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THE NUMERICAL RANGE: A TOOL FOR ROBUST STABILITY STUDIES?

by

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ABSTRACT

The use of numerical range concepts for assessing robust stability of multivariable feedback systems is investigated. The characterization of perturbations by their numerical range allows a more detailed description of their gains and phases and allows a robust stability theorem similar in structure to that of Postlethwaite et al with the possibility of arbitrary large perturbations.

1. Introduction

The problem of robust stability of incompletely known feedback systems has merited much attention recently [1] - [8]. The problem considered is exemplified by a consideration of the closed-loop feedback system shown in Fig.1. where it is assumed that the plant G is described by the strictly proper $m\times\ell$ transfer function matrix G(s) and K(s) is the proper $\ell\times m$ forward path controller. The controller is assumed to be designed to ensure closed-loop stability for the configuration when G is replaced by a nominal model $G_o(s)$. It will be assumed that G and G_o are related by the multiplicative perturbation rule.

$$G(s) = (I_m + \Delta(s))\, G_o(s) \tag{1}$$

where $\Delta(s)$ is a stable perturbation representing modelling errors. The robust stability problem is to ensure the stability of the closed-loop system for all possible Δ in a defined set chosen by the designer to contain all possible perturbations met by the system in practice.

The general approach taken in previous work comes under the heading of application of small gain methods based on the intuition that small enough perturbations will not violate the stability requirement. Typical of these methods is the use of singular values (ℓ_2 induced norms) to represent gains as described in [1] and [2] where the class of allowable perturbation is described by an eigenstructure bounding relation of the form

$$\bar{\sigma}\,(\Delta\,(s)) \leq \ell(s), s \in \Omega \tag{2}$$

on the Nyquist contour Ω , where $\ell(s)$ is a known or postulated scalar upper bound. The derived sufficient conditions for robust stability then take the form (assuming invertibility of Q).

$$\underline{\sigma}\,(I + \hat{Q}(s)) > \ell(s), \quad s \in \Omega \tag{3}$$

where $Q=G_oK$ and $\hat{M}$ is used to denote the inverse of the matrix M. Other studies such as [3] and the related work in [4] and [8] use matrix valued versions of (2) to identify explicitly the form of variation of elements of Δ and can lead to less conservative results.

A universal problem in 'norm-based' methods is the neglect of the phase information in Δ. This is unfortunate as classical control design tells us that the answer to the question of whether a perturbation is stabilizing or destabilizing depends crucially on both the gain (norm) and phase of that perturbation.

In fact, if the phase is advantageous, the gain could be arbitrarily large but stability retained. Some useful results have been provided in [2] which include phase information but these also require boundedness conditions on the condition numbers of the perturbation and limit the phase spread to less than π radians.

It is the purpose of this paper to present some new results on robust stability using a new perturbation characterization (the numerical range) that allows the incorporation of both gain and phase information. Other advantages of the approach are the graphical nature of the results which make them eminently suitable for CAD and the possibility of coping with situations where perturbation gains are unbounded. The price to be paid for these powerful features is the need for the nominal closed-loop system to satisfy phase spread conditions as expressed in terms of conditions on a numerical range.

2. Perturbation and the Numerical Range.

The numerical range [9], [10] of an mxm matrix M on C^m regarded as a Hilbert space with inner product $\langle x,y\rangle = x^* y$ is defined to be the compact, convex set in C

$$V(M) \triangleq \{ z \in C: z = x^* Mx, \quad x^* x = 1\} \tag{4}$$

It is easily computed to be the intersection of all half-planes containing V(M) and can be expressed as

$$V(M) = \bigcap_{0 \leq \theta < \pi} V(M,\theta) \tag{5}$$

where

$$V(M,\theta) \triangleq \{z \in C: \alpha_1(\theta) \leq \operatorname{Re}(e^{-i\theta} z) < \alpha_m(\theta)\} \tag{6}$$

and

$\alpha_1(\theta) \leq \alpha_2(\theta) \leq \ldots \leq \alpha_m(\theta)$ are the (real) eigenvalues of $(e^{-i\theta} M + e^{i\theta} M^*)/2$.

Alternatively it can be bounded (with respect to set inclusion) by evaluating the intersection in (5) over a finite number of values of 'rotation' θ, with consequent benefits for computational complexity.

The location and shape of V(M) provides information on the location of the eigenvalues of M as its spectrum sp(M) satisfies [9].

$$sp(M) \subset V(M) \tag{7}$$

This indicates that $V(M)$ is usefully sensitive to the (eigenvalue) gains and phases of M as it is trivially verified that $V(gM) = g\,V(M)$ for any complex scalar g ie $V(gM)$ is just the set $V(M)$ rotated through arg g and scaled by $|g|$. The potential gain and phase variation of eigenvalues of M are described more explicitly by the size and shape of $V(M)$ in the sense that

(a) Let $P(M) \triangleq \{z = \alpha e^{i\theta_1} + \beta e^{i\theta_2},\ \alpha \geq o,\ \beta \geq o\},\ o \leq \theta_2 - \theta_1 \leq \pi$,

or $P(M) = C$ (when we let the 'phase spread' $\theta_2 - \theta_1 = 2\pi$) be the smallest closed cone of vertex the origin containing $V(M)$, then the phases of the eigenvalues of M lie in the closed interval $[\theta_1, \theta_2]$.

(b) If the upper and lower numerical radii of $V(M)$ are defined by

$$V_+(M) \triangleq \sup\{|z| : z \in V(M)\},\ V_-(M) = \inf\{|z| : z \in V(M)\} \quad (8)$$

respectively, then the gains of the eigenvalues of M lie in $[V_-(M), V_+(M)]$.

(c) For any potential eigenvalue phase $\theta \in [\theta_1, \theta_2]$, the corresponding eigenvalue gain must be in the range $[V_-(M,\theta),\ V_+(M,\theta)]$ where

$$V_+(M,\theta) = \sup\{a \geq o,\ ae^{i\theta} \in V(M)\},\ V_-(M,\theta) = \inf\{a \geq o,\ ae^{i\theta} \in V(M)\} \quad (9)$$

To a certain degree, the shape of $V(M)$ also provides information on the eigenstructure of M. More precisely, the eigenvectors of M are orthonormal if, and only if, M is normal when [9] $V(M)$ is just the closed convex hull $\overline{co}\,(sp(M))$ of the spectrum $sp(M)$ of M. Notes that $\overline{co}\,(sp(M))$ is a polygon with, at most, m sides.

Motivated by the above discussion and the conclusion that the numerical range is a gain/phase sensitive measure of important spectral dynamic properties of any transfer function matrix, it will be assumed in this paper that the allowable plant perturbations are represented by the set inclusion relation,

$$V(\Delta(s)) \subset \delta(s) \quad , \quad s \in \Omega \quad (10)$$

where $\delta(s)$ is a subset of the complex phase specified by the designer. In essence, the location and shape of $\delta(s)$ represents the designers view of the potential variation of the eigenvalues of $\Delta(s)$ and its deviations from normality.

To illustrate the calculation of a suitable $\delta(s)$, suppose that G and Δ are expressed as

$$G(s) = \text{diag}\ \{\frac{1}{1+sT_j}\}_{1\leq j\leq m} G_o(s) \quad ,\Delta\ (s) = \text{diag}\{\frac{-sT_j}{1+sT_j}\}_{1\leq j\leq m} \qquad (11)$$

representing G as the nominal model with first order measurement dynamics that are to be ignored during the design phase. If the measurement time constants are uncertain but known to be in the range $o \leq T_j \leq T_o$, $1\leq j\leq m$, then a suitable choice of $\delta(s)$ is just

$$\delta(s) = \bigcup_{\substack{o\leq T_j\leq T_o \\ 1\leq j\leq m}} \overline{co}\ \left(\{\frac{-sT_k}{1+sT_k}\}_{1\leq k\leq m}\right) \qquad (12)$$

and is illustrated in Fig.2. for s on the imaginary axis in terms of the Nyquist locus of $-s/(1+s)$. Note the explicit gain/phase information in the representation in the fact that $\delta(s)$ lies in the phase range $-\theta_o \leq \theta \leq -\pi/_2$ with gains in the range $o \leq g \leq |sT_o/(1+sT_o)|$ and the observation that perturbation gains decrease to zero if the phases move to $-\pi/_2$. This description of perturbation contrasts sharply with the singular value approach (2) (interpreted as locating the spectrum of $\Delta(s)$ in $|z|\leq \ell(s)$) which essentially represents uncertainty by the circle of centre the origin and radius $|sT_o/(1+sT_o)|$ and ignores phase structure entirely. The work described in [2] includes some phase information but locates the spectrum of Δ in the sector $S_{sv}(s)$ illustrated in Fig.2. and cannot describe the gain reduction in the vicinity of $-\pi/_2$.

3. Robust Stability Analysis

Assuming that K stabilizes the nominal plant G_o and noting that

$$|I_m+G(s)K(s)| \equiv |I_m+Q(s)|\ |I_m+(I_m+Q(s))^{-1}Q(s)\ \Delta(s)| \qquad (13)$$

it follows that K stabilizes $G = (I+\Delta)\ G_o$ if

$$|I_m+p(I_m+Q(s))^{-1}Q(s)\ \Delta(s)| \neq o\ ,\ s \in \Omega,\ p \in [o,1] \qquad (14)$$

The derivation of robust stability conditions is, in effect, the derivation of conditions on the design Q and perturbation Δ such that (14) holds true. For example, (2) and (3) imply the validity of (14). Our concern here is with the use of numerical range descriptions, however:

Theorem 1: If the control K stabilizes the nominal plant G_o, it will also stabilize all perturbed plants (1) with modelling error Δ satisfying (10) if

$$(1 + V(\hat{Q}(s)) \cap (-\delta_o(s)) = \emptyset \quad , \; s \in \Omega \tag{15}$$

where $\delta_o(s) =]o,1] \;\; \delta(s) \triangleq \{z = p\,\mu: \;\; p \in]o,1], \mu \in \delta(s)\}$.

Proof: We show that (14) holds true, for, if not, we can choose $p \in]o,1]$ and $s \in \Omega$ and a vector x satisfying $x^* x = 1$ such that

$$x^*(I+\hat{Q}(s))x + p\, x^* \Delta\,(s)\; x = o \tag{16}$$

The first term lies in $1+V(\hat{Q}(s))$ whilst the second lies in $pV(\Delta(s)) \subset \delta_o(s)$ which contradicts (15).

Condition (15) is easily checked graphically and hence is well suited for CAD. It has a number of equivalent forms, namely,

$$o \notin 1 + V(\hat{Q}(s)) + \delta_o(s) \quad , \; s \in \Omega \tag{17}$$

$$-1 \notin V(\hat{Q}(s)) + \delta_o(s) \quad , \; s \in \Omega \tag{18}$$

The final form seems to be most consistent with classical methodologies by its inclusion of the (-1,o) point and states simply that the (-1,o) point must not (in a manner reminiscent of the INA method [11] , [12]) lie in the set generated by the algebraic sums of all points in $V(\hat{Q}(s))$ and $\delta_o(s)$. One important immediate observation is that the robust stability condition can permit the analysis of arbitrarily large perturbations as represented by situations where $\delta(s)$ and $\delta_o(s)$ is unbounded. Although obvious by a graphical argument, it is of theoretical interest to underline the point by the following special case satisfying positivity conditions:

Corollary 1.1: The plant G_o is robust stable with respect to the perturbation (10) in the presence of the control K if there exists real valued scalar functions $\psi(s)$ and $\varepsilon(s)$ defined on Ω such that

$$\text{(a)} \quad \tfrac{1}{2}\,(e^{i\,\psi(s)}\hat{Q}(s) + e^{-i\psi(s)}\hat{Q}^*(s)) > (-\cos\,\psi(s)+\varepsilon(s))I_m \quad , s \in \Omega \tag{19}$$

and

$$\text{(b)} \quad \delta_o(s) \subset \{z: \operatorname{Re}\, e^{i\psi(s)}z \geq -\varepsilon(s)\} \quad , \; s \in \Omega \tag{20}$$

Proof: We use contradiction by supposing that (15) is violated i.e. $o = 1 + \alpha+\beta$ with $\alpha \in V(\hat{Q}\,(s))$ and $\beta \in \delta_o(s)$. Multiplying by $e^{i\psi(s)}$ and taking real parts yields a contradiction as $\mathrm{Re}\; e^{i\psi(s)}\alpha > -\cos\psi(s) + \varepsilon(s)$ and $\mathrm{Re}\; e^{i\psi(s)}\beta \geq -\varepsilon(s)$ from (a) and (b).

A more familiar form of this result is obtained when we can choose $\psi(s) \equiv o$ and $\varepsilon(s) \equiv 1$. More precisely, if Q (and hence $\hat{Q}$) is (strictly) positive real [13] in the sense that

$$Q(s) + Q^*(s) > o \quad , s \in \Omega \tag{21}$$

then robust stability is guaranteed with respect to any perturbations $\Delta(s)$ generating a set $\delta(s)$ (and hence $\delta_o(s)$) located in the closed half space $\{z: \mathrm{Re}\; z \geq -1\}$. A similar conclusion holds when Q (and hence $\hat{Q}$) is positive-real with $\delta(s)$ located in the open half-space $\{z: \mathrm{Re}\; z > -1\}$ but the proof is omitted for brevity. A typical example of the application of the ideas can be obtained by considering the situation described by (11) where G_o and K are such that $Q = G_o K$ is positive real on Ω. The robustness of the design to the uncertainty in measurement dynamics is guaranteed as an inspection of (12) indicates that $\delta(s)$ (and hence $\delta_o(s)$) lies in $\{z: \mathrm{Re} z > -1\}$ for all $s \in \Omega$.

4. Discussion.

The potential usefulness of the numerical range analysis can be highlighted by comparing the results with the singular value approach. The two main areas to be considered are the benefits to be gained by use of numerical ranges in perturbation characterization and the benefits and implicit constraints in the robust stability theorem.

The use of the numerical range in perturbation characterization has been motivated in section 2 in terms of its ability to represent gain-phase structures and the example (11) indicates that it can be a less conservative representation of gain-phase structure by allowing phase information and variations of gain with phase to be included. These results can be extended by examination of the relationships between the maximum and minimum gains predicted by numerical range and singular value methods.

Theorem 2: For any mxm complex matrix M, we have the interlacing property

$$o \leq V_-(M) \leq \underline{\sigma}(M) \leq V_+(M) \leq \bar{\sigma}(M) \tag{22}$$

Proof: If $\lambda = x^* Mx$, $x^* x=1$, then $|\lambda| \leq \bar{\sigma}(M)$ proving the last inequality. If $M=UH$, $UU^* = I$, $H=H^*$ is a polar decomposition of M and $Hx=\underline{\sigma}(M)x$, $x^* x = 1$, we have $\lambda = x^* Mx = \underline{\sigma}(M)\, x^* Ux$ so that $V_-(M) \leq |\lambda| \leq \underline{\sigma}(M)$. Finally we prove that $\underline{\sigma}(M) \leq v_+(M)$ by the observation that any eigenvalue λ of M satisfies $|\lambda| \leq V_+(M)$ (by(7)) and $\underline{\sigma}(M) \leq |\lambda|$ [2].

We interpret this result as stating that the numerical range is a less conservative estimate of largest eigenvalue gain than the singular value but a more conservative estimate of the smallest gain. The conservatism at the lower gain is not regarded as a limitation in error characterization as it is expected that $\Delta = o$ is a possible perturbation when $V_-(M) = \underline{\sigma}(M) = o$ and $o \in \delta(s) \subset \delta_o(s)$. The more interesting situation occurs in estimation of the largest gain where $V_+(M) \leq \bar{\sigma}(M)$. There is here a substantial benefit. It is, however, limited as $\bar{\sigma}(M)$ and $V_+(M)$ are topologically equivalent norms on $L(C^m)$ satisfying [10].

$$\frac{1}{2}\, \bar{\sigma}(M) \leq V_+(M) \leq \bar{\sigma}(M) \tag{23}$$

i.e. a benefit of, at most, 100% is available, the actual benefit depending on the eigenstructure of M as $V_+(M) = \bar{\sigma}(M)$, if M is normal [10], [11].

It is also possible to produce relationships between the phase-spreads predicted by numerical range and singular value methods [2] :

Theorem 3: If M=UH is a polar decomposition of M and U has eigenvalues $e^{i\phi_j}$, $1 \leq j \leq m$, satisfying $\phi_1 \leq \phi_2 \leq \dots \leq \phi_m$, then

$$\Theta_1 \leq \phi_1 \leq \phi_m \leq \Theta_2 \tag{24}$$

Proof: Let $z^* U = e^{i\phi} z^*$, $z^* z=1$ and note that $z^* Mz = e^{i\phi} z^* Hz \in V(M)$.

In effect, the numerical range is more conservative that singular value methods [2] in predicting maximum phase spreads. In the case of $\phi_m - \phi_1 \leq \pi$, this may have noticeable effects (Fig.3(a)) but more generally, with $\phi_m - \phi_1 > \pi$, both techniques can only predict a phase spread of 2π. In such situations, the overall shape of V(M) will carry much more information than singular values (Fig.3(b)), by limiting gain variations as a function of phase direction.

Turning now to the stability theorem 1, note that the origin lies in the closure of $\delta_o(s)$ and, in the commonly expected case of $o \in \delta(s)$, lies in $\delta_o(s)$. An implicit constraint in the theorem therefore is that the control K should be such that (equation (18),

$$-1 \notin V(\hat{Q}(s)) \quad , \quad s \in \Omega \tag{25}$$

This can easily be checked graphically and is strongly related to the principal phase structure in the polar decomposition [2]:

<u>Theorem 4:</u> With the notation of theorem 3 and $0 \notin V(M)$, we have $\phi_m - \phi_1 < \pi$

<u>Proof:</u> By convexity, if $0 \notin V(M)$, we have $\Theta_2 - \Theta_1 < \pi$, the result following from (24)

Condition (25) is now seen to require that the spread of the principal phases of $I + \hat{Q}$ is less than π. The result is hence closely related to that of [2] but, in contrast, places no constraint on the gain/condition number and phase of the perturbations. It seems therefore that it is more generally applicable. It is however necessary to investigate other potential problems.

It is expected, in general, that the results of application of theorem 1 will be, at least, comparable with other approaches with the in-built advantage of a more detailed, gain-phase sensitive error characterization. However, in the <u>worst case</u> of phase distribution, (15) will require that

$$V_{-}(I_m + \hat{Q}(s)) > V_{+}(\delta(s)) \quad , \quad s \in \Omega \tag{26}$$

(an inequality that is the numerical range analogue of (3)). Bearing in mind theorem 2, the relative merits of numerical range and singular value methods will depend upon the degree to which $V_{+}(\delta(s))$ improves upon $\ell(s)$ as a representation of gain variations and the degree to which $V_{-}(I+\hat{Q})$ underestimates

$\underline{\sigma}(I+\hat{Q})$. In the worst case of $V_+(\delta(s)) \equiv \ell(s)$, the numerical range prediction will be more conservative. Remember, however, that this is a worst case analysis and will not be met in many applications. In fact, Corollary 1.1. indicates that the singular value method can be infinitely more conservative than the numerical range as it cannot cope with unbounded perturbation sets.

5. Conclusions

A conceptually new method of approaching robust stability theory has been described using the numerical range as a basis for both error characterization and stability studies. The numerical range is seen to be a less conservative description of modelling error variation in terms of gain and permits a more detailed representation of gain and phase structure when compared with singular value methodologies. In the application of the stability theorem, the approach is capable of dealing with arbitrary large modelling errors in situations where the phase is not destabilizing. It can hence be significantly less conservative than singular value methods but, in worst case situations it can be more conservative. The results are hence best viewed as an additional tool in the designers armoury and used in conjunction with singular value methods by, for example, application of different methods over different frequency ranges covering Ω. Finally, noting the work in [14], it may be possible to extend the ideas to cover nonlinear perturbations.

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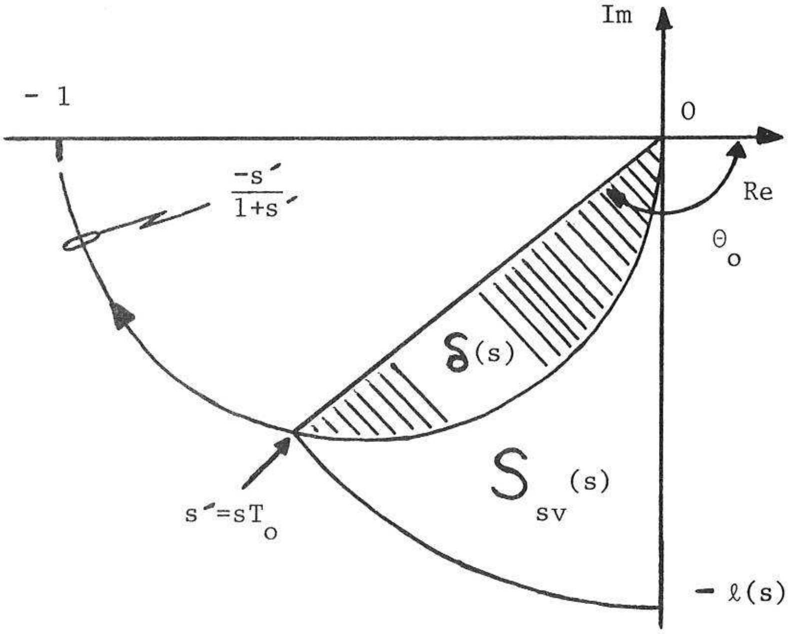

Fig.2. δ(s) for diagonal measurement dynamics.

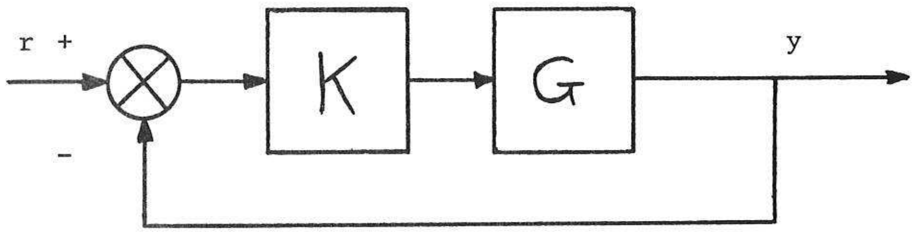

Fig.1. Multivariable Feedback Scheme

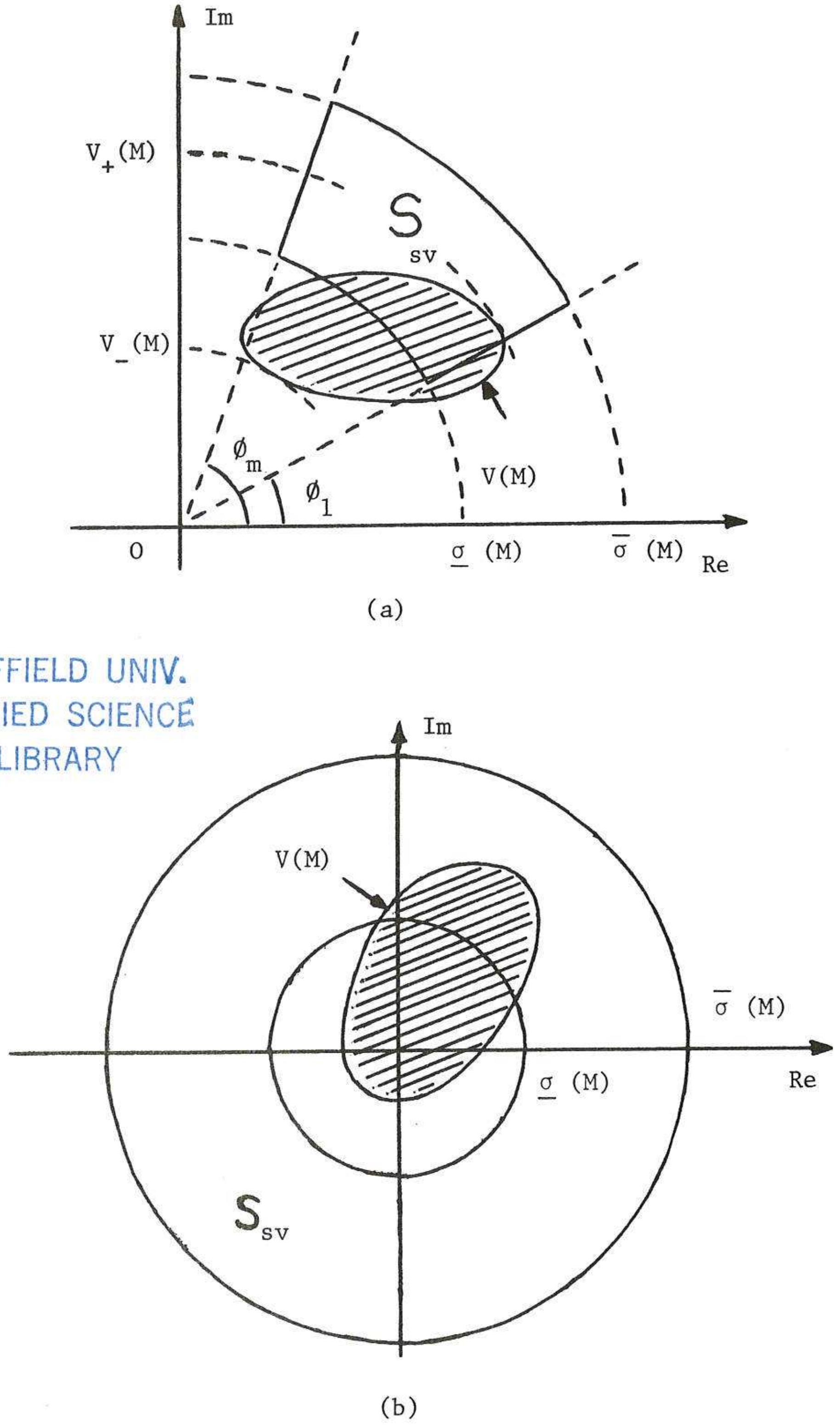

Fig.3. Relative form of numerical range and singular regions.