Some non-stability results for geometric Paneitz–Branson type equations

Laurent Bakri, Jean-Baptiste Casteras

Some non-stability results for geometric Paneitz-Branson type equations.

Laurent Bakri *† Jean-Baptiste Casteras ‡§

Abstract

Let (M,g) be a compact riemannian manifold of dimension $n \geq 5$. We consider two Paneitz-Branson type equations with general coefficients

$$\Delta_g^2 u - div_g(A_g du) + hu = |u|^{2^*-2-\varepsilon} u \text{ on } M, \tag{E1}$$

and

$$\Delta_g^2 u - div_g((A_g + \varepsilon B_g)du) + hu = |u|^{2^*-2} u \text{ on } M, \tag{E2}$$

where A_g and B_g are smooth symmetric $(2,0)$-tensors, $h \in C^\infty(M)$, $2^* = \dfrac{2n}{n-4}$ and ε is a small positive parameter. Under suitable assumptions, we construct solutions u_ε to (**??**) and (**??**) which blow up at one point of the manifold when ε tends to 0. In particular, we extend the result of Deng and Pistoia 2011 (to the case where A_g is the one defined in the Paneitz operator) and the result of Pistoia and Vaira 2013 (to the case $n = 8$ and (M,g) locally conformally flat).

Keywords: Paneitz-Branson type equations, blow up solutions, Liapunov-Schmidt reduction procedure.

Mathematics Subject Classification (2010) : 35J30, 35J60, 35B33, 35B35.

*Departamento de Matemática, Universidad Técnica Federico Santa María, 1680 Av. españa Valparaíso, Chile. E-mail :laurent.bakri@gmail.com.

†The first author was supported by PROYECTO BASAL PFB03 CMM, Universidad de Chile, Santiago (Chile).

‡UFRGS, Instituto de Matemática, Av. Bento Goncalves 9500, 91540-000 Porto Alegre-RS, Brasil Phone : (55) 51 3308-6208. E-mail Jean-Baptiste.Casteras@univ-brest.fr

§The second author was supported by the CNPq (Brazil) project 501559/2012-4.

1 Introduction and statements of the results

In this paper, we will study the stability of Paneitz type equations in the geometric case for two kinds of perturbations. The Paneitz operator, which is a conformally covariant fourth order operator defined on any pseudo-Riemannian manifold, has been introduced by Paneitz in [?]. Branson [?] discovered that this operator describes the conformal transformation of a curvature quantity, the Q-curvature. It turns out that this curvature appears in a lot of geometric and physics problems. We refer to the articles of Branson and Gover [?], Chang [?], [?], Chang and Yang [?], and Gursky [?] (and the references therein) for more details on the geometric and physics aspects associated to the notion of Q-curvature. Let (M, g) be a compact riemannian manifold of dimension $n \geq 5$. We will be interested in solutions $u \in C^{4,\theta}(M)$, $\theta \in (0,1)$, of the following equation

$$P_g u := \Delta_g^2 u - div_g(A_g du) + hu = |u|^{2^*-2}u, \tag{1.1}$$

where $h \in C^\infty(M)$, $2^* = \dfrac{2n}{n-4}$ and A_g a smooth symmetric $(2,0)$-tensor given by

$$A_g := \frac{(n-2)^2+4}{2(n-1)(n-2)} R_g g - \frac{4}{n-2} Ric_g, \tag{1.2}$$

where R_g (resp. Ric_g) stands for the scalar curvature (resp. Ricci curvature) with respect to the metric g. When h is given by $h = \dfrac{n-4}{2} Q_g$ where Q_g is the Q-curvature with respect to the metric g then P_g is the so-called Paneitz-Branson operator and equation (**??**) is refered to as the Paneitz-Branson equation. We recall that the Q-curvature is defined by

$$Q_g = \frac{1}{2(n-1)} \Delta_g R_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2} |Ric_g|_g^2.$$

It is well known that the Paneitz operator is conformally invariant, i.e. if $\tilde{g} = \varphi^{\frac{4}{n-4}} g$ then, for all $u \in C^\infty(M)$, we have

$$P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(u).$$

We point out that if (M, g) is Einstein ($Ric_g = \lambda g$, $\lambda \in \mathbb{R}$), then the Paneitz-Branson operator takes the form

$$P_g u = \Delta_g^2 u + b\Delta_g u + cu, \tag{1.3}$$

where $b = \dfrac{n^2-2n-4}{2(n-1)} \lambda$ and $c = \dfrac{n(n-4)(n^2-4)}{16(n-1)^2} \lambda$. More generally, following the terminology introduced in [?], when P_g is of the form given by (**??**)

(respectively by (**??**)) for arbitrary smooth $(2,0)$ tensor A_g and $h \in C^\infty(M)$ (respectively for arbitrary real numbers b and c), the operator P_g is referred to as a Paneitz-Branson type operator with general coefficients (respectively Paneitz-Branson type operator with constant coefficients). A lot of attention has been devoted to the study of existence and compactness of solution to (**??**) (see for example [**?**, **?**, **?**, **?**, **?**, **?**] and the references therein). Here, we will be interested in the stability of (**??**).
In this paper, we will consider two kinds of stability for (**??**) : the stability with respect to the tensor A_g and the stability with respect to the power of the right-side term of (**??**). More precisely, we say that (**??**) is exponent-stable if, for any sequences of real positive numbers $(\varepsilon_\alpha)_\alpha$ such that $\varepsilon_\alpha \underset{\alpha\to\infty}{\longrightarrow} 0$ and for any sequences of solutions $(u_\alpha)_\alpha \in C^{4,\theta}(M)$, $\theta \in (0,1)$, of

$$\Delta_g^2 u_\alpha - div_g(A_g du_\alpha) + hu_\alpha = |u_\alpha|^{2^*-2-\varepsilon_\alpha} u_\alpha, \tag{1.4}$$

bounded in $H^2(M)$, then up to a subsequence, u_α converges in $C^4(M)$ to some smooth function u solution of (**??**). Respectively we say that (**??**) is A_g-stable if the functions u_α are in fact solutions of

$$\Delta_g^2 u_\alpha - div_g((A_g + \varepsilon_\alpha B)du_\alpha) + hu_\alpha = |u_\alpha|^{2^*-2} u_\alpha, \tag{1.5}$$

where B is a smooth symmetric $(2,0)$ tensor. We point out that a related notion of A_g-stability has been first introduced by Hebey and Robert in [**?**]. Before, stating more precisely their results, we introduce some notations. We let $\lambda_i(A_g)_x$, $x \in M$, $i = 1, \dots, n$, be the eigenvalues of $A_g(x)$ (with respect to the metric g) repeated with their multiplicity. We define $\lambda_1 = \inf_{x,i} \lambda_i(A_g)_x$, $\lambda_2 = \max_{x,i} \lambda_i(A_g)_x$ and $S_w = [\lambda_1, \lambda_2]$. In particular, it is proved in [**?**] that if (M,g) is locally conformally flat (l.c.f.) and P_g is a Paneitz-Branson type operator with strictly positive constant coefficients satisfying $c - \dfrac{b^2}{4} < 0$, then (**??**) is A_g-stable whenever

a. $b \notin S_w$ and $n = 6$,

b. $b \neq \dfrac{TrA_g}{n}$ if $n \neq 9$ or $n = 7$,

c. $b < \dfrac{TrA_g}{n}$ if $n = 8$.

We point out that the results obtained in [**?**] are stronger than the ones quoted above. In fact, they show stability of the equation with respect to both A_g and h. They also obtained the stability when $n = 5$ under the hypothesis that the mass of the Green function associated to P_g is strictly positive. To the authors' knowledge, it is the most refined positive stability result known presently. Concerning non-stability, the first result has been obtained by Deng and Pistoia in [?]. There, they show that, when A_g is replaced by some arbitrary smooth $(2,0)$ tensor B_g, equation (**??**) is not exponent-stable if

a. $n \geq 7$, $Tr_g(B_g - A_g)$ is not constant and $\min_M Tr_g(B_g - A_g) > 0$,

b. or $n \geq 8$ and $\xi_0 \in M$ a C^1 stable critical point of $Tr_g(B_g - A_g)$ such that $Tr_g(B_g - A_g)(\xi_0) > 0$.

A related result has been obtained by the authors in [?] where sign changing blowing-up solutions have been constructed in arbitrary dimensions. We refer to [?] for more details. Recently, Pistoia and Vaira [?] studied the A_g-stability of (**??**) when P_g is the Paneitz-Branson operator. They proved that equation (**??**) is not A_g-stable, under the following conditions : (M,g) is not conformally flat, $n \geq 9$ and there exists $\xi_0 \in M$ a C^1 stable critical point (see below for the definition) of the function $\xi \to \dfrac{Tr_g B(\xi)}{|Weyl_g(\xi)|_g}$, such that $Tr_g B(\xi_0) > 0$. For a function $\phi \in C^1(M)$, we recall that a critical point ξ_0 of ϕ is said C^1 stable if there exists an open neighborhood Ω of ξ_0 such that, for any point $\xi \in \bar{\Omega}$, there holds $\nabla_g \phi(\xi) = 0$ if and only if $\xi = \xi_0$ and such that the Brower degree $deg(\nabla_g \phi, \Omega, 0) \neq 0$. Our first theorem extends [?] to the case where $B_g = A_g$. More precisely, we have

Theorem 1.1. *Let (M,g) be a compact riemannian manifold of dimension n, the function h be such that P_g is coercive and let Φ be defined by*

$$\Phi := -\frac{n^2 - 4n - 4}{96(n-1)(n-3)} |Weyl_g|_g^2 + \frac{1}{n-4}\Big(h - \frac{n-4}{2} Q_g\Big). \tag{1.6}$$

Assume either that :

a. $n \geq 8$ and that Φ is such that $\min_{x \in M} \Phi(x) > 0$.

b. or $n \geq 11$ and there exists $\xi_0 \in M$ a C^1 stable critical point of Φ.

*Then (**??**) is not exponent-stable.*

As usual for this kind of result, we obtain the previous theorem by constructing a family of solutions $(u_\varepsilon)_\varepsilon$ of (**??**) which blows-up at some point $\xi \in M$ when ε goes to 0. More precisely, the family of solutions we construct is of the form

$$u_\varepsilon = \varphi BBl_\varepsilon + o(1),$$

where $o(1) \underset{\varepsilon\to 0}{\longrightarrow} 0$, and φ is a conformal factor, the purpose of which will be precised later (see (**??**)), and

$$BBl_\varepsilon(x) = [n(n-4)(n^2-4)]^{\frac{n-4}{8}} \left(\frac{\mu_\varepsilon}{\mu_\varepsilon + d_g(x,x_\varepsilon)^2}\right)^{\frac{n-4}{2}},$$

where $x, x_\varepsilon \in M$ and $\mu_\varepsilon \in \mathbb{R}^+$ is such that $\mu_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$. This form has been introduced by Esposito and Robert in [**?**]. In our second result we extend the result of [**?**] to dimension $n = 8$ using the same family of blowing-up solutions as the one described above. More precisely, we have

Theorem 1.2. *Let (M, g) be a compact riemannian manifold of dimension $n = 8$. Assume that $\min_M \{|Weyl_g(\xi)|_g : Tr_g(B)(\xi) > 0\} > 0$. Then (**??**) is not A_g-stable.*

We would like to make some comments on this theorem. As it was pointed out in [**?**] (see Remark 3.1), with our approach we are also able to recover the case $n > 8$. The approach used in [**?**] consisted in taking $\varphi \equiv 1$ but adding an higher-order term to the standard bubble BBl_ε. The method we use here is more simple but on the other hand, it seems more rigid. Finally, in our last result, we investigated the A_g-stability when (M, g) is l.c.f. Before stating more precisely our theorem, we introduce some notations. We let i_g be the injectivity radius of (M, g) and $r_0 \in \mathbb{R}_+^*$ such that $r_0 < i_g$. Since we are assuming that (M, g) is l.c.f., there exists a family $(g_\xi)_{\xi\in M}$ of smooth conformal metrics to g such that g_ξ is flat in the geodesic ball $B_\xi(r_0)$. We let G_g be the Green's function of the Paneitz operator P_g. We will assume that G_g is of the form

$$G_{g_\xi}(\exp_\xi y, \xi) = \frac{1}{\beta_n |y|^{n-4}} + A_\xi + 0^{(4)}(|y|), \tag{1.7}$$

where $\beta_n = (n-2)(n-4)\omega_{n-1}$, $\omega_{n-1} = |S^{n-1}|$, $A_\xi > 0$ depending only (M, g) and on ξ (being smooth with respect to ξ) and $f = O^{(k)}(r^m)$ denotes any quantity satisfying

$$|\nabla^j f(\exp_\xi y)| \le C_j |y|^{m-j},$$

for $1 \le j \le k$. We have :

Theorem 1.3. *Let (M, g) be a locally conformally flat manifold of dimension $n \ge 6$. Assume that $h = Q_g$, (**??**) holds and that $\max_M Tr_g(B) > 0$. Then (**??**) is not A_g-stable. More precisely, for $\varepsilon > 0$, there exists a family of solutions u_ε of (**??**) which blows-up, when $\varepsilon \to 0$, at some point ξ_0 so that*

$E(\xi_0) = \max_M E(\xi)$ *where* $E(\xi) = \dfrac{h(\xi)}{A_\xi^{\frac{2}{n-4}}}$. *Moreover if* $n \geq 7$, *for any isolated critical point* ξ_0 *of* E *with non-trivial degree and* $Tr_g(B)(\xi_0) > 0$, *for* $\varepsilon > 0$, *there exists a family of solutions* u_ε *of* (**??**) *which blows-up, when* $\varepsilon \to 0$, *at*ξ_0.

The method used in order to prove the previous theorem is inspired by the one of Esposito, Pistoia and Vétois [**?**] where a similar result has been proved for the Yamabe equation. The main idea consists in modifying slightly the shape of the family $(u_\varepsilon)_\varepsilon$ of blowing-up solutions we are looking for by multiplying the standard bubble BBl_ε by a function depending on the Green function. Finally, we point out that the assumption (**??**) is very natural. Gursky and Malchiodi in [**?**] (see Theorem 2.9) proved that if Q_g is semi positive, $R_g \geq 0$, (M, g) locally conformally flat but not conformally equivalent to the round sphere, then (**??**) holds (see also some recent preprints of Hang and Yang for improved results [**?**]).

The proof of the theorems relies on a well known Lyapunov-Schmidt reduction procedure which permits to reduce the problem to a finite dimensional one for which we defined a reduced energy. The solutions to (**??**) will then be obtained as critical points of this reduced energy. We refer to [**?**] and the references therein for more information on the Lyapunov-Schmidt reduction procedure.

The plan of this paper is the following : in section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorems **??** and **??** where the proofs of these two theorems are done in parallel. We begin by giving an estimate of the error and then give an estimate of the reduced energy. Finally, in Section 4, we prove Theorem **??**.

2 Preliminaries.

Let $(\xi_\alpha)_\alpha$ be a sequence of points of M. In all the following, we will suppose up to extracting a subsequence that, for α large enough, all the points ξ_α belong to a small open set Ω of M in which there exists a smooth orthogonal frame. Thus, we will identify the tangent spaces $T_\xi M$ with $\mathbb{R}^n$ for all $\xi \in \Omega$. We recall that we suppose that P_g is coercive.

In all the following, we will denote by $\langle ., . \rangle_{P_g}$, the scalar product, for $u, v \in H^2(M)$,

$$\langle u, v \rangle_{P_g} = \int_M \Delta_g u \Delta_g v dV + \int_M A_g(\nabla_g u, \nabla_g v) dV + \int_M huv dV,$$

where here and in the following dV stands for the volume element with respect to the metric g. We will denote $\|.\|_{P_g}$ the associated norm which is equivalent to the standard norm of $H^2(M)$. We denote by $i^* : L^{\frac{2n}{n+4}}(M) \to H^2(M)$ the adjoint operator of the embedding $i : H^2(M) \to L^{\frac{2n}{n-4}}(M)$, i.e. for all $w \in L^{\frac{2n}{n+4}}(M)$, the function $u = i^*(w) \in H^2(M)$ is the unique solution of $\Delta_g^2 u - div_g(A_g du) + hu = w$. Using this notation, we see that equations (**??**) and (**??**) can be rewritten as, for $u \in H^2(M)$,

$$u = i^*(f_\varepsilon(u)),$$

where $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$ for (**??**) and $f_\varepsilon(u) = |u|^{2^*-2}u - \varepsilon div_g(B(\nabla u)$ for (**??**). Before proceeding we recall some basic facts. It is well known (see [**?**]) that all solutions $u \in H^2(\mathbb{R}^n)$ of the equation

$$\Delta_{eucl}^2 u = u^{2^*-1} = u^{\frac{n+4}{n-4}} \ in \ \mathbb{R}^n$$

are given by

$$U_{\delta,y}(x) = \delta^{\frac{4-n}{2}} U\Big(\frac{x-y}{\delta}\Big), \ \delta > 0, \ y \in \mathbb{R}^n$$

where

$$U(x) = [n(n-4)(n^2-4)]^{\frac{n-4}{8}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}} = \alpha_n \left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}}. \quad (2.1)$$

It is also well known (see [**?**]) that all solutions $v \in H^2(\mathbb{R}^n)$ of

$$\Delta_{eucl}^2 v = (2^*-1)U^{2^*-2}v$$

are linear combinations of

$$V_0(x) = \alpha_n \frac{n-4}{2} \frac{|x|^2-1}{(1+|x|^2)^{\frac{n-2}{2}}}$$

and

$$V_i(x) = \alpha_n (n-4) \frac{x_i}{(1+|x|^2)^{\frac{n-2}{2}}}, \ i = 1, \dots, n.$$

Let us fix $N > n$ and $\xi \in M$, it is well known that there exists $\tilde{g} = \varphi^{\frac{4}{n-2}} g$, $\varphi > 0$ is a smooth function on M, such that

$$Ric_{\tilde{g}}(\xi) = 0, \ \nabla R_{\tilde{g}}(\xi) = 0, \ \Delta_{\tilde{g}} R_{\tilde{g}}(\xi) = \frac{1}{6} |Weyl_g(\xi)|_g^2 \quad (2.2)$$

and

$$dV_{\tilde{g}} = \big(1 + O(r^N)\big)\, dV_{\mathbb{R}^n}.$$

We notice that, in this system of coordinates, we have

$$Q_g(\xi) = \frac{1}{2(n-1)}\Delta_g R_g(\xi) = \frac{1}{12(n-1)}|Weyl_g(\xi)|_g^2.$$

We also give the expression of the Paneitz operator for radial function in the previous metric which will be useful in the sequel :

Lemma 2.1. *[?, lemma 2.6 p.13] If u is a radial function, then we have*

$$\begin{aligned} P_{\tilde{g}}(u) = \Delta^2 u + \frac{1}{n-2}\nabla_l\nabla_k R_g(0)x^k x^l A(u) + \frac{n-4}{24(n-1)}\left|Weyl_g\right|_g^2(0)u \\ + O\big(r^3\big)|u''| + O\big(r^2\big)|u'| + O(r)u + O\big(r^{N-1}\big)u''' \\ + O\big(r^{N-2}\big)u'' + O\big(r^{N-3}\big)u', \end{aligned}$$

where $u' = \dfrac{\partial u}{\partial r}$ and

$$A(u) = \frac{u'}{r}\left(\frac{2(n-1)}{n-2} - \frac{(n-1)(n-2)}{2} + 6 - n\right) - u''\left(\frac{n-2}{2} + \frac{2}{n-2}\right).$$

3 Proofs of Theorems ?? and ??.

This section will be devoted to the proof of Theorems **??** and **??**. Since the form of the solution we construct will be the same in both cases, we will prove Theorems **??** and **??** in parallel . We define, for any real δ strictly positive, $\xi \in M$ and $x \in M$,

$$W_{\delta,\xi}(x) = \chi\left(d_g(x,\xi)\right)\varphi(x)\delta^{\frac{4-n}{2}}U\big(\delta^{-1}\exp_\xi^{-1}(x)\big),$$

where $d_g(x,\xi)$ stands for the distance from x to ξ with respect to the metric g, $\exp_\xi$ is the exponential map with respect to the metric g and $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth cutoff function such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $x \in \left[-\frac{r_0}{2}, \frac{r_0}{2}\right]$ and $\chi(x) = 0$ if $x \in \mathbb{R}\backslash(-r_0, r_0)$. We also define, for any real δ strictly positive, $\xi \in M$ and $x \in M$,

$$Z_{\delta,\xi}(x) = \chi(d_g(x,\xi))\delta^{\frac{n-4}{2}}\frac{d_g(x,\xi)^2 - \delta^2}{(\delta^2 + d_g(x,\xi)^2)^{\frac{n-2}{2}}},$$

and, for $\omega \in T_\xi M$,

$$Z_{\delta,\xi,\omega}(x) = \chi(d_g(x,\xi))\delta^{\frac{n-2}{2}}\frac{\left\langle \exp_\xi^{-1} x, \omega\right\rangle_g}{(\delta^2 + d_g(x,\xi)^2)^{\frac{n-2}{2}}}.$$

We denote by $\Pi_{\delta,\xi}$ respectively $\Pi^{\perp}_{\delta,\xi}$ the projection of $H^2(M)$ onto

$$K_{\delta,\xi} = \text{span}\left\{Z_{\delta,\xi}, (Z_{\delta,\xi,e_i})_{i=1..n}\right\}$$

respectively

$$K^{\perp}_{\delta,\xi} = \left\{\phi \in H^2(M) / \left\langle \phi, Z_{\delta,\xi}\right\rangle_{P_g} = 0 \text{ } and \text{ } \left\langle \phi, Z_{\delta,\xi,\omega}\right\rangle_{P_g} = 0, \ \forall \omega \in T_\xi M\right\}. \tag{3.1}$$

We are looking for solution u to (**??**) of the form

$$u = W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon},$$

where $\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} \in K^{\perp}_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ and $\delta_\varepsilon(t_\varepsilon) \in \mathbb{R}^+$ is defined below. It is easy to see that equations (**??**) (respectively (**??**)) are equivalent to the following system

$$\Pi_{\delta_\varepsilon(t),\xi}\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} - i^*\left(f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})\right)\right) = 0, \tag{3.2}$$

and

$$\Pi^{\perp}_{\delta_\varepsilon(t),\xi}\left(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} - i^*\left(f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})\right)\right) = 0, \tag{3.3}$$

where f_ε is defined by $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$ (respectively by $f_\varepsilon(u) = |u|^{2^*-2}u - \varepsilon div_g(B(\nabla u))$). We define $\delta_\varepsilon(t_\varepsilon)$, for $t_\varepsilon > 0$ by

$$\delta_\varepsilon(t_\varepsilon) = \begin{cases} (t_\varepsilon \varepsilon)^{\frac{1}{4}}, & if\ n \geq 9, \\ t_\varepsilon l^{-1}(\varepsilon), & if\ n = 8 \end{cases} \tag{3.4}$$

where $l : (0, e^{-\frac{1}{2}}) \to (0, e^{-\frac{1}{2}})$ is defined by $l(\delta) = -\delta^4 \ln \delta$ if $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$ and by $l(\delta) = -\delta^2 \ln \delta$ if $f_\varepsilon(u) = |u|^{2^*-2}u - \varepsilon div_g(B(\nabla u))$.

Let us now give the plan of this section. We will begin by solving (**??**) in Section 3.1. In Section 3.2, we will solve (**??**) by proving an estimate of the reduced energy (see Propositions **??** and **??**) and give the proof of the theorems. Finally, in Section 3.3, we finish the proof of Proposition **??** by showing that the estimate of the reduced energy holds C^1-uniformly when $n \geq 11$.

3.1 Finite dimensional reduction.

We begin by solving (**??**). The following proposition is well known and we refer to [**?**] and [**?**] for a proof of it.

Proposition 3.1. *Given two real numbers $a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any $t \in [a,b]$ and any $\xi \in M$, there exists a unique function $\phi_{\delta_\varepsilon(t),\xi} \in K^{\perp}_{\delta_\varepsilon(t),\xi}$ which solves equation (**??**) and satisfies*

$$\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g} \leq C_{a,b}\left\|i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}\right\|_{P_g}. \tag{3.5}$$

Moreover, $\phi_{\delta_\varepsilon(t),\xi}$ is continuously differentiable with respect to t and ξ.

The next two lemma are devoted to estimate $\left\|i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}\right\|_{P_g}$ in term of ε. We begin by the case $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$ (i.e. by (**??**)).

Lemma 3.2. *Assume that $n \geq 8$ and $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$. Given two positive real numbers $a < b$, there exists a positive constant $C'_{a,b}$ such that for ε small, for any real number $t \in [a,b]$ and any point $\xi \in M$, there holds*

$$\begin{aligned}&\left\|i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}\right\|_{P_g}\\ &\qquad \leq C'_{a,b}\left(\varepsilon|\ln\delta_\varepsilon(t)| + \begin{cases}\delta_\varepsilon(t)^{\frac{n-4}{2}}, & if\ n < 12,\\ \delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)|, & if\ n \geq 12\end{cases}\right).\end{aligned}$$

Proof. All the estimates will be uniform in t, ξ and ε. Since i^* is continuous, we have

$$\begin{aligned}&\left\|i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}\right\|_{P_g}\\ &\qquad = O\left(\left\|(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - P_g(W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{2n}{n+4}}}\right).\end{aligned} \tag{3.6}$$

The triangular inequality yields to

$$\begin{aligned}&\left\|i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}\right\|_{P_g}\\ &\qquad \leq C\left\|f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - f(W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{2n}{n+4}}}\\ &\qquad\quad + C\left\|f(W_{\delta_\varepsilon(t),\xi}) - P_g(W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{2n}{n+4}}}\\ &\qquad \leq C(I_1 + I_2).\end{aligned} \tag{3.7}$$

It is easy to see (see for instance inequality (3.28) of [**?**]) that

$$I_1 = \left\|f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - f_0(W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{2n}{n+4}}} = O(\varepsilon|\ln\delta_\varepsilon(t)|). \tag{3.8}$$

Using that the Paneitz operator $P^{paneitz}$, (i.e. for which $h = \dfrac{n-4}{2}$) is a conformal operator and denoting $\tilde{W}_{\delta_\varepsilon(t),\xi} = \dfrac{W_{\delta_\varepsilon(t),\xi}}{\varphi}$, we have

$$f(W_{\delta_\varepsilon(t),\xi}) - P_g^{paneitz}(W_{\delta_\varepsilon(t),\xi}) = \varphi^{2^*-1}(f(\tilde{W}_{\delta_\varepsilon(t),\xi}) - P_{\tilde{g}}^{paneitz}(\tilde{W}_{\delta_\varepsilon(t),\xi})),$$

where $\tilde{g}$ is the metric defined in (**??**). Now, since $\tilde{W}_{\delta_\varepsilon(t),\xi}$ is a radial function, using Lemma **??**, we have

$$\begin{aligned}
f(W_{\delta_\varepsilon(t),\xi}) - P_g(W_{\delta_\varepsilon(t),\xi}) &= f(W_{\delta_\varepsilon(t),\xi}) - P_g^{paneitz}(W_{\delta_\varepsilon(t),\xi}) \\
&\quad + \left(\frac{n-4}{2}Q - h\right) W_{\delta_\varepsilon(t),\xi} \\
&= \varphi^{2^*-1}(f(\tilde{W}_{\delta_\varepsilon(t),\xi}) - \Delta^2_{\mathbb{R}^n}(\tilde{W}_{\delta_\varepsilon(t),\xi})) \\
&\quad + O(r^2\partial_{rr}\tilde{W}_{\delta_\varepsilon(t),\xi} + r\partial_r\tilde{W}_{\delta_\varepsilon(t),\xi} + \tilde{W}_{\delta_\varepsilon(t),\xi}) \\
&= O(r^2\partial_{rr}\tilde{W}_{\delta_\varepsilon(t),\xi} + r\partial_r\tilde{W}_{\delta_\varepsilon(t),\xi} + \tilde{W}_{\delta_\varepsilon(t),\xi}).
\end{aligned}$$

Direct computations give

$$\begin{aligned}
\max\left\{\left\|\tilde{W}_{\delta_\varepsilon(t),\xi}\right\|_{L^{\frac{2n}{n+4}}}, \left\|r\partial_r\tilde{W}_{\delta_\varepsilon(t),\xi}\right\|_{L^{\frac{2n}{n+4}}}, \left\|r^2\partial_{rr}\tilde{W}_{\delta_\varepsilon(t),\xi}\right\|_{L^{\frac{2n}{n+4}}}\right\} \\
\leq C\begin{cases} \delta_\varepsilon(t)^4, & if\ n \geq 13, \\ \delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)|, & if\ n = 12, \\ \delta_\varepsilon(t)^{\frac{n-4}{2}}, & if\ 5 \leq n \leq 11. \end{cases}
\end{aligned}$$

Therefore, we deduce that

$$|I_2| \leq C\begin{cases} \delta_\varepsilon(t)^4, & if\ n \geq 13, \\ \delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)|, & if\ n = 12, \\ \delta_\varepsilon(t)^{\frac{n-4}{2}}, & if\ 5 \leq n \leq 11. \end{cases} \tag{3.9}$$

Combining (**??**), (**??**) and (**??**), the proof of the lemma follows. □

Next we prove the equivalent of the previous lemma for equation (**??**).

Lemma 3.3. *Assume that $n = 8$ and $f_\varepsilon(u) = |u|^{2^*-2}u - \varepsilon div_g(B(\nabla u))$. Given two positive real numbers $a < b$, there exists a positive constant $C'_{a,b}$ such that for ε small, for any real number $t \in [a,b]$ and any point $\xi \in M$, there holds*

$$\left\|i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}\right\|_{P_g} \leq C'_{a,b}(\delta_\varepsilon(t)^2 + \varepsilon\delta_\varepsilon(t)^2|\ln\delta_\varepsilon(t)|^{\frac{3}{4}}).$$

Proof. Using the continuity of i^* and the triangular inequality, we have

$$\begin{aligned}
&\left\|i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}\right\|_{P_g} \\
&\quad \leq \varepsilon\left\|div_g(B(\nabla W_{\delta_\varepsilon(t),\xi}))\right\|_{L^{\frac{2n}{n+4}}} \\
&\quad\quad + C\left\|f(W_{\delta_\varepsilon(t),\xi}) - P_g(W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{2n}{n+4}}} \\
&\quad \leq C(I_1 + I_2).
\end{aligned} \tag{3.10}$$

From (??), we know that $I_2 = \delta_\varepsilon(t)^2$. Since

$$\left\| \frac{1}{r} W'_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}} + \left\| W''_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}} \leq O(\delta_\varepsilon(t)^2 |\ln \delta_\varepsilon(t)|^{\frac{3}{4}}),$$

we deduce that

$$I_1 = \varepsilon \left\| div_g(B(\nabla W_{\delta_\varepsilon(t),\xi})) \right\|_{L^{\frac{2n}{n+4}}} = O(\varepsilon \delta_\varepsilon(t)^2 |\ln \delta_\varepsilon(t)|^{\frac{3}{4}})).$$

The proof follows from the previous estimates. □

3.2 The reduced problem.

In this section, we will solve equation (??) and give the proof of Theorems ?? and ??. We begin with the case $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$ (i.e. equation (??)). For $\varepsilon > 0$ small enough, we define the energy associated to (??) by, for $u \in H^2(M)$,

$$\begin{aligned} J_\varepsilon(u) =& \frac{1}{2}\int_M (\Delta_g u)^2 + \frac{1}{2}\int_M A_g(\nabla_g u, \nabla_g u) dV + \frac{1}{2}\int_M hu^2 dV \\ & - \int_M F_\varepsilon(u) dV, \end{aligned} \tag{3.11}$$

where $F_\varepsilon(u) = \displaystyle\int_0^u f_\varepsilon(s)ds$. We set $I_\varepsilon(t,\xi) = J_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})$, $t \in \mathbb{R}_+^*$ and $\xi \in M$ where $\phi_{\delta_\varepsilon(t),\xi} \in K^\perp_{\delta_\varepsilon(t),\xi}$ is the function defined in Proposition ??. In the next proposition, we give the expansion of I_ε with respect to ε.

Proposition 3.4. *Assume that $n \geq 8$ and $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$. There exist constants $c_i(n)$, $i = 1, \dots, 5$ depending on n, such that*

$$I_\varepsilon(t,\xi) = c_5(n) + c_2(n)\varepsilon + c_3(n)\varepsilon\ln\varepsilon - c_4(n)\varepsilon\ln(t) + c_1(n)\Phi(\xi)\varepsilon t + o(\varepsilon) \tag{3.12}$$

as $\varepsilon \to 0$, C^0 uniformly with respect to t in compact subsets of $\mathbb{R}_+^*$ and with respect to $\xi \in M$ and C^1 uniformly if $n \geq 11$. Moreover, we have that $c_4(n) > 0$, $c_1(n) > 0$ and

$$\Phi = -\frac{n^2 - 4n - 4}{96(n-1)(n-3)} |Weyl_g|_g^2 + \frac{1}{n-4}(h - \frac{n-4}{2} Q_g).$$

Proof. We begin by proving that

$$I_\varepsilon(t,\xi) = J_\varepsilon(W_{\delta_\varepsilon(t),\xi}) + o(\varepsilon), \tag{3.13}$$

as $\varepsilon \to 0$, uniformly with respect to t in compact subsets of $\mathbb{R}_+^*$ and points $\xi \in M$ (we will show in Lemma **??** that this estimate holds C^1 uniformly with respect to t and ξ when $n \geq 12$). Indeed, we have

$$\begin{aligned} &I_\varepsilon(t,\xi) - J_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \\ &\quad = \left\langle W_{\delta_\varepsilon(t),\xi} - i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})), \phi_{\delta_\varepsilon(t),\xi} \right\rangle_{P_g} + O(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g}^2) \end{aligned} \tag{3.14}$$

when $\varepsilon \to 0$. Using Lemma **??**, Proposition **??** and the definition of $\delta_\varepsilon(t)$ (**??**), we get

$$\begin{aligned} &\left\langle W_{\delta_\varepsilon(t),\xi} - i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})), \phi_{\delta_\varepsilon(t),\xi} \right\rangle_{P_g} + O(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g}^2) \\ &\quad = O(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g}^2) \\ &\quad = O\left(\varepsilon^2|\ln\delta_\varepsilon(t)|^2 + \begin{cases} \delta_\varepsilon(t)^{n-4} & if\ n < 12 \\ \delta_\varepsilon(t)^8|\ln\delta_\varepsilon(t)|^2 & if\ n \geq 12 \end{cases}\right) \\ &\quad = o(\varepsilon). \end{aligned}$$

Now, the proposition is reduced to estimate $J_\varepsilon(W_{\delta_\varepsilon(t),\xi})$. We will focus on C^0-estimates. The C^1-estimates can be obtained using the same argument as in Lemma 4.1 of [**?**]. We use the computations of section 6 of [**?**] and the estimate (4.2) of [**?**] to estimate $I_{1,\varepsilon,t,\delta}$. Using that J_ε is conformally invariant and using (**??**), we have

$$\begin{aligned} &\int_M P_g(W_{\delta_\varepsilon(t),\xi})W_{\delta_\varepsilon(t),\xi}dV = \alpha_n^2\left[\frac{n(n-4)(n^2-4)\omega_n}{2^n}\right] \\ &+ \begin{cases} \dfrac{\omega_n(n-1)(n-3)(n-4)\Phi(\xi)}{2^{n-4}(n-6)(n-8)}\delta_\varepsilon(t)^4 + o(\delta_\varepsilon(t)^4), & if\ n \geq 9, \\ \omega_{n-1}(n-4)\Phi(\xi)\delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)| + o(\delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)|), & if\ n = 8, \end{cases} \end{aligned} \tag{3.15}$$

where Φ is given by

$$\Phi = -\frac{n^2-4n-4}{96(n-1)(n-3)}|Weyl_g|_g^2 + \frac{1}{n-4}(h - \frac{n-4}{2}Q_g).$$

Next, we define $I_p^q = \int_0^\infty \frac{r^q}{(1+r)^p}dr$ for any p, q integers such that $q < p-1$. In the sequel, we will use that

$$I_p^q = \frac{q}{p-q-1}I_p^{q-1} = \frac{p}{p-q-1}I_{p+1}^q,$$

and

$$I_n^{n-2} = \frac{1}{n-1},\ I_n^{\frac{n-2}{2}} = \frac{\omega_n}{2^{n-1}\omega_{n-1}}.$$

We also have that

$$I^{\frac{n-2}{2}}_{n-\frac{n-4}{2}\varepsilon} = I^{\frac{n-2}{2}}_{n} + \frac{n-4}{2}\tilde{I}^{\frac{n-2}{2}}_{n}\varepsilon + O(\varepsilon^2),$$

where $\tilde{I}^q_p = \int_0^\infty \frac{r^q \ln(1+r)}{(1+r)^p} dr$. Therefore, we obtain that

$$\begin{aligned}
\frac{1}{2^*-\varepsilon}&\int_M W^{2^*-\varepsilon}_{\delta_\varepsilon(t),\xi} dV \\
=&\frac{\alpha_n^{2^*-\varepsilon}}{2^*-\varepsilon}(\delta_\varepsilon(t))^{\frac{n-4}{2}\varepsilon}\omega_{n-1}\int_0^{\frac{r_0}{\delta_\varepsilon(t)}}\left(\frac{1}{1+r^2}\right)^{n-\frac{n-4}{2}\varepsilon} r^{n-1}(1+O(r^n))dr \\
=&\frac{n-4}{2n}K_n^{-n/4}\Bigg[1+\frac{n-4}{2}\varepsilon\ln(\delta_\varepsilon(t)) \\
&+\frac{n-4}{2n}\left(\frac{n\tilde{I}^{\frac{n-2}{2}}_n}{I^{\frac{n-2}{2}}_n} + \frac{n(1-\frac{n}{2}\ln\sqrt{n(n-4)(n^2-4)})}{n-2}\right)\varepsilon\Bigg] + o(\delta_\varepsilon(t)^5).
\end{aligned} \tag{3.16}$$

Then the proposition follows from (??) and (??). □

Next we consider equation (??). For $\varepsilon > 0$ small enough, we define the energy associated to (??) by, for $u \in H^2(M)$,

$$\begin{aligned}
\tilde{J}_\varepsilon(u) =&\frac{1}{2}\int_M (\Delta_g u)^2 + \frac{1}{2}\int_M (A_g+\varepsilon B)(\nabla_g u, \nabla_g u)dV + \frac{1}{2}\int_M hu^2 dV \\
&-\int_M F_0(u)dV,
\end{aligned} \tag{3.17}$$

where $F_0(u) = \int_0^u f_0(s)ds$. We set $\tilde{I}_\varepsilon(t,\xi) = \tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})$, $t \in \mathbb{R}^*_+$ and $\xi \in M$ where $\phi_{\delta_\varepsilon(t),\xi} \in K^\perp_{\delta_\varepsilon(t),\xi}$ is the function defined in Proposition ??. In this case, we obtain :

Proposition 3.5. *Assume that $n = 8$ and $f_\varepsilon(u) = |u|^{2^*-2}u - \varepsilon div_g(B(\nabla u))$. There exist constants $c_i(n)$, $i = 1,\dots,3$ depending on n, such that*

$$\begin{aligned}
\tilde{I}_\varepsilon(t,\xi) =&c_1(n) + \varepsilon^2|\ln\varepsilon|^{-1}\left(c_2(n)t^2 Tr_g B(\xi) - c_3(n)t^4|Weyl_g|^2_g\right) \\
&+ o(\varepsilon^2|\ln\varepsilon|^{-1}),
\end{aligned} \tag{3.18}$$

as $\varepsilon \to 0$ C^0 uniformly with respect to t in compact subsets of $\mathbb{R}^*_+$ and with respect to $\xi \in M$. Moreover, we have that $c_2(n) > 0$ and $c_3(n) > 0$.

Proof. Using Lemma **??**, Proposition **??** and (**??**), we get

$$\left\langle W_{\delta_\varepsilon(t),\xi} - i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})), \phi_{\delta_\varepsilon(t),\xi} \right\rangle_{P_g} + O\left(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g}^2\right) = O\left(\delta_\varepsilon(t)^4\right).$$

We have

$$\frac{1}{2^*}\int_M W_{\delta_\varepsilon(t),\xi}^{2^*} dV = \frac{\omega_n}{2^n} + o(\delta_\varepsilon(t)^5),$$

and

$$\begin{aligned}
&\frac{1}{2}\varepsilon\int_M B(\nabla W_{\delta_\varepsilon(t),\xi}, \nabla W_{\delta_\varepsilon(t),\xi}) dV \\
&= \frac{(n-4)^2}{2n}\varepsilon\delta_\varepsilon(t)^2\alpha_n^2\omega_{n-1}Tr_g B(\xi)\int_0^{\frac{r_0}{\delta_\varepsilon(t)}} \frac{r^{n+1}}{(1+r^2)^{n-2}} dr + O(\delta_\varepsilon(t)^5) \\
&= \varepsilon\delta_\varepsilon(t)^2 \frac{2(n-1)}{n(n-6)(n^2-4)} K_n^{-\frac{n}{4}} Tr_g B(\xi) + O(\delta_\varepsilon(t)^5).
\end{aligned}$$

Therefore, from the two previous estimates and (**??**), we have

$$\begin{aligned}
\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi}) = \alpha_n^2 &\left[\frac{n(n-4)(n^2-4)\omega_n}{2^n}\right] \\
&+ \varepsilon\delta_\varepsilon(t)^2 \frac{2(n-1)}{n(n-6)(n^2-4)} K_n^{-\frac{n}{4}} Tr_g B(\xi) \\
&- \frac{n^2-4n-4}{96(n-1)(n-3)} |Weyl_g|_g^2 \delta_\varepsilon(t)^4 |\ln\delta_\varepsilon(t)| + o\left(\delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)|\right).
\end{aligned}$$

□

Finally, we recall the following proposition (see [?] and [?] for a proof) which shows that to obtain a solution of (**??**) (respectively (**??**)), we only need to find a critical point for $\tilde{I}_\varepsilon$ (respectively I_ε).

Proposition 3.6. *Given two positive real numbers $a < b$, for ε small, if $(t_\varepsilon, \xi_\varepsilon) \in (a,b) \times M$ is a critical point of $\tilde{I}_\varepsilon$ (respectively I_ε), then the function $W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution of* (**??**) *(respectively* (**??**)*).*

We are now in position to prove Theorems **??** and **??**. We restrict ourselves to prove Theorem **??** (Theorem **??** can be obtained in the same way).

Proof of Theorem **??**. We set $\mathcal{G} : \mathbb{R}_+^* \times M \to \mathbb{R}$ the function defined by

$$\mathcal{G}(t,\xi) = -c_4(n)\ln t + c_1(n)\varphi(\xi)t,$$

where $c_4(n)$, $c_1(n)$ and $\varphi(\xi)$ are defined in (**??**). From Proposition **??**, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(I_\varepsilon(t,\xi)-c_5(n)-c_2(n)\varepsilon-c_3(n)\varepsilon\ln\varepsilon)=\mathcal{G}(t,\xi), \tag{3.19}$$

C^0 (and C^1 if $n\geq 11$) uniformly with respect to $\xi\in M$ and t in compact subset of $\mathbb{R}_+^*$. We will consider two cases depending on the dimension of the manifold.

First case : $n\geq 11$.

We argue as in [**?**]. Let ξ_0 be the C^1 stable critical point of φ such that $\varphi(\xi_0)>0$ and set

$$t_0=\frac{c_4(n)}{c_1(n)\varphi(\xi_0)}>0.$$

Identifying the tangent space at ξ with $\mathbb{R}^n$ we define the map H from $[0,1]\times\mathbb{R}^+\times\mathbb{R}^n$into $\mathbb{R}^{n+1}$ by

$$\begin{aligned}H(s,t,\xi)=s&\left(\frac{\partial\mathcal{G}(t,\exp_\xi(y))}{\partial t},\frac{\partial\mathcal{G}(t,\exp_\xi(y))}{\partial y_1}|_{y=0},\dots,\frac{\partial\mathcal{G}(t,\exp_\xi(y))}{\partial y_n}|_{y=0}\right)\\&+(1-s)\left(t-t_0,\frac{\partial(\varphi\circ\exp_\xi(y))}{\partial y_1}|_{y=0},\dots,\frac{\partial(\varphi\circ\exp_\xi(y))}{\partial y_n}|_{y=0}\right).\end{aligned}$$

By the invariance of the Brower degree via homotopy, we have that (t_0,ξ_0) is a C^1 stable critical point of $\mathcal{G}$. From Proposition **??** and standard properties of the Brower degree (see *e.g.* [**?**]), there exists a couple $(t_\varepsilon,\xi_\varepsilon)$ of critical points of I_ε converging to (t_0,ξ_0).

Second case : $8\leq n\leq 10$.

Since $c_4(n)$ and $c_1(n)$ are positive, we have

$$\lim_{t\to 0^+}\mathcal{G}(t,\xi)=\lim_{t\to\infty}\mathcal{G}(t,\xi)=+\infty,$$

uniformly in $\xi\in M$. Therefore, from (**??**) we deduce that, for ε small enough,there exists a couple $(t_\varepsilon,\xi_\varepsilon)$ which is a minimum for the functional I_ε in $(a,b)\times M$ where a,b are positive constants not depending on ε. This implies from Proposition **??** that $W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}+\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution of (**??**). Thus Theorem **??** is established. □

3.3 C^1 uniform estimate for the reduced energy.

Finally, we end this section by proving that the estimate (**??**) holds C^1 uniformly if $n\geq 11$.

Lemma 3.7. *If $n \geq 11$, we have*

$$I_\varepsilon(t,\xi) = J_\varepsilon(W_{\delta_\varepsilon(t),\xi}) + o(\varepsilon),$$

C^1 uniformly with respect to t in compact subsets of $\mathbb{R}_+^*$ and $\xi \in M$.

Proof. To simplify notations, we set, for $i = 1, \dots, n$,

$$Z_0 = Z_{\delta_\varepsilon(t),\xi} \text{ and } Z_i = Z_{\delta_\varepsilon(t),\xi,e_i}.$$

We recall that

$$\frac{\partial}{\partial t}(W_{\delta_\varepsilon(t),\xi}) = \frac{\tilde{C}_n}{t} Z_0 = \frac{\tilde{C}_n \delta_\varepsilon'(t)}{\delta_\varepsilon(t)} Z_0,$$

where $\tilde{C}_n = \dfrac{\alpha_n(n-4)}{4}$ (see (**??**) for the definition of α_n). Taking the derivative with respect to t to $I_\varepsilon(t,\xi) - J_\varepsilon(W_{\delta_\varepsilon(t),\xi})$, we obtain

$$\begin{aligned}
&\frac{\partial I_\varepsilon}{\partial t}(t,\xi) - \frac{\partial J_\varepsilon}{\partial t}(W_{\delta(t),\xi}) \\
&= \int_M P_g(\phi_{\delta_\varepsilon(t),\xi}) \frac{\partial}{\partial t} W_{\delta_\varepsilon(t),\xi} dV \\
&\quad - \int_M (f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) \frac{\partial W_{\delta_\varepsilon(t),\xi}}{\partial t} dV \\
&\quad + DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t} \right] \\
&= \frac{\tilde{C}_n}{t} \Bigg(\int_M (P_g(Z_0) - f_\varepsilon'(W_{\delta_\varepsilon(t),\xi}) Z_0) \phi_{\delta_\varepsilon(t),\xi} dV \\
&\qquad\qquad - \int_M \big(f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \\
&\qquad\qquad\qquad - f_\varepsilon'(W_{\delta_\varepsilon(t),\xi}) \phi_{\delta_\varepsilon(t),\xi} \big) Z_0 dV \Bigg) \\
&\quad + DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t} \right] \\
&= I_1 + I_2 + I_3, \qquad (3.20)
\end{aligned}$$

where

$$I_1 = \frac{\tilde{C}_n}{t} \int_M (P_g(Z_0) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi}) Z_0) \phi_{\delta_\varepsilon(t),\xi} dV, \tag{3.21}$$

$$I_2 = -\frac{\tilde{C}_n}{t} \int_M (f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \phi_{\delta_\varepsilon(t),\xi}) Z_0 dV, \tag{3.22}$$

$$I_3 = DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t} \right]. \tag{3.23}$$

In the same way, recalling that

$$\frac{\partial}{\partial y_i} (W_{\delta_\varepsilon(t),\exp_\xi(y)}) \Big|_{y=0} = \frac{\alpha_n(n-4)}{\delta_\varepsilon(t)} Z_i + R_{\delta_\varepsilon(t),\xi},$$

where $\left\| R_{\delta_\varepsilon(t),\xi} \right\|_{P_g} = O(\delta_\varepsilon(t)^2)$ (see (6.13) of [?]) and using (**??**), we find

$$\begin{aligned}
&\frac{\partial I_\varepsilon}{\partial y_i}(t, \exp_\xi(y)) \Big|_{y=0} - \frac{\partial J_\varepsilon}{\partial y_i}(W_{\delta(t),\exp_\xi(y)}) \Big|_{y=0} \\
&= \frac{\alpha_n(n-4)}{\delta_\varepsilon(t)} \left(\int_M (P_g(Z_i) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi}) Z_i) \phi_{\delta_\varepsilon(t),\xi} dV \right. \\
&\qquad - \int_M \left(f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \right. \\
&\qquad \left. \left. - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \phi_{\delta_\varepsilon(t),\xi} \right) Z_i dV \right) \\
&\quad + DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i} \right] \Big|_{y=0} \\
&\quad + O\left(\left\| R_{\delta_\varepsilon(t),\xi} \right\|_{P_g} \left\| \phi_{\delta_\varepsilon(t),\xi} \right\|_{P_g} \right) \\
&= I_4 + I_5 + I_6 + o(\varepsilon),
\end{aligned} \tag{3.24}$$

where

$$I_4 = \frac{\alpha_n(n-4)}{\delta_\varepsilon(t)} \int_M (P_g(Z_i) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi}) Z_i) \phi_{\delta_\varepsilon(t),\xi} dV,$$

$$I_5 = -\frac{\alpha_n(n-4)}{\delta_\varepsilon(t)} \int_M (f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \phi_{\delta_\varepsilon(t),\xi}) Z_i dV,$$

$$I_6 = DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i} \right] \Big|_{y=0}.$$

We begin by estimating the terms I_3 and I_6. By Proposition **??**, there exist real numbers λ_i, $i = 0, \dots, n$ such that

$$DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[.] = \sum_{i=0}^{n} \lambda_i \left\langle Z_i, . \right\rangle_{P_g}.$$

Arguing in the same way as in Proposition 2.2 of [**?**] (see in particular (4.23) and (4.24)), we have

$$DJ(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t} \right] = O \left(\left\| \phi_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}} \sum_{i=0}^{n} |\lambda_i| \right),$$

and

$$DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i} \right]_{\big| y=0} = O \left(\frac{\left\| \phi_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}} \sum_{i=0}^{n} |\lambda_i|}{\delta_\varepsilon(t)} \right).$$

We claim that $|\lambda_i| = O\big(\left\| P_g(W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \big)$, for all $i = 0, \dots, n$. Using that

$$\left\langle Z_i, Z_j \right\rangle_{P_g} \to \| \Delta_{eucl} V_i \|^2_{L^2(\mathbb{R}^n)} \delta_{ij},$$

to prove the claim, we just need to show that

$$DJ(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[Z_i] = O(\left\| P_g(W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}}),$$

for all $i = 0, \dots, n$. Since $\phi_{\delta_\varepsilon(t),\xi} \in K^\perp_{\delta_\varepsilon(t),\xi}$, using Hölder inequality, (**??**), Lemma **??** and rough estimates, we have

$$\begin{aligned}
&DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[Z_i] \\
&\qquad = \int_M P_g(W_{\delta_\varepsilon(t),\xi}) Z_i dV - \int_M f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) Z_i dV \\
&\qquad = \int_M (P_g(W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) Z_i dV \\
&\qquad\quad - \int_M (f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) Z_i dV \\
&\qquad \leq \left\| P_g(W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \| Z_i \|_{L^{2^*}} \\
&\qquad\quad + \left\| f_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \| Z_i \|_{L^{2^*}} \\
&\qquad \leq O \left(\left\| P_g(W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \right) \\
&\qquad\quad + O \left(\left\| \phi_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n-4}}} \left(\left\| W_{\delta_\varepsilon(t),\xi} \right\|^{2^*-2-\varepsilon}_{L^{\frac{2n}{n-4}}} + \left\| \phi_{\delta_\varepsilon(t),\xi} \right\|^{2^*-2-\varepsilon}_{L^{\frac{2n}{n-4}}} \right) \right) \\
&\qquad \leq O \left(\left\| P_g(W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \right).
\end{aligned}$$

Combining the previous estimates, we get

$$DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi})\left[\frac{\partial\phi_{\delta_\varepsilon(t),\xi}}{\partial t}\right] = O\left(\left\|P_g(W_{\delta_\varepsilon(t),\xi})-f_\varepsilon(W_{\delta_\varepsilon(t),\xi})\right\|^2_{L^{\frac{2n}{n+4}}}\right), \tag{3.25}$$

and

$$DJ_\varepsilon(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi})\left[\frac{\partial\phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i}\right]|_{y=0} = O\left(\frac{\left\|P_g(W_{\delta_\varepsilon(t),\xi})-f_\varepsilon(W_{\delta_\varepsilon(t),\xi})\right\|^2_{L^{\frac{2n}{n+4}}}}{\delta_\varepsilon(t)}\right). \tag{3.26}$$

Now let us estimate I_2 and I_5. Noticing that, if $n\geq 11$,

$$\left\|(W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon}Z_i\right\|_{L^{\frac{n}{4}}}=O(1),$$

we obtain, for $i=0,\ldots,n$,

$$\begin{aligned}
&\int_M (f_\varepsilon(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi})-f_\varepsilon(W_{\delta_\varepsilon(t),\xi})-f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi})Z_i dV\\
&\leq C\begin{cases}\int_M (W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon}\phi^2_{\delta_\varepsilon(t),\xi}Z_i dV, & if\ n\geq 12,\\ \int_M((W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon}\phi^2_{\delta_\varepsilon(t),\xi}+\phi^{2^*-1-\varepsilon}_{\delta_\varepsilon(t),\xi}Z_i dV), & if\ n=11,\end{cases}\\
&\leq C\begin{cases}\left\|(W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon}Z_i\right\|_{L^{\frac{n}{4}}}\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^2_{L^{\frac{2n}{n-4}}}, & if\ n\geq 12,\\ \left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^2_{L^{\frac{2n}{n-4}}}\left\|(W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon}Z_i\right\|_{L^{\frac{n}{4}}} & \\ \quad +\left\|Z_i\right\|_{L^{\frac{2n}{n-4}}}\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^{2^*-1-\varepsilon}_{L^{\frac{2n}{n-4}}}, & if\ n=11,\end{cases}\\
&\leq O(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^2_{L^{\frac{2n}{n-4}}})\ when\ n\geq 11.
\end{aligned} \tag{3.27}$$

Finally, let us estimate I_1 and I_4. Since

$$\left\|P_g(Z_i)-f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i\right\|_{L^{\frac{2n}{n+4}}}=O(\delta_\varepsilon(t)^2)$$

(one can argue as in [?], inequality (4.17)), we obtain

$$\begin{aligned}
&\int_M (P_g(Z_i)-f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i)\phi_{\delta_\varepsilon(t),\xi}dV\\
&\leq C\left\|P_g(Z_i)-f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i\right\|_{L^{\frac{2n}{n+4}}}\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{L^{2^*}}\\
&\leq O\left(\delta_\varepsilon(t)^2\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{L^{2^*}}\right).
\end{aligned} \tag{3.28}$$

The lemma now follows from (??), (??), (??), (??), (??) and (??).

□

4 Case M is l.c.f. : Proof of Theorem ??.

In all this section, we assume that M is l.c.f. We will also assume that G_{g_ξ}, the Green function of P_{g_ξ}, is of the form

$$G_{g_\xi}(\exp_\xi y, \xi) = \frac{1}{\beta_n |y|^{n-4}} + A + 0^{(4)}(|y|),$$

where $\beta_n = (n-2)(n-4)\omega_{n-1}$ (see the introduction for the definition of $0^{(4)}(|y|)$). In the following, with an abuse of notation, we will identify the metric g and g_ξ. In this section, we will modify the notation of the function $W_{\delta_\varepsilon(t),\xi}$ defined in the previous one. Here, we will be looking for a solution of the form

$$W_{\delta_\varepsilon(t),\xi}(x) = G_g(x,\xi)\hat{W}_{\delta_\varepsilon(t),\xi}(x),$$

where $\hat{W}_{\delta_\varepsilon(t),\xi}$ is defined by

$$\hat{W}_{\delta_\varepsilon(t),\xi} = \begin{cases} \hat{W}^{in}_{\delta_\varepsilon(t),\xi}(x) := \beta_n \delta_\varepsilon(t)^{\frac{4-n}{2}} d(x,\xi)^{n-4} U\left(\dfrac{d(x,\xi)}{\delta_\varepsilon(t)}\right), & if\ d(x,\xi) \le r_0, \\ \hat{W}^{out}_{\delta_\varepsilon(t),\xi}(x) := \beta_n \delta_\varepsilon(t)^{\frac{4-n}{2}} r_0^{n-4} U\left(\dfrac{r_0}{\delta_\varepsilon(t)}\right) & \\ \quad + \gamma_{\tilde{\varepsilon}}(d(x,\xi) - r_0)(\hat{W}^{in}_{\delta_\varepsilon(t),\xi})'(r_0), & if\ d(x,\xi) > r_0. \end{cases}$$

In the previous definition, $\gamma_{\tilde{\varepsilon}} : \mathbb{R} \to \mathbb{R}$, $\tilde{\varepsilon} \in \mathbb{R}^+$ is a smooth function satisfying the following properties :

$$supp(\gamma_{\tilde{\varepsilon}}) \subset [0, \tilde{\varepsilon}],$$

$$\gamma(0) = 0,\ \gamma'(0) = 1,\ |\nabla^i \gamma(r)| \le \tilde{\varepsilon}^{-i}, \forall r \ge 0\ and\ i = 1, \dots, 4.$$

It is easy to check that $\hat{W}_{\delta_\varepsilon(t),\xi} \in H^2(M)$. We also define, for any real δ strictly positive, $\xi \in M$ and $x \in M$,

$$Z_{\delta,\xi}(x) = G_g(x,\xi)\hat{Z}_{\delta,\xi}(x), \tag{4.1}$$

and, for $\omega \in T_\xi M$,

$$Z_{\delta,\xi,\omega}(x) = G_g(x,\xi)\hat{Z}_{\delta,\xi,\omega}(x), \tag{4.2}$$

where

$$\hat{Z}_{\delta,\xi}(x) = d(x,\xi)^{n-4}\chi(d_g(x,\xi))\delta^{\frac{n-4}{2}} \frac{d(x,\xi)^2 - \delta^2}{(\delta^2 + d(x,\xi)^2)^{\frac{n-2}{2}}},$$

and, for $\omega \in T_\xi M$,

$$\hat{Z}_{\delta,\xi,\omega}(x) = d(x,\xi)^{n-4}\chi(d_g(x,\xi))\delta^{\frac{n-2}{2}} \frac{\left\langle \exp_\xi^{-1} x, \omega \right\rangle_g}{(\delta^2 + d(x,\xi)^2)^{\frac{n-2}{2}}},$$

χ is a smooth cut-off function such that $0 \leq \chi \leq 1$, $\chi \equiv 0$ in $[r_0, \infty)$ and $\chi \equiv 1$ in $[0, \frac{r_0}{2}]$. In this section, we choose

$$\delta_\varepsilon(t) = \begin{cases} t\varepsilon^{\frac{1}{n-6}} & if\ n \geq 7 \\ e^{-\frac{t}{\varepsilon}} & if\ n = 6 \end{cases}. \tag{4.3}$$

In view of the results of the previous section, it is easy to see that we only need to obtain an estimate of the error and of the reduced energy in order to prove Theorem **??**. We begin with the error estimate.

Lemma 4.1. *Assume that M is l.c.f. and $f_\varepsilon(u) = |u|^{2^*-2}u - \varepsilon div_g(B(\nabla u))$. Given two positive real numbers $a < b$, there exists a positive constant $C'_{a,b}$ such that for ε small, for any real number $t \in [a, b]$ and any point $\xi \in M$, there holds*

$$\begin{aligned} &\left\| i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi} \right\|_{P_g} \\ &\leq C'_{a,b} \begin{cases} \delta_\varepsilon(t)^{n-4} + \varepsilon\delta_\varepsilon(t)^{\frac{n-4}{2}}, & if\ n < 8, \\ \delta_\varepsilon(t)^4 |\ln \delta_\varepsilon(t)|^{\frac{3}{4}} + \varepsilon\delta_\varepsilon(t)^2 |\ln \delta_\varepsilon(t)|^{\frac{3}{4}}, & if\ n = 8, \\ \delta_\varepsilon(t)^{\frac{n}{2}} + \varepsilon\delta_\varepsilon(t)^2 & if\ n > 8. \end{cases} \end{aligned} \tag{4.4}$$

Proof. Integrating by parts, we find

$$\begin{aligned} \left\langle i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi}, \phi \right\rangle &= \int_M (f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - P_g W_{\delta_\varepsilon(t),\xi})\phi dV \\ &- \int_{\partial B_\xi(r_0)} (\Delta_g W^{out}_{\delta_\varepsilon(t),\xi} \partial_{in}\phi + \Delta_g W^{in}_{\delta_\varepsilon(t),\xi} \partial_{out}\phi) d\sigma \\ &+ \int_{\partial B_\xi(r_0)} \big(\partial_{in}(\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}) + \partial_{out}(\Delta_g W^{in}_{\delta_\varepsilon(t),\xi})\big)\phi d\sigma \end{aligned}$$

where $\partial B_\xi(r_0)$ is the boundary of the geodesic ball with respect to g_ξ of radius r_0 centered in ξ and $\partial_{\nu_{in}}$ (resp. $\partial_{\nu_{out}}$) stands for the derivatives with respect to the inward (resp. outward), unit normal vectors of $\partial B_\xi(r_0)$ and $d\sigma_g$ is the volume element of $\partial B_\xi(r_0)$. Now, straight forward computations give, on $\partial B_\xi(r_0)$,

$$\begin{aligned} (\partial_{in}\Delta_g W^{out}_{\delta_\varepsilon(t),\xi} + \partial_{out}\Delta_g W^{in}_{\delta_\varepsilon(t),\xi}) &= G'_g(\Delta_g \hat{W}^{in}_{\delta_\varepsilon(t),\xi} - \Delta_g \hat{W}^{out}_{\delta_\varepsilon(t),\xi}) \\ &+ G_g\big((\Delta_g \hat{W}^{in}_{\delta_\varepsilon(t),\xi})' - (\Delta_g \hat{W}^{out}_{\delta_\varepsilon(t),\xi})'\big) \\ &+ 2G'_g\big((\nabla \hat{W}^{in}_{\delta_\varepsilon(t),\xi})' - (\nabla \hat{W}^{out}_{\delta_\varepsilon(t),\xi})'\big) \\ &= O(\delta_\varepsilon(t)^{\frac{n}{2}}), \end{aligned} \tag{4.5}$$

and

$$(\Delta W^{in}_{\delta_\varepsilon(t),\xi} - \Delta W^{out}_{\delta_\varepsilon(t),\xi}) = G_g \Delta_g (\hat{W}^{in}_{\delta_\varepsilon(t),\xi} - \hat{W}^{out}_{\delta_\varepsilon(t),\xi}) = O(\delta_\varepsilon(t)^{\frac{n}{2}}). \tag{4.6}$$

We observe, since $\hat{W}_{\delta_\varepsilon(t),\xi} = O(\delta_\varepsilon(t)^{\frac{n-4}{2}})$ on $M \backslash B_\xi(r_0)$ and $P_g G_g(.,\xi) = 0$, that

$$\begin{aligned} f_\varepsilon(W_{\delta_\varepsilon(t),\xi} - G_g(.,\xi)\Gamma) - P_g(W_{\delta_\varepsilon(t),\xi} - G_g(.,\xi)\Gamma) \\ = O(\delta_\varepsilon(t)^{\frac{n+4}{2}} + \varepsilon\delta_\varepsilon(t)^{\frac{n-4}{2}}),\ \ on\ M\backslash B_\xi(r_0), \end{aligned} \tag{4.7}$$

where $\Gamma(x) = \gamma_{\tilde{\varepsilon}}(d(x,\xi) - r_0)(\hat{W}^{in}_{\delta_\varepsilon(t),\xi})'(r_0)$. Next, using the properties of $\gamma_{\tilde{\varepsilon}}$ and the fact that $n \geq 5$, it is easy to see that

$$\int_{M\backslash B_\xi(r_0)} P_g(G_g(.,\xi)\Gamma)\phi dV = \int_{B_\xi(r_0+\tilde{\varepsilon})\backslash B_\xi(r_0)} P_g(G_g(.,\xi)\Gamma)\phi dV = O(\tilde{\varepsilon}), \tag{4.8}$$

and

$$\int_{M\backslash B_\xi(r_0)} f_\varepsilon(G_g(.,\xi)\Gamma)\phi dV = O(\tilde{\varepsilon}).$$

Therefore, by Sobolev's and trace's embeddings, and choosing $\tilde{\varepsilon} \leq \delta_\varepsilon(t)^{\frac{n}{2}}$, we deduce that

$$\begin{aligned} &\left\| i^*(f_\varepsilon(W_{\delta_\varepsilon(t),\xi})) - W_{\delta_\varepsilon(t),\xi} \right\|_{H^2(M)} \\ &\qquad \leq \left\| f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - P_g W_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}(B_\xi(r_0))} + O(\delta_\varepsilon(t)^{\frac{n}{2}}). \end{aligned}$$

Since M is l.c.f., using the conformal invariance of P_g and doing some computations, we find

$$\begin{aligned} P_g(W_{\delta_\varepsilon(t),\xi}) = {} & \beta_n G_g r^{n-4} \delta_\varepsilon(t)^{-\frac{n+4}{2}} U(\delta^{-1}y)^{2^*-1} \\ & + \frac{4A_\xi \delta_\varepsilon(t)^2 \beta_n \alpha_n \delta_\varepsilon(t)^{\frac{n-4}{2}} (n-4)^2 |y|^{n-8}}{(\delta_\varepsilon(t)^2 + |y|^2)^{\frac{n+2}{2}}} \\ & \times \left(\delta_\varepsilon(t)^4(18 - 9n + n^2) - 8\delta_\varepsilon(t)^2(n-3)|y|^2 - (n-6)|y|^4\right) \\ & + h.o.t. \end{aligned} \tag{4.9}$$

Here and in the following, $h.o.t.$ stands for a term which is asymptotically smaller than one of the previous terms in the expansion as ε goes to 0. Therefore, we deduce that

$$\begin{aligned} &\left\| f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - P_g W_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}(B_\xi(r_0))} \\ &= O\Bigg(\left\| \frac{\delta_\varepsilon(t)^{\frac{n}{2}} |y|^{n-8}}{(\delta_\varepsilon(t)^2 + |y|^2)^{\frac{n+2}{2}}} \left(\delta_\varepsilon(t)^4 + \delta_\varepsilon(t)^2|y|^2 + |y|^4\right) \right\|_{L^{\frac{2n}{n+4}}(B_\xi(r_0))} \\ &\qquad\qquad + \varepsilon \left\| div_g\left(B(\nabla W_{\delta_\varepsilon(t),\xi})\right) \right\|_{L^{\frac{2n}{n+4}}(B_\xi(r_0))} . \end{aligned}$$

Now, rough estimates give

$$\left\| \frac{\delta_\varepsilon(t)^{\frac{n}{2}}|y|^{n-8}}{(\delta_\varepsilon(t)^2+|y|^2)^{\frac{n+2}{2}}} \left(\delta_\varepsilon(t)^4+\delta_\varepsilon(t)^2|y|^2+|y|^4\right) \right\|_{L^{\frac{2n}{n+4}}(B_\xi(r_0))}$$

$$\leq C \begin{cases} \delta_\varepsilon(t)^{n-4}, & if\ n<8, \\ \delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)|^{\frac{3}{4}}, & if\ n=8, \\ \delta_\varepsilon(t)^{\frac{n}{2}}, & if\ n>8. \end{cases}$$

and

$$\varepsilon \left\| div_g(B(\nabla W_{\delta_\varepsilon(t),\xi})) \right\|_{L^{\frac{2n}{n+4}}(B_\xi(r_0))} \leq C \begin{cases} \varepsilon\delta_\varepsilon(t)^{\frac{n-4}{2}}, & if\ n<8, \\ \varepsilon\delta_\varepsilon(t)^2|\ln\delta_\varepsilon(t)|^{\frac{n+4}{2n}}, & if\ n=8, \\ \varepsilon\delta_\varepsilon(t)^2, & if\ n>8. \end{cases}$$

Combining all the previous estimates, we conclude that

$$\left\| f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - P_g W_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}(B_\xi(r_0))}$$

$$= \begin{cases} \delta_\varepsilon(t)^{n-4}+\varepsilon\delta_\varepsilon(t)^{\frac{n-4}{2}}, & if n<8, \\ \delta_\varepsilon(t)^4|\ln\delta_\varepsilon(t)|^{\frac{3}{4}}+\varepsilon\delta_\varepsilon(t)^2|\ln\delta_\varepsilon(t)|^{\frac{3}{4}}, & if\ n=8, \\ \delta_\varepsilon(t)^{\frac{n}{2}}+\varepsilon\delta_\varepsilon(t)^2, & if n>8. \end{cases}$$

□

Finally, we give an estimate of the reduced energy $\tilde{I}_\varepsilon(t,\xi)$.

Proposition 4.2. *Assume that M is l.c.f. and $f_\varepsilon(u) = |u|^{2^*-2}u - \varepsilon div_g(B(\nabla u))$. There exist constants $c_i(n)$, $i = 1,2,3$ depending on n, such that*

$$\tilde{I}_\varepsilon(t,\xi)$$
$$= c_1(n) + \begin{cases} e^{-\frac{2t}{\varepsilon}}(c_2(n)tTr_gB(\xi)-c_3A_\xi)+o(e^{-\frac{2t}{\varepsilon}}), & if\ n=6, \\ \varepsilon^{\frac{n-4}{n-6}}(c_2(n)t^2Tr_gB(\xi)-c_3(n)t^{n-4}A_\xi)+o(\varepsilon^{\frac{n-4}{n-6}}), & if\ n\geq 7 \end{cases} \tag{4.10}$$

as $\varepsilon \to 0$ C^0 uniformly with respect to t in compact subsets of $\mathbb{R}_+^*$ and with respect to $\xi \in M$ and C^1 uniformly if $n \geq 7$. Moreover, we have that $c_2(n) > 0$ and $c_3(n) > 0$.

Proof. As previously, using Proposition **??**, Lemma **??** and the definition of $\delta_\varepsilon(t)$ (given in (**??**)), it is easy to see that

$$\tilde{I}_\varepsilon(t,\xi) = \tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi}) + o(\delta_\varepsilon(t)^{n-4}).$$

Using (??) and (??), we have

$$\int_M (\Delta_g W_{\delta_\varepsilon(t),\xi})^2 dV + \int_M A_g(\nabla W_{\delta_\varepsilon(t),\xi}, \nabla W_{\delta_\varepsilon(t),\xi}) dV + \int_M h W^2_{\delta_\varepsilon(t),\xi} dV$$
$$= \int_{B_\xi(r_0)} P_g(W_{\delta_\varepsilon(t),\xi}) W_{\delta_\varepsilon(t),\xi} dV + O(\delta_\varepsilon(t)^{n-2}).$$

Here we also used (??), (??) and $W_{\delta_\varepsilon(t),\xi} = O(\delta_\varepsilon(t)^{\frac{n-4}{2}})$ on $\partial B_\xi(r_0)$. Now using (??), we find

$$\begin{aligned} P_g(W_{\delta_\varepsilon(t),\xi}) W_{\delta_\varepsilon(t),\xi} &= \alpha_n^{2^*} \frac{\delta_\varepsilon(t)^n}{(\delta_\varepsilon(t)^2 + r^2)^n} + 2A_\xi \beta_n \alpha_n^{2^*} r^{n-4} \frac{\delta_\varepsilon(t)^n}{(\delta_\varepsilon(t)^2 + r^2)^n} \\ &\quad + 4(n-4)^2 A_\xi \beta_n \alpha_n^2 r^{n-8} \frac{\delta_\varepsilon(t)^{n-2}}{(\delta_\varepsilon(t)^2 + r^2)^{n-1}} \\ &\qquad \times ((n^2 - 9n - 18)\delta^4 - 8(n-3)\delta_\varepsilon(t)^2 r^2 - (n-6) r^4) \\ &\quad + \ h.o.t. \end{aligned}$$

Integrating the previous formula, we find

$$\begin{aligned} &\int_{B_\xi(r_0)} P_g(W_{\delta_\varepsilon(t),\xi}) W_{\delta_\varepsilon(t),\xi} dV \\ &= \frac{\alpha^{2^*} \omega_{n-1}}{2} I_n^{\frac{n-2}{2}} + A_\xi \beta_n \alpha_n^{2^*} \omega_{n-1} \delta_\varepsilon(t)^{n-4} I_n^{n-3} \\ &\quad + \Big(\left(n^2 - 9n - 18\right) I_{n-1}^{n-5} - 8(n-3) I_{n-1}^{n-4} - (n-6) I_{n-1}^{n-3} \Big) \\ &\qquad \times 4(n-4)^2 A_\xi \beta_n \alpha_n^2 \frac{\delta_\varepsilon(t)^{n-4} \omega_{n-1}}{2} + o(\delta_\varepsilon(t)^{n-4}) \\ &= \alpha^{2^*} \frac{\omega_n}{2^n} + \frac{A_\xi \beta_n \alpha_n^{2^*} \omega_{n-1}}{(n-1)(n-2)} \delta^{n-4} \\ &\quad + \left(\frac{2(n^2 - 9n - 18)}{(n-2)(n-3)(n-4)} - \frac{n+2}{n-2} \right) \\ &\qquad \times 4(n-4)^2 A_\xi \beta_n \alpha_n^2 \frac{\delta_\varepsilon(t)^{n-4} \omega_{n-1}}{2} + o(\delta_\varepsilon(t)^{n-4}). \end{aligned}$$

We also have

$$\begin{aligned} &\int_{B_\xi(r_0)} W^{2^*}_{\delta_\varepsilon(t),\xi} dV \\ &= \alpha_n^{2^*} \omega_{n-1} \int_0^{r_0} (1 + \frac{2n}{n-4} A_\xi \beta_n r^{n-4}) \frac{\delta_\varepsilon(t)^n r^{n-1}}{(\delta_\varepsilon(t)^2 + r^2)^n} dr + o(\delta_\varepsilon(t)^{n-4}) \\ &= \frac{1}{2} \alpha_n^{2^*} \omega_{n-1} I_n^{\frac{n-2}{2}} + \frac{1}{2} \frac{2n}{n-4} A_\xi \alpha_n^{2^*} \omega_{n-1} \beta_n \delta_\varepsilon(t)^{n-4} I_n^{n-3} + o(\delta_\varepsilon(t)^{n-4}) \end{aligned}$$

and

$$\varepsilon \int_{B_\xi(r_0)} B(\nabla W_{\delta_\varepsilon(t),\xi}, \nabla W_{\delta_\varepsilon(t),\xi}) dV$$

$$= \frac{(n-4)^2}{2n} \varepsilon \delta_\varepsilon(t)^2 \alpha_n^2 \omega_{n-1} Tr_g\ (B) \int_0^{\frac{r_0}{\delta_\varepsilon(t)}} \frac{r^{n+1}}{(1+r^2)^{n-2}} dr + o(\delta_\varepsilon(t)^{n-4})$$

$$= \frac{1}{2}\frac{(n-4)^2}{2n} \varepsilon \delta_\varepsilon(t)^2 \alpha_n^2 \omega_{n-1} Tr_g\ (B) \begin{cases} \dfrac{4n(n-1)}{(n-4)(n-6)}, & if\ n > 6, \\ \dfrac{1}{2}\ln(1+\dfrac{1}{\delta_\varepsilon(t)^2}), & if\ n = 6 \end{cases}$$

$$+ o(\delta_\varepsilon(t)^{n-4}).$$

Therefore, combining the three previous estimates, we finally obtain

$$\begin{aligned}
\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi}) =& \frac{4}{n}\alpha^{2^*}\omega_n + \frac{A_\xi \beta_n \alpha_n^{2^*}\omega_{n-1}}{4(n-1)(n-2)} \delta_\varepsilon(t)^{n-4} \\
&+ 2(n-4)^2 A_\xi \beta_n \alpha_n^2 \frac{\delta_\varepsilon(t)^{n-4}\omega_{n-1}}{2} \\
&\qquad \times \left(\frac{2(n^2-9n-18)}{(n-2)(n-3)(n-4)} - \frac{(n+2)}{(n-2)} \right) \\
&+ \frac{(n-4)^2}{8n} \varepsilon \delta_\varepsilon(t)^2 \alpha_n^2 \omega_{n-1} Tr_g\ (B) \begin{cases} \dfrac{4n(n-1)}{(n-4)(n-6)}, & if\ n > 6, \\ \ln|\delta_\varepsilon(t)|, & if\ \ = 6 \end{cases} \\
&+ o(\delta_\varepsilon(t)^{n-4}) \\
=& C_1 - C_2 A_\xi \delta_\varepsilon(t)^{n-4} + C_3 Tr_g B \varepsilon \delta_\varepsilon(t)^2 + o(\delta_\varepsilon(t)^{n-4}),
\end{aligned}$$

where $C_3 > 0$ and $C_2 > 0$. □

The rest of the proof is devoted to prove that (**??**) holds C^1-uniformly if $n \geq 7$. This will be an immediate consequence of the following three lemmata. To simplify notation, in the following, we denote $Z_0 = Z_{\delta,\xi}$ and $Z_i = Z_{\delta,\xi,e_i}$, where $(e_i)_i$ is a base of $T_\xi M$ (see (**??**) and (**??**) for the definitions of $Z_{\delta,\xi}$ and Z_{δ,ξ,e_i}).

Lemma 4.3. *For any* $i = 1, \ldots, n$, *we have*

$$\frac{d}{d\eta_i}\tilde{I}_\varepsilon(t, \exp_\xi \eta)|_{\eta=0} = \frac{C_n \lambda_i}{\delta_\varepsilon(t)} \|\Delta V_i\|_{L^2}^2 + o\left(\sum_{j=0}^n \lambda_j\right), \tag{4.11}$$

and

$$\frac{d}{dt}\tilde{I}_\varepsilon(t,\xi) = \frac{C_n \lambda_0 \delta_\varepsilon(t)'}{4\delta_\varepsilon(t)} \|\Delta V_0\|_{L^2}^2 + o\left(\sum_{j=0}^n \lambda_j\right). \tag{4.12}$$

*where $C_n = \alpha_n(n-4)$ (see (**??**) for the definition of α_n and see below for the definition of the λ_i's).*

Proof. We only prove (**??**) (the proof of (**??**) following along the same line). For any $i = 1, \dots, n$, using Proposition **??**, we have that there exist λ_i, $i = 0, \dots, n$, such that

$$\frac{d}{d\eta_i} \tilde{I}_\varepsilon(t, \exp_\xi \eta)|_{\eta=0} = \sum_{j=0}^{n} \lambda_j \left\langle Z_j, \frac{d}{d\eta_i}(W_{\delta_\varepsilon(t),\exp_\xi \eta} + \phi_{\delta_\varepsilon(t),\exp_\xi \eta})|_{\eta=0} \right\rangle .$$

To simplify notation, we denote $\frac{d}{d\eta_i}(W_{\delta_\varepsilon(t),\exp_\xi \eta})|_{\eta=0} = \frac{d}{d\eta_i} W$ (and we adopt the same convention for all functions). Integrating by parts, we have, for all $i = 1, \dots, n$ and $j = 0, \dots n$,

$$\left\langle Z_j, \frac{d}{d\eta_i} W \right\rangle = \int_M P_g Z_j \Big(\frac{d}{d\eta_i} W - \frac{C_n}{\delta_\varepsilon(t)} Z_i \Big) dV + \frac{C_n}{\delta_\varepsilon(t)} \langle Z_i, Z_j \rangle .$$

Using Hölder's inequality and since $\left\| \frac{d}{d\eta_i} W - \frac{C_n}{\delta_\varepsilon(t)} Z_i \right\|_{L^{2^*}} = o(1)$, we get

$$\left| \int_M P_g Z_j \Big(\frac{d}{d\eta_i} W - \frac{C_n}{\delta_\varepsilon(t)} Z_i \Big) dV \right| \le \| P_g Z_j \|_{L^{\frac{2n}{n+4}}} \left\| \frac{d}{d\eta_i} W - \frac{C_n}{\delta_\varepsilon(t)} Z_i \right\|_{L^{2^*}} = o(1).$$

Therefore, we deduce from the two previous lines that

$$\left\langle Z_j, \frac{d}{d\eta_i} W \right\rangle = \frac{C_n}{\delta_\varepsilon(t)} \| \Delta V_i \|_{L^2}^2 \delta_{ij} + o(1). \tag{4.13}$$

On the other hand, using that $\phi_{\delta_\varepsilon(t),\xi} \in K^\perp_{\delta_\varepsilon(t),\xi}$, Hölder inequality and Lemma **??**, we deduce that

$$\left| \left\langle Z_j, \frac{d}{d\eta_i} \phi \right\rangle \right| \le \left\| \frac{d}{d\eta_i} Z_j \right\|_{P_g} \| \phi_{\delta_\varepsilon(t),\xi} \|_{P_g} = O\left(\frac{\| \phi_{\delta_\varepsilon(t),\xi} \|_{P_g}}{\delta_\varepsilon(t)} \right) = o(1). \tag{4.14}$$

Combining (**??**) and (**??**), we get (**??**). □

Lemma 4.4. *We have*

$$\frac{d}{dt} \tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi}) = C_n \frac{\delta_\varepsilon(t)'}{4\delta_\varepsilon(t)} D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi}) Z_0 + \begin{cases} o(\delta_\varepsilon(t)^2 \ln \delta_\varepsilon(t)), & if\ n = 6, \\ O(\delta_\varepsilon(t)^{n-2} + \varepsilon \delta_\varepsilon(t)^{n-4}), & if\ n > 6, \end{cases} \tag{4.15}$$

and

$$\frac{d}{d\eta_i}\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\exp_\xi\eta})|_{\eta=0} = \frac{C_n}{\delta_\varepsilon(t)}D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i + \begin{cases} o(\delta_\varepsilon(t)^2), & if\ n=6, \\ O(\delta_\varepsilon(t)^{n-3}+\varepsilon\delta_\varepsilon(t)^3\ln\delta_\varepsilon(t)), & if\ n>6. \end{cases} \tag{4.16}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - C_n\frac{\delta_\varepsilon(t)'}{4\delta_\varepsilon(t)}D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_0 \\
&= \int_M \Big(P_gW_{\delta_\varepsilon(t),\xi} - f_\varepsilon(W_{\delta_\varepsilon(t),\xi})\Big)\Big(\frac{d}{dt}W_{\delta_\varepsilon(t),\xi} - C_n\frac{\delta_\varepsilon(t)'}{4\delta_\varepsilon(t)}Z_0\Big)dV \\
&- \int_{\partial B_\xi(r_0)}(\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}\partial_{in}\frac{d}{dt}W_{\delta_\varepsilon(t),\xi} + \Delta_g W^{in}_{\delta_\varepsilon(t),\xi}\partial_{out}\frac{d}{dt}W_{\delta_\varepsilon(t),\xi})d\sigma \\
&+ \int_{\partial B_\xi(r_0)}\big(\partial_{in}(\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}) + \partial_{out}(\Delta_g W^{in}_{\delta_\varepsilon(t),\xi})\big)\frac{d}{dt}W_{\delta_\varepsilon(t),\xi}d\sigma.
\end{aligned}$$

Since $\frac{d}{dt}W_{\delta_\varepsilon(t),\xi} - C_n\frac{\delta_\varepsilon(t)'}{4\delta_\varepsilon(t)}Z_0 = O(\delta_\varepsilon(t)^{\frac{n-6}{2}}\delta_\varepsilon(t)'1_{M\setminus B_\xi(\frac{r_0}{2})})$, using estimates obtained in Lemma **??**, we find that

$$\begin{aligned}
&\int_M (P_gW_{\delta_\varepsilon(t),\xi} - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}))(\frac{d}{dt}W_{\delta_\varepsilon(t),\xi} - C_n\frac{\delta_\varepsilon(t)'}{2\delta_\varepsilon(t)}Z_0)dV \\
&= O\left(\delta_\varepsilon(t)'(\delta_\varepsilon(t)^{n-3}+\varepsilon\delta_\varepsilon(t)^{n-5})\right),\ if\ n\geq 6.
\end{aligned}$$

We also have, on $\partial B_\xi(r_0)$,

$$\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}\partial_{in}\frac{d}{dt}W_{\delta_\varepsilon(t),\xi} + \Delta_g W^{in}_{\delta_\varepsilon(t),\xi}\partial_{out}\frac{d}{dt}W_{\delta_\varepsilon(t),\xi} = O(\delta_\varepsilon(t)^{n-3}\delta_\varepsilon(t)'),$$

and

$$\big(\partial_{in}(\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}) + \partial_{out}(\Delta_g W^{in}_{\delta_\varepsilon(t),\xi})\big)\frac{d}{dt}W_{\delta_\varepsilon(t),\xi} = O(\delta_\varepsilon(t)^{n-3}\delta_\varepsilon(t)').$$

This proves (**??**). Now, let us show (**??**). Integrating by parts, we have

$$\begin{aligned}
&\frac{d}{d\eta_i}\tilde{J}_\varepsilon(W) - C_n\frac{1}{\delta_\varepsilon(t)}D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i \\
&\qquad = \int_M (P_g W_{\delta_\varepsilon(t),\xi} - f_\varepsilon(W_{\delta_\varepsilon(t),\xi}))(\frac{d}{d\eta_i}W - C_n\frac{1}{\delta_\varepsilon(t)}Z_i)dV \\
&\qquad\quad - \int_{\partial B_\xi(r_0)} (\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}\partial_{in}\frac{d}{d\eta_i}W + \Delta_g W^{in}_{\delta_\varepsilon(t),\xi}\partial_{out}\frac{d}{d\eta_i}W)d\sigma \\
&\qquad\quad\quad + \int_{\partial B_\xi(r_0)} \big(\partial_{in}(\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}) + \partial_{out}(\Delta_g W^{in}_{\delta_\varepsilon(t),\xi})\big)\frac{d}{d\eta_i}W d\sigma.
\end{aligned}$$

Proceeding exactly as in Lemma 6.1 of [**?**], we have for any $y \in B_0(r_0)$,

$$\frac{d}{d\eta_i}W_{\delta_\varepsilon(t),\exp_\xi \eta}(\exp_\xi y)|_{\eta=0} = \frac{C_n}{\delta_\varepsilon(t)}Z_i(\exp_\xi y) + O\left(\frac{|y|\delta_\varepsilon(t)^{\frac{n-4}{2}}}{(\delta_\varepsilon(t)^2+|y|^2)^{\frac{n-4}{2}}}\right). \tag{4.17}$$

For $y \in M\backslash\overline{B}_\xi(r_0)$ we have

$$\frac{d}{d\eta_i}W_{\delta_\varepsilon(t),\exp_\xi \eta}(\exp_\xi y)|_{\eta=0} = O\Big(\delta_\varepsilon(t)^{\frac{n-4}{2}}\Big), \tag{4.18}$$

here we assumed to simplify computations that the function $\gamma_{\tilde{\varepsilon}} \equiv 0$, it is easy to check that this assumption is harmless since $n \geq 6$. Using (**??**), one can show that

$$\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}\partial_{in}\frac{d}{d\eta_i}W + \Delta_g W^{in}_{\delta_\varepsilon(t),\xi}\partial_{out}\frac{d}{d\eta_i}W = O\left(\delta_\varepsilon(t)^{n-2}\right),$$

and

$$\left(\partial_{in}(\Delta_g W^{out}_{\delta_\varepsilon(t),\xi}) + \partial_{out}(\Delta_g W^{in}_{\delta_\varepsilon(t),\xi})\right)\frac{d}{d\eta_i}W_{\delta_\varepsilon(t),\xi} = O\left(\delta_\varepsilon(t)^{n-2}\right).$$

Next, using (**??**) and (**??**), we obtain

$$\begin{aligned}
&\int_{B_\xi(r_0)} \left(P_g W_{\delta_\varepsilon(t),\xi} - f_0(W_{\delta_\varepsilon(t),\xi})\right)\left(\frac{d}{d\eta_i}W - C_n\frac{1}{\delta_\varepsilon(t)}Z_i\right)dV \\
&= O\left(\delta_\varepsilon(t)^{n-2}\int_0^{r_0}\frac{r^{2n-8}}{(\delta_\varepsilon(t)^2+r^2)^{n-1}}\left(\delta_\varepsilon(t)^4 + \delta_\varepsilon(t)^2 r^2 + r^4\right)dr\right) \\
&= O(\delta_\varepsilon(t)^{n-3}),
\end{aligned}$$

and

$$\int_{B_\xi(r_0)} B(\nabla W_{\delta_\varepsilon(t),\xi}, \nabla\left(\frac{d}{d\eta_i}W - C_n\frac{1}{\delta_\varepsilon(t)}Z_i\right)dV$$
$$= O\left(\left\{\begin{array}{ll} \delta_\varepsilon(t)^2, & if\ n=6, \\ \delta_\varepsilon(t)^3\ln\delta_\varepsilon(t), & if\ n=7, \\ \delta_\varepsilon(t)^3, & if\ n>7 \end{array}\right.\right)$$

Finally, using (**??**), we get

$$\int_{M\setminus B_\xi(r_0)} \left(P_g W_{\delta_\varepsilon(t),\xi} - f_\varepsilon(W_{\delta_\varepsilon(t),\xi})\right)\left(\frac{d}{d\eta_i}W - C_n\frac{1}{\delta_\varepsilon(t)}Z_i\right)dV$$
$$= O\left(\delta_\varepsilon(t)^{n-2} + \varepsilon\delta_\varepsilon(t)^{n-4}\right).$$

Combining the previous estimates, we obtain (**??**). □

Lemma 4.5. *For any $i = 0,\dots,n$, we have*

$$\lambda_i = \frac{D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi}).Z_i}{\|\Delta V_i\|_{L^2}^2} + \left\{\begin{array}{ll} o(\varepsilon\delta_\varepsilon(t)^2) & if\ i=0\ and\ n\geq 7 \\ o(\varepsilon\delta_\varepsilon(t)^3) & if\ i\neq 0\ and\ n\geq 7 \end{array}\right. . \tag{4.19}$$

In particular, we have, for all $i = 0,\dots,n$ and $n \geq 7$,

$$\lambda_i = O(\varepsilon\delta_\varepsilon(t)^2). \tag{4.20}$$

Proof. For any $i = 0,\dots,n$, we have

$$D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})Z_i = \sum_{j=0}^n \lambda_j \langle Z_i, Z_j\rangle = \lambda_i \|\nabla V_i\|_2^2 + o(\sum_{j=0}^n \lambda_i). \tag{4.21}$$

Independently, we obtain that

$$\begin{aligned} D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})Z_i &= D\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i \\ &+ \left\langle Z_i - i^*(f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i), \phi_{\delta_\varepsilon(t),\xi}\right\rangle \\ &- \int_M (f_0(W_{\delta_\varepsilon(t),\xi} + \phi) - f_0(W_{\delta_\varepsilon(t),\xi}) - f'_0(W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi})Z_i dV. \end{aligned} \tag{4.22}$$

Using Hölder inequality, we get

$$\left\langle Z_i - i^*(f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i), \phi_{\delta_\varepsilon(t),\xi}\right\rangle \leq \left\|Z_i - i^*(f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i)\right\|_{P_g}\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g},$$

and, as in (**??**),

$$\int_M (f_0(W_{\delta_\varepsilon(t),\xi}+\phi_{\delta_\varepsilon(t),\xi}) - f_0(W_{\delta_\varepsilon(t),\xi}) - f_0'(W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi})Z_i dV \leq O\left(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^2_{L^{\frac{2n}{n-4}}}\right) \ when\ 6\leq n. \tag{4.23}$$

To conclude, we only have to estimate $\left\|Z_i - i^*(f_\varepsilon'(W_{\delta_\varepsilon(t),\xi})Z_i)\right\|_{P_g}$. First, by Sobolev's embedding, we have

$$\left\|Z_i - i^*(f_\varepsilon'(W_{\delta_\varepsilon(t),\xi})Z_i)\right\|_{P_g} \leq C\left\|P_gZ_i - f_\varepsilon'(W_{\delta_\varepsilon(t),\xi})Z_i\right\|_{L^{\frac{2n}{n+4}}}.$$

Straight forward computations using that M is l.c.f. and P_g is conformally invariant give

$$P_gZ_0 - f_0'(W_{\delta_\varepsilon(t),\xi})Z_0 = O\Bigg(\delta_\varepsilon(t)^{\frac{n-4}{2}} + \frac{|y|^{n-8}\delta_\varepsilon(t)^{\frac{n-4}{2}}}{(|y|^2+\delta_\varepsilon(t)^2)^{\frac{n+4}{2}}}(\delta_\varepsilon(t)^8 + \delta_\varepsilon(t)^6|y|^2 + \delta_\varepsilon(t)^4|y|^4 + \delta_\varepsilon(t)^2|y|^6 + |y|^8)\Bigg),$$

and, for $i=1,\dots,n$,

$$P_gZ_i - f_0'(W_{\delta_\varepsilon(t),\xi})Z_i = O\Bigg(\delta_\varepsilon(t)^{\frac{n-2}{2}} + \frac{|y|^{n-8}\delta_\varepsilon(t)^{\frac{n-2}{2}}}{(|y|^2+\delta_\varepsilon(t)^2)^{\frac{n+4}{2}}}(\delta_\varepsilon(t)^6 + \delta_\varepsilon(t)^4|y|^2 + \delta_\varepsilon(t)^2|y|^4 + |y|^6)\Bigg).$$

Using the two previous estimates and that

$$\varepsilon\left\|div(B(\nabla Z_i))\right\|_{L^{\frac{2n}{n+4}}} = \begin{cases} \varepsilon\delta_\varepsilon(t)^2(\ln\delta_\varepsilon(t))^{\frac{5}{6}} & if\ i\neq 0,\ n=6,\\ \varepsilon\delta_\varepsilon(t)^2, & if\ i\neq 0,\ n>6,\\ \varepsilon\delta_\varepsilon(t)^2, & if\ i=0,\ n>8,\\ \varepsilon\delta_\varepsilon(t)^2(\ln\delta_\varepsilon(t))^{\frac{3}{4}}, & if\ i=0,\ n=8,\\ \varepsilon\delta_\varepsilon(t)^{\frac{n-4}{2}}, & if\ i=0,\ n<8 \end{cases}$$

one deduces that

$$\left\|Z_i - i^*(f_\varepsilon'(W_{\delta_\varepsilon(t),\xi})Z_i)\right\|_{P_g} \leq \begin{cases} \delta_\varepsilon(t)^{\frac{5}{2}} + \varepsilon\delta_\varepsilon(t)^{\frac{5}{2}} & if\ i=0,\ n=7,\\ \delta_\varepsilon(t)^2 + \varepsilon\delta_\varepsilon(t)^2(\ln\delta_\varepsilon(t))^{\frac{3}{4}} & if\ i=0,\ n=8,\\ \delta_\varepsilon(t)^{\frac{n-4}{2}} + \varepsilon\delta_\varepsilon(t)^2 & if\ i=0,\ n>8,\\ \delta_\varepsilon(t)^{\frac{n-2}{2}} + \varepsilon\delta_\varepsilon(t)^2, & if\ i\neq 0,\ n\geq 7. \end{cases} \tag{4.24}$$

Combining (**??**), (**??**), (**??**), (**??**) and using Lemma **??**, we obtain (**??**). Finally (**??**) follows from the fact that the estimates we obtain for $\tilde{J}_\varepsilon(W_{\delta_\varepsilon(t),\xi})$ in Lemma **??**, are C^1-uniform with respect to t in compact sets of $\mathbb{R}^*_+$ and $\xi \in M$. $\square$