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Singularity resolution from polymer quantum matter

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We study the polymeric nature of quantum matter fields using the example of a Friedmann-Lemaître-Robertson-Walker universe sourced by a minimally coupled massless scalar field. The model is treated in the symmetry reduced regime via deparametrization techniques, with the scale factor playing the role of time. Subsequently the remaining dynamic degrees of freedom are polymer quantized. The analysis of the resulting dynamic shows that the big bang singularity is resolved, although with the form of the resolution differing significantly from that of the models with matter clocks: dynamically, the singularity is made passable rather than avoided. Furthermore, the results of the genuine quantum analysis expose crucial limitations to the so-called effective dynamics in loop quantum cosmology when applied outside of the simplest isotropic settings.

I. INTRODUCTION

Einstein’s theory of general relativity (GR) successfully describes gravitational phenomena, predicting with high precision all large scale observations made to date. It is however expected to fail in the ultraviolet regime due to the quantum nature of the reality at the Planck energy scales. To obtain accurate predictions for such situations one has to resort to quantum gravity (QG).

Despite many attempts [1–6], no general, complete and working (quantitatively) formulation of QG exists. In particular, in the context of the canonical quantization programs, so far it was possible to complete the quantization program only in certain situations, where gravity is coupled to specific matter fields (for irrotational dust see [7, 8] and also the earlier, well defined implicit constructions in [9, 10]). In order to generalize these frameworks (or complete the alternative approaches) it is crucial to first study in detail the simplified mini- and midisuperspace settings.

In this paper we consider the simplest minisuperspace model, which represents a Friedmann-Lemaître-Robertson-Walker (FLRW) universe – an isotropic, flat spacetime admitting a massless scalar field as source. This model is widely used as a testing ground for QG methods and is at the same time of particular interest in cosmology.

In the context of QG, the model has been studied in detail using tools of loop quantum cosmology (LQC – see the references in the second paragraph and [11–17]). In LQC, an application of the polymer quantization [18–22] to the geometric degrees of freedom results in a dynamical singularity resolution [17], whereby the big bang is replaced by a big bounce. This result was later confirmed (at the genuine quantum level) for different matter fields, in particular the Maxwell field [23] and dust [24].

However, the abovementioned big bounce result was obtained via a somewhat “hybrid” approach: the geometry is quantized via loop techniques, while the matter (the scalar field) is treated by methods of standard quantum mechanics (Schrödinger representation).¹ Furthermore, the system was analyzed by methods dedicated to theories with a time reparametrization freedom. Namely, the evolution was implicitly defined by means of the formalism of partial observables [26]. The above approach was also applied outside of the isotropic settings, both for homogeneous models (like various Bianchi models [27]) and inhomogeneous spacetimes (in particular Gowdy models [28]) as well as in the context of perturbation theory about the cosmological sectors [29–31].² The results for the inhomogeneous settings are however based on heuristic methods and the dynamic is not systematically investigated.

An alternative approach is presented in [32] (see also [33]), where the investigated model is the one considered here but including a nonvanishing cosmological constant. The analysis is carried out in the context of quantum geometrodynamics, which is based on the standard Schrödinger quantization of the metric Hamiltonian formulation of Arnowitt, Deser, and Misner (ADM). More specifically, the system is treated via the so-called deparametrization technique (for an example starting from the full theory see [9]): one of the dynamical variables – in this case the scale factor – is selected as clock at the classical level, after which the system can be quantized and regarded as freely evolving with respect to this clock. The obtained results are (by the majority) consistent with those of the studies of the same systems in the

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¹ Recently, the results of [17] were confirmed [25] through the analysis of the same model, where both the geometric and the matter degrees of freedom are quantized via polymer techniques. This work used one of several possible in this context loop quantization schemes (see [19] and the discussion in Sec. III).

² The list of references given here contains only selected examples representing the current state of development for each model. For a more complete list we refer the reader to [13, 15].

framework of geometrodynamics in [34–36], where the partial variable formalism has been applied [26].

A consistent treatment requires the quantization of the geometry and the matter in the same way. In the context of LQC the intermediate step towards this goal is the analysis of a loop quantized scalar field coupled to gravity quantized via standard techniques. This step is necessary to identify the physical effects arising specifically due to the polymer nature of matter.

In full loop quantum gravity (LQG) a consistent quantization of the scalar field was proposed in [37], where one of two possible (and inequivalent) implementations of the polymer representation [19] was used. The same choice was later made in [25] to derive the symmetry reduced description and to determine the LQC dynamics of (this form of) the polymer scalar field.

The alternative (in a certain sense dual to the above) consistent quantization prescription was applied in [38], where again the FLRW isotropic universe is investigated by scalar quantum mechanics on a classical cosmological background. Elements of a semiclassical analysis led to the construction of an effective approximation of the dynamics, of which the study showed that the big bang singularity is replaced by a past-eternal de Sitter phase (“eternal inflation”) with graceful exit.

The mathematical formalism characteristic to this prescription was later successfully extended to the inhomogeneous setting in the context of quantum field theory on Minkowski space and on a cosmological background [39]. Both of these extensions were built via Bojowald’s lattice refinement techniques [40].

Since the nature of the studies mentioned just now is semi-heuristic, a comparison with the genuine quantum dynamic is indispensable. This is exactly the goal of the work presented here. To provide the precise quantum theory we first perform a deparametrization analogous to the one in [32], choosing the scale factor cubed as clock. Then, we quantize the scalar field via loop techniques, applying the prescription originally provided in [38].

In our work we focus on the precise construction of the quantum model, that is in particular the correct definition of the Hilbert space, and the analysis of the physical consequences: the dynamic, the existence of a semiclassical sector, and the correct GR limit. Surprisingly, the requirement of the latter will have a critical impact on the form of the Hilbert space and, consequently, on the domain of applicability of the heuristic construction of the so-called effective dynamics from LQC.

The numerical analysis of the dynamic shows that, yet again, there is no quantum big bang. However, instead of bouncing back, the quantum state (the wave packet) transits deterministically through the point marking the singularity in GR. The quantum evolution picture appearing here resembles thus the one advertised in the “early LQC epoch” [11].³ Consequently, this work (see

also the results in [25]) suggests that the big bounce is an effect arising solely due to the polymeric (discrete) quantum nature of the *geometry*.

The paper is structured as follows: in Sec. II we introduce the details of the classical FLRW model that we analyze. Then we proceed in Sec. III with the construction of the precise quantum theory. Finally, in Sec. IV we analyze the physical results and conclude in Sec. V with a general discussion.

In our studies we select the natural units $c \equiv \hbar \equiv 1$ and introduce the abbreviation $L^2 \equiv 12\pi G_N$ for a length scale. Later in the paper we further restrict our attention to the case $L = 1$ corresponding to a form of Planck units.

II. CLASSICAL THEORY

In this paper we focus on the case of an isotropic and flat FLRW universe with a minimally coupled massless scalar field $\phi \equiv \phi(T) \in \mathbb{R}$ as source. The metric tensor of such a universe can be expressed as

$$g \equiv -(NdT)^2 + a^2 \delta_{ij} dX^i dX^j, \quad (2.1)$$

where $N \equiv N(T) \in \mathbb{R}^+$ is the lapse function and $a \equiv a(T) \in \mathbb{R}$ is the scale factor. Since the “symmetric criticality principle” [3, 41] is valid in the present situation, the canonical action

$$A = \int_{T_i}^{T_f} (P_a \dot{a} + P_\phi \dot{\phi} - H[N]) dT, \quad (2.2)$$

can be directly and conveniently derived from the reduced Einstein-Hilbert action. The canonical momenta appearing in (2.2) are

$$\frac{P_a}{V_C} = \left(\frac{3}{L}\right)^2 \frac{|a|\dot{a}}{N}, \quad \frac{P_\phi}{V_C} = \frac{|a|^3 \dot{\phi}}{N}, \quad (2.3a)$$

$$\{a, P_a\} = 1, \quad \{\phi, P_\phi\} = 1, \quad (2.3b)$$

whereas the scalar Hamiltonian constraint takes the form

$$H[N] = N \frac{V_C |a|^3}{2L^2} \left[-\left(\frac{L^2 P_a}{3V_C a^2}\right)^2 + \left(\frac{L P_\phi}{V_C a^3}\right)^2 \right]. \quad (2.4)$$

Note that to arrive at the form (2.2) of the action we had to first introduce the 3+1 splitting $\mathcal{M} \simeq \mathbb{R} \times \mathcal{N}$, where for the flat universe (considered here) $\mathcal{N} = \mathbb{R}^3$. Due to the noncompactness of the spatial slices we had to introduce in the process of deriving (2.2) the infrared regulator – a cube or “cell” $\mathcal{V} \subset \mathbb{R}^3$ of finite size (see [34, 42] and the discussion in [43]). Its physical volume is $V = V_C |a|^3$, where $V_C \equiv \int_{\mathcal{V}} d^3 X$ is the comoving coordinate volume of \mathcal{V} .

³ We note that the early results were derived for a different system,

where the geometry instead of the matter was polymeric.

This step introduces an additional complication into the treatment as one has to make sure that the resulting model has a well defined (unambiguous) regulator removal limit. The classical FLRW theory is invariant under the rescaling $\vec{X} \mapsto \zeta \vec{X}$, $\zeta \in \mathbb{R}^+$, which increases V by a factor of ζ^3 (an *active* diffeomorphism). This invariance is a natural requirement for the description of the model to remain well defined when the regulator is removed [44]. We stress that this requirement is however by no means sufficient in the quantum theory (see in particular [43]). Furthermore, in the quantum theory the ζ -invariance is not given trivially [13, 34, 42] and hence imposing it as a condition for consistency affects the choice of the canonical variables for the quantization [see (2.11), (2.12), and the paragraph prior to them].

The *first-class* [45–47] Hamiltonian constraint $H[N] \approx 0$ generates *infinitesimal* transformations of T . As can be seen from (2.1), there also is the possibility of a reparametrization of the scale factor a and the Euclidean metric δ by an $\eta \in \mathbb{R}^\pm$ such that $a \mapsto a/\eta$ and $\delta \mapsto \eta^2 \delta$, respectively. This residual η -symmetry corresponds to the freedom of fixing the coordinate scale (a *passive* diffeomorphism) and orientation (a “large gauge transformation” [13, 17, 34, 42, 48]). Just like it is required for the ζ -transformation, the physics has to be invariant under an η -transformation. Among the η -invariant quantities are the action A , the volume V and ratios of the scale factor such as the Hubble parameter $h \equiv \dot{a}/(Na)$.

In the next step we fix the time reparametrization freedom by implementing the *second-class* [45–47] gauge

$$G \equiv T - \frac{V_C a^3}{L^2}, \quad (2.5)$$

Given the equation of motion

$$\dot{a} = \{a, H[N]\} = -N \frac{L^2 P_a}{9V_C |a|}, \quad (2.6)$$

the form of G implies in particular that P_a is negative. This and the form of the constraint $H[N]$ then lead to the reduced canonical action

$$A = \int_{T_1}^{T_F} (P_\phi \dot{\phi} - H_R) dT, \quad (2.7)$$

where again $\{\phi, P_\phi\} = 1$. The reduced Hamiltonian takes the form

$$H_R \equiv -P_T = -\frac{L^2 P_a}{3V_C a^2} = \left| \frac{L^2 P_a}{3V_C a^2} \right| = \left| \frac{P_\phi}{LT} \right|. \quad (2.8)$$

Finally, the consistency condition

$$\partial_T G + \{G, H[N]\} = 0 \quad (2.9)$$

uniquely determines the lapse function

$$N = \frac{L}{|P_\phi|}. \quad (2.10)$$

We emphasize that the time-gauge G becomes $T - \text{sig}(\eta)V_C a^3/L^2$ under an η -transformation. This means that the orientation of a relative to T changes if $\eta \in \mathbb{R}^-$. Replacing T by $-T$ has no impact on the space of solutions to the Wheeler-DeWitt equation resulting from (2.7), and thus amounts to a “time-reversal” operation [49]. However, the *reduced* classical and quantum formalism based on (2.5) and derived here is the result of singling out one of $\pm T$, and is therefore no longer time-reversal invariant. If (2.5) defines a future-directed clock, the past orientation would be given by the gauge constraint $G = T + V_C a^3/L^2$. These considerations are relevant in the construction of the initial state for the quantum evolution (see Subsec. IV B).

The canonical formalism we constructed here has the deficiency of explicitly depending on the infra-red regulator \mathcal{V} since, according to (2.5), the clock variable T scales like ζ^3 . This would make the removal of the infra-red regulator \mathcal{V} from the resulting quantum theory a rather tedious task. As the initial step in addressing this problem we replace T by the dimensionless variable $t \equiv T/|T^\varnothing|$, where $T^\varnothing \in \mathbb{R}^\pm$ is some fixed but otherwise arbitrary reference value. Furthermore, since P_ϕ also scales like ζ^3 , we analogously define $p_\phi \equiv P_\phi/|P_\phi^\varnothing|$ with $P_\phi^\varnothing \in \mathbb{R}^\pm$ being a fixed reference value for P_ϕ . Technically, the replacement of T by t can be brought about by a simple change of the integration variable in (2.8), whereas the replacement of P_ϕ by p_ϕ is realized by an “extended canonical” or “scale transformation” [50]. Altogether, this procedure yields the canonical action

$$A = \frac{|P_\phi^\varnothing|}{L} \int_{|T^\varnothing|t_1}^{|T^\varnothing|t_F} (L p_\phi \phi' - H_S) dt, \quad (2.11)$$

where $\phi' \equiv d\phi/(dt)$, $\{\phi, p_\phi\} = 1/L$, and

$$H_S \equiv \left| \frac{p_\phi}{t} \right|, \quad \left| \frac{p_\phi}{t} \right| dt = \frac{L}{|P_\phi^\varnothing|} H_R dT, \quad (2.12)$$

is the Hamiltonian related to H_R by a *scaling*. We stress that t and p_ϕ are dimensionless variables.

At this point it is necessary to mention that the above modification does not yet remove the dependence of the theory on the infrared regulator. Indeed, while the constant T^\varnothing and P_ϕ^\varnothing are fixed, no particular value of \mathcal{V} can be distinguished on physical grounds. In consequence, the particular “physical” universe is represented by classes of solutions rather than by single ones. This nonuniqueness can be easily shown on the level of specifying the initial data. There, the single universe regulated by different cells \mathcal{V} will correspond to the entire set (class of equivalence) of the initial data at the chosen initial time t^* . This dependence will propagate through to the quantum theory.

For initial $t^*, p_\phi^* \in \mathbb{R}^\pm$ the solutions to Hamilton’s equations of motion derived from (2.11) and (2.12) are

$$p_\phi(t) = p_\phi^*, \quad \phi(t) = \phi^* + \frac{\text{sig}(t p_\phi^*)}{L} \ln \left(\left| \frac{t}{t^*} \right| \right). \quad (2.13)$$

Therefore, the lapse function $N = |T^\nabla|/|P_\phi^\nabla p_\phi|$ is a constant [positive since we fixed the time orientation by the chosen gauge constraint (2.5)]. The canonical representation of the spacetime Ricci scalar takes then the form $R = -6h^2$, which in turn implies

$$-\frac{3R(t)}{2R^\nabla} = \left[\frac{p_\phi(t)}{t} \right]^2 = \left(\frac{p_\phi^*}{t} \right)^2, \quad R^\nabla \equiv \left(\frac{P_\phi^\nabla}{LT^\nabla} \right)^2. \quad (2.14)$$

The form of the reference value R^∇ suggests the natural and simplifying choice $|P_\phi^\nabla| = |T^\nabla|$, which corresponds to fixing the initial curvature value to be $R(t^*) = -(2/3)[p_\phi^*/(Lt^*)]^2$. This partially fixes the freedom of rescaling t and p_ϕ but still does not remove the implicit ξ -dependence discussed earlier.

In order to simplify the expressions, from now on we will use the Planck units normalized by $L = 1$. With this choice the energy density ρ and the pressure p are

$$\rho(t) = p(t) = -\frac{3}{4}R(t), \quad (2.15)$$

and the spacetime singularity occurs at $t = 0^\pm$. The positivity of N implies that the future-pointing evolution of the scalar field is “into” a *big crunch* for $t \in \mathbb{R}^-$ and “away from” a *big bang* for $t \in \mathbb{R}^+$. From (2.13) it is evident that for an element of the branch of the solution space admitting a big crunch, the value of $\phi(t) - \phi^*$ for $p_\phi^* \in \mathbb{R}^-$ is related to the analogous value for $p_\phi^* \in \mathbb{R}^+$ by an overall sign-change. The same holds for for an element of the big bang branch of the solution space. Furthermore, we have the large time-inversion symmetry

$$(\text{sig}(p_\phi^*)[\phi(t) - \phi^*])_- = -(\text{sig}(p_\phi^*)[\phi(t) - \phi^*])_+, \quad (2.16)$$

which relates big crunches (the left-hand side for $t \in \mathbb{R}^-$) with big bangs (the right-hand side for $t \in \mathbb{R}^+$). In the canonical formalism at hand these identities are manifestations of the invariance of the *covariant* action under a replacement of ϕ with $-\phi$.

Finally, we note again that the variables t and p_ϕ are dimensionless but still not invariant under a ζ -transformation. The observable scalar field in (2.13) and the spacetime Ricci scalar in (2.14) – along with its related scalars in (2.15) – are inheriting this implicit non-invariance. This fact will play a crucial role in singling out the correct regularization scheme in the quantum theory.

III. QUANTUM THEORY

Our goal here is to build the precise quantum mechanical representation of the model introduced above. This means in particular the construction of a suitable Hilbert space \mathbb{H} and the representation of H_S as a self-adjoint operator acting on the suitable domain in \mathbb{H} . The quantum evolution will then be determined by some Schrödinger

equation

$$i\frac{\partial}{\partial t}\psi = \left[\frac{p_\phi}{t} \right] \psi = \hat{H}_S \psi \quad (3.1)$$

for $\psi \equiv \psi(t, \phi)$.

A. The scalar field momentum operator

To start with, let us recall that the canonical formalism introduced in the previous section describes a freely evolving, isotropic and flat FLRW model. The evolution is governed by the Hamiltonian H_S in (2.12), which by (2.5) depends on the scale factor clock $t = V_C a^3/|T^\nabla|$. The scalar field ϕ is thus the only object subject to a quantization. Here, we have several possibilities to proceed.

The most obvious way to construct the quantum description is to apply the Schrödinger representation, as in [32]. As shown there, this representation does not lead to a singularity resolution, as the semiclassical wave packets simply follow the classical trajectories into the singularity.

An alternative approach, which is pursued here, is the implementation of the polymer representation. The requirement of the existence of an infra-red regulator removal limit (see the previous section) forces this representation to be *time-dependent*.⁴

To begin the detailed specification of the polymer quantization procedure, let us briefly recall the standard Schrödinger representation. It is characterized by the Stone-von Neumann uniqueness theorem [19, 20, 52], which implies that among all the irreducible regular realizations of the Weyl form

$$\hat{I}_\lambda \hat{J}_\mu = e^{i\lambda\mu} \hat{J}_\mu \hat{I}_\lambda, \quad \lambda, \mu \in \mathbb{R}^+, \quad (3.2)$$

of the canonical commutation relation

$$[\hat{\phi}, \hat{p}_\phi] = i\hat{1} \quad (3.3)$$

on the space $L^2(\mathbb{R}, d\phi)$ of Lebesgue square-integrable functions, the Schrödinger representation

$$\hat{I}_\lambda \equiv e^{i\lambda\hat{\phi}}, \quad \hat{J}_\mu \equiv e^{-i\mu\hat{p}_\phi}, \quad (3.4)$$

is unique. The regularity property says that the mappings of λ to \hat{I}_λ and μ to \hat{J}_μ are continuous, which holds if for $\psi, \omega \in L^2(\mathbb{R}, d\phi)$ the mappings

$$\lambda \mapsto \langle \psi | \hat{I}_\lambda | \omega \rangle, \quad \mu \mapsto \langle \psi | \hat{J}_\mu | \omega \rangle, \quad (3.5)$$

are continuous.

⁴ This situation is analogous to the one in the loop quantization of the geometric degrees of freedom, where consistency requirements label the improved dynamics construction as the correct one [51].

To generalize the above formalism, let us now consider the space of exponentiated operator labels, further parametrized in the following way

$$\lambda \equiv \lambda_t \equiv \nu \nu_t, \quad \mu \equiv \mu_t \equiv \rho \rho_t, \quad \nu, \nu_t, \rho, \rho_t \in \mathbb{R}^+. \quad (3.6)$$

The subscript t can be seen as parametrizing the parameters λ and μ of the groups of unitary operators \hat{I}_λ and \hat{J}_μ , respectively. *That is, the Weyl algebra given in (3.2) depends now on time and so do the unitary operators defined in (3.4).* However, we observe that because of the continuity the operators $\hat{\phi}$ and \hat{p}_ϕ are

$$\hat{\phi} \equiv i \lim_{\lambda_t \rightarrow 0} \frac{\hat{1} - \hat{I}_{\lambda_t}}{\lambda_t} = i \lim_{\nu \rightarrow 0} \frac{\hat{1} - \hat{I}_{\lambda_t}}{\lambda_t} = \phi \hat{1}, \quad (3.7a)$$

$$\hat{p}_\phi \equiv -i \lim_{\mu_t \rightarrow 0} \frac{\hat{1} - \hat{J}_{\mu_t}}{\mu_t} = -i \lim_{\rho \rightarrow 0} \frac{\hat{1} - \hat{J}_{\mu_t}}{\mu_t} = -i \frac{\partial}{\partial \phi}, \quad (3.7b)$$

thus they are independent of t . To conclude, in the Schrödinger quantization the operators defined in (3.7) are not changing if the group parameters are themselves parametrized by t .

The situation changes drastically if the regularity condition in the Stone-von Neumann uniqueness theorem is dropped. In this case a possible faithful realization of (3.2) is given by the so-called “polymer representation”. For the sake of generality we will further allow it to be time-dependent (in a yet unspecified way). The non-separable Hilbert space \mathbf{H} for this representation consists of functions $\psi \in l^2(\mathbb{R}, \#_\phi)$ satisfying the square-summation requirement

$$\|\psi\|^2 \equiv \sum_{\phi \in \mathbb{D}_{\mu_t}(\psi)} |\psi(\phi)|^2 < \infty. \quad (3.8)$$

The inner product on \mathbf{H} providing this norm is

$$\langle \psi | \omega \rangle \equiv \sum_{\phi \in \mathbb{D}_{\mu_t}(\psi, \omega)} \overline{\psi(\phi)} \omega(\phi), \quad (3.9)$$

where ω is another element of $l^2(\mathbb{R}, \#_\phi)$. We denote by $\#_\phi$ the measure that maps a subset of \mathbb{R} to its cardinality (the so-called “counting measure”) and by

$$\mathbb{D}_{\mu_t}(\psi) \equiv \bigcup_{\phi_0 \in \mathbb{S}(\psi)/\cong} \mathbb{L}_{\mu_t}(\phi_0), \quad \mathbb{D}_{\mu_t}(\psi, \omega) \equiv \bigcup_{\phi_0 \in (\mathbb{S}(\psi) \cap \mathbb{S}(\omega))/\cong} \mathbb{L}_{\mu_t}(\phi_0), \quad (3.10)$$

domains defined in terms of the necessarily countable support \mathbb{S} of ψ or ω . For $\phi, \chi \in \mathbb{R}$ we have the equivalence $\phi \approx \chi$ if and only if there is an integer $i \in \mathbb{Z}$ such that $\phi = \chi + i\mu_t$ [recall (3.6)]. The domains \mathbb{D}_{μ_t} are then disjoint unions of uniform lattices

$$\mathbb{L}_{\mu_t}(\phi_0) \equiv \{\phi_0\} + \mathbb{Z}\mu_t, \quad \phi_0 \in [0, \mu_t). \quad (3.11)$$

Orthonormal basis states of \mathbf{H} are “half-deltas”

$$\delta_\phi : \chi \mapsto \delta_\phi(\chi) \equiv \delta_{\phi\chi} \equiv \begin{cases} 1, & \phi = \chi, \\ 0, & \text{otherwise,} \end{cases} \quad (3.12)$$

extending the definition of the Kronecker delta symbol to the real line.

The time-dependent polymer representation is now given by

$$\hat{I}_{\lambda_t} \delta_\phi \equiv e^{i\lambda_t \phi} \delta_\phi, \quad \hat{J}_{\mu_t} \delta_\phi \equiv \delta_{\phi + \mu_t}, \quad (3.13)$$

which characterizes again a *multiplication* and a *translation* operator, respectively. The “ λ -mapping” given in (3.5) is once more continuous for λ_t so that by Stone’s theorem [19, 52] the scalar field multiplication operator remains to be given by (3.7). The difference to the Schrödinger representation in (3.4) is that the “ μ -mapping” in (3.5) is no longer continuous in μ_t . There is therefore *no* self-adjoint momentum operator generating infinitesimal translations. On $l^2(\mathbb{R}, \#_\phi)$ there is only an operator generating *finite* translations. We are therefore forced to regularize it, for which we employ the technique introduced by Thiemann in the context of full LQG [5, 53]. In essence this technique is approximating the undefined \hat{p}_ϕ by well-defined translation operators. Following [20, 39, 54] we choose

$$\hat{p}_{\phi\mu_t} \equiv -\frac{i}{2\mu_t} (\hat{J}_{\mu_t}^\dagger - \hat{J}_{\mu_t}), \quad (3.14a)$$

$$\widehat{p_{\phi\mu_t}^2} \equiv \frac{2}{\mu_t^2} \left(\hat{1} - \frac{\hat{J}_{\mu_t}^\dagger + \hat{J}_{\mu_t}}{2} \right). \quad (3.14b)$$

The action of the former on a state $\psi \in \mathbf{H}$ is

$$\hat{p}_{\phi\mu_t} \psi(\phi) = -\frac{i}{2\mu_t} [\psi(\phi + \mu_t) - \psi(\phi - \mu_t)] \quad (3.15)$$

so that, *if* we could send μ_t to 0 [or according to (3.6) send ρ to 0, thereby taking the limit at the *kinematic* level], we would get back the differential operator $-i\partial/(\partial\phi)$. We observe that the representation of the momentum operator $\hat{p}_{\phi\mu_t}$ is highly non-unique, in the same way the representation of finite difference operators in numerical analysis is. We stress that, unlike in the Schrödinger representation, because of (3.6) the momentum operator is now time-dependent.

At this point it is necessary to emphasize that the presented polymer quantization is not the only possible one. Essentially, by replacing the roles of ϕ and p_ϕ we arrive at another polymer representation, *inequivalent* (and in a sense “dual”) to ours (see the discussion in [19]). Such a dual representation was used in the quantization of the scalar field in full LQG [37]. Its symmetry reduced version was applied to the LQC model of an FLRW universe [25] filled with a massless scalar field. The subsequent analysis of the spectral decomposition of the evolution operator (playing the role of the Hamiltonian) shows that the dynamic of such a system is *exactly the same* as the one of the system with the scalar field quantized via standard methods of quantum mechanics [34]. Both approaches, ours and the one of [25], are equally viable from a mathematical point of view. Therefore, choosing one of them requires a physical input.

B. The Hamiltonian

The next step is the construction (including the determination of the action) of the quantum Hamiltonian \hat{H}_S generating the unitary evolution. To do so, we switch to the scalar field momentum space, which is again the Pontryagin dual of the real line but this time the latter is equipped with the discrete topology. In short, it is the Bohr-compactified real line \mathbb{R}_B . The Hilbert space defined in the previous subsection is then equivalent to the space $\mathbb{H}^{\mathfrak{h}}$, which consists of Bohr square-measurable functions

$$\psi^{\mathfrak{h}}(p_\phi) \equiv \sum_{\phi \in \mathbb{D}_{\mu_t}(\psi)} \psi(\phi) e^{-i\phi p_\phi} \in L^2(\mathbb{R}_B, (dp_\phi)_B) \quad (3.16)$$

satisfying

$$\begin{aligned} \|\psi^{\mathfrak{h}}\|^2 &\equiv \int |\psi^{\mathfrak{h}}(p_\phi)|^2 (dp_\phi)_B \\ &\equiv \lim_{C \rightarrow \infty} \frac{1}{2C} \int_{-C}^C |\psi^{\mathfrak{h}}(p_\phi)|^2 dp_\phi < \infty. \end{aligned} \quad (3.17)$$

The inner product (between $\psi^{\mathfrak{h}}$ and another $\omega^{\mathfrak{h}} \in \mathbb{H}^{\mathfrak{h}}$) generating this norm is

$$\langle \psi^{\mathfrak{h}} | \omega^{\mathfrak{h}} \rangle \equiv \int \overline{\psi^{\mathfrak{h}}(p_\phi)} \omega^{\mathfrak{h}}(p_\phi) (dp_\phi)_B. \quad (3.18)$$

The basis orthonormal with respect to it is formed by the plane waves

$$e_\phi : p_\phi \mapsto e_\phi(p_\phi) \equiv e^{-i\phi p_\phi} = \delta_\phi^{\mathfrak{h}}(p_\phi). \quad (3.19)$$

We observe that for a uniform lattice $\mathbb{D}_{\mu_t}(\psi) = \mathbb{L}_{\mu_t}(\phi_0)$ [see (3.11)] the Bohr measure $(dp_\phi)_B$ becomes the Lebesgue measure dp_ϕ with an integration over the fixed interval $(-\pi/\mu_t, \pi/\mu_t]$. The momentum space polymer theory defined here *would* then be that of Fourier with discreteness in position rather than momentum space.

In the general polymer theory at hand, the action of the multiplication and translation operator on the basis states e_ϕ is unchanged in comparison to (3.13) so that the scalar field operator is given by $i\partial/(\partial p_\phi)$. On the other hand, the operator \hat{p}_ϕ is now the regularized $\hat{p}_{\phi\mu_t}$ (see the previous subsection), which is approximated by translation operators. Indeed, since $e_{\phi+\mu_t} = e_{\mu_t} e_\phi$ we have

$$\hat{J}_{\mu_t} = e^{-i\mu_t p_\phi} \hat{1}, \quad \hat{p}_{\phi\mu_t} = \frac{\sin(\mu_t p_\phi)}{\mu_t} \hat{1}, \quad (3.20)$$

because of which the action of the Hamiltonian operator can be explicitly given by

$$\hat{H}_S \psi^{\mathfrak{h}} = \left| \frac{\sin(\mu_t p_\phi)}{\mu_t t} \right| \psi^{\mathfrak{h}}, \quad (3.21)$$

where $\psi^{\mathfrak{h}} \equiv \psi^{\mathfrak{h}}(t, p_\phi)$. Given this, the fact that $\|\psi\| = \|\psi^{\mathfrak{h}}\|$ allows us to immediately prove the conservation of the

norm under an action of \hat{H}_S . To show this we write explicitly the time derivative of the norm

$$\begin{aligned} i \frac{\partial}{\partial t} \|\psi\|^2 &= i \frac{\partial}{\partial t} \|\psi^{\mathfrak{h}}\|^2 = \int i \frac{\partial}{\partial t} |\psi^{\mathfrak{h}}|^2 (dp_\phi)_B \\ &= \sum_{\phi, \chi \in \mathbb{D}_{\mu_t}(\psi)} \overline{\psi(t, \phi)} \psi(t, \chi) \int \hat{H}_S e^{-ip_\phi(\phi-\chi)} (dp_\phi)_B. \end{aligned} \quad (3.22)$$

To evaluate the right-hand side we first observe that the integral can be expressed as the contour integral

$$\begin{aligned} \lim_{C \rightarrow \infty} \frac{1}{2C} \int_{-C}^C &= \lim_{n \rightarrow \infty} \frac{\mu_t}{4n\pi} \sum_{i=1}^n \left(\int_{-2i\pi/\mu_t}^{-2(i-1)\pi/\mu_t} \right. \\ &\quad \left. + \int_{-(2i-1)\pi/\mu_t}^{-2(i-1)\pi/\mu_t} + \int_{2(i-1)\pi/\mu_t}^{2i\pi/\mu_t} + \int_{(2i-1)\pi/\mu_t}^{2i\pi/\mu_t} \right). \end{aligned} \quad (3.23)$$

The specific form of this integral allows us to drop the absolute value \hat{H}_S in (3.21) replacing it instead with a sign appropriate for each integration domain. Next, we apply some simple trigonometric identities, the μ_t -translation invariance of $\mathbb{D}_{\mu_t}(\psi)$, and (see [55])

$$\sum_{i=1}^n \cos\left(\frac{(2i+1)\pi}{\mu_t}(\phi-\chi)\right) = \frac{\sin\left(\frac{2n\pi}{\mu_t}(\phi-\chi)\right)}{2 \sin\left(\frac{\pi}{\mu_t}(\phi-\chi)\right)}. \quad (3.24)$$

Finally, if we divide this by n and take the limit $n \rightarrow \infty$ [see (3.23)], we get $i\partial\|\psi\|/\partial t = 0$.

Up to now, the shift function μ_t in the approximated \hat{p}_ϕ operator is an arbitrary function of time. At the mathematical level the situation is analogous to the one in the loop quantization of the geometry (see [56]), where the fiducial holonomy length could be an arbitrary function on the phase space). There, however, the physical consistency requirements restricted the possible choices to just one class of functions [51]. We expect that the same situation occurs in our model. To show that this expectation is indeed realized let us recall the following facts.

The particular moment of the universe's evolution can be represented by various points on the phase space corresponding to different choices of the regulator cell. Furthermore, once we ask about the locally measurable properties of the universe (observables) at this moment, there has to exist their nontrivial limit as we remove the regulator.

One such local observable is the energy density (2.15) determined by (2.14). The quantum operator corresponding to it is related to the Hamiltonian \hat{H}_S in the following way

$$\hat{\rho} = \frac{1}{2} \hat{H}_S^2. \quad (3.25)$$

From (3.21) it follows that at the fixed time t the spectrum of this operator equals

$$\text{Sp}(\hat{\rho}) = \left[0, \frac{1}{2\mu_t^2 t^2} \right]. \quad (3.26)$$

The most natural way to satisfy the consistency requirements discussed in the previous paragraph is to require that the spectrum of \hat{q} be *time independent*. This implies $\mu_t \propto 1/t$ so that we can fix the function ρ_t in (3.6) as

$$\rho_t \equiv \frac{1}{|t|}. \quad (3.27)$$

This in turn gives $\mu_t = \rho/|t|$, which for the “volume clock” $v \equiv t/\rho$ results in the momentum space Schrödinger equation

$$i \frac{\partial}{\partial v} \psi^{\natural} = \left| \sin \left(\frac{p_\phi}{v} \right) \right| \psi^{\natural} \quad (3.28)$$

for $\psi^{\natural} \equiv \psi^{\natural}(v, p_\phi)$.

Note, that the above method of fixing μ_t is almost a full analog of the conditions used for the geometry degrees of freedom in [51]. There, however, the reasoning exploited the existence of “nicely” behaving semiclassical sectors through the use of the so called *effective dynamics*. Here, as we have not yet investigated the dynamical sector implementing that reasoning directly would be risky. Instead, we managed to fix the ambiguity through the considerations on the genuine quantum level.

In the next section we solve the Schrödinger equation (3.28) in order to analyze the dynamics of the system and to discuss the physical properties of the solutions.

IV. THE DYNAMIC

The Hilbert space and the explicit action of the Hamiltonian operator constructed just now allow us to easily determine the system’s dynamic. At this level the requirement of the theory to be physically meaningful becomes crucial. The principal requirement is that the theory must have the proper low energy limit. Here, this means that in the distant past and future the quantum evolution ought to agree with the predictions of GR. In our case an inability of the model-description to realize this property would imply that the formulation should be further and adequately corrected. In fact, as we will see below, this is precisely what is required here.

To begin, let us investigate the dynamic of the theory exactly as specified in the previous section.

A. Single Lattice Hilbert space

Once we select p_ϕ as the configuration variable, the Schrödinger equation given in (3.28) becomes a simple ordinary differential equation, which is easy to solve on

the domains $t > 0$ and $t < 0$. Its solution reads

$$\begin{aligned} \psi^{\natural}(v, p_\phi) &\equiv \hat{E}_{vv^*} \psi^{\natural}(v^*, p_\phi) \\ &\equiv e^{-i[F(v, p_\phi) - F(v^*, p_\phi)]} \psi^{\natural}(v^*, p_\phi), \end{aligned} \quad (4.1a)$$

$$F(v, p_\phi) \equiv v S(v, p_\phi) \left[\sin \left(\frac{p_\phi}{v} \right) - \text{Ci} \left(\left| \frac{p_\phi}{v} \right| \right) \frac{p_\phi}{v} \right], \quad (4.1b)$$

$$S(v, p_\phi) \equiv \text{sig} \left(\sin \left(\frac{p_\phi}{v} \right) \right), \quad (4.1c)$$

where we set $v^* = t^*/\rho$. For $|\arg(z)| < \pi$ the cosine integral function is

$$\text{Ci}(z) \equiv \gamma + \ln(z) + \int_0^z \frac{\cos(y) - 1}{y} dy \quad (4.2)$$

with γ the Euler-Mascheroni number [57]. The definition implies $\text{Ci}(z) \sim \gamma + \ln(z)$ for $z \rightarrow 0$, suggesting semiclassical behavior of sufficiently sharply peaked initial states in the limit $|v| \rightarrow \infty$.

At the point $v = 0$ the operator \hat{H}_S is not well defined so that the Cauchy-Peano theorem, which normally ensures the existence and uniqueness of the solution, cannot be applied there. Thus, in principle one cannot extend the solution to (3.28) through that point. However, the solutions (4.1) admit well defined limits $v \rightarrow 0^+$ for $v > 0$ and $v < 0$, and thus one can define an extension to that point by taking these limits. Since

$$\lim_{v \rightarrow 0} F(v, p_\phi) = 0, \quad (4.3)$$

there exists a unique unitary operator evolving states to the instant $v = 0$. Furthermore, it has the very simple form

$$\hat{E}_{0v^*} \equiv e^{iF(v^*, p_\phi)} \hat{1}. \quad (4.4)$$

As a consequence *there exists a preferred extension* of the evolution through $v = 0$, defined by the requirement of continuity of ψ^{\natural} at $v = 0$. The global solution is again given by (4.1).

It appears that the existence of such a preferred extension is sufficient for singularity resolution. However, as we will see below, this is not the case. To explain what is missing, we consider any normalized initial state $\psi^{\natural}(v^*, p_\phi)$ such that the expectation value of the scalar field operator is finite

$$\langle \psi^{\natural}, v^* | \hat{\phi} | \psi^{\natural}, v^* \rangle = \phi^*, \quad |\phi^*| < \infty. \quad (4.5)$$

Then the expectation value of ϕ at any volume v is given by the following formula

$$\begin{aligned} \phi(v) &= \langle \psi^{\natural}, v | \hat{\phi} | \psi^{\natural}, v \rangle \\ &= \left\langle \psi^{\natural}, v^* \left| \hat{\phi} - \left[S(v, p_\phi) \text{Ci} \left(\left| \frac{p_\phi}{v} \right| \right) \right. \right. \right. \\ &\quad \left. \left. \left. - S(v^*, p_\phi) \text{Ci} \left(\left| \frac{p_\phi}{v^*} \right| \right) \right] \hat{1} \right| \psi^{\natural}, v^* \right\rangle. \end{aligned} \quad (4.6)$$

Since the cosine integral function belongs to $L^2(\mathbb{R}, dp_\phi)$, the above expectation value is in fact equal to ϕ^* . That

is to say *the evolution is frozen*. This result is then in direct disagreement with the predictions of GR. In consequence, our states exhibit an unphysical behavior in the low energy (large v) limit.

Our model then still lacks an appropriate *physical* Hilbert space. To explore the possibilities of constructing it, let us first go back to analyzing the solutions to (3.28) but this time by considering the wave functions on the configuration space as opposed to the momentum one used in (4.1). Then, the evolution of a state $\psi_v \equiv \psi(v, \cdot)$ can be viewed as an assignment $v \mapsto \psi_v \in \mathbf{H}_v$, where $v \in \mathbb{R}$. The Hilbert space \mathbf{H}_v is spanned by eigenstates of \hat{H}_S for a fixed value of v . However, per analogy with the loop quantization of the geometry [34] we can distinguish sectors that are invariant with respect to the action of \hat{H}_S at v . These sectors consist of functions that are supported on the lattices $\mathbb{L}(\varphi_0)\mu_v$, where $\mu_v \equiv 1/|v|$ and

$$\mathbb{L}(\varphi_0) \equiv \mathbb{L}_{\mu_v}(\phi_0)/\mu_v \equiv \{\varphi_0\} + \mathbb{Z}, \quad \varphi_0 \in [0, 1). \quad (4.7)$$

We can then attempt to consider at the initial $v = v^*$ the subspaces $\mathbf{H}_{v\varphi_0} \equiv \mathbf{H}_v|_{\mu_v\mathbb{L}(\varphi_0)}$ as the superselection sectors and evolve them independently. Such a decomposition can be performed at each v independently. Let us now probe whether there exists any relation between the spaces $\mathbf{H}_{v\varphi_0}$ for different values of v . The answer is given immediately by the form of the solution (4.1): since the cosine integral function is non-periodic, the unitary evolution to any $v \neq v^*$ immediately couples an infinite number of these lattices. In consequence, the sectors $\mathbf{H}_{v\varphi_0}$ of \mathbf{H}_v are *not* true superselection sectors in the sense of [13, 17, 34, 42, 48]. Therefore, we are forced to work with the original *non-separable* Hilbert space \mathbf{H}_v without access to previously available tools that allow to distinguish separable subspaces. Furthermore, the form of (4.6) suggests that in order to provide a nontrivial evolution, the physical Hilbert space needs to be equipped with a continuous rather than a discrete inner product.

B. Integral Hilbert space

A similar situation appeared in LQC already in a different context during the studies of the FLRW universe with a massless scalar field and a positive cosmological constant [36]. There, following the choice of a lapse adopted to using the scalar field as time variable, the evolution operator admitted a family of self-adjoint extensions, each with a discrete spectrum. However, a different choice of the lapse – corresponding to parametrizing the evolution by the cosmic time variable – led to a unique self-adjoint generator of the evolution with continuous spectrum [58]. The physical Hilbert space corresponding to the latter case appeared, furthermore, to be an integral of all the Hilbert spaces corresponding to the particular self-adjoint extensions of the former case, with the Lebesgue measure determined by the group averaging procedure.

Motivated by this observation, we introduce the analog of the integral Hilbert space in our case. First, we

note that on the domain $[0, 1)$ of φ_0 one can introduce a natural (quite general and time dependent) Lebesgue measure $M(v, \varphi_0)d\varphi_0$. Next, we introduce a decomposition of the non-separable Hilbert space \mathbf{H} onto spaces $\mathbf{H}_{v\varphi_0}$ at the initial time v^* . This defines the decomposition of the initial data at $v = v^*$

$$\mathbf{H} \ni \psi(v^*, \phi) \mapsto \psi_{\varphi_0}(v^*, \phi) \equiv \psi(v^*, \phi)|_{\mathbb{L}_{\mu_{v^*}}(\phi_0)} \in \mathbf{H}_{v^*\varphi_0}. \quad (4.8)$$

This initial data is then extended to the solutions to (3.28) via (4.1). We thus have a decomposition of the physical Hilbert space onto explicitly separable (at least at $v = v^*$) subspaces.

Now, we compose the new Hilbert space via an integral

$$\mathbf{H}_{Pv^*} \equiv \int_0^1 \mathbf{H}_{v^*\varphi_0} M(v^*, \varphi_0) d\varphi_0, \quad (4.9)$$

equipping it with the inner product

$$\langle \psi_v | \omega_v \rangle \equiv \int_0^1 \langle \psi_{v\varphi_0} \hat{E}_{vv^*} | \hat{E}_{v^*v} \omega_{v\varphi_0} \rangle M(v^*, \varphi_0) d\varphi_0, \quad (4.10)$$

where $\psi_{v\varphi_0} = \psi_{\varphi_0}(v, \phi) \in \mathbf{H}_{v\varphi_0}$. This is our candidate for the physical inner product: between each pair of solutions it is evaluated on the initial data slice at $v = v^*$. On that initial slice it can be written in a very simple form, namely,

$$\langle \psi_{v^*} | \omega_{v^*} \rangle \equiv \int_{\mathbb{R}} \overline{\psi(v^*, \phi)} \omega(v^*, \phi) M(v^*, \varphi_0(\phi)) / \mu_{v^*} d\phi. \quad (4.11)$$

It is by definition time-independent but a priori it may not have a simple local form analogous to (4.11) at $v \neq v^*$, which can potentially complicate the evaluations of the expectation values of the observables.

We note, however, that the construction performed for $v = v^*$ can be repeated at each value of v , giving rise to potentially inequivalent constructions of the candidate Hilbert space. One can then consider a function

$$P(\psi_v | \omega_v) \equiv \int_{\mathbb{R}} \overline{\psi(v, \phi)} \omega(v, \phi) M(v, \varphi_0(\phi)) / \mu_v d\phi. \quad (4.12)$$

On each slice of constant v this function equals the inner product of the candidate Hilbert space constructed with respect to this slice. One can then ask under which condition these Hilbert spaces will be equivalent and their inner products equal. A condition necessary and sufficient for it is that $\partial P(\psi_v | \omega_v) / \partial v = 0$. The form of the unitary evolution operator (4.1a) implies however that this condition will be satisfied if and only if

$$M(v, \varphi_0) \equiv \mu_v m(\varphi_0). \quad (4.13)$$

Following this choice, our candidate Hilbert space becomes (up to a rescaling $\psi_v \mapsto \psi_v / \sqrt{m}$ on \mathbf{H}_v) the space $\mathbf{L}^2(\mathbb{R}, d\phi)$ with the standard \mathbf{L}^2 -inner product. Also, the *momentum* space is now $\mathbf{L}^2(\mathbb{R}, dp_\phi)$ with the corresponding Lebesgue measure.

Using (4.4), we can now consider an initial state

$$\psi^{\text{h}}(v^*, p_\phi) \equiv \hat{E}_{v^*0} \psi^{\text{h}}(0, p_\phi) \quad (4.14)$$

with a *real*, L^2 -normalized Gaussian

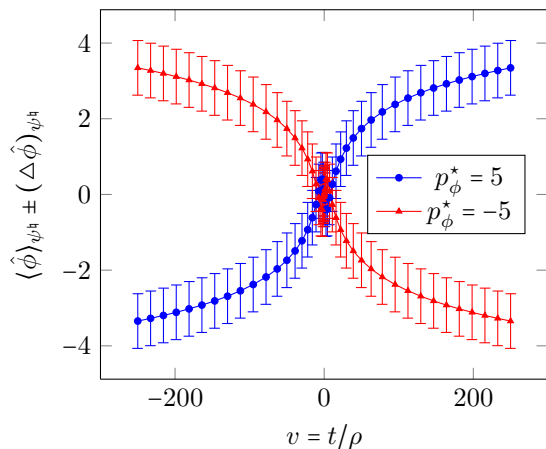
$$\psi^{\text{h}}(0, p_\phi) \equiv \sqrt{\frac{w}{\sqrt{\pi}}} e^{-w^2(p_\phi - p_\phi^*)^2/2}. \quad (4.15)$$

This class of states is “special” in the sense that the quantum evolution they undergo is semiclassical both for $v \sim v^*$ and $v \sim -v^*$ (see below). Furthermore, the states $\hat{E}_{v0} \psi^{\text{h}}(0, p_\phi)$ with unit L^2 -normalized $\psi^{\text{h}}(0, p_\phi)$ also span the solution space of the Wheeler-DeWitt equation defined by the Hamiltonian constraint in (2.4) and with $V_C a^3 = T$ identified as clock. They are thus particularly convenient in comparing the evolution in the Schrödinger and polymer quantizations. The set of the complex conjugate of these states represents the analogous states for the clock $-V_C a^3 = T$ (see [49] and recall that we set $L = 1$).

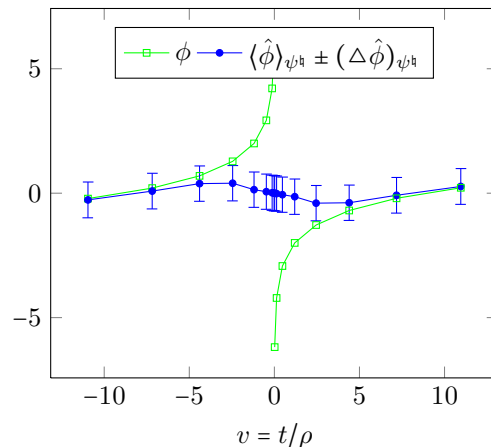
We can now try to evaluate once more the expectation value of the scalar field. For that we chose an example of a “semiclassical” Gaussian state peaked at $v^* = 250$ with $w = 1$. This choice results in a relatively small initial value of both the scalar field and its momentum fluctuations (see below). We further set $p_\phi^* = 5$ to prevent any significant portion of the Gaussian initial state from overlapping with the momentum space origin, as this is where the cosine integral function has an integrable singularity. The quantum evolution of such initial state was then calculated numerically. Fig. 1 shows the quantum trajectory corresponding to this evolution. From there, it is evident that the evolution is semiclassical for $|v| \gg 1$. In fact, the classical solution that is well approximating the quantum trajectory is characterized by

$$\phi^* = \mp \left\langle S(v^*, p_\phi) \left[\text{Ci} \left(\left| \frac{p_\phi}{v^*} \right| \right) + \ln(|v^*|) \right] \hat{1} \right\rangle_{\psi^{\text{h}}}, \quad (4.16)$$

where the overall sign “ \mp ” corresponds to $v \in \mathbb{R}^\pm$ and, where v^* is taken to be the largest plotted value of v (250 in the present case). Given the definition of the function S , we observe that $|v^*| \gg |p_\phi^*|/\pi$ is a necessary requirement for semiclassicality. As we can see, the physical state is indeed passing in a continuous manner *through* the point $v = 0$, corresponding in the classical theory to the big bang singularity [which is particularly clear from Fig. 1(a) and also from (4.3)]. This happens regardless of the sign of the initial momentum so that the quantum evolution is effectively respecting (2.16), which in the classical theory specifies the relation between the solutions for negative and positive t .



(a) Quantum evolution for two values of p_ϕ^* .



(b) Classical and quantum evolution for $p_\phi^* = 5$.

FIG. 1. An illustration of the quantum evolution of the scalar field is presented in Fig. 1(a). In Fig. 1(b) we display the $p_\phi^* = 5$ quantum evolution along with the corresponding classical evolution over a subset of the v -interval used in Fig. 1(a). The circles, triangles and squares correspond to actual measurements made using the software MATLAB and OCTAVE. The time-interval between two consecutive measurements becomes smaller as $|v|$ approaches 0. In this plot, the smallest value of $|v|$ is 0.25 but values as small as 10^{-4} have been considered with the same outcome. Namely, there is semiclassicality for $|v| \gg 1$ and the big bang singularity at $v = 0$ is resolved.

To examine more closely the issue of the singularity resolution we also analyzed the expectation values of the operator corresponding to the spacetime Ricci scalar. The quantum trajectory is presented in Fig. 2. The measurements are independent of the overall sign of p_ϕ^* , thus only the case $p_\phi^* > 0$ has been plotted. We see that the spacetime curvature remains finite for all v . This confirms the analytical result of (3.26) and, thus, implies the global boundedness of the spectrum of the Ricci scalar operator once μ_v is fixed via (3.27). One can thus conclude that the big bang singularity is resolved.

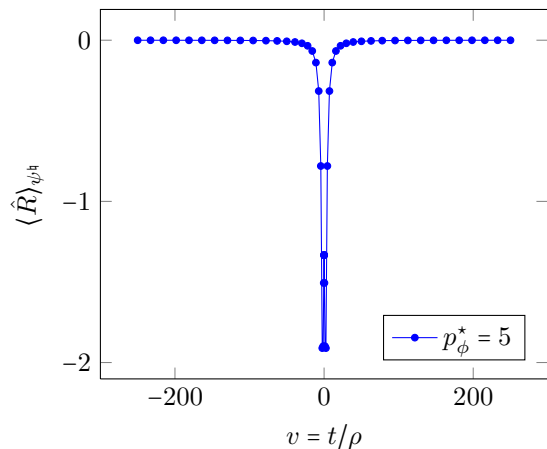


FIG. 2. Expectation values of the spacetime Ricci operator for various values of v are plotted in this figure. As in Fig. 1, measurements are taken the more often the closer $|v|$ gets to 0 and the smallest value is $|v| = 0.25$. It follows that the big bang curvature singularity is resolved and the branches $v \in \mathbb{R}^\pm$ are connected.

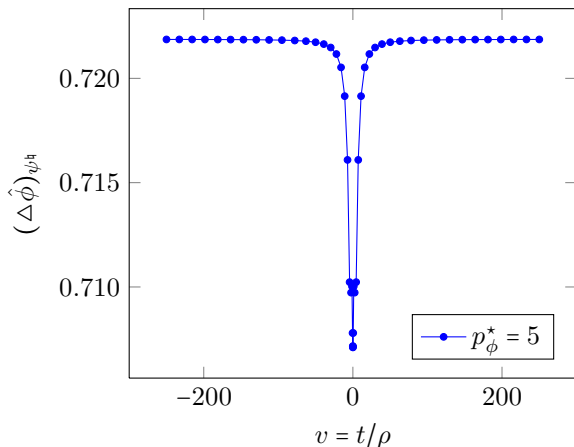


FIG. 3. In this figure an illustration of the fluctuations of the scalar field operator is presented. For $|v| \rightarrow 0$ the fluctuations increase but remain nonetheless finite.

Finally, in Figs. 3 and 4 the fluctuations of the scalar field and polymer momentum operator are depicted. They both quickly approach a constant value (for the presented example approximately equal to $2^{-1/2}$ for both quantum fluctuations) as $|v|$ increases, which is the value expected for a Gaussian with $w = 1$. This confirms the semiclassical nature of the state for $|v| \sim v^*$. What is interesting in the near-singularity region is the fact that the fluctuations of the scalar field operator are in fact decreasing for $|v| \rightarrow 0$. This may happen because the state gets “squeezed” towards the origin in order to “fit through” the point $v = 0$. This however requires a more detailed analysis of the nature of the state there, which may be the subject of a subsequent investigation.

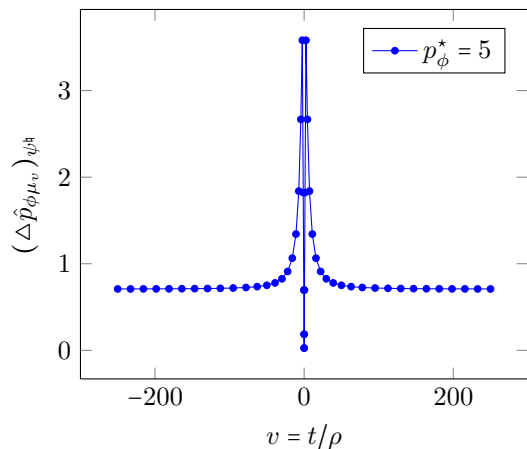


FIG. 4. This figure provides an illustration of the fluctuations of the polymeric momentum operator. Just like for the scalar field fluctuations in Fig. 3, the behavior for $|v| \gg 1$ indicates semiclassicality.

V. DISCUSSION

In this paper we investigated the quantum dynamics of the isotropic and flat FLRW universe of infinite extent and sourced by a minimally coupled massless scalar field. Our focus was on the modifications to the dynamics following from the polymeric nature of the matter and in particular the issue of the singularity resolution. To single out these effects we implemented one of two possible loop quantization schemes of the scalar field. This scheme is the analog of the one used so far in LQC to quantize the geometry degrees of freedom. Unlike in the most previous works in LQC, instead of implementing the Dirac program to solve the Hamiltonian constraint, we performed a complete deparametrization of the system by choosing a scale factor dependent time variable. As a result, the physical evolution is described by a free Hamiltonian. The quantization of such a deparametrized system is implicitly equivalent to selecting the Schrödinger quantization for the geometry when applying the Dirac program. Therefore, the effects of the geometry discreteness are not featured in our model.

In the process of constructing the correct description of the quantum system we encountered several obstacles:

First, the noncompactness of the universe’s spatial slices forced us to introduce an infrared regulator. The necessary consistency condition that the theory has to admit a well defined and nontrivial regulator removal limit restricted then the Hamiltonian to particular form, which happened to be explicitly time-dependent.

Second, the Hilbert space to which the physical states belong occurred to be non-separable. This is a standard (and treatable) problem in LQC. Here however, the explicit time dependence of the Hamiltonian prevented us from implementing the known technique of subdividing the (too big) Hilbert space onto separable superselection sectors. An idea to naively proceed with determining

the dynamics on that space led to a model significantly disagreeing with GR predictions at the low curvature limit. Indeed the quantum evolution of the scalar field was frozen.

To cure this defect we performed a specific construction of the separable Hilbert space out of the nonseparable one, taking as the guideline the relation between Hilbert spaces corresponding to the models with different choices of the lapse function in LQC in the presence of a positive cosmological constant. As a result, we were able to construct a certain integral Hilbert space equipped with continuous rather than discrete (as usually in LQC) inner product.

Such a construction of the Hilbert space was then used to investigate the dynamics. To do so we selected a class of Gaussian initial states and evolved them numerically. The resulting quantum trajectories showed a good convergence to the classical trajectories predicted by GR at low energies. At high curvatures (small $|v|$) however we observed a significant departure from GR. Indeed, the most critical feature of the model is the existence of a unique unitary evolution operator evolving to/from the time slice $v = 0$ corresponding to the classical singularity. This and the regularity of the wave function describing the physical state allowed us to select a naturally preferred extension of the evolution, thus ensuring a *deterministic evolution through the classical singularity*. Furthermore, the quantum counterparts of the Ricci scalar, energy density, or pressure are explicitly bounded operators. In consequence, the listed quantities remain finite throughout the entire evolution, including in particular $v = 0$.

At this moment it is important to note that, *unlike in previous contributions to the literature on this model, here the quantum features responsible for singularity resolution originate from the matter rather than the geometric sector*. Therefore, the form of the singularity resolu-

tion is different than in the literature: Instead of being avoided, the surface $v = 0$ is made passable and all the standard locally measurable quantities remain finite.

Finally, let us comment on an important lesson learned from this model: the predicted dynamics depends critically on the construction of the physical Hilbert space of the model, even though the regularized form of the Hamiltonian remains the same. This implies in particular that the regularized form of the classical Hamiltonian or Hamiltonian constraint *is not sufficient* to robustly determine or even well approximate the quantum evolution. This issue is particularly critical in all the studies of models in LQC performed via the so called *effective dynamics* techniques without prior specification of the elements of the genuine quantum system.

In a further project we intend to take a closer look at the behavior of the state near the singularity. Why do the quantum fluctuations of the scalar field decrease towards the origin of the time-axis? Of interest is also the inclusion of a non-zero cosmological constant. Finally, and this is most intriguing, we would like to address the question of how the quantization procedure presented here can be combined with that of the geometric sector discussed in the LQC works [13, 17, 34, 42, 48].

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