightarrow{\bar{g}} \hat{k}$, j belongs to the latter set. If $j = \hat{k}$ then obviously $j \in N(i; g_{-i} \oplus \hat{g}_i)$ as well. Finally, suppose $j \notin K \cup \{\hat{k}\}$. If $g_{i,j} = 1$ then $\hat{g}_{i,j} = 1$ by definition. Otherwise there exist $j_1, \ldots, j_m$ distinct from each other and i and j such that $\bar{g}_{j_1,j} = \ldots = \bar{g}_{j_m,j_{m-1}} = 1$ and $g_{i,j_m} = 1$. There are three cases: (a) If $\{j_1, \ldots, j_m\} \cap (K \cup \{\hat{k}\}) = \emptyset$ then since $\hat{g}_{i,j_m} = g_{i,j_m}$ we have $j \in N(i; g_{-i} \oplus \hat{g}_i)$. (b) If $\hat{k} = j_p$ for some $j_p \in \{j_1, \ldots, j_m\}$ then since $j \xrightarrow{\bar{g}} j_p = \hat{k}$ and $\hat{g}_{i,\hat{k}} = 1$ we have $j \in N(i; g_{-i} \oplus \hat{g}_i)$. (c) Finally if $k = j_p$ for some $k \in K$ and $j_p \in \{j_1, \ldots, j_m\}$, then $j \in N(i; g_{-i} \oplus \hat{g}_i)$ since $j \xrightarrow{\bar{g}} j_p = k$, $k \xrightarrow{\bar{g}} \hat{k}$ and $\hat{g}_{i,\hat{k}} = 1$. In all cases $N(i; g_{-i} \oplus g_i) \subset N(i; g_{-i} \oplus \hat{g}_i)$ from which the result follows. □

We now come to the main steps required to prove Theorem 3.1. It is easy to see that the wheel is an absorbing state of the Markov chain. The strategy of the proof is to show that every network other than the wheel is transient. This is proved by showing that given an arbitrary network g different from the wheel, there is a positive probability of a transition to the wheel in finitely many periods.

Recall that the network g is the initial state of the Markov chain. Consider agent n first. As above, suppose that $\mathbf{E}$ is the partition of maximal components induced by agent

n in the network $\bar{g} = g'_{-n}$ formed by deleting all links to and from agent n. Recall that $\mathbf{T} \subset \mathbf{E}$ is the subclass of 'top' maximal components. We now provide an alternative classification of the sets in $\mathbf{E}$. Let $\mathbf{B}_1 \subset \mathbf{E}$ consist of all the smallest elements of $\mathbf{E}$ in the partial ordering '$\stackrel{\bar{g}}{\longrightarrow}$' i.e. $E \in \mathbf{B}_1$ if there does not exist $E' \in \mathbf{E}$ such that $E' \stackrel{\bar{g}}{\longrightarrow} E$. Since $\mathbf{E}$ is finite, $\mathbf{B}_1$ is non-empty. The class $\mathbf{B}_1$ consists of the 'bottom' maximal components, whose agents do not have links with any agent outside their components in the network $\bar{g}$. For $p \geq 1$, having defined $\mathbf{B}_p$, we then define the class $\mathbf{B}_{p+1}$ as:

$$\mathbf{B}_{p+1} = \{E \in \mathbf{E} \backslash \mathbf{E}_p \mid \exists E' \in \mathbf{B}_p \text{ with } E' \stackrel{\bar{g}}{\longrightarrow} E, \text{ and } \nexists E'' \in \mathbf{E} \backslash \mathbf{E}_p \text{ with } E'' \stackrel{\bar{g}}{\longrightarrow} E\}. \tag{20}$$

where $\mathbf{E}_p = \cup_{1 \leq q \leq p} \mathbf{B}_q$. Note that if $\mathbf{E} \backslash \mathbf{E}_p$ is non-empty then so is $\mathbf{B}_{p+1}$. We proceed recursively until all sets $E \in \mathbf{E}$ are exhausted. Let $\mathbf{B}_1, \ldots, \mathbf{B}_s$ be the resulting collection of classes. The classes $\mathbf{B}_1$ to $\mathbf{B}_s$ are pairwise disjoint and their union is $\mathbf{E}$. We can regard the sets in $\mathbf{B}_1$ as being 'bottom' sets or on the lowest 'level', those in $\mathbf{B}_2$ as the 'second-lowest' level and so on. The reason for our nomenclature is that in $\bar{g}$, by construction, an agent in a set $E \in \mathbf{B}_p$ can only be observed by agents in sets E' of level $\mathbf{B}_{p+1}$, $\mathbf{B}_{p+2}$ etc., and never by agents in the class $\mathbf{B}_p$ (apart possibly from other agents in E) or agents in levels lower than $\mathbf{B}_p$. Formally, we write:

Remark: If j lies in some set $E \in \mathbf{B}_p$, and for some $k \in E'$ we have $j \stackrel{\bar{g}}{\longrightarrow} k$ then either $E' = E$ or $E' \in \mathbf{B}_{p'}$ for some $p' > p$.

Note that for $p \geq 1$, the class $\mathbf{B}_p \cap \mathbf{T}$ may be non-empty, i.e. a component in $\mathbf{E}$ may be both a 'top' and belong to the p^{th} level. Furthermore since $\mathbf{B}_s$ is the highest class in the hierarchy, we must have $\mathbf{B}_s \subset \mathbf{T}$.

Now, by Lemma 3.3, agent n will choose a best response $\hat{g}_n$ such that for all $E \in \mathbf{T}$, $\hat{g}_{n,j(E)} = 1$ for exactly one $j(E) \in E$ and $\hat{g}_{n,j} = 0$ for all other $j \in N$. Let $g^1 = g_{-n} \oplus \hat{g}_n$. We note that due to the inertia assumption, g^1 occurs with strictly positive probability. This is because each agent other than n independently maintains his original strategy with positive probability, and agent n has a positive probability of choosing his best response $\hat{g}_n$. In what follows, since it is quite difficult to characterize the resulting network if more than one agent chooses his best response simultaneously, we shall exploit this idea repeatedly: we shall 'pick' a particular agent, have him choose a best response (with certain properties), and construct the network in the next period assuming that every other agent has displayed inertia. By the rules of the process, the resulting network occurs with positive probability given its predecessor.

The additional classification of the sets in $\mathbf{E}$ into the classes $\{\mathbf{B}_q\}_{q=1}^s$ can now be used to establish a special case: for a certain class of networks there is a positive probability of converging to the wheel in finitely many periods. We define a *hyperwheel* to be a network which contains the wheel as a sub-network.

Lemma 3.5 *Suppose that the sets $E \in \mathbf{E}$ are all singletons and that $g_{j,n} = 0$ for all j, i.e. no agent has a link with n in g (and hence also in g^1). Then with positive probability, the network g^1 converges to a hyperwheel.*

Proof: Let $\mathbf{B}_1 \subset \mathbf{E}$ consist of sets $\{B_1^1, \ldots, B_1^{q_1}\}$. By assumption, each $B_1^k \in \mathbf{B}_1$ consists of a single agent. Refer to the agent in B_1^1 as j_1^1. Consider the best response of j_1^1. Since $V > c$, we have $N(n; g^1) = N$, i.e. n observes every agent in N. Thus if $k \in N\backslash\{j_1^1, n\}$ then $k \xrightarrow{g^1} n$. In fact, since $j_1^1 \in B_1^1 \in \mathbf{B}_1$, it must be the case that $k \xrightarrow{g^1_{-j_1^1}} n$ as well. This follows because the network g_1 is the same as g except for n's choice. Since by construction j_1^1 is a 'bottom' level agent and therefore does not observe anyone in g, the same is true in g^1. Thus any path from k to n must exist in the network g^1 independently of j_1^1, i.e. $k \xrightarrow{g^1_{-j_1^1}} n$ as required. Now note that since k is arbitrary, there is a path from every agent k to agent n in $g^1_{-j_1^1}$. Hence by Lemma 3.4, we can choose agent j_1^1's best response $\hat{g}_{j_1^1}$ to g^1 to be simply $\hat{g}_{j_1^1,n} = 1$, and $\hat{g}_{j_1^1,k} = 0$ for all $k \neq n$. In other words, j_1^1 need only form a link with agent n to obtain all the information in the society. Let $g^2 = g^1_{-j_1^1} \oplus \hat{g}_{j_1^1}$ be the network formed when j_1^1 chooses his best response in this way, with all other agents exhibiting inertia. By the rules of the process, g^2 occurs with positive probability.

Next consider $B_1^2 \in \mathbf{B}_1$. Refer to the agent in B_1^2 as j_1^2. Note that in g^2 the structure of the network g^1 is unaltered except for j_1^1's choice, which in turn is unaltered from g except for n's choice. In particular, if $k \in N\backslash\{j_1, n\}$, then $k \xrightarrow{g^2_{-j_1^1}} n$. However, since $j_1^2 \in B_1^2 \in \mathbf{B}_1$, j_1^2 does not observe any agent in g, and hence, since he has displayed inertia throughout, in g^1 and g^2 as well. Thus, for every $k \in N\backslash\{j_1, n\}$ there is a path in g_2 from k to n independent of j_1^2 as well, i.e. $k \xrightarrow{g^2_{-j_1^2}} n$. It follows from Lemma 3.4 that j_1^2 has a best response $\hat{g}_{j_1^2}$ satisfying $\hat{g}_{j_1^2,k} = 0$ for all $k \notin \{j_1, n\}$. Furthermore, since $g^2_{j_1^1,n} = 1$ as well, applying Lemma 3.4 again, j_1^2's best response can be chosen simply as $\hat{g}_{j_1^2,j_1^1} = 1$ with $\hat{g}_{j_1^2,k} = 0$ for every other k. Let $g^3 = g^2_{-j_1^2} \oplus \hat{g}_{j_1^2}$. Once again, there is a positive probability of getting to g^3 given g^2.

We now proceed in the same fashion until all sets in $\mathbf{B}_1$ are exhausted. The resulting network (call it g^4) has the property that the structure of the network for all levels above $\mathbf{B}_1$ are the same as in g^1, and $g^4_{j_1^1,n} = g^4_{j_1^2,j_1^1} = \cdots = g^4_{j_1^{q_1},j_1^{q_1-1}} = 1$. We now consider sets in $\mathbf{B}_2$. Let them be numbered as $\{B_2^1, \ldots, B_2^{q_2}\}$. Denote the agent in B_2^1 as j_2^1. Since all agents in $\mathbf{B}_2$ or higher levels have exhibited inertia, the network g^4 has the same structure for the sets in $\mathbf{B}_2$ and higher as in g^1. In particular, given any $k \in E$ for

$$E \in \left\{B_2^2, \ldots, B_2^{q_2}\right\} \cup \bigcup_{\{E' \in \mathbf{B}_p :\, p \geq 3\}} E' \tag{21}$$

we have $k \overset{g^4_{-j_2^1}}{\longrightarrow} n$ for the same reasons as before. Applying Lemma 3.4 again, we can choose j_2^1's best response $\hat{g}_{j_2^1}$ to satisfy $\hat{g}_{j_2^1,j_1^{q_1}} = 1$ with $\hat{g}_{j_2^1,k} = 0$ for all other k. The new network (which again occurs with positive probability) is $g^5 = g^4_{-j_2^1} \oplus \hat{g}_{j_2^1}$, assuming as before that all other agents exhibit inertia.

We repeat the process in the same way for all the remaining sets in $\mathbf{B}_2$ and then for the sets in each higher level in turn until all levels and each set in each level is exhausted. The resulting network g^6 satisfies

$$g^6_{j_1^1,n} = g^6_{j_1^2,j_1^1} = \cdots = g^6_{j_1^{q_1},j_1^{q_1-1}} = g^6_{j_2^1,j_1^{q_1}} = \cdots = g^6_{j_s^1,j_{s-1}^{q_{s-1}}} = \cdots = g^6_{j_s^{q_s},j_s^{q_s-1}} = 1. \tag{22}$$

Furthermore, recall that $\mathbf{B}_s \subset \mathbf{T}$. Since agent n is assumed to display inertia from g^1 to g^6, and in g^1 we have $g^1_{n,j(E)} = 1$ for each $E \in \mathbf{T}$ and some $j(E) \in E$, we have (in particular) $g^6_{n,j_s^{q_s}} = g^1_{n,j_s^{q_s}} = 1$. Thus the network g^6 contains a wheel, i.e. it is a hyperwheel. The result follows. $\square$

The following lemma establishes convergence to the wheel with positive probability for the special case above.

Lemma 3.6 *Suppose $V > c$ and g^6 is a hyperwheel as above. Then there is a strictly positive probability that g^6 will transit in one period to a wheel.*

Proof: The only agent who potentially has superfluous links is agent n, since in g^6 (as in g^1) he has a link with an agent in each $E \in \mathbf{T}$. We now assume that n alone chooses a best response to g^6. Note that by equation (22), there is a path from every $j \notin \{n, j_s^{q_s}\}$ to $j_s^{q_s}$ in g^6. Hence by Lemma 3.4, we can assume without loss of generality that agent n chooses $\hat{g}_n$ as his best response to g^6 defined as $\hat{g}_{n,j_s^{q_s}} = 1$ with $\hat{g}_{n,k} = 0$ for all other k. The resulting network $g^7 = g^6_{-n} \oplus \hat{g}_n$ is a wheel. $\square$

In the more general case, if g is the original network, the maximal components $E \in \mathbf{E}$ of the network $\bar{g} = g'_{-n}$ may consist of many agents linked together. In addition, individual agents $j \in E$ may have links with agent n in g, i.e. $g_{j,n} = 1$. The proof now needs to be extended to take into account these possibilities.

Recall that the network $g^1 = g_{-n} \oplus \hat{g}_n$ is obtained from the original network g after n chooses his best response. As before $\mathbf{E}$ is the partition of maximal components induced by g'_{-n} and $\mathbf{T}$ is the collection of 'top' maximal components. In addition, $\mathbf{B}_1$, $\mathbf{B}_2$, ..., $\mathbf{B}_s$ are exactly as defined prior to Lemma 3.5. Lastly, each class $\mathbf{B}_p$ consists of the maximal components $\{B_p^1, \ldots, B_p^{q_p}\}$; unlike the special case considered earlier, these sets may not be singletons.

We now proceed inductively. For some $p \geq 1$ and some $m \geq 1$ consider the set $B_p^m \in \mathbf{B}_p$. Suppose that the current state of the Markov chain is a network g^2 with the following properties:

(a) $g_n^2 = g_n^1$.

(b) For all $k \in F$, where

$$F = \bigcup_{r=m+1}^{q_p} B_p^r \cup \bigcup_{\{E \in \mathbf{B}_{p'} : p' > p\}} E \tag{23}$$

we have $g_k^2 = g_k^1 = g_k$.

(c) Property (b) also holds for $k \in B_p^m$, i.e. $g_k^2 = g_k^1 = g_k$.

(d) The agents in

$$J \equiv \bigcup_{\{E \in \mathbf{B}_{p'} : p' < p\}} E \cup \bigcup_{r=1}^{m-1} B_p^r. \tag{24}$$

are arranged as a 'linear chain', i.e. $J = \{j_1, \ldots, j_w\}$ where $g_{j_1,n}^2 = g_{j_2,j_1}^2 = \cdots = g_{j_w,j_{w-1}}^2 = 1$ and the agents in J do not have any other links.

We can then prove that the network g^2 will transit with positive probability to a network g^3 where the chain J will be extended by the agents in B_p^m. The proof uses a technique we label as 'geodesic descent'. Consider a network $\tilde{g}$ and recall that given two agents i and j, a geodesic from j to i is a path of the shortest length from j to i in $\tilde{g}$. The length of a geodesic from j to i is denoted $d(i, j; \tilde{g})$. (If no path exists from j to i, then $d(i, j; \tilde{g}) = \infty$ by convention). Furthermore, if $E \subset N$, and $i,j \in E$, an E-geodesic from

j to i is a path of the shortest length, when only paths from j to i entirely within E are considered. Furthermore, let $d(i, j; \tilde{g}, E)$ denote the length of an E-geodesic from j to i. If there is no path in E from j to i we write $d(i, j; \tilde{g}, E) = \infty$ as in the earlier case. We now show:

Lemma 3.7 *Let the state of the system be g^2, where g^2 satisfies properties (a) to (d). Furthermore, denote the agents in B_p^m as $\{k_1, \ldots, k_r\}$. Then there is a positive probability that the system will move to a network g^3 where properties (a) and (b) continue to hold, and there is a linear chain $J' = \{j_1, \ldots, j_w, k_1, \ldots, k_r\}$ containing J which satisfies $g^3_{j_1,n} = g^3_{j_2,j_1} = \cdots = g^3_{j_w,j_{w-1}} = g^3_{k_1,j_w} = \cdots = g^3_{k_r,k_{r-1}} = 1$. Furthermore, the agents in J' do not have any other links in g^3.*

Proof: Suppose (a) to (d) hold in g^2. Since $V > c$, $g^2_n = g^1_n$, and g^1_n is a best response to g_{-n}, we must have $B_p^m \subset N(n; g^2)$. Hence there exists some $k_u \in B_p^m$ and some $i' \in F \cup \{n\}$ such that $g^2_{i',k_u} = 1$. Relabelling the agents for convenience, suppose k_u is k_r, i.e. $g^2_{i',k_r} = 1$. By (a) and (b) and the above argument, there is a path in g^2 from k_r to n which does not involve any agent in B_p^m.

Recall that B_p^m is a component of $\bar{g} = g'_{-n}$; by virtue of (c), it continues to be a component of g^2. Hence for every $k \in B_p^m \backslash \{k_r\}$ we have $k \overset{g^2, B_p^m}{\longrightarrow} k_r$. Choose an agent $k \in B_p^m \backslash \{k_r\}$ who maximizes $d(k_r, k; g^2, B_p^m)$, i.e. with whom k_r has the longest B_p^m-geodesic. Relabelling the agents again if necessary, suppose without loss of generality that k_1 is this agent. We now note that if $k \in \{k_2, \ldots, k_{r-1}\}$ then by the choice of k_1, we have $k \overset{g^2_{-k_1}, B_p^m}{\longrightarrow} k_r$. (If this were not true, then the shortest path within B_p^m from k to k_r would have to pass through agent k_1, in which case $d(k_r, k; g^2, B_p^m) > d(k_r, k_1; g^2, B_p^m)$, which contradicts the definition of k_1).

Since for each $k \in \{k_2, \ldots, k_{r-1}\}$ we have $k \overset{g^2_{-k_1}, B_p^m}{\longrightarrow} k_r$, and there is a path from k_r to n independent of the agents in B_p^m, this implies $k \overset{g^2_{-k_1}, B_p^m}{\longrightarrow} n$ as well. Next note that from (a) and (b), there is a path from every agent $i' \in F$ to n independently of the agents in B_p^m. Finally, note from (d) that there is a path from n to j_w also independent of the agents in B_p^m. Using all these observations, we see that there is a path in $g^2_{-k_1}$ from every $k \neq k_1$ to j_w. Hence, applying Lemma 3.4, agent k_1 has a best response $\hat{g}_{k_1}$ which is simply $\hat{g}_{k_1,j_w} = 1$ and $\hat{g}_{k_1,j} = 0$ for any other agent. Let $\dot{g} = g^2_{-k_1} \oplus \hat{g}_{k_1}$ be the network formed when k_1 chooses this best response and all other agents show inertia.

Next consider the remaining agents in B_p^m. Fix $k \in \{k_2, \ldots, k_{r-1}\}$. By the choice of k_1 we have $k \overset{g^2_{-k_1}, B_p^m}{\longrightarrow} k_r$. Since $\dot{g}_{-k_1} = g^2_{-k_1}$ by construction, $k \overset{\dot{g}_{-k_1}, B_p^m}{\longrightarrow} k_r$ as well, i.e. there is a path from k to k_r in $\dot{g}$ independent of k_1. In particular, $d(k_r, k; \dot{g}, B_p^m) < \infty$. Furthermore, note that in $\dot{g}$ the B_p^m-geodesic from k to k_r cannot involve k_1 since k_1 has no longer any links within B_p^m. Now choose an agent in $\{k_2, \ldots, k_r\}$ to maximize $d(k_r, k; \dot{g}, B_p^m)$. Relabelling the agents if necessary let k_2 be such an agent. Now, if $k \in \{k_3, \ldots, k_{r-1}\}$ then $k \overset{\dot{g}_{-k_2}, B_p^m}{\longrightarrow} k_r$ as well. If not, all paths from k to k_r (at least one exists since $k \overset{\dot{g}, B_p^m}{\longrightarrow} k_r$) must pass through k_2. But then $d(k_r, k; \dot{g}, B_p^m) > d(k_r, k_2; \dot{g}, B_p^m)$, which contradicts the choice of k_2.

Since in $\dot{g}$ there is a path from every agent $k \in \{k_3, \ldots, k_{r-1}\}$ to k_r independently of k_2 or k_1, the same logic as used earlier with k_1 leads to the conclusion that k_2 can obtain all the information in the society by forming a link with k_1 alone; formally, he has a best response $\hat{g}_{k_2}$ which is $\hat{g}_{k_2,k_1} = 1$, $\hat{g}_{k_2,j} = 0$ for all other j. Let $\ddot{g} = \dot{g}_{-k_2} \oplus \hat{g}_{k_2}$ be the new network formed in this way. We can then repeat the above steps with all the agents $\{k_3, \ldots, k_r\}$ in succession to arrive a network g^3 which satisfies the conditions of the lemma. $\square$

Note that the situation of $B_p^{m+1} \in \mathbf{B}_p$ in g^3 is identical to that of $B_p^m \in \mathbf{B}_p$ in g^2. Hence we can continue the inductive step. In this way we exhaust all the maximal components in $\mathbf{B}_p$ before moving on to the next level and so on until all levels are exhausted. The end result is a hyperwheel g^4, as in the special case of Lemma 3.5. Thus applying Lemma 3.6 to the hyperwheel g^4, we see that every network has a positive probability of converging to the wheel, which is an absorbing state. Theorem 3.1 now follows from standard results on Markov chains.

Proof of Theorem 3.2: (Sketch) When $V < c$ there exist networks g such that the best response of every agent i to g_{-i} is to form no links. Let the set of such networks be given by $\mathcal{G}^1$. Also define $\mathcal{G}^2 = \mathcal{G} \backslash \mathcal{G}^1$ to the set of networks such that there is at least one agent whose best response involves forming some links. It is easily verified that if $g \in \mathcal{G}^1$ then the Markov process starting from g converges to the empty network g^e with probability 1. From now on we therefore concentrate our attention on $\mathcal{G}^2$.

Step 1 Consider a network $g \in \mathcal{G}^2$. By definition, there is some agent n whose best response to $\bar{g} = g'_{-n}$ involves forming some links. It is not difficult to show the following characterization of agent n's best response:

$\hat{g}_n \in BR_n(\bar{g})$ only if for the network g'_{-n} and $\mathbf{T}$ as defined above, we have (i) if $j \notin T = \cup_{E \in \mathbf{T}} E$ then $\hat{g}_{n,j} = 0$ and (ii) if for some $E \in \mathbf{T}$, and $j(E) \in E$ we have $\hat{g}_{n,j(E)} = 1$, then $j(E)$ is unique.

As before, allow agent n to play his best response, with the remaining agents displaying inertia. Define $g^1 = g_{-n} \oplus \hat{g}_n$.

Step 2 There are two cases to be considered: (1) for all j such that $n \xrightarrow{g^1} j$ we also have $j \xrightarrow{g^1} n$, and (2) there exists a j such that $n \xrightarrow{g^1} j$ but not $j \xrightarrow{g^1} n$.

We consider case (1) first. Define the classes $\{\mathbf{B}_q\}_{q=1}^s$ as before. Let $\mathbf{B}_1 = \{B_1^1, \ldots, B_1^{q_1}\}$ be the bottom components and start with B_1^1. Assume initially that it is a singleton, and let j_1^1 be this agent. There are two subcases: (i) $j_1^1 \xrightarrow{g^1} n$ and (ii) there does not exist a path in g^1 from j_1^1 to n.

In subcase (i) we show that j_1^1 has a best response which involves forming a link with some agent. If he forms no link he will obtain V. If he forms a link with n then he obtains a payoff of $|N(n; g^1)|V - c$. However,

$$|N(n; g^1)|V - c \geq |N(n; g^1)|V - |N_d(n; g^1)|c \geq V. \tag{25}$$

The first inequality is obvious since $|N_d(n; g^1)| \geq 1$ by assumption. The second follows because agent n must be obtaining at least V with his best response. Thus agent j_1^1 has a best response with a non-zero number of links. We next argue that j_1^1 has a best response $\hat{g}_{j_1^1}$ such that

$$\hat{g}_{j_1^1,n} = 1 \text{ and } \hat{g}_{j_1^1,j'} = 0 \text{ for all other } j'. \tag{26}$$

The proof is as follows. Suppose j_1^1 has a best response which involves forming a link with an agent j' different from n and not forming one with n. If $j' \in N(n; g^1)$ then an application of Lemma 3.4 shows that the link with j' can be replaced by a link with n instead. Suppose $j' \notin N(n; g^1)$. Since when n chose his best response he did not form a link with j', it must be the case that

$$|N(n; g^1)|V - |N_d(n; g^1)|c \geq |N(j'; g_{-n'})| - c. \tag{27}$$

The left hand side is n's payoff by playing his best response, while the right is his payoff from forming a link with j' instead in g_{-n}. Since by (25) agent j_1^1 obtains $|N(n; g^1)|V - c$ by forming a link with n alone, which is at least as large as the left-hand side of (27), we can assume that forming a link with n is as good as forming one with j' instead. The last situation to consider is when j_1^1 has a best response which involves links with both n and some $j' \notin N(n; g^1)$. This can also be ruled out, because if n did not find it worthwhile to form a link with j', then j_1^1 cannot do so either. Thus we can assume without loss of generality that (26) holds. Let the new network formed when j_1^1 chooses his best response in this way be given by $g^2 = g^1_{-j_1^1} \oplus \hat{g}_{j_1^1}$.

In the more general situation of subcase (i), B_1^1 may not be a singleton. Let $B_1^1 = \{j_1^1, \ldots, j_1^r\}$ and suppose there is a path from j_1^r to n in g^1. In this case we employ the method of geodesic descent as in Lemma 3.7. Following the same logic, we can show that the agents in B_1^1 will align themselves with positive probability in a chain, i.e. there will be a new network g^2 formed where every agent outside B_1^1 is unchanged from g^1, and the agents in B_1^1 satisfy $g^2_{j_1^1,n} = g^2_{j_1^2,j_1^1} = \cdots = g^2_{j_1^r,j_1^{r-1}} = 1$.

We now come to subcase (ii), which is simpler. When B_1^1 is a singleton $\{j_1^1\}$, this is the situation where there is no path from j_1^1 to n in g^1. Here a similar argument to (i) establishes that j_1^1 has a best response which involves forming a link with n alone. In the more general situation where $B_1^1 = \{j_1^1, \ldots, j_1^r\}$ we consider j_1^r and arrange the remaining agents in B_1^1 in terms of decreasing geodesic distance. Starting with the agent having the maximum distance, we show that he has a best response which involves forming a link with n alone. The remaining agents are chosen as in the method of geodesic descent, to link up as in subcase (i) to form the network g^2.

Finally, we note that the above arguments apply in case (1). Recall that this is the case where the best response of n is such that if there is a path from some j to n in g^1 the same is true in the opposite direction. The analysis of the complementary case proceeds as follows: we start by noting that agent $j_1^1 \in B_1^1$ has a best response in which he forms a link with the component which is "furthest" away from B_1^1 rather than forming a link with agent n. The arguments developed above can now be applied with only slight modification to allow for some relabelling of the ordering of the components.

Step 3 We now proceed as in Theorem 3.1 to carry out the above operations on the remaining components in $\mathbf{B}_1$ and then with the components in $\mathbf{B}_2$, $\mathbf{B}_3$, ..., $\mathbf{B}_s$. The final outcome is a hyperwheel, after which we apply Lemma 3.6 to obtain the result. □

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