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Theoretical Note

Additive Representation for Equally Spaced Structures

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It is shown that additive conjoint measurement theory can be considerably generalized and simplified in the equally spaced case. © 1991 Academic Press, Inc.

1. INTRODUCTION

The major step forward in additive conjoint measurement may have been Debreu's (1960) contribution. Debreu showed how for three or more essential coordinates mainly an independence condition, abbreviated "CI" in this paper, is necessary and sufficient for additive representability of preferences. For the case of two essential coordinates, Debreu showed that a condition, the "hexagon condition," had to be added, and he showed that this condition is weaker than both the Thomsen condition and triple cancellation (see his Fig. 1). Debreu used restrictive nonnecessary conditions that were of a topological nature; i.e., he required continuity with respect to a connected separable topology. A next step forward was obtained in the psychological literature; the results have been gathered mainly in Chapter 6 of Krantz, Luce, Suppes, & Tversky (1971), hereafter abbreviated KLST. They used an alternative nonnecessary condition of an algebraic nature, "restricted solvability." It was shown that this condition is less restrictive than Debreu's topological condition. Wakker (1988) argued that the algebraic conditions are not only mathematically, but also intuitively preferable to the topological conditions. Necessary and sufficient conditions for additive representability in full generality have been provided by Jaffray (1974b) for two dimensions, and by Jaffray (1974a) in full generality. These conditions are very complicated, and because of that the simpler results which use nonnecessary conditions remain of interest.

The approach of KLST allows one to deal not only with spaces isomorphic to a continuum, but also with discrete spaces. Discrete spaces may be "densely spaced" as are the spaces isomorphic to a continuum, but they do not have to be. If they

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are not, then they turn out to be "equally spaced," as will be shown. KLST gave proofs that apply simultaneously to the densely spaced and to the equally spaced cases, building upon a generalization of a lemma of Hölder that applies to each of these cases. In subsequent contributions in the literature, authors have usually avoided the equaly spaced case by simply adding an assumption of dense spacedness. The equally spaced case may have been considered complicated because it is of a combinatorial nature.

This paper demonstrates that the equally spaced case in fact is much simpler, and allows much stronger theorems, than has usually been thought. We shall strengthen the results of KLST by showing that only the weakest implication of CI, called "weak separability" in this paper, is necessary and sufficient for additive representability, in the presence of the other conditions. In particular for the two-dimensional case no additional condition (such as the Thomsen condition) is needed. Under a more restrictive definition of equal spacedness, such an observation was made for two dimensions in KLST, Theorem 2.1. For the case of difference measurement (i.e., the two-dimensional case where the second additive value function is minus the first) on finite subsets of Cartesian products a related observation was made in Doignon and Falmagne (1974, Theorem 1 and the last sentence of Section II). Also the proofs for the equally spaced case are fairly simple. It turns out that the conditions of equal spacedness, restricted solvability, and weak separability combine nicely. The crucial idea is described in Remark 6 and the last sentence above Formula (8).

A first impression may be obtained from Corollary 4, and the fact that strict increasingness there implies weak separability. The latter illustrates the generality of weak separability.

2. The Main Result

We concisely repeat the standard definitions of conjoint measurement. Let $\mathsf{X}_{i=1}^{n} \Gamma_i$ be a finite Cartesian product, with elements $x = (x_1, ..., x_n)$ called *alternatives*. Let $\succcurlyeq$ be a binary relation on $\mathsf{X}_{i=1}^{n} \Gamma_i$, with $\succ$, $\preccurlyeq$, $\prec$, $\sim$ as usual. Let $n \geqslant 2$. We write $x_{-i}v_i$ for x with x_i replaced by v_i, and, for $i \neq j$, $x_{-i,j}v_i, w_j$ for x with x_i replaced by v_i and x_j replaced by w_j. $\succcurlyeq$ is a *weak order* if it is *complete* ($\forall x, y \in \mathsf{X}_{i=1}^{n} \Gamma_i : x \succcurlyeq y$ or $y \succcurlyeq x$) and transitive. $\succcurlyeq$ satisfies *restricted solvability* if for each $x_{-i}a_i \succ y \succ x_{-i}c_i$ there exists b_i such that $x_{-i}b_i \sim y$. For the definition of the technical *Archimedean axiom* the reader is referred to Definition 6.7 and Section 6.11 of KLST. Coordinate i is *inessential* if always $x_{-i}v_i \sim x_{-i}w_i$; otherwise it is *essential*. A function $V\colon \mathsf{X}_{i=1}^{n} \Gamma_i \to \Re$ *represents* $\succcurlyeq$ if $[x \succcurlyeq y \Leftrightarrow V(x) \geqslant V(y)]$. The function V is *additive* if $V\colon x \mapsto \sum_{j=1}^{n} V_j(x_j)$ for some functions V_j. If V is additive and representing, then the involved functions V_j are called *additive value functions*. V is an *interval scale* if it can be replaced by another function if and only if the other function is obtained from V by adding a real number, and multiplying with a positive number.

The main condition used by KLST to characterize additive representability is "independence." Since the term independence occurs in many different contexts with many different meanings, we prefer to rename the condition as follows. We say $\succcurlyeq$ is independent of common coordinates, or *coordinate independent* (CI) for short, if for all i, x, y, v_i, w_i

$$x_{-i}v_i \succcurlyeq y_{-i}v_i \Leftrightarrow x_{-i}w_i \succcurlyeq y_{-i}w_i. \tag{1}$$

CI says that in a preference any common coordinate may be replaced by another common coordinate. Obviously by repetition this implies that any set of common coordinates can be replaced by any other set of common coordinates. The latter is the formulation of KLST, and is equivalent to ours.

The discrete case which deviates most from the continuum is the equally spaced case, the topic of this paper. The terminology in the definition below does not seem to reflect an idea of equal spacedness, and may not be perfectly suited for general cases. For additive representations as considerd in this paper, it will nevertheless be equivalent to equal spacedness as in KLST, as the last sentence in Theorem 3 will show. This explains our choice of term. Also Theorem 3 will show that there are not many equally spaced models that are not additive, i.e., for which the terminology below is not suited.

DEFINITION 1. We say $\mathsf{X}_{i=1}^n \Gamma_i$, $\succcurlyeq$ satisfies the *equally spaced condition* if there exist i, x, a_i, c_i such that

$$\begin{gathered} x_{-i}a_i \succ x_{-i}c_i \\ \nexists b_i : x_{-i}a_i \succ x_{-i}b_i \succ x_{-i}c_i. \end{gathered} \tag{2}$$

As we shall show, the cancellation axioms and independence can be weakened to the following condition. We are not aware of an explicit term for the condition in psychological literature, though it did occur as condition 3′ in KLST (Theorem 7.1) and condition A2 in Fishburn and Roberts (1988). "Weak independence" might be a good term. We shall however not introduce a new term, but instead use the term customary in economic literature.

DEFINITION 2. The binary relation $\succcurlyeq$ satisfies *weak separability* if for all i, x, y, v_i, w_i

$$x_{-i}v_i \succcurlyeq x_{-i}w_i \Leftrightarrow y_{-i}v_i \succcurlyeq y_{-i}w_i.$$

Whereas CI implies that every subset of common coordinates in a preference can be replaced by other common coordinates, weak separability implies this only for subsets of size $n-1$. This makes it much weaker. For instance, if $\forall i : \Gamma_i = \Re$, then the usual strict increasingness of $\succcurlyeq$ (the higher a coordinate the better) already implies weak separability, whereas it does not at all imply CI.

THEOREM 3. *Let $\succcurlyeq$ be a binary relation on a finite Cartesian product $\times_{i=1}^{n} \Gamma_i$. Let at least two coordinates be essential, and let restricted solvability and the equally spaced condition hold. Then the following two statements are equivalent:*

(i) *There exists an additive representation for $\succcurlyeq$.*

(ii) *The binary relation $\succcurlyeq$ is a weak order satisfying the Archimedean axiom and weak separability.*

Further, the additive representation in (i) *is an interval scale. The range of every additive value function can be taken as an interval within the set of integers.*

A proof will be given in the next section. The claim about the range of the additive value functions could have been taken as a characterization of the case of equal spacedness and restricted solvability: one can include these conditions in statement (ii), rather than as structural presupposition, if one includes the claim about the range in statement (i). Obviously, if $\times_{i=1}^{n} \Gamma_i$ contains finitely many elements, then the Archimedean axiom in statement (ii) may be omitted. A direct consequence of the above theorem is that the conditions in statement (ii) imply all the cancellation axioms, including CI and the Thomsen condition. The following corollary illustrates the little restrictiveness of weak separability. An *accumulation point* of a set is a point that is a limit of points of the set different from the point itself. Given that strict increasingness implies weak separability, the corollary follows straightforwardly from the above theorem.

COROLLARY 4. *Let $\forall i : \Gamma_i \subset \Re$. Suppose the weak order $\succcurlyeq$ on $\times_{i=1}^{n} \Gamma_i$ $(n \geqslant 2)$ is strictly increasing in each coordinate, and satisfies restricted solvability, the Archimedean axiom, and the equally spaced condition. Then there exists an additive representation for $\succcurlyeq$, strictly increasing in each coordinate. The Γ_i's are discrete and have no accumulation points.*

3. PROOF OF THEOREM 3

Necessity of the conditions in statement (ii) is elementary and well-known. So we turn to sufficiency. Suppose the conditions in statement (ii) hold. Say that the i in the definition of equal spacedness is 1. So there exist a_1^0 and a_1^1 such that, for some x, $x_{-1}a_1^1 \succ x_{-1}a_1^0$ but $x_{-1}a_1^1 \succ x_{-1}b_1 \succ x_{-1}a_1^0$ for no b_1. By weak separability we can in the usual way define $\succcurlyeq_i$ for each coordinate i, i.e., $v_i \succcurlyeq_i w_i$ if and only if there exists x so that $x_{-i}v_i \succcurlyeq x_{-i}w_i$, which by weak separability holds if and only if $y_{-i}v_i \succcurlyeq y_{-i}w_i$ for all y. Also from weak separability it follows that $\succcurlyeq_i$ is a weak order, that the relations $\succ_i$, $\preccurlyeq_i$, $\prec_i$, $\sim_i$ can be derived from $\succcurlyeq_i$ in the usual way, and that we have the following monotonicities:

$$[\forall i : x_i \succcurlyeq_i y_i] \Rightarrow [x \succcurlyeq y], \tag{3}$$

$$\text{if further besides (3) also } [\exists i : x_i \succ_i y_i] \text{ then } x \succ y, \tag{4}$$

$$[\forall i : x_i \sim_i y_i] \Rightarrow [x \sim y]. \tag{5}$$

We call b_i *strictly between* a_i and c_i if $a_i \succ_i b_i \succ_i c_i$ or $c_i \succ_i b_i \succ_i a_i$. Equal spacedness says there does not exist b_1 strictly between a_1^0 and a_1^1. The following lemma straightforwardly follows from application of restricted solvability to the implied $w_{-i,j}a_i, q_j \succ w_{-i,j}c_i, p_j \succ w_{-i,j}c_i, q_j$.

LEMMA 5. *Let $a_i \succ_i c_i$, $p_j \succ_j q_j$. Let $w_{-i,j}a_i, q_j \succ w_{-i,j}c_i, p_j$. Then there exists b_i strictly between a_i and c_i so that $w_{-i,j}b_i, q_j \sim w_{-i,j}c_i, p_j$.*

We have

$$\nexists p_j \succ_j q_j : w_{-1,j}a_1^1, q_j \succ w_{-1,j}a_1^0, p_j \tag{6}$$

because that would imply, by Lemma 5, existence of b_1 strictly between a_1^0 and a_1^1. Next we construct standard sequences on each coordinate. First this is done for all $j \neq 1$. We fix any arbitrary $a_j^0 \in \Gamma_j$, $j = 2, ..., n$. Say, for any $j \neq 1$, there is $b_j \succ_j a_j^0$. By (6) $w_{-1,j}a_1^0, b_j \succcurlyeq w_{-1,j}a_1^1, a_j^0$; by Lemma 5 there must exist a_j^1 to give $w_{-1,j}a_1^0, a_j^1 \sim w_{-1,j}a_1^1, a_j^0$. There can be no b_j strictly between a_j^0 and a_j^1; it would by Lemma 5 imply existence of a b_1 strictly between a_1^0 and a_1^1. Analogously, if there exists $b_j \succ_j a_j^1$, we can construct a_j^2 so that $w_{-1,j}a_1^0, a_j^2 \sim w_{-1,j}a_1^1, a_j^1$ and that there exists no b_j strictly between a_j^1 and a_j^2. We continue inductively by constructing a_j^{k+1} so that $w_{-1,j}a_1^0, a_j^{k+1} \sim w_{-1,j}a_1^1, a_j^k$ as long as there exists $b_j \succ_j a_j^k$. Either this gives an infinite sequence, or the sequence stops at a maximal a_j^k. If the sequence is infinite, then by the Archimedean axiom there can be no upper bound for the standard sequence. Further there is no b_j strictly between any a_j^k and a_j^{k+1}.

Again analogously, if there exists $b_j \prec_j a_j^0$, we can construct an a_j^{-1} so that $w_{-1,j}a_1^0, a_j^0 \sim w_{-1,j}a_1^1, a_j^{-1}$, and so that there is no b_j strictly between a_j^0 and a_j^{-1}. We can again continue and define inductively a_j^{-k} with $w_{-1,j}a_1^0, a_j^{-k+1} \sim w_{-1,j}a_1^1, a_j^{-k}$. Either this gives an infinite sequence, or the sequence stops at a minimal a_j^{-k}. Again there is no b_j strictly between any a_j^{-k+1} and a_j^{-k}. From the above one sees

$$\text{Each } b_j \in \Gamma_j \text{ is } (\sim_j) \text{ equivalent to } a_j^z \text{ for some } z. \tag{7}$$

To construct the standard sequence on the first coordinate, note that there are at least two coordinates essential, so that there must be a $j \neq 1$ for which there exists not only a_j^0, but also a_j^1 or a_j^{-1}. Say a_j^1 exists. Then we can use the "measure stick" a_j^1, a_j^0 on coordinate j to construct the standard sequence $a_1^0, a_1^1, ...$ and $a_1^0, a_1^{-1}, ...$ on coordinate 1, exactly as we used the "measure stick" a_1^1, a_1^0 on coordinate 1 to construct the standard sequences $a_i^0, a_i^1, ...$ and $a_i^0, a_i^{-1}, ...$ for all $i \neq 1$.

The construction of the standard sequences is not unique. Any a_j^k could be replaced by any $b_j^k \sim_j a_j^k$. It should be noted that, by weak separability and its implication (5), nothing in our reasonings, constructions, or definitions would have altered if we had carried out a replacement as just mentioned.

Remark 6. The important property of the standard sequences constructed above, showing independence of the particular way they have been constructed, and facilitating additive representation, is that there are no coordinates strictly between them, so, loosely speaking, a_j^{z+1} is the direct follower in preference of a_j^z.

For the case of difference measurement ($n=2$, $\Gamma_1=\Gamma_2$, $V_1=-V_2$) on a finite subset of a Cartesian product, a result related to Remark 6 was obtained in Lemma 1 in Doignon and Falmagne (1974).

We define the additive value functions as follows, in accordance with the claim about the range in the theorem: For any j and $b_j \in \Gamma_j$, $V_j(b_j)=z$ with z so that $a_j^z \sim_j b_j$. By (5) and (7) it suffices to show that these functions are additive value functions on the *grid* alternatives, i.e., the alternatives with all coordinates of the form a_j^z. We first show that the "trade-offs of size one" are alright. The following equivalence in the present setup simply follows from Lemma 5 and Remark 6.

$$w_{-i,j}a_i^z, a_j^{z'+1} \sim w_{-i,j}a_i^{z+1}, a_j^{z'} \tag{8}$$

for all grid alternatives $w_{-i,j}a_i^z, a_j^{z'+1}$, $w_{-i,j}a_i^{z+1}, a_j^{z'}$. In general additive conjoint measurement the derivation of the above equation is the only step where more implications of the cancellation axioms are needed than weak separability. This is explicated in Wakker (1989, Section III.5, comment at Stage 3); there only the derivation of Fomula (III.5.3),the analogue of the present Formula (8), requires CI in full strength and besides that also the Thomsen condition (or the weaker hexagon condition) if no more than two coordinates are essential. It is remarkable that Formula (8) follows, in the presence of restricted solvability and the equally spaced condition, from merely weak separability, and does not need anything else from CI and/or other cancellation axioms.

Completion of the demonstration that the V_j's are additive value functions is done as in Wakker (1989, Section III.5, Step 3.3). In short, the above trade-offs of size one can be repeated, showing that any two alternatives with the same sum of additive value functions must be equivalent. Increasing the sum of the additive value functions leads obviously to a higher equivalence class.

That the additive representation is an interval scale is standard. For example, one can see that any pair of values $\mu > \nu$ instead of 1, 0 would have been possible for $V_1(a_1^1)$, $V_1(a_1^0)$, and that after those choices the additive representation is uniquely determined. For the case of finitely many equivalence classes this uniqueness result is a special case of Corollary 1 in Fishburn and Roberts (1988), which gives necessary and sufficient conditions for additive represenations on finite structures to be interval scales.

4. Conclusion

We have shown that additive conjoint measurement for the equally spaced case is simpler, and that stronger results can be obtained, than has usually been thought. Only weak separability, a very weak version of cancellation, is needed.

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