Problems in Convex Geometry

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I, Edgardo Roldán Pensado confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

We deal with five different problems from convex geometry, each on its own chapter of this Thesis. These problems are the following.

Random copies of a convex body: We study the probability that a random copy of a convex body intersects the integer lattice in a certain way.

A conjecture by Erdős: We study the statement by Erdős "On every convex curve there exists a point P such that every circle with centre P intersects the curve in at most 2 points."

A Yao-Yao type theorem: Given a nice measure in $\mathbb{R}^d$, we show that there is a partition $\mathcal{P}$ of $\mathbb{R}^d$ into $3 \cdot 2^{d/2}$ convex pieces of equal measure such that every hyperplane avoids at least 2 elements of $\mathcal{P}$.

Line transversals: Given a family $\mathcal{F}$ of balls in $\mathbb{R}^d$ such that every three have a transversal line, we bound the blow-up factor λ needed so that $\lambda\mathcal{F}$ has a transversal line.

Longest lattice convex chains: Given a triangle with two specified vertices $v_1, v_2 \in \mathbb{Z}^2$, we bound the size of the largest lattice convex chain from v_1 to v_2.

The techniques used to tackle these problems are very diverse and include results from analysis, combinatorics, number theory and topology, as well as the use of computers.

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Chapter 1

Introduction

During the three years of my PhD I have come across several problems in the area of Convex Geometry. I have worked on them and I consider that there are five problems in which I have made non-trivial progress. I include these problems together with the advancements made in this thesis. The proof methods are diverse, results are used from analysis, geometry, topology and at some point a computer was used to analyse a large amount of cases.

Every following chapter is dedicated to one of these problems, below is a short summary of each.

Some of the work presented here has been done in collaboration with other people, I will indicate when this is the case.

Random copies of a convex body and the integer lattice

The first two problems involve the integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Let K be a fixed convex body and ρ be a randomly chosen isometry on $\mathbb{R}^d$. Bárány and Matoušek proved in [7, 8] that the probability that $\rho(K)$ does not intersect $\mathbb{Z}^d$ is at most $C/|K|$ for some $C > 0$ that depends only on the dimension d. They also showed that this result is asymptotically correct by exhibiting a family of convex bodies with a probability of intersecting $\mathbb{Z}^d$ larger than

$c/|K|$ for some constant $c > 0$.

This left several questions open, for example: What is the smallest possible value of the constant C? Which convex body gives rise to this constant? What can be said if $\rho(K) \cap \mathbb{Z}^d$ is some larger set?

We answer the first two questions in an asymptotic sense, and the third question is tackled when $\rho(K) \cap \mathbb{Z}^d$ consists of k points and when it has a given dimension. These results are published and can be found in [32].

A conjecture by Erdős

In [14], P. Erdős conjectured that for every convex body K there is a point $P \in \partial K$ such that every circle centred at P intersects ∂K in at most 2 points.

This turned out to be false, but there is still something to be done here. Given a convex body K, consider the smallest number N so that there is a point $P \in \partial K$ with the property that every circle centred at P intersects ∂K in at most N points.

There is no known global upper bound for N. We show that no convex body has $N = \infty$ and that there are convex bodies for which $N \geq 6$. Furthermore, we prove that a typical point P in the boundary of a typical convex body K (in the Baire sense) satisfies that every circle centred at P intersects ∂K in an infinite amount of points. This is joint work with I. Bárány.

An extension of the Yao & Yao theorem

A theorem by Yao & Yao (see [40, 25]) states that given a nice measure in $\mathbb{R}^d$ there exists a convex partition of $\mathbb{R}^d$ into 2^d parts of equal measure such that every hyperplane avoids one of these parts. This is useful in computation for developing fast algorithms for some geometric queries.

Let k be a positive integer. We work on the problem of determining the smallest n so that the following holds: For any nice measure in $\mathbb{R}^d$, there is a partition of $\mathbb{R}^d$ into n convex parts of equal measure such that every hyperplane avoids at least k elements of the partition.

We apply our results in a problem regarding separation of points and hyperplanes.

This joint work P. Soberón and can be found in [33]. It was presented in EuroCG 2012.

Line transversals

This is joint work with J. Jerónimo-Castro. The results here also appear in [21].

Given a family $\mathcal{F} = \{K_1, \ldots, K_n\}$ of convex bodies in $\mathbb{R}^d$, we say that they have property T if there is a line intersecting every member of $\mathcal{F}$. Also, if k is a positive integer we say that $\mathcal{F}$ has property $T(k)$ if every subset of $\mathcal{F}$ with k elements has property T.

Assuming some conditions on the family $\mathcal{F}$, we want to determine the minimum $\lambda > 0$ such that the family $\lambda\mathcal{F} = \{\lambda K_1, \ldots, \lambda K_n\}$ satisfies property T.

In [20, 22] bounds for λ are given when $\mathcal{F}$ consists of translates of a convex body $K \subset \mathbb{R}^2$ and has property $T(3)$ or $T(4)$. Special interest has been given when K is the unit ball in $\mathbb{R}^2$.

Now we consider the case when $\mathcal{F}$ consists of closed balls in $\mathbb{R}^d$ with property $T(k)$ and write $\lambda_d(k) = \lambda$. We prove that $\lambda_d(d+1) \le 4$, $\lambda_2(4) \le 2\sqrt{2}$ and $\lambda_2(3) < 2.88$.

Longest lattice convex chains

Let $\mathbb{Z}_t = \frac{1}{t}\mathbb{Z}^2 \subset \mathbb{R}^2$. In [5, 6] Bárány showed that if $K \subset \mathbb{R}^2$ is a convex body then, as $t \to \infty$, almost all convex polytopes contained in K with vertices in $\mathbb{Z}_t$ are close in Hausdorff metric to a certain convex body $Q \subset K$.

It is also proved that Q is characterised as the convex body contained in K with maximal affine perimeter. Let $Q_t \subset K$ be a $\mathbb{Z}_t$-lattice polytope with maximum amount of vertexes. Bárány and Prodromou showed in [9] that, with respect to the Hausdorff metric, the sequence of convex bodies Q_t converges to the convex body Q. Furthermore, they showed that if t is large, the amount of vertexes Q_t has is essentially equal to

$$\frac{3t^{2/3}}{(2\pi)^{2/3}} A(K),$$

where $A(K)$ is the supremum of the affine perimeter of all convex sets contained in K.

To prove these theorems it is necessary to find large $\mathbb{Z}_t$-lattice convex chains contained in a given triangle. Let $a, b \in \mathbb{R}^2$ be the vertexes of a triangle with $a \in \mathbb{Z}^2$, the parabola of Oab is the parabola that passes through the origin O and a such that Ob and ab are tangent to the parabola at O and a, respectively. If t is large enough then the longest convex $\mathbb{Z}_t$-lattice chain $O = p_0, p_1, \ldots, p_n = a$ contained in the triangle Oab consists of points close to this parabola and $n \leq c|Oab|^{1/3}$ for some constant c.

This gives raise to a new question: Given a convex $\mathbb{Z}^2$-lattice chain with $n + 1$ vertexes $p_0, \ldots, p_n$, there is a unique minimal area triangle that has p_0 and p_n as vertexes and contains $\{p_0, \ldots, p_n\}$. If we fix the area A of such a triangle, how large can n be?

Together with I. Bárány, we answer this question assuming A is large enough.

Chapter 2

Random copies of a convex body

This chapter contains research which has been published in [32].

Let $\mathbb{Z}^d \subset \mathbb{R}^d$ denote the d-dimensional lattice of integer points. An isometry ρ of $\mathbb{R}^d$ is a pair (r,t) given by a rotation $r \in SO(d)$ and a translation vector $t \in \mathbb{R}^d$. For a given $K \subset \mathbb{R}^d$ and $\rho = (r,t)$, we write $\rho(K) = r(K) + t$.

In this chapter we are interested in properties of the set $\rho(K) \cap \mathbb{Z}^d$. If $t' \in \mathbb{Z}^d$, then $(\rho(K)+t') \cap \mathbb{Z}^d$ and $\rho(K) \cap \mathbb{Z}^d$ are essentially the same. Because of this it makes sense to consider exactly one vector in each equivalence class of $\mathbb{R}^d/\mathbb{Z}^d$. One way to do this is to consider only the vectors in the unit cube $[0,1)^d$, however it is useful not to fix this set.

The set $SO(d) \times (\mathbb{R}^d/\mathbb{Z}^d)$ has a natural probability measure given by the product of the normalised Haar measures in $SO(d)$ and $\mathbb{R}^d/\mathbb{Z}^d$. To any element $(r,\bar{t}) \in SO(d) \times (\mathbb{R}^d/\mathbb{Z}^d)$ we can assign an isometry $\rho = (r,t)$ such that $t \in \bar{t}$. In this way we can assign a probability measure to a subset of the isometries of $\mathbb{R}^d$. The probabilities we deal with in this paper are independent of the assignment of t.

To avoid unnecessary long statements, in this chapter we use the letter C to represent an appropriate constant each time it is used. Thus, C need not represent the same value each time it appears. If C is dependent on other variables, they will appear as subscripts, for example, C_d represent a suitable constant dependent on d.

In [7, 8] it is shown that

$$\text{Prob}\left\{\rho(K) \cap \mathbb{Z}^d = \emptyset\right\} \leq \frac{C_d}{|K|} \tag{2.1}$$

for all convex bodies $K \subset \mathbb{R}^d$, and that

$$\text{Prob}\left\{\rho(K) \cap \mathbb{Z}^d = \emptyset\right\} \geq \frac{C_d}{|K|}$$

for all convex bodies $K \subset \mathbb{R}^d$ with small enough width.

A natural question now is to compute the probability that $\rho(K)$ contains a certain number of integer lattice points. This seems to be hard in dimensions $d \geq 3$, but for $d = 2$ we obtained the following theorem.

Theorem 2.1. *For every positive integer n and every convex body $K \subset \mathbb{R}^2$ with $|K| \geq Cn^{\frac{3}{2}}$,*

$$\mathit{Prob}\left\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\right\} \leq C\frac{n^2}{|K|^2}.$$

For every rectangle K with small enough width and $|K| > n$, we have

$$\mathit{Prob}\left\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\right\} \geq C\frac{1}{|K|^2}.$$

It would be interesting to know how large $|K|^2\text{Prob}\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\}$ can be for each n. We must be careful here if n is not fixed, for example, there are families of convex bodies such that this probability is larger than $\frac{n^{0.3}}{|K|^{1.5}}$ for infinitely many values of n. Without the hypothesis on the size of $|K|$ in the first part of Theorem 2.1, then a bound of $\frac{n^3}{|K|^2}$ can be obtained using the same proof method.

The next theorem tells us that the upper bound cannot be lowered too much.

Theorem 2.2. *For every $\varepsilon > 0$, $c > 0$ and $N > 0$ there is a rectangle K and $n > N$ such that*

$$Prob\left\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\right\} > c\frac{n^{1-\varepsilon}}{|K|^2}.$$

Returning to the general case, we may also consider the probability that $\rho(K) \cap \mathbb{Z}^d$ has dimension k.

Theorem 2.3. *Let $k < d$ be a non-negative integer, then*

$$Prob\left\{\dim\left(\rho(K) \cap \mathbb{Z}^d\right) = k\right\} \leq C_d\frac{1}{|K|}.$$

If $k = d - 1$ then this is best possible.

We believe that this bound is not best possible if $k \neq d-1$, this is indeed the case when $d = 2$ as Theorem 2.1 implies.

Finally, we return to the probability that $\rho(K)$ does not contain any lattice points. If $d = 2$ we can obtain a near-optimal bound.

Theorem 2.4. *For every $\varepsilon > 0$, there exist constants k_0 and w_0 such that if K is a planar convex body with $k_0 < |K|$ and width $w < w_0$, then*

$$Prob\left\{\rho(K) \cap \mathbb{Z}^2 = \emptyset\right\} < \frac{1}{4|K|}(1 + \varepsilon).$$

Furthermore, the constant $\frac{1}{4}$ is best possible.

We conjecture that $\frac{1}{4}$ is actually the best possible value the constant C_2 from (2.1) can have for all bodies K with large enough area.

Theorem 2.4 can also be stated as

$$\lim_{\substack{w\to 0\\ |K|\to\infty}} \sup_{K} \left\{|K| \operatorname{Prob}\left\{\rho(K)\cap\mathbb{Z}^2=\emptyset\right\}\right\} = \frac{1}{4},$$

where the supremum is taken over all convex bodies K with width w. If we take the supremum over the family of ellipses we still have this equality, but if we take it over the family of rectangles we obtain $\frac{2}{\pi^2}$.

From the proof we can see that the main reason for this is the Blaschke-Santaló inequality. If we take a family $\mathcal{F}$ of convex bodies that are not similar to the ellipse (in the sense that $\left|K\right|\left|K^P\right| < \pi^2-\varepsilon$ for all $K\in\mathcal{F}$), then by taking the supremum over $\mathcal{F}$ the result is smaller than $\frac{1}{4}$ (see section 2 for statements and definitions).

Bárány asked in [7] for which convex body of fixed volume is this probability largest. In dimension 2, the proof of the last theorem shows that thin ellipses have relatively high probability of not containing lattice points. It is likely that the body that maximises this probability does not exist but with a thin enough ellipse we can get arbitrarily close.

2.1 Preliminaries

The lattice-width of a convex body K is defined as

$$W(K) = \min_{\substack{z\in\mathbb{Z}^d\\ z\neq 0}} \max\{z\cdot(x-y) : x,y\in K\}.$$

Any vector $z \in \mathbb{Z}^d$ that minimises this quantity is called a lattice-width vector of K. Note that the lattice-width vectors come in pairs, if z is one then so is $-z$.

The set of primitive vectors $\mathbb{P}^d$ is the set of vectors in $\mathbb{Z}^d$ such $\frac{1}{m}z$ is not an integer-lattice point for any positive integer m. This set is also referred to as the set of lattice points visible from the origin. If z is a lattice-width vector of K then z must be a primitive vector.

Given $z \in \mathbb{P}^d$, the z-lattice hyperplanes are the hyperplanes perpendicular to z that pass through some integer-lattice point. The distance between two consecutive z-lattice hyperplanes is $\frac{1}{|z|}$. If z is a lattice-width vector of K, then the number of z-lattice hyperplanes intersecting K is at most $W(K) + 1$.

An essential tool for our results is a generalisation of the Flatness Theorem which can be found in [23].

Theorem (Generalised Flatness Theorem)**.** *If $L \subset \mathbb{R}^d$ is a d-dimensional lattice and $K \subset \mathbb{R}^d$ is a convex body with $\#(K \cap \mathbb{Z}^d) \leq n$, then there exists a nonzero element z of the dual L^* such that* $\max\{z \cdot (x - y) : x, y \in K\} \leq C(n+1)^{\frac{1}{d}} d^2$.

This is stronger than the usual Flatness Theorem (see [24]) which only deals with the case $n = 0$.

If $L = \mathbb{Z}^d$ then what the Generalised Flatness Theorem states is that

$$W(K) \leq C(n+1)^{\frac{1}{d}} d^2.$$

In the case $d = 2$, if K contains no lattice points and its area is larger than some fixed constant c, then K can only have one pair of lattice-width vectors. Otherwise there is a parallelogram containing K with sides perpendicular to two linearly independent lattice-width vectors. The Flatness Theorem implies that the area of this parallelogram is bounded by some constant c.

If $d > 2$ this is not always the case. However, if the projection of K onto any 2-dimensional plane has area larger than some constant c_d, then K can only have one pair of lattice-width vectors.

If $d = 2$, then we can strengthen the conclusion of the Generalised Flatness Theorem by making $|K|$ large.

Lemma 2.5. *Let n be a non-negative integer. There exists a constant c such that if $K \subset \mathbb{R}^2$ is a convex body that contains at most n integer lattice points and $|K| > c(n+1)^{\frac{3}{2}}$, then there is a unique pair $\{z, -z\} \subset \mathbb{P}$ such that K intersects at most 2 z-lattice lines.*

Proof. By the Generalised Flatness Theorem there exists a constant c' and $z \in \mathbb{P}^2$ such that K intersects at most $c'\sqrt{n+1}$ z-lines. Let s be a section of K perpendicular to z with largest length. Note that $|K| \leq |s|\frac{c'\sqrt{n+1}}{|z|}$. Therefore if $|K| > c(n+1)^{\frac{3}{2}} > \frac{c'\sqrt{n+1}(2n+4)}{|z|}$ for some large enough c, then $|s| > 2n+4$. From here it follows that if K intersects more than one z-lattice line on the same side of s, then K must contain at least $n+1$ lattice points in some z-lattice line. We conclude that K intersects at most two z-lattice lines.

For the uniqueness, if there are two independent lattice-width directions, then K is contained in a parallelogram of area 9. □

In the proofs of Lemma 4.4 in [8] and Theorem 1.1 in [7] what is basically proved is the following.

Lemma 2.6. *Let $W_0 > 0$ and $K \subset \mathbb{R}^d$ be a convex body. Then*

$$Prob\left\{W\left(\rho(K) \cap \mathbb{Z}^d\right) < W_0\right\} \leq C_{W_0,d}\frac{1}{|K|}.$$

We also need some properties of the distribution of $\mathbb{P}^2$ in $\mathbb{Z}^2$, several of these are described in [18]. The following lemma is well known, but we could not find a reference for it, so we include a sketch of the proof.

Lemma 2.7. *If $m \geq -1$ is an integer, then*

$$\sum_{\substack{z\in\mathbb{P}^2\\|z|\leq R}} |z|^m = \frac{12\,R^{m+2}}{\pi(m+2)}\left(1+O\left(\frac{\log(R)}{R}\right)\right).$$

Here the implicit constant depends on m.

Proof. Let μ be the Möbius function. If $z\in\mathbb{Z}^2$ and d is a positive integer, we write $d|z$ if d divides both coordinates of z. All the implicit constants in this proof will depend on m. Using standard arguments we have

$$\begin{aligned}\sum_{\substack{z\in\mathbb{P}^2\\|z|\leq R}} |z|^m &= \sum_{\substack{z\in\mathbb{P}^2\\|z|\leq R}}\sum_{d|z}\mu(d)|z|^m = \sum_{d=1}^{R}\sum_{\substack{w\in\mathbb{Z}^2\\|dw|\leq R}}\mu(d)|dw|^m\\ &= \sum_{d=1}^{R} d^m\mu(d)\sum_{\substack{w\in\mathbb{Z}^2\\|w|\leq R/d}}|w|^m. \qquad (1)\end{aligned}$$

To deal with the last term, let $B(r)\subset\mathbb{R}^2$ be the disc centred at 0 with radius r and set $c=\sqrt{2}/2$. Then

$$\begin{aligned}\sum_{\substack{z\in\mathbb{Z}^2\\|z|\leq r}} |z|^m &= \int_{B(r)}|x|^m dx + O\left(\int_{B(r+c)\setminus B(r-c)}|x|^m dx\right)\\ &= \frac{2\pi r^{m+2}}{(m+2)} + O\left(r^{m+1}\right).\end{aligned}$$

Using this in (1) and the well-known identity

$$\sum_{d=1}^{R}\frac{\mu(d)}{d^2} = \frac{6}{\pi^2}+O\left(\frac{1}{R}\right)$$

we obtain

$$\begin{aligned}\sum_{\substack{z\in\mathbb{P}^2\\|z|\le R}} |z|^m =& \frac{2\pi R^{m+2}}{m+2}\sum_{d=1}^{R}\left(\frac{\mu(d)}{d^2}+\frac{1}{d}O\left(\frac{1}{R}\right)\right)\\ =& \frac{12R^{m+2}}{\pi(m+2)}\left(1+O\left(\frac{\log(R)}{R}\right)\right).\end{aligned}$$

□

The polar reciprocal of a convex body K relative to a point P is defined as

$$K^P = \{y : x\cdot(y-P)\le 1 \text{ for all } x\in K\}.$$

The Santaló point of a convex body K is the point $P\in\mathbb{R}^d$ that minimises the volume $\left|K^P\right|$. If K is centrally symmetric then its centre coincides with its Santaló point. The properties of K^P have been widely studied. We need the following theorem which can be found in [29].

Theorem (Blaschke-Santaló inequality)**.** *If K is a convex body in $\mathbb{R}^2$ with Santaló point P, then*

$$\left|K\right|\left|K^P\right| \le \pi^2,$$

with equality if and only if K is an ellipse.

2.2 Proof method

Let $K\subset\mathbb{R}^d$ be a convex body and let $\mathcal{P}$ be a property of K which is invariant under translations by vectors in $\mathbb{Z}^d$. Most proofs in this chapter involve estimating some probability of the form

$$p = \text{Prob}\left\{\rho(K) \text{ has property } \mathcal{P}\right\}.$$

For example, Theorem 2.1 gives bounds for p when $\mathcal{P}$ is the property "intersects $\mathbb{Z}^2$ in n points".

We now describe a general method that will be used several times below to bound p.

Assume that for every isometry ρ, $\rho(K)$ has only one pair of lattice-width vectors of $\rho(K)$. This is done by considering K with large enough area when $d = 2$.

For a fixed $z \in \mathbb{P}^d$, let $\mathcal{P}_z$ be the property "has property $\mathcal{P}$ and has z as a lattice-width vector" and define

$$p_z = \text{Prob}\left\{\rho(K) \text{ has property } \mathcal{P}_z\right\}.$$

In the cases we consider, $p_z = 0$ if $\frac{1}{|z|}$ is small compared to the width of K. To compute p_z more easily, we fix a starting isometry ρ_z and define the set

$$\mathcal{A} = \{\alpha \in SO(d) : \exists t \text{ such that } \alpha(\rho_z(K)) + t \text{ satisfies } \mathcal{P}_z\}.$$

If $d = 2$, we also think of α as a angle.

For every $\alpha \in \mathcal{A}$ we define

$$T(\alpha) = \{t \in \mathbb{R}^d/\mathbb{Z}^d : \alpha(\rho_z(K)) + t \text{ satisfies } \mathcal{P}_z\}.$$

Then we have

$$p_z = \int_{\mathcal{A}} |T(\alpha)| d\alpha.$$

Since every $\rho(K)$ has exactly two lattice-width vectors (z and $-z$) and the property $\mathcal{P}_z$ is identical to $\mathcal{P}_{-z}$, we may compute p by adding p_z over all primitive vectors and dividing the result by 2. We do this to obtain

$$\text{Prob}\,\{\rho(K) \text{ has property } \mathcal{P}\} = \frac{1}{2}\sum_{z\in\mathbb{P}^d} p_z.$$

In the proofs where this method is used we give bounds for $|\mathcal{A}|$ and $|T(\alpha)|$.

2.3 Proof of Theorems 2.1 and 2.2

In this section we will only deal with bodies in $\mathbb{R}^2$, so we refer to the z-lattice hyperplanes as z-lattice lines.

First we prove Theorem 2.1, but the proof we give also gives us another useful fact.

Lemma 2.8. *Let $K \subset \mathbb{R}^2$ be a convex body. For every isometry ρ fix a lattice-width vector z_ρ of $\rho(K)$. Then*

$$\mathit{Prob}\{\rho(K)\cap\mathbb{Z}^2 = \emptyset \text{ and } \rho(K) \text{ intersects a } z_\rho\text{-lattice line}\} \le C\frac{1}{|K|^2}.$$

Note that it is enough to prove this for K with large enough area.

Proof of Lemma 2.8 and the first part of Theorem 2.1. Let $n \ge 0$ be an integer. By Lemma 2.5 there is a $z \in \mathbb{P}^2$ for every isometry ρ such that $\rho(K)$ intersects at most 2 z-lines.

Let R be a rectangle containing K with smallest possible width and such that all of its sides touch K. Let w and l be the lengths of the sides of R with $w \le l$. Then we have that $\frac{1}{2}wl \le |K| \le wl$.

Now we use the method described in Section 2.2 with property $\mathcal{P}$ being "intersects a z_ρ-lattice line and contains at most n of its lattice points". If

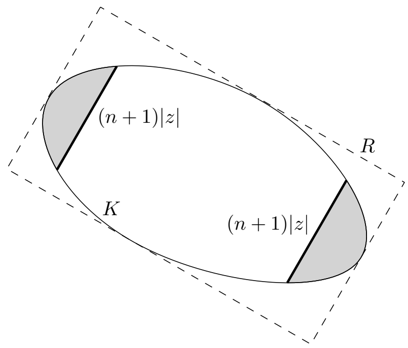

Figure 2.1: The caps of $\rho(K)$.

$n = 0$ this is exactly what we need for Lemma 2.8 and if $n \geq 1$ it is a weaker condition than the one needed in Theorem 2.1.

Fix $z \in \mathbb{P}^2$ and choose ρ_z such that the long side of $\rho_z(R)$ is perpendicular to z. We may assume that $w < \frac{3}{|z|}$, otherwise $\rho_z(R)$ intersects more than one z_ρ-lattice line and $p_z = 0$. If $\alpha \in \mathcal{A}$, then

$$|\sin(\alpha)| \leq \frac{3}{\sqrt{l^2 + w^2}|z|} < \frac{3}{l|z|}.$$

Therefore, we can think of $\mathcal{A}$ as a subset of

$$\left[-\frac{3\pi}{2l|z|}, \frac{3\pi}{2l|z|}\right] \cup \left[\pi - \frac{3\pi}{2l|z|}, \pi + \frac{3\pi}{2l|z|}\right].$$

Now fix $\alpha \in \mathcal{A}$ and consider the chords of $\alpha(\rho_z(K))$ perpendicular to z. Let s be the length of the longest of these chords, then we have $|K| \leq \frac{3}{|z|}s$. Since the area of K is large, this implies $s \geq \frac{|K||z|}{3} > (n+1)|z|$.

To measure $T(\alpha)$ if $n > 0$, it is easier to think of $\alpha(\rho_z(K))$ as being fixed and translating $\mathbb{Z}^2$. We can bound this by the area of the region where O can be translated to. This region must be contained in the union of the caps of $\rho(K)$ cut off by chords perpendicular to z of length $(n+1)|z|$ (see Figure 2.2). If O is outside of these caps, then the chord perpendicular to

z through O would have length larger that $(n+1)|z|$ and therefore would contain at least $n+1$ points. If $n=0$ the same holds by a similar argument.

Lemma 2.9. *The area of any of these caps is at most* $(n+1)^2|z|^2\left(\alpha+\frac{l}{w}\alpha^2\right)$.

This lemma is proved below. With this bound for $|T(\alpha)|$ we have

$$p_z \le Cn^2\int_0^{\frac{\pi}{2l|z|}}\left(|z|^2\alpha^2\frac{l}{w}+|z|^2\alpha\right)d\alpha \le Cn^2\left(\frac{1}{|z|wl^2}+\frac{1}{l^2}\right).$$

Summing over $z\in\mathbb{P}^2$ with $|z|\le\frac{3}{w}$ and using Lemma 2.7 we obtain

$$\operatorname{Prob}\left\{\#(\rho(K)\cap\mathbb{Z}^2)=n\right\}\le Cn^2\sum_{\substack{z\in\mathbb{P}^2\\ z\le\frac{3}{w}}}\left(\frac{1}{|z|wl^2}+\frac{1}{l^2}\right)\le C\frac{n^2}{w^2l^2}\le C\frac{n^2}{|K|^2}.$$

$\square$

Proof of Lemma 2.9. Let D be one of the caps and let A be the point of D farthest away from the line determined by the chord of D perpendicular to z with length $(n+1)|z|$. Set h as the distance between A and this line.

The convexity of K implies that any chord perpendicular to z inside the cap has length at most $(n+1)|z|$, therefore the area of the cap is at most $(n+1)|z|h$. It only remains to bound h.

Choose new coordinates so that the vertices of $\rho(R)$ are $(0,0)$, $(l,0)$, (l,w), $(0,w)$, and $z=|z|(\sin(\alpha),\cos(\alpha))$. Let $A=(a,b)$ (see Figure 2.2). For simplicity, we write $\mathsf{s}=\sin(\alpha)$ and $\mathsf{c}=\cos(\alpha)$.

The line generated by z is the given by the equation $\mathsf{s}x+\mathsf{c}y=0$. Now consider lines perpendicular to z passing through A and $(l,0)$. The distance between these lines is $\mathsf{s}l-\mathsf{s}a-\mathsf{c}b$. Note that the line through A must separate the origin from the points $(l,0)$ and $(0,w)$.

Let P be the point in the line joining A and (l,w) such that the vector

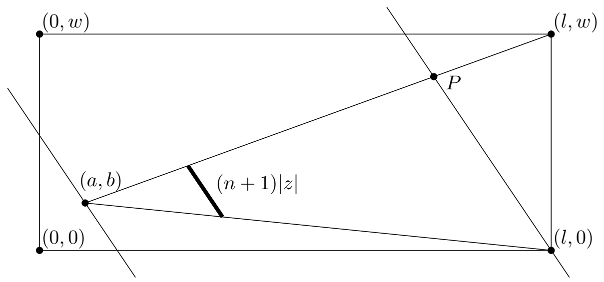

Figure 2.2: Figure for Lemma 2.9.

$Q = (l,0) - P$ is perpendicular to z. By measuring the area of the triangle with vertices $(l,0)$, (l,w) and (a,b) in two different ways we obtain $\frac{1}{2}|Q|(l-a, w-b)\cdot(\mathsf{s},\mathsf{c}) = \frac{1}{2}w(l-a)$. Therefore

$$|Q| = \frac{w(l-a)}{\mathsf{s}l + \mathsf{c}w - \mathsf{s}a - \mathsf{c}b}.$$

The convexity of K implies that h must be smaller than the distance between A and the marked line in the picture. This line is the chord of the angle $\angle(l,w)(a,b)(l,0)$ with length $(n+1)|z|$ perpendicular to z. This distance can be computed using similarity to obtain the following:

$$\begin{aligned}
h \leq & (\mathsf{s}l - \mathsf{s}a - \mathsf{c}b)\frac{(n+1)|z|}{|Q|} \\
= & (n+1)|z|\frac{(\mathsf{s}l - \mathsf{s}a - \mathsf{c}b)(\mathsf{s}l + \mathsf{c}w - \mathsf{s}a - \mathsf{c}b)}{w(l-a)} \\
\leq & (n+1)|z|\frac{(\mathsf{s}l - \mathsf{s}a)(\mathsf{s}l + \mathsf{c}w)}{w(l-a)} < (n+1)|z|\frac{\alpha(\alpha l + w)}{w}.
\end{aligned}$$

$\square$

Proof of the second part of Theorem 2.1. Let $z \in \mathbb{P}^2$ be a vector with $|z| < \frac{1}{w}$.

Choose ρ_z such that the width side of $\rho_z(K)$ is parallel to z. Let α be an angle such that $0 < \sin(\alpha) < \frac{1}{|z|l}$. We can rotate $\rho_z(K)$ by this angle and translate so that it does not touch any z-lattice line, therefore $\alpha \in \mathcal{A}$.

By thinking of $\alpha(\rho_z(K))$ as fixed and translating $\mathbb{Z}^2$, it is easy to see that $T(\alpha)$ contains a rectangle with sides parallel to those of K with lengths $|z|\sin(\alpha)$ and $|z|\cos(\alpha)$. This gives

$$p_z \geq \int_0^{\frac{1}{|z|l}} |z|^2 \sin(\alpha)\cos(\alpha) d\alpha \geq C\frac{1}{l^2}.$$

Finally, we sum over $|z| < \frac{1}{w}$ and use Lemma 2.7 to obtain

$$\text{Prob}\left\{\#\left(\rho(K)\cap\mathbb{Z}^2\right) = n\right\} \geq C \sum_{|z|<\frac{1}{w}} \frac{1}{l^2} \geq C\frac{1}{|K|^2}.$$

$\square$

Proof of Theorem 2.2. It is well known that if $t \in [0,1]^2$ is chosen with uniform probability, then the expected value of the number of integer lattice points in $K + t$ is $|K|$. This implies that the expected value of the number of integer lattice points in a random isometry of K is also $|K|$.

We may assume that c is large compared to $N^{1+\varepsilon}$, then Theorem 2.1 implies

$$\text{Prob}\left\{\#\left(\rho(K)\cap\mathbb{Z}^2\right) = n\right\} \leq c\frac{n^{1-\varepsilon}}{|K|^2}$$

for every convex body K and $n \leq N$. For the sake of contradiction, assume that this is also true for all $n > N$.

Let $l > 0$ be large and K be a rectangle with side lengths l and $l^{-\varepsilon/6}$. Fix an isometry ρ. Since $l^{-\varepsilon/6} < \frac{1}{\sqrt{2}}$ then every vertical chord or every horizontal chord of $\rho(K)$ has length less than 1. Assume this happens in the vertical direction, then clearly $\rho(K)$ can contain at most one lattice point in each vertical lattice line. Since there can be at most $l+1$ such lines intersecting $\rho(K)$, $\text{Prob}\left\{\#\left(\rho(K)\cap\mathbb{Z}^2\right) = n\right\} = 0$ for every $n > l+1$. We then have

$$
\begin{aligned}
l^{3-\frac{\varepsilon}{2}} = |K|^3 =& |K|^2 \sum_{n=1}^{\infty} n \mathrm{Prob}\left\{\#\left(\rho(K) \cap \mathbb{Z}^2\right) = n\right\} \\
\leq& C \sum_{n=1}^{\lfloor l+1 \rfloor} n^{2-\varepsilon} \leq C l^{3-\varepsilon},
\end{aligned}
$$

a contradiction. □

2.4 Proof of Theorem 2.3

Proof of the first part of Theorem 2.3. Let $K \subset \mathbb{R}^d$ be a convex body such that $\dim(K \cap \mathbb{Z}^d) = k$. There exists a lattice vector $z \in \mathbb{P}^d$ perpendicular to the affine space generated by $K \cap \mathbb{Z}^d$. Consider the family $\{H_n : n \in \mathbb{Z}\}$ of z-lattice hyperplanes ordered in the natural way such that all the points of $\mathbb{Z}^d \cap K$ are in H_0. Let

$$
L = \bigcup_{n \text{ odd}} \left(\mathbb{Z}^d \cap H_n\right).
$$

The set L is the translate of a lattice of determinant 2 and K does not intersect any point of L. By using the Flatness Theorem it can be shown that there is a constant W smaller than twice of the implicit constant in the Flatness Theorem such that $W(K) \leq W$. We conclude by using Lemma 2.6. □

Now we prove that this bound is best possible if $k = d - 1$.

Proof of the second part of Theorem 2.3. Given $r, w > 0$ such that w is small but rw is large, let K be the cylinder $B^{d-1}(r, 0) \times [-\frac{w}{2}, \frac{w}{2}] \subset \mathbb{R}^d$, where $B^{d-1}(r, 0)$ is the closed $(d-1)$-dimensional ball with radius r and centre 0. The symmetry of the cylinder is useful here because measuring on

$SO(d)$ can be reduced to measuring on the sphere S^{d-1}. The natural way to do this is by identifying an element $\lambda \in S^{d-1}$ with the elements of $SO(d)$ that send K to a cylinder with its axis parallel to λ.

Since rw is large, the projection of K onto any 2-dimensional plane has large area. Therefore, if K contains no lattice points, it has a unique pair of lattice-width vectors. For every $z \in \mathbb{P}^d$, we define $L_z \subset \mathbb{Z}^d$ as the $d-1$ dimensional lattice of vectors in $\mathbb{Z}^d$ perpendicular to z. The determinant of the lattice L_z is $|z|$.

Assume that the first coordinate of $z = (z_1, \ldots, z_d)$ is not zero. If $\{e_i\}$ is the canonical basis of $\mathbb{R}^d$, then the family $\{z_i e_1 - z_1 e_i\}_{i=2}^d \subset L_z$ is linearly independent and all its elements have norm smaller than $|z|$. Therefore we can find a basis $\mathcal{B}$ of L_z such that all its elements have norm smaller than $|z|$.

Let $\lambda_z = \frac{z}{|z|} \in S^{d-1}$. If $r > |z|$ and $w < \frac{1}{|z|}$, then $\lambda_z(K) \cap \mathbb{Z}^2$ has dimension $d-1$. Note that if $|K|$ is large enough then $w < \frac{1}{|z|}$ implies $r > |z|$.

Let α be an angle such that $\sin(\alpha) = \frac{1}{2|z|r}$. Consider the set $\mathcal{A} \subset S^{d-1}$ of elements that form an angle smaller than α from λ_z, then the $(d-1)$-dimensional measure of $\mathcal{A}$ is

$$|\mathcal{A}| \geq C_d \left(\frac{1}{|z|r} \right)^{d-1}.$$

At this point we choose a representative for every element of $\mathbb{R}^d/\mathbb{Z}^d$. The set of representatives is the parallelotope determined by $\mathcal{B}$ and $\frac{z}{|z|^2}$.

By considering projections of K onto 2-dimensional planes that contain $\lambda \in \mathcal{A}$ it can be easily seen that if $w < \frac{1}{|z|}$, then $\lambda(K)$ can be translated by a set of vectors with volume at least $C_d|z|w\cos(\alpha) \geq C_d|z|w$ so that the resulting body intersects $\mathbb{Z}^d$ in a set of dimension $d-1$.

Using the method described in section 2.2 with $\mathcal{P}$ being "intersects $\mathbb{Z}^d$ in a set of dimension $d-1$", we obtain

$$\begin{aligned} p = \sum_{|z|<\frac{1}{w}} p_z &\geq C_d \sum_{|z|<\frac{1}{w}} \frac{w}{r^{d-1}|z|^{d-2}} \\ &\geq C_d \frac{1}{r^{d-1}w} \geq C_d \frac{1}{|K|}. \end{aligned}$$

□

2.5 Proof of Theorem 2.4

Proof of Theorem 2.4. First we express the probability as

$$\mathrm{Prob}\left\{\rho(K) \cap \mathbb{Z}^2 = \emptyset\right\} = p + q,$$

where

$$p = \mathrm{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset \text{ and } \rho(K) \text{ intersects no } z_\rho\text{-lattice line}\}$$

and

$$q = \mathrm{Prob}\{\rho(K) \cap \mathbb{Z}^2 = \emptyset \text{ and } \rho(K) \text{ intersects a } z_\rho\text{-lattice line}\}.$$

Lemma 2.8 states that

$$q \leq C \frac{1}{|K|^2},$$

so we only need to bound p. Once again, we do this using the method described in section 2.2.

Let w be the width of K and assume $w < 1$. Clearly $p_z > 0$ if and only if $|z| < \frac{1}{w}$.

For a given z, let ρ_z be a rotation such that the width of $\rho_z(K)$ is attained in the horizontal direction. Let $w(\alpha)$ be the width in direction α of $\rho_z(K)$, then $w = w(0)$. Set $l = w(\frac{\pi}{2})$. Note that

$$|T(\alpha)| = |z| \left(\frac{1}{|z|} - w(\alpha) \right).$$

Proposition 2 in [27] states that for every convex body $K \subset \mathbb{R}^d$,

$$\int_0^{2\pi} \frac{1}{(w(\alpha)/2)^{2d}} d\alpha \leq 2 \left| K^P \right| \tag{2.2}$$

with equality if and only if K is centrally symmetric with centre P. We use this below for $d = 2$.

Now we can compute p very accurately. The following computations are explained below. For small enough w we have

$$\begin{aligned}
p =& \frac{1}{4\pi} \sum_{|z|<\frac{1}{w}} \int_{\left\{\alpha : w(\alpha) < \frac{1}{|z|}\right\}} (1 - |z| w(\alpha)) d\alpha \\
=& \frac{1}{4\pi} \int_{\{\alpha : w(\alpha) < 1\}} \sum_{|z| < \frac{1}{w(\alpha)}} (1 - |z| w(\alpha)) d\alpha \\
=& \frac{1}{4\pi} \int_{\{\alpha : w(\alpha) < 1\}} \frac{2}{\pi} \frac{1}{w(\alpha)^2} + O\left(\frac{\log(w(\alpha))}{w(\alpha)} \right) d\alpha & (2.3) \\
=& \frac{1}{8\pi^2} \int_0^{2\pi} \frac{1}{(w(\alpha)/2)^2} d\alpha + O\left(\frac{w \log(w)^2}{|K|} \right) & (2.4) \\
\leq& \frac{1}{4\pi^2} \left| K^P \right| + O\left(\frac{w \log(w)^2}{|K|} \right) & (2.5) \\
\leq& \frac{1}{4|K|} + O\left(\frac{w \log(w)^2}{|K|} \right) & (2.6)
\end{aligned}$$

where P is the Santaló point of K. The equality (2.3) is by Lemma 2.7, (2.4) is easy to obtain by using the fact that $w + l\alpha \leq Cw(\alpha) \leq C(w + l\alpha)$

for $0 \leq \alpha \leq \pi/2$. The inequality (2.5) is a direct consequence of (2.2), and (2.6) is the Blaschke-Santaló inequality.

From here it follows that

$$\operatorname{Prob}\left\{\rho(K) \cap \mathbb{Z}^2 = \emptyset\right\} \leq \frac{1}{4|K|}\left(1 + Cw\log(w)^2 + \frac{C}{|K|}\right).$$

To see that the $\frac{1}{4|K|}$ is best possible, notice that if K is an ellipse, then (2.5) and (2.6) are equalities. $\square$

Chapter 3

A conjecture by Erdős

This is joint work with I. Bárány and has been submitted to Discrete and Computational Geometry.

In his celebrated paper [14] *"On sets of distances of n points"*, Paul Erdős makes the following conjecture:

"On every convex curve there exists a point P such that every circle with centre P intersects the curve in at most 2 points."

This conjecture turned out to be false, for any point P on the boundary of a regular triangle there is a circle centred at P that intersects the boundary of the triangle 4 times. In fact, any regular $(2k+1)$-gon has this property.

Perhaps the number 2 in Erdős's conjecture can be replaced by some other number independent of the convex curve. We wish to determine how large this number can be.

Let K be a planar convex body. We define $N = N(K) \in \mathbb{N} \cup \{\infty\}$ as the smallest number for which there is a point $P \in \partial K$ such that every circle with centre P intersects ∂K in at most N points. With this notation, Erdős's original conjecture states that $N(K) \leq 2$ for every convex body K. We conjecture that $N(K)$ is indeed bounded by some finite constant independent of K, probably by 6.

Theorem 3.1. *There is a planar convex body K with $N(K) = 6$.*

The simplest example we found for this is a 15-gon and it is constructed in Section 3.2. On the other hand we can show the following.

Theorem 3.2. *For every planar convex body K, $N(K) < \infty$.*

A stronger version of this theorem is proved in Section 3.1. So far we have not been able to find a finite upper bound that works for all K. Part of the difficulty of improving this bound may come from the following two theorems.

For $n \in \mathbb{N} \cup \{\infty\}$, let $J(K, n)$ be the set of points $P \in \partial K$ such that there is a circle centred at P that intersects ∂K in at least n points. Note that, in view of Theorem 3.2, $N = N(K)$ is the largest N such that $J(K, N) = \partial K$.

We denote by $|X|$ the 1-dimensional Hausdorff measure (perimeter) of a set $X \subset \mathbb{R}^2$.

Theorem 3.3. *Let $\epsilon > 0$, then there is a convex body K_ϵ such that*

$$\frac{|J(K_\epsilon, \infty)|}{|\partial K_\epsilon|} > 1 - \epsilon.$$

If K_0 is a segment or an acute triangle, then we can construct K_ϵ as in Theorem 3.3 so that $\lim_{\epsilon \to 0} K_\epsilon = K_0$ in the Hausdorff metric. These examples are also constructed in Section 3.2.

The next theorem is in the Baire category sense (see Section 3.3 and Chapter 20 of [35] for notions and definitions).. Let $\mathcal{K}$ be the set of planar convex bodies together with the Hausdorff metric.

Theorem 3.4. *For most convex bodies $K \in \mathcal{K}$, the set $\bigcap_{n=1}^{\infty} J(K, n)$ contains most points of ∂K.*

We give the proof of this theorem in Section 3.3.

3.1 The finiteness of N

First we fix some notation. We write $\mathcal{B}$ for the closed unit disk and $\mathcal{B}(Q, r)$ resp. $\mathcal{S}(Q, r)$ for disk and the circle centred at Q with radius $r > 0$.

If K is a convex body and $P \in \partial K$, then a line l is a *normal of K at P* if $P \in l$ and the line orthogonal to l through P supports K at P.

Fix a convex body K and define the set

$$\Gamma = \{(Q, l) : Q \in \partial K,\ l \text{ is a normal of } K \text{ at } Q\}.$$

The set Γ is actually a curve, this can be seen by considering the smooth convex body $K' = K + \mathcal{B}$. The set Γ is in bijective correspondence with $\partial K'$ in the following way: For every point $Q' \in \partial K'$, let l be the normal line of K' at Q' and let Q be the point in $l \cap \partial K$ at distance 1 from Q'. Then the pair $(Q, l) \in \Gamma$ corresponds to the point $Q' \in \partial K'$.

The distance between two points $Q'_1, Q'_2 \in \partial K'$ is the length of the shortest arc of $\partial K'$ bounded by these points. We use the above bijection to measure the distance between points in Γ and the Euclidean metric to measure distances between points in the plane.

Now we go back to the problem in question. Take $P \in \partial K$ and assume that there are two different points $Q_1, Q_2 \in S(P, r) \cap \partial K$. Let $H \subset \partial K$ be the closed arc bounded by Q_1 and Q_2 that does not contain P. Consider the function $g(Q) = \text{dist}(P, Q)$ for $Q \in H$. Since $g(Q_1) = g(Q_2)$, there exists Q in the relative interior of H such that g attains either its maximum or its minimum on Q. For this Q there is a line l so that $(Q, l) \in \Gamma$, and $P \in l$.

Hence, if $P \in \partial K$ is a point such that there are exactly M pairs $(Q, l) \in \Gamma$

with $P \in l$ and $P \neq Q$, then any circle centred at P intersects ∂K in at most $M+1$ points. This implies $N(K) \leq M+1$, therefore to prove Theorem 3.2 it is enough to show the following.

Theorem 3.5. *Given a convex body K, there is a point $P \in \partial K$ such that the number M of pairs $(Q,l) \in \Gamma$ with $P \neq Q$ and $P \in l$ is finite.*

It may even be possible that M is bounded by some constant independent of K. From the proof it can be seen that M is finite in a positive fraction of the perimeter of K.

To prove this theorem we define $\Gamma_0 \subset \Gamma$ as the set of pairs (Q,l) for which $l \cap \partial K$ contains exactly one point besides Q, let $f(Q,l)$ be this point. We shall study the function $f : \Gamma_0 \to \partial K$.

If $(Q,l) \in \Gamma \setminus \Gamma_0$ then $l \cap \partial K$ contains either one point (namely Q) or an infinite number of points (namely an edge of K with Q as an endpoint). In either case, ∂K is not smooth at Q and the internal angle formed at this point is at most $\frac{\pi}{2}$. We call such a point Q a *small angle of K*. Since there are at most 4 small angles on a closed convex curve, $\Gamma \setminus \Gamma_0$ contains at most 8 connected components.

Given $(Q,l) \in \Gamma_0$, define $\alpha(Q,l)$ as the smallest angle between the line l and a supporting line of K at $f(Q,l)$. Note that $\alpha(Q,l) > 0$ and that $\alpha : \Gamma_0 \to \mathbb{R}$ is a lower semi-continuous function and therefore the sets

$$\Delta_t = \{(Q,l) \in \Gamma_0 : \alpha(Q,l) > t\}$$

are open in Γ_0.

Lemma 3.6. *For every $t > 0$, the function $f|\Delta_t : \Delta_t \to \partial K$ is locally Lipschitz. If D is the diameter of K, then $Lip(f) \leq \frac{\max\{1,D\}}{\sin(\frac{t}{2})}$ in any small-enough open set of Δ_t.*

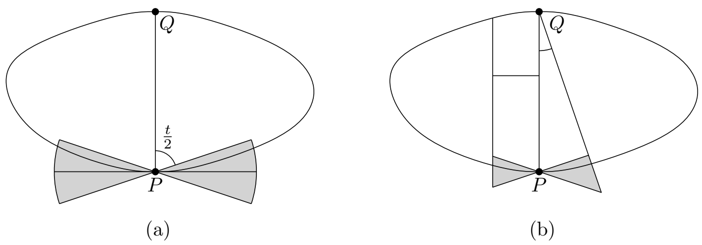

Figure 3.1: Lemma 3.6.

Proof. Let $(Q, l) \in \Delta_t$ and $P = f(Q, l)$. Since K is convex, there exists $\epsilon > 0$ such that any point $R \in \mathcal{B}(P, \epsilon) \cap \partial K$ satisfies $\angle QPR > \frac{t}{2}$ (see Figure 3.1(a)).

It is not difficult to see that there is a $\delta > 0$ such that if the pair $(Q', l') \in \Delta_t$ is at distance less than δ from (Q, l), then the point $P' = f(Q', l')$ is in $\mathcal{B}(P, \epsilon)$.

Since $\operatorname{dist}(Q, Q') \leq \operatorname{dist}((Q, l), (Q', l'))$ and the angle between l and l' is at most $\operatorname{dist}((Q, l), (Q', l'))$, the region where P' is can be further bounded. If we assume in Figure 3.1(a) that Q' is to the left of Q, then we have $\angle P'QP \leq \operatorname{dist}((Q, l), (Q', l'))$ if P' is right of P, and $\operatorname{dist}(P', l) \leq \operatorname{dist}((Q, l), (Q', l'))$ if P' is to the left of P. This determines the marked region in Figure 3.1(b). Thus,

$$\operatorname{dist}(P, P') \leq \frac{1}{\sin(\frac{t}{2})} \operatorname{dist}((Q, l), (Q', l'))$$

if P' is right of P, and

$$\operatorname{dist}(P, P') \leq \frac{\operatorname{dist}(Q, P')}{\sin(\frac{t}{2})} \operatorname{dist}((Q, l), (Q', l'))$$

if P' is to the left of P. In both cases we have

$$\operatorname{dist}(P, P') \leq \frac{\max\{1, D\}}{\sin(\frac{t}{2})} \operatorname{dist}((Q, l), (Q', l')).$$

This implies for the Lipschitz constant of f that

$$\mathrm{Lip}(f) \leq \frac{\max\{1, D\}}{\sin(\frac{t}{2})}$$

in any small-enough open set of Δ_t. $\square$

Lemma 3.7. *If the convex body K is not a polygon with at most 6 sides, there is a set $F \subset \partial K$ with $|F| > 0$ and a number $t > 0$ such that $f^{-1}(F) \subset \Delta_t$.*

Proof. For every small angle $Q \in \partial K$ the set of pairs $(Q, l) \in \Gamma$ is a closed arc, let (Q, l_+) and (Q, l_-) be its boundary points. Define the set $L = \bigcup(l_+ \cup l_-)$, where the union is taken over all small angles of K.

Since K is not a polygon with at most 6 sides, then $\partial K \setminus L$ is open relative to ∂K and non-empty. Therefore, there is a closed set $F \subset \partial K \setminus L$ with non-empty interior relative to ∂K (and hence, with positive perimeter).

Suppose that there is a sequence of pairs $\{(Q_i, l_i)\}_{i=1}^{\infty}$ with $f(Q_i, l_i) \in F$ and satisfying $\lim_{i\to\infty} \alpha(Q_i, l_i) = 0$. Let l_i' be a supporting line of K at $f(Q_i, l_i)$ that forms an angle of $\alpha(Q_i, l_i)$ with l_i. By taking a subsequence if necessary, we may assume that (Q_i, l_i) converges to a pair (Q, l) and that l_i' converges to a line l'. Then l' must support K at (Q, l), thus $Q \in L$. This contradicts the definition of F, therefore there exists $t > 0$ such that $\alpha(Q_i, l_i) > t$ for all $(Q, l) \in F$ and hence $f^{-1}(F) \subset \Delta_t$. $\square$

Proof of Theorem 3.5. If K is a polygon with at most 6 sides, then for any $P \in \partial K$ the set $f^{-1}(P)$ contains at most 12 points, so $M \leq 12$ there.

Assume now that K is not a polygon with at most 6 sides and take F as in Lemma 3.7. By Lemma 3.6, f is locally Lipschitz on $f^{-1}(F)$ and the coarea formula (see [15]) gives

$$\int_F \# f^{-1}(P)\, dP = \int_{f^{-1}(F)} |\nabla f(Q,l)|\; d(Q,l) \leq \left|f^{-1}(F)\right| \mathrm{Lip}(f).$$

Therefore, there is a point $P \in F$ which is taken only finitely many times by $f_{|f^{-1}(F)}$. Since no other pair $(Q,l) \in \Gamma$ with $Q \neq P$ can have $P \in l$, we are done. □

3.2 Examples

In this section we give examples for Theorems 3.1 and 3.3. First we need a couple of lemmas.

Lemma 3.8. *Fix $N \in \mathbb{N} \cup \{\infty\}$. Let A, B, C, D be points in convex position ordered counter-clockwise such that the angle $\angle ABC \in (0, \frac{\pi}{2})$. For any neighbourhood V of B there is a sequence of points $\{C_i\}_{i=1}^N$ such that:*

i) The points $D, C, C_1, C_2, \ldots B, A$ are all extreme points of their convex hull and are ordered clockwise.

ii) For every $P \in [A, B]$ outside of V, there is a circle centred at P that intersects the broken line $CC_1C_2 \ldots BA$ in at least $2N + 2$ points.

Proof. Given a point P on the line AB, let $\mathcal{S}_P$ be the circle centred at P that passes through B. Let $B' \in [A, B] \cap V$ so that C is outside of $\mathcal{S}_{B'}$.

We construct the points C_i inductively starting with $C_0 = C$. Once C_{i-1} is constructed, let C_i be a point such that:

- The points $D, C_0, \ldots, C_i, B, A$ are all extreme points of their convex hull and are ordered clockwise,

- C_i is outside of the circle $\mathcal{S}_A$,

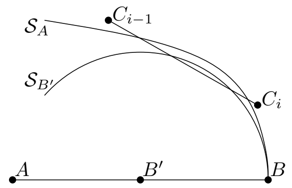

Figure 3.2: Lemma 3.8.

- $\angle ABC_i < \frac{\pi}{2}$,
- the segment (C_{i-1}, C_i) intersects $\mathcal{S}_{B'}$ twice.

See Figure 3.2 for a non-realistic example of this construction. Clearly condition (i) holds.

For a given $P \in [A, B']$ the circle $\mathcal{S}_P$ is between the circles $\mathcal{S}_{B'}$ and $\mathcal{S}_A$, therefore $\mathcal{S}_P$ intersects each of the segments (C, C_1) and $(A, B]$ at least once and each of the segments (C_i, C_{i+1}) twice, giving an infinite number of intersections when $N = \infty$. If $N < \infty$, then a circle slightly smaller that $\mathcal{S}_P$ will, in addition, intersect (C_N, B) twice giving a total of $2N + 2$ intersections. $\square$

Lemma 3.9. *Let A_1, B, A_2 be points in the plane. For $i = 1, 2$ let C_i be the midpoint of A_iB and S_i be the set of points P such that the orthogonal projection of P on A_iB is contained in the segment $(B, C_i]$. Then for any point $P \in S_1 \cap S_2$ there is a circle centred at P that intersects each of the segments (A_i, B) twice.*

Proof. Let $P \in S_1 \cap S_2$ and assume that $\mathrm{dist}(P, A_1B) \leq \mathrm{dist}(P, A_2B)$. It is easy to see that there is a real number r larger than $\mathrm{dist}(P, A_2B)$ and smaller than $\mathrm{dist}(P, B)$, $\mathrm{dist}(P, A_1)$ and $\mathrm{dist}(P, A_2)$. Therefore, the circle centred at P with radius r intersects each of the segments (A_i, B) twice. $\square$

We note that the set of points P that satisfy the above lemma is actually larger. The regions we use are simple and enough for our purposes.

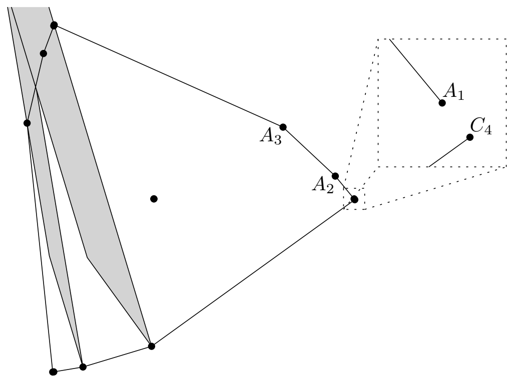

Figure 3.3: Construction for Theorem 3.1 with the regions from Lemma 3.9.

Now we are ready to construct the examples which prove Theorems 3.3 and 3.4.

Proof of Theorem 3.1. Consider the points with coordinates

$$A_1 = (1000, 0),\ A_2 = (906, 114),\ A_3 = (645, 359),\ A_4 = (-498, 871).$$

For $i = 1, \ldots 4$, let B_i and C_i be the rotation around the origin of A_i by an angle of $2\pi/3$ and $4\pi/3$, respectively. The 12 points A_i, B_i, C_i are in convex position (see Figure 3.3).

Using Lemma 3.9 on the triples C_1, C_2, C_3 and C_2, C_3, C_4, it can be shown by direct computation that for any point P in some neighbourhood V of the broken line $A_4B_1B_2B_3$ there is a circle centred at P that intersects the broken line $C_1C_2C_3C_4$ in at least 6 points. This direct computation amounts to checking that the two shaded strips in Figure 3.3 together contain the broken line $B_3B_2B_1A_4$ in their interior.

The angle $\angle A_3A_4B_1$ is acute. This is again a simple computation. Lemma 3.8 implies the existence of points $A_5 \in V$ such that for any point P on $[A_3, A_4] \setminus V$ there is a circle centred at P that intersects $A_3A_4A_5B_1$ in at least 4 points. Define B_5 and C_5 as above to obtain a 15-gon K having

A_i, B_iC_i as its vertices. The radius of the circle $\mathcal{S}_P$ is close to $|PA_4|$ and therefore intersects ∂K an additional 2 times, once between C_3 and P and once again between A_5 and B_3.

By the rotational symmetry of the figure, K has the desired property.

It can also be verified that there are points $P \in \partial K$ that are not in $J(K, 7)$, for example the midpoint of $[A_3A_4]$. □

An interactive version of Figure 3.3 made with GeoGebra[1] can be found on-line at `http://www.geogebratube.org/student/m33469`.

Proof of Theorem 3.3. As mentioned before, the convex body K_ϵ can be constructed so that it is close to any triangle or a straight line segment.

Fix a triangle $A_1A_2A_3$ and let $\epsilon > 0$. Choose points B_1, B_2 and B_3 so that $A_1B_1A_2B_2A_3B_3$ is a convex 6-gon, each B_i is ϵ-close to A_i and the angles $\angle A_iB_iA_{i+1}$ are acute. Using Lemma 3.8 with $N = \infty$ on the points $A_iB_iA_{i+1}B_{i+1}$, we obtain three families of points that together with the points A_i and B_i determine the required convex body.

For a straight segment $[A, B]$ a similar thing is done. Choose points C close to A and D close B such that $ACBD$ is a convex 4-gon and the angles $\angle ACB$ and $\angle ADB$ are acute, then Lemma 3.8 on $BCAD$ and $ADBC$ gives the required convex body. □

3.3 Generic behaviour

The set of planar convex bodies $\mathcal{K}$ with the Hausdorff metric is a complete metric space, thus, it is a Baire space.

The defining property of Baire spaces is that the intersection of countably many dense open sets is also dense. The intersection of countably many open

[1] `http://www.geogebra.org`

sets is called a G_δ set. Such sets are considered large. It is said that *most points in a Baire space* satisfy a property if the set of points satisfying this property contains a dense G_δ set. These notions can be found in Chapter 20 of [35] and similar techniques are applied in [36].

We prove Theorem 3.4 here, but we need some definitions and lemmas first. Let K be a convex body and assume the circle $\mathcal{S}$ intersects ∂K at Q. If for every $\epsilon > 0$ there are points $Q_1, Q_2 \in \mathcal{B}(Q, \epsilon) \cap \mathcal{S}$ such that $Q_1 \in \mathrm{int}K$ and $Q_2 \notin K$, then we say that *$\mathcal{S}$ intersects ∂K transversally at Q*.

To make things simpler, we work with the set $J_0(K, n) \subset J(K, n)$ of points $P \in \partial K$ such that there is a circle centred at P that intersects ∂K transversally in at least n points. If $n < \infty$ then the sets $J_0(K, n)$ are clearly open relative to ∂K.

Remark. It can be shown that if $n < \infty$ and ∂K contains no circle-arcs (which is true for most convex bodies) then $J_0(K, n) = J(K, n)$, but we do not need this.

Instead of proving Theorem 3.4 we prove the following stronger statement.

Theorem 3.10. *For most convex bodies $K \in \mathcal{K}$, the set $\bigcap_{n=1}^{\infty} J_0(K, n)$ contains most points of ∂K.*

Let $\mathcal{K}_{n,m}$ be the set of convex bodies $K \in \mathcal{K}$ such that for every point $P \in \partial K$, the set $J_0(K, n) \cap \mathcal{B}(P, \frac{1}{m})$ is non-empty.

Lemma 3.11. *The set $\mathcal{K}_{n,m}$ is open and dense in $\mathcal{K}$.*

Proof. First we prove that $\mathcal{K}_{n,m}$ is open. Let $K \in \mathcal{K}_{n,m}$ and choose a finite family $\{P_i\}$ such that $\{\mathcal{B}(P_i, \frac{1}{2m})\}$ covers ∂K. From the definition of $J_0(K, n)$ and the finiteness of $\{P_i\}$, it follows that there exists $\epsilon > 0$ such that whenever $\mathrm{dist}(K, K') < \epsilon$ the following hold:

- $\{\mathcal{B}(P_i, \frac{1}{2m})\}$ covers $\partial K'$,

- if $Q \in J_0(K, n)$ and $Q' \in \partial K' \cap \mathcal{B}(Q, \epsilon)$ then $Q' \in J_0(K', n)$.

This implies that $\mathcal{K}_{n,m}$ is open.

To show that it is dense, let $K \in \mathcal{K}$ and $\epsilon > 0$. We construct a convex body $K' \in \mathcal{K}_{n,m}$ such that $\mathrm{dist}(K, K') < \epsilon$.

Let K_0 be a polygon such that $\mathrm{dist}(K, K_0) < \epsilon$ and the distance between any two consecutive vertices of K_0 is less than $\frac{1}{4m}$. Let $\{P_1, \ldots, P_M\}$ be the set of midpoints of the sides of K_0. Given these points we construct new polygons $K_1, \ldots, K_M$ recursively, the following way.

Once K_{i-1} has been constructed, let Q, R, S be consecutive vertices of K_{i-1} such that R is a vertex of K_{i-1} farthest away from P_i. Now we remove the vertex R from K_{i-1} and add vertices $R_1, \ldots, R_n$ to form a new polygon K_i with the following properties:

- The points $R_1, \ldots, R_n$ are between Q and S,

- the distance between P_i and any R_j is some $r > 0$,

- the points $P_1, \ldots, P_M$ belong to ∂K_i and are not vertices of K_i,

- $\mathrm{dist}(K, K_i) < \epsilon$.

Note that any circle centred at P_i with radius slightly smaller than r will intersect ∂K_{i-1} transversally in at least n points.

It clear that the polygon obtained at the end of this process belongs to $\mathcal{K}_{n,m}$. $\square$

Proof of Theorem 3.10. By Lemma 3.11 and since $\mathcal{K}$ is a Baire space, the set $\bigcap_{n,m=1}^{\infty} \mathcal{K}_{n,m}$ is a dense G_δ subset of $\mathcal{K}$. Let $K \in \bigcap_{n,m=1}^{\infty} \mathcal{K}_{n,m}$, then each $J_0(K, n)$ is open and dense relative to ∂K. Therefore $\bigcap_{n=1}^{\infty} J_0(K, n)$ is a dense G_δ subset of ∂K. $\square$

Chapter 4

A Yao-Yao type theorem

This is joint work with P. Soberón, it was presented in EuroCG 2012 and has been submitted for publication to Discrete and Computational Geometry.

A convex partition of $\mathbb{R}^d$ into n parts is a covering $\mathcal{P} = \{C_1, \dots, C_n\}$ of $\mathbb{R}^d$ consisting of closed convex bodies with pairwise disjoint interiors. We say that a hyperplane $H \subset \mathbb{R}^d$ avoids a set C if it does not intersect its interior.

The classical Yao-Yao theorem [40] states the following.

Theorem (Yao and Yao)**.** *Let μ be a nice measure in $\mathbb{R}^d$, then there is a convex partition $\mathcal{P}$ of $\mathbb{R}^d$ into 2^d parts of equal μ-measure such that every hyperplane in $\mathbb{R}^d$ avoids at least one element of $\mathcal{P}$.*

In the original proof of this theorem, the measure has to have a continuous density function bounded away from the origin. Later, in [25], this was weakened by J. Lehec to the condition that the measure of every hyperplane be 0. We require the measure μ to satisfy the original conditions, as in [40]. We call a measure that satisfies these conditions a nice measure.

This theorem gives a partition in which all hyperplanes, except for those passing through a certain point, intersects exactly $2^d - 1$ pieces. The proof gives a unique partition for each ordered orthonormal basis $(u_1, \ldots, u_d)$. We give an extension of this theorem to the case when every hyperplane is required to avoid 2 pieces.

Theorem 4.1. *Let μ be a nice measure in $\mathbb{R}^d$, then there is a convex partition $\mathcal{P}$ of $\mathbb{R}^d$ into $3 \cdot 2^{d-1}$ parts of equal μ-measure such that every hyperplane in $\mathbb{R}^d$ avoids at least two elements of $\mathcal{P}$.*

For $d = 2$ this follows from a theorem by Buck and Buck [11] which states that any nice measure μ in $\mathbb{R}^2$ can be divided into six parts of equal measure by three concurrent lines. Our method gives in this case a partition by three lines, two of which are parallel. The proof of Theorem 4.1 is given in section 4.3. Some details of Yao and Yao's proof are necessary, so we give a sketch of their proof in section 4.2.

Let $N_d(k)$ be the smallest positive integer such that the following holds: For every nice measure μ on $\mathbb{R}^d$ there exists a partition of $\mathbb{R}^d$ into $N_d(k)$ convex parts of equal measure such that every hyperplane avoids at least k parts. We call such a partition a k-equipartition. The Yao-Yao Theorem and Theorem 4.1 are equivalent to the bounds $N_d(1) \leq 2^d$ and $N_d(2) \leq 3 \cdot 2^{d-1}$.

There is another number which seems useful, although it is less natural. Let $M_d(k, \alpha)$ be the smallest positive integer satisfying that for every nice measure μ on $\mathbb{R}^d$ there is a family of $M_d(k, \alpha)$ convex sets, each with measure at least α, such that every hyperplane avoids at least k of them. Then we have

$$N_d(k) \geq M_d\left(k, \frac{1}{N_d(k)}\right). \tag{4.1}$$

Lemma 4.2. *Let $p \leq q$ be non-negative integers, then $M_2(q-p, \frac{p}{2q}) \leq 2q$.*

From the proof of this lemma (found in section 4.4) we also obtain a bound for N_2.

k	$N_2(k)$	k	$N_2(k)$	k	$N_2(k)$	k	$N_2(k)$
1	4	6	14	11	24	16	32
2	6	7	16	12	24	17	32
3	8	8	18	13	28	18	36
4	10	9	20	14	30	19	36
5	12	10	22	15	32	20	36

Table 4.1: Upper bounds for N_2.

Corollary 4.3. $N_2(k) \leq 2k + 2$.

This improves the bounds that can be obtained by using the Yao-Yao Theorem, Theorem 4.1, and equations 4.3 and 4.4 from section 4.4 for all $k \leq 15$ except $k = 1, 2, 7, 12$. In Table 4 we give the best bounds obtainable by these methods.

For a fixed d, we can determine the asymptotic behaviour of $N_d(k)$.

Theorem 4.4. $\lim\limits_{k \to \infty} \dfrac{N_d(k)}{k} = 1.$

This means that the condition of equipartitioning a given measure is not very strong in the sense that, if there are enough parts, it can be done so that every hyperplane avoids almost all of them. This is the same behaviour as one would expect from a generic partition. In section 4.1 these results are applied to an apparently simple problem regarding separation of points and hyperplanes.

Problem (The (α, β)-problem)**.** *Determine all pairs $(\alpha, \beta) \in \mathbb{R}^2_+$ such that for any finite set X of points in $\mathbb{R}^d$ and any finite set Y of hyperplanes in $\mathbb{R}^d$, there are sets $A \subset X$ and $B \subset Y$ such that:*

- $|A| \geq \alpha |X|$,
- $|B| \geq \beta |Y|$,
- *no two points in A are separated by a hyperplane in B.*

We note that this problem is similar to Theorem 18 in [12] by Bukh and Hubard, the proof method used for that problem also involves the Yao-Yao Theorem. Trying to solve the question above we found a lower bound for $N_d(1)$ in terms of $h_d(t)$, the measure of a spherical cap of S^d with central angle t computed with the usual probability measure in S^d.

Theorem 4.5. *Let $\alpha > 0$ be such that there is a family A of convex sets in $\mathbb{R}^d$ with the following properties:*

- *Every set in A has measure at most α,*
- *every hyperplane avoids at least one set of A,*
- *the sum of the measures of the sets in A is not greater than 1.*

Then

$$\alpha \leq h_d\left(\frac{\pi}{4}\right) \approx C \cdot 2^{-\frac{d}{2}}$$

for some universal constant $C > 0$.

The hypotheses of this theorem can be written simply as $1 \geq \alpha \cdot M_d(1, \alpha)$. Note that if A is a 1-equipartition, the conditions above hold. Thus, we can set $\alpha = \frac{1}{N_d(1)}$. This is also implied by (4.1).

Corollary 4.6.

$$N_d(1) \geq C \cdot 2^{\frac{d}{2}}.$$

For the approximation of the cap measure see [3], for example. This answers a question by B. Bukh on whether the number of pieces needed is indeed super-polynomial. Similar bounds can be obtained for $N_d(k)$ for any k in terms of spherical caps of S^d. However, explicit approximations are hard to find. These results are a consequence of Theorems 4.7 and 4.10 below.

The Yao-Yao theorem can be generalised in the following way:
Given $a_1, a_2, \ldots, a_{2^d} > 0$ such that $\sum a_i = \mu(\mathbb{R}^d)$, there is a partition of $\mathbb{R}^d$ into 2^d convex parts $\{C_1, C_2, \ldots, C_{2^d}\}$ such that $\mu(C_i) = a_i$ for all i and every hyperplane avoids the interior of at least one C_i. Theorem 4.1 can also be generalised in the same way. This is made clear in the next sections, but a complete proof is not included.

4.1 The (α, β)-problem

The so called (α, β)-problem deals with how well behaved points and hyperplanes are with each other in terms of separation. It has the difficulty that it is not self-dual, so we work with a second version which does have this property. Namely,

Problem (Second version). *Find all pairs $(\alpha, \beta) \in \mathbb{R}^2_+$ such that for any two nice centrally symmetric probability measures μ_1, μ_2 in S^d there are sets $A, B \subset S^d$ with $\mu_1(A) \geq \alpha$, $\mu_2(B) \geq \beta$ such that either*

$$a \cdot b \geq 0 \quad \textit{for all} \quad a \in A,\, b \in B$$
$$\textit{or}$$
$$a \cdot b \leq 0 \quad \textit{for all} \quad a \in A,\, b \in B. \tag{4.2}$$

Here the condition of the measures being centrally symmetric is not really needed. If they are not then we can consider the measures given by $\mu_1'(A) = \frac{1}{2}(\mu_1(A) + \mu_1(-A))$ and $\mu_2'(A) = \frac{1}{2}(\mu_2(A) + \mu_2(-A))$ and obtain the same pairs (α, β) for these measures.

Let $\mathcal{C}_d'$ be the set of pairs (α, β) satisfying the conditions of the original problem and $\mathcal{C}_d$ be the set of pairs satisfying the conditions of the second version of the problem. Then $(\alpha,\beta)\in\mathcal{C}'_d$ if and only if $(\frac{\alpha}{2},\frac{\beta}{2})\in\mathcal{C}_d$.

This is done simply by embedding $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ as a hyperplane not containing the origin O. Then every point $a\in\mathbb{R}^d$ corresponds to the pair of points $\{a',-a'\}$ in S^d in the line Oa and every hyperplane $H\in\mathbb{R}^d$ corresponds to the pair of points $\{b',-b'\}$ in S^d such that b' is orthogonal to every line Ob with $b\in H$. With this transformation we see that the original problem is essentially equivalent to one similar to the second version but with symmetric finite sets of points instead of probability measures. The change to measures follows from the fact that every nice measure can be approximated by linear combinations of Dirac measures and vice versa.

The sets A and B in this problem can (and will) be taken as the intersection of S^d and a convex cone in $\mathbb{R}^{d+1}$ with apex at the origin.

We shall denote by M^d the usual probability measure on S^d. Given $0\le t<\pi$ and $x\in S^d$, let $C_d(x,t)$ be the spherical cap of S^d with centre x and central angle t. Denote by $h_d(t)$ its M^d-measure.

For $A\subset S^d$, let $A^\perp$ be the set of points $x\in S^d$ such that there exists $a\in A$ with $a\cdot x=0$. Note that if A is connected, the largest set $B\subset S^d$ that satisfies (4.2) is one of the connected components of the complement of $A^\perp$.

Given a set A of fixed measure, in order to bound the measure of B we need a variant of Lévy's isoperimetric inequality [26]. With our notation, Theorem 2.1 in [16] states the following.

Theorem. *Let A be a closed subset of S^d and set $t>0$ so that $M^d(A)=h_d(t)$. Then for every $\varepsilon>0$, $M^d(A_\varepsilon)\ge h_d(t+\varepsilon)$, where A_ε is the set of points $x\in S^d$ with geodesic distance smaller than ε from x (i.e. the set of points for which there exists $a\in A$ with* $\arccos(a\cdot x)<\varepsilon$*).*

If $\varepsilon=\frac{\pi}{2}$ and A is connected, then $S^d\setminus A_\varepsilon$ is one of the two connected components of $S^d \setminus A^\perp$. Therefore, if $A, B \subset S^d$ satisfy (4.2) and $M^d(A) = h_d(t)$ for some $t > 0$, then $M^d(B) \le h_d(\frac{\pi}{2} - t)$. This is the following theorem.

Theorem 4.7. *All points in $\mathcal{C}_d$ lie on or below the curve*

$$\left\{ \left(h_d(t), h_d\left(\frac{\pi}{2} - t\right) \right) : 0 \le t \le \frac{\pi}{2} \right\}.$$

This turns out to be best possible if $d = 1$.

Theorem 4.8.

$$\mathcal{C}_1 = \left\{ (\alpha, \beta) : \alpha + \beta \le \frac{1}{2} \right\}.$$

The Yao-Yao type partition theorems can be used to find pairs in the (α, β) problem. The following lemma is the main tool for this purpose.

Lemma 4.9. *Let $0 \le \rho \le 1$. Suppose that for any nice measure μ_1 on S^d there exists a family F of closed connected subsets of S^d and a probability measure μ_F on F such that*

- *$\mu_1(A) \ge \alpha$ for all $A \in F$,*
- *For every $b \in S^d$, the set $F_b = \{A \in F : A \cap \{b\}^\perp \neq \emptyset\}$ is μ_F-measurable and $\mu_F(F_b) \le \rho$.*

Then $(\alpha, \frac{1-\rho}{2}) \in \mathcal{C}_d$.

Using this with $N_d(k)$ and $M_d(k, \alpha)$, we obtain the following.

Theorem 4.10. *For any two positive integers k and d,*

$$\left(\frac{1}{2N_d(k)}, \frac{k}{2N_d(k)} \right) \in \mathcal{C}_d.$$

More generally, if $\alpha > 0$,

$$\left(\frac{\alpha}{2}, \frac{k}{2M_d(k, \alpha)} \right) \in \mathcal{C}_d.$$

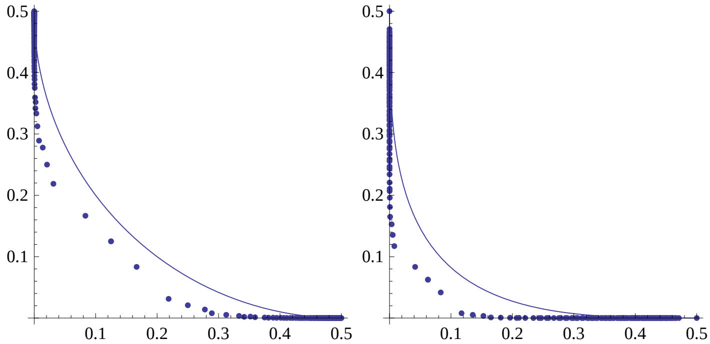

Figure 4.1: Bounds for $\mathcal{C}_2$ and $\mathcal{C}_3$.

With this theorem and Theorem 4.7 we obtain the lower bounds in Theorem 4.5. It should be noted that this also implies lower bounds for $N_d(k)$ for any k.

Applying the results obtained for $N_d(k)$, we can show the following.

Corollary 4.11. *For any two non-negative integers k_1 and k_2, not both equal to* 0*, we have*

$$\frac{1}{2}\left(\left(\frac{1}{2^d}\right)^{k_1}\left(\frac{1}{3\cdot 2^{d-1}}\right)^{k_2}, 1-\left(1-\frac{1}{2^d}\right)^{k_1}\left(1-\frac{1}{3\cdot 2^{d-2}}\right)^{k_2}\right) \in \mathcal{C}_d.$$

This gives in particular that $\left(\frac{1}{2^{d+1}}, \frac{1}{2^{d+1}}\right) \in \mathcal{C}_d$ and $\left(\frac{1}{3\cdot 2^d}, \frac{1}{3\cdot 2^{d-1}}\right) \in \mathcal{C}_d$. The fact that $\left(\frac{1}{2^{d+1}}, \frac{1}{2^{d+1}}\right) \in \mathcal{C}_d$ was obtained earlier in [1] using a similar method. In Fig. 4.1 there are plots of these points together with the bound obtained in Theorem 4.7 in dimensions 2 and 3.

Corollary 4.12. *There are pairs $(\alpha, \beta) \in \mathcal{C}_d$ arbitrarily close to $(0, \frac{1}{2})$.*

This last corollary comes from the fact that $\lim N_d(k)/k = 1$. However, as the pairs given by Corollary 4.11 get close to $(0, \frac{1}{2})$, they are significantly smaller than what Theorem 4.7 gives.

We can obtain better bounds if further conditions are imposed on one of the measures. Let $\mathcal{C}_d(\Delta)$ be the set of pairs (α, β) such that for any measure

μ_1 on S^d with density function f respect to M^d satisfying $\mathrm{Lip}(f) \leq \Delta$, and every nice measure μ_2 on S^d, the following holds: There are sets A, B in S^d with $\mu_1(A) = \alpha$, $\mu_2(B) = \beta$ and either $a \cdot b \geq 0$ for all $a \in A$, $b \in B$ or $a \cdot b \leq 0$ for all $a \in B$, $b \in B$.

Theorem 4.13. *For every* $0 < \lambda \leq 1$ *and* $0 < r < \frac{1-\lambda}{\Delta}$ *we have,*

$$\left(\lambda h_d(r), h_{d-1}\left(\frac{\pi}{2} - 2\arcsin\left(\frac{\sin(r)}{\sin(\frac{1-\lambda}{\Delta} - r)}\right)\right)\right) \in \mathcal{C}_d(\Delta).$$

If r is close to 0, then the pairs obtained are close to

$$\left(\lambda h_d(r), h_{d-1}\left(\frac{\pi}{2} - c_2 r\right)\right)$$

for some constant c_2 depending on λ and Δ. That is, the difference in dimension with respect to the bounds of Theorem 4.7 is compensated by the constants. The idea of the proof is to use a small S^{d-1} in S^d and the Lipschitz condition to construct sets as in the proof of Lemma 4.9. If instead of this we use a hypercube (of dimension $d-1$) in S^d, we can obtain bounds of the type

$$\left(\frac{c_1}{m^{d-1}}, \frac{1}{2} - \frac{c_2 d}{m}\right).$$

These are worse than the ones in Theorem 4.13 but are easier to grasp.

4.2 Yao-Yao's original proof

We give a sketch of the original proof by Yao and Yao since its spirit is followed in the proof of our main theorem. This will also allow us to note some additional properties and fix notation.

Let $O(d)$ be the space consisting of $d \times d$ matrices u such that $u^T u = I$, the space $SO(d) \subset O(d)$ consists of matrices with determinant 1. A matrix

$u \in O(d)$ can be expressed as $u = (u_1, \ldots, u_d)$ where u_i is the i-th row vector of u. In this way every u can be identified with an ordered orthonormal base of $\mathbb{R}^d$.

Fix a base $u = (u_1, \ldots, u_d)$ of $\mathbb{R}^d$. If H is a hyperplane orthogonal to u_1, define the open half-spaces

$$H^+ = \{x + tu_1 : x \in H, t > 0\},$$
$$H^- = \{x - tu_1 : x \in H, t > 0\}.$$

Let v be a unit vector in $\mathbb{R}^d$ not orthogonal to u_1 and let $p_v : \mathbb{R}^d \to H$ be the projection such that $p_v(x + tv) = x$ for all $x \in H$ and $t \in \mathbb{R}$. We can identify H with $\mathbb{R}^{d-1}$ by means of the base $u_2, \ldots, u_d$. There is a natural way to define measures μ_v^+ and μ_v^- in H: For any measurable $S \subset H$, set $\mu^\pm(S) = \mu(p_v^{-1}(S) \cap H^\pm)$.

In Yao and Yao's proof [40], a centre $c \in \mathbb{R}^d$ for μ relative to the base $u_1, \ldots, u_d$ is defined as follows:

- If $d = 1$ then c is the point that splits $\mathbb{R}$ into two parts of equal μ-measure.
- If $d > 1$, let H be the hyperplane orthogonal to u_1 that splits $\mathbb{R}^d$ into two parts of equal μ-measure. Then c lies on H and there exists a unit vector v (with $u_1 \cdot v > 0$) such that c is a centre for both μ_v^+ and μ_v^- relative to $u_2, \ldots, u_d$.

This induces a partition into 2^d parts, if a hyperplane intersects the line through c parallel to v in H^+ then it avoids one of the elements of the partition contained in H^- and vice versa.

It is then proved that c exists and is unique. Note that since v is unitary and because c exists, v must be contained in some fixed hyperplane orthogonal to u_2. Otherwise μ_v^+ and μ_v^- would not be equipartitioned by the corresponding hyperplane. This argument can be continued up to u_d to obtain that the projection vector v is unique. Using the uniqueness of v and c we can also obtain that they vary continuously with u in a similar way.

If we want each element of the partition to have a pre-described value, then the same proof works by changing the choice of H appropriately.

4.3 Proof of Theorem 4.1

Problems involving partitions of measures are topological in nature. There is a standard way to approach such problems, known as the test map scheme. First, we parametrise a subset of possible partitions by a space X (called phase space), and construct a space Y (called target space) of parameters of a partition. These spaces are related by a natural function $f : X \to Y$ (called test map). Ideally, there is group acting on both X and Y such that f is equivariant. The existence of the target partition is then reduced to showing that any equivariant function on those spaces always takes some value (e.g. [28], [37]). We follow this sketch and reduce the problem to showing that some equivariant functions always have a zero.

In our construction, we parametrise a set of partitions by $O(d)$, the space of all orthonormal basis of $\mathbb{R}^d$. There is an action of $(\mathbb{Z}_2)^d$ on $O(d)$, such that given $g \in (\mathbb{Z}_2)^d$ and $u \in O(d)$, gu is the result of changing the sign of some elements of u, depending on g. The target space and the action on it are much more elaborate and will be shown in the proof.

We start with the geometrical part of the proof of Theorem 4.1 and continue with the topological part.

Let $u = (u_1, \ldots, u_d)$ be an orthonormal base of $\mathbb{R}^d$, we think of u_1 as the upwards direction. If H is a hyperplane orthogonal to u_1, define the open

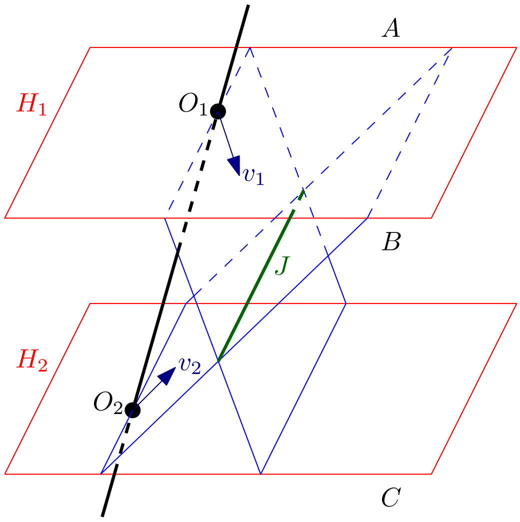

Figure 4.2: The hyperplanes, centres and projection vectors.

half-spaces

$$H^+ = \{x + tu_1 : x \in H, t > 0\},$$
$$H^- = \{x - tu_1 : x \in H, t > 0\}.$$

Let H_1 and H_2 be the hyperplanes orthogonal to u_1 such that the sets $A = H_1^+$, $B = H_1^- \cap H_2^+$ and $C = H_2^-$ have equal μ-measure. Let $\mu_1 = \mu|A \cup B$ and $\mu_2 = \mu|B \cup C$.

Yao-Yao's theorem applied to μ_1 gives a unique centre $O_1 \in H_1$ and a unique projection vector v_1 pointing downwards (i.e. $u_1 \cdot v_1 < 0$).

Let $J_1 \subset H_1$ be the $(d-2)$-dimensional flat through O_1 orthogonal to u_2. Note that the hyperplane $K_1 = \{J_1 + tv_1 : t \in \mathbb{R}\}$ splits B into two parts of equal μ_1-measure.

Define analogously $O_2 \in H_2$, v_2 pointing upwards, J_2 and K_2 (see Fig. 4.2). Since K_1 and K_2 each divide B into two parts of equal μ-measure, they intersect in a $(d-2)$-dimensional flat $J \subset B$ parallel to J_1 and J_2.

The centres O_1 and O_2 as well as the vectors v_1 and v_2 vary continuously with u.

Our aim is to find u such that the vectors v_1 and v_2 are parallel to the line O_1O_2. We first show how, using this, we can construct the desired partition. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the corresponding Yao-Yao partition to μ_1 and μ_2.

Then we can use the partition $\mathcal{P}$ consisting of the elements of $\mathcal{P}_1$ contained in A, the elements of $\mathcal{P}_2$ contained in C and the non-empty elements $K \cap B$ such that $K \in \mathcal{P}_1$. Every hyperplane avoids at least two elements of $\mathcal{P}$. This is because if it hits the line O_1O_2 in section A, it misses a section contained in B and one contained in C, if it hits O_1O_2 in B it misses a section in A and one in C and if it hits O_1O_2 in C it misses a section in A and one in B.

Now we use topology to search for the base u. First we need some definitions.

Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$, we define $g_i(x)$ as the result of changing the sign of the i-th coordinate of x. We always use g_i to denote this function independently of the target space, since it causes no confusion. We denote the j-th coordinate of $x_i \in \mathbb{R}^{d_i}$ by $x_i^{(j)}$. Given $v \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$, define $v^{(j)} = (v_1^{(j)}, \ldots, v_{d-j}^{(j)})$ and $v^T \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$ as $v^T = (v^{(1)}, \ldots, v^{(d-1)})$.

An easier way to visualise this last construction is to consider a $(d-1) \times (d-1)$ matrix V induced by v in the following way. In the k-th row write the coordinates of v_k followed by $k-1$ signs "$\times$",

$$V = \begin{pmatrix} v_1^{(1)} & v_1^{(2)} & \cdots & v_1^{(d-2)} & v_1^{(d-1)} \\ v_2^{(1)} & v_2^{(2)} & \cdots & v_2^{(d-2)} & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{d-2}^{(1)} & v_{d-2}^{(2)} & \cdots & \times & \times \\ v_{d-1}^{(1)} & \times & \cdots & \times & \times \end{pmatrix}.$$

Then v^T is the set of vectors induced in the same way by the transpose V^T of V, namely

$$V^T = \begin{pmatrix} v_1^{(1)} & v_2^{(1)} & \cdots & v_{d-2}^{(1)} & v_{d-1}^{(1)} \\ v_1^{(2)} & v_2^{(2)} & \cdots & v_{d-2}^{(2)} & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_1^{(d-2)} & v_2^{(d-2)} & \cdots & \times & \times \\ v_1^{(d-1)} & \times & \cdots & \times & \times \end{pmatrix}.$$

Let $r : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection such that $r(O_1) = r(O_2) = O$ and $\{r(u_2), \dots, r(u_d)\}$ is the canonical basis.

The affine hyperplane $r(J)$ is orthogonal to u_2, so it is of the form $\{v : v \cdot u_2 = \lambda\}$ for some $\lambda \in \mathbb{R}$. Let $x \in \mathbb{R}^{d-2}$ and $y \in \mathbb{R}^{d-2}$ be the vectors consisting of the last $d-2$ coordinates of $r(v_1)$ and $r(v_2)$, respectively.

Let $h(u) = (x, y, \lambda) \in \mathbb{R}^{d-2} \times \mathbb{R}^{d-2} \times \mathbb{R}$, note that if $h(u) = 0$ for some u, then the vectors v_1 and v_2 are parallel to the line O_1O_2 and we are done.

The map h satisfies the following conditions:

- $h(g_1(u)) = (y, x, \lambda)$,
- $h(g_2(u)) = (x, y, -\lambda)$,
- $h(g_{i+2}(u)) = (g_i(x), g_i(y), \lambda)$ for $i = 1, \dots, d-3$.

Where for any z, $g_i(z)$ is the result of changing the sign on the i-th coordinate.

Let $f : O(d) \to \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$ be defined by

$$f(u) = ((x + y, \lambda), x - y, 0, \ldots, 0).$$

Finding a zero of h is equivalent to finding a zero of f. We will prove something more general.

Claim. *Assume $f : O(d) \to \mathbb{R}^{d-1} \times \cdots \times \mathbb{R}^1$ is a continuous function such that whenever $f(u) = v$, the following conditions hold:*

- $f(g_1(u)) = g_2(v)$,
- $f(g_2(u)) = g_{d-1}(v^T)^T$,
- $f(g_{i+2}(u)) = g_i(v^T)^T$ *for* $i = 1, \ldots, d - 3$.

Then there exists $u \in O(d)$ such that $f(u) = 0$.

We use a similar proof method to the one Bárány used to prove the Borsuk-Ulam theorem in [4]. This method is thoroughly explained in Chapter 2.2 of [28] and in [30].

Together with composition, we can think of $\{g_1, \ldots g_d\}$ as a set of generators of the group $\mathbb{Z}_2^d$. Given the conditions on f, there are natural group actions of $\mathbb{Z}_2^d$ on $O(d)$ and $\mathbb{R}^{d-1} \times \cdots \times \mathbb{R}^1$ such that f is equivariant. However, the space $O(d)$ is too large for our needs, so instead we consider the restriction $f_1 = f|SO(d)$ and the group actions of $\mathbb{Z}_2^{d-1}$ on $SO(d)$ and $\mathbb{R}^{d-1} \times \cdots \times \mathbb{R}^1$ obtained by taking the group generated by $\{g_1 \circ g_d, \ldots, g_{d-1} \circ g_d\}$.

The main idea is of the proof is to show that, given two equivariant functions on these spaces, the parity of the number of orbits (of $\mathbb{Z}_2^{d-1}$ in

$SO(d)$) that are sent to 0 is invariant. This is done via an equivariant homotopy, and is the last step in the proof. Let us first find a function $f_0 : SO(d) \to \mathbb{R}^{d-1} \times \ldots, \mathbb{R}^1$ that sends exactly one orbit of $\mathbb{Z}_2^{d-1}$ to 0.

Define $f_0 : SO(d) \to \mathbb{R}^{d-1} \times \ldots, \mathbb{R}^1$ as the function given by

$$f_0(u) = (v_1, \ldots, v_{d-1}),$$

where

- $v_1 = (u_3^{(1)}, \ldots, u_d^{(1)}, u_2^{(1)})$,
- $v_2 = u_1^{(1)} \cdot (u_3^{(2)}, \ldots, u_d^{(2)})$,
- $v_{i+2} = (u_3^{(i+2)}, \ldots, u_{d-i}^{(i+2)})$ for $i = 1, \ldots, d-3$.

This function is continuous and equivariant. Furthermore, if $f_0(u) = 0$, then u_i has to be the i-th element of the canonical basis or its negative. Therefore f_0 has exactly 2^{d-1} zeros in $SO(d)$. Note as well that the differential Df_0 is non-degenerate in its zeros.

Let $F : SO(d) \times I \to \mathbb{R}^{d-1} \times \cdots \times \mathbb{R}^1$ be the equivariant homotopy given by $F(u,t) = tf_1(u) + (1-t)f_0(u)$ that takes f_0 to f_1.

If we assume that F is generic enough, then the set $F^{-1}(0)$ consists of paths and cycles. If f_1 has no zeros, then all the paths of $F^{-1}(0)$ have their endpoints at points of the form $(u, 0)$ where u is a zero of f_0. Therefore there must be a path connecting two such points. This last statement is impossible; for each t the zeros of $F(u,t)$ come in sets of 2^{d-1}, each being an orbit of the group action.

Since f_0 has 0 as a regular value, if F is not generic enough, then a small perturbation F' of F can be found with the following properties:

- F' is an homotopy between two maps f'_0 and f'_1, and 0 is a regular value of F',
- f'_1 has no zeros.
- f'_0 has only one orbit of points that give 0.

Thus, we may reach a contradiction using F' instead of F. □

4.4 Other proofs

Proof of Lemma 4.2. Set a parameter $t \geq 0$, we proceed inductively. Let ℓ_1 be an oriented halving line (i.e. a line that splits $\mathbb{R}^2$ into two parts of equal μ-measure that has fixed right and left sides). Once that ℓ_i has been constructed, let ℓ_{i+1} be the oriented halving line such that the regions

$$\begin{aligned} A_i =& \{x \in \mathbb{R}^2 : x \text{ is right of } \ell_{i+1} \text{ and left of } \ell_i\} \\ A_{q+i} =& \{x \in \mathbb{R}^2 : x \text{ is left of } \ell_{i+1} \text{ and right of } \ell_i\} \end{aligned}$$

have μ-measure $\frac{p}{2q}+t$, for $i = 1, \ldots, q$ (see left part of Fig. 4.3). If $t = 0$ then the sum of the measures of these regions up to $i = q$ is p, but the regions may overlap and not cover almost every point of $\mathbb{R}^2$ at least p times. Let t be the smallest real number such that almost every point of $\mathbb{R}^2$ is covered at least p times. For this choice of t, the non-oriented lines determined by ℓ_1 and ℓ_{q+1} are equal. In total there are q lines and the boundary of every A_i is contained in the union of two of them. If $\ell \subset \mathbb{R}^2$ is a line not parallel to any ℓ_i, then it intersects each of the lines $\ell_1, \ldots, \ell_q$ once, at every intersection ℓ enters a new region. Since every point is covered p times, ℓ intersects exactly $p+q$ elements of F. This implies that it avoids at least $q-p$, as we wanted. □

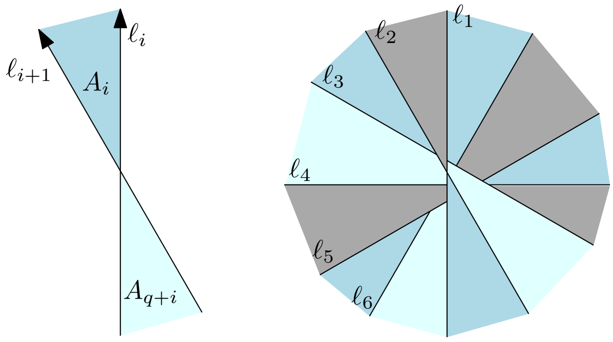

Figure 4.3: Regions for Lemma 4.2 and Corollary 4.3.

Proof of Corollary 4.3. This is similar to the previous proof with $p = 1$ and $q = k + 1$, but this time we define the lines ℓ_i such that the regions

$$A_i = \{x \in \mathbb{R}^2 : x \text{ is right of } \ell_{i+1} \text{and left of } \ell_i\} \setminus \bigcup_{j<i} A_j$$

$$A_{q+i} = \{x \in \mathbb{R}^2 : x \text{ is left of } \ell_{i+1} \text{ and right of } \ell_i\} \setminus \bigcup_{j<i} A_{q+j}$$

have μ-measure $\frac{1}{2k+2}$. To see that it is possible to find such l_i, consider μ' the measure μ restricted to $\mathbb{R}^2 \setminus \bigcup_{j<i} (A_j \cup A_{q+j})$. Note that l_{i-1} is a halving line of μ', so we need l_i to be a halving line of μ' such that $\mu'(A_i) = \frac{1}{2k+2}$. This implies that $\mu'(A_{q+i}) = \frac{1}{2k+2}$. Since μ and μ' coincide in A_i and A_{q+i}, we obtain the desired line.

We end up with something like the right side of Fig. 4.3. This is a partition as, once again, ℓ_1 and ℓ_{q+1} are equal as non-oriented lines. Since $\mathbb{R}^2 \setminus \bigcup_{j<i} A_j$ consists of two convex components for all i, every A_i is convex. The same argument as above gives that every line avoids at least k regions.

$\square$

Proof of Theorem 4.4. Clearly $N_d(k) \geq d + k$, as there is a hyperplane through any given d points. The function N_d also satisfies

$$N_d(k_1 + k_2) \le N_d(k_1) + N_d(k_2). \tag{4.3}$$

To see this, partition $\mathbb{R}^d$ by a hyperplane H that divides its measure in proportions $N_d(k_1) : N_d(k_2)$. We can find a k_1-equipartition of one side and a k_2-equipartition of the other side. We are left with $N_d(k_1) + N_d(k_2)$ parts of equal measure such that every hyperplane avoids k_1 parts on one side of H and k_2 parts in the other side of H.

We also have the asymptotically stronger equation

$$N_d(k_1 N_d(k_2) + k_2 N_d(k_1) - k_1 k_2) \le N_d(k_1) N_d(k_2). \tag{4.4}$$

This can be shown by finding a k_1-equipartition of $\mathbb{R}^d$ and further partition each of its pieces by k_2-equipartitions. We are left with $N_d(k_1)N_d(k_2)$ parts of equal measure such that every hyperplane intersects at most $(N_d(k_1) - k_1)(N_d(k_2) - k_2)$ of them.

Starting with Yao-Yao's theorem and iterating (4.4), a sequence of partitions can be found such that in the i-th step we have 2^{di} parts of equal measure and every hyperplane intersects at most $(2^d - 1)^i$ of them. Therefore, a sequence k_i can be found in which $N_d(k_i)/k_i$ tends to 1. Then (4.3) implies

$$\lim_{k \to \infty} \frac{N_d(k)}{k} = 1.$$

$\square$

Proof of Theorem 4.8. Take α, β with $\alpha + \beta \le \frac{1}{2}$. Suppose that for every arc segment A with $\mu_1(A) = \alpha$ we have that $\mu_2(A^{\perp}) > 2\alpha$. Let ϕ be the rotation of S^1 by an angle of $\frac{\pi}{2}$ and note that $A^{\perp} = \phi(A) \cup \phi^{-1}(A)$.

Since μ_1 and μ_2 are centrally symmetric then $\phi(A) = \phi^{-1}(A)$ and therefore $\mu_1(A) < \mu_2(\phi(A))$. Since this happens for every arc A with μ_1-measure α, $\mu_1(S^1) < \mu_2(S^1)$, which is a contradiction. Therefore there exists an arc segment A that satisfies $\mu_2(A^\perp) \leq 2\alpha$. If B is any of the two components of $S^1 \setminus A^\perp$, then $\mu_2(B) \geq \beta$. This proves one of the inclusions, Theorem 4.7 gives us the other. $\square$

Proof of Lemma 4.9. Let μ_1 and μ_2 be nice measures. Suppose that we can find μ_F as above, then by Fubini's Theorem

$$\begin{aligned}\int_F \mu_2(A^\perp) d\mu_F &= \int_F \int_{S^d} \chi(A^\perp) d\mu_2 d\mu_F = \int_{S^d} \int_F \chi(F_b) d\mu_F d\mu_2 \\ &= \int_{S^d} \mu_F(F_b) d\mu_2 \leq \rho,\end{aligned}$$

where χ is the characteristic function of a set. Thus, we can find A_0 such that $\mu_2(A_0^\perp) \leq \rho$. This means that there is a set B such that $\mu_1(B) \geq \frac{1-\rho}{2}$ and the sign of $a \cdot b$ is constant for all $a \in A_0$ and $b \in B$. $\square$

Proof of Theorem 4.10. Given a hyperplane in $\mathbb{R}^{d+1}$ that does not pass through the origin, consider the radial projection $R^d \to S^d$. Since the measure of every great circle in S^d has null μ-measure, this induces a probability measure $\mu^\perp$ in $\mathbb{R}^d$. Fix a family of $M_d(k, \alpha)$ convex sets of measure α such that every hyperplane (in $\mathbb{R}^d$) avoids the interior of at least k of them. If we bring this family back to S^d, we obtain a new family $\mathcal{F}$ of $2M_d(k, \alpha)$ convex sets in S^d, each of measure $\frac{\alpha}{2}$. By choosing μ_F to be the uniform probability measure on F and applying Lemma 4.9, we are done. $\square$

Proof of Theorem 4.13. Since $\mu_1(S^d) = 1$, there is a point $x_0 \in S^d$ such that $f(x_0) \geq 1$. Let $\lambda \leq 1$ and $R = \min(\frac{1-\lambda}{\Delta}, \frac{\pi}{2})$. Recall that $C_d(x, t)$ is defined as the spherical cap of S^d with centre x and central angle t. From

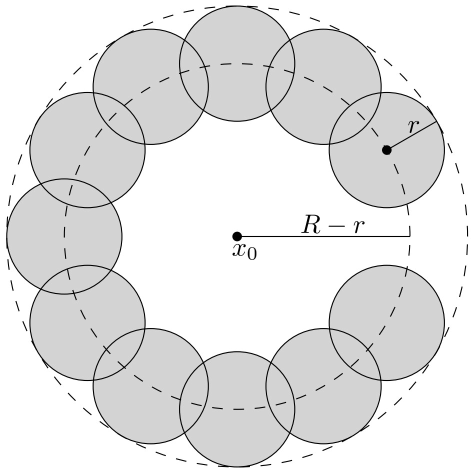

Figure 4.4: The family for Theorem 4.13.

the fact that $\mathrm{Lip}(f) \leq \Delta$ it follows that $f \geq \lambda$ on $C_d(x_0, R)$.

Set $r \leq \frac{R}{4}$, we consider the family F of caps $C_d(x, r)$ with x in the boundary of $C_d(x_0, R-r)$ as shown in Fig. 4.4. Each of these caps has measure at least $\lambda h_d(r)$.

Here we need some observations:

- $C_d(x, r)$ is the intersection of S^d with a ball with centre x and radius $\sin(r)$.
- $\partial C_d(x_0, R-r)$ is a $(d-1)$-sphere with radius $\sin(R-r)$.

Let μ_F be the usual probability measure on $\partial C_d(x_0, R-r)$. From this we can see that any hyperplane intersects a portion of F with size at most

$$1 - 2h_{d-1}\left(\frac{\pi}{2} - 2\arcsin\left(\frac{\sin(r)}{\sin(R-r)}\right)\right).$$

We conclude the proof in the same way as Lemma 4.9. $\square$

Chapter 5

Line transversals

This is joint work with J. Jerónimo-Castro and the results here can be found in [21].

Let $\mathcal{F}$ be a family of convex bodies in $\mathbb{R}^d$. We say that the family $\mathcal{F}$ has property T if there exists a 1-dimensional line in $\mathbb{R}^d$ that intersects all the members of $\mathcal{F}$. Furthermore, if $k \in \mathbb{N}$ then $\mathcal{F}$ has property $T(k)$ if every subfamily of $\mathcal{F}$ with at most k members has property T.

In 1935, P. Vincensini [38] posed the problem of finding conditions on $\mathcal{F}$ so that property $T(k)$ would imply property T. The first result of this type was due to Santaló [34] who showed the following.

Theorem (Santaló)**.** *If $\mathcal{F}$ is a family of parallelotopes in $\mathbb{R}^d$ with edges parallel to the coordinate axes and $\mathcal{F}$ has property $T(2^{d-1}(2d-1))$, then $\mathcal{F}$ has property T.*

Afterwards, several variations of this problem emerged. We are interested in one posed by B. Grünbaum [17] in 1964:

Problem. *Let K be a convex body in $\mathbb{R}^d$ and $\mathcal{F} = \{x_1 + K, \ldots, x_n + K\}$ be a family of translates of K with property $T(k)$. Determine the smallest*

$\lambda = \lambda(K, k) > 0$ *such that the family* $\lambda\mathcal{F} = \{x_1 + \lambda K, \ldots, x_n + \lambda K\}$ *has property* T.

There have been several results on this problem such as the following.

Theorem (Eckhoff [13]). *Let* D *be a disk in* $\mathbb{R}^2$*, then* $\lambda(D, 3) \leq 2$.

Theorem (Heppes [19]). *Let* $\mathcal{F}$ *be a family of disjoint translates of a disc in* $\mathbb{R}^2$ *with property* $T(3)$*, then* $1.65\mathcal{F}$ *has property* T.

Theorem (Jerónimo-Castro [20, 22] and Roldán-Pensado [22]). *Let* $K \subset \mathbb{R}^2$ *be any convex body, then*

$$\lambda(K, 4) \leq \frac{1 + \sqrt{5}}{2} \approx 1.618.$$

If K *is a disk then the equality holds.*

By considering a family of five disks in $\mathbb{R}^2$ of appropriate size with centres on the vertices of a regular pentagon, the inequality $\lambda(D, 3) \geq \frac{1+\sqrt{5}}{2}$ is obtained (see Figure 5.1). J. Eckhoff conjectured that this is actually the correct value for $\lambda(D, 3)$ and, further more, that $\lambda(D, 2k - 1) = \lambda(D, 2k)$ for $k \geq 2$. These two conjectures remain open problems.

Let K be a convex body in $\mathbb{R}^2$, the number $\mu = \mu(K)$ is defined as the smallest $\mu > 0$ such that the following holds: If x_1, x_2, x_3 and x_4 form the vertices of a parallelogram, and the family $\mathcal{F} = \{x_1 + K, \ldots, x_4 + K\}$ has property $T(3)$, then the family $\mu\mathcal{F}$ has property T. For simple convex bodies $\mu(K)$ is easy to compute, for example, $\mu(D) = \sqrt{2}$ if D is a disk and $\mu(S) = 2$ if S is a square.

Theorem 5.1 (Jerónimo-Castro and Roldán-Pensado [22]). *There exists*

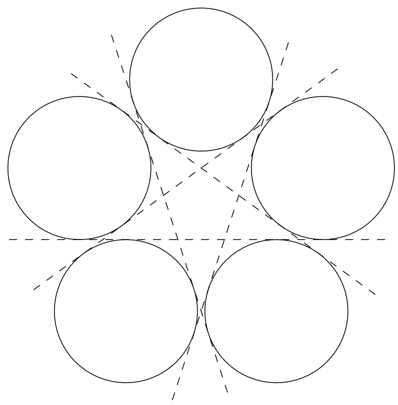

Figure 5.1: Five equal circles with property $T(4)$.

$\varepsilon > 0$ such that for every convex body $K \subset \mathbb{R}^2$,

$$\frac{4}{3} + \varepsilon \le \lambda(K, 3) \le \max\left\{\frac{2 + \sqrt{1 + 4\mu(K)}}{2}, \rho\right\},$$

where $\rho \approx 1.76$ is the real root of the polynomial $x^3 - 2x^2 - x + 1$.

Corollary 5.2. *If D is a disk, then*

$$\frac{1 + \sqrt{5}}{2} \le \lambda(D, 3) \le \frac{1 + \sqrt{1 + 4\sqrt{2}}}{2} \approx 1.79.$$

So far this is the best known bound towards Eckhoff's conjecture.

5.1 The problem and the results

A variant of Grünbaum's problem is obtained by replacing translates of K with homothetic copies of K.

Problem. *Let K be a convex body in $\mathbb{R}^d$ and let $\mathcal{F} = \{x_1 + t_1K, \ldots, x_n + t_nK\}$ be a family of homothets of K with property $T(k)$. Determine the smallest $\lambda_h = \lambda_h(K, k) > 0$ such that the family $\lambda_h\mathcal{F} = \{x_1 + \lambda_h t_1 K, \ldots, x_n + \lambda_h t_n K\}$ has property T.*

There is very little done on this problem. In this chapter we focus on finding bounds for $\lambda_h(K,k)$ in the case when K is an euclidean ball. Here we state the results and prove them in the next section.

Theorem 5.3. *Let B be an euclidean ball in $\mathbb{R}^d$, then*

$$\lambda_h(B, d+1) \leq 4.$$

This theorem is far from being optimal. The main idea used here is to fix the smallest ball in $\mathcal{F}$ and shrink it to a point P while expanding the others so that the $T(d+1)$ property is preserved. Then we represent a subset of the lines through P as $\mathbb{R}^{d-1}$ and use Helly's theorem to find a line transversal. Using similar ideas we derive two more results in $\mathbb{R}^2$.

Theorem 5.4. *If $D \subset \mathbb{R}^2$ is a disk, then*

$$\frac{1+\sqrt{5}}{2} \leq \lambda_h(D, 4) \leq 2\sqrt{2}.$$

The example that gives the lower bound here is the same as the one for $\lambda(D, 4)$ shown in Figure 5.1.

Unfortunately the proof argument for the upper bound of this theorem does not work for $d > 2$. It can be shown in a similar manner that $\lambda_h(D, 3) \leq 2\sqrt{3}$, however, a more refined argument improves this.

Theorem 5.5. *If $D \subset \mathbb{R}^2$ is a disk, then*

$$\sqrt{3} \leq \lambda_h(D, 3) \leq \rho \approx 2.875,$$

where ρ is the real root of the polynomial $x^3 - x^2 - 4x - 4$.

We conjecture that $\sqrt{3}$ is actually the correct value for $\lambda_h(D, 3)$. It is worth noting that if one of the circles in $\mathcal{F}$ has radius 0, then $\sqrt{3}\mathcal{F}$ always

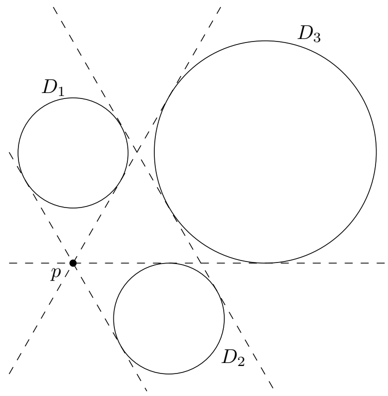

Figure 5.2: Example for the lower bound of Theorem 5.5.

has a transversal line. This can be shown in a similar (but easier) way to Theorem 5.5.

The example shown in Figure 5.2 gives the lower bound in Theorem 5.5. The example consists of three discs and a point. The discs D_1 and D_2 have radii equal to 1 and D_3 has radius equal to 2, the dashed lines are common tangents and the triangle in the middle is equilateral. The four circles p, D_1, D_2 and D_3 form a family with property $T(3)$. In order to have a common transversal for all of them it is necessary to multiply the radii of D_1, D_2 and D_3 by $\sqrt{3}$.

5.2 Proofs

Throughout this section we fix the family $\mathcal{F} = \{B_1, B_2, \ldots, B_n\}$ of translates of a Euclidean ball and set r_i as the radius of B_i. Furthermore, we assume that $r_1 \leq r_2 \leq \cdots \leq r_n$ and that no two balls in $\mathcal{F}$ are concentric.

If B_i and B_j are two distinct closed balls in $\mathbb{R}^d$, we define the distance between them as the smallest $\mu \geq 0$ such that $(\mu B_i) \cap (\mu B_j) \neq \emptyset$. Note that this notion of distance does not define a metric space. It will be most useful to us when B_1 is a point (has radius 0). In this case we can identify B_i with

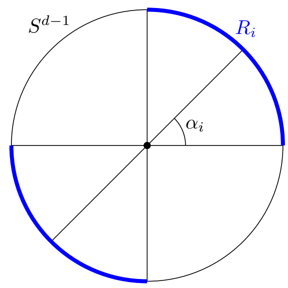

Figure 5.3: The set R_i.

a subset R_i of the sphere S^{d-1}, this is done by considering the set of unit vectors v such that the line through B_1 parallel to v intersects B_i.

If the point B_1 is not contained in B_i then R_i is the union of two disjoint, closed and opposite caps in S^{d-1}. We refer to the axis of revolution of R_i simply as the axis of R_i. Let α_i be the angle such that $2\alpha_i$ is the diameter (measured over S^{d-1}) of each connected component of R_i (Figure 5.3 shows the planar case). This angle and the distance μ_i between B_1 and B_i are related by $\mu_i \sin \alpha_i = 1$.

Notice that if B_1 is a point, the family $\mathcal{F}$ has property $T(k)$ only if every $k-1$ members of $\{R_2, \dots, R_n\}$ intersect. In particular we have that $\mathcal{F}$ has property T if and only if $R_2 \cap \cdots \cap R_n \neq \emptyset$.

In order to use the above, we shall need the following lemma which establishes a useful reduction.

Lemma 5.6. *For every B_i, let $t_i \geq 1$ be the smallest number such that the distance between $0B_1$ and t_iB_i is the same as the distance between B_1 and B_i. If $\mathcal{F}$ has property $T(k)$, the family $\mathcal{G} = \{0B_1, t_2B_2, \dots, t_nB_n\}$ also has property $T(k)$ and for every i we have $t_i \leq 2$.*

Proof. It is easy to see that $t_i = 1 + \frac{r_1}{r_i}$ which implies that $t_i \leq 2$. Let $\mathcal{G}'$ be a subfamily of $\mathcal{G}$ with k elements. If $0B_1 \notin \mathcal{G}'$ then clearly $\mathcal{G}'$ has a line transversal. If $0B_1 \in \mathcal{G}' = \{0B_1, t_{i_2}B_{i_2}, \dots t_{i_k}B_{i_k}\}$, let ℓ be a line transversal to $\{B_1, B_{i_2}, \dots B_{i_k}\}$ and ℓ' be the line parallel to ℓ passing through $0B_1$. Then the distance between ℓ' and the centre of B_i is at most $r_i + r_1 = t_i r_i$, therefore ℓ' is a line transversal to $\mathcal{G}'$. $\square$

Proof of Theorem 5.3. First assume that $r_1 = 0$. Let $\mathcal{F}_1 \subset \mathcal{F}$ be the set of balls with distance greater than 2 from B_1.

Fix an element $B_j \in \mathcal{F}_1$ and let H be a half-space with boundary hyperplane Γ passing through the origin and perpendicular to the axis of R_j. Set R'_j as the component of R_j in H. Note that for every $B_i \in \mathcal{F}_1$, $\alpha_i \leq \frac{\pi}{6}$ and therefore the component R'_i of R_i that intersects R'_j is completely contained in H. Since $\mathcal{F}_1$ has property $T(d+1)$, then every d members of $\mathcal{R} = \{R'_i : B_i \in \mathcal{F}_1\}$ have non-empty intersection. Now project each set $R'_i \in \mathcal{R}$ into another plane Γ' parallel to Γ with centre of projection at the origin. Let $S_i \subset \Gamma'$ be the projection of R'_i for every $R'_i \in \mathcal{R}$. By Helly's theorem in $\mathbb{R}^{d-1}$, there exists a point $x \in \bigcap_i S_i$. Hence, there is a line ℓ transversal to $\mathcal{F}_1$. If $B_i \notin \mathcal{F}_1$ then $2B_i$ contains B_1, therefore the whole family $2\mathcal{F}$ intersects ℓ.

The general case can be reduced to this one. Lemma 5.6 implies that the family $\mathcal{G} = \{0B_1, 2B_2, \dots, 2B_n\}$ has property $T(d+1)$. Then the family $2\mathcal{G} = \{0B_1, 4B_2, \dots, 4B_n\}$ has a line transversal, that is, $4\mathcal{F}$ has a line transversal. $\square$

Proof of Theorem 5.4. For the upper bound we start almost exactly the same as in the previous proof. First assume that $r_1 = 0$ and let $\mathcal{F}_1 \subset \mathcal{F}$ be the set of balls with distance greater than $\sqrt{2}$ from B_1.

Consider two elements B_j, B_k of $\mathcal{F}_1$ such that $R_j \cup R_k$ is as large as possible. Note that $R_j \cup R_k$ does not cover S^1 because $\alpha_i < \frac{\pi}{4}$ for every $B_i \in \mathcal{F}_1$. Let R'_j and R'_k be components of R_j and R_k, respectively, such that $R'_j \cap R'_k \neq \emptyset$. For every other $B_i \in \mathcal{F}_1$, let $R'_i = R_i \cap \left(R'_j \cup R'_k\right)$. The $T(4)$ property implies that every 3 members of $\mathcal{R} = \{R'_i : B_i \in \mathcal{F}_1\}$ have a non-empty intersection. This, together with the maximality of $R'_j \cup R'_k$, implies that every R'_i is a connected arc. We have that every two members of $\mathcal{R}$ have non-empty intersection within the arc $R'_j \cup R'_k$. Now we may apply Helly's theorem to obtain $\bigcap_{R'_i \in \mathcal{R}} R'_i \neq \emptyset$. This implies that there is a line transversal to $\mathcal{F}_1$ which passes through B_1.

By the same argument as in the previous proof we find that the family $\sqrt{2}\mathcal{F}$ has a line transversal through B_1 and conclude for the general case that $2\sqrt{2}\mathcal{F}$ has a line transversal. □

Proof of Theorem 5.5. This is done by induction on the number of circles. If $\#\mathcal{F} \leq 3$, the result is trivial. Assume that $\#\mathcal{F} = n$ and that the result is true for every family with less than n elements. The proof is divided in two cases.

Case 1: Every B_i is at distance less than or equal to ρ from B_1.

Consider a line ℓ closest to ρB_1 and transversal to $\rho\mathcal{F} \setminus \{\rho B_1\}$. If ℓ does not intersect ρB_1, then there is an element $\rho B_j \in \rho F$ tangent to ℓ and not on the side of ℓ that ρB_1 is on, otherwise ℓ would not be closest to ρB_1. This is a contradiction because ρB_1 intersects every element of $\rho\mathcal{F}$ but ℓ separates ρB_1 from ρB_j.

Case 2: There exists $B_j \in \mathcal{F}$ with distance greater than ρ from B_1.

Assume that B_j has largest distance from B_1. Because of Lemma 5.6 we can assume that $r_1 = 0$ and prove instead that $\frac{\rho}{2}\mathcal{F}$ has property T. We may further assume that the distance between any B_i and B_1 is greater than $\frac{\rho}{2}$, because otherwise $\frac{\rho}{2}B_i$ contains B_1.

Let $t \geq 1$ and let $R \subset S^1$ be a component of the region corresponding to a ball B, if tB does not contain the point B_1 then define $b_t(R)$ as the

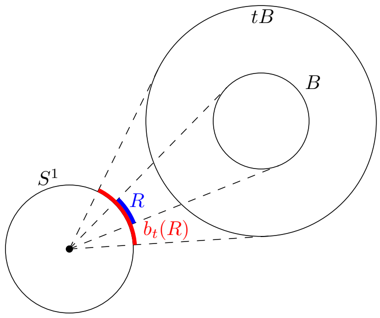

Figure 5.4: The arcs R and $b_t(R)$.

component of the region corresponding to the ball tB that contains R (see Figure 5.4). Note that if R covers an angle $2\theta < \pi$, then $b_t(R)$ covers an angle $2\arcsin(t\sin(\theta))$.

Let R'_j be a component of R_j and for each other i let R'_i be a component of R_i that intersects R'_j. We shall show that every two elements of the family $\left\{b_{\rho/2}(R'_i) \cap R'_j\right\}_i$ intersect. Take B_k and B_l different from B_j, if R'_k and R'_l intersect in R_j we are done. If not, then because of property $T(3)$ we can choose a semicircle $H \subset S^1$ containing R'_j that has its boundary contained in $R_k \cap R_l$.

Set $Q_k = \overline{(R'_k \cap H) \setminus R_j}$ and $Q_l = \overline{(R'_l \cap H) \setminus R_j}$, let $2\alpha_k$ and $2\alpha_l$ be the angles covered by the arcs Q_k and Q_l, respectively. We now have something like Figure 5.5. It is enough to show that $b_{\rho/2}(Q_k)$ and $b_{\rho/2}(Q_l)$ intersect in R_j. The set $b_{\rho/2}(Q_k)$ covers all of H or a region of H with size $\arcsin\left(\frac{\rho}{2}\sin(\alpha_k)\right) + \alpha_k$ and the same holds for $b_{\rho/2}(Q_l)$. Therefore we must show

$$\arcsin\left(\frac{\rho}{2}\sin(\alpha_k)\right) + \arcsin\left(\frac{\rho}{2}\sin(\alpha_l)\right) \geq \pi - \alpha_k - \alpha_l.$$

The function $\alpha \mapsto \arcsin(t\sin(\alpha))$ is convex for any $t \geq 1$, so

$$\arcsin\left(\frac{\rho}{2}\sin(\alpha_k)\right) + \arcsin\left(\frac{\rho}{2}\sin(\alpha_l)\right) \geq 2\arcsin\left(\frac{\rho}{2}\sin\left(\frac{\alpha_k+\alpha_l}{2}\right)\right).$$

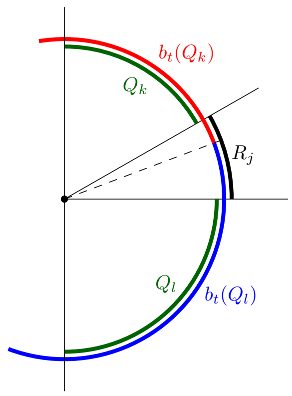

Figure 5.5: The arcs $b_t(Q_k)$ and $b_t(Q_l)$ intersecting in R_j

Let $\gamma = \frac{\alpha_k+\alpha_l}{2}$, it is enough to prove that $\arcsin\left(\frac{\rho}{2}\sin(\gamma)\right) \geq \frac{\pi}{2} - \gamma$, or equivalently, that $\frac{\rho}{2}\tan(\gamma) \geq 1$. Since the distance between B_1 and B_j is larger than ρ, then $\rho\sin(\frac{\pi}{2} - 2\gamma) < 1$. This implies that $\gamma \geq \frac{1}{2}\arccos(\frac{1}{\rho})$, or equivalently $\tan(\gamma) \geq \frac{\sqrt{\rho^2-1}}{(\rho+1)}$. This and the definition of ρ imply

$$\frac{\rho}{2}\tan(\gamma) \geq \frac{\rho\sqrt{\rho^2-1}}{2(\rho+1)} \geq 1.$$

The result now follows after applying Helly's theorem. $\square$

Chapter 6

Longest lattice convex chains

This is joint work with I. Bárány and has been submitted to Computational Geometry: Theory and Applications.

Given a convex body $K \subset \mathbb{R}^2$ and $t > 0$, let n be the largest possible number of vertices a lattice polygon contained in tK can have. In [9], I. Bárány and M. Prodromou study the number n and determine its asymptotic behaviour as $t \to \infty$. In order to do this, they define $m(T)$ as the maximum number of vertices that a *convex lattice chain* within a triangle T can have (see [9] for precise definitions). The behaviour of $m(tT)$ is described in terms of $|T|$ as $t \to \infty$. We ask a similar question here, but remove the factor t completely.

Define $\mathcal{G}$ as the set of triangles T in the plane with two specified vertices, v_0 and v_1, belonging to $\mathbb{Z}^2$. Distinct points $p_0, p_1, \ldots, p_n \in \mathbb{Z}^2 \cap T$ form a convex lattice chain in T (from v_0 to v_1) if $p_0 = v_0$, $p_n = v_1$ and the convex hull of $\{p_0, \ldots, p_n\}$ has exactly $n+1$ vertices, namely $p_0, \ldots, p_n$. The length of this convex chain is n. Let $\ell(T)$ denote the largest n such that T contains a convex lattice chain of length n (from v_0 to v_1). We are interested in the maximal value of $\ell(T)$ when the area, $|T|$, of T is fixed. Here is our main result, which is made more precise in Theorem 6.3.

Theorem 6.1. *There is $t_0 > 0$ such that for all triangles $T \in \mathcal{G}$ with $|T| > t_0$*

$$\frac{1}{8}(\ell(T) - 1)\ell(T)^2 \leq |T|,$$

and this estimate cannot be improved.

A few things are known about $\ell(T)$. Andrews [2] showed in 1963 that the area of a convex lattice n-gon is at least cn^3 for some constant c. This result is in fact more general and applies in any dimension. It has been proved in [31] and [10] that the value of c is at least $1/(8\pi^2)$, implying in our case that

$$|T| \geq |\mathrm{conv}\{p_0, p_1, \ldots, p_n\}| \geq \frac{(n+1)^3}{8\pi^2}.$$

Consequently

$$\frac{1}{8\pi^2}\ell(T)^3 \leq |T|.$$

Another bound on $\ell(T)$ comes when using the lattice-width $W(T)$ of T (see Section 2.1 for definition). It is clear that at most $\lfloor W(K) + 1 \rfloor$ consecutive lattice lines orthogonal to the lattice-width direction of K intersect K. As every lattice line contains at most two points from a convex lattice chain, the bound

$$\ell(T) \leq 2W(T) + 1$$

is immediate.

Just like $|T|$ and $W(T)$, $\ell(T)$ is invariant under lattice preserving affine transformations. Thus the use of the lattice width is very natural here. This invariance is important and will be used later. For instance, we assume from now on that one specified vertex of T, namely v_0, coincides with the origin.

Here is another result from [9] concerning the typical behaviour of $\ell(T)$.

Let $T \in \mathcal{G}$ (with $v_0 = (0,0)$) and assume that $\lambda \to \infty$ in such a way that $\lambda v_1 \in \mathbb{Z}^2$. Theorem 4.1 from [9] says that

$$\lim_{\lambda \to \infty} \lambda^{-2/3} \ell(\lambda T) = \frac{6}{(2\pi)^{2/3}} \sqrt[3]{|T|}.$$

This result can be strengthened.

Theorem 6.2. *There are constants $C, D > 0$ such that if $T \in \mathcal{G}$ with $W(T) > C|T|^{1/3}$, then*

$$\left| \ell(T) - \frac{6}{(2\pi)^{2/3}} \sqrt[3]{|T|} \right| \leq D \frac{\log \left(W(T)|T|^{-1/3} \right)}{W(T)|T|^{-1/3}}.$$

Note that for a typical "fat" triangle T, $W(T)$ is of order $\sqrt{|T|}$. For the rest $\ell(T)$ is of order $W(T)$. The proof of Theorem 6.2 is almost identical with that of Theorem 4.1 in [9] and is therefore omitted.

6.1 Reformulation

We can turn around the question by asking the following minimization problem, to be called Min(n):

$$\text{minimize } |T| \text{ subject to } T \in \mathcal{G},\ \ell(T) = n.$$

Let T_n ($n \geq 3$) be the triangle with vertices

$$v_0 = (0,0),\ v_1 = \left(\frac{1}{2} n(n-1), n \right) \text{ and } v_2 = \left(\frac{1}{2} n(n-1), \frac{1}{2} n \right),$$

then $|T_n| = \frac{1}{8}(n-1)n^2$. For a fixed n define

$$p_k = \left(\frac{1}{2} n(n-1) - \frac{1}{2}(n-k)(n-k-1), k \right)$$

for $k = 0, \ldots, n$. It is easy to check that $p_0, \ldots, p_n$ is a convex lattice chain of length n in T_n from v_0 to v_1.

Theorem 6.3. *There exists $n_0 > 0$ such that for all $n > n_0$ the following holds: If the triangle $T \in \mathcal{G}$ contains a convex lattice chain of length n, then*

$$\frac{1}{8}(n-1)n^2 \le |T|.$$

Equality holds here if and only if T is the image of T_n under a lattice preserving affine transformation.

There are only two cases we know of where T_n is not the minimizer for Min(n), namely:

- When $n = 3$. Let p_0, p_1, p_2, p_3 be equal to $(0,0)$, $(1,0)$, $(2,1)$, $(2,2)$ respectively. Then $|T| = 2$ which is smaller by $\frac{1}{4}$ than the expected $|T_3| = \frac{9}{4}$.
- When $n = 5$. Let p_0, p_1, p_2, p_3, p_4 and p_5 be equal to $(0,0)$ $(1,0)$, $(3,1)$, $(4,2)$, $(6,5)$ and $(7,7)$ respectively, then $|T| = \frac{49}{4}$ which is smaller than $|T_5| = \frac{25}{2}$, again by $\frac{1}{4}$.

The following sections of this chapter are devoted to proving Theorem 6.3.

6.2 Reduction

We assume from now on that the points $p_0, p_1, \ldots, p_n$ lie in this order on the perimeter of their convex hull. Let $z_i = p_i - p_{i-1}$, $i = 1, \ldots, n$, these are the edge vectors of the convex lattice chain and determine the convex lattice chain completely. For T_n this is just the vectors $(0,1), (1,1), (2,1), \ldots, (n-$

1, 1). By ordering the vectors z_i by increasing slope, we can construct a convex lattice chain having them as edge vectors and every convex lattice chain defines the minimal area triangle T that contains it.

We consider the set $\mathcal{H}_n$ of triangles $\triangle$ satisfying the conditions

- the origin is a vertex of $\triangle$,
- $|\triangle \cap \mathbb{P}^2| = n$,
- each side of $\triangle$ contains a point from $\mathbb{P}^2$,
- there is no smaller triangle $\triangle' \subset \triangle$ that satisfies the above three.

Every $\triangle \in \mathcal{H}_n$ gives rise to a convex lattice chain with n edges, and every convex lattice chain defines the minimal area triangle that contains it. This way every $\triangle \in \mathcal{H}_n$ gives rise to a triangle $T(\triangle)$. For example, the triangle $\triangle_n = \mathrm{conv}\{(0,0),(0,1),(n-1,1)\}$ gives $T(\triangle) = T_n$.

A useful reduction for Min(n) is given by the following lemma.

Lemma 6.4. *There is $\triangle \in \mathcal{H}_n$ with $\triangle \cap \mathbb{P}^2 = \{z_1,\ldots,z_n\}$ such that the corresponding triangle $T(\triangle)$ is a minimizer for the problem* Min(n).

Proof. Let T be a minimizer of Min(n) with vertices $v_0 = (0,0), v_1, v_2$ and having $z_1,\ldots,z_n$ as the edge vectors of its longest lattice convex chain. All the edge vectors $z_1,\ldots,z_n$ are in $\mathbb{P}^2$ as otherwise the area of T can be decreased.

Let P be the parallelogram with vertices $v_0, v_2, v_1, v_3 = v_1 - v_2$ and let u_2 and u_3 be points on the segments $[(0,0), v_2]$ and $[(0,0), v_3]$ so that $[u_2, u_3]$ is parallel with $[v_2, v_3]$, the triangle $\triangle = \mathrm{conv}\{(0,0), u_2, u_3\}$ contains $z_1,\ldots,z_n$, and some edge vector is on the segment $[u_2, u_3]$ (see Figure 6.1).

Let z_k be the edge vector on $[u_2, u_3]$ closest to u_2 and assume that there exists $z \in \triangle \cap \mathbb{P}^2 \setminus \{z_1,\ldots,z_n\}$.

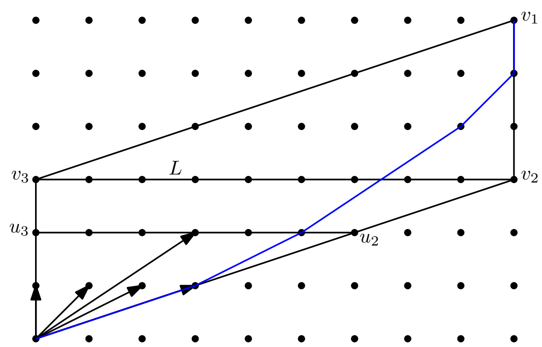

Figure 6.1: Lemma 6.4

If $z \notin [u_2, u_3]$, we replace the edge vector z_k for z. The resulting new lattice convex chain can be fitted in a triangle having $(0,0)$ and $(\sum z_i)-z_k+z$ as two of its vertices and two of its sides parallel to v_2 and v_3. The area of this triangle is less than $|T|$, which contradicts our original assumption. Therefore we can assume $z \in [u_2, u_3]$.

If z is closer to u_3 than z_k then we replace z_k for z, doing this does not change $|T|$. By repeating this process several times if necessary, we may assume that $[z_k, u_3] \cap \mathbb{P}^2 \subset \{z_1, \dots, z_n\}$. Now we can choose $\epsilon, \epsilon' \geq 0$ small enough so that the new triangle $\tilde{\triangle} = \text{conv}\{(0,0), (1-\epsilon)u_2, (1+\epsilon')u_3\}$ satisfies $\triangle \cap \mathbb{P}^2 = \tilde{\triangle} \cap \mathbb{P}^2$ and $[u_2, u_3] \cap \mathbb{P}^2 = \{z_k\}$. The triangle $\tilde{\triangle}$ satisfies the required conditions and $T = T(\tilde{\triangle})$. $\square$

Consider now the following problem, to be called Red(n):

$$\text{minimize } |T(\triangle)| \text{ subject to } \triangle \in \mathcal{H}_n.$$

Theorem 6.5. *For $n > n_0$ the triangle $\triangle_n$ is a solution to* Red(n). *This solution is unique apart from a lattice preserving affine transformation.*

It suffices to prove this theorem only. The plan for the proof is given next.

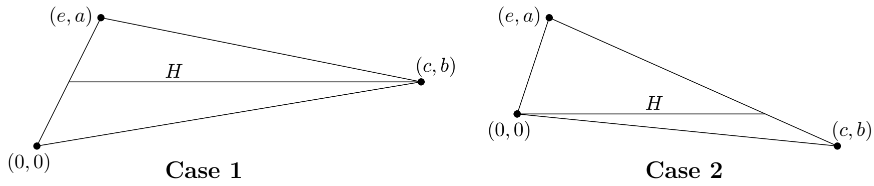

Figure 6.2: The triangle $\triangle$

6.3 Plan of proof

First we bring $\triangle \in \mathcal{H}_n$ into standard position by a lattice preserving affine transformation as follows. Set $W(\triangle) = w$ and choose a lattice preserving affine transformation so that the lattice-width vector of $\triangle$ is $(0, 1)$. Now w also represents the ordinary width in direction $(0, 1)$. Let $(0, 0), (e, a), (c, b)$ be the vertices of $\triangle$ (see figure 6.2). We can assume that $0 \le e \le a$, $|b| \le a$ and $ac - be = 2|\triangle| > 0$, by applying a suitable lattice preserving affine transformation.

Let h be the length of the longest horizontal chord, H, of $\triangle$. Then $|\triangle| = \frac{1}{2}ch$. We have to consider two separate cases.

Case 1. When $b \ge 0$. Then $W = a$ and (c, b) is an endpoint of H.

Case 2. When $b < 0$. Then $W = a - b$ and $(0, 0)$ is an endpoint of H.

It is not difficult to see that in both cases $c/2 \le h \le c$, so

$$\frac{cw}{4} \le |\triangle| \le \frac{cw}{2}. \tag{6.1}$$

Now let

$$S = (S_x, S_y) = \sum_{z \in \triangle \cap \mathbb{P}^2} z.$$

The area of $T = T(\triangle)$ can be determined in terms of $\triangle$ by

$$|T| = \frac{\det((c,b), S)\det(S, (e,a))}{2\det((c,b),(e,a))} = \frac{(cS_y - bS_x)(aS_x - eS_y)}{4|\triangle|}. \tag{6.2}$$

It is well known that the density of $\mathbb{P}^2$ in $\mathbb{Z}^2$ is $\frac{6}{\pi^2}$ (e.g. Theorem 459 in [18]). So in a typical triangle $\triangle$, we expect the number of primitive lattice points in $\triangle$ to be close to $\frac{6}{\pi^2}|\triangle|$ and their sum S to be close to $\frac{6}{\pi^2}|\triangle|g$, where g is the centre of gravity of $\triangle$. If this is the case, it follows from (6.2) that $|T|$ is close to $\frac{4|\triangle|^3}{\pi^4}$ and $\frac{|T|}{n^3} \approx \frac{\pi^2}{54} > \frac{1}{8}$.

In the first step of the proof we formalize this argument for triangles with large lattice width. Namely, we show the existence of a finite w_0 such that for $w > w_0$ the inequality $\frac{|T|}{n^3} > \frac{1}{8}$ holds.

In the second step we assume that $w \leq w_0$, and show, by subtle though lengthy and technical estimates, that $\frac{|T|}{n^3} > \frac{1}{8}$ for $w \geq 250$ if n, and then c are large enough.

After this we are left with finitely many cases, roughly 250^2 of them. Here we suppose again that c is large enough. In each case the limit when $c \to \infty$ of $T(\triangle)/n^3$ can be exactly expressed as a rational function of the parameters a, b. The third step of the proof is carried out by a computer using Mathematica [39], and consists of careful checking of these cases. The outcome is, again, that $\frac{|T|}{n^3} > \frac{1}{8}$, apart from three special cases that are treated in the last step of the proof separately.

6.4 Large lattice width

Here we prove that $\triangle \in \mathcal{H}_n$ does not solve Red(n) if the lattice width of $\triangle$ is large enough.

Lemma 6.6. *There is $w_0 > 0$ so that if $\triangle \in \mathcal{H}_n$ and $W(\triangle) > w_0$ then $|T| = |T(\triangle)| > \frac{1}{8}n^3$.*

Proof. We assume that $w = W(\triangle)$ is large. For this section we use Vinogradov's convenient $f(c, w) \ll g(c, w)$ notation meaning, in our case, the existence of constants $D_1, D_2 > 0$ such that $f(c,w) \le D_1 g(c,w)$ for all $c \ge w \ge D_2$. Here $c \ge w$ follows from $W(\triangle) = w$. For instance $\sum_{d=1}^{w} \frac{|\mu(d)|}{d} \ll \log w$, since $\sum_{d=1}^{w} \frac{|\mu(d)|}{d} < 1 + \log w \ll \log w$.

We apply a commonly used method involving the Möbius function.

$$n = \sum_{z \in T \cap \mathbb{P}^2} 1 = \sum_{z \in \triangle \cap \mathbb{Z}^2} \sum_{d|z} \mu(d) = \sum_{d=1}^{w} \mu(d) \# \left(\mathbb{Z}^2 \cap \frac{1}{d} \triangle \right).$$

Here the term $\# \left(\mathbb{Z}^2 \cap \frac{1}{d}\triangle \right)$ is approximately equal to $\left| \frac{1}{d} \triangle \right| = \frac{1}{d^2} |\triangle|$, so we may write $\left| \mathbb{Z}^2 \cap \frac{1}{d} \triangle \right| = \frac{1}{d^2} |\triangle| + E(d)$ where $E(d)$ is an error term. Then

$$n = |\triangle| \sum_{d=1}^{w} \frac{\mu(d)}{d^2} + \sum_{d=1}^{w} \mu(d) E(d). \tag{6.3}$$

The target is to estimate the error term $E = \sum_{d=1}^{w} \mu(d) E(d)$. To this end, for every $z \in \mathbb{Z}^2$, define Q_z to be the square of side length 1 with centre z and sides parallel to the axes. We define the sets

$$\Gamma_d^+ = \left\{ z \in \mathbb{Z}^2 : z \notin \frac{1}{d} \triangle \text{ and } Q_z \cap \frac{1}{d} \triangle \neq \emptyset \right\},$$
$$\Gamma_d^- = \left\{ z \in \mathbb{Z}^2 : z \in \frac{1}{d} \triangle \text{ and } Q_z \setminus \frac{1}{d} \triangle \neq \emptyset \right\}.$$

Thus Γ_d^+ and Γ_d^- are the centres of the *boundary squares* Q_z that intersect the boundary of $\frac{1}{d}\triangle$.

Claim 6.7. $|\Gamma_d^+| + |\Gamma_d^-| \le 2\lceil \frac{c}{d} \rceil + 2 \lceil \frac{w}{d} \rceil + 4 \ll \frac{c+w}{d}$.

Proof. The sides of the smallest axis parallel rectangle containing $\frac{1}{d}\triangle$ have lengths $\frac{c}{d}$ and $\frac{w}{d}$, which gives the bound on the number of boundary squares. □

Define now A_d^+ to be the union over $z \in \Gamma_d^+$ of the sets $Q_z \cap \frac{1}{d}\triangle$ and

A_d^- be the union over $z \in \Gamma_d^-$ of the sets $Q_z \setminus \frac{1}{d}\triangle$. Clearly $|A_d^+| \le \#\Gamma_d^+$ and $|A_d^-| \le \#\Gamma_d^-$.

Since we have $\#\left(\mathbb{Z}^2 \cap \frac{1}{d}\triangle\right) = \left|\frac{1}{d}\triangle\right| + |A_d^+| - |A_d^-|$, it follows that

$$E(d) = |A_d^+| - |A_d^-| \text{ and so } |E(d)| = \big||A_d^+| - |A_d^-|\big| \ll \frac{c+w}{d}. \tag{6.4}$$

Consequently

$$\left|\sum_{d=1}^{w} \mu(d)E(d)\right| \ll (c+w)\log w \ll c\log w.$$

As $cw \ll |\triangle| \ll cw$ this implies that

$$\left|n - \sum_{d=1}^{w} \frac{\mu(d)}{d^2}|\triangle|\right| \ll |\triangle|\frac{\log w}{w}. \tag{6.5}$$

Estimating the sum of the primitive vectors in $\triangle$ is similar, just a little more involved. Let $g = \frac{1}{3}(e+c, a+b)$ be the centre of gravity of $\triangle$. Then

$$\begin{aligned} S = \sum_{z \in \triangle \cap \mathbb{P}^2} z = \sum_{z \in \triangle \cap \mathbb{Z}^2} \sum_{d|z} \mu(d) z = \sum_{d=1}^{w} d\mu(d) \sum_{z \in \frac{1}{d}\triangle} z \\ = |\triangle| \sum_{d=1}^{w} \frac{\mu(d)}{d^2} g + \sum_{d=1}^{w} d\mu(d)\vec{E}(d), \end{aligned} \tag{6.6}$$

where $\vec{E}(d) = (E_x(d), E_y(d)) \in \mathbb{R}^2$ represents the error here. Since

$$\int_{\frac{1}{d}\triangle} z dz = \frac{1}{d^2}|\triangle| g,$$

we have, similarly as in Claim 6.7, that

$$|E_x(d)| \le \left|\int_{A_d^+} x dz - \int_{A_d^-} x dz\right| \ll \frac{c(c+w)}{d^2}$$

and

$$|E_y(d)| \leq \left| \int_{A_d^+} y dz - \int_{A_d^-} y dz \right| \ll \frac{w(c+w)}{d^2}.$$

For simpler writing we define

$$\sigma_w = \sum_{d=1}^{w} \frac{\mu(d)}{d^2},\ E_x = \sum_{d=1}^{w} d\mu(d) E_x(d), \text{ and } E_y = \sum_{d=1}^{w} d\mu(d) E_y(d).$$

Thus with notation $S = (S_x, S_y)$ and $g = (g_x, g_y)$, $S_x = \sigma_w |\triangle| g_x + E_x$ and $S_y = \sigma_w |\triangle| g_y + E_y$. Then

$$|E_x| = \left| \sum_{d=1}^{w} d\mu(d) E_x(d) \right| \ll c^2 \log w$$

and

$$|E_y| = \left| \sum_{d=1}^{w} d\mu(d) E_y(d) \right| \ll cw \log w.$$

We use (6.2) to compute $|T|$. First

$$\begin{aligned} cS_y - bS_x =& c(|\triangle|\sigma_w g_y + E_y) - b(|\triangle|\sigma_w g_x + E_x) \\ =& |\triangle|\sigma_w(cg_y - bg_x) + (cE_y - bE_x) = \sigma_w \frac{2}{3}|\triangle|^2 + (cE_y - bE_x), \end{aligned}$$

and similarly

$$\begin{aligned} aS_x - eS_y =& a(|\triangle|\sigma_w g_x + E_x) - e(|\triangle|\sigma_w g_y + E_y) \\ =& |\triangle|\sigma_w(ag_x - eg_y) + (aE_x - eE_y) = \sigma_w \frac{2}{3}|\triangle|^2 + (aE_x - eE_y), \end{aligned}$$

where we used the fact that $\frac{2}{3}|\triangle| = cg_y - bg_x = ag_x - eg_y$. So we have

$$|T| = |\triangle|^3 \left(\frac{1}{3}\sigma_w + \frac{cE_y - bE_x}{2|\triangle|^2} \right) \left(\frac{1}{3}\sigma_w + \frac{aE_x - eE_y}{2|\triangle|^2} \right). \tag{6.7}$$

Here $|cE_y|, |aE_x|, |bE_x| \ll c^2 w \log w$ and $|eE_y| \ll cw^2 \log w$, thus

$$|cE_y - bE_x| \ll c^2 w \log w \text{ and } |aE_x - eE_y| \ll c^2 w \log w.$$

Using (6.1) it follows that

$$\left| |T| - \frac{\sigma_w^2}{9} |\triangle|^3 \right| \ll |\triangle|^3 \left(\frac{\log w}{w} + \frac{\log^2 w}{w^2} \right) \ll |\triangle|^3 \frac{\log w}{w}.$$

This inequality, together with (6.5) finishes the proof quickly. For suitable positive constants D_1, D_2, D_3 we have

$$\frac{|T|}{n^3} \geq \frac{\left(\frac{\sigma_w^2}{9} - D_1 \frac{\log w}{w} \right) |\triangle|^3}{\left(\sigma_w + D_2 \frac{logw}{w} \right)^3 |\triangle|^3} \geq \frac{1}{9\sigma_w} - D_3 \frac{\log w}{w}.$$

Since σ_w tends to $\frac{6}{\pi^2}$ as $w \to \infty$, the right hand here tends to $\frac{\pi^2}{54} = 0.18277 \dots > \frac{1}{8}$. This shows that $\frac{|T|}{n^3} > \frac{1}{8}$ if w is large enough. $\square$

6.5 Auxiliary lemmas

We need some preparations for the case $w \leq w_0$. Recall that we keep the parameters a, b, e fixed and wish to show that $\lim |T|/n^3 > 1/8$ as $n \to \infty$, or equivalently, as $c \to \infty$.

First we get rid of the parameter e, we simply change the triangle $\triangle$ by replacing its vertex (e, a) by $(0, a)$. It is clear that the change in $\# \left(\triangle \cap \mathbb{P}^2 \right)$ is at most w^2, and the change in S_x, S_y respectively, is at most wc and w^2 which is of smaller order than the corresponding error terms (as we shall see). We keep the notation $\triangle$ for the new triangle.

We also have in both Case 1 (when $b \geq 0$) and Case 2 (when $b < 0$) that $|\triangle| = \frac{ac}{2}$, which will work better than (6.1).

We show now that $|b| \geq 1$. Since the edge vector $z_1 \in \mathbb{Z}^2$ of the convex lattice chain lies on the segment $[(0,0),(c,b)]$, $|b| < 1$ implies $b = 0$, and then $z_1 = (1,0)$ is the only possibility. Removing this vector from the convex lattice chain can only decrease $\lim|T|/n^3$ and does not affect the lattice width direction, as one can check easily. We assume further that $a - b \geq 1$. This is evident in Case 2, and if $a - b < 1$ in Case 1, then one can change a and b a little so that $a - b \geq 1$ while $\triangle \cap \mathbb{P}^2$ remains unchanged.

Recall that in Case 1 $w = a$ and in Case 2 $a < w = a - b \leq 2a$ since $|b| \leq a$. So w and a are comparable, and in the next section it will be more convenient to work with a instead of w.

We will need a simple bound on $\sum_1^a |\mu(d)|$ and on $\sum_1^a \frac{|\mu(d)|}{d}$.

Lemma 6.8. *If $a \geq 44$ then*

$$\sum_{d=1}^{a} |\mu(d)| \leq \frac{2}{3}a,$$

and if $a \geq 126$ then

$$\sum_{d=1}^{a} \frac{|\mu(d)|}{d} \leq \frac{6}{10} + \frac{7}{10}\log a.$$

Proof. Let I be a set containing 36 consecutive positive integers and I' be equal to I after removing the multiples of 4 and 9. Since $\mu(d) = 0$ if d is divisible by a square,

$$\sum_{d \in I} |\mu(d)| \leq \sum_{d \in I'} 1 = \frac{2}{3}\#I.$$

To prove the first inequality, it only remains to verify it for $a \in [44, 80]$. This is easily done with help of a computer.

For the other inequality we first show that

$$\sum_{d\in I'}\frac{1}{d}<\frac{7}{10}\sum_{d\in I}\frac{1}{d}$$

if all the elements of I are larger than 125. Note that this inequality is of the form

$$\sum_{d\in I''}\frac{1}{d+m}<\frac{7}{10}\sum_{d=0}^{36}\frac{1}{d+m},$$

where $I''\subset[0,35]$ is some set of integers and $m>125$. There are 36 different possibilities for the set I'' depending on the value of m mod 36. For each of these cases, the inequality reduces to showing that a polynomial on m of degree at most 36 is positive for $m>125$. After this the only thing left is to verify the original inequality for $a\in[126,162]$. Both can be confirmed easily with the help of a computer. $\square$

6.6 Medium lattice width

Here we prove a strengthening of Lemma 6.6 for the case when $W(\triangle)$ is not too small but at most w_0. More precisely we show the following.

Lemma 6.9. *There is $n_0>0$ so that if $\triangle\in\mathcal{H}_n$, $n>n_0$ and $a>250$ then $|T|=|T(\triangle)|>\frac{1}{8}n^3$.*

Proof. By Lemma 6.6 we can assume $w\le w_0$, and so $c\to\infty$ as $n\to\infty$. We show that for some $\epsilon>0$, $\lim_{c\to\infty}\frac{|T(\triangle)|}{n^3}>\frac{1}{8}+\epsilon$ when $a>250$ and $w\le w_0$. Since here both $|T(\triangle)|$ and n^3 are of order c^3 we can ignore smaller order terms during the computations.

We want to have sharper and explicit estimates instead of those in section 6.4. For simpler notation set $\widetilde{\triangle}=\frac{1}{d}\triangle$, let $\widetilde{a}=\frac{a}{d}$, $\widetilde{b}=\frac{b}{d}$, and $\widetilde{c}=\frac{c}{d}$.

We begin with Case 1, so $a=w$. We are going to estimate $E(d)=$

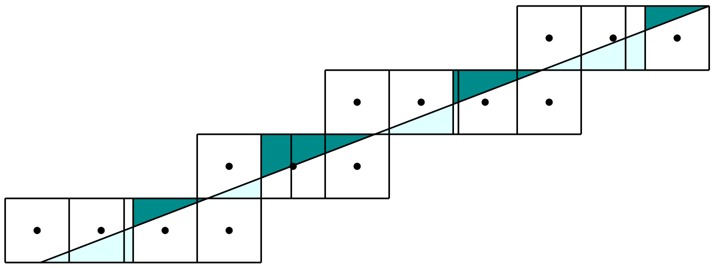

Figure 6.3: A_d^+ and A_d^- near L and correction lines

$|A_d^+| - |A_d^-|$ again. The triangle $\widetilde{\triangle}$ has a lower side $L = [(0,0),(\widetilde{c},\widetilde{b})]$ and upper one $U = [(0,\widetilde{a}),(\widetilde{c},\widetilde{b})]$. We ignore the boundary cells on its vertical side since they cause only a minimal ($O(1)$) error. Figure 6.3 shows that $A_d^+ \cup A_d^-$ near L (respectively U) consists of triangles, bounded by L (and U), and horizontal segments (on the lines $y = m + \frac{1}{2}$, m an integer) and vertical segments of unit length (on the lines $x = m + \frac{1}{2}$, m an integer). These triangle alternately belong to A_d^+ and A_d^- and two consecutive triangles have almost the same area. We modify these triangles by moving the unit segment containing their vertical side so that L (respectively U) halves the new unit segment. We call this a *correction*. Each correction changes the sum of the signed area of the two triangles it affects by at most $\frac{1}{2}$. After the correction consecutive triangles have the same area so they cancel in $E(d) = |A_d^+| - |A_d^-|$.

Even more generally the following holds. Call a *valid period* any vertical strip of width $\frac{\widetilde{c}}{\widetilde{b}}$ between the lines $x = 0$ and $x = \widetilde{c}$. Then the sum of the signed areas of the triangles in A_d^+ and A_d^- near L in a valid strip equals zero.

Consequently the contribution of $|A_d^+| - |A_d^-|$ from triangles near L is at most the area of one of the triangles, which is $\frac{\widetilde{c}}{8\widetilde{b}}$ if $\widetilde{b} \geq 1$. There is no valid period if $\widetilde{b} < 1$, and then the contribution of $|A_d^+| - |A_d^-|$ near L is at most $\frac{\widetilde{c}}{2}$.

Similarly, the contribution of $|A_d^+| - |A_d^-|$ near U is at most $\frac{\widetilde{c}}{8(\widetilde{a}-\widetilde{b})}$ if $\widetilde{a} - \widetilde{b} \geq 1$ and $\frac{\widetilde{c}}{2}$ if $\widetilde{a} - \widetilde{b} < 1$. Then, ignoring the correction terms,

$$\begin{aligned}
|E| &= \left| \sum_1^a \mu(d)E(d) \right| \leq \sum_1^a |\mu(d)| \left| |A_d^+| - |A_d^-| \right| \\
&\leq \sum_1^b \frac{c}{8b}|\mu(d)| + \sum_b^a \frac{c}{2}\frac{|\mu(d)|}{d} + \sum_1^{a-b} \frac{c}{8(a-b)}|\mu(d)| + \sum_{a-b}^a \frac{c}{2}\frac{|\mu(d)|}{d} \\
&\leq \left[\frac{c}{8}\left(\frac{1}{b}\sum_1^b |\mu(d)| + \frac{1}{a-b}\sum_1^{a-b}|\mu(d)| \right) + \frac{c}{2}\left(\sum_b^a \frac{|\mu(d)|}{d} + \sum_{a-b}^a \frac{|\mu(d)|}{d} \right) \right].
\end{aligned}$$

Here we can assume by symmetry that $b \leq a - b$. The first term in the square brackets is bounded using Lemma 6.8 by $\frac{c}{2}(1 + \frac{2}{3})$. For the second term we can use the method as in the proof of Lemma 6.8 as follows: Let $\mu^*(d) = 0$ if 4 or 9 divides d and $\mu^*(d) = 1$ otherwise. Then

$$\begin{aligned}
\sum_b^a \frac{|\mu(d)|}{d} + \sum_{a-b}^a \frac{|\mu(d)|}{d} &\leq \sum_b^a \frac{\mu^*(d)}{d} + \sum_{a-b}^a \frac{\mu^*(d)}{d} \\
&\leq \sum_{b-36m}^a \frac{\mu^*(d)}{d} + \sum_{a-b+36m}^a \frac{\mu^*(d)}{d},
\end{aligned}$$

for every positive integer m such that $b - 36m > 0$. Choose m so that $1 \leq b_0 = b - 36m \leq 36$. Then

$$\begin{aligned}
&\sum_b^a \frac{|\mu(d)|}{d} + \sum_{a-b}^a \frac{|\mu(d)|}{d} \leq \sum_{b_0}^a \frac{\mu^*(d)}{d} + \sum_{a-b_0}^a \frac{\mu^*(d)}{d} \\
&\leq \sum_1^a \frac{\mu^*(d)}{d} + \sum_{a-36}^a \frac{\mu^*(d)}{d} \leq \left(\frac{6}{10} + \frac{7}{10}\log a \right) + \frac{24}{250-36}.
\end{aligned} \tag{6.8}$$

Therefore we have the bound

$$|E| < \frac{c}{2}\left(2.3789 + \frac{7}{10}\log a \right).$$

The same general method applies to $E_y = \sum_1^a d\mu(d)E_y(d)$ where $E_y(d) = \int_{A_d^+} ydz - \int_{A_d^-} ydz$. The integral on the corrections is $\frac{\widetilde{a}}{2}$, small again. On a valid period the contribution in absolute value of the integral near L is at most $\frac{\widetilde{c}}{6}$ and the contribution of the final part is at most $\frac{\widetilde{c}}{4}$ if $\widetilde{b} \geq 1$. The same contribution near U is at most $\frac{\widetilde{c}}{24} + \frac{\widetilde{c}}{8(\widetilde{a}-\widetilde{b})}(\widetilde{b}+1)$ if $\widetilde{a} - \widetilde{b} \geq 1$, and is $\frac{1}{2}\widetilde{c}\widetilde{a}$ if $\widetilde{a} - \widetilde{b} < 1$. This way we obtain, ignoring correction terms and using Lemma 6.8 again,

$$\begin{aligned}
|E_y| = \left|\sum_1^a d\mu(d)E_y(d)\right| &\leq \frac{c}{6}\sum_1^b |\mu(d)| + \frac{c}{4}\sum_b^a |\mu(d)| \\
&+ \left(\frac{c}{24} + \frac{bc}{8(a-b)}\right)\sum_1^{a-b}|\mu(d)| + \frac{c}{8(a-b)}\sum_1^{a-b} d|\mu(d)| + \frac{ca}{2}\sum_{a-b}^a \frac{|\mu(d)|}{d} \\
&< \frac{c}{2}\left(\frac{a}{3} + \frac{a+2b}{18} + \frac{a-b}{6} + a\sum_{a-b}^a \frac{|\mu(d)|}{d}\right) < \frac{c}{2}\left(1.1556a + \frac{7}{10}a\log a\right).
\end{aligned}$$

In this last part we used $b > 0$ and Lemma 6.8.

The estimate for E_x is similar. The correction term is $O(c)$ this time. For the integral on the triangles near L on a given valid period we get the bound $\frac{\widetilde{c}^2}{6\widetilde{b}}$ if $\widetilde{b} \geq 1$ and $\frac{\widetilde{c}^2}{6}$ if $\widetilde{b} < 1$. For those near U we have $\frac{\widetilde{c}^2}{6(\widetilde{a}-\widetilde{b})}$ if $\widetilde{a} - \widetilde{b} \geq 1$ and $\frac{\widetilde{c}^2}{6}$ if $\widetilde{a} - \widetilde{b} < 1$. This gives, ignoring the correction terms again,

$$\begin{aligned}
|E_x| &\leq \frac{c^2}{6b}\sum_1^b |\mu(d)| + \frac{c^2}{6}\sum_b^a \frac{|\mu(d)|}{d} + \frac{c^2}{6(b-a)}\sum_1^{a-b}|\mu(d)| + \frac{c^2}{6}\sum_{a-b}^a \frac{|\mu(d)|}{d} \\
&\leq \frac{c^2}{2}\left[\frac{2}{9} + \frac{1}{3}\sum_b^a \frac{|\mu(d)|}{d} + \frac{2}{9} + \frac{1}{3}\sum_{a-b}^a \frac{|\mu(d)|}{d}\right] \\
&< \frac{c^2}{2}\left(0.6819 + \frac{7}{30}\log a\right).
\end{aligned}$$

Here we used Lemma 6.8 and (6.8).

Recall that $\sigma_a = \sum_1^a \frac{\mu(d)}{d^2}$. We use equation (6.7), which is simpler this time as $e = 0$:

$$|T| = |\triangle|^3 \left(\frac{1}{3}\sigma_a + \frac{cE_y - bE_x}{2|\triangle|^2} \right) \left(\frac{1}{3}\sigma_a + \frac{aE_x}{2|\triangle|^2} \right). \tag{6.9}$$

If $a \geq 250$ then $|\sigma_a - \frac{6}{\pi^2}| < \sum_{250}^{\infty} \frac{1}{d^2} < 0.004$. Finally we use $|cE_y - bE_x| \leq |cE_y| + |bE_x|$ to obtain

$$\lim_{n\to\infty} \frac{|T|}{n^3} \geq \frac{\left(\frac{1}{3}\sigma_a - \frac{1.8374+0.9334\log a}{a}\right)\left(\frac{1}{3}\sigma_a - \frac{0.6819+0.2334\log a}{a}\right)}{\left(\sigma_a + \frac{2.3789+0.7\log a}{a}\right)^3} > \frac{1}{8} + 10^{-5}$$

when $a \geq 250$. This finishes the proof of Case 1.

The proof in Case 2 is almost identical. There are some necessary changes, but no new idea or method. This time b is negative, so $w = a - b$ and $a \geq -b \geq 1$. This means that in (6.3) for instance, d runs from 1 to a instead of w and $\widetilde{a} - \widetilde{b}$ is never smaller than 1. It is easy to see that in this case we can obtain smaller bounds for E, E_y and E_x than in Case 1 and so $\lim \frac{|T|}{n^2} > \frac{1}{8} + 10^{-5}$ when $a \geq 250$. $\square$

6.7 Small lattice width

We have reduced the problem to a relatively small amount of cases. To deal with all of them we use a computer. Once again we assume that $e = 0$. Using the Euler totient function φ we are able to compute n and S, this is done below.

We determine n in Case 1 the following way. Given an integer $k \in [1, b]$, the number of primitive points on the line $y = k$ in $\triangle$ is $\varphi(k)\frac{b}{c} + O(k)$. The same number for an integer $k \in (b, a]$ is $\frac{\varphi(k)}{k}\frac{ca-ck}{a-b} + O(k)$. The $O(k)$ terms are small, and so is their sum.

$$
\begin{aligned}
n &= \sum_{k=\lfloor b+1 \rfloor}^{a} \frac{\varphi(k)}{k} \left(\frac{ca-ck}{a-b} + O(k) \right) + \sum_{k=1}^{b} \frac{\varphi(k)}{k} \left(\frac{c}{b}k + O(k) \right) \\
\frac{n}{c} &= \frac{a}{a-b} \sum_{k=b}^{a} \frac{\varphi(k)}{k} - \frac{1}{a-b} \sum_{k=b}^{a} \varphi(k) + \frac{1}{b} \sum_{k=1}^{b} \varphi(k) + O\left(\frac{1}{c}\right),
\end{aligned}
$$

The computation for S_x, S_y is similar:

$$
\begin{aligned}
(S_x, S_y) = &\sum_{k=\lfloor b+1 \rfloor}^{a} \frac{\varphi(k)}{2k} \left(\frac{ca-ck}{a-b} + O(k) \right) \left(\frac{ca-ck}{a-b} + O(k), 2k \right) \\
&+ \sum_{k=1}^{b} \frac{\varphi(k)}{2k} \left(\frac{c}{b}k + O(k) \right) \left(\frac{c}{b}k + O(k), 2k \right), \\
\frac{S_x}{c^2} =& \frac{a^2}{2(a-b)^2} \sum_{k=b}^{a} \frac{\varphi(k)}{k} - \frac{a}{(a-b)^2} \sum_{k=b}^{a} \varphi(k) \\
&+ \frac{1}{2(a-b)^2} \sum_{k=1}^{a} k\varphi(k) + \frac{1}{2b^2} \sum_{k=1}^{b} k\varphi(k) + O\left(\frac{1}{c}\right), \\
\frac{S_y}{c} =& \frac{a}{a-b} \sum_{k=b}^{a} \varphi(k) - \frac{1}{a-b} \sum_{k=b}^{a} k\varphi(k) + \frac{1}{b} \sum_{k=1}^{b} k\varphi(k) + O\left(\frac{1}{c}\right).
\end{aligned}
$$

The area of T is $\frac{\left(\frac{c}{b}S_y - S_x\right)S_x}{2c/b} = \frac{1}{2}\left(S_y - \frac{b}{c}S_x\right)S_x$, so we want to bound

$$
F_1 = \frac{\left(\frac{S_y}{c} - b\frac{S_x}{c^2}\right)\frac{S_x}{c^2}}{2\left(\frac{n}{c}\right)^3}
$$

from below.

In Case 2 b is negative, but we change its sign and work with it. So $(c, -b)$ is a vertex of $\triangle$ and $1 \leq b \leq a$. Doing a similar computation as in

Case 1 we obtain

$$
\begin{aligned}
\frac{n}{c} =& \frac{a}{a+b}\sum_{k=1}^{a}\frac{\varphi(k)}{k} - \frac{1}{a+b}\sum_{k=1}^{a}\varphi(k) \\
&+ \frac{a}{a+b}\sum_{k=1}^{b}\frac{\varphi(k)}{k} - \left(\frac{1}{b} - \frac{1}{a+b}\right)\sum_{k=1}^{b}\varphi(k) + O\left(\frac{1}{c}\right), \\
\frac{S_x}{c^2} =& \frac{a^2}{2(a+b)^2}\sum_{k=1}^{a}\frac{\varphi(k)}{k} - \frac{a}{(a+b)^2}\sum_{k=1}^{a}\varphi(k) \\
&+ \frac{1}{2(a+b)^2}\sum_{k=1}^{a}k\varphi(k) + \frac{a^2}{2(a+b)^2}\sum_{k=1}^{b}\frac{\varphi(k)}{k} \\
&+ \frac{a}{(a+b)^2}\sum_{k=1}^{b}\varphi(k) + \left(\frac{1}{2(a+b)^2} - \frac{1}{2b^2}\right)\sum_{k=1}^{b}k\varphi(k) + O\left(\frac{1}{c}\right), \\
\frac{S_y}{c} =& \frac{a}{a+b}\sum_{k=1}^{a}\varphi(k) - \frac{1}{a+b}\sum_{k=1}^{a}k\varphi(k) \\
&- \frac{a}{a+b}\sum_{k=1}^{b}\varphi(k) + \left(\frac{1}{b} - \frac{1}{a+b}\right)\sum_{k=1}^{b}k\varphi(k) + O\left(\frac{1}{c}\right).
\end{aligned}
$$

The area of T is $\frac{\left(\frac{c}{b}S_y+S_x\right)S_x}{2c/b} = \frac{1}{2}\left(S_y + \frac{b}{c}S_x\right)S_x$, so we want to bound

$$
F_2 = \frac{\left(\frac{S_y}{c} + b\frac{S_x}{c^2}\right)\frac{S_x}{c^2}}{2\left(\frac{n}{c}\right)^3}
$$

from below.

As $c \to \infty$ we can ignore the terms $O\left(\frac{1}{c}\right)$ and fix the values $\bar{a} = \lfloor a \rfloor$ and $\bar{b} = \lfloor b \rfloor$. Then for $i = 1, 2$, F_i is a rational function of the variables a and b. Not all pairs (a, b) of real numbers come from one of these triangles, but we treat F_i as a function defined on all real numbers. The infimum of F_i in the square $(a, b) \in [\bar{a}, \bar{a}+1] \times [\bar{b}, \bar{b}+1]$ can be computed exactly using the Mathematica function `MinValue`.

If the infimum of F_i is larger than $\frac{1}{8}$ and the infimum of $\frac{n}{c}$ is positive, then it follows that F_i is larger than $\frac{1}{8}$ for all $a \in [\bar{a}, \bar{a}+1]$, $b \in [\bar{b}, \bar{b}+1]$

when c is large enough.

This was verified for all but three of the pairs $(\bar{a}, \bar{b})$ determined by triangles $\triangle$ in standard position with $a \leq 250$ and $1 \leq |b| \leq a$. The pairs on which this could not be verified are $(\bar{a}, \bar{b}) = (1,1), (2,1)$ in Case 1 and $(\bar{a}, \bar{b}) = (1,1)$ in Case 2. We now deal with these last three pairs.

If $(\bar{a}, \bar{b}) = (1,1)$ in Case 1, then $\triangle \cap \mathbb{P}^2$ consists of the vectors $(k,1)$ for $k = 0, \ldots, n-1$. This is the only example for which $\frac{|T|}{n^2(n-1)} = \frac{1}{8}$.

If $(\bar{a}, \bar{b}) = (2,1)$ in Case 1, let $(k,1)$ be the rightmost point on the line $y = 1$ and let $(l,2)$ be the rightmost point on the line $y = 2$. Note that l must be an odd integer and that $2k > l$. Then,

$$\frac{|T|}{n^2(n-1)} = \frac{(2k(1+k) + (1+l)^2)(2k^2 - (1+l)^2 + k(6+4l))}{4k(1+2k+l)(3+2k+l)^2}$$

and it can be verified with Mathematica that under these conditions this is larger than $\frac{1}{8}$ if $n \geq 6$.

In Case 2, when $(\bar{a}, \bar{b}) = (1,1)$, let $(k,1)$ be the rightmost lattice point of $\triangle$ on the line $y = 1$ and let $(m,-1)$ and $(m+l,-1)$ be the first and last lattice points in $\triangle$ on the line $y = -1$. Then,

$$\frac{|T|}{n^3} = \frac{(2+k+k^2+l+l^2+2(1+k)m)(2+k+k^2+l+l^2+2(1+l)m)}{8(3+k+l)^3 m}$$

which can be verified, again with Mathematica, to be larger than $\frac{1}{8}$ if $n \geq 9$.

Bibliography

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić, and M. Sharir, *Crossing patterns of semi-algebraic sets*, Journal of Combinatorial Theory, Series A **111** (2005), no. 2, 310–326.

[2] G. E. Andrews, *A lower bound for the volume of strictly convex bodies with many boundary lattice points*, Transactions of the American Mathematical Society **106** (1963), no. 2, 270–279.

[3] K. Ball, *An elementary introduction to modern convex geometry*, in Flavors of Geometry, 1997, pp. 1–58.

[4] I. Bárány, *Borsuk's theorem through complementary pivoting*, Mathematical Programming **18** (1980), 84–88.

[5] ______, *The limit shape of convex lattice polygons*, Discrete Comput. Geom. **13** (1995), no. 3-4, 279–295.

[6] ______, *Affine perimeter and limit shape*, J. Reine Angew. Math. **484** (1997), 71–84.

[7] ______, *The chance that a convex body is lattice-point free: a relative of Buffon's needle problem*, Random Structures Algorithms **30** (2007), no. 3, 414–426.

[8] I. Bárány and J. Matousek, *The randomized integer convex hull*, Discrete Comput. Geom. **33** (2005), 3–25.

[9] I. Bárány and M. Prodromou, *On maximal convex lattice polygons inscribed in a plane convex set*, Israel Journal of Mathematics **154** (2006), 337–360.

[10] I. Bárány and N. Tokushige, *The minimum area of convex lattice n-gons*, Combinatorica **24** (2004), 171–185.

[11] R. C. Buck and E. F. Buck, *Equipartition of convex sets*, Mathematics Magazine **22** (1949), no. 4, pp. 195–198.

[12] B. Bukh and A. Hubard, *Space crossing numbers*, Combinatorics Probability and Computing **21** (2012), no. 3, 358.

[13] J. Eckhoff, *Transversalenprobleme in der Ebene*, Arch. Math. (Basel) **24** (1973), 195–202.

[14] P. Erdős, *On sets of distances of n points*, The American Mathematical Monthly **53(5)** (1946), 248–250.

[15] H. Federer, *Geometric measure theory*, Springer-Verlag, Berlin, 1969.

[16] T. Figiel, J. Lindenstrauss, and V. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Mathematica **139** (1977), 53–94.

[17] B. Grünbaum, *Common secants for families of polyhedra*, Arch. Math. (Basel) **15** (1964), 76–80.

[18] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 1979.

[19] A. Heppes, *New upper bound on the transversal width of $T(3)$-families of discs*, Discrete Comput. Geom. **34** (2005), no. 3, 463–474.

[20] J. Jerónimo Castro, *Line transversals to translates of unit discs*, Discrete Comput. Geom. **37** (2007), no. 3, 409–417.

[21] J. Jerónimo-Castro and E. Roldán-Pensado, *Line transversals to blown up closed balls*, Journal of Geometry **100** (2011), no. 1-2, 79–84.

[22] ———, *Line transversals to translates of a convex body*, Discrete Comput. Geom. **45** (2011), 329–339.

[23] R. Kannan and L. Lovász, *Covering minima and lattice-point-free convex bodies*, The Annals of Mathematics **128** (1988), no. 3, 577–602.

[24] A. I. Khinchin, *A quantitative formulation of Kronecker's theory of approximation*, Izv. Akad. Nauk SSR, Ser. Mat. **12** (1948), 113–122.

[25] J. Lehec, *On the Yao-Yao partition theorem*, Archiv der Mathematik **92** (2009), 366–376.

[26] P. Lévy and F. Pellegrino, *Problemes concrets d'analyse fonctionnelle*, vol. 8, Gauthier-Villars Paris, 1951.

[27] E. Lutwak, *Width-integrals of convex bodies*, Proc. Amer. Math. Soc. **53** (1975), no. 2, 435–439.

[28] J. Matoušek, *Using the Borsuk-Ulam theorem*, Universitext, Springer-Verlag, Berlin, 2003.

[29] M. Meyer and A. Pajor, *On the Blaschke-Santaló inequality*, Archiv der Mathematik **55** (1990), no. 1, 82–93.

[30] Oleg R Musin, *Borsuk-Ulam type theorems for manifolds*, Proc. Amer. Math. Soc, vol. 140, 2012, pp. 2551–2560.

[31] S. Rabinowitz, *$O(n^3)$ bounds for the area of a convex lattice n-gon*, Geombinatorics **2** (1993), no. 4, 85–88.

[32] E. Roldán-Pensado, *The probability that a convex body intersects the integer lattice in a k-dimensional set*, Discrete Comput. Geom. **47** (2011), no. 2, 288–300.

[33] E. Roldán-Pensado and P. Soberón, *An extension of a theorem by Yao & Yao*, Arxiv preprint arXiv:1112.5737 (2011).

[34] L. Santaló, *Un teorema sobre conjuntos de paralelepipedos de aristas paralelas*, Publ. Inst. Mat. Univ. Nac. Litoral **2** (1940), 49–60.

[35] E. Schechter, *Handbook of analysis and its foundations*, Academic Pr, 1997.

[36] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, vol. 44, Cambridge Univ Pr, 1993.

[37] P. Soberón, *Balanced convex partitions of measures in $\mathbb{R}^d$*, Mathematika **58** (2012), 71–76.

[38] P. Vincensini, *Figures convexes et variétés linéaires de l'espace euclidienan à n dimensions*, Bull. Sci. Math **59** (1935), 163–174.

[39] Wolfram Research, Inc., *Mathematica, Version 8.0*, Champaign, IL, 2010.

[40] A. C. Yao and F. F. E. Yao, *A general approach to d-dimensional geometric queries*, Proceedings of the seventeenth annual ACM symposium on Theory of computing, ACM, 1985, pp. 163–168.