Representations and Completions for Ordered Algebraic Structures

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I, Rob Egrot confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

The primary concerns of this thesis are *completions* and *representations* for various classes of poset expansion, and a recurring theme will be that of *axiomatizability*. By a representation we mean something similar to the Stone representation whereby a Boolean algebra can be homomorphically embedded into a field of sets. So, in general we are interested in order embedding posets into fields of sets in such a way that existing meets and joins are interpreted naturally as set theoretic intersections and unions respectively.

Our contributions in this area are an investigation into the ostensibly *second order* property of whether a poset can be order embedded into a field of sets in such a way that arbitrary meets and/or joins are interpreted as set theoretic intersections and/or unions respectively. Among other things we show that unlike Boolean algebras, which have such a 'complete' representation if and only if they are atomic, the classes of bounded, distributive lattices and posets with complete representations have no first order axiomatizations (though they are pseudoelementary). We also show that the class of posets with representations preserving arbitrary joins is pseudoelementary but not elementary (a dual result also holds).

We discuss various completions relating to the canonical extension, whose classical construction is related to the Stone representation. We claim some new results on the structure of classes of poset meet-completions which preserve particular sets of meets, in particular that they form a weakly upper semimodular lattice. We make explicit the construction of Δ_1-completions using a two stage process involving meet- and join-completions.

Linking our twin topics we discuss canonicity for the representation classes we deal with, and by building representations using a meet-completion construction as a base we show that the class of representable ordered domain algebras is finitely axiomatizable. Our method has the advantage of representing finite algebras over finite bases.

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Chapter 1

Introduction

Ordered structures appear in many branches of mathematics. At a very basic level, ordering is an important feature of the natural numbers, and more generally, the more than/less than relation is common in everyday reasoning. Abstracting from this relation we arrive at the concept of the partially ordered set (poset), and structures based on posets find practical application in computer science in a number of ways. For example, special kinds of posets appear as algebraic correspondents to information systems (see e.g. [27, Chapter 9]), and there are deep connections between algebras with a poset structure and non-classical logics, which themselves can be used to model the behaviour of programs (see e.g. [107, 86]).

Connections between algebra and logic have been known for well over a century, via the work of Boole with his inchoate algebraization of what would later be known as propositional logic, and the works of DeMorgan, Schröder, and Peirce on relations (see e.g. [117] for a survey of historical and recent developments in algebraic logic).

More recently, algebra has found a new role in the model theory of non-classical logic, with Boolean algebras with operators (BAOs) providing algebraic semantics for modal logics, and similar duties being performed by algebras based on weaker structures such as Heyting algebras and distributive lattices for Intuitionistic and distributive modal logics respectively (see e.g. [55]).

Via duality theories extending that of Stone [127], algebraic models can be tied with *relational* semantics for these logics, and the developing theory of canonical extensions provides the modern logician with algebraic techniques for reasoning about these relational models (see Section 4.4.1). In particular, as we explain later, closure of a variety of BAOs under canonical extension implies completeness of the corresponding modal logic with respect to its class of frames.

Canonical extensions are not the only poset completion of interest to the logician, and in this document we shall also examine the MacNeille completion, which has applications in modal predicate logic (see e.g. [106]), meet-completions (of which the MacNeille completion is an example), and Δ_1-completions [51], a recent development generalizing both the MacNeille completion and the canonical extension. These structures are interrelated, and we shall make some of the links explicit in later chapters.

Also of relevance in the study of poset structures is the concept of *representation*. Here one seeks to understand the circumstances under which posets are equivalent to systems of sets where existing joins and meets are interpreted as unions and intersections respectively. Work in this area goes back to Stone, who demonstrated that every Boolean algebra is isomorphic to a field of sets, and Birkhoff, who showed that a lattice is isomorphic to a ring of sets if and only if it is distributive. Indeed, a poset has a representation as a partial sublattice of a ring of sets if and only if it embeds into a distributive lattice via an embedding preserving all existing finite meets and joins (see Theorem 3.3.6), so distributivity is the key property, which is unsurprsing given that by its very nature a ring of sets must satisfy the distributivity laws.

In the lattice case it turns out that representations, when they occur, are equivalent to representations whose underlying sets have as elements prime filters of the original lattice (this is where Birkhoff's representation theorem comes from - a lattice is distributive if and only if it has a separating set of prime filters), and a similar result holds for posets more generally.

It is known that a Boolean algebra has a representation as a field of sets where *arbitrary* joins and meets are represented as unions and meets if and only if it is atomic [2], and part of this thesis explores the possibilty of similar results in more general settings (see Section 1.1 for a guide to the later chapters).

This concept of representation can be extended to ordered structures augmented by algebraic operations intended to capture the manipulation of relations, for example relation algebras (see e.g. [81]). Here the elements of the set underlying the constructed ring of sets are ordered pairs (or ordered n-tuples for relations of higher order), and thus the elements of the representation are concrete relations, and in addition to suitable preservation of the poset structure it is a requirement that the additional operations be interpreted as some concrete manipulations of relations. Later in the thesis we apply some theory from our discussion of meet-completions to prove finite axiomatizability for the class of ordered domain algebras (that is, domain algebras with an underlying poset structure). An advantage of our construction is that finite algbras are represented over a finite base.

1.1 A guide to the document

We provide a guide to the contents of this thesis. Roughly speaking the ratio of original work to background material in each chapter increases as the thesis progresses, with care being taken to arrange the exposition logically around the order in which new ideas are introduced. We have tried to keep forward referencing to a minimum, though at times we are not able to avoid it.

The mathematics of interest to us here has resonances in many areas. We have tried to provide context and background for our work without going into extraneous detail and writing text book style expositions on subjects to which we do not add anything original. The policy has been to provide mathematical details only when they are required for understanding our own results, and even then not when they are deemed to be sufficiently common knowledge. For our review passages we have tried to provide a relatively informal overview, and the aim has been to make up for the lack of detail in our exposition with vigorous referencing.

1.1.1 Chapter 2

We present some basic concepts from logic and order theory, and also some deeper results we shall use frequently. The main purpose is primarily to introduce our notation.

1.1.2 Chapter 3

Here we formally introduce the concepts of representation we shall be using, and characterize these in terms of separation properties with prime filter like structures. We also provide a brief discussion of the related Stone and Priestley dualities. In addition to this we discuss algebras intended to capture the behaviour of relations, in particular relation algebras, with a focus on the representation problem for these structures.

Technical summary of main original results

- (Theorem 3.3.8). For a poset P the following are equivalent:

  (R) P is representable,

  (F) if γ is a weak-filter of P and $b \in P \setminus \gamma$ then there is a prime filter γ' of $F_\omega(P)$ with $\iota[\gamma] \subseteq \gamma'$ and $\iota(b) \notin \gamma'$,

  (F') suppose $n \in \omega$, and $p_i \in P$ for each $i \in \{1, ..., n\}$ with $\bigvee_{i=1}^n p_i$ defined in P. Then for all finite $S_i \subseteq P$ we have

  $$\bar{\bigvee}_{i=1}^n \bar{\bigwedge}(\iota[S_i \cup \{p_i\}]) \leq \iota(b) \implies \bar{\bigwedge}(\bigcup_{j=1}^n \iota[S_j] \cup \{\iota(\bigvee_{i=1}^n p_i)\}) \leq \iota(b)$$

  where $\bar{\wedge}$ and $\bar{\vee}$ denote the meet and join in $F_\omega(P)$ respectively.

- (Corollary 3.3.9). The class of representable posets is elementary.

1.1.3 Chapter 4

In this chapter we present the theory of poset completions with a uniform approach based on characteristic density and compactness properties. First we define meet-completions, canonical extensions, MacNeille completions and Δ_1-completions in abstract terms, then in Section 4.2 we develop sufficient theory for lifting montone maps and operations to these structures. In Sections 4.3 and 4.4 we go into more detail concerning the MacNeille completion and canonical extension respectively, and Sections 4.5 and 4.6 do similar for more general meet-completions and Δ_1-completions.

In particular, in Section 4.5.2 we investigate meet-completions preserving specified existing meets, and provide a characterization of the sets of meets where such a construction exists. We show that when a poset does admit a meet-completion preserving only certain specified meets then the set of such completions naturally forms a lattice that is not necessairly bottomed. In Section 4.5.3 we show that this lattice is always weakly upper semimodular, with stronger results holding when the original poset is finite.

In Section 4.6.1 we relate the Δ_1-completion to meet- and join-completions explicitly by showing how any Δ_1-completion can be realized as a join/meet-completion of a meet/join-completion.

Technical summary of main original results

- (Theorem 4.5.29). If P is a poset and $\mathscr{S} \subseteq P^*$ is regular then the set $\mathbb{S}_{\mathscr{S}}$ of $\mathscr{S}$-closures is a lattice with bottom element $\Gamma_{\mathscr{S}}$ when ordered pointwise (i.e. $\Gamma_1 \leq \Gamma_2 \iff \Gamma_1(S) \leq \Gamma_2(S) \iff \Gamma_1(S) \subseteq \Gamma_2(S)$ for all $S \in P^*$). Moreover, arbitrary non-empty meets are defined in $\mathbb{S}_{\mathscr{S}}$.

- (Proposition 4.5.38). If $\mathscr{S} \subseteq P^*$ is regular then $\mathbb{S}_{\mathscr{S}}$ is weakly upper semimodular.

- (Corollary 4.6.4). Every Δ_1-completion Q' of a poset P can be obtained using a pair (e, e'), where $e\colon P \to Q$ is a meet-completion, and $e'\colon Q \to Q'$ is a join-completion.

- (Theorem 4.6.6). Given meet-completions $e_1\colon P \to Q_1$ and $e_2'\colon Q_2 \to Q_2'$, and join-completions $e_2\colon P \to Q_2$ and $e_1'\colon Q_1 \to Q_1'$, there is an isomorphism $f_1\colon Q_1' \leftrightarrow Q_2'\colon f_2$ such that the diagram in Figure 4.9 commutes if and only if ϕ_1 and ϕ_2 are join- and meet-completions respectively ($\phi_1\colon Q_1 \to Q_2'$ and $\phi_2\colon Q_2 \to Q_1'$ are natural maps) and $e_1'(q_1) \leq \phi_2(q_2) \iff e_2'(q_2) \geq \phi_1(q_1)$ for all $q_1 \in Q_1$ and $q_2 \in Q_2$, in which case f_1 is the minimal lift of the identity on Q_1 along e_1' and ϕ_1, and f_2 is the maximal lift of the identity on Q_2 along e_2' and ϕ_2 (see Proposition 4.2.2).

1.1.4 Chapter 5

Here we develop the concept of complete representation in settings beyond the Boolean. In the absence of complementation the situation is more complex, as structures can have representations preserving arbitrary meets but not joins, and vice versa. In Section 5.1 we characterize complete, and meet/join-complete representability for distributive lattices in terms of separation properties involving complete, and completely-prime filters. Using this we show that the classes of completely, and meet/join-completely representable distributive lattices (**CRL**, **mCRL**, and **jCRL** respectively) are all pseudoelementary, but that **CRL** is not elementary. Whether the same is true for **mCRL** and **jCRL** is equivalent to their being closed under ultraroots and remains an open question.

In Section 5.2 we generalize the characterization from Section 5.1 to posets, and show that pseudoelementarity still holds for all the relevant classes. Moreover, the failure of elementarity of the class of completely representable distributive lattices carries through trivially to the poset setting. The significant departure from Section 5.1 is that we are able to construct a counterexample to closure under ultraroots for the class of join-completely representable posets, thus showing that neither it, nor, by duality, the class of meet-completely representable posets are elementary.

Finally, in Section 5.3 we investigate the complete representability of canonical extensions of distributive lattices and posets. The result here is that the canonical extension of a poset is completely representable if and only if it is distributive, and thus canonical extensions of distributive lattices are always completely representable.

Technical summary of main original results

- (Theorem 5.1.6). Let L be a bounded, distributive lattice. Then:

    1. L has a meet-complete representation iff L has a distinguishing set of complete, prime filters,
    2. L has a join-complete representation iff L has a distinguishing set of completely-prime filters,
    3. L has a complete representation iff L has a distinguishing set of complete, completely-prime filters,

- (Theorem 5.1.16). **CRL** is not closed under elementary equivalence.
- (Theorem 5.1.18). **mCRL**, **jCRL**, and **CRL** are all pseudoelementary.
- (Theorem 5.2.4). Let P be a poset. Then:

1. P has a meet-complete representation if and only if the set of complete, prime, weak-filters of P is separating over P,
2. P has a join-complete representation if and only if the set of completely-prime, weak-filters of P is separating over P,
3. P has a complete representation if and only if the set of complete, completely-prime, weak-filters of P is separating over P,

- (Theorem 5.2.10). The classes of meet-completely representable, join-completely representable and completely representable posets are all pseudoelementary.
- (Theorem 5.2.18). The classes of join-completely and meet-completely representable posets are not closed under ultraroots and thus are not elementary.
- (Theorem 5.3.4). Given a poset P the following are equivalent

1. P^σ is distributive,
2. P^σ is completely representable,
3. P^σ is representable.

1.1.5 Chapter 6

Using a meet-completion construction we prove finite axiomatizability for the class of representable ordered domain algebras. In particular our representation construction represents finite algebras over finite bases.

Technical summary of main original results

- The class $\mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, ^\smile, 0, \mathbf{id}, \subseteq)$ is finitely axiomatizable and has the finite representation property.

1.1.6 Chapter 7

Here we reiterate the main contributions of the thesis and give some suggestions for further work.

1.2 Other appearances of work from this thesis, including joint work

Material concerning complete representations for distributive lattices from Sections 5.1 and 5.3 appears as [41]. The work emerged from discussion between the two authors of this paper and the exact attribution of individual results is not clear. Both authors feel that the end product is the result of an even division of labour and inspiration. We note that there is an error in the statement of Proposition 2.16(2), and in the proof of Lemma 2.18 of the published version of this paper. These mistakes do not undermine the main results, and corrected versions appear here as Proposition 5.1.12 and Lemma 5.3.2 respectively. The author also gave a presentation on much of this material at the workshop on lattices, relations and Kleene algebras held in September 2010 at UCL.

Material on meet-completions from Section 4.5 was submitted as [40] and is currently under review. The work on ordered domain algebras extends work appearing as [42], and is the content of a note being developed by the same authors. The contribution to this work of the author of this document is the explicit use of closure operators in the construction, and also results relating to the preservation of identities. The representation result itself is the work of Hirsch and Mikulás.

Chapter 2

Technical preliminaries

In this chapter we introduce the basics of model theory, universal algebra and order theory we shall need for the more advanced theory in later chapters. The intention is not to provide a comprehensive overview, but rather to formally define the foundations on which the later work is based. Those with any more than a passing familiarity with these subjects will likely find nothing new here, and the purpose is primarily to introduce the reader to our terminology and notation. References to standard texts will be given when appropriate.

2.1 Logic and model theory

We deal here with first order logic, and we assume the reader has at least a basic understanding of this area. We quickly introduce the definition of a language, and of formula construction within a language. We define satisfiability and the concept of a model, and proceed quickly to the concepts of elementarity and pseudoelementarity. We have attempted to keep our notation in line with standard practice; for further information see e.g. [23, 83].

Definition 2.1.1 (Language)**.** A language $\mathscr{L}$ is the union of sets (possibly empty) of *non-logical* symbols standing for relations, functions and constants. When dealing with abstract languages we primarily use (indexed versions of) R to denote relations, f functions, and c constants, and each function and relation symbol is implicitly associated with a natural number defining its arity. For indexing sets I_R, I_f and I_c, we may for example define $\mathscr{L} = \{R_i : i \in I_R\} \cup \{f_i : i \in I_f\} \cup \{c_i : i \in I_c\}$.

Definition 2.1.2 (Language expansion/reduction)**.** Given languages $\mathscr{L}_1$ and $\mathscr{L}_2$ we say $\mathscr{L}_2$ is an expansion of $\mathscr{L}_1$, or equivlently that $\mathscr{L}_1$ is a reduction of $\mathscr{L}_2$, when $\mathscr{L}_1 \subseteq \mathscr{L}_2$.

Definition 2.1.3 (Structure)**.** Given a language $\mathscr{L}$, a structure for $\mathscr{L}$ is a pair $\mathfrak{A} = (A, \mathscr{I})$, where A is some non-empty set and $\mathscr{I}$ assigns to each relation or function symbol an actual relation or function on A with the appropriate arity, and to each constant symbol an element of A.

Definition 2.1.4 (Structure isomorphism)**.** If $\mathscr{L}$ is a language and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $\mathscr{L}$-structures we say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic if there is a bijction ψ between A_1 and A_2 such $R_i^1(x_1, ..., x_n) \iff R_i^2(\psi(x_1), ..., \psi(x_n))$ for all $i \in I_R$, $\psi(f_i^1(x_1, ..., x_m)) = f_i^2(\psi(x_1), ..., \psi(x_m))$ for all $i \in I_f$, and $\psi(c_i^1) = c_i^2$ for all $i \in I_c$. In this case we say ψ is an *isomorphism* and we write $\mathfrak{A}_1 \cong \mathfrak{A}_2$.

Given a map $f\colon X \to Y$ and $S \subseteq X$ we use the notation $f[S]$ to denote $\{f(x) : x \in S\}$.

Definition 2.1.5 (Reduct)**.** Given languages $\mathscr{L}_1$ and $\mathscr{L}_2$ as in Definition 2.1.2, with $\mathscr{L}_1 \subseteq \mathscr{L}_2$, and an $\mathscr{L}_2$-structure $\mathfrak{A} = (A, \mathscr{I})$, the $\mathscr{L}_1$-reduct of $\mathfrak{A}$ is the pair $(A, \mathscr{I}')$, where $\mathscr{I}'$ is the restriction of $\mathscr{I}$ to the relations, functions and constants of $\mathscr{L}_1$.

Given a language $\mathscr{L}$ we construct formulas using the symbols of $\mathscr{L}$ and the additional symbols $\{), (, \wedge, \neg, \forall, =\} \cup \{v_i : i \in I\}$ for some indexing set I. We use the standard rules for constructing terms, atomic formulas and formulas, and these can be found in, for example, [23, Chapter 1]. We define the additional symbols $\{\vee, \rightarrow, \leftrightarrow, \exists\}$ in the normal way, and we define occurences of variables as either *free* or *bound* as standard. We define a *sentence* to be a formula with no free variables, and we define a *theory* to be a set of sentences. Given a language $\mathscr{L}$, a model $\mathfrak{A}$ for $\mathscr{L}$, and an $\mathscr{L}$-sentence ϕ, we say $\mathfrak{A}$ satisfies ϕ, and we write $\mathfrak{A} \models \phi$, if ϕ is a true statement in A under the interpretation defined by $\mathscr{I}$. Given a theory τ we say $\mathfrak{A}$ is a model of τ and write $\mathfrak{A} \models \tau$ if $\mathfrak{A}$ satisfies ϕ for all $\phi \in \tau$. We will not need formal deduction rules here, so we do not need to specify a particular system and instead just assume something equivalent to a standard Hilbert style system. In particular we assume a theory τ is consistent if and only if it has a model.

Definition 2.1.6 (Elementary class)**.** A class $\mathscr{C}$ of $\mathscr{L}$-structures is elementary if it is the class of all models of some $\mathscr{L}$-theory τ. Equivalently we say $\mathscr{C}$ is *axiomatizable* (in first order logic), and in the special case where we can choose τ to be a single sentence we say $\mathscr{C}$ is *finitely axiomatizable* (in first order logic).

Definition 2.1.7 (Elementary equivalence)**.** Two $\mathscr{L}$-structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are elementarily equivalent if for every $\mathscr{L}$-sentence ϕ we have $\mathfrak{A}_1 \models \phi \iff \mathfrak{A}_2 \models \phi$.

Many important classes are not elementary, so we also have the following more general notion.

Definition 2.1.8 (Pseudoelementary class)**.** A class $\mathscr{C}$ of $\mathscr{L}$-structures is pseudoelementary if there is a language $\mathscr{L}'$ with $\mathscr{L} \subseteq \mathscr{L}'$, and an $\mathscr{L}'$-theory τ such that $\mathscr{C}$ is the class of all $\mathscr{L}$-reducts of the class of all models of τ.

Proposition 2.1.9. ([81, Proposition 9.9]). *A class $\mathscr{C}$ of $\mathscr{L}$-structures is pseudoelementary if and only if there are*

1. *a two-sorted language $\mathscr{L}^+$, with disjoint sorts $\mathbb{A}$ and $\mathbb{S}$, containing $\mathbb{A}$-sorted copies of all symbols of $\mathscr{L}$, and*
2. *an $\mathscr{L}^+$ theory τ*

with $\mathscr{C} = \{\mathfrak{A}^{\mathbb{A}} \upharpoonright_{\mathscr{L}} : \mathfrak{A} \models \tau\}$, where $\mathfrak{A}$ is an $\mathscr{L}^+$-structure, $\mathfrak{A}^{\mathbb{A}}$ is the structure in the sublanguage of $\mathscr{L}^+$ containing only $\mathbb{A}$-sorted symbols whose domain contains the $\mathbb{A}$-sorted elements of $\mathfrak{A}$, and $\mathfrak{A}^{\mathbb{A}} \upharpoonright_{\mathscr{L}}$ is the $\mathscr{L}$ reduct of $\mathfrak{A}^{\mathbb{A}}$ obtained by identifying the symbols of $\mathscr{L}$ with their $\mathbb{A}$-sorted counterparts in $\mathscr{L}^+$.

2.2 Universal algebra

An *algebra*, in the sense of e.g. [20], is defined to be a non-empty $\mathscr{L}$-structure for a language $\mathscr{L}$ that contains only function symbols. We say two algebras have the same *type* if they are structures for the same functional language $\mathscr{L}$. Of particular interest is the concept of a *variety*, that is, the class of models for a functional $\mathscr{L}$-theory τ where τ contains only *identities* (which we define as being $\mathscr{L}$-sentences of form $\forall x_1 ... \forall x_n \phi$, where ϕ is an atomic formula and $\{x_1, ..., x_n\}$ is the set of variables appearing in ϕ). Since constants can be treated as nullary functions the signatures of many familiar algebraic structures like groups, rings, and lattices (see Definition 2.3.2) can be treated with this methodology, and in fact turn out to be varieties.

While there is a substantial amount of theory in the literature concerning this topic we will only need the following result (for a proof see e.g. [20, Theorem 11.9]):

Theorem 2.2.1 (Birkhoff). *A class of algebras of the same type is a variety if and only if it is closed under homomorphic images, subalgebras and direct products.*

Over the course of this document we shall discuss several kinds of algebras and posets (see the next section). Unless stated otherwise we assume throughout that these structures are non-empty.

2.3 Lattices and ordered sets

Here we introduce partially ordered sets and lattices. In Section 2.3.1 we define posets, lattices and Boolean algebras, along with homomorphisms, and concepts relating to filters and ideals. Following this in Section 2.3.2 we return to model theory and introduce some deep theory relating to ultraproducts that we shall rely on for many of our results. More detailed introductions to this subject can be found in e.g. [67, 27], and standard texts for more advanced topics are [68, 69].

2.3.1 Basic definitions

A partially ordered set (poset) P is a set equipped with a binary relation $\leq$ with the following properties:

P1 $p \leq p$ for all $p \in P$ (reflexivity),

P2 $p \leq q$ and $q \leq r \implies p \leq r$ for all $p, q, r \in P$ (transitivity), and

P3 $p \leq q$ and $q \leq p \implies p = q$ for all $p, q \in P$ (antisymmetry).

When for every $p, q \in P$ we have either $p \leq q$ or $q \leq p$ we say P is *totally ordered*. When P is such that $p \leq q \implies p = q$ for all $p, q \in P$ we say P is an *antichain*. In this document we shall write e.g. $a < b$ to denote the situation where $a \leq b$ and $a \neq b$. When $S \subseteq P$ and $p \in P$ we write $p \leq S$ when $p \leq s$ for all $s \in S$ (and similar for $p \geq S$).

Definition 2.3.1 (Up-set, down-set)**.** $S \subseteq P$ is an up-set if $S = S^{\uparrow} = \{p \in P : p \geq s \text{ for some } s \in S\}$, dually, S is a down-set if $S = S^{\downarrow} = \{p \in P : p \leq s \text{ for some } s \in S\}$. Given $p \in P$ we define $p^{\uparrow} = \{p\}^{\uparrow}$ and $p^{\downarrow} = \{p\}^{\downarrow}$.

Given a poset P and a set $S \subseteq P$, when S has a supremum p with respect to $\leq$ we say p is the *join* of S and write $\bigvee S = p$. Similarly when S has an infimum q with respect to $\leq$ we say q is the *meet* of S and write $\bigwedge S = q$. A *lattice* is a poset in which every pair of elements (and thus every finite subset) has both a meet and a join. Equivalently, we can define a lattice algebraically as follows.

Definition 2.3.2 (Lattice)**.** A lattice is a set L equipped with binary operations $\vee$ and $\wedge$ satisfying the following:

L1 $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for all $a, b \in L$ (commutativity),

L2 $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ for all $a, b, c \in L$ (associativity),

L3 $a \vee (a \wedge b) = a \wedge (b \vee a) = a$ for all $a, b \in L$ (absorption), and

L4 $a \vee a = a$ and $a \wedge a = a$ for all $a \in L$ (idempotence).

A lattice by this definition is also a lattice under the ordering defined by $a \leq b \iff a \vee b = b$, or equivalently $a \leq b \iff a \wedge b = a$. Note that the set of axioms of Definition 2.3.2 is not minimal as L4 follows from L3.

When a poset (which may be a lattice) has unique top and bottom elements (usually denoted 1 and 0 respectively) we say it is *bounded*, and when $\bigwedge S$ and $\bigvee S$ exist for all $S \subseteq P$ (including $\emptyset$) we say P is *complete*. Note that the supremum of the empty set will be the bottom element, and the infimum of the empty set will be the top element, when they exist. It is a well known fact of order theory that if $\bigwedge S$ exists for all $S \subseteq P$ then P is complete (and thus is a complete lattice), and similarly for $\bigvee S$ (see e.g. [27, Theorem 2.3.1].

Given a poset P we define the *order dual* (or just the dual when the context is clear) P^δ to be the poset with the same underlying set as P but ordered by $\leq_\delta$, where $p \leq_\delta q \iff q \leq p$. Taking order duals is self-inverse, and it's easy to show that if L is a lattice then L^δ is also a lattice and can be obtained algebraically from L by swapping $\vee$ and $\wedge$. Boundedness and completeness are also preserved under taking order duals. For every sentence in the language of posets (or lattices) there is a dual sentence obtained by reversing inequalities (or swapping joins and meets). By considering the order duals of posets (or lattices) it's easy to see that if a sentence is true in every poset (or lattice) then so too must be its dual.

Definition 2.3.3 (Semilattice)**.** A meet-semilattice is an ordered set where every pair of elements has an infimum, and join-semilattices are defined dually. Algebraically they can be defined using a single associative, commutative, idempotent binary operation (either $\wedge$ or $\vee$). Semilattices may have top and bottom elements like any other poset.

Definition 2.3.4 (Distributive lattice)**.** A lattice L is distributive when either

DL1 $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$, or

DL2 $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.

Note that DL2 is obtained from DL1 by swapping $\vee$ and $\wedge$, so DL1 and DL2 are dual. It can be shown that DL1 $\iff$ DL2 and thus distributivity is also preserved under taking order duals (see e.g. [27, Lemma 4.3]).

Definition 2.3.5 (Boolean algebra)**.** A Boolean algebra B is a bounded, distributive lattice where every element has a unique *complement*, that is, for all $a \in B$ there is a unique a' such that $a \vee b = 1$ and $a \wedge b = 0$. We can treat Boolean algebras algebraically by augmenting the signature of bounded lattices with the unary operation $-$ and taking the axioms of distributive lattices plus the following:

B1 $a \vee -a = 1$ for all $a \in B$,

B2 $a \wedge -a = 0$ for all $a \in B$,

B3 $a \vee 0 = a$ for all $a \in B$, and

B4 $a \wedge 1 = a$ for all $a \in B$.

A canonical example of a Boolean algebra is the powerset of a set under set theoretic union, intersection and complementation. We say a Boolean algebra B is *atomic* if every element of B is above an element that is minimal in $B \setminus \{0\}$. Note that Boolean algebras are often presented with the signature $(0, 1, +, -)$, where $+$ stands for $\vee$. This is possible as $a \wedge b$ can be defined as $-(-a \vee -b)$.

Given posets P_1 and P_2, a map $f \colon P_1 \to P_2$ is *order preserving* if $p \leq q \implies f(p) \leq f(q)$ for all $p, q \in P_1$, and it is an *order embedding* if $p \leq q \iff f(p) \leq f(q)$ for all $p, q \in P_1$. A *lattice homomorphism* between lattices L_1 and L_2 is a map $f \colon L_1 \to L_2$ such that $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in L_1$. If L_1 and L_2 are bounded we say f is a *bounded lattice homomorphism* if $f(0) = 0$ and $f(1) = 1$. In the case where L_1 and L_2 are Boolean algebras we say f is a *Boolean homomorphism* if it is a bounded lattice homomorphism and $f(-a) = -f(a)$ for all $a \in B$. We define the various kinds of *isomorphism* to be bijective versions of the appropriate homomorphism. Note that this definition is a special case of Definition 2.1.4.

Using the concept of homomorphism we can prove the following well known characterization theorem for distributive lattices (see e.g. [27, Theorem 4.10] for a proof).

Theorem 2.3.6. *A lattice L is distributive if and only if neither M_3 (see figure 2.1), nor N_5 (see figure 2.2) embeds into L via a lattice homomosprhism (not necessarily bounded).*

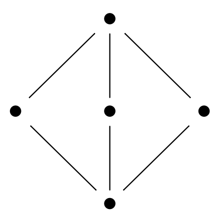


Figure 2.1: The lattice M_3

Definition 2.3.7 (Filter/ideal)**.** We define a filter in each of our structures as follows (ideals are defined dually):

- In a poset P a filter is an upward closed set (up-set) $\emptyset \subset F \subset P$ that is also *down directed* (for all $a \neq b \in F$ there is $c \in F$ with $c \leq a$ and $c \leq b$).

- In a lattice (such as a Boolean algebra) a filter is an up-set that is closed under $\wedge$.

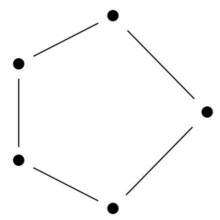


Figure 2.2: The lattice N_5

In a Boolean algebra B we define an *ultrafilter* to be a filter U where for every $a \in B$, either $a \in U$ or $-a \in U$. In all our structures a single element defines a filter or ideal by taking the up-set or down-set respectively generated by that element. For an element p we call these sets the *principal filter* (respectively, *ideal*) generated by p.

Definition 2.3.8 (Prime filter/ideal)**.** Given a filter $F \subset P$ we say F is prime if whenever $\bigvee_J p_i$ is defined in P for finite, non-empty J with $\bigvee_J p_i \in F$ we have $p_i \in F$ for some $i \in J$. Prime ideals are defined dually.

Prime filters and ideals will be very important in the representation theory we discuss in Chapters 3 and 5.

Definition 2.3.9 (Irreducible and prime elements)**.** Given a poset P and $p \in P$ such that p is not the bottom of P, we say p is

1. *join-prime* if $\bigvee X \geq p \implies p \leq q$ for some $q \in X$ whenever $\bigvee X$ exists for finite $\emptyset \subset X \subseteq P$, and we denote the set of join-primes of P by $J_p(P)$,

2. *join-irreducible* if $\bigvee X = p \implies p = q$ for some $q \in X$ whenever $\bigvee X$ exists for finite $\emptyset \subset X \subseteq P$, and we denote the set of join-irreducibles of P by $J(P)$.

We make definitions for *completely join-prime* and *completely join-irreducible* by removing the demand that X be finite (note that we still exclude the bottom element), and we denote the sets of these elements by $J^\infty_p(P)$ and $J^\infty(P)$ respectively. We make dual definitions for *meet-prime* etc., denoting the corresponding classes by subsitituing M for J.

Note that χ-primality implies χ-irreducibility, and complete χ-primality implies complete χ-irredicubility (where $\chi \in \{join, meet\}$). The following is also well known and straightforward to show:

Lemma 2.3.10. *Let $\chi \in \{join, meet\}$. Then when L is a distributive lattice χ-irreducibility implies χ-primality.*

Remark 2.3.11. In the literature on posets it is common to use χ-irreducible where we would use χ-prime. We have chosen not to do this, as we would prefer to have a single consistent notation to cover all the structures we are concerned with.

Definition 2.3.12 (Monotonicity type)**.** Given a natural number n we define an n-monotonicity type to be a map $\eta\colon \{1, ..., n\} \to \{a, b\}$.

Definition 2.3.13 (Isotone/monotone map)**.** If P is a poset and $f\colon P^n \to P$ then we say f is isotone if whenever $(x_1, ..., x_n) \leq (y_1, ..., y_n) \in P^n$ (using the product ordering; i.e. $(x_1, ..., x_n) \leq (y_1, ..., y_n) \iff x_i \leq y_i$ for all $i \in \{1, ..., n\}$) we have $f(x_1, ..., x_n) \leq f(y_1, ..., y_n)$. We say f is η-monotone if there is an n-monotonicity type $\eta\colon \{1, ..., n\} \to \{a, b\}$ with f isotone when considered as a map $f\colon \prod_{i=1}^n P_{\eta(i)} \to P$, where we define $P_a = P$ and $P_b = P^\delta$.

Lemma 2.3.14. *Let P be a poset, and let $f\colon P^n \to P$ be an n-ary operation on P. Suppose f is η-monotone and μ-monotone for some n-monotonicity types η and μ. Then for all $i \in \{1, ..., n\}$ with $\eta(i) \neq \mu(i)$, if $\{p, q\}$ has either an upper or a lower bound in P, then $f(x_1, ...x_{i-1}, p, x_{i+1}, ..., x_n) = f(x_1, ...x_{i-1}, q, x_{i+1}, ..., x_n)$.*

Proof. Let z be either an upper bound or a lower bound for $p, q \in P$ and suppose $\eta(i) \neq \mu(i)$. Then $f(x_1, ...x_{i-1}, z, x_{i+1}, ..., x_n) \leq f(x_1, ...x_{i-1}, p, x_{i+1}, ..., x_n) \leq f(x_1, ...x_{i-1}, z, x_{i+1}, ..., x_n)$, as f is both order preserving and reversing at its ith coordinate, so $f(x_1, ...x_{i-1}, z, x_{i+1}, ..., x_n) = f(x_1, ...x_{i-1}, p, x_{i+1}, ..., x_n)$. Similarly $f(x_1, ...x_{i-1}, z, x_{i+1}, ..., x_n) = f(x_1, ...x_{i-1}, q, x_{i+1}, ..., x_n)$ so we are done. □

Corollary 2.3.15. *If P has either an upper or a lower bound, or if P is a semilattice, and if $f\colon P^n \to P$ is η-monotone and μ-monotone for some n-monotonicity types η and μ, then whenever $\eta(i) \neq \mu(i)$ f is independent of the value of its ith argument.*

Definition 2.3.16 (Poset expansion)**.** A poset expansion $\mathcal{P}$ is a structure $(P, f_i : i \in I)$, where P is a poset and f_i is a function for each $i \in I$ for some ordinal I. We say $\mathcal{P}$ is an *isotone expansion* if the interpretation of f_i is an isotone map for all $i \in I$. We make a similar definition for η-monotone expansions, and we can define isotone and monotone (or otherwise) lattice and Boolean algebra expansions analogously (applying Corollary 2.3.15 we can drop the η in these cases). Note that in the lattice and Boolean algebra cases the expansion can be considered to be an *algebra* in the sense of [20].

Definition 2.3.17 (BAO)**.** A BAO (Boolean algebra with operators) is a poset expansion $\mathcal{B}$ where the underlying poset is a Boolean algebra, and the additional operations are *normal* and *additive*, i.e.

$$f(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) = 0,$$

and

$$\begin{aligned} f(x_1, ..., x_{i-1}, a \vee b, x_{i+1}, ..., x_n) =& f(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n) \\ & \vee f(x_1, ..., x_{i-1}, b, x_{i+1}, ..., x_n) \end{aligned}$$

whenever f is an n-ary operation in the signature of $\mathcal{B}$ and $i \in \{1, ..., n\}$. Operations that are normal and additive are known as *operators*, and additonally are *complete* if the additivity property holds for arbitrary joins.

2.3.2 Model theory revisited: ultraproducts and ultraroots

Definition 2.3.18 (Ultraproduct)**.** Given a language $\mathscr{L}$, an ordinal $I \neq \emptyset$, a set of $\mathscr{L}$-structures $\{\mathfrak{A}_i : i \in I\}$, and an ultrafilter $U \subseteq \wp(I)$, we define an equivalence relation $\sim$ on $\prod_I \mathfrak{A}_i$ by $\bar{x} \sim \bar{y} \iff \{i \in I : \bar{x}(i) = \bar{y}(i)\} \in U$. We define the ultraproduct of $\{\mathfrak{A}_i : i \in I\}$ over U to be the set of $\sim$ equivalence classes of $\prod_I \mathfrak{A}_i$ and we denote it with $\prod_U \mathfrak{A}_i$. We can use $\prod_U \mathfrak{A}_i$ as an $\mathscr{L}$-structure by making interpretations as follows:

- $\prod_U \mathfrak{A}_i \models R([\bar{x_1}], ..., [\bar{x_n}]) \iff \{i \in I : \mathfrak{A}_i \models R_i(\bar{x_1}(i), ..., \bar{x_n}(i))\} \in U$ for all n-ary relation symbols R of $\mathscr{L}$, where R_i is the interpretation of that relation in $\mathfrak{A}_i$,
- $f([\bar{x_1}], ..., [\bar{x_n}]) = [f_i(\bar{x_1}(i), ..., \bar{x_1}(i))]$ for arbitrary $i \in I$ for all n-ary function symbols f of $\mathscr{L}$, where f_i is the interpretation of that function in $\mathfrak{A}_i$,
- $c = [\bar{c}]$, where $\bar{c}(i)$ is the interpretation of c in $\mathfrak{A}_i$ for all $i \in I$ for all constant symbols c of $\mathscr{L}$.

It's easy to check that this interpretation defines an $\mathscr{L}$-structure (see e.g. [23, Proposition 4.1.7]). When $\mathfrak{A}_i = \mathfrak{A}_j = \mathfrak{A}$ for all $i, j \in I$ we say $\prod_U \mathfrak{A}_i$ is the *ultrapower* of $\mathfrak{A}$ over U, and we write $\prod_U \mathfrak{A}$. In this case we say $\mathfrak{A}$ is an *ultraroot* of $\prod_U \mathfrak{A}$.

Theorem 2.3.19 (Łoś)**.** *Let $\mathscr{L}$ be a language, let ϕ be an $\mathscr{L}$-sentence, let $I \neq \emptyset$ be an ordinal, let $\{\mathfrak{A}_i : i \in I\}$ be a set of $\mathscr{L}$-structures, and let U be an ultrafilter of $\wp(I)$. Then $\prod_U \mathfrak{A}_i \models \phi \iff \{i \in I : \mathfrak{A}_i \models \phi\} \in U$.*

Theorem 2.3.19 is sometimes referred as the fundamental theorem of ultraproducts (see e.g. [23, Theorem 4.1.9] for a proof). This result is crucial to our work here as it allows us to prove axiomatization results with largely algebraic methods. In particular we shall use the following well known results.

Theorem 2.3.20 (Shelah [122])**.** *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $\mathscr{L}$-structures then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are elementarily equivalent if and only if there is an ordinal I and an ultrafilter $U \subseteq \wp(I)$ with $\prod_U \mathfrak{A}_1 \cong \prod_U \mathfrak{A}_2$.*

Theorem 2.3.21. ([81, Theorem 3.32])**.** *Given a language $\mathscr{L}$ and a class $\mathscr{C}$ of $\mathscr{L}$-structures, $\mathscr{C}$ is elementary if and only if $\mathscr{C}$ is closed under taking isomorphic copies, ultraproducts, and ultraroots (recall Definition 2.3.18).*

Proposition 2.3.22. *$\mathscr{C}$ is pseudoelementary $\implies$ $\mathscr{C}$ is closed under ultraproducts.*

Chapter 3

Representation and duality

The theme of representation is common in mathematics, though the general concept is not precisely defined. In practice a representation theorem usually involves showing a class of mathematical structures can be reduced to a subclass that is, in some sense, simpler, or more intuitive. A famous example is what is known as Cayley's theorem; that every group is isomorphic to some permutation group on a set. A strong representation theorem allows insights from familiar structures to be transferred to a much wider class, but representation can also play a converse role. Suppose we have a class of structures satisfying some suitably intuitive notion of 'concreteness' about which we wish to reason abstractly (for example the algebras of relations considered in Section 3.4). A representation theorem can tell us to what extent abstractions intended to capture properties of these structures are succesful.

Related to representation is the concept of duality; two categories (that is, classes of mathematical object equipped with additional information about the maps between them, see e.g. [97] for more information) are dual if, roughly speaking, there is a correspondence between their objects that reverses the direction of all maps. A famous example of duality is Stone's theorem [127] whereby every Boolean algebra is shown to be isomorphic to a field of sets picked out topologically from the powerset algebra of its set of ultrafilters (see Section 3.1). Duality will not play much of a part in our work here, but Stone's theorem and related duality results play an important historical role in the completion theory we discuss in Section 4.4.

The plan for this chapter is first to introduce Stone's theorem, then in the following two sections to consider generalizations of this 'representation as set systems' concept to distributive lattices and posets respectively. Finally in Section 3.4 we sketch out some of the issues surrounding the representation of algebras of binary relations.

3.1 Stone's theorem

We introduced Boolean algebras (named for Boole [18]) in Definition 2.3.5. Originally intended to provide an abstract framework for reasoning about logic they can be regarded from a modern perspective as being an algebraization of propositional logic, an idea made precise via Lindenbaum–Tarski algebras (see e.g. [27, Section 11], or [17, Section 5.1]). The correspondence between Boolean algebras and Lindenbaum–Tarski algebras is a kind of representation theorem, another, as we have mentioned before, is Stone's theorem, which we state below as Theorem 3.1.2.

Definition 3.1.1 ($B_\circ$)**.** Given a Boolean algebra B define $B_\circ$ to be the topological space $(U(B), \tau_B)$, where $U(B)$ is the set of ultrafilters of B and τ_B is the topology generated by the basis $\{\{u \in U(B) : u \in x\} : x \in B\}$.

Given a Boolean algebra B it can be shown (see e.g. [27, Chapter 11] or [68, Appendix B]) that $B_\circ$ is compact and totally disconnected, and conversely, whenever $T = (X, \tau)$ is a topological space, $T^\circ = (Cl_\tau(X), \emptyset, X, \cup, \setminus)$ is a Boolean algebra (where $Cl_\tau(X)$ is the set of clopen subsets of X).

Theorem 3.1.2 (Stone [127])**.** *If B is a Boolean algebra and T is a compact, totally disconnected topological space then $(B_\circ)^\circ \cong B$ and $(T^\circ)_\circ \approx T$ (here $\approx$ denotes homeomorphism).*

Futher to this, if we define a Stone space to be a compact, totally disconnected space, it can be shown that Boolean homomorphisms of the form $f\colon A \to B$ correspond to continuous maps of form $f_\circ\colon B_\circ \to A_\circ$ between Stone spaces, and vice versa, in the form of a duality between the category of Boolean algebras with Boolean homomorphisms and the category of Stone spaces with continuous maps.

Stone duality can be extended by the addition of operators; Jónsson and Tarski [90] showed a relationship between BAOs and certain relational frames as part of their development of the canonical extension (see Section 4.4), and this can be extended to a full categorical duality between the category of BAOs (of some fixed type) and their homomorphisms and the category of *descriptive general frames* (of appropriate relational signature) and bounded morphisms [64]. Descriptive general frames can be thought of as Stone spaces equipped with *point closed* relations.

The relationship between classes of BAOs and classes of frames is of particular interest to modal logicians as it allows for the completeness of modal logics with respect to the algebraic semantics provided by their corresponding class of algebras (BAOs) to be transformed into completeness with respect to relational semantics over a class of general frames. The reader is directed towards [17] for background information on modal logic, and to Chapter 5 of the same for a more detailed discussion of algebraic correspondence and general frames.

3.2 Representations for lattices

Definition 3.2.1 (Representation)**.** Let L be a bounded lattice. A *representation* of L is a bounded lattice embedding $h\colon L \to \wp(X)$ for some set X, where $\wp(X)$ is considered as a ring of sets, under the operations of set union and intersection. When such a representation exists we say that L is *representable*.

Note that a representable lattice must be distributive, as a ring of sets has this property. For simplicity we shall assume that our representations $h\colon L \to \wp(X)$ are *irredundant*, that is, for all $x \in X$ there is some $a \in L$ with $x \in h(a)$. For irredundant representations $h\colon L \to \wp(X)$ the 'inverse image' $h^{-1}[x] = \{a \in L\colon x \in h(a)\}$ of any point $x \in X$ is a prime filter, with closure under finite meets coming from finite meet preservation by the representation, and primality coming from finite join preservation. Upward closure can be derived from either of these preservation properties using the equivalent definitions of the order relation in a lattice. Conversely, any set K of prime filters of L with the property that for every pair $a \neq b \in L$ there exists $f \in K$ with either $a \in f$ and $b \notin f$ or vice versa determines a representation $h_K\colon L \to \wp(K)$ using $h_K(a) = \{f \in K\colon a \in f\}$ (note that for $f \in K$ we have $h_K^{-1}[f] = f$). For ease of exposition later we introduce a definition for sets of sets generalizing the condition for filters given above.

Definition 3.2.2 (Distinguishing set)**.** A set $S \subseteq \wp(L)$ is distinguishing over L iff for every pair $a \neq b \in L$ there exists $s \in S$ with either $a \in s$ and $b \notin s$ or vice versa.

Note that when S is a set of prime filters, saying that S is distinguishing over L is equivalent to saying that L has a familiar separation property, i.e. that whenever $a \not\leq b \in L$ there is $f \in S$ with $a \in f$ and $b \notin f$. Using this definition we state the results of the preceding discussion as a simple theorem.

Theorem 3.2.3. *A bounded distributive lattice L is representable if and only if it has a distinguishing set of prime filters.*

Theorem 3.2.4 (Birkhoff)**.** *Let L be a distributive lattice, let F be a filter of L, and let $a \in L\backslash F$. Then there is a prime filter $F' \subseteq L$ with $F \subseteq F'$ and $a \notin F'$.*

As a simple corollary to Theorem 3.2.3 and Theorem 3.2.4 we have:

Theorem 3.2.5. *A bounded lattice is representable if and only if it is distributive.*

There are also a number of duality results for lattices. Stone himself adapted his result for Boolean algebras to provide a duality between the category $\mathbf{DLAT}$ of bounded, distributive lattices with bounded lattice homomorphisms and the category $\mathbf{SPEC}$ of spectral spaces with spectral maps [128]. A drawback with this method is that the axioms defining a spectral space appear less natural than those defining a Stone space, and spectral maps are required to not only be continuous but also to have the property that inverse images of compact open sets are compact. Priestley [108, 109] showed that much of the unnatural feel of the spectral duality could be overcome by considering topological spaces equipped with a poset structure.

The objects in this new category (which we shall denote $\mathbf{PS}$) are compact, totally order-disconnected topological spaces (known as *Priestley spaces*), and the maps are continuous and order preserving. We have a categorical duality between $\mathbf{DLAT}$ and $\mathbf{PS}$, and the subcategory of $\mathbf{PS}$ of Priestley spaces with trivial order and the continuous maps between them is precisely the category of Stone spaces with continuous maps (so the corresponding subcategory of $\mathbf{DLAT}$ is the category of Boolean algebras with Boolean homomorphisms). As a consequence of the dual equivalences between $\mathbf{PS}$ and $\mathbf{DLAT}$, and $\mathbf{SPEC}$ and $\mathbf{DLAT}$, $\mathbf{PS}$ and $\mathbf{SPEC}$ must be equivalent, and in fact are isomorphic [25]. $\mathbf{SPEC}$ and $\mathbf{PS}$ are both isomorphic to a category, $\mathbf{PSTONE}$, of certain bitopological spaces whose objects are *pairwise Stone spaces* and whose maps are *bicontinuous* [92, 10].

Hansoul [71] extends Stone's duality for distributive lattices to give a spectral type duality for distributive lattices with operators. Here the dual objects are similar to those in $\mathbf{SPEC}$ but are additionally equipped with $n+1$-ary relations corresponding to n-ary operators (these relational correspondents are based on those appearing in [90, 91]). The maps are spectral with respect to the topology and also satisfy some additional conditions with respect to the additional relational structures of their domain and codomain. Goldblatt [65] provides a similar extension for Priestley duality.

Dualities have also been provided for lattices that are not necessarily distributive. In particular [135, 76] provide dualities for general bounded lattices that reduce to Priestley duality in the distributive case. A constructive duality is provided in [75] in which the dual objects are certain pairs of ordered Stone spaces with a binary relation between. This approach has been further 'topologized' to give a dual category that is a strict subcategory of the category of topological spaces with continuous maps [87].

3.3 Representations for posets and semilattices

Given a poset P we can consider $\wp(P)$ as a poset ordered by inclusion. The map $h\colon P \to \wp(P)$ defined by $h(p) = \{q \in p : q \leq p\}$ is an embedding (which maps the top and bottom elements of P to P and $\emptyset$ respectively, whenever they exist), so any poset can be thought of as a system of sets ordered by inclusion. Indeed, this representation has the benefit of interpreting existing meets as set theoretic intersection, and we can define a similar such representation for existing joins (see Lemma 3.3.1).

Lemma 3.3.1. *Given a poset P define $h\colon P \to \wp(P)$ by $h(p) = \{q \in P : q \leq p\}$. Then h is an embedding and $h(\bigwedge_I p_i) = \bigcap_I h(p_i)$ whenever $\bigwedge_I p_i$ is defined in P. Similarly, the map $g\colon P \to \wp(P)$ defined by $g(p) = \{q \in P : q \not\geq p\}$ is an embedding such that $g(\bigvee_I p_i) = \bigcup_I g(p_i)$ whenever $\bigvee_I p_i$ is defined in P. Moreover, both h and g map the top and bottom elements of P to P and $\emptyset$ respectively, whenever they exist.*

Proof. This is straightforward, and first appears in [1]. □

Lemma 3.3.1 shows that semilattices can be represented in such a way that their binary operation is modelled by union or intersection appropriately, though in general we cannot construct representations where both existing joins and meets are interpreted as unions and intersections respectively.

Definition 3.3.2 (Representation)**.** A representation of a poset P is an embedding $h\colon P \to \wp(X)$ for some set X such that for all finite $F \subseteq P$ we have $h(\bigvee F) = \bigcup_{p\in F} h(p)$ whenever $\bigvee F$ is defined, and $h(\bigwedge F) = \bigcap_{p\in F} h(p)$ whenever $\bigwedge F$ is defined. When P has a top and/or bottom, we demand that h maps them to X and/or $\emptyset$ respectively. When P has a representation we say it is *representable*.

The concept of representability is equivalent to a separation property. We make this precise in Theorem 3.3.6, though first we require some preliminary definitions. Note that we could have defined a notion of representability requiring only that existing *binary* meets and joins be represented. Example 3.3.10 below demonstrates that this is a strictly weaker notion.

Definition 3.3.3 (Weak-filter)**.** $S \subseteq P$ is a weak-filter of P if it is closed upwards and for all finite $\emptyset \subset X \subseteq S$ and we have $\bigwedge X \in S$ whenever $\bigwedge X$ is defined.

Note that we formally demand only closure under non-empty finite meets, but since the meet of the empty set is the top element, the closure of weak-filters under the empty meet follows from upward closure. We define *weak-ideals* dually as being the sets that are weak-filters in the order dual P^δ. Note that filters/ideals are automatically weak-filters/ideals. A weak-filter $F \subseteq P$ is prime if whenever $\emptyset \subset X \subseteq P$ is finite with $\bigvee X$ defined in P we have $\bigvee X \in F \implies x \in F$ for some $x \in X$, and prime weak-ideals are defined dually. In a lattice the prime weak-filters are precisely the prime filters, and the prime weak-ideals are precisely the prime ideals.

Definition 3.3.4 (Separating)**.** $S \subseteq \wp(P)$ is separating over P if whenever $p \not\leq q \in P$ there is $X \in S$ with $p \in X$ and $q \notin X$. We say S is *dually-separating* over P if whenever $p \not\leq q$ there is $X \in S$ with $q \in S$ and $p \notin X$.

Definition 3.3.5 (α-morphism)**.** Given an ordinal α, and posets P_1 and P_2 we say an order preserving map $f\colon P_1 \to P_2$ is an α-morphism if $f(\bigwedge_I p_i) = \bigwedge_I f(p_i)$ whenever $I < \alpha$ and $\bigwedge_I p_i$ is defined, and $f(\bigvee_I p_i) = \bigvee_I f(p_i)$ whenever $I < \alpha$ and $\bigvee_I p_i$ is defined. If f is also an embedding we say it is an α-embedding.

Theorem 3.3.6. *For a poset P the following are equivalent:*

1. *The prime weak-filters of P are separating over P,*
2. *P is representable,*
3. *there is a distributive lattice L and an ω-embedding $e\colon P \to L$.*

Proof.

1. $\Rightarrow$ 2. Let X be the set of prime, weak-filters of P and define h by $h(p) = \{S \in X : p \in S\}$. If X is separating over P it's easy to see h will be a representation.

2. $\Rightarrow$ 3. Let $h\colon P \to \wp(X)$ be a representation of P. Then the sublattice of $\wp(X)$ generated by $h[P]$ (defining join and meet by union and intersection respectively) is a distributive lattice. Moreover the natural embedding induced by h preserves joins and meets as required.

3. $\Rightarrow$ 1. If L is a distributive lattice and $e\colon P \to L$ is an embedding preserving existing joins and meets, then by identifying P with $e[P]$, the restrictions of the separating set of prime filters of L that exist by Theorem 3.2.4 to $e[P]$ give us a separating set of prime, weak-filters for P as required.

□

We note that in the meet-semilattice case we have representability if and only if a necessarily infinite family of first order axioms demanding that whenever $a \wedge (b_1 \vee ... \vee b_n)$ is defined, then $(a \wedge b_1) \vee ... \vee (a \wedge b_n)$ is also defined and the two are equal is satisfied [6, 118]. We can generalize this idea to produce a first order axiomatization for the class of representable posets, though we require some technical preliminaries.

Definition 3.3.7 (ω-free lattice generated by P)**.** Given a poset P we define the poset extension $\iota\colon P \to F_\omega(P)$ by the following properties:

1. ι is an ω-embedding,
2. $F_\omega(P)$ is a lattice,
3. $F_\omega(P)$ is exactly the sublattice of itself generated by $\iota[P]$, and
4. if L is a lattice, and $e\colon P \to L$ is an ω-morphism then there is a unique lattice homomorphism $h\colon F_\omega(P) \to L$ such that the following commutes

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[dr, "e"'] & F_\omega(P) \arrow[d, "h"] \\ & L \end{tikzcd}$$

Associating P with $\iota[P]$ we say $F_\omega(P)$ is the ω-free lattice generated by P (uniqueness up to isomorphism lifting the identity on P follows from the universal property we have just described).

Given P, existence of the ω-free lattice generated by P is shown by Dean [29], generalizing results on free lattices generated by a poset from [37, 28]. In the same paper Dean also addresses the word problem for such free lattices, generalizing results from [138, 139]. We will not go into details here, but we shall refer to results therein as necessary.

Theorem 3.3.8. *For a poset P the following are equivalent:*

- (R) *P is representable,*
- (F) *if γ is a weak-filter of P and $b \in P \setminus \gamma$ then there is a prime filter γ' of $F_\omega(P)$ with $\iota[\gamma] \subseteq \gamma'$ and $\iota(b) \notin \gamma'$,*
- (F') *suppose $n \in \omega$, and $p_i \in P$ for each $i \in \{1, ..., n\}$ with $\bigvee_{i=1}^n p_i$ defined in P. Then for all finite $S_i \subseteq P$ we have*

$$\bar{\bigvee}_{i=1}^n \bar{\bigwedge}(\iota[S_i \cup \{p_i\}]) \leq \iota(b) \implies \bar{\bigwedge}(\bigcup_{j=1}^n \iota[S_j] \cup \{\iota(\bigvee_{i=1}^n p_i)\}) \leq \iota(b)$$

where $\bar{\wedge}$ and $\bar{\vee}$ denote the meet and join in $F_\omega(P)$ respectively.

Proof. By Theorem 3.3.6 if P is representable there is an ω-embedding $e\colon P \to L$ where L is the ring of sets whose base is the set of prime weak-filters of P, and where $e(p)$ is the set of prime weak-filters of P containing p. Let γ be a weak-filter of P, and let $b \in P \setminus \gamma$. Define $\zeta = \{\bigcap S : S \text{ is a finite subset of } e[\gamma]\}^{\uparrow}$. Then $\gamma \in e(p)$ for all $p \in \gamma$ and $\gamma \notin e(b)$, so ζ is a filter of L with $e[\gamma] \subseteq \zeta$ and $e(b) \notin \zeta$. As L is distributive there is a prime filter ζ' of L with $\zeta \subseteq \zeta'$ and $e(b) \notin \zeta'$. By the universal property of $F_\omega(P)$ there is a homomorphism $h\colon F_\omega(P) \to L$ lifting the identity on P, and $h^{-1}[\zeta']$ is a prime filter of $F_\omega(P)$ with $\iota[\gamma] \subseteq h^{-1}[\zeta']$ and $\iota(b) \notin h^{-1}[\zeta']$, so $(R) \implies (F)$.

Now, suppose $n \in \omega$, and $p_i \in P$ for each $i \in \{1, ..., n\}$ with $\bigvee_{i=1}^n p_i$ defined in P. Suppose also that $b \in P$ and $\bar{\bigwedge}(\bigcup_{j=1}^n \iota[S_j] \cup \{\iota(\bigvee_{i=1}^n p_i)\}) \not\leq \iota(b)$, where $S_i \subseteq P$ is finite for each $i \in \{1, ..., n\}$. Then $\iota^{-1}[\bar{\bigwedge}(\bigcup_{j=1}^n \iota[S_j] \cup \{\iota(\bigvee_{i=1}^n p_i)\})^{\uparrow}]$ is a weak-filter of P not containing b, and thus, assuming (F), there is a prime filter γ of $F_\omega(P)$ with $\bigcup_{j=1}^n \iota[S_j] \cup \{\iota(\bigvee_{i=1}^n p_i)\} \subseteq \gamma$ and $\iota(b) \notin \gamma$. By the properties of prime filters, for some $i \in \{1, ..., n\}$ we must have $\iota[\{p_i\} \cup S_i] \subseteq \gamma$, and thus $\bar{\bigwedge}\iota[\{p_i\} \cup S_i] \in \gamma$. We must have $\bar{\bigwedge}\iota[S_i \cup \{p_i\}] \not\leq \iota(b)$ and thus $\bar{\bigvee}_{i=1}^n(\bar{\bigwedge}\iota[S_i \cup \{p_i\}]) \not\leq \iota(b)$, and so $(F) \implies (F')$.

Finally, suppose P satisfies (F'), and let $a \not\leq b \in P$. Then $\iota(a) \not\leq \iota(b)$ so we define $X = \{S \subseteq F_\omega(P) : S$ is an up-set, $\iota(a) \in S$ and $\iota(b) \notin S$, and S is closed under finite meets from $S \cap \iota[P]\}$, and a Zorn's lemma argument says that X must have a maximal element, which we call γ. Suppose there are $n \in \omega$ and $p_i \in P$ for $i \in \{1, ..., n\}$ with $\bigvee_{i=1}^n p_i$ defined in P. Suppose also that $\iota(\bigvee_{i=1}^n p_i) \in \gamma$ but $\iota(p_i) \notin \gamma$ for all $i \in \{1, ..., n\}$. Then for each i the set $\gamma_i = (\{\bar{\bigwedge}(\{\iota(p_i)\} \cup T_i) : T_i$ is a finite subset of $\gamma \cap \iota[P]\} \cup \gamma)^{\uparrow}$ is a member of X if and only if it does not contain $\iota(b)$. As γ is maximal and for each i we have $\gamma \subseteq \gamma_i$ we must have $\iota(b) \in \gamma_i$, and thus for each i there is a finite $T_i \subseteq \gamma \cap \iota[P]$ with $\bar{\bigwedge}(T_i \cup \{\iota(p_i)\}) \leq \iota(b)$. For each i define $S_i = \iota^{-1}[T_i]$. Then we have $\bar{\bigvee}_{i=1}^n \bar{\bigwedge}(\iota[S_i \cup \{p_i\}]) \leq \iota(b)$, and thus by (F') we also have $\bar{\bigwedge}(\bigcup_{j=1}^n \iota[S_j] \cup \{\iota(\bigvee_{i=1}^n p_i)\}) \leq \iota(b)$, which is a contradiction as $\bigcup_{j=1}^n \iota[S_j] \cup \{\iota(\bigvee_{i=1}^n p_i)\} \subseteq \gamma$. Thus $\iota^{-1}[\gamma]$ is a prime weak-filter of P, and by Theorem 3.3.6 P is representable. $\square$

Corollary 3.3.9. *The class of representable posets is elementary.*

Proof. The solution to the word problem for ω-free lattices over P from [29] can be used in conjunction with property (F') from Theorem 3.3.8 to give an explicit, though complex and infinite first axiomatization. Alternatively, a closure under ultraproducts/ultraroots argument gives the existence of such an axiomatization indirectly. $\square$

Example 3.3.10. *To show the existence of a binary only representation for a poset is strictly weaker than the existence of a representation.* Define S_{ab}, S_{ac}, S_{ad}, S_{bc}, S_{bd}, and S_{cd} to be distinct copies of ω (for $x \neq y \in \{a, b, c, d\}$ we will identify S_{xy} with S_{yx}). Let a be a previously unused element and define $A = S_{ab} \cup S_{ac} \cup S_{ad} \cup \{a\}$. Make analogous definitions for B, C, and D (so e.g. $B = S_{ba} \cup S_{bc} \cup S_{bd} \cup \{b\}$ for some b). Define $X_{ab} = \{\{a, b, c, d\} \cup A \cup B \cup S_{cd} \setminus \{k : k \leq n\} : n \geq 1\}$, and define $Y = A \cup B \cup C \cup D = S_{ab} \cup S_{ac} \cup S_{ad} \cup S_{bc} \cup S_{bd} \cup S_{cd} \cup \{a, b, c, d\}$. Let z be a previously unused element and define $Z = S_{ab} \cup S_{ac} \cup S_{ad} \cup S_{bc} \cup S_{bd} \cup S_{cd} \cup \{z\}$. Let P be the poset whose set of elements is $\{S_{ab}, S_{ac}, S_{ad}, S_{bc}, S_{bd}, S_{cd}, A, B, C, D, Y, Z\} \cup X_{ab} \cup X_{ac} \cup X_{ad} \cup X_{bc} \cup X_{bd} \cup X_{cd}$ ordered by inclusion. A routine though tedious check confirms that existing binary meets and joins in P correspond correctly to binary unions and intersections, however, if γ is a prime, weak-filter containing Y, then as $A \vee B \vee C = Y$, we have (wlog) $A \in \gamma$, and similarly we have $B \vee C \vee D = Y$ and thus (wlog) $B \in \gamma$. Thus we have $A \wedge B = A \cap B = S_{ab} \in \gamma$, and $Z \in \gamma$ follows from the fact that $S_{ab} \leq Z$. So there is no prime, weak-filter containing Y but not Z, and P is therefore not representable by Theorem 3.3.6.

Duality theories for distributive semilattices have also been studied (a join-semilattice is distributive when the lattice of its ideals ordered by inculsion is distributive, and a dual definition is made for meet-semilattices). A spectral duality for distributive meet-semilattices with top elements is given in [21], and steps are made toward a Priestley style duality for bounded, distributive join-semilattices in [72] (the defect being that no correspondent is provided for general semilattice homomorphisms). More recently a full Priestley style duality for bounded, distributive meet-semilattices has been developed [13]. Duality theory remains an active area of research.

3.4 Algebras of relations

Algebras of relations, perhaps unsurprisingly, allow for algebraic reasoning about relations (predicates) and can capture significant fragments of first order logic. As a somewhat degenerate example we have seen via Stone's theorem that Boolean algebras capture exactly the essential properties of unary relations on a set, as a unary relation is merely a subset and the 'first order' ways for obtaining new unary predicates from old can be reconstructed using set theoretic union, intersection and complementation.

One may wonder whether higher order predicates can be similarly algebraized, and the answer, to an extent, is yes. Here we will primarily deal with *relation algebras* (introduced in [131]) for binary relations.

3.4.1 Relation algebras

Although our work does not directly concern them, we shall define relation algebras carefully as they provide useful context for the kind of representation issues we shall meet in Chapter 6. Further to this, binary relations are a central issue in foundational mathematics, and their behaviour can be formalized in such a way as to provide a variable-free deductive system powerful enough to develop ZFC (see e.g. [133, 62]). Our approach to exposition here is similar to that in [81], where the subject is covered in much greater depth. First we describe the kind of system a relation algebra was originally intended to model.

Definition 3.4.1 (Proper relation algebra with base B)**.** Let $B \neq \emptyset$ be a set, let $S \subseteq \wp(B \times B)$ (so the elements of S are binary relations on B), let $U = \bigcup S$, let $\cup$ and $\setminus$ be the usual set theoretic union and complementation (relative to U) respectively, let $\mathbf{id}_B = \{(b, b) : b \in B\}$ (so $\mathbf{id}_B$ is the identity relation on B), let ${}^{-1}$ denote the unary operation of taking converses (so for $s \in S$ we have $s^{-1} = \{(b, a) : (a, b) \in s\}$), and let $|$ denote the binary operation of relation composition (so for $s, t \in S$ we have $s|t = \{(a, b)$: there is $c \in B$ with $(a, c) \in s$ and $(c, b) \in t\}$. Then $\mathcal{S} = (S, \emptyset, U, \cup, \setminus, \mathbf{id}_B, {}^{-1}, |)$ is a proper relation algebra with base B whenever $(S, \emptyset, U, \cup, \setminus, \mathbf{id}_B, {}^{-1}, |)$ is a field of sets, S is closed under composition and taking converses, and $\mathbf{id}_B \in S$.

For algebraic counterparts to such structures we make the following definition (primarily due to Tarski [24, 91]).

Definition 3.4.2 (Relation algebra)**.** A relation algebra is an algebra of form

$$\mathcal{A} = (A, 0, 1, +, -, \mathbf{id}, {}^{\smile}, ;)$$

where 0, 1, and $\mathbf{id}$ are nullary, $-$ and ${}^{\smile}$ are unary, $+$ and ; are binary, the system $(A, 0, 1, +, -)$ is a Boolean algebra, and the following additional axioms hold for all $a, b, c \in A$:

R1 $(a\,;b)\,;c = a\,;(b\,;c)$

R2 $(a + b)\,;c = a\,;c + b\,;c$

R3 $a\,;\mathbf{id} = a$

R4 $(a^{\smile})^{\smile} = a$

R5 $(a + b)^{\smile} = a^{\smile} + b^{\smile}$

R6 $(a\,;b)^{\smile} = b^{\smile}\,;a^{\smile}$

R7 $a^{\smile};(-(a;b)) \leq -b$

We note that each of axioms (R1)-(R7), and the Boolean algebra axioms, either are or can be written as identities, so the class **RA** of relation algebras is a variety, moreover, its defining identities are *Sahlqvist*, in the sense of [114], and thus it is closed under canonical extensions, and as its operators are *conjugated* it is thus also closed under MacNeille completions (see Chapter 4 for discussion of canonical extensions and MacNeille completions, and see [63, Corollary 34] for proof of the claim of closure under MacNeille completions). Conjugation of the **RA** operators also implies that they are complete (see e.g. [81, Theorem 2.40]).

It is straightforward to check that proper relation algebras are relation algebras, but do the relation algebra axioms capture all true properties of proper relation algebras? The answer is no, though this is by no means the end of the story, as we shall see in Section 3.4.2.

3.4.2 The representation problem

We asked in the last section whether the axioms given in Definition 3.4.2 captured the essential properties of the proper relation algebras of Definition 3.4.1. We make this question precise in Definition 3.4.3 below.

Definition 3.4.3 (Representable relation algebra)**.** A relation algebra $\mathcal{R}$ is representable if there is an isomorphism $h\colon\ \mathcal{R} \to R(B)$, where $R(B)$ is a proper relation algebra over some set B. In this case the map h is referred to as the *representation* of $\mathcal{R}$. The class of representable relation algebras is denoted by **RRA**.

Lyndon [95] constructed a relation algebra that was not representable, thus proving the strict inclusion **RRA** $\subset$ **RA**. Given this discovery a natural question was whether **RRA** could be axiomatized by a finite set of identities. Tarski [132] showed that **RRA** can be axiomatized by an infinite set of equations, and thus is a variety, but Monk [102] showed that no finite axiomatization exists for **RRA** in first order logic, and Andréka demonstrated that any equational axiomatization of **RRA** must contain infinitely many equations containing more than k variables, for arbitrary $k < \omega$ (see [4]). Further to this, it is not decidable whether a finite relation algebra is representable [80], so **RRA** can not be finitely axiomatized in higher order logic either (as we could use such an axiomatization to define a decision procedure for finite algebras).

More positively Monk proved that **RRA** is *canonical*, i.e. the canonical extension of a representable relation algebra is also representable (see Definition 4.1.7 and Section 4.4 for definition and discussion of the canonical extension). The first published proof of Monk's result (which we present as Theorem 5.3.1) is attributed in [81, Section 3.4.4] to [99], though a statement of the result is also said to be given in [100, Theorem 2.12], where the proof is attributed to Monk. More recent work [85] has shown that though **RRA** is canonical it is only barely so, in that any axiomatization must contain an infinite number of non-canonical sentences (this result also implies that no finite axiomatization exists for **RRA**).

Given the difficulties with classical representability, a number of alternative approaches have been considered. For example, the **RA** signature has been modified, both by adding and subtracting operations, and representability for these alternative signatures investigated (see e.g. [119]). The representation problems for these signatures can also be interesting in their own right, for example in the model theory of non-classical logics (see e.g. [5, 101]). Alternative approaches include altering the definition of representability, for example by considering *weak representations* [88]. We will not go into details here, and the interested reader is directed to [81] for further information.

3.4.3 Relations of higher order

Relation algebras are not the only algebraizations of systems of relations; in particular *cylindric algebras* and *polyadic algebras* algebraize higher order predicates, and are again examples of BAOs with similar representation problems. A discussion of these structures would take us well behind the scope of this thesis so we restrict ourselves to noting in passing that classic references for these subjects can be found in [78] and [70] respectively, and a more general survey of algebraizations of logic can be found in [105].

Chapter 4

Completions of ordered structures

In any ordered set we have the notions of supremum and infimum, and even in a lattice, where the suprema and infima of finite sets are necessarily defined, we may have infinite subsets for which these limits do not exist. A simple example of such a structure is the set of rational numbers $\mathbb{Q}$ with its natural ordering. Possibly the first, and certainly the most famous example of a kind of completion process for a poset is Dedekind's construction of the reals from the rationals. This construction, generalized to arbitrary posets by MacNeille [98], not only provided a completeness property (the existence of suprema and infima for all non-empty bounded subsets) but also extended to $\mathbb{R}$ the basic arithmetic operations (addition, subtraction, multiplication) from $\mathbb{Q}$ in a natural way.

This is a common theme; one frequently wishes to complete a poset in such a way that certain algebraic properties are preserved, and there are many completion methods, each with advantages and disadvantages depending on the situation. Here we discuss MacNeille completions, canonical extensions, meet-completions, and Δ_1-completions [51], and also the relationships between them. We begin with a brief algebraic overview, then use this to describe the methods for lifting maps and monotone operations that allow us to extend these completion methods to poset expansions, before proceeding to more detailed examinations of each construction in turn.

4.1 Completions via density conditions: an overview

Definition 4.1.1 (Completion)**.** Given a poset P we define a completion of P to be a complete lattice Q and an order embedding $e\colon P \to Q$. We may also make the following definitions in cases where we are interested in the lattice or Boolean structure of P:

1. if P is considered as a (bounded) lattice then e is a (bounded) lattice homomorphism,

2. if P is considered as a Boolean lattice then Q is a Boolean lattice and e is a Boolean homomorphism.

When our meaning is clear from context we may abuse our notation by referring to either Q or e alone as 'the completion' of P.

The manner in which $e[P]$ (the image of P under e) 'sits' in Q provides us with a useful method for characterizing the completions we are interested in.

Definition 4.1.2 (Density and compactness)**.** Given a complete lattice Q, a set $S \subseteq Q$, a set $\mathcal{F}$ of non-empty up-sets of S, and a set $\mathcal{I}$ of non-empty down-sets of S we make the following definitions:

- S is *meet-dense* in Q if $q = \bigwedge\{p \in S : p \geq q\}$ for all $q \in Q$, join-density is defined dually,
- S is *doubly-dense* in Q if it is both meet-dense and join-dense in Q,
- $q \in Q$ is $\mathcal{F}$*-closed* if it is the infimum of some $F \in \mathcal{F}$,
- $q \in Q$ is $\mathcal{I}$*-open* if it is the supremum of some $I \in \mathcal{I}$,
- $K_{\mathcal{F}}(Q)$ is the set of $\mathcal{F}$-closed elements of Q,
- $O_{\mathcal{I}}(Q)$ is the set of $\mathcal{I}$-open elements of Q,
- Q is $(\mathcal{F},\mathcal{I})$*-dense* if $q = \bigwedge\{x : q \leq x \in O_{\mathcal{I}}(Q)\} = \bigvee\{x : q \geq x \in K_{\mathcal{F}}(Q)\}$ for all $q \in Q$,
- S is *dense* in Q if Q is $(\mathcal{U}(S), \mathcal{D}(S))$-dense, where $\mathcal{U}(S)$ is the set of all non-empty up-sets of S, and $\mathcal{D}(S)$ is the set of all non-empty down-sets of S,
- Q is $(\mathcal{F},\mathcal{I})$*-compact* if whenever $F \in \mathcal{F}$ and $I \in \mathcal{I}$ with $\bigwedge F \leq \bigvee I$ we have $F \cap I \neq \emptyset$.

Henceforth we assume $\mathcal{F}$ and $\mathcal{F}'$, and $\mathcal{I}$ and $\mathcal{I}'$ are subsets of $\mathcal{U}(e[P])$ and $\mathcal{D}(e[P])$ respectively.

Lemma 4.1.3. *If $S \subseteq Q$, $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{I} \subseteq \mathcal{I}'$ then Q is $(\mathcal{F},\mathcal{I})$-dense $\implies$ Q is $(\mathcal{F}',\mathcal{I}')$-dense.*

Proof. $q \leq \bigwedge\{x : q \leq x \in O_{\mathcal{I}'}(Q)\} \leq \bigwedge\{x' : q \leq x' \in O_{\mathcal{I}}(Q)\} = q$, so $\bigwedge\{x : q \leq x \in O_{\mathcal{I}'}(Q)\} = q$, and similar for $\bigvee\{y : q \leq y \in K_{\mathcal{F}'}(Q)\}$. ☐

In view of Lemma 4.1.3, if P is a poset then the class of completions $e\colon P \to Q$ for which $e[P]$ is dense in Q contains every $(\mathcal{F},\mathcal{I})$-dense completion for which $\mathcal{F} \subseteq \mathcal{U}(e[P])$ and $\mathcal{I} \subseteq \mathcal{D}(e[P])$. When the sets $\mathcal{I}$ and $\mathcal{F}$ are closed under arbitrary (non-empty) intersections they generate as many open and closed elements as any pair $(\mathcal{F}',\mathcal{I}')$ with $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{I} \subseteq \mathcal{I}'$, in a sense we make precise in Lemma 4.1.4 below.

Lemma 4.1.4. *Let $e\colon P \to Q$ be an $(\mathcal{F},\mathcal{I})$-dense completion, and let $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{I} \subseteq \mathcal{I}'$. Then*

1. *$\mathcal{F}$ is closed under arbitrary non-empty intersections $\implies K_{\mathcal{F}}(Q) = K_{\mathcal{F}'}(Q)$, and*

2. *$\mathcal{I}$ is closed under arbitrary non-empty intersections $\implies O_{\mathcal{I}}(Q) = O_{\mathcal{I}'}(Q)$.*

Proof. We prove 1, and 2 is similar. Let $x' \in K_{\mathcal{F}'}(Q)$. Then $x' = \bigwedge F_{x'}$ for some $F_{x'} \in K_{\mathcal{F}'}(Q)$. By $(\mathcal{F},\mathcal{I})$-density we also have $x' = \bigvee Y$, where $Y = \{y : x' \geq y \in K_{\mathcal{F}}(Q)\}$. For each $y \in Y$ pick $F_y \in K_{\mathcal{F}}(Q)$ so that $y = \bigwedge F_y$. Now, $x' \geq y \iff F_{x'} \subseteq F_y$, so $F_{x'} \subseteq \bigcap_Y F_y$, and thus $x' \geq \bigwedge(\bigcap_Y F_y)$. Conversely, since $\bigcap_Y F_y \subseteq F_y$ for all $y \in Y$ we have $\bigwedge(\bigcap_Y F_y) \geq y$ for all $y \in Y$, and so $\bigwedge(\bigcap_Y F_y) \geq \bigvee Y = x'$, and consequently $x' = \bigwedge(\bigcap_Y F_y)$. Since $\mathcal{F}$ is closed under arbitrary interections this says that $x' \in K_{\mathcal{F}}(Q)$, and since $K_{\mathcal{F}}(Q) \subseteq K_{\mathcal{F}'}(Q)$ (as $\mathcal{F} \subseteq \mathcal{F}'$) this means $K_{\mathcal{F}}(Q) = K_{\mathcal{F}'}(Q)$. □

Remark 4.1.5. Lemma 4.1.3 tells us that any $(\mathcal{F},\mathcal{I})$-dense completion is also $(\mathcal{F}',\mathcal{I}')$-dense whenever $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{I} \subseteq \mathcal{I}'$. However, it is not the case that an $(\mathcal{F},\mathcal{I})$-compact completion must also be $(\mathcal{U}(e[P])), \mathcal{D}(e[P]))$-compact.

Definition 4.1.6 (Regularity)**.** A completion $e\colon P \to Q$ is *meet-regular* if for all $\emptyset \subset S \subseteq P$, if $\bigwedge S$ exists in P then $e(\bigwedge S) = \bigwedge e[S]$. *Join-regular* is defined dually, and a completion is *regular* if it is both meet- and join-regular.

We can use the terminology from Definition 4.1.2 to characterize the completions we are concerned with. We use a system based on that of [50]. The characterization of the MacNeille completion is attributed to [8] and [120] independently, and that of the canonical extension is a generalization to posets of the characterization of the canonical extension of a bounded lattice from [49]. This generalization first appears explicitly in [39], though we adopt terminology closer to the more general framework of [48, 59, 66, 51].

Definition 4.1.7. A completion $e\colon P \to Q$ is a

- *MacNeille completion* when $e[P]$ is doubly dense in Q. We shall see in Section 4.3 that every poset has a MacNeille completion, and that it is unique up to isomorphism fixing $e[P]$. The MacNeille completion is regular.

- *canonical extension* when it is $(\mathcal{F},\mathcal{I})$-dense and $(\mathcal{F},\mathcal{I})$-compact, where $\mathcal{F}$ and $\mathcal{I}$ are the sets of filters and ideals (see Definition 2.3.7) of $e[P]$ respectively. It is unique up to isomorphism fixing $e[P]$, and we shall see various constructions in Section 4.4.

- *meet-completion* when $e[P]$ is meet-dense in Q. Join-completion is defined dually. Note that meet-completion (and join-completion) is not determined up to isomorphism; we shall see in Section 4.5 that the sets of (isomorphism classes of) meet-completions and join-completions of a poset form a complete lattice that is in general non-trivial. The MacNeille completion is an example of a meet-completion (and a join-completion), unlike the canonical extension which in general is not (though when P is finite its MacNeille completion and canonical extension are isomorphic). Meet-completions are join-regular, and join-completions are meet-regular.

- Δ_1-*completion* when $e[P]$ is dense in Q. It is not in general unique up to isomorphism, and both the MacNeille completion and the canonical extension of a poset are examples of Δ_1-completions.

Note that a Δ_1-completion $e\colon P \to Q$ is defined in [51] as being a completion such that every element $q \in Q$ is both a meet of joins and a join of meets of elements of $e[P]$. This is clearly equivalent to saying that $e\colon P \to Q$ is an $(\mathcal{F}, \mathcal{I})$-dense completion for some $\mathcal{F} \subseteq \mathcal{U}(e[P])$ and $\mathcal{I} \subseteq \mathcal{D}(e[P])$, and so our definition of the Δ_1-completion from Definition 4.1.7 is equivalent by Lemma 4.1.3.

4.2 Lifting isotone maps and monotone operations using density

We can use information about the way a poset P embeds into a completion Q of P to naturally extend maps with domain P to maps with domain Q. In this section we deal mainly with order preserving maps, as this is sufficient to lift the operations used in Chapter 6.

4.2.1 Lifting maps

Definition 4.2.1 (Lift)**.** Given posets P_1, P_2, Q_1, and Q_2, embeddings $e_1\colon P_1 \to Q_1$ and $e_2\colon P_2 \to Q_2$, and order preserving map $f\colon P_1 \to P_2$, we say a lift of f along e_1 and e_2 (or just a lift of f when e_1 and e_2 are clear from context) is an order preserving map $f'\colon Q_1 \to Q_2$ such that the following commutes:

$$\begin{tikzcd} P_1 \arrow[r, "e_1"] \arrow[d, "f"'] & Q_1 \arrow[d, "f'"] \\ P_1 \arrow[r, "e_2"'] & Q_2 \end{tikzcd}$$

If $e_1\colon P_1 \to Q_1$ is a meet-completion, $e_2\colon P_2 \to Q_2$ is any completion of P, and $f\colon P_1 \to P_2$ is an order preserving map, there is a natural method (introduced in [103]) for lifting f to an order preserving map $\hat{f}\colon Q_1 \to Q_2$, given by

$$\hat{f}(q) = \bigwedge \{e_2(f(p)) : e_1(p) \geq q\} \tag{4.1}$$

Similarly if e_1 is a join-completion we can lift f to $\check{f}\colon Q_1 \to Q_2$ using

$$\check{f}(q) = \bigvee\{e_2(f(p)) : e_1(p) \geq q\} \tag{4.2}$$

$\hat{f}$ and $\check{f}$ are by no means unique in lifting f in this way, and in fact, in the meet-completion case, the possible lifts of f form a complete lattice with $\hat{f}$ as a top element (see Proposition 4.2.2). A dual argument says that in the join-completion case the possible lifts of f form a complete lattice with $\check{f}$ as a bottom element.

Proposition 4.2.2. *Given meet-completion $e_1\colon P_1 \to Q_1$, completion $e_2\colon P_2 \to Q_2$, and order preserving map $f\colon P_1 \to P_2$, the possible lifts of f to maps from Q_1 to Q_2 form a complete lattice (ordered pointwise) with top element $\hat{f}$.*

Proof. To see that $\hat{f}$ is maximal suppose k is another lift of f and let $q = \bigwedge e_1[S] \in Q_1$, for some $S \in \mathcal{U}(P)$. Then $k(q) \leq k(e_1(p)) = e_2(f(p))$ for all $p \in S$, so $k(q) \leq \bigwedge e_2[f[S]] = \hat{f}(q)$. To see that the lifts of f form a complete lattice let $I \neq \emptyset$ be an ordinal and let k_i be a lift of f for each $i \in I$. Define $k\colon Q_1 \to Q_2$ by $k(q) = \bigwedge_I k_i(q)$ for all $q \in Q_1$. Then k is well defined as Q_2 is a complete lattice. To see that k is order preserving note that if $q_1 \leq q_2$ then, as k_i is order preserving for all $i \in I$, we must have $\bigwedge_I k_i(q_1) \leq \bigwedge_I k_i(q_2)$, and commutativity follows from the fact that k_i is a lift for each $i \in I$. Since the 'meet' of the empty set is the top element $\hat{f}$ we are done. $\square$

The following example shows that this lattice of lifts can be non-trivial.

Example 4.2.3. *To show the lattice of lifts of a map can be non-trivial.* Let P be the antichain on $\{a, b\}$, and let $e\colon P \to Q$ and $f\colon P \to L$ be as in Figure 4.1 (both e and f map a and b to a and b as marked in the diagrams). Then $e\colon P \to Q$ is the MacNeille completion of P, and L is the MacNeille completion of itself via the identity map. The lift $\hat{f}\colon Q \to L$ takes q to r, the lift $\check{f}\colon Q \to L$ takes q to t, and there is a third lift $f'\colon Q \to L$ defined by $f' = \hat{f} = \check{f}$ on $Q \setminus \{q\}$, and $f'(q) = s$.

The following lemma will be useful later:

Lemma 4.2.4. *Let P_1 and P_2 be posets, let $e_1\colon P_1 \to Q_1$ be a meet-completion, $e_2\colon P_2 \to Q_2$ be a completion, and let $g\colon Q_1 \to Q_2$ be an isomorphism lifting the identity on P. Then $g = \widehat{g|_{e_1[P]}}$, where $\widehat{g|_{e_1[P]}}$ is the restriction of g to $e_1[P]$.*

Proof. Let $q \in Q_1$. Then by Proposition 4.2.2 we must have $g(q) \leq \widehat{g|_{e_1[P]}}(q)$, and as g is onto there is $q' \in Q_1$ such that $g(q') = \widehat{g|_{e_1[P]}}(q)$. Now, by definition $\widehat{g|_{e_1[P]}}(q) = \bigwedge\{g(e_1(p)) :$

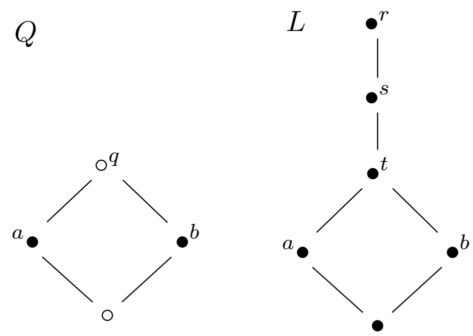


Figure 4.1: The lattice of lifts may be non-trivial

$e_1(p) \geq q\}$, so $g(q') \leq g(e_1(p))$ for all $p \in P$ with $e_1(p) \geq q$. As g is an embedding this means $q' \leq e_1(p)$ for all $p \in P$ with $e_1(p) \geq q$, and thus we must have $q' \leq q$, and so $g(q') \leq g(q)$, which means $\hat{g}|_{e_1[P]}(q) \leq g(q)$ and we are done. $\square$

Proposition 4.2.2 and its dual together tell us that when e_1 defines the MacNeille completion the possible lifts of f along e_1 and e_2 form a complete lattice bounded by $\hat{f}$ and $\check{f}$, and Example 4.2.3 showed that this lattice can be a strict superset of $\{\hat{f}, \check{f}\}$. The picture for $(\mathcal{F}, \mathcal{I})$-dense completions is less clear.

If $e_1 \colon P_1 \to Q_1$ is an $(\mathcal{F}, \mathcal{I})$-dense completion, $e_2 \colon P_2 \to Q_2$ is a completion of P, and $f \colon P_1 \to P_2$ is order preserving we can define lifts f^σ and f^π by

$$f^\sigma(q) = \bigvee\{\bigwedge\{e_2(f(p)) : x \leq e_1(p)\} : q \geq x \in K_{\mathcal{F}}(Q_1)\} \tag{4.3}$$

$$f^\pi(q) = \bigwedge\{\bigvee\{e_2(f(p)) : x \geq e_2(p)\} : q \leq x \in O_{\mathcal{I}}(Q_1)\} \tag{4.4}$$

From Lemma 4.1.3 we know that if $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{I} \subseteq \mathcal{I}'$ then $e_1 \colon P_1 \to Q_1$ is also an $(\mathcal{F}', \mathcal{I}')$-dense completion, and so we could also define lifts

$$f'^\sigma(q) = \bigvee\{\bigwedge\{e_2(f(p)) : x \leq e_1(p)\} : q \geq x \in K_{\mathcal{F}'}(Q_1)\} \tag{4.5}$$

$$f'^\pi(q) = \bigwedge\{\bigvee\{e_2(f(p)) : x \geq e_2(p)\} : q \leq x \in O_{\mathcal{I}'}(Q_1)\} \tag{4.6}$$

This calls into question the 'naturalness' of the lifts f^σ and f^π, as we saw in Lemma 4.1.3 that an $(\mathcal{F}, \mathcal{I})$-dense completion is always a $(\mathcal{U}(e[P]), \mathcal{D}(e[P]))$-dense completion. It is always the case that $f^\sigma(q) \leq f'^\sigma(q)$ and $f'^\pi(q) \leq f^\pi(q)$, as we have $K_{\mathcal{F}}(Q) \subseteq K_{\mathcal{F}'}(Q)$ and $O_{\mathcal{I}}(Q) \subseteq O_{\mathcal{I}'}(Q)$. When $\mathcal{F}$ and $\mathcal{I}$ are closed under arbitrary intersections (as they are in the definition of the canonical extension) it follows trivially from Lemma 4.1.4 that $f^\sigma(q) = f'^\sigma(q)$ and $f'^\pi(q) = f^\pi(q)$ (we state this result as Lemma 4.2.5). Example 4.2.6 below shows that when we are given more freedom in our choice for $\mathcal{F}$ this may not be the case, and we may have $f^\sigma < f'^\sigma$ (with a dual example giving $f'^\pi < f^\pi$).

Lemma 4.2.5. *Let $e\colon P_1 \to Q_1$ be an $(\mathcal{F},\mathcal{I})$-dense completion, and let $e_2\colon P_2 \to Q_2$ be any completion. Let $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{I} \subseteq \mathcal{I}'$. Then*

1. *$\mathcal{F}$ is closed under arbitrary intersections $\implies$ $f^\sigma(q) = f'^\sigma(q)$, and*

2. *$\mathcal{I}$ is closed under arbitrary intersections $\implies$ $f^\pi(q) = f'^\pi(q)$*

for all $q \in Q_1$.

Example 4.2.6. *To show we may have $f^\sigma < f'^\sigma$.* Let P, $e_1\colon P \to Q_1$ and $e_2\colon P \to Q_2$ be as in Figure 4.2 (e_1 and e_2 are implicit, for ease of exposition we identify points of P, which are marked with dots, with their images under these embeddings. For clarity we mark the new points of Q_1 and Q_2 with unfilled dots). Define $\mathcal{F} = \{\{a,b,c\}, \{x,y,z\}^\uparrow\} \cup \{p^\uparrow : p \in P\}$, define $\mathcal{F}' = \mathcal{U}(P)$, and define $\mathcal{I} = \mathcal{I}' = \mathcal{D}(P)$. It's easy to see that Q_1 is both $(\mathcal{F},\mathcal{I})$-dense and $(\mathcal{F}',\mathcal{I}')$-dense, and that both Q_1 and Q_2 are complete lattices (note that Q_2 is also $(\mathcal{F}',\mathcal{I}')$-dense, but not $(\mathcal{F},\mathcal{I})$-dense as r cannot be written as a join of $\mathcal{F}$-closed elements).

Let f be the identity on P. Then with f^σ and f'^σ defined as in equations (4.3) and (4.5) respectively we have $f^\sigma(q) = \bigvee\{\bigwedge\{e_2(f(p)) : x \le e_1(p)\} : q \ge x \in K_{\mathcal{F}}(Q_1)\} = \bigvee\{x, y, z, \bigwedge\{a,b,c\}\} = s$, and $f'^\sigma(q) = \bigvee\{\bigwedge\{e_2(f(p)) : x \le e_1(p)\} : q \ge x \in K_{\mathcal{F}'}(Q_1)\} = \bigvee\{x,y,z,\bigwedge\{a,b,c\},\bigwedge\{b,c\}\} = r$, and thus $f^\sigma < f'^\sigma$.

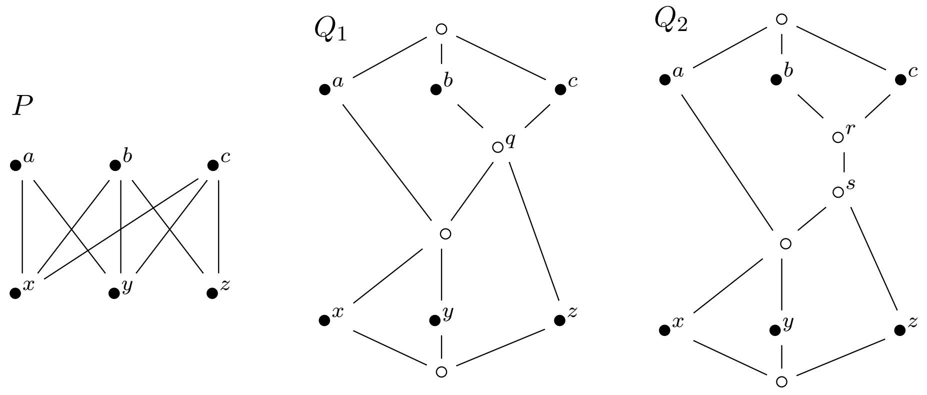


Figure 4.2: $f^\sigma < f'^\sigma$

In the presence of $(\mathcal{F},\mathcal{I})$-compactness we can say something about the relationship between f^σ and f^π, as we demonstrate in the following minor adaptation of [39, Lemma 3.4].

Proposition 4.2.7. *If $e_1\colon P_1 \to Q_1$ is an $(\mathcal{F},\mathcal{I})$-dense and $(\mathcal{F},\mathcal{I})$-compact completion of P_1, and if $e_2\colon P_2 \to Q_2$ is any completion of P_2, then for any order preserving map $f\colon P_1 \to P_2$ we have $f^\sigma(x) = f^\pi(x)$ for all $x \in K_{\mathcal{F}}(Q_1) \cup O_{\mathcal{I}}(Q_1)$.*

Proof. Let $x \in K_{\mathcal{F}}(Q_1)$. Then

$$\begin{aligned} f^\sigma(x) &= \bigwedge\{e_2(f(p)) : e_1(p) \geq x\} \\ &= \bigwedge\{f^\pi(e_1(p)) : e_1(p) \geq x\} \end{aligned}$$

and, by $(\mathcal{F},\mathcal{I})$-compactness, if $x \leq y \in O_{\mathcal{I}}(Q_1)$ then there is $p \in P$ with $x \leq e_1(p) \leq y$, so

$$\begin{aligned} f^\pi(x) &= \bigwedge\{f^\pi(y) : x \leq y \in O_{\mathcal{I}}(Q_1)\} \\ &= \bigwedge\{f^\pi(e_1(p)) : e_1(p) \geq x\} \\ &= f^\sigma(x). \end{aligned}$$

By duality we also have $f^\sigma(y) = f^\pi(y)$ for all $y \in O_{\mathcal{I}}(Q_1)$ so we are done. $\square$

Corollary 4.2.8. *If $e_1\colon P_1 \to Q_1$ is an $(\mathcal{F},\mathcal{I})$-dense and $(\mathcal{F},\mathcal{I})$-compact completion of P_1, and if $e_2\colon P_2 \to Q_2$ is any completion of P_2, then for any order preserving map $f\colon P_1 \to P_2$ we have $f^\sigma \leq f^\pi$.*

Proof. This follows easily from Proposition 4.2.7 and the fact that $f^\sigma(q) = \bigvee\{f^\sigma(x) : q \geq x \in K_{\mathcal{F}}(Q_1)\}$ and $f^\pi(q) = \bigwedge\{f^\pi(y) : q \leq y \in O_{\mathcal{I}}(Q_1)\}$. $\square$

Absence of $(\mathcal{F},\mathcal{I})$-compactness does not guarantee that $f^\sigma \not\leq f^\pi$, as [51, Example 5.2] produces a Δ_1-completion $e'\colon P' \to Q'$ which cannot be $(\mathcal{F},\mathcal{I})$-compact for any $\mathcal{F}$ and $\mathcal{I}$ that generate Q' as a Δ_1-completion of P', and it's easy to see that for any $(\mathcal{F},\mathcal{I})$-dense completion $e\colon P \to Q$, if f is the identity on P and $f^\sigma\colon Q \to Q$ and $f^\pi\colon Q \to Q$ are the lifts as defined in (4.3) and (4.4) then we always have $f^\sigma = f^\pi$.

Lemma 4.2.9 below gives a necessary and sufficient condition on an $(\mathcal{F},\mathcal{I})$-dense completion for $f^\sigma \not\leq f^\pi$, and Example 4.2.10 shows that it is satisfiable.

Lemma 4.2.9. *Let $e_1\colon P_1 \to Q_1$ be an $(\mathcal{F},\mathcal{I})$-dense completion, let $e_2\colon P_2 \to Q_2$ be a completion, and let $f\colon P_1 \to P_2$ be an order preserving map. Then $f^\sigma \not\leq f^\pi$ if and only if there are $q \in Q_1$, $x \in K_{\mathcal{F}}(Q_1)$, and $y \in O_{\mathcal{I}}(Q_1)$ with $x \leq q \leq y$ and $f^\sigma(x) \not\leq f^\pi(y)$.*

Proof. Since $f^\sigma(q) = \bigvee\{f^\sigma(x) : q \geq x \in K_{\mathcal{F}}(Q_1)\}$ and $f^\pi(q) = \bigwedge\{f^\pi(y) : q \leq y \in O_{\mathcal{I}}(Q_1)\}$ for all $q \in Q$, if there are $q \in Q$, $x \in K_{\mathcal{F}}(Q_1)$, and $I \in O_{\mathcal{I}}(Q_1)$ with $x \leq q \leq y$ and $f^\sigma(x) \not\leq f^\pi(y)$ then we must have $f^\sigma(q) \not\leq f^\pi(q)$. Conversely, suppose for all $q \in Q_1$, for all $x \in K_{\mathcal{F}}(Q_1)$, and for all $y \in O_{\mathcal{I}}(Q_1)$ we have $x \leq q \leq y \implies f^\sigma(x) \leq f^\pi(y)$. Take any $q \in Q_1$, define $X = \{f^\sigma(x) : q \geq x \in K_{\mathcal{F}}(Q_1)\}$, and define $Y = \{f^\pi(y) : q \leq y \in O_{\mathcal{I}}(Q_1)\}$. Let $y' \in Y$. Then $y' \wedge \bigvee X$ is an upper bound for X, and thus we must have $y' \wedge \bigvee X \geq \bigvee X$, which in turn implies $\bigvee X$ is a lower bound for Y, and thus $\bigvee X \leq \bigwedge Y$, but this is exactly the statement that $f^\sigma(q) \leq f^\pi(q)$, and so $f^\sigma \leq f^\pi$. $\square$

Example 4.2.10. *To show we may have* $f^\sigma \not\leq f^\pi$. Let P be the antichain on $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$, and let $e\colon P \to Q$ be as in Figure 4.3 (the filled dots denote the embeded image of P, to simplify the notation we identify elements of P with their images under e), let $e'\colon P \to L$ be as in Figure 4.4, and let f be the identity on P. Then $x \in K_{\mathcal{D}(e[P])}(Q)$, and $y \in O_{\mathcal{U}(e[P])}(Q)$, and $x \leq q \leq y$, but $f^\sigma(x) \not\leq f^\pi(y)$, so by Lemma 4.2.9 we must have $f^\sigma \not\leq f^\pi$ (in fact we have $f^\pi < f^\sigma$ as $f^\sigma(p) = f^\pi(p)$ for all $p \in Q \setminus \{q\}$ and $f^\pi(q) = 0 < 1 = f^\sigma(q)$).

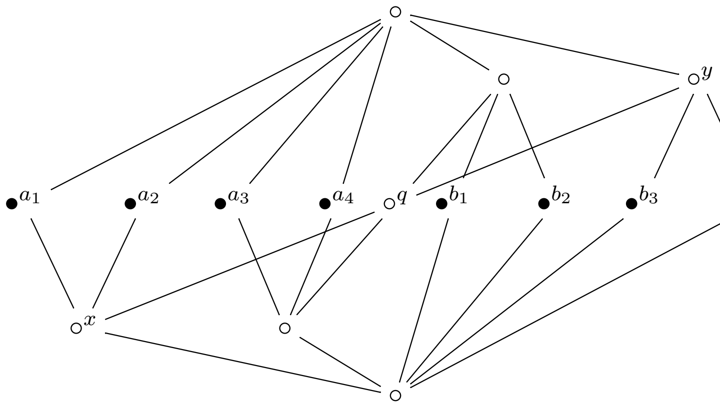


Figure 4.3: $f^\sigma \not\leq f^\pi$, part 1

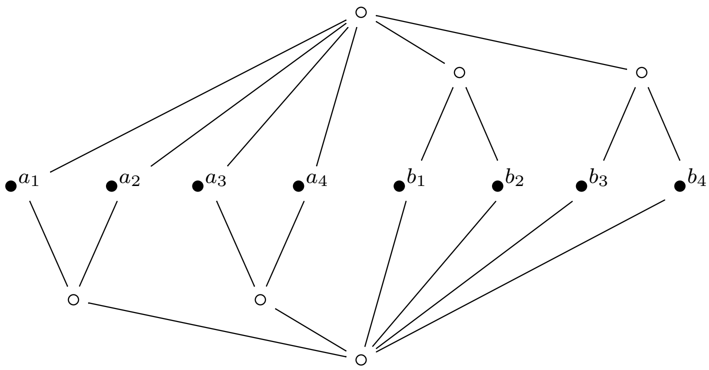


Figure 4.4: $f^\sigma \not\leq f^\pi$, part 2

Remark 4.2.11. The approach taken in [54] and [49] for bounded distributive lattices and bounded lattices respectively is more general than the one taken here, and applies to arbitrary maps rather than only to those that are order preserving. We have not gone into the details of this more general method as we shall only be concerned with order preserving maps, and in the special case where maps *are* order preserving the general method is equivalent to the one we use.

Remark 4.2.12. The technical machinery used to lift maps can be expressed in terms of topology (see e.g. [54, 134, 136]), with many results being phrased in terms of continuity with respect to various topologies. While this approach has conceptual benefits we have chosen not to go into details, as results are inter-translatable between topological and non-topological methods, and taking the topological approach would add to the technical burden.

4.2.2 Lifting operations

It is known that taking canonical extensions commutes with finite products and order duals [39, Theorem 2.8], and similar for the MacNeille completion [134, Proposition 2.5] (the latter result is for lattices but the proof for posets is similar), so the method given in the previous section for lifting order preserving maps extends easily to lifting η-monotone operations, as these can be considered to be isotone operations as indicated by their monotonicity type (see Definition 2.3.12). This approach is taken in [52, 53, 49, 54, 39, 134].

There is an important difference in the meet-completion case however, as taking meet-completions does not necessarily commute with either finite products or duals (see Examples 4.5.3 and 4.2.14 respectively). Lemma 4.2.13 below allows us to overcome the problem with products and extend our methodology to *isotone* operations on meet- and join-completions (see Figure 4.5, this approach is taken in [74, Section 3.2]). It is not clear whether the problem with duals can be overcome and the method extended to *η-monotone* operations on meet- and join-completions. Fortunately, our application in Chapter 6 only involves isotone operations.

Lemma 4.2.13. *If P is a poset, $e\colon P \to Q$ is a meet-completion, and $n \in \omega$, then $e^n\colon P^n \to Q^n$ is a meet-completion of P^n, where we define $e^n((p_1, ..., p_n)) = (e(p_1), ..., e(p_n))$.*

Proof. Since a finite product of complete lattices is again a complete lattice it remains only to check that $e^n[P^n]$ is meet-dense in Q^n. Given $(q_1, ..., q_n) \in Q^n$ we claim that $(q_1, ..., q_n) = \bigwedge\{e^n((p_1, ..., p_n)) : e(p_i) \geq q_i \text{ for all } i \in \{1, ..., n\}\}$. Now, $(q_1, ..., q_n)$ is clearly a lower bound, so suppose $(q'_1, ..., q'_n)$ is another such lower bound. Then for $i \in \{1, ..., n\}$ $q'_i \leq e(p_i)$ for all $p_i \in P$ with $q_i \leq e(p_i)$, so by meet-density of $e[P]$ in Q we have $q'_i \leq q_i$, and so $(q'_1, ..., q'_n) \leq (q_1, ..., q_n)$ as required. $\square$

$$\begin{tikzcd}
P^n \arrow[r, "e^n"] \arrow[d, "f"'] & Q^n \arrow[d, "f'"] \\
P \arrow[r, "e"'] & Q
\end{tikzcd}$$

Figure 4.5: Lifting isotone poset operations

Example 4.2.14. . Let $e\colon P \to Q$ be a meet-completion as in Figure 4.6. Then Q^δ cannot be a meet-completion of P^δ as meet-completions necessarily preserve all joins (see Proposition 4.5.18).

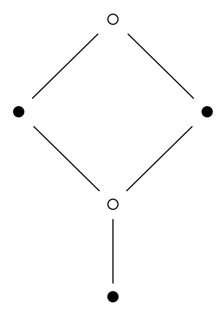


Figure 4.6: The dual of a meet-completion may not be a meet-completion

4.3 The MacNeille completion

As mentioned in the introduction to this chapter, the MacNeille completion originated as a generalization of Dedekind's 'completion by cuts' of the rationals (in 1858, published 1872 [30]) to arbitrary posets. We gave a definition of the MacNeille completion of a poset P in Definition 4.1.7, and we claimed that it always existed, that it was unique up to isomorphism lifting the identity on P, and that it was regular. Regularity follows from Proposition 4.5.18, and uniqueness follows easily from regularity and Proposition 4.5.12. We sketch an existence proof, details can be found in e.g. [15, Chapter V.9]:

Given $S \subseteq P$ define $L(S) = \{t \in P : t \leq S\}$, and define $U(S) = \{t \in P : t \geq S\}$, define an up-set S of P to be *normal* if $S = U(L(S))$ (so S is normal if it is equal to the set of upper bounds of the set of lower bounds of itself), and define $\mathcal{F}_N$ to be the set of normal filters of P. We define $\mathbf{DM}(P) = (\mathcal{F}_N, \supseteq)$ to be the set of normal filters of P ordered by reverse inclusion, and it's not difficult to show that $\mathbf{DM}(P)$ is a complete lattice with $\bigvee_I F_i = \bigcap_I F_i$ and $\bigwedge_I F_i = U(L(\bigcup_I F_i))$. Moreover, P embeds into $\mathbf{DM}(P)$ via the embedding $\iota\colon p \mapsto p^\uparrow$. Finally, given $F \in \mathcal{F}_N$ it's easy to see that $F = \bigcup\{p^\uparrow : p \in F\} = \bigcup\{p^\uparrow : p^\uparrow \subseteq F\} = \bigwedge\{\iota(p) : \iota(p) \geq F\}$, and if $q \in p^\uparrow$ for all p with $p^\uparrow \supseteq F$ then q is an upper bound for the set of lower bounds of F, and thus $q \in F$, so since clearly $F \subseteq \bigcap\{p^\uparrow : p^\uparrow \supseteq F\}$ we have $F = \bigcap\{p^\uparrow : p^\uparrow \supseteq F\} = \bigvee\{\iota(p) : \iota(p) \leq F\}$ and thus $\iota[P]$ is doubly dense in $\mathbf{DM}(P)$.

It is known that the MacNeille completion is minimal, in the sense that if $e\colon P \to L$ is a completion of P then there is an embedding $e'\colon \mathbf{DM}(P) \to L$ such that $e = e' \circ \iota$, and this minimality property can also be used to characterize it [9].

The MacNeille completion is notoriously badly behaved with respect to the preservation of identities, even those only involving the basic lattice operations. For example, the MacNeille completion of a distributive lattice need not be distributive [47, 73], though it was shown in [98] that the class of Boolean algebras *is* closed under taking MacNeille completions, as is the class of Heyting algebras [7] (though, as a further example of the bad identity preservation properites of the MacNeille completion, the *only* varieties of Heyting algebras that are closed under this completion are the trivial variety, the class of all Boolean algebras, and the class of all Heyting algebras [11]).

This kind of closure can be important, as the MacNeille completion is regular, and the closure of certain classes under a regular completion can be used to show completeness of various predicate calculi with respect to their algebraic semantics [112, 111, 113].

The first systematic study of MacNeille completions for expanded ordered structures is Monk [103] for BAOs, where it is shown, among other things, that all identities not involving negation are preserved (modulo a particular lifting process; see Remark 4.3.1 below). More general Sahlqvist style preservation results for this class are obtained in [63]. The concept of MacNeille completion is generalized to monotone lattice expansions in [134], where the relatively good behaviour of completely additive operators is put into context.

Remark 4.3.1. We note at this point that it is not altogether clear what is meant by 'the MacNeille completion' of a lattice expansion, as we saw in Section 4.2 that an order preserving map can be extended in at least two ways (unless it is *smooth*, by which we mean that the lifts in equations (4.1) and (4.2) are equal), and thus there are often a number of ways we can complete a lattice expansion based on choosing a particular lift for each operation. For obvious reasons the literature has concentrated on the lifts $\hat{f}$ and $\check{f}$, with completion often defined by uniformly taking the *upper* ($\hat{f}$) and *lower* ($\check{f}$) lift for each additional operation ([103], for example, uses the lower lift).

The uniform approach may be overly simplistic in some cases; for example, in [12] a variety of isotone Boolean algebra expansions with only one additional unary operation is produced that is closed under neither the lower nor upper MacNeille completion, but where, nevertheless, each algebra can be regularly embedded in a doubly dense completion that is also a member of that variety and whose additional operation lifts the operation of the original algebra.

4.4 The canonical extension

In [90, 91], Jónsson and Tarski introduced a process by which a BAO (a concept they also introduce in these papers) could be embedded into a complete, atomic BAO with complete operarators. This extended the known process by which a Boolean algebra could be embedded into a complete and atomic Boolean algebra by taking the powerset of the elements used in its Stone representation.

In the Jónsson-Tarski method, the additional operations are lifted using an intermediate translation of $\mathcal{B}$ into a relational structure over the set of ultrafilters of $\mathcal{B}$. While certainly useful, for example canonical extensions of BAOs have a strong connection to classical modal logic (see Section 4.4.1), the completion via Stone representation approach has some limitations. Most obviously it applies only to BAOs, and for applications in certain non-classical logics we may prefer that our underlying poset were not complemented (e.g. [38, 22]) and/or distributive [3, 96], and maybe even that it were not a lattice at all [39]. Further to this the restriction to operators is often inadequate.

A more subtle complaint is that the construction via Stone representation relies on the Boolean prime ideal theorem, which is know to be a weaker form of the axiom of choice (see [44] for a general discussion of this and related issues), and thus is non-constructive.

Fortunately, all of these issues can be overcome; the need for negation was removed in [52], the restriction to operators overcome in [53, 49, 54], reliance on distributivity abandoned in [49], and the concept extended to arbitrary monotone poset expansions in [39]. Moreover, the modern construction (which we explain below) is closely related the constructive methods of [60, 61], and does not require any form of the axiom of choice.

Better still, many applications of ordered algebras in logic had relied on the various duality theories, which become increasingly complex as one departs further from the Boolean, while the general theory of canonical extensions reamains relatively neat, even in the poset case. We now sketch proofs of the construction of the canonical extension of a poset P, which we shall label P^σ, and its uniqueness up to isomorphism lifting the identity on P, details can be found as [39, Theorems 2.5 and 2.6].

Definition 4.4.1 (Polarity)**.** A polarity is a triple (X, Y, R) such that X and Y are non-empty sets and R is a binary relation from X to Y.

A polarity (X, Y, R) gives rise to a Galois connection between $\wp(X)$ and $\wp(Y)$ defined by

$$\phi\colon \wp(X) \leftrightarrow \wp(Y)\colon \psi$$

$$S \mapsto \{y \in Y : sRy \text{ for all } s \in S\}$$

$$\{x \in X : yRt \text{ for all } t \in T\} \leftarrow\!\shortmid T$$

Let $G(X, Y, R)_X$ be the set of Galois closed subsets of X, that is, the set of sets $S \subseteq X$

such that $\psi(\phi(S)) = S$. Then it's not difficult to show that $G(X, Y, R)_X$ is a complete lattice when ordered by inclusion.

Given a poset P let $\mathcal{F}$ and $\mathcal{I}$ be the sets of filters and ideals of P respectively, and let R_l be the relation defined by $FR_lI \iff F \cap I \neq \emptyset$ for $F \in \mathcal{F}$ and $I \in \mathcal{I}$. Define a pre-order on the disjoint union $\mathcal{F} + \mathcal{I}$ by

1. $F_1 \leq F_2 \iff F_1 \supseteq F_2$, for $F_1, F_2 \in \mathcal{F}$,

2. $I_1 \leq I_2 \iff I_1 \subseteq I_2$, for $I_1, I_2 \in \mathcal{I}$,

3. $I \leq F \iff p \leq q$ for all $p \in I, q \in F$, for $I \in \mathcal{I}$, $F \in \mathcal{F}$,

4. $F \leq I \iff F \cap I \neq \emptyset$, for all $I \in \mathcal{I}$, $F \in \mathcal{F}$.

Then a little working reveals that $G(\mathcal{F}, \mathcal{I}, R_l)_{\mathcal{F}}$ is exactly the MacNeille completion of $\mathcal{F} \oplus \mathcal{I}$, where $\mathcal{F} \oplus \mathcal{I}$ is defined to be the partial order naturally induced from $\mathcal{F} + \mathcal{I}$ by identifying F and I whenever $F \leq I$ and $I \leq F$ (the poset $\mathcal{F} \oplus \mathcal{I}$ is often refered to as the *intermediate structure*). We note that $p^\uparrow$ and $p^\downarrow$ are identified in $\mathcal{F} \oplus \mathcal{I}$, and thus that the embedding $e\colon P \to \mathcal{F} \oplus \mathcal{I}$, $p \mapsto p^\uparrow$ is well defined. For this embedding it's easy to see that the set of $\mathcal{F}$-closed elements of $\mathcal{F} \oplus \mathcal{I}$ is precisely $\mathcal{F}$, and that the set of $\mathcal{I}$-open elements is $\mathcal{I}$, and that we have $(\mathcal{F}, \mathcal{I})$-density and $(\mathcal{F}, \mathcal{I})$-compactness of $e\colon P \to \mathcal{F} \oplus \mathcal{I}$ as an extension of P (note though that $\mathcal{F} \oplus \mathcal{I}$ is not necessarily complete). If ι is the natural embedding of $\mathcal{F} \oplus \mathcal{I}$ into its MacNeille completion $\mathbf{DM}(\mathcal{F} \oplus \mathcal{I})$, then that $\iota \circ e\colon P \to \mathbf{DM}(\mathcal{F} \oplus \mathcal{I}) = G(\mathcal{F}, \mathcal{I}, R_l)_{\mathcal{F}}$ is an $(\mathcal{F}, \mathcal{I})$-dense and $(\mathcal{F}, \mathcal{I})$-compact completion of P follows from the definition and regularity of the MacNeille completion.

Conversely, if $e\colon P \to Q$ is an $(\mathcal{F}, \mathcal{I})$-dense and $(\mathcal{F}, \mathcal{I})$-compact completion (where $\mathcal{F}$ and $\mathcal{I}$ are the sets of filters and ideals of P respectively), then there are natural surjective maps $F \mapsto \bigwedge F$ and $I \mapsto \bigvee I$ from $\mathcal{F}$ to $K_{\mathcal{F}}(Q)$ and from $\mathcal{I}$ to $O_{\mathcal{I}}(Q)$ respectively. Suppose $x_F = \bigwedge F$, $x_G = \bigwedge G$, and $x_F \leq x_G$. Then given $p \in G$ we have $\bigwedge F \leq \bigvee p^\downarrow$, and thus by $(\mathcal{F}, \mathcal{I})$-compactness $p \in F$. A dual result also holds and it follows that $K_{\mathcal{F}}(Q)$ is order isomporphic to $(\mathcal{F}, \supseteq)$, and $O_{\mathcal{I}}(Q)$ is order isomorphic to $(\mathcal{I}, \subseteq)$ ($K_{\mathcal{F}}(Q)$ and $O_{\mathcal{I}}(Q)$ are ordered naturally by the restriction of the ordering on Q).

Moreover, if $x = \bigwedge F$ and $y = \bigvee I$ it follows from $(\mathcal{F}, \mathcal{I})$-compactness that $x \leq y \iff F \cap I \neq \emptyset$, and clearly $y \leq x \iff p \leq q$ for all $p \in I$ and $q \in F$, so $K_{\mathcal{F}}(Q) \cup O_{\mathcal{I}}(Q)$ is order isomorphic to $\mathcal{F} \oplus \mathcal{I}$. It's easy to see that $K_{\mathcal{F}}(Q) \cup O_{\mathcal{I}}(Q)$ is doubly dense in Q, and so Q is (up to isomorphism) the MacNeille completion of $K_{\mathcal{F}}(Q) \cup O_{\mathcal{I}}(Q)$, and from uniqueness up to isomorphism of the MacNeille completion we conclude that $Q \cong G(\mathcal{F}, \mathcal{I}, R_l)$, and thus that the canonical extrension of P is unique up to isomorphism lifting the identity on P.

Remark 4.4.2. We saw in Section 4.2 that given a monotone poset operation $f\colon P^n \to P$ there are a number of ways to lift f to a map from the canonical extension of P^n to the canonical extension of P, with the lifts f^σ and f^π from (4.3) and (4.4) being of particular interest (noting that by Corollary 4.2.8 we have $f^\sigma \leq f^\pi$). Maps $f\colon P_1 \to P_2$ for which $f^\sigma = f^\pi\colon P_1^\sigma \to P_2^\sigma$ are called *smooth* (we note the potential for confusion in using the same word to describe the analogous concept for MacNeille completions), and display particularly good behaviour (see Section 4.4.2).

As in Remark 4.3.1, given a monotone poset expansion there is usually a choice in how the additional operations are to be lifted, and the particular method used will be determined by context. We note that in the lattice setting if $f\colon L \to M$ preserves either binary joins or meets then it is smooth (see [49, Lemma 4.4 and Corollary 4.7]), so, in particular, unary lattice operators are smooth.

4.4.1 The modal connection

A significant motivation for the study of canonical extensions comes from the study of relational semantics for modal logics, and in particular from the desire for frame-completeness results. Curiously, although many important steps in this direction were present in Jónsson and Tarski [90], and Tarski himself had published some work in modal logic, the connection was not apparantly realized till much later [89]. We do not intend to go into detail about modal logic here, rather we aim to provide a brief overview so as to give the canonical extension construction some context in the broader tapestry of mathematics. The interested reader is referred to [17] for a textbook introduction to this material.

Suitable classes of BAOs readily provide sound and complete algebraic semantics for classical modal logics (by which we mean classical propositional logic augmented with one or more n-ary modal operators), via the fact that if Σ is a consistent set of modal formulas (for some fixed modal language), and if K_Σ is the normal modal logic generated by Σ, then given any formula ϕ in the (modal) language of Σ we have $\vdash_{K_\Sigma} \phi \iff \mathbf{V}_\Sigma \models \phi' = 1$, where ϕ' is the natural algebraic interpretation of ϕ, and $\mathbf{V}_\Sigma$ is the variety of BAOs of appropriate type axiomatized by $\{\sigma' = 1 : \sigma \in \Sigma\}$ (see e.g. [17, Theorem 5.27]). Algebraic completeness is interesting in itself, but algebraic thinking can get us even more.

Given a normal modal logic Λ we can construct the *canonical frame* $\mathfrak{F}^\Lambda$ for Λ using Lindenbaum–Tarski algebras and Stone duality (see e.g. [17, Section 4.2 and Theorem 5.42]).

A formula ϕ is *canonical* if $\phi \in \Lambda \implies \mathfrak{F}^\Lambda \Vdash \phi$ for all normal modal logics Λ. Canonical frames have many uses, but the key result here is that if Λ is axiomatized by canonical formulas then it is complete with respect to the class of frames it generates, which, since it will always be sound with respect to this class, gives us a sound and complete relational semantics for Λ. The algebraic link here is that given any set Σ of modal formulas for some fixed modal language, if V_Σ is canonical then Σ is canonical (see e.g. [17, Proposition 5.45]) (we say a variety of monotone poset expansions is *canonical* if it is closed under taking canonical extensions).

This connection was, for example, exploited by [89] to provide an alternative proof of the cannonicity of *Sahlqvist formulas* (see Section 4.4.2 for a brief discussion of Sahlqvist theory).

4.4.2 Identity preservation and functoriality

We have just seen that canonicity of a variety of BAOs can be a useful property, so it is natural to ask when identities are preserved by this construction. Indeed, results to this effect were given in [90], even before the modal connection had become clear. Since it is known that there are normal modal logics that are not complete with respect to any class of frames (see e.g. [45]), it follows that there are BAO identities that are not canonical, i.e. that are not preserved by taking canonical extensions (see [56] for an example in the case of MV algebras).

The preservation of BAO identities is closely related to Sahlqvist theory, named for Sahlqvist [116], where sufficient syntactic conditions are given for modal formulas to correspond effectively to first order conditions on the associated frame class. Valuable in itself, this correspondence can also be used to prove canonicity of Sahlqvist formulas, either directly as in [17, Theorem 5.91] or via general considerations from [65], and thus axiomatization of a modal logic via Sahlqvist formulas is a rather desirable property.

Exploring the algebraic angle, and exploiting developments in the canonical extension construction, algebraic Sahlqvist type theories have been given for BAOs [114], and recently in the lattice [129] and poset [130] settings, seeking to generalize many existing results from beyond the BAO setting (e.g. [61, 121, 39, 55]). Algebraic canonicity in settings beyond the Boolean is particularly useful as it can be used to provide dualities with which to prove completeness results in situations where suitable direct duality theories are unwieldy or even nonexistent (see e.g. [124, 125, 39, 57, 58]).

We note that the canonical extension is generally better behaved with respect to identity preservation than the MacNeille completion. In particular, varieties of monotone lattice expansions that are closed under canonical extensions are also closed under MacNeille completions (for suitable choices of lift for each operation) [50].

Another issue that has received attention is that of functoriality, i.e. under which circumstances the canonical extension construction gives rise to a functor. More precicely, given a category $\mathscr{C}$ of poset expansions of a fixed type, and given a rule for assigning lifts to maps and operations, we would like to know whether the map taking each object of $\mathscr{C}$ to its canonical extension, and each arrow to its lift as per the lifting rule, can be thought of as a functor between $\mathscr{C}$ and some category of complete lattice expansions.

Recalling from Remark 4.4.2 that lattice homomorphisms are smooth this question is fairly straightforward in the pure lattice (i.e. no additional operations) case, and the canonical extension with lifts given by ${}^{\sigma}$ (or equivalently by ${}^{\pi}$) gives rise to a functor from the category of bounded lattices and lattice homomorphisms to the category of complete lattices with complete homomorphisms [49, 59], and similar results also hold for monotone lattice expansions [49]. When we allow additional operations that are not monotonic we do not necessarily get functoriality from canonical extension constructions, however, at least in the distributive case, functoriality does hold in a broad range of cases [54].

4.5 Meet-completions

Here we are concerned with meet-completions, that is, completions where each element is expressible as a meet of some subset of the (embedded image of) original poset. It is well known (see Proposition 4.5.18) that a meet-completion must preserve all existing joins from the original poset, and the fundamental observation here is that a meet-completion is characterized by the subsets of P to which it assigns the same infimum (meet) (Proposition 4.5.12). In particular, under certain circumstances (examined in Section 4.5.2) we may choose a meet-completion of a poset that preserves specific existing meets and destroys all others.

Given a poset P, we define a condition we call *regularity* (definition 4.5.21) on a set $\mathscr{S} \subseteq P^*$ (where P^* is the set of all up-sets of P) and show that it is equivalent to the property that every $S \in \mathscr{S}$ has an infimum in P and there is at least one meet-completion of P preserving all and only the meets of the sets in $\mathscr{S}$. In fact, we shall see that the set $\mathbf{M}_{\mathscr{S}}$ of these meet-completions forms a topped lattice (ordered via embedding lifting the identity on P) that is, in general, not bottomed (Example 4.5.31). A necessary and sufficient condition on $\mathscr{S}$ for $\mathbf{M}_{\mathscr{S}}$ to have a bottom element, and thus be a complete lattice, is given (Proposition 4.5.30).

It has been shown [104] that every finite interval of the set of all join-completions of a poset P (considered as a complete lattice ordered by embedding lifting the identity on P) is upper semimodular, an upper bounded homomorphic image of a free lattice, and thus meet-semidistributive. In Section 4.5.3 we adapt and apply these results to show that $\mathbf{M}_{\mathscr{S}}$ is always weakly lower semimodular, and that when P is finite and $\mathscr{S}$ is regular, $\mathbf{M}_{\mathscr{S}}$ is also lower semimodular, a lower bounded homomorphic image of a free lattice, and thus join-semidistributive.

In Section 4.5.4 we give some results regarding the preservation of inequalities in meet-completions of poset expansions.

Note that there is a well known duality between meet-completions of P and certain closure operators on P^*, which we explain in Section 4.5.1. For ease of exposition we phrase our results largely in terms of closure operators, appealing to this duality to justify our claims.

Before continuing we pause to note the following relationship between meet-completions and join-completions.

Proposition 4.5.1. *If $e\colon P \to Q$ is a meet-completion then $e^\delta\colon P^\delta \to Q^\delta$ is a join-completion (where e^δ is the natural map induced by e).*

Proof. Trivial. □

4.5.1 Meet-completions and closure operators

Definition 4.5.2 (P^*)**.** If P is a poset define P^* to be the complete lattice of up-sets (including $\emptyset$) of P ordered by reverse inclusion (so $P^{*\delta}$ is the lattice of up-sets ordered by inclusion).

It's easy to see the map $\iota\colon P \to P^*$ defined by $\iota(p) = p^\uparrow$ defines a meet-completion of P (note though that ι will not map the top element of P (if it exists) to the top element of P^*, as the top element of P^* will be $\emptyset$). It turns out that this particular completion plays an important role in the theory of meet-completions, but first we use it to show that taking a meet-completion need not commute with direct products:

Example 4.5.3. Let P be the singleton poset. Then $P \times P \cong P$ and P^* is the two element chain, so $P^* \times P^*$ is the four element diamond while $(P \times P)^*$ is the two element chain, and so $(P \times P)^* \not\cong P^* \times P^*$.

Definition 4.5.4 (Closure operator)**.** Given a poset P a closure operator on P is a map $\Gamma\colon P \to P$ such that

1. $p \leq \Gamma(p)$ for all $p \in P$,
2. $p \leq q \implies \Gamma(p) \leq \Gamma(q)$ for all $p, q \in P$, and
3. $\Gamma(\Gamma(p)) = \Gamma(p)$ for all $p \in P$.

Following [43] we say a closure operator Γ on P^* or $P^{*\delta}$ is *standard* when $\Gamma(p^\uparrow) = p^\uparrow$ for all $p \in P$.

It is well known that a meet-completion $e\colon P \to Q$ defines a standard closure operator $\Gamma_e\colon P^{*\delta} \to P^{*\delta}$ by $\Gamma_e(S) = \{p \in P : e(p) \geq \bigwedge e[S]\}$ (we take the dual of P^* as otherwise condition 1 of Definition 4.5.4 fails). In this case Q is isomorphic to the lattice $\Gamma_e[P^*]$ of Γ_e-closed subsets of P^* (note we are purposefully taking P^* rather than $P^{*\delta}$ here as we want to order by reverse inclusion, this is technically an abuse of notation as Γ_e is originally defined on $P^{*\delta}$, but as these structures have the same carrier hopefully our meaning is clear). The isomorphism is given by the map $h_e\colon Q \to \Gamma_e[P^*]$ defined by $h_e(q) = \{p \in P : e(p) \geq q\}$. Conversely, whenever Γ is a standard closure operation on $P^{*\delta}$ it induces a meet-completion $e_\Gamma\colon P \to \Gamma[P^*]$ defined by $e_\Gamma(p) = p^\uparrow$. For $S \in P^*$ we have $\Gamma_{e_\Gamma}(S) = \{p \in P : p^\uparrow \geq \bigwedge\{p^\uparrow : p \in S\}\} = \{p : p^\uparrow \subseteq \Gamma(S)\} = \Gamma(S)$, so $\Gamma_{e_\Gamma} = \Gamma$, and, for all $p \in P$, $e_{\Gamma_e}(p) = p^\uparrow = h_e \circ e(p)$ so the following commutes:

$$\begin{tikzcd} P \arrow[r, "e"] \arrow[d, "e_{\Gamma_e}"'] & Q \arrow[dl, "h_e"] \\ \Gamma_e[P^*] & \end{tikzcd}$$

We state the results of the preceeding discussion as a theorem.

Theorem 4.5.5. *If $e\colon P \to Q$ is a meet-completion then there is a unique isomorphism between Q and $\Gamma_e[P^*]$ such that the following commutes:*

$$\begin{tikzcd} P \arrow[r, "e"] \arrow[d, "e_{\Gamma_e}"'] & Q \arrow[dl, leftrightarrow, "\cong"] \\ \Gamma_e[P^*] & \end{tikzcd}$$

Moreover, if $e_1\colon P \to Q_1$ and $e_2\colon P \to Q_2$ are meet-completions such that there is an isomorphism $h\colon Q_1 \to Q_2$ with $h \circ e_1 = e_2$ then $\Gamma_{e_1} = \Gamma_{e_2}$.

Proof. The existence of the required isomorphism has been established, and uniqueness follows from Lemma 4.5.6 below. If $h\colon Q_1 \to Q_2$ with $h \circ e_1 = e_2$ then $\Gamma_{e_2}(S) = \{p \in P : e_2(p) \geq \bigwedge e_2[S]\} = \{p \in P : h \circ e_1(p) \geq \bigwedge h \circ e_1[S]\} = \{p \in P : h \circ e_1(p) \geq h(\bigwedge e_1[S])\} = \{p \in P : e_1(p) \geq \bigwedge e_1[S]\} = \Gamma_{e_1}(S)$. □

Lemma 4.5.6. *If $e_1\colon P \to Q_1$ and $e_2\colon P \to Q_2$ are meet-completions of P and $g\colon Q_1 \to Q_2$ is an isomorphism such that $g \circ e_1 = e_2$, then g is unique with this property.*

Proof. Suppose h is another such isomorphism. Then for all $p \in P$, and for all $q \in Q$, we have $e_1(p) \geq q \iff g \circ e_1(p) \geq g(q) \iff h \circ e_1(p) \geq h(q)$, and $g \circ e_1(p) \geq g(q) \iff e_2(p) \geq g(q)$, and similarly $h \circ e_1(p) \geq h(q) \iff e_2(p) \geq h(q)$, so $\{p \in P : e_2(p) \geq g(q)\} = \{p \in P : e_2(p) \geq h(q)\}$ and thus by meet-density we are done. □

The meet-completions of a poset P can be characterized up to isomorphism by a generalized concept of meet-primality, which we explain below. In particular the algebra P^* is the unique (up to isomorphism lifting the identity on P) meet-completion of P whose set of completely meet-prime elements is the embedded image of P (Corollary 4.5.11).

Definition 4.5.7 (Relatively meet-prime)**.** Let L be a lattice, let $\emptyset \subset \mathscr{X} \subseteq \wp(L)$, and let $a \in L$. Then a is meet-prime relative to $\mathscr{X}$ provided it is not the top element and whenever $\emptyset \subset X \in \mathscr{X}$ and $a \geq \bigwedge X$ there is $x \in X$ with $a \geq x$.

Definition 4.5.8 ($\mathscr{X}_\Gamma$)**.** Given a closure operator $\Gamma\colon P^{*\delta} \to P^{*\delta}$ define $\mathscr{X}_\Gamma = \{\emptyset \subset X \in \wp(\Gamma[P^*]) : \Gamma(\bigcup X) = \bigcup X\}$.

Lemma 4.5.9. *Let $e\colon P \to Q$ be a meet-completion. Then the set of elements of Q that are meet-prime relative to $\mathscr{X} = \{\emptyset \subset X \in \wp(Q) : h_e[X] \in \mathscr{X}_{\Gamma_e}\}$ is precisely $e[P] \setminus \{\top\}$ where $\top$ is the top element of Q.*

Proof. Let $q \in Q \setminus \{\top\}$. Then q is meet-prime relative to $\mathscr{X}$ if and only if, for all $X \in \mathscr{X}$, we have $q \geq \bigwedge X \implies q \geq x$ for some $x \in X$, which is equivalent to saying that $h_e(q) \subseteq \bigcup h_e[X] \implies h_e(q) \subseteq h_e(x)$ for some $x \in X$. If $q \in e[P]$ then this clearly holds as $h_e(e(p)) = p^\uparrow$ for all $p \in P$. Conversely, suppose $q \notin e[P]$ and let $X = \{e(p) : p \in h_e(q)\}$. Suppose first that $X \neq \emptyset$. Then $\Gamma_e(\bigcup X) = \bigcup X$, and clearly $h_e(q) = \bigcup h_e[X]$, but $h_e(q) \not\subseteq h_e(x)$ for all $x \in X$, and thus q cannot be meet-prime relative to $\mathscr{X}$. If $X = \emptyset$ then q is the top element, which is a contraction. $\square$

Lemma 4.5.10. *If $e\colon P \to Q$ is a meet-completion such that the set of completely meet-prime elements of Q is precisely $e[P]$ then $\Gamma_e(S) = S$ for all $S \in P^*$.*

Proof. Suppose the set of completely meet-prime elements of Q is precisely $e[P]$ and let $\emptyset \subset S \in P^*$. Then $\Gamma_e(S) = \{p \in P : e(p) \geq \bigwedge e[S]\} = \{p \in P : e(p) \geq e(s)$ for some $s \in S\} = S$. We must also have $\Gamma_e(\emptyset) = \emptyset$ as otherwise for some $p \in P$ we would have $e(p)$ being the top for Q, which would contradict it being completely meet-prime. $\square$

Corollary 4.5.11. *$\iota : P \to P^*$ can be characterized abstractly as the unique (up to isomorphism lifting the identity on P) meet-completion of P whose set of completely meet-prime elements is the embedded image of P.*

Proof. P^* corresponds to the closure operator on $P^{*\delta}$ that is the identity map, and for this closure operator the set $\mathscr{X}_\Gamma$ is the set of non-empty subsets of P^*. That the set of completely meet-prime elements of P^* is $\iota[P]$ follows from Lemma 4.5.9.

If $e\colon P \to Q$ is a meet-completion of P whose set of completely meet-prime elements is $e[P]$, then by Lemma 4.5.10 we have $\Gamma_e(S) = S$ for all $S \in P^*$, and the result follows from Theorem 4.5.5. $\square$

We make the simple observation that a meet-completion is defined (up to isomorphism lifting the identity on P) by the subsets of P to which it assigns the same infimum.

Proposition 4.5.12. *If $e_1\colon P \to Q_1$ and $e_2\colon P \to Q_2$ are meet-completions then there is a unique isomorphism $h\colon Q_1 \to Q_2$ lifting the identity on P if and only if for all $S, T \subseteq P$ we have $\bigwedge e_1[S] = \bigwedge e_1[T] \iff \bigwedge e_2[S] = \bigwedge e_2[T]$ (defining $e[\emptyset]$ to be $\emptyset$).*

Proof. That the existence of a unique isomorphism $h\colon Q_1 \to Q_2$ with $h \circ e_1 = e_2$ implies for all $S, T \subseteq P$ we have $\bigwedge e_1[S] = \bigwedge e_1[T] \iff \bigwedge e_2[S] = \bigwedge e_2[T]$ is trivial. For the converse note that to say $\bigwedge e_1[S] = \bigwedge e_1[T] \iff \bigwedge e_2[S] = \bigwedge e_2[T]$ is equivalent to saying that $\Gamma_{e_1}(S) = \Gamma_{e_1}(T) \iff \Gamma_{e_2}(S) = \Gamma_{e_2}(T)$. Let $S \in P^*$ and suppose that $\Gamma_1(S) \neq S$. Suppose also that $\Gamma_{e_1}(S) = \Gamma_{e_1}(T) \iff \Gamma_{e_2}(S) = \Gamma_{e_2}(T)$. Then $\Gamma_1(S) = C = \Gamma_1(C) \supset S$ for some $C \in P^*$, and so $\Gamma_2(S) = \Gamma_2(C) \supset S$, and thus $\Gamma_2[P^*] \subseteq \Gamma_1[P^*]$. By symmetry we have $\Gamma_1[P^*] = \Gamma_2[P^*]$, hence $\Gamma_1 = \Gamma_2$ and the result follows from Theorem 4.5.5. $\square$

We can say more about relationship between meet-completions and standard closure operators.

Definition 4.5.13 ($\mathbb{S}_P$)**.** Define $\mathbb{S}_P$ to be the complete lattice of standard closure operators on $P^{*\delta}$ (ordered pointwise, i.e. $\Gamma_1 \leq \Gamma_2 \iff \Gamma_1(S) \leq \Gamma_2(S)$ for all $S \in P^* \iff \Gamma_1(S) \subseteq \Gamma_2(S)$ for all $S \in P^*$).

Definition 4.5.14 ($\mathbf{M}_P$)**.** Define $\mathbf{M}_P$ to be the set of meet-completions of P, ordered by defining $e_1\colon P \to Q_1 \leq e_2\colon P \to Q_2 \iff$ there is an embedding $e\colon Q_1 \to Q_2$ lifting the identity on P. Since we have $e_1 \leq e_2$ and $e_2 \leq e_1 \iff Q_1 \cong Q_2$ we can consider $\mathbf{M}_P$ to be a poset.

Lemma 4.5.15. *For all $\Gamma_1, \Gamma_2 \in \mathbb{S}_P$, $\Gamma_1 \leq \Gamma_2 \iff \Gamma_2[P^*]$ embeds into $\Gamma_1[P^*]$ via a map lifting the identity on P.*

Proof. Suppose $\Gamma_1 \leq \Gamma_2$ and define $f\colon \Gamma_2[P^*] \to \Gamma_1[P^*]$ by $f(C) = \Gamma_1(C) = C$. Then f is well defined as $\Gamma_1 \leq \Gamma_2$, and certainly is an embedding lifting the identity on P. Conversely, if $e\colon \Gamma_2[P^*] \to \Gamma_1[P^*]$ is an appropriate embedding then for all $S \in P^*$ and for all $p \in P$ we have $p \in e(\Gamma_2(S)) \iff p^{\uparrow} \geq e(\Gamma_2(S)) \iff p^{\uparrow} \geq \Gamma_2(S) \iff p \in \Gamma_2(S)$, so $e(\Gamma_2(S)) = \Gamma_2(S)$, and thus $\Gamma_2(S)$ is Γ_1-closed. Since $\Gamma_1(S)$ must be the smallest Γ_1-closed set containing S we must have $\Gamma_1(S) \subseteq \Gamma_2(S)$. $\square$

Theorem 4.5.16. *M_P and $\mathbb{S}_P$ are dually isomorphic.*

Proof. This follows easily from Theorem 4.5.5 and Lemma 4.5.15. $\square$

Finally we give an example of two meet-completions of a poset P that are isomorphic but with no isomorphism lifting the identity on P (Example 4.5.17).

Example 4.5.17. Let $P = \{a, b, c\}$ be the three element antichain, and let Q_1 and Q_2 be as in Figure 4.7. Suppose the image of P in Q_1 and Q_2 is as marked. Then Q_1 and Q_2 are clearly order isomorphic, but there is no such isomorphism that is a lift of the identity on P.

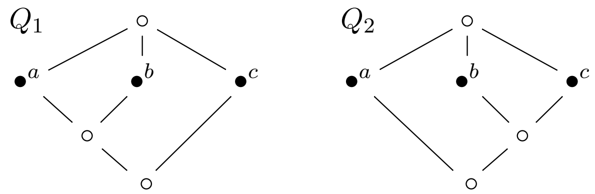


Figure 4.7: Meet completions of P that are isomorphic but not via any isomorphism lifting the identity on P

4.5.2 Preserving meets in meet-completions

It is well known that meet-completions preserve all existing joins (see Proposition 4.5.18), existing meets however are not necessarily preserved, though subject to certain constraints we can construct meet-completions preserving only specific meets, in a sense we shall make precise in this section.

Proposition 4.5.18. *Let $e\colon P \to Q$ be a meet-completion of P. Then for all $S, T \subseteq P$, we have $\bigvee S = \bigvee T \implies \bigvee e[S] = \bigvee e[T]$ (defining $e[\emptyset] = \emptyset$). Conversely, if either $\bigvee S$ or $\bigvee T$ exists in P then $\bigvee e[S] = \bigvee e[T] \implies$ they both exist and are equal.*

Proof. Suppose $\bigvee e[S] \not\leq \bigvee e[T]$ (note that this implies $S \neq \emptyset$). Then by meet-density there is $p \in P$ with $e(p) \geq \bigvee e[T]$ and $e(p) \not\geq \bigvee e[S]$. So $p \geq t$ for all $t \in T$, and there is $s \in S$ with $p \not\geq s$, and so $p \geq \bigvee T$ but $p \not\geq \bigvee S$ and so $\bigvee S \not\leq \bigvee T$. The result follows by symmetry. Conversely, let $\bigvee S = p \in P$ and suppose $\bigvee e[S] = \bigvee e[T]$. Then $e(p) = \bigvee e[S] = \bigvee e[T]$, so either $T = \emptyset$ (in which case $\bigvee S = \bigvee T$), or $e(p) \geq e(t)$ for all $t \in T$, and as e is an embedding this means $p \geq t$ for all $t \in T$. If $z \geq t$ for all $t \in T$ then $e(z) \geq e(t)$ for all $t \in T$, but then $e(z) \geq \bigvee e[T] = \bigvee e[S] = e(p)$, and so $z \geq p$, and thus $\bigvee T = p = \bigvee S$ as required. $\square$

Lemma 4.5.19. *Let $e\colon P \to Q$ be a meet-completion, let $S \in P^*$, and suppose $\bigwedge S = p$ in P. Then $\bigwedge e[S] = e(p)$ in Q if and only if $\Gamma_e(S) = p^\uparrow$.*

Proof. When $S = \emptyset$ we must have $\bigwedge S = \top_P$ and $\bigwedge e[S] = \top_Q$, and $\top_Q = e(p) \iff \Gamma_e(\emptyset) = p^\uparrow$ so we are done. Suppose $S \neq \emptyset$. Then since $e(p)$ is clearly a lower bound for $e[S]$ we know $\bigwedge e[S] = e(p)$ if and only if whenever q is a lower bound for $e[S]$ we have $q \leq e(p)$, but this occurs if and only if whenever $C \in \Gamma_e[P^*]$ and $C \leq h_e(s)$ for all $s \in S$ we have $C \leq h_e(p)$ (by Theorem 4.5.5), but this is just saying that whenever $C \in \Gamma_e[P^*]$ and $S \subseteq C$ we have $p \in C$, which in turn is equivalent to saying that $p \in \Gamma_e(S)$. Since $\Gamma_e(p^\uparrow) = p^\uparrow$ and $S \subseteq p^\uparrow$ we have $p \in \Gamma_e(S) \iff \Gamma_e(S) = p^\uparrow$. □

Definition 4.5.20 ($\mathscr{S}$-regular)**.** Given $\mathscr{S} \subseteq P^*$, we say a meet-completion $e\colon P \to Q$ is $\mathscr{S}$-regular if $\bigwedge S$ is defined in P and $e(\bigwedge S) = \bigwedge e[S]$ for all $S \in \mathscr{S}$. If $\mathscr{S}$ contains every set S for which $\bigwedge S$ is defined in P, then we say that an $\mathscr{S}$-regular meet-completion is *regular*.

Definition 4.5.21 (Regular)**.** Let P be a poset. Then $\mathscr{S} \subseteq P^*$ is regular if

1. $p^\uparrow \in \mathscr{S}$ for all $p \in P$,
2. $\bigwedge S$ exists in P for all $S \in \mathscr{S}$, and
3. whenever $T \in P^* \setminus \mathscr{S}$, there is $T' \in P^*$ with $T \subseteq T'$ such that
   (a) $p < T \implies p < T'$ for all $p \in P$, and
   (b) for all $S \in \mathscr{S}$, $S \subseteq T' \implies \bigwedge S \in T'$.

Note that in the above definition if $T = \emptyset$ we must also have $T' = \emptyset$. This definition appears rather technical, but the intuition behind it is quite simple. If $e\colon P \to Q$ is a meet-completion then $\{p \in P : e(p) \geq q\}$ is clearly an up-set for every $q \in Q$, and thus every meet-completion can be thought of as being based on a subset of P^* (this is implicit in our definition via closure operators). A connection between preservation of finite meets and the closure of the up-sets it is based on under certain finite infima is known (see for example [51, Proposition 6.10]), and it is not hard to show that this extends to arbitrary meets too. The key point is that a meet-completion will preserve the infimum of a set S if and only if every up-set on which the completion is based contains that infimum whenever it contains S. Condition 3 of Definition 4.5.21 essentially demands that every non-empty set T whose infimum we wish to destroy can be contained in an up-set which also does not contain the infimum of T but does not interfere with the preservation of the infima we *do* wish to preserve.

Definition 4.5.22 ($\mathscr{S}$-closure)**.** If $\mathscr{S} \subseteq P^*$ we say a standard closure operator $\Gamma\colon P^{*\delta} \to P^{*\delta}$ is an $\mathscr{S}$-closure if for all $S \in P^*$ we have $\Gamma(S) = p^\uparrow$ for some $p \in P \iff S \in \mathscr{S}$.

Definition 4.5.23 ($\mathcal{F}_{\mathscr{S}}, \Gamma_{\mathscr{S}}$)**.** If $\mathscr{S} \subseteq P^*$ is regular define

$$\mathcal{F}_{\mathscr{S}} = \{f \in P^* : S \subseteq f \implies \bigwedge S \in f \text{ for all } S \in \mathscr{S}\},$$

and define $\Gamma_{\mathscr{S}} \colon\ P^{*\delta} \to P^{*\delta}$ by

$$\Gamma_{\mathscr{S}}(S) = \bigcap \{f \in \mathcal{F}_{\mathscr{S}} : S \subseteq f\}.$$

Lemma 4.5.24. *If $\mathscr{S} \subseteq P^*$ is regular then $\Gamma_{\mathscr{S}}$ is an $\mathscr{S}$-closure.*

Proof. If $\emptyset \in \mathscr{S}$ then $\top_P \in f$ for all $f \in \mathcal{F}_{\mathscr{S}}$, and if $\emptyset \notin \mathscr{S}$ then $\emptyset \in \mathcal{F}_{\mathscr{S}}$. In both cases $\mathcal{F}_{\mathscr{S}}$ is closed under arbitrary intersections so it's easy to see that $\Gamma_{\mathscr{S}}$ is a standard closure operator on $P^{*\delta}$. It's similarly easy to see that if $S \in \mathscr{S}$ then $\Gamma_{\mathscr{S}}(S) = p^{\uparrow}$ for some $p \in P$. If $T \in P^* \setminus \mathscr{S}$ then by definition of regularity there is $T' \in \mathcal{F}_{\mathscr{S}}$ with $T \subseteq T' \subset p^{\uparrow}$ whenever p is a lower bound for T, and since we must have $\Gamma_{\mathscr{S}}(T) \subseteq \Gamma_{\mathscr{S}}(T') = T' \subset p^{\uparrow}$ for all lower bounds p of T we cannot have $\Gamma_{\mathscr{S}}(T) = p^{\uparrow}$ for any $p \in P$, as $\Gamma_{\mathscr{S}}(T) = p^{\uparrow} \implies p < T$. □

Proposition 4.5.25. *Let P be a poset, and let $\mathscr{S} \subseteq P^*$. Then $\mathscr{S}$ is regular if and only if there exists an $\mathscr{S}$-closure.*

Proof. Suppose first that Γ is an $\mathscr{S}$-closure. Then by definition we must have $p^{\uparrow} \in \mathscr{S}$ for all $p \in P$, and if $S \in \mathscr{S}$ then $\Gamma(S) = p^{\uparrow}$ for some $p \in P$. In the case where $S = \emptyset$ then $\bigwedge S = \top_P$, so assume that $S \neq \emptyset$. If p is not the greatest lower bound of S there must be $p' \in P$ with $p' \leq S$ and $p' \not\leq p$. But then $S \subseteq \Gamma(p'^{\uparrow})$ and thus $S \subseteq \Gamma(p'^{\uparrow}) \cap \Gamma(p^{\uparrow}) \subset p^{\uparrow}$, which contradicts the facts that the intersection of Γ-closed sets is closed and $\Gamma(S)$ must be the smallest Γ-closed set containing S. We have shown that $\bigwedge S$ exists for every $S \in \mathscr{S}$.

Now suppose $T \in P^* \setminus \mathscr{S}$ and let $p \leq T$. Then by definition of $\mathscr{S}$-closure we have $\Gamma(T) \neq p^{\uparrow}$, so $T \subseteq \Gamma(T) \subset p^{\uparrow}$, and for all $S \in \mathscr{S}$ we have $S \subseteq \Gamma(T) \implies \Gamma(S) \subseteq \Gamma(T) \implies \bigwedge S \in \Gamma(T)$ as required. So for each $T \in P^* \setminus \mathscr{S}$ we can use $\Gamma(T)$ for T' and thus satisfy Definition 4.5.21 (note that when $T = \emptyset$ we must also have $\Gamma(T) = \emptyset$). We have shown that the existence of an $\mathscr{S}$-closure implies $\mathscr{S}$ is regular, and the converse follows trivially from Lemma 4.5.24. □

Corollary 4.5.26. *Let P be a poset and let $\mathscr{S} \subseteq P^*$. Then $\mathscr{S}$ is regular if and only if there is at least one $\mathscr{S}$-regular meet-completion.*

Proof. If $\mathscr{S}$ is regular just take $\Gamma_{\mathscr{S}}$ and Lemmas 4.5.19 and 4.5.24 give the result. Conversely, an $\mathscr{S}$-regular meet-completion $e \colon P \to Q$ gives rise to a standard closure operator Γ_e, which Lemma 4.5.19 says must be an $\mathscr{S}$-closure. The result then follows from Proposition 4.5.25. □

Lemma 4.5.27. *Let $\mathscr{S} \subseteq P^*$ be regular, and let Γ be an $\mathscr{S}$-closure. Then for all $T \in P^{*^\delta}$ we have $\Gamma_{\mathscr{S}}(T) \leq \Gamma(T)$.*

Proof. Let $T \in P^{*^\delta}$, and let $S \in \mathscr{S}$. Then $S \subseteq \Gamma(T) \implies \Gamma(S) \subseteq \Gamma(T) \iff \bigwedge S \in \Gamma(T)$. So $\Gamma(T) \in \mathcal{F}_{\mathscr{S}}$ and thus $\Gamma_{\mathscr{S}}(T) \subseteq \Gamma(T)$ by definition of $\Gamma_{\mathscr{S}}$. □

Lemma 4.5.28. *If Γ_1 and Γ_2 are $\mathscr{S}$-closures then $\Gamma_2 \circ \Gamma_1$ is also an $\mathscr{S}$-closure.*

Proof. For all $S \in P^*$ we have $\Gamma_1(\Gamma_2(S)) = p^\uparrow$ for some $p \in P \iff \Gamma_2(S) \in \mathscr{S}$ and $\bigwedge \Gamma_2(S) = p \iff \Gamma_2(\Gamma_2(S)) = p^\uparrow \iff \Gamma_2(S) = p^\uparrow$ for some $p \in P \iff S \in \mathscr{S}$. □

Theorem 4.5.29. *If P is a poset and $\mathscr{S} \subseteq P^*$ is regular then the set $\mathbb{S}_{\mathscr{S}}$ of $\mathscr{S}$-closures is a lattice with bottom element $\Gamma_{\mathscr{S}}$ when ordered pointwise (i.e. $\Gamma_1 \leq \Gamma_2 \iff \Gamma_1(S) \leq \Gamma_2(S) \iff \Gamma_1(S) \subseteq \Gamma_2(S)$ for all $S \in P^*$). Moreover, arbitrary non-empty meets are defined in $\mathbb{S}_{\mathscr{S}}$.*

Proof. Lemma 4.5.27 tells us $\Gamma_{\mathscr{S}}$ is mimimal in the set of $\mathscr{S}$-closures, and the meet operation on $\mathbb{S}_{\mathscr{S}}$ is defined by set intersection. For $\Gamma_1, \Gamma_2 \in \mathbb{S}_{\mathscr{S}}$, $\Gamma = \Gamma_2 \circ \Gamma_1$ is an upper bound for $\{\Gamma_1, \Gamma_2\}$ in $\mathbb{S}_{\mathscr{S}}$ (using Lemma 4.5.28), so by closure under arbitrary meets the join of Γ_1 and Γ_2 must be defined in $\mathbb{S}_{\mathscr{S}}$. □

In light of Theorem 4.5.29, $\mathbb{S}_{\mathscr{S}}$ will be a complete lattice if and only if it has a top element, which it may not (see Example 4.5.31). Proposition 4.5.30 gives a necessary and sufficient condition for $\mathbb{S}_{\mathscr{S}}$ to have a top.

Proposition 4.5.30. *Let P be a poset and let $\mathscr{S} \subseteq P^*$ be regular. For each $T \in P^* \setminus \mathscr{S}$ define $\bar{T} = \{T' \in \mathcal{F}_{\mathscr{S}} : T \subseteq T'$, and $p < T \implies p < T'$ for all $p \in P\}$. Then $\mathbb{S}_{\mathscr{S}}$ has a top element if and only if for each $T \in P^* \setminus \mathscr{S}$ we can choose $T^m \in \bar{T}$ in such a way that*

1. *for all $T, T_0 \in P^* \setminus \mathscr{S}$ we have $T \subseteq T_0^m \implies T^m \subseteq T_0^m$, and*
2. *for every $\emptyset \subset Y \subseteq \bigcup\{\bar{U} : U \in P^* \setminus \mathscr{S}\}$ with $Y \cap \bar{U} \neq \emptyset$ for all $\emptyset \subset U \in P^* \setminus \mathscr{S}$ we have $\bigcap\{f \in Y : T \subseteq f\} \subseteq T^m$.*

Proof. Suppose such T^m exist. Define $\mathcal{F} = \{p^\uparrow : p \in P\} \cup \{T^m : T \in P^* \setminus \mathscr{S}\}$, and define $\Gamma\colon P^{*^\delta} \to P^{*^\delta}$ by $\Gamma(U) = \bigcap\{f \in \mathcal{F} : U \subseteq f\}$. Then Γ is a standard closure operator on P^{*^δ}. If $T \in P^* \setminus \mathscr{S}$ then $\Gamma(T) = T^m$, and $T^m \neq p^\uparrow$ for all $p \in P$. If $S \in \mathscr{S}$ then for all $f \in \mathcal{F}$ we have $S \subseteq f \implies (\bigwedge S)^\uparrow \subseteq f$. So $\Gamma(S) = (\bigwedge S)^\uparrow$, and thus Γ is an $\mathscr{S}$-closure.

If Γ' is another $\mathscr{S}$-closure then $\Gamma(S) = (\bigwedge S)^\uparrow = \Gamma'(S)$ for all $S \in \mathscr{S}$. Define $Y = \{\Gamma'(U) : \emptyset \subset U \in P^* \setminus \mathscr{S}\}$. Then, since from the proof of Proposition 4.5.25 we deduce that

$\Gamma'(U) \in \bar{U}$ for all $\emptyset \subset U \in P^* \setminus \mathscr{S}$, we have $Y \subseteq \bigcup\{\bar{U} : U \in P^* \setminus \mathscr{S}\}$ and $Y \cap \bar{U} \neq \emptyset$ for all $\emptyset \subset U \in P^* \setminus \mathscr{S}$. So, given $T \in P^* \setminus \mathscr{S}$ we have $\Gamma'(T) = \bigcap\{f \in Y : T \subseteq f\} \subseteq T^m = \Gamma(T)$, and thus $\Gamma' \leq \Gamma$ as required.

Conversely, suppose Γ is a top element for $\mathbb{S}_{\mathscr{S}}$ and let Y be any subset of $\bigcup\{\bar{T} : T \in P^* \setminus \mathscr{S}\}$ such that $\bar{T} \cap Y \neq \emptyset$ for all $\emptyset \subset T \in P^* \setminus \mathscr{S}$. Then the closure under intersections of $\mathcal{F} = \{p^\uparrow : p \in P\} \cup Y$ defines an $\mathscr{S}$-closure, Γ_Y, and $\Gamma_Y(T) = \bigcap\{f \in Y : T \subseteq f\}$ for all $T \in P^* \setminus \mathscr{S}$. By maximality of Γ we have $\Gamma_Y(T) \subseteq \Gamma(T)$ for all $\emptyset \subset T \in P^* \setminus \mathscr{S}$, which means $\bigcap\{f \in Y : T \subseteq f\} \subseteq \Gamma(T)$, so we take $T^m = \Gamma(T)$. Since if $T \subseteq \Gamma(T_0)$ we must have $\Gamma(T) \subseteq \Gamma(T_0)$ we are done. $\square$

Example 4.5.31. *$\mathbb{S}_{\mathscr{S}}$ may have no top.* Let $P' = \{p_n : n \in \omega\}$ be an antichain, let $P = P' \cup \{0\}$, where 0 is a a bottom element, and let $\mathscr{S} = \{p^\uparrow : p \in P\} \cup \{S \in P^* : |S| = |\omega|\}$ (it's easy to check that $\mathscr{S}$ is regular). Let $U = \{p_1, p_2\}$ and let $U' \in \bar{U}$. Since U' must be finite we can choose $p \in P' \setminus U'$ and define $Y = \{f \in P^* : f \text{ is finite and } U' \cup \{p\} \subseteq f\}$. Then $Y \subseteq \bigcup\{\bar{T} : T \in P^* \setminus \mathscr{S}\}$, and $Y \cap \bar{T} \neq \emptyset$ for all $\emptyset \subset T \in P^* \setminus \mathscr{S}$. However, $\bigcap\{f \in Y : U \subseteq f\} \supseteq U' \cup \{p\} \supset U'$, so by Proposition 4.5.30 $\mathbb{S}_{\mathscr{S}}$ has no top element.

4.5.3 The structure of $\mathbb{S}_{\mathscr{S}}$

We can say a little more about the structure of $\mathbb{S}_{\mathscr{S}}$, though first we require some preliminary definitions and results.

Definition 4.5.32 (Weakly lower/upper semimodular)**.** A poset P is weakly lower semimodular if whenever $a, b, c \in P$ with $a \neq b$, $a \prec c$ and $b \prec c$ there is d with $d \prec a$ and $d \prec b$ (where $\prec$ is the 'covers' relation defined by $x \prec y \iff x < y$ and there is no z with $x < z < y$). Weakly upper semimodularity is defined dually

This definition appears as a definition for lower/upper semimodularity in [77]. In the special case where P is a lattice, weak upper semimodularity is Birkhoff's condition, and thus is implied by upper semimodularity (see Definition 4.5.33 below). In an upper continuous, strongly atomic lattice the converse also holds (see e.g. [126, Theorem 1.7.1], a lattice is *strongly atomic* if every subinterval is atomic). Note that a finite lattice is always upper continuous and strongly atomic. In a lattice weak lower semimodularity is the dual to Birkhoff's condition, and dual results hold regarding its relationship with lower semimodularity.

Definition 4.5.33 (Lower/upper semimodular)**.** A lattice L is lower semimodular if for all $a, b, c \in L$, if $a \parallel b$ and $a < b < a \vee c$ then there is some $d \in L$ with $c \leq d < a \vee c$ and $a \vee (b \wedge d) = b$. Upper semimodularity is defined dually.

Definition 4.5.34 (Upper/lower bounded homomorphism)**.** A lattice homomorphism $h\colon K \to L$ is upper bounded if $\{b \in K : h(b) \leq a\}$ is either empty or has a greatest element for all $a \in L$. Lower bounded homomorphisms are defined dually.

Given a complete lattice L we define $\mathbb{CL}(L)$ to be the complete lattice of closure operators on L (ordered pointwise), and we define $\mathbb{SUB}_\wedge(L)$ to be the complete lattice of meet subsemilattices of L (regarded as a meet-semilattice) containing 1.

Theorem 4.5.35 (originating in [137], reproduced here as it appears in [104])**.** *If L is a complete lattice then $\mathbb{CL}(L)$ is dually isomorphic to $\mathbb{SUB}_\wedge(L)$.*

Lemma 4.5.36 (this is Lemma 2 of [104])**.** *Let L be a complete lattice. For all $U, V \in \mathbb{SUB}_\wedge(L)$, if $U \prec V$ then $U = V \setminus \{a\}$ for some $a \in V$.*

Corollary 4.5.37. *If $S \subseteq \mathbb{SUB}_\wedge(L)$ has the following properties*

1. *the interval $[y, x] \subseteq S$ for all $y \leq x \in S$, and*

2. *S is closed under finite meets (inherited from $\mathbb{SUB}_\wedge(L)$)*

then S is weakly lower semimodular.

Proof. Let $U, V, W \in S$, and suppose $U \prec W$ and $V \prec W$ and $U \neq V$. Then $U = W \setminus \{a\}$, and $V = W \setminus \{b\}$ for some $a \neq b \in W$ (by Lemma 4.5.36), so $U \wedge V = U \cap V = W \setminus \{a, b\}$, and so clearly $U \wedge V \prec U$ and $U \wedge V \prec V$ so we are done. □

Proposition 4.5.38. *If $\mathscr{S} \subseteq P^*$ is regular then $\mathbb{S}_\mathscr{S}$ is weakly upper semimodular.*

Proof. This follows easily from Theorem 4.5.35, Corollary 4.5.37, and the fact that $\mathbb{S}_\mathscr{S}$ is a sublattice of $\mathbb{CL}(P^{*\delta})$. □

When P is finite we can go further, as in this case $\mathbb{S}_\mathscr{S}$ is a finite interval of $\mathbb{CL}(P^{*\delta})$, and we can apply [104, Theorem 6] (reproduced for convenience as Theorem 4.5.40 below) directly to obtain the following.

Theorem 4.5.39. *If P is finite, and if $\mathscr{S} \subseteq P^*$ is regular, then $\mathbb{S}_\mathscr{S}$ is upper semimodular, an upper bounded homomorphic image of a free lattice, and thus meet semidistributive.*

See e.g. [46, Chapter 2] for further discussion of these ideas. The duality between $\mathbf{M}_P$ and $\mathbb{S}_P$ from Theorem 4.5.16 restricted to $\mathbb{S}_\mathscr{S}$ gives dual results to those in this and the last section for $\mathbf{M}_\mathscr{S}$, the set of meet-completions of P preserving those, and only those, meets that are defined in $\mathscr{S}$.

Theorem 4.5.40. [104, Theorem 6] *Let L be a complete lattice. Then every finite interval of $\mathbb{CL}(L)$ is upper semimodular, an upper bounded homomorphic image of a free lattice, and meet semidistributive.*

4.5.4 Preserving inequalities in meet-completions of isotone poset expansions

We saw with Lemma 4.2.13 a recipe for lifting isotone poset operations to isotone operations on meet-completions, and given any standard closure operator $\Gamma\colon\ {P^*}^\delta \to {P^*}^\delta$ there is a natural map $f^\bullet_\Gamma\colon \Gamma[P^*]^n \to \Gamma[P^*]$ defined by

$$f^\bullet_\Gamma(C_1, ..., C_n) = \Gamma(f[C_1 \times ... \times C_n]^\uparrow)$$

and when $\Gamma = \Gamma_e$ for some meet-completion $e\colon P \to Q$ the diagram in Figure 4.5 commutes (where $\hat{f}$ in this diagram is defined as in (4.1) from Section 4.2.1). We can use this to define lifts of isotone (order preserving) operations $P^n \to P$ to order preserving operations $\Gamma[P^*]^n \to \Gamma[P^*]$. This means that given an isotone poset expansion $\mathcal{P}$ (i.e. a structure $\mathcal{P} = (P, \leq, f_i : i \in I)$ where f_i is an n_i-ary isotone operation $P^{n_i} \to P$ for each $i \in I$, where I is some ordinal), such as the ODAs in Chapter 6, we can use Γ to define a completion of $\mathcal{P}$ with the corresponding signature of operations. We note that frequently inequalities that hold with respect to the operations of $\mathcal{P}$ will fail in this completion. The remainder of this section is devoted to an examination of some conditions which guarantee inequality preservation. We will see an application of this theory in Chapter 6.

$$\begin{tikzcd}
P^n \arrow[r, "e^n"'] \arrow[rr, bend left, "(e_{\Gamma_e})^n"] \arrow[d, "f"'] & Q^n \arrow[r, leftrightarrow, "\cong"'] \arrow[d, "\hat{f}"] & \Gamma_e[P^*]^n \arrow[d, "f^\bullet_{\Gamma_e}"] \\
P \arrow[r, "e"] \arrow[rr, bend right, "e_{\Gamma_e}"'] & Q \arrow[r, leftrightarrow, "\cong"] & \Gamma_e[P^*]
\end{tikzcd}$$

Figure 4.8: Lifting operations in terms of closure operators

Definition 4.5.41 (Γ_ι)**.** Define Γ_ι to be the identity on ${P^*}^\delta$

Lemma 4.5.42. *Let $\mathcal{P} = (P, \leq, f_i : i \in I)$ be an isotone poset expansion, let $x_1, ..., x_n$ be distinct variables, and let $\phi(x_1, ..., x_n)$ be a term in the language of $\mathcal{P}$ such that x_i does not appear more than once in ϕ for all $1 \in \{1, ..., n\}$. Define $\phi^\bullet_{\Gamma_\iota}(x_1, ..., x_n)$ by replacing each occurence of f_i in ϕ with $f^\bullet_{i\Gamma_\iota}$. Then, for all $(C_1, ..., C_n) \in {P^*}^n$, $\phi^\bullet_{\Gamma_\iota}(C_1, ..., C_n) = \phi[C_1 \times ... \times C_n]^\uparrow$, where $\phi[C_1 \times ... \times C_n]^\uparrow$ is defined to be $\{\phi(x_1, ..., x_n) : x_i \in C_i$ for all $i \in \{1, ..., n\}\}^\uparrow$.*

Proof. We use induction on the construction of ϕ. In the case where $\phi(x_1,...,x_n) = f_i(x_1,...,x_n)$ for some $i \in I$ the result follows trivially from the definition of $f^\bullet_{i\Gamma_\iota}$ and the fact that no variable occurs more than once in ϕ, so suppose instead that $\phi(x_1,...,x_n) = f(\phi_1(\bar{x}_1),...,\phi_m(\bar{x}_m))$ for some $f = f_k$ with $k \in I$, where, for each $i \in \{1,...,m\}$, $\bar{x}_i \subseteq \{x_1,...,x_n\}$, and $\phi_i(\bar{x}_i)$ is a term such that the appropriate induction hypothesis holds. If we let $\prod$ stand for the usual direct product of sets then for all $\bar{C} = (C_1,...C_n) \in P^{*n}$ with $C_1 = C_j \implies i = j$ we have

$$\begin{aligned}
\phi^\bullet_{\Gamma_\iota}(C_1,...,C_n) &= f^\bullet_{\Gamma_\iota}(\phi^\bullet_{1\Gamma_\iota}(\bar{C}_1),...,\phi^\bullet_{m\Gamma_\iota}(\bar{C}_m)) \\
&= f^\bullet_{\Gamma_\iota}(\phi_1[\prod \bar{C}_1]^\uparrow,...,\phi_m[\prod \bar{C}_m]^\uparrow) \\
&= \Gamma_\iota(f[\phi_1[\prod \bar{C}_1]^\uparrow \times ... \times \phi_m[\prod \bar{C}_m]^\uparrow]^\uparrow) \\
&= f[\phi_1[\prod \bar{C}_1]^\uparrow \times ... \times \phi_m[\prod \bar{C}_m]^\uparrow]^\uparrow \\
&= \{\phi(x_1,...,x_n) : x_i \in C_i \text{ for all } i \in \{1,...,n\}\}^\uparrow \\
&= \phi[\prod \bar{C}]^\uparrow
\end{aligned}$$

□

Note that the condition that no variable occurs more than once in ϕ is required in Lemma 4.5.42, as otherwise even the base case fails. For example if $\phi(x) = f(x,x)$ for binary operation f then $f[C \times C]^\uparrow \neq \{f(x,x) : x \in C\}^\uparrow$ in general.

Definition 4.5.43. Given poset expansion $\mathcal{P} = (P, \leq, f_i : i \in I)$ define $\mathcal{P}^* = (P^*, \supseteq, f^\bullet_{i\Gamma_\iota} : i \in I)$.

Proposition 4.5.44. *Let $\mathcal{P} = (P, \leq, f_i : i \in I)$ be an isotone poset expansion, and let $\phi(x_1,...,x_n)$ and $\psi(x_1,...,x_n)$ be terms in the language of $\mathcal{P}$ such that $x_i = x_j \implies i = j$ for all $i, j \in \{1,...,n\}$. Define $\phi^\bullet_{\Gamma_\iota}(x_1,...,x_n)$ and $\psi^\bullet_{\Gamma_\iota}(x_1,...,x_n)$ as in Lemma 4.5.42. Then*

$$\mathcal{P} \models \phi(x_1,...,x_n) \leq \psi(x_1,...,x_n) \iff \mathcal{P}^* \models \phi^\bullet_{\Gamma_\iota}(x_1,...,x_n) \leq \psi^\bullet_{\Gamma_\iota}(x_1,...,x_n).$$

Proof. Let $(C_1,...,C_n) \in P^{*n}$ and suppose $\mathcal{P} \models \phi(x_1,...,x_n) \leq \psi(x_1,...,x_n)$. Then

$$\begin{aligned}
p \in \psi^\bullet_{\Gamma_\iota}(C_1,...,C_n) &\iff p \geq \psi(x_1,...,x_n) \text{ for some } (x_1,...,x_n) \in C_1 \times ... \times C_n \\
&\implies p \geq \phi(x_1,...,x_n) \\
&\iff p \in \phi^\bullet_{\Gamma_\iota}(C_1,...,C_n)
\end{aligned}$$

So $\psi^\bullet_{\Gamma_\iota}(C_1,...,C_n) \subseteq \phi^\bullet_{\Gamma_\iota}(C_1,...,C_n)$, and thus $\mathcal{P}^* \models \phi^\bullet_{\Gamma_\iota}(x_1,...,x_n) \leq \psi^\bullet_{\Gamma_\iota}(x_1,...,x_n)$ as required. Conversely, if $\mathcal{P}^* \models \phi^\bullet_{\Gamma_\iota}(x_1,...,x_n) \leq \psi^\bullet_{\Gamma_\iota}(x_1,...,x_n)$ then in particular, for all

$(p_1, ..., p_n) \in P^n$, $\phi[p_1^{\uparrow} \times ... \times p_n^{\uparrow}]^{\uparrow} \supseteq \psi[p_1^{\uparrow} \times ... \times p_n^{\uparrow}]^{\uparrow}$, and this can happen only when $\phi(p_1, ..., p_n) \geq \psi(p_1, ..., p_n)$, so $\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n)$. $\square$

Corollary 4.5.45. *Let $\mathcal{P}$, $\phi(x_1, ..., x_n)$, and $\psi(x_1, ..., x_n)$, be as in Proposition 4.5.44 and suppose $\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n)$. Let Γ be a standard closure operator on $P^{*\delta}$ and define $\Gamma[\mathcal{P}] = (\Gamma[P^*], \supseteq, f^{\bullet}_{i\Gamma} : i \in I)$. Define $\phi^{\bullet}_{\Gamma}(x_1, ..., x_n)$ and $\psi^{\bullet}_{\Gamma}(x_1, ..., x_n)$ in a similar manner to Lemma 4.5.42, and suppose for all $(C_1, ..., C_n) \in \Gamma[P^*]^n$ we have $\psi^{\bullet}_{\Gamma}(C_1, ..., C_n) = \Gamma(\psi[C_1 \times ... \times C_n]^{\uparrow}])$. Then*

$$\mathcal{P} \models \phi(x_1, ..., x_n) \leq \psi(x_1, ..., x_n) \implies \Gamma[\mathcal{P}] \models \phi^{\bullet}_{\Gamma}(x_1, ..., x_n) \leq \psi^{\bullet}_{\Gamma}(x_1, ..., x_n).$$

Proof. By Proposition 4.5.44 we have $\mathcal{P}^* \models \phi^{\bullet}_{\Gamma_\iota}(x_1, ..., x_n) \leq \psi^{\bullet}_{\Gamma_\iota}(x_1, ..., x_n)$, so in particular $\psi[C_1 \times ... \times C_n]^{\uparrow} \subseteq \phi[C_1 \times ... \times C_n]^{\uparrow}$ for all $(C_1, ..., C_n) \in \Gamma[P^*]^n$. We must always have $\Gamma(\phi[C_1 \times ... \times C_n]^{\uparrow}) \subseteq \phi^{\bullet}_{\Gamma}(C_1, ..., C_n)$, and similar for ψ, so $\Gamma(\psi[C_1 \times ... \times C_n]^{\uparrow}) \subseteq \Gamma(\phi[C_1 \times ... \times C_n]^{\uparrow}) \subseteq \phi^{\bullet}_{\Gamma}(C_1, ..., C_n)$. If $\psi^{\bullet}_{\Gamma}(C_1, ..., C_n) = \Gamma(\psi[C_1 \times ... \times C_n]^{\uparrow}])$ then $\psi^{\bullet}_{\Gamma}(C_1, ..., C_n) \subseteq \phi^{\bullet}_{\Gamma}(C_1, ..., C_n)$, and thus $\Gamma[\mathcal{P}] \models \phi^{\bullet}_{\Gamma}(x_1, ..., x_n) \leq \psi^{\bullet}_{\Gamma}(x_1, ..., x_n)$ as required. $\square$

Corollary 4.5.46. *With all notation as in Corollary 4.5.45, suppose $\mathcal{P} \models \phi(x_1, ..., x_n) = \psi(x_1, ..., x_n)$. Then if*

1. $\psi^{\bullet}_{\Gamma}(C_1, ..., C_n) = \Gamma(\psi[C_1 \times ... \times C_n]^{\uparrow}])$*, and*

2. $\phi^{\bullet}_{\Gamma}(C_1, ..., C_n) = \Gamma(\phi[C_1 \times ... \times C_n]^{\uparrow}])$

for all $(C_1, ..., C_n) \in \Gamma[P^*]^n$, then

$$\Gamma[\mathcal{P}] \models \phi^{\bullet}_{\Gamma}(x_1, ..., x_n) = \psi^{\bullet}_{\Gamma}(x_1, ..., x_n)$$

Proof. It is always true that $\psi^{\bullet}_{\Gamma}(C_1, ..., C_n) \supseteq \Gamma(\psi[C_1 \times ... \times C_n]^{\uparrow}) = \Gamma(\phi[C_1 \times ... \times C_n]^{\uparrow}) \subseteq \phi^{\bullet}_{\Gamma}(C_1, ..., C_n)$ and the result follows. $\square$

4.6 Δ_1-completions

Introduced in [51], Δ_1-completions encompass meet- and join-completions, and both the canonical extension and the MacNeille completion. Construction of Δ_1-completions is similar to that of the canonical extension. It is well known (see e.g. [27, Section 7.2.2]) that a polarity (see Definition 4.4.1, and note that in [27] it is referred to as a *context*) gives rise to a Galois connection given by the maps $f\colon \wp(X) \to \wp(Y)$ and $g\colon \wp(Y) \to \wp(X)$ defined by $f(A) = \{y : \forall x(x \in A \implies xRy)\}$ and $g(B) = \{x : \forall y(y \in B \implies xRy)\}$, and that the Galois closed sets of this connection form a complete lattice. Moreover, a connection between Δ_1-completions and certain polarities is given in [51, Theorem 3.4], which we reproduce (without proof) for convenience as Theorem 4.6.1 below.

Theorem 4.6.1. *Let P be a poset. There is a 1-1 correspondence between Δ_1-completions of P and polarities $(\mathcal{F}, \mathcal{I}, R)$ where*

1. *$\mathcal{F}$ is a standard closure system of up-sets of P,*

2. *$\mathcal{I}$ is a standard closure system of down-sets of P, and*

3. *the relation R satisfies the following four conditions:*

   (a) *for all $p \in P$, and for all $x \in \mathcal{F}$ we have $p \in x \iff xR\,p^{\downarrow}$,*

   (b) *for all $p \in P$, and for all $y \in \mathcal{I}$ we have $p \in y \iff p^{\uparrow}\,Ry$,*

   (c) *for all $x, x' \in \mathcal{F}$, and for all $y \in \mathcal{I}$ we have $(x \supseteq x'$ and $x'Ry) \implies xRy$,*

   (d) *for all $x \in \mathcal{F}$, and for all $y, y' \in \mathcal{I}$ we have $(y \subseteq y'$ and $xRy) \implies xRy'$,*

The MacNeille completion plays a role here too, as a completion $e\colon P \to Q$ is a Δ_1-completion if and only if it is the MacNeille completion of $K_{\mathcal{D}(e[P])}(Q) \cup O_{\mathcal{U}(e[P])}(Q)$ (recall Definition 4.1.2, ordering is the restriction of the order on Q) [51, Proposition 2.1].

Being a recent development, the literature on Δ_1-completions is limited, so rather than reproducing it here we direct the reader to [51] for what is currently the state of the art. We note that due to the connection between standard closure systems of up- and down-sets of P (which are precisely the closed sets induced by standard closure operators with appropriate domain and codomain) and meet/join-completions the theorem above could be stated in terms of these completions, and in fact the proof in [51] takes this approach. In the next section we claim to add something original to the proceedings by making explicit another connection between Δ_1-completions and meet- and join-completions. First we make the following definition.

Definition 4.6.2 (P_*)**.** Given a poset P we define P_* to be the complete lattice of down-sets ordered by inclusion.

This definition is very close to that of P^* from Definition 4.5.2, and we make an analogous definition for standard closure operators on P_*. It is well known that the complete lattice of standard closure operators on P_* is isomorphic to the complete lattice of (isomorphism classes of) join-completions of P, via similar considerations to those of Section 4.5.1.

4.6.1 Δ_1-completions via alternating meet- and join-completions

The key observation is that a Δ_1-completion of a poset P can be thought of as a system

$$e_1 \colon P \to Q_1,$$
$$e_1' \colon Q_1 \to Q_1',$$
$$e_2 \colon P \to Q_2,$$
$$e_2' \colon Q_2 \to Q_2'$$

such that e_1 and e_2' are meet-completions, e_2 and e_1' are join-completions, and there is an isomorphism between Q_1' and Q_2' such that the diagram in Figure 4.9 commutes.

$$\begin{tikzcd}
 & Q_1 \arrow[r, "e_1'"] & Q_1' \arrow[dd, leftrightarrow, "\cong"] \\
P \arrow[ur, "e_1"] \arrow[dr, "e_2"'] & & \\
 & Q_2 \arrow[r, "e_2'"] & Q_2'
\end{tikzcd}$$

Figure 4.9: Meet and join-completions defining a Δ_1-completion

Proposition 4.6.3. *If $(\mathcal{F}, \mathcal{I}, R)$ is a standard polarity on P, then we can define standard closure operators Γ_1, Γ_2, Γ_1', and Γ_2' so that the diagram in Figure 4.10 commutes (here $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{F}}$ is the set of Galois closed subsets of $\mathcal{F}$, $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{I}}$ is the set of Galois closed subsets of $\mathcal{I}$, both ordered by inclusion, and the dual order isomorphism between them is given by the restriction of the maps f and g that define the Galois connection).*

Proof. We need to check that $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{F}}$ and $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{I}}$ induce standard closure operators on $\mathcal{F}_* = (\mathcal{F}, \supseteq)_*$ and $\mathcal{I}^{*\delta} = (\mathcal{I}, \subseteq)^{*\delta}$ respectively, and as $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{F}}$ and $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{I}}^{\delta}$ are both complete lattices it is sufficient to show that $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{F}} \subseteq \mathcal{F}_*$ and $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{I}} \subseteq \mathcal{I}^*$, and that $x^{\downarrow} = \{x' \in \mathcal{F} : x' \supseteq x\}$ and $y^{\uparrow} = \{y' \in \mathcal{I} : x' \supseteq x\}$ are Galois closed for all $x \in \mathcal{F}$ and $y \in \mathcal{I}$.

First, that $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{F}}$ is a subset of of $\mathcal{F}_*$ follows from condition 3.c of Theorem 4.6.1, and that $G(\mathcal{F}, \mathcal{I}, R)_{\mathcal{I}}$ is a subset of $\mathcal{I}^*$ follows from condition 3.b of the same theorem. Given $x \in \mathcal{F}$, to show that $x^{\downarrow}$ is Galois closed it is sufficient to show that $x'' \in g(f(\{x' \in \mathcal{F} : x' \supseteq x\})) \implies x'' \supseteq x$, and if $p \in x$ then by condition 3.a we have $x R\, p^{\downarrow}$, and thus $x' R\, p^{\downarrow}$ whenever $x' \supseteq x$ (by 3.c). By definition of g we must have $x'' R\, p^{\downarrow}$ whenever $x'' \in g(f(\{x' \in \mathcal{F} : x' \supseteq x\}))$, and another application of 3.a says $p \in x''$. Given $y \in \mathcal{I}$ a similar proof shows $y^{\uparrow}$ is Galois closed.

It is well known, and straightforward to show, that the maps f and g form a dual order isomorphism between $G(\mathcal{F},\mathcal{I},R)_{\mathcal{F}}$ and $G(\mathcal{F},\mathcal{I},R)_{\mathcal{I}}$, and commutativity of the diagram comes from the equalities $\{y \in \mathcal{I} : p \in y\} = \{y \in \mathcal{I} : \forall x(p \in x \implies xRy)\}$ and $\{x \in \mathcal{F} : p \in x\} = \{x \in \mathcal{F} : \forall y(p \in y \implies xRy)\}$, which are easily proved using condition 3 of Theorem 4.6.1. $\square$

Corollary 4.6.4. *Every Δ_1-completion Q' of a poset P can be obtained using a pair (e, e'), where $e\colon P \to Q$ is a meet-completion, and $e'\colon Q \to Q'$ is a join-completion.*

Proof. This follows easily from Proposition 4.6.3, Theorem 4.6.1, and the correspondence between meet- and join-completions and certain standard closure operators. A dual result also holds. $\square$

$$\begin{tikzcd}
& \Gamma_1[P^*] = (\mathcal{F},\supseteq) \arrow[r, "\iota_1'"] & \Gamma_2'[\Gamma_1[P^*]_*] = G(\mathcal{F},\mathcal{I},R)_{\mathcal{F}} \arrow[dd, leftrightarrow, "\cong"] \\
P \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & \\
& \Gamma_2[P_*] = (\mathcal{I},\subseteq) \arrow[r, "\iota_2'"] & \Gamma_1'[\Gamma_2[P_*]^*] = G(\mathcal{F},\mathcal{I},R)^{\delta}_{\mathcal{I}}
\end{tikzcd}$$

Figure 4.10: Closure operators from the Galois connection induced by a polarity

Given a pair $(\mathcal{F},\mathcal{I})$ of standard closure systems of up- and down-sets respectively, it is known from [51, Section 4] that there is always at least one relation R such that $(\mathcal{F},\mathcal{I},R)$ is standard polarity. In particular, the relation R_l defined by $xR_ly \iff x \cap y \neq \emptyset$ always gives rise to a standard polarity for $(\mathcal{F},\mathcal{I})$, and is in fact the least relation that does so. As a consequence of this we have the following.

Proposition 4.6.5. *Given meet-completion $e_1\colon P \to Q_1$, and join-completion $e_2\colon P \to Q_2$, there is a join-completion $e_1'\colon Q_1 \to Q_1'$, and a meet-completion $e_2'\colon Q_2 \to Q_2'$ such that the diagram in Figure 4.9 commutes.*

Proof. We can use the correspondence between meet- and join-completions of P and standard closure operators on $P^{*\delta}$ and P_* respectively to give us a suitable pair $(\mathcal{F},\mathcal{I})$ of standard closure systems. We can take any relation R such that $(\mathcal{F},\mathcal{I},R)$ is a standard polarity, and Proposition 4.6.3 and the correspondence between meet/join- completions and closure operators give the result. $\square$

Given meet-completions $e_1 \colon P \to Q_1$, and $e'_2 \colon Q_2 \to Q'_2$, and join-completions $e_2 \colon P \to Q_2$ and $e'_1 \colon Q_1 \to Q'_1$, there are natural maps $\phi_1 \colon Q_1 \to Q'_2$, and $\phi_2 \colon Q_2 \to Q'_1$ defined by

$$\phi_1(q) = \bigwedge\{e'_2 \circ e_2(p) : e_1(p) \geq q\} \tag{4.7}$$

and

$$\phi_2(q) = \bigvee\{e'_1 \circ e_1(p) : e_2(p) \leq q\}. \tag{4.8}$$

We show in Theorem 4.6.6 that these maps play an important role in determining when Q'_1 and Q'_2 are isomporhic via an isomorphism lifting the identity on P, and thus define a Δ_1-completion of P.

Theorem 4.6.6. *Given meet-completions $e_1 \colon P \to Q_1$ and $e'_2 \colon Q_2 \to Q'_2$, and join-completions $e_2 \colon P \to Q_2$ and $e'_1 \colon Q_1 \to Q'_1$, there is an isomorphism $f_1 \colon Q'_1 \leftrightarrow Q'_2 \colon f_2$ such that the diagram in Figure 4.9 commutes if and only if ϕ_1 and ϕ_2 are join- and meet-completions respectively and $e'_1(q_1) \leq \phi_2(q_2) \iff e'_2(q_2) \geq \phi_1(q_1)$ for all $q_1 \in Q_1$ and $q_2 \in Q_2$, in which case f_1 is the minimal lift of the identity on Q_1 along e'_1 and ϕ_1, and f_2 is the maximal lift of the identity on Q_2 along e'_2 and ϕ_2 (see Proposition 4.2.2).*

Proof. If there is an isomorphism $f_1 \colon Q'_1 \leftrightarrow Q'_2 \colon f_2$ such that the diagram in Figure 4.9 commutes then for all $q \in Q_1$ we have $f_1 \circ e'_1(q) = f_1 \circ e'_1(\bigwedge\{e_1(p) : e_1(p) \geq q\}) = f_1(\bigwedge\{e'_1 \circ e_1(p) : e_1(p) \geq q\}) = \bigwedge\{f_1 \circ e'_1 \circ e_1(p) : e_1(p) \geq q\} = \bigwedge\{e'_2 \circ e_2(p) : e_1(p) \geq q\} = \phi_1(q)$, and similarly $f_2 \circ e'_2 = \phi_2$. Since $f_1 \circ e'_1 = \phi_1$ and $f_2 \circ e'_2 = \phi_2$ must be join- and meet-completions respectively so too are ϕ_1 and ϕ_2, moreover, $e'_1(q_1) \leq \phi_2(q_2) \iff f_1(e'_1(q_1)) \leq f_1(\phi_2(q_2)) \iff \phi_1(q_1) \leq e'_2(q_2)$ as required.

Conversely, if ϕ_1 is a join-completion of P then we can define f_1 to be the minimal lift of the identity on Q_1 along e'_1 and ϕ_1, and similarly if ϕ_2 is a meet-completion we can define f_2 to be the maximal lift of the identity on Q_2 along e'_2 and ϕ_2.

Suppose that $e'_1(q_1) \leq \phi_2(q_2) \iff e'_2(q_2) \geq \phi_1(q_1)$ for all $q_1 \in Q_1$ and $q_2 \in Q_2$ for all $q \in Q'_1$. Then for $q' \in Q'_1$ we have

$$\begin{aligned}
f_2 \circ f_1(q') &= f_2(\bigvee\{\phi_1(q_1) : e'_1(q_1) \leq q'\}) \\
&= \bigwedge\{\phi_2(q_2) : e'_2(q_2) \geq \bigvee\{\phi_1(q_1) : e'_1(q_1) \leq q'\}\} \\
&= \bigwedge\{\phi_2(q_2) : e'_2(q_2) \geq \phi_1(q_1) \text{ for all } q_1 \in Q_1 \text{ with } e'_1(q_1) \leq q'\} \\
&= \bigwedge\{\phi_2(q_2) : \phi_2(q_2) \geq e'_1(q_1) \text{ for all } q_1 \in Q_1 \text{ with } e'_1(q_1) \leq q'\} \\
&= \bigwedge\{\phi_2(q_2) : \phi_2(q_2) \geq q'\} \\
&= q'.
\end{aligned}$$

We also have $f_1 \circ f_2$ being the identity on Q'_2, and since f_1 and f_2 are order preserving by the definition of a lift, and we have just shown they must be mutual inverses, they must form an isomorphism. That these lifts are the only possible candidates for an isomorphism between Q'_1 and Q'_2 follows from Lemma 4.2.4. □

If e_1, e_2, e'_1, e'_2 Q_1, Q_2, Q'_1 and Q'_2 are as in Figure 4.9, then a consequence of Theorem 4.6.1 is that Q'_1 and Q'_2 can be constructed using some standard polarity $(\mathcal{F}, \mathcal{I}, R)$. We can make this explicit in terms of the maps ϕ_1 and ϕ_2; the sets of down- and up-closed sets defined by the closure operators corresponding to e_1 and e_2 are $\mathcal{F}$ and $\mathcal{I}$ respectively, and R is the relation defined by $aRb \iff \big(\phi_1(\bigwedge e_1[a]) \leq e'_2(\bigvee e_2[b]) \iff \phi_2(\bigvee e_2[b]) \geq e'_1(\bigwedge e_1[a])\big)$.

Not every pair (e_1, e'_1) where $e_1\colon P \to Q_1$ is a meet-completion and $e'_1\colon Q_1 \to Q'_1$ is a join-completion defines a Δ_1-completion, as we see in Example 4.6.7.

Example 4.6.7. *A two stage meet-then-join-completion that is not a join-then-meet-completion.* Let P be the antichain with two elements $\{a, b\}$. Then all meet- and join-completions of P are isomorphic to the four element diamond D. We can choose a meet-completion of $e'\colon D \to D'$ that fails to preserve the only non-trivial meet, but any join-completion preserves this necessarily. Thus D' cannot be obtained as the join-completion of a meet-completion of P.

Given posets P_1 and P_2, meet-completions $e_1\colon P_1 \to Q_1$ and $e'_1\colon P \to Q'_1$, join-completions $e_2\colon Q_1 \to Q_2$ and $e'_2\colon Q'_1 \to Q'_2$ and a map $f\colon P_1 \to P_2$ we can lift f along e_1 and e'_1 to some $f'\colon Q_1 \to Q'_1$, and then lift f' along e_2 and e'_2 to some $f''\colon Q_2 \to Q'_2$ (see Figure 4.11). However, Example 4.6.8 demonstrates that not all lifts $g\colon Q_2 \to Q'_2$ of f along $e_2 \circ e_1$ and $e'_2 \circ e'_1$ can be obtained in this way.

$$\begin{tikzcd}
P_1 \arrow[r, "e_1"] \arrow[d, "f"] & Q_1 \arrow[r, "e_2"] \arrow[d, "f'"] & Q_2 \arrow[d, "f''"] \\
P_2 \arrow[r, "e'_1"] & Q'_1 \arrow[r, "e'_2"] & Q'_2
\end{tikzcd}$$

Figure 4.11: Lifting maps to Δ_1-completions in two stages

Example 4.6.8. *Not every map can be lifted to a Δ_1-completion in two stages.* Let P be the two element antichain $\{a, b\}$, let $e\colon P \to Q$ be a meet-completion (so Q is the four element diamond), define $e'\colon Q \to Q'$ as in Figure 4.12, so $e\colon Q \to Q'$ is a join-completion. Define f to be the identity map on P. Then the map $g\colon Q' \to Q'$ that maps x to y and is the identity everywhere else is a lift of f, but the only lift of f along e is the identity map on Q, and g is not a lift of the identity on Q as $g(x) \neq x$.

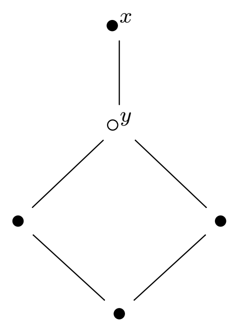


Figure 4.12: To show that not every map can be lifted to a Δ_1-completion in two stages

We saw in Corollary 4.6.4 that every Δ_1-completion of a poset can be obtained by taking first a meet-completion, then a join-completion (with a dual result also holding). However, this is not the case with Δ_1-completions of isotone poset expansions, where we demand that completions specify a lift for each operation, as, while we can obtain lifts of operations on a poset P to operations on a Δ_1-completion of P by lifting first to a meet-completion and then again to a join-completion of that meet-completion, a consequence of Example 4.6.8 is that there are lifts of poset operations to Δ_1-completions that cannot be obtained in this way.

Chapter 5

Complete representations

We saw in Chapter 3 the circumstances under which various classes of posets have a representation as a system of sets ordered by inclusion, and where existing finite joins and meets are interpreted as set theoretic unions and intersections respectively. To recap, we know that an arbitrary poset is representable if and only if it satisifes a certain separation property (Theorem 3.3.6), and thus a lattice is representable if and only if it is distributive, and a Boolean algebra is always representable. Here we investigate the subclasses whose members admit representation by a set system where one or both of *arbitrary* joins and meets are interpreted as union and intersection respectively. We begin with the Boolean algebra case, where the answers are simple.

An *atomic representation* h of a Boolean algebra B is a representation $h\colon B \to \wp(X)$ (some set X) where $h(1) = \bigcup\{h(a)\colon a \text{ is an atom of } B\}$. It is known that a representation of a Boolean algebra is a complete representation (in the sense of a complete embedding into a field of sets) if and only if it is an atomic representation and hence that the class of completely representable Boolean algebras is precisely the class of atomic Boolean algebras, and thus is elementary [2, 79]. This result is not obvious as the usual definition of a complete representation is thoroughly second order. The work in this chapter was inspired by a desire to extend this result to the class of bounded, distributive lattices [41], and, more generally, to posets.

In the lattice case the situation is a little more complex, as in the absence of Boolean complementation a representation of a (distributive) lattice may be complete with respect to one of the lattice operations but not the other. The poset case has even more complications, as, in addition to the problems of the general distributive lattice case, a poset may not have a representation preserving both existing *finite* infima and suprema.

In Section 5.1 we investigate the distributive lattice question, the results of which appear in [41]. In particular it is shown that if **CRL** is the class of bounded distributive lattices (DLs) which have representations preserving arbitrary joins and meets, **jCRL** is the class of DLs which have representations preserving arbitrary joins, **mCRL** is the class of DLs which have representations preserving arbitrary meets, and **biCRL** is defined to be **jCRL** $\cap$ **mCRL**, then

$$\mathbf{CRL} \subset \mathbf{biCRL} = \mathbf{mCRL} \cap \mathbf{jCRL} \subset \mathbf{mCRL} \neq \mathbf{jCRL} \subset \mathbf{DL}$$

where the marked inclusions are proper. Each of the classes above is shown to be *pseudoelementary* hence closed under ultraproducts, and the class **CRL** is shown not to be closed under elementary equivalence, and thus not elementary.

Section 5.2 examines the more general poset situation. In particular we show that the classes of posets with representations as fields of sets preserving any existing partial lattice structure and either or both existing arbitrary infima and/or suprema are pseudoelementary but not elementary. Finally, in Section 5.3 we give some results concerning complete representation and the canonical extension.

5.1 Complete representations for distributive lattices

Unlike the Boolean algebra situation, it turns out (Theorem 5.1.16) that the class **CRL** of completely representable bounded, distributive lattices is *not* elementary, however, building on early work in lattice theory by Birkhoff [14], and Birkhoff and Frink [16] it is possible to characterize complete representability of a lattice in terms of the existence of certain prime filters (or dually using prime ideals). Using this characterization an alternative proof of the identification of the completely representable Boolean algebras with the atomic ones is provided. It is also shown that **CRL**, and the classes of (bounded, distributive) lattices that have representations respecting either or both arbitrary infima and suprema *are* pseudoelementary, and thus closed under ultraproducts. Using the well known fact that a class is elementary if and only if it is closed under isomorphism, ultraproducts and ultraroots it follows that **CRL** is not closed under ultraroots. The question of whether this holds for the other classes of lattices under consideration, and thus whether they are elementary, remains open.

5.1.1 Complete representations for lattices

Definition 5.1.1 (Meet-complete map)**.** A lattice map $f\colon L_1 \to L_2$ is *meet-complete* if for all $\emptyset \subset S \subseteq L_1$ where $\bigwedge S$ exists in L_1 we have $f(\bigwedge_{L_1} S) = \bigwedge_{L_2} f[S]$.

A similar definition is made for *join-complete*. When a map is both *meet-complete* and *join-complete* we say it is *complete*. When a bounded, distributive lattice has a meet-complete representation we say it is *meet-completely representable*, and we make similar definitions for join-complete and complete representations. We shall call the class of all bounded, distributive lattices **DL**, the class of all completely representable lattices **CRL**, the classes of meet- and join-completely representable lattices **mCRL** and **jCRL** respectively, and the class of lattices with both a meet-complete and a join-complete representation **biCRL**.

Theorem 5.1.2. *A lattice L has a meet-complete representation iff its order dual L^δ has a join-complete representation.*

Proof. If $h\colon L \to \wp(P)$ is a representation, where P is some distinguishing set of prime filters of L, then the map $\bar{h}\colon L^\delta \to \wp(P)$, $a \mapsto -h(a)$ is also a representation. If h is meet-complete then by De Morgan $\bar{h}(\bigvee_\delta S) = -h(\bigwedge S) = -\bigcap h[S] = -\bigcap -\bar{h}[S] = \bigcup \bar{h}[S]$ (here '$-$' denotes set theoretic complement). □

Definition 5.1.3 (Complete ideal/filter)**.** An ideal I of a lattice L is complete if whenever $\bigvee S$ exists in L for $\emptyset \subset S \subseteq I$ then $\bigvee S \in I$. Similarly a filter F of L is complete if whenever $\bigwedge T$ exists in L for $\emptyset \subset T \subseteq F$ then $\bigwedge T \in F$.

Definition 5.1.4 (Completely-prime ideal/filter)**.** A prime ideal I of L is completely-prime if whenever $\bigwedge T \in I$ for some $\emptyset \subset T \subseteq L$ then $I \cap T \neq \emptyset$. Similarly, a prime filter F of L is completely-prime if whenever $\bigvee S \in F$ for some $\emptyset \subset S \subseteq L$ then $F \cap S \neq \emptyset$.

Lemma 5.1.5. *If F is a prime filter of L and $I = L \setminus F$ is its prime ideal complement then F is complete iff I is completely-prime, and I is complete iff F is completely-prime.*

Proof. Using $I = L \setminus F$ we can rewrite the definition of completeness of I as $\bigvee S \in F \implies F \cap S \neq \emptyset$. Similarly we can write completeness for F as $\bigwedge T \in I \implies T \cap I \neq \emptyset$. □

Theorem 5.1.6. *Let L be a bounded, distributive lattice. Then:*

1. *L has a meet-complete representation iff L has a distinguishing set of complete, prime filters,*
2. *L has a join-complete representation iff L has a distinguishing set of completely-prime filters,*
3. *L has a complete representation iff L has a distinguishing set of complete, completely-prime filters,*

Proof. This follows easily from the more general Theorem 5.2.4. □

In the light of Lemma 5.1.5 it's straightforward to prove an analogous result to Theorem 5.1.6 using ideals in place of filters.

We briefly turn our attention to the special case of Boolean algebras. Recall that a bounded lattice $(L, 0, 1, \wedge, \vee)$ is *complemented* iff for all $s \in L$ there is $s' \in L$ such that $s \vee s' = 1$ and

$s \wedge s' = 0$. Since there can be at most one complement to an element, we may write $-s$ instead of s'.

Lemma 5.1.7. *If L is complemented then its prime filters are precisely its ultrafilters, moreover the following are equivalent:*

1. *U is a principal ultrafilter of L,*
2. *U is a complete ultrafilter of L,*
3. *U is a completely-prime ultrafilter of L.*

Proof. It's easy to see that the ultrafilters of a BA are precisely its prime filters. Clearly $1) \implies 2)$. Let U be an ultrafilter. If U is complete it must contain a non-zero lower bound s and thus be principal (otherwise it would contain the complement of that lower bound, but $s \leq -s \Rightarrow s = 0$), so $2) \implies 1)$. For any $\emptyset \subset S \subseteq L$ we write $-S$ for $\{-s\colon s \in S\}$. The infinite De Morgan law for Boolean algebras (see e.g. [123, Section 19]) gives $-\bigvee S = \bigwedge -S$ so if U is complete then $S \cap U = \emptyset \implies -\bigvee S \in U \implies \bigvee S \notin U$, so $2) \implies 3)$. Similarly, if U is completely-prime then $\bigwedge S \notin U \implies -\bigwedge S \in U \implies \bigvee -S \in U \implies -s \in U$ for some $s \in S \implies S \not\subseteq U$, so $3) \implies 2)$. □

We have as a corollary the following result.

Corollary 5.1.8. *For a Boolean algebra B the following are equivalent:*

1. *B is atomic,*
2. *B is completely representable,*
3. *B is meet-completely representable,*
4. *B is join-completely representable.*

Turning our attention back to the lattice case we now give some examples to illustrate the relationships between the classes we have defined.

Example 5.1.9. *A distributive lattice both meet-completely representable and join-completely representable but not completely representable.* Let $L = [0,1] \subseteq \mathbb{R}$. Then by taking $\{[x,1]\colon x \in L\}$ we obtain a distinguishing set of complete, prime filters, and by taking $\{(x,1]\colon x \in L\}$ we obtain a distinguishing set of completely-prime filters.
However, if F is a complete filter of L then $\bigwedge F \in F$ (by completeness properties of L and F) and, since $\bigwedge F = \bigvee\{x \in L\colon x < \bigwedge F\}$, F cannot be completely-prime.

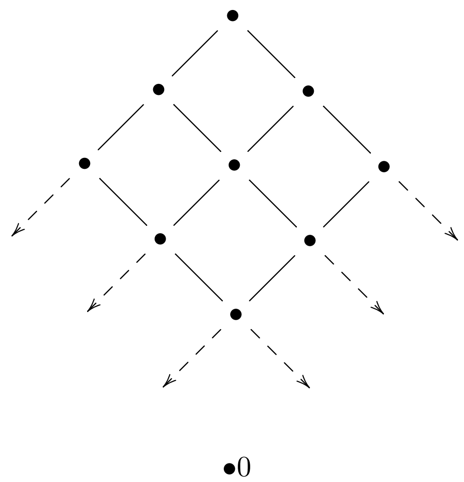


Figure 5.1: The lattice $(\overline{\mathbb{N}} \times \overline{\mathbb{N}}) \cup \{0\}$

Example 5.1.10. *A distributive lattice neither meet nor join-completely representable.* In view of Corollary 5.1.8 we can take any Boolean algebra that fails to be atomic.

Example 5.1.11. *A distributive lattice join-completely representable but not meet-completely representable.* Let L be the lattice $(\overline{\mathbb{N}} \times \overline{\mathbb{N}}) \cup \{0\}$ shown in Figure 5.1, where $\overline{\mathbb{N}}$ is the set of non-positive integers under the usual ordering and the element 0 is a lower bound for the whole lattice. Then L has no complete, prime filters, but all its filters are completely-prime, hence by Theorem 5.1.6 it has a join-complete representation but no meet-complete representation.

Examples 5.1.9, 5.1.10 and 5.1.11 (and its dual) give us the following:

$$\mathbf{CRL} \subset \mathbf{biCRL} = \mathbf{mCRL} \cap \mathbf{jCRL} \subset \mathbf{mCRL} \neq \mathbf{jCRL} \subset \mathbf{DL} \tag{$\dagger$}$$

There is a relationship between the existence of types of complete representation and the density of sets of join- and meet-irredicibles and primes in L (recall Definition 2.3.9). We make this precise in Proposition 5.1.12 and Corollary 5.1.13 below.

Proposition 5.1.12. *Let L be a bounded, distributive lattice. Recall the definitions of $J(L)$, $M(L)$, $J_p^\infty(L)$, and $M_p^\infty(L)$ from Definition 2.3.9. Then*

1. *If the set $J(L)$ is join-dense in L then L has a meet-complete representation, dually if the set $M(L)$ is meet-dense in L then L has a join-complete representation. When L is complete then if L has a meet/join-complete representation the sets $J(L)/M(L)$ are join/meet-dense in L.*

2. *If either $J_p^\infty(L)$ is join-dense in L or $M_p^\infty(L)$ is meet-dense in L then L has a complete representation. When L is complete it is also true that whenever L has a complete representation $J_p^\infty(L)$ and $M_p^\infty(L)$ are join- and meet-dense in L respectively.*

Proof. For the first part of 1, we just take the sets of principal filters/ideals generated by the join/meet-irreducibles respectively, for the second we note that the generator of each filter/ideal must be join/meet-irreducible. For the first part of 2 we note that if we take the sets of principal filters/ideals generated by $J_p^\infty(L)$ and $M_p^\infty(L)$ respectively we obtain distinguishing sets of completely-prime filters/ideals, and for the second part the generator of each filter/ideal will be completely join/meet-prime. $\square$

We note that Proposition 5.1.12(2) is a minor extension of [110, Theorem 2], though there the term 'completely join-irreducible' is used where we would use 'completely join-prime'.

Corollary 5.1.13. *Let L be complete. Then the following are equivalent*

1. *L is completely representable,*
2. *L completely distributive and $J^\infty(L)$ is join-dense in L,*
3. *L is completely distributive and $M^\infty(L)$ is meet-dense in L.*

Proof. We show that $1 \iff 2$, and $1 \iff 3$ is similar: If L is completely representable it must be completely distributive, and by Proposition 5.1.12 the set $J_p^\infty(L)$ must be join-dense in L. Since complete primality implies complete irreducibility, $J^\infty(L)$ must be join-dense in L. Conversely, suppose L is completely distributive, let $p \in L$ be completely join-irreducible, and suppose $p \leq \bigvee X$ for $\emptyset \subset X \subseteq L$. Then $p \wedge \bigvee X = p$, so $\bigvee\{p \wedge x : x \in X\} = p$ by complete distributivity, and thus $p = x \wedge p$ for some $x \in X$. So in L the completely join-irreducibles are completely join-prime, and the result follows from Proposition 5.1.12(2).

$\square$

Note that the full converses to Proposition 5.1.12 (i.e. when L is not complete) do not hold, so e.g. in a completely representable lattice L, $J^\infty(L)$ need not be join-dense, as the following example illustrates.

Example 5.1.14. *A completely representable lattice where $J^\infty(L)$ is not join-dense.* L is the lattice with domain $(\overline{\mathbb{N}} \times \overline{\mathbb{N}}) \cup \mathbb{N}$ as shown in Figure 5.2, where $\overline{\mathbb{N}}$ is the set of non-positive integers under their usual ordering and each element of $\mathbb{N}$ is less than each element of $(\overline{\mathbb{N}} \times \overline{\mathbb{N}})$. For $-n \in \overline{\mathbb{N}}$, the set $[-n, 0] \times \overline{\mathbb{N}}$ is a complete, completely-prime filter (with no infimum) and

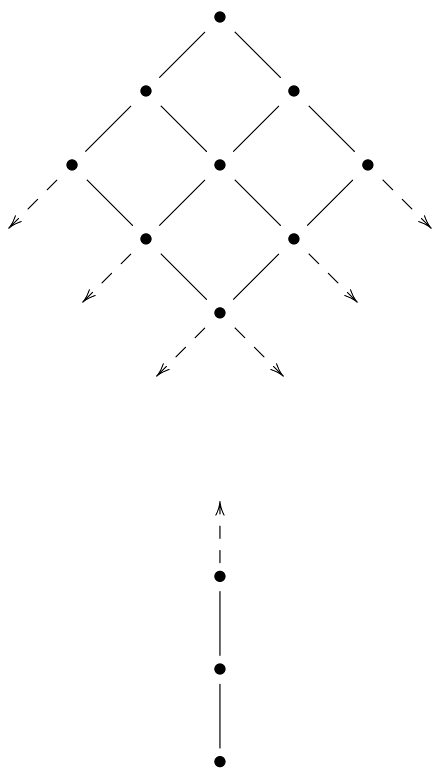


Figure 5.2: The lattice $(\overline{\mathbb{N}} \times \overline{\mathbb{N}}) \cup \mathbb{N}$

similarly $\overline{\mathbb{N}} \times [-n, 0]$ is also complete and completely-prime. Hence L has a distinguishing set of complete, completely-prime filters but $J^{\infty}(L) = J(L) = \mathbb{N}$ is not join dense in L.

5.1.2 HSP, elementarity and pseudoelementarity

Since a subalgebra of an atomic Boolean algebra need not be atomic we know that none of the classes in (†) is closed under subalgebras, and thus cannot be varieties, or even quasi-varieties. Similarly, given an atomic Boolean algebra B we can define an equivalence relation R on B by $xRy \iff |\{a \in At(B) \colon a \leq x\} \triangle \{a \in At(B) \colon a \leq y\}| < |\omega|$ (where $\triangle$ denotes the symmetric difference). So $xRy \iff (\{a \in At(B) \colon a \leq x\} \cup \{a \in At(B) \colon a \leq y\}) \setminus (\{a \in At(B) \colon a \leq x\} \cap \{a \in At(B) \colon a \leq y\})$ is finite. It can easily be shown that R is a congruence, and in the case where B is the complete, atomic Boolean algebra on ω generators the resulting $\frac{B}{R}$ is isomorphic to the countable atomless Boolean algebra, and thus none of classes in (†) can be closed under homomorphic images. We can say something positive about closure under direct products, which we express in the following lemma.

Lemma 5.1.15. *The classes in* (†) *are all closed under taking direct products.*

Proof. We do the proof for **mCRL**, the others are similar. Suppose $\{L_i\}_I$ is a non-empty family of lattices in **mCRL**. Let $f \neq g \in \prod_I L_i$. Then we can choose $j \in I$ with $f(j) \neq g(j)$, and by the assumption of meet-complete representability there is a complete, prime filter γ distinguishing $f(j)$ and $g(i)$. Define sets $S_i \subseteq L_i$ by $S_j = \gamma$ and $S_i = L_i$ for all $i \neq j$. Then $S = \prod_I S_i$ is a complete, prime filter distinguishing f and g. □

As 'being atomic' is a first order property for Boolean algebras, it follows immediately from Corollary 5.1.8 that the class of completely representable Boolean algebras is elementary. The aim here is to investigate to what extent similar results hold for the classes in (†). Our first result is negative.

Theorem 5.1.16. ***CRL*** *is not closed under elementary equivalence.*

Proof. The lattice $L = [0,1] \subseteq \mathbb{R}$ from Example 5.1.9 is not in **CRL**, however the lattice $L' = [0,1] \cap \mathbb{Q}$ *is* in **CRL** as for every irrational r the set $\{a \in L' \colon a > r\}$ is a complete, completely-prime filter. L and L' are elementarily equivalent as $\mathbb{R}$ and $\mathbb{Q}$ are. □

We can, however, show that all the classes in (†) are at least *pseudoelementary*. In particular we shall demonstrate that **mCRL** is precisely the first order reduct of the class of models of a theory in two-sorted FOL, and thus is pseudoelementary (the proof can be readily adapted for the other classes). We proceed as follows:

Let $\mathscr{L} = \{+, \cdot, 0, 1\}$ be the language of bounded, distributive lattices in FOL. Define the two-sorted language $\mathscr{L}^+ = \mathscr{L} \cup \{\in\}$, where $\in$ is a binary predicate whose first argument takes variables of the $\mathbb{A}$ sort and whose second takes variables of the $\mathbb{S}$ sort. Let the original functions of $\mathscr{L}$ be wholly $\mathbb{A}$-sorted in $\mathscr{L}^+$ (the $\mathbb{A}$ sort is meant to represent lattice elements and the $\mathbb{S}$ sort sets of these elements). Define binary $\mathbb{A}$-sorted predicates $\leq$ and $\geq$, and binary $\mathbb{S}$-sorted predicate $\subseteq$ in $\mathscr{L}^+$ in the obvious way. For simplicity we will write $x^{\mathbb{A}} \in s^{\mathbb{S}}$ for $\in (x^{\mathbb{A}}, s^{\mathbb{S}})$, and similar for $\leq$, $\subseteq$ etc.

Define additional predicates P, I and C as follows:

- $P(s^{\mathbb{S}})$ if and only if each of the following properties hold:
    1. $\forall\, x^{\mathbb{A}}\, y^{\mathbb{A}} \Big(\big((x^{\mathbb{A}} \in s^{\mathbb{S}}) \wedge (y^{\mathbb{A}} \geq x^{\mathbb{A}}) \big) \to (y^{\mathbb{A}} \in s^{\mathbb{S}}) \Big)$
    2. $\forall\, x^{\mathbb{A}}\, y^{\mathbb{A}} \Big(\big((x^{\mathbb{A}} \in s^{\mathbb{S}}) \wedge (y^{\mathbb{A}} \in s^{\mathbb{S}}) \big) \to (x^{\mathbb{A}} \cdot y^{\mathbb{A}} \in s^{\mathbb{S}}) \Big)$
    3. $\forall\, x^{\mathbb{A}}\, y^{\mathbb{A}} \Big((x^{\mathbb{A}} + y^{\mathbb{A}} \in s^{\mathbb{S}}) \to \big((x^{\mathbb{A}} \in s^{\mathbb{S}}) \vee (y^{\mathbb{A}} \in s^{\mathbb{S}}) \big) \Big)$
    4. $\forall\, s^{\mathbb{S}}\, \exists\, x^{\mathbb{A}} (x^{\mathbb{A}} \in s^{\mathbb{S}})$

  P is meant to capture the property of being a prime filter.

- $I(x^{\mathbb{A}}, s^{\mathbb{S}})$ if and only if

  $$\forall\, y^{\mathbb{A}} \left((y^{\mathbb{A}} \in s^{\mathbb{S}}) \to (x^{\mathbb{A}} \leq y^{\mathbb{A}})\right) \land \forall\, z^{\mathbb{A}} \left(((y^{\mathbb{A}} \in s^{\mathbb{S}}) \to (z^{\mathbb{A}} \leq y^{\mathbb{A}})) \to (z^{\mathbb{A}} \leq x^{\mathbb{A}})\right).$$

  I corresponds to the notion of an element being the infimum of a set.

- $C(s^{\mathbb{S}})$ if and only if $\forall\, t^{\mathbb{S}}\, \forall\, x^{\mathbb{A}} \left(((t^{\mathbb{S}} \subseteq s^{\mathbb{S}}) \land I(x^{\mathbb{A}}, t^{\mathbb{S}})) \to (x^{\mathbb{A}} \in s^{\mathbb{S}})\right)$, so C specifies a limited form of completeness.

Now, let T be the $\mathscr{L}$ theory of bounded, distributive lattices. Define T^+ as the natural translation of T into the language $\mathscr{L}^+$ plus the following additional axioms:

I. $\forall\, x^{\mathbb{A}}\, y^{\mathbb{A}} \left(x^{\mathbb{A}} \neq y^{\mathbb{A}} \to \exists\, s^{\mathbb{S}} \left((P(s^{\mathbb{S}}) \land C(s^{\mathbb{S}})) \land (((x^{\mathbb{A}} \in s^{\mathbb{S}}) \land (y^{\mathbb{A}} \notin s^{\mathbb{S}})) \lor ((y^{\mathbb{A}} \in s^{\mathbb{S}}) \land (x^{\mathbb{A}} \notin s^{\mathbb{S}})))\right)\right)$

II. $\forall\, x^{\mathbb{A}}\, \exists\, s^{\mathbb{S}}\, \forall\, y^{\mathbb{A}} \left((y^{\mathbb{A}} > x^{\mathbb{A}}) \leftrightarrow (y \in s^{\mathbb{S}})\right)$

III. $\forall\, s^{\mathbb{S}}\, t^{\mathbb{S}}\, \exists\, u^{\mathbb{S}}\, \forall\, x^{\mathbb{A}} \left(((x^{\mathbb{A}} \in s^{\mathbb{S}}) \land (x^{\mathbb{A}} \in t^{\mathbb{S}})) \leftrightarrow (x^{\mathbb{A}} \in u^{\mathbb{S}})\right)$

The first of these axioms forces the $\mathbb{S}$ sort into providing a distinguishing set of 'complete' (with respect to $\mathbb{S}$) prime filters, and the second and third force the existence of sufficiently many elements of $\mathbb{S}$ that this notion of completeness is equivalent to actual completeness, as the lemma below demonstrates.

Lemma 5.1.17. *The class $\{M^{\mathbb{A}} \restriction_{\mathscr{L}} : M \models T^+\}$ of $\mathscr{L}$-reducts of models of T^+ is precisely the class of meet-completely representable bounded, distributive lattices.*

Proof. Clearly if L is in **mCRL** its elements satisfy T, and $(L, \mathscr{P}(L), \in)$ satisfy T^+, where $\in$ is ordinary set membership. Conversely, if $A = M^{\mathbb{A}} \restriction_{\mathscr{L}}$ for some model M of T^+ then by axiom I of T^+ the (interpretation of) the $\in$ predicate naturally defines a distinguishing set K of prime filters of A. We claim that each prime filter in K is complete. For the claim, let $p \in K$ and $s \subseteq p$ with $x = \inf(s)$. We must show that $x \in p$. If $x \in s$ then this is immediate, so we suppose not: $x \notin s$. We consider the following cases:

1. $x = \inf\{y : y > x\}$: then $s \subseteq \{y : y > x\} \cap p \subseteq \{y : y > x\}$ so $x = \inf(s) \geq \inf(\{y : y > x\} \cap p) \geq x$ and thus $\inf(\{y : y > x\} \cap p) = x$, but clearly $\{y : y > x\} \cap p \subseteq p$ and by axioms II and III of T^+ also corresponds to an element of the $\mathbb{S}$ sort. Therefore, by definition of the predicate C we have $x \in p$, as required.

2. $x \neq \inf\{y : y > x\}$: Let z be a lower bound for $\{y : y > x\}$, suppose $z \not\leq x$. Then $x \lor z$ is a lower bound for $\{y : y > x\}$ and is contained in $\{y : y > x\}$. In light of this assume

wlog that $\inf\{y\colon y > x\} = z > x$. Then, as $x = \inf(s)$, we have $s \subseteq \{y\colon y > x\}$ and thus s has z as a lower bound, but this a contradiction as $x < z$, so this case cannot arise.

We deduce that $x \in p$, so p is complete, as claimed. Since T^+ demands A be a bounded, distributive lattice we have $A \in$ **mCRL**, by Theorem 5.1.6. □

By Lemma 5.1.17 and Proposition 2.1.9 we have:

Theorem 5.1.18. ***mCRL*** *is pseudoelementary.*

It is not difficult to see how analogous results can also be proved for **jCRL**, **biCRL** and **CRL** using a similar method.

5.1.3 Elementarity

In view of Theorem 2.3.21, Proposition 2.3.22, and the material in the preceding section, since **CRL** is pseudoelementary, and closed under isomorphism, but is not elementary, it cannot be closed under ultraroots. **mCRL**, **jCRL** and **biCRL** will be elementary if and only if they are closed under ultraroots. Note that **mCRL** is elementary iff **jCRL** is elementary (by duality), and therefore **mCRL** is elementary $\implies$ **biCRL** is elementary (as **biCRL** $=$ **mCRL** $\cap$ **jCRL**). It is not known which, if any, of **biCRL**, **mCRL** and **jCRL** are closed under ultraroots but it is possible to state some conditions on a lattice L which must necessarily hold if $L \notin X$ but an ultrapower of L belongs to X (where $X =$ **biCRL**, **mCRL** or **jCRL**).

First of all in order for the ultraproduct $\prod_U L$ to be meet-completely representable L must be $\vee(\bigwedge)$-distributive, i.e. for $a \in L,\ S \subseteq L$ if both sides of the equation below are defined then they are equal

$$a \vee \bigwedge S = \bigwedge_{s\in S}(a \vee s)$$

as we shall see in the next proposition. Note that the converse to this is false as, for example, every Boolean algebra is $\vee(\bigwedge)$-distributive (see e.g. [115, Theorem 5.13] for a proof) but not necessarily atomic, so not necessarily meet-completely representable by Corollary 5.1.8. We will use the following notation and lemma:

- For $a \in L$ define $\bar{a} \in \prod_I L$ by $\bar{a}(i) = a$ for all $i \in I$.
- Fix some ultrafilter U over I. For $x \in \prod_I L$ we write $[x]$ for $\{y \in \prod_I L\colon : \{i\colon x(i) = y(i)\} \in U\}$.
- For $S \subseteq L$ define $S^* = \{[x] \in \prod_U L\colon \{i \in I\colon x(i) \in S\} \in U\}$.
- For $T \subseteq \prod_U L$ define $T_* = \{a \in L\colon [\bar{a}] \in T\}$.

Lemma 5.1.19. *Let $S \subseteq L$ and suppose $\bigwedge S$ exists in L. Then $\bigwedge(S^*)$ exists in $\prod_U L$ and equals $[\overline{\bigwedge S}]$.*

Proof. This can be proved by defining an additional predicate 'S' in the language of lattices meant to correspond to 'being an element of the set S', the result then following easily from Łoś' theorem. An alternative algebraic proof is as follows: Clearly $[\overline{\bigwedge S}]$ is a lower bound for S^*. Suppose $[z]$ is another such lower bound and $[z] \not\leq [\overline{\bigwedge S}]$. Then $\{i \in I \colon z(i) \not\leq \bigwedge S\} \in U$, so $\{i \in I \colon \exists s_i \in S \text{ with } z(i) \not\leq s_i\} \in U, = u$ say (as $\bigwedge S$ is the greatest lower bound of S). Define x by $x(i) = s_i$ for $i \in u$ and $x(i) = \bigwedge S$ otherwise. Then $[x] \in S^*$ but $[z] \not\leq [x]$, but this contradicts the assertion that $[z]$ is a lower bound. □

Corollary 5.1.20. *The class of $\vee(\bigwedge)$-distributive bounded lattices is closed under ultraroots.*

Proof. By Lemma 5.1.19 if there is some $A \cup \{b\} \subseteq L$ with $b \vee \bigwedge A \neq \bigwedge(b \vee A)$ then $\bigwedge A^* \vee [\bar{b}] = [\bar{\bigwedge A}] \vee [\bar{b}] = [\overline{(\bigwedge A) \vee b}] \neq [\overline{\bigwedge(A \vee b)}] = \bigwedge(A^* \vee [\bar{b}])$, so if L is not $\vee(\bigwedge)$-distributive then neither is $\prod_U L$. □

Proposition 5.1.21. *If $\prod_U L$ has a meet-complete representation then L is $\vee(\bigwedge)$-distributive.*

Proof. This follows from Corollary 5.1.20 and the fact that when $\prod_U L$ is in **mCRL** it inherits $\vee(\bigwedge)$-distributivity from its representation. □

By duality a similar result holds for **jCRL**, and hence for **biCRL**. In order for $\prod_U L$ to be in **mCRL** but L not to be it turns out L must satisfy an infinite density property, which we make precise in the next proposition.

Proposition 5.1.22. *If $\prod_U L$ has a meet-complete representation but L does not then there is a pair $x < y$ such that for every pair $a < b \in [x, y]$ there is some c with $a < c < b$.*

Proof. If L is not in **mCRL** then there is a pair $x, y, \in L$ that cannot be distinguished by a complete, prime filter. Wlog assume $x < y$. Since $\prod_U L$ is in **mCRL**, for each pair $a < b \in [x, y]$ there is a complete, prime filter γ distinguishing $[\bar{a}]$ and $[\bar{b}]$. It's easy to show that γ_* is a prime filter of L with $b \in \gamma_*$ and $a \notin \gamma_*$ (and thus $y \in \gamma_*$ and $x \notin \gamma_*$). Let $a < b$ and $(a, b) = \emptyset$ and suppose $S \subseteq \gamma_*$. Then for each $[z] \in S^*$ we have must have $[z] \vee [\bar{a}] = [\bar{b}]$, and thus by primality $S^* \subseteq \gamma$. So by Lemma 5.1.19 $\bigwedge S \in \gamma_*$, and so γ_* is complete, which is a contradiction as we assumed x and y could not be distinguished by a complete, prime filter. □

Again by duality the same result holds for join-complete representations. Note that if we could find a counterexample $(L, \prod_U L)$ where $\prod_U L \in$ **mCRL**, $L \notin$ **mCRL**, we could restrict to the sublattice bounded by x and y, so we lose nothing by assuming that x and y are the lower and upper bounds respectively, and that the whole lattice therefore has this density property.

We have seen that the class of completely representable Boolean algebras is atomic (indeed finitely axiomatisable) and that the class **CRL** of completely representable lattices is not. We are left with the following conjecture.

Conjecture 5.1.23. *None of the classes* ***jCRL***, ***mCRL***, ***biCRL*** *is elementary.*

An attempt has been made at this by the author. In particular it was hoped that using Propositions 5.1.21 and 5.1.22 as a guide a lattice L could be constructed with $L \notin$ **mCRL** and $\prod_U L \in$ **mCRL** for some ultrafilter U. While this may yet prove to be a profitable approach, so far there has been little success regarding the lattice problem. We have been able solve an analogous problem for posets using this method, which we demonstrate in Section 5.2.

5.2 Complete representations for posets

Given the (elementary) class $\mathbf{RP}$ of representable posets and the class $\mathbf{RS}$ of representable (meet) semilattices (which is also elementary by Theorem 2.2 of [6]) we are interested in the subclasses containing those posets and semilattices which have representations preserving arbitrary meets/and or joins whenever they occur. Following Section 5.1.1 we make the following definitions.

Definition 5.2.1. We say $h\colon P \to \wp(X)$ is a *meet-complete representation* if h is a representation and $h(\bigwedge S) = \bigcap h[S]$ whenever $\emptyset \subset S \subseteq P$ and $\bigwedge S$ is defined in P. Similarly we say $h\colon P \to \wp(X)$ is a *join-complete representation* if it is a representation and $h(\bigvee S) = \bigcup h[S]$ whenever $\emptyset \subset S \subseteq P$ and $\bigvee S$ is defined in P. We say h is a *complete representation* if it is both meet- and join-complete. We say P is *meet-completely representable* if it has a meet-complete representation, and we make similar definitions for join-complete and complete representability.

Examples 5.1.9, 5.1.10, and 5.1.11 provided posets (distributive lattices and Boolean algebras to be precise) that are, respectively:

- meet-completely representable and join-completely representable but not completely representable,
- neither join- nor meet-completely representable,
- join-completely representable but not meet-completely representable.

The following definitions generalize Definitions 5.1.3 and 5.1.4 respectively.

Definition 5.2.2 (Complete filter/weak-filter)**.** A filter, or a weak-filter, $F \subseteq P$ is complete if for all $S \subseteq F$ we have $\bigwedge S \in F$ whenever $\bigwedge S$ is defined in P.

Definition 5.2.3 (Completely-prime filter/weak-filter)**.** A filter, or a weak-filter, $F \subseteq P$ is completely-prime if for all $\emptyset \subset S \subseteq P$ we have $S \cap F \neq \emptyset$ whenever $\bigvee S$ is defined in P and $\bigvee S \in F$.

Using these definitions we obtain the following generalization of Theorem 5.1.6.

Theorem 5.2.4. *Let P be a poset. Then:*

1. *P has a meet-complete representation if and only if the set of complete, prime, weak-filters of P is separating over P,*
2. *P has a join-complete representation if and only if the set of completely-prime, weak-filters of P is separating over P,*
3. *P has a complete representation if and only if the set of complete, completely-prime, weak-filters of P is separating over P,*

Proof. We prove 1), the rest is similar: For $\Rightarrow$ suppose $h\colon P \to \wp(X)$ is a meet-complete representation for P. We prove that for all $x \in X$ the set $h^{-1}[x] = \{p \in P : x \in h(p)\}$ is a complete, prime, weak-filter. Since $h^{-1}[x]$ is clearly a prime, weak-filter suppose $S \subseteq h^{-1}[x]$ and $\bigwedge S$ exists in P. Then $x \in \bigcap h[S]$ by definition of $h^{-1}[x]$, so by completeness of h we have $x \in h(\bigwedge S)$, which is equivalent to saying $\bigwedge S \in h^{-1}[x]$. Since $\{h^{-1}[x] : x \in X\}$ is separating over P we are done. Conversely, let X be the separating set of complete, prime, weak-filters of P and define $h\colon P \to \wp(X)$ by $h(p) = \{x \in X : p \in x\}$. Then h is a representation, and furthermore, since each $x \in X$ is a complete, prime, weak-filter, if $S \subseteq P$ and $\bigwedge S$ exists then for all $x \in X$,

$$\begin{aligned} x \in h(\bigwedge S) &\iff \bigwedge S \in x \\ &\iff S \subseteq x \\ &\iff x \in \bigcap h[S] \end{aligned}$$

so $h(\bigwedge S) = \bigcap h[S]$ as required. □

Theorem 5.2.4 dualizes to ideals in an entirely predictable way.

Lemma 5.2.5. *The complement γ^c of a complete, prime, weak-filter γ of P is a completely-prime, weak-filter in the order dual P^δ. Similarly the complement of a completely-prime, weak-filter is a complete, prime, weak-filter in the order dual.*

Proof. This is straightforward. □

Corollary 5.2.6. *$\gamma \subset P$ is a complete, prime, weak-filter $\iff$ γ^c is a completely-prime, weak-ideal.*

Corollary 5.2.7. *P has a meet-complete representation if and only if P^δ has a join-complete representation. Similarly P has a join-complete representation if and only if P^δ has a meet-complete representation*

Proof. This follows easily from Theorem 5.2.4 and Lemma 5.2.5. □

In view of Corollary 5.2.6 we can easily formulate a version of Theorem 5.2.4 using dual-separation and weak-ideals.

To close this section we note that from Theorem 5.2.4 we can obtain as a corollary the main result of [93], which we state and prove as Theorem 5.2.8 below.

Theorem 5.2.8. *Let P be a poset where*

1. *every non-empty chain has an infimum, and*
2. *for all $a, b \in P$, if $\{a, b\}$ has an upper bound then $a \wedge b$ exists in P.*

Then P has a complete representation if and only if whenever $p, q \in P$ with $p \not\leq q$ there is $r \in J^\infty_p(P)$ with $r \leq p$ and $r \not\leq q$ (recall Definition 2.3.9).

Proof. If P has a complete representation then given $p \not\leq q \in P$, by Theorem 5.2.4 there is a complete, completely-prime weak-filter $F \subseteq P$ with $p \in F$ and $q \notin F$. Let $X = p^\downarrow \cap F$. Then using the condition that non-empty chains in P have an infimum and the completeness of F we apply the dual of Zorn's lemma to say that X contains minimal elements. Furthermore, the minimal element of X is unique, and is in fact the infimum of X, as by the second condition on P we must have $x \wedge y \in X$ whenever $x, y \in X$.

Defining $r = \bigwedge X$, clearly $r \leq p$ and $r \not\leq q$, and if $r \leq \bigvee S$ then by complete primality of F we have $s \in F$ for some $s \in S$. Moreover we have $s \geq r$, as $r \wedge s$ must exist and be in X, $r \wedge s = r$ by definition of r. Thus r is completely join-prime.

The converse follows trivially from Theorem 5.2.4 and the fact that if r is completely join-prime then $r^\uparrow$ is a complete, completely-prime weak-filter. □

5.2.1 Pseudoelementarity of complete representation in the poset and semilattice settings

We will show that the class of meet-completely representable posets is pseudoelementary. The basic strategy is similar to that used in Section 5.1.2, though there are some significant differences. We note that it is straightforward to adapt this proof to give pseudoelementarity for the classes of join-complete and complete posets, and also for the various classes of completely representable semilattices.

Recall Definition 2.1.8 and Proposition 2.1.9 regarding pseudoelementarity. Now, define $\mathscr{L}$ to be the language of posets in first order logic, so $\mathscr{L} = \{\leq\}$, and define $\mathscr{L}^+$ to be the two-sorted language $\{\leq, \in, \mathbf{fin}\}$, with sorts $\mathbb{A}$ and $\mathbb{S}$, and $\leq$ and $\in$ being binary, and $\mathbf{fin}$ being unary. In $\mathscr{L}^+$ the first argument of $\in$ is $\mathbb{A}$ sorted and the second is $\mathbb{S}$ sorted, while $\leq$ takes both its arguments from $\mathbb{A}$, and $\mathbf{fin}$ takes an $\mathbb{S}$ sorted argument. Define additional predicates as follows:

- $\subseteq(s^{\mathbb{S}}, t^{\mathbb{S}}) \iff \forall\, a^{\mathbb{A}}((a^{\mathbb{A}} \in s^{\mathbb{S}}) \to (a^{\mathbb{A}} \in t^{\mathbb{S}}))$
- $\mathbf{glb}(a^{\mathbb{A}}, s^{\mathbb{S}}) \iff \Big(\forall\, b^{\mathbb{A}}\left((b^{\mathbb{A}} \in s^{\mathbb{S}}) \to (a^{\mathbb{A}} \leq b^{\mathbb{A}})\right) \wedge \forall\, c^{\mathbb{A}}\left(\forall\, d^{\mathbb{A}}((d^{\mathbb{A}} \in s^{\mathbb{S}}) \to (c^{\mathbb{A}} \leq d^{\mathbb{A}})) \to (c^{\mathbb{A}} \leq a^{\mathbb{A}})\right)\Big)$
- $\mathbf{lub}(a^{\mathbb{A}}, s^{\mathbb{S}}) \iff \Big(\forall\, b^{\mathbb{A}}\left((b^{\mathbb{A}} \in s^{\mathbb{S}}) \to (b^{\mathbb{A}} \leq a^{\mathbb{A}})\right) \wedge \forall\, c^{\mathbb{A}}\left(\forall\, d^{\mathbb{A}}((d^{\mathbb{A}} \in s^{\mathbb{S}}) \to (d^{\mathbb{A}} \leq c^{\mathbb{A}})) \to (a^{\mathbb{A}} \leq c^{\mathbb{A}})\right)\Big)$
- $\mathbf{F}(s^{\mathbb{S}})$ if and only if each of the following holds:
  1. $\forall\, a^{\mathbb{A}}\, b^{\mathbb{A}}\Big(\left((a^{\mathbb{A}} \in s^{\mathbb{S}}) \wedge (a^{\mathbb{A}} \leq b^{\mathbb{A}})\right) \to (b^{\mathbb{A}} \in s^{\mathbb{S}})\Big)$
  2. $\forall\, a^{\mathbb{A}}\, t^{\mathbb{S}}\Big(\left((t^{\mathbb{S}} \subseteq s^{\mathbb{S}}) \wedge \mathbf{fin}(t^{\mathbb{S}})\right) \to \left(\mathbf{glb}(t^{\mathbb{S}}, a^{\mathbb{A}}) \to (a^{\mathbb{A}} \in s^{\mathbb{S}})\right)\Big)$
  3. $\forall\, a^{\mathbb{A}}\, t^{\mathbb{S}}\Big(\left((\mathbf{fin}(t^{\mathbb{S}}) \wedge \mathbf{lub}(a^{\mathbb{A}}, t^{\mathbb{S}}) \wedge (a^{\mathbb{A}} \in s^{\mathbb{S}})\right) \to \exists\, b^{\mathbb{A}}\left((b^{\mathbb{A}} \in t^{\mathbb{S}}) \wedge (b^{\mathbb{A}} \in s^{\mathbb{S}})\right)\Big)$
  4. $\forall\, s^{\mathbb{S}}\, \exists\, a^{\mathbb{A}}(a^{\mathbb{A}} \in s^{\mathbb{S}})$
- $\mathbf{C}(s^{\mathbb{S}}) \iff \forall\, t^{\mathbb{S}}\, a^{\mathbb{A}}\Big(\left((t^{\mathbb{S}} \subseteq s^{\mathbb{S}}) \wedge \mathbf{glb}(a^{\mathbb{A}}, t^{\mathbb{S}})\right) \to (a^{\mathbb{A}} \in s^{\mathbb{S}})\Big)$

So $\subseteq$ corresponds to set inclusion, **glb** and **lub** to the concepts of greatest lower bound and least upper bound respectively, and **C** to a limited form of completeness of a set under existing infima. We aim to define a theory so that **F** corresponds to the idea of a prime, weak-filter.

Let T be the $\mathscr{L}$ theory defining partially ordered sets and define an $\mathscr{L}^+$ extension T^+ of T by adding the following axioms:

T_1^+: $\forall\, a^{\mathbb{A}}\, b^{\mathbb{A}}\Big((a^{\mathbb{A}} \not\leq b^{\mathbb{A}}) \to \exists\, s^{\mathbb{S}}\left(\mathbf{F}(s^{\mathbb{S}}) \wedge \mathbf{C}(s^{\mathbb{S}}) \wedge (a^{\mathbb{A}} \in s^{\mathbb{S}}) \wedge (b^{\mathbb{A}} \notin s^{\mathbb{S}})\right)\Big)$

T_2^+: $\forall\, a^{\mathbb{A}}\, \exists\, s^{\mathbb{S}}\, \forall\, b^{\mathbb{A}}\Big((a^{\mathbb{A}} < b^{\mathbb{A}}) \leftrightarrow (b^{\mathbb{A}} \in s^{\mathbb{S}})\Big)$

T_3^+: $\forall\, s^{\mathbb{S}}\, t^{\mathbb{S}}\, \exists\, u^{\mathbb{S}}\, \forall\, a^{\mathbb{A}}\Big(\left((a^{\mathbb{A}} \in s^{\mathbb{S}}) \wedge (a^{\mathbb{A}} \in t^{\mathbb{S}})\right) \leftrightarrow (a^{\mathbb{A}} \in u^{\mathbb{S}})\Big)$

and also adding for each $n \in \omega$ the axiom defined by

F_n: $\forall a_1^{\mathbb{A}} ... \forall a_n^{\mathbb{A}} \exists\, s^{\mathbb{S}} \left(\bigwedge_{i=1}^{n} (a_i^{\mathbb{A}} \in s^{\mathbb{S}}) \wedge \mathbf{fin}(s^{\mathbb{S}}) \wedge \forall\, b^{\mathbb{A}} (b^{\mathbb{A}} \in s^{\mathbb{S}} \to \bigvee_{i=1}^{n} (a_i^{\mathbb{A}} = b^{\mathbb{A}}))\right)$.

The first of these axioms when taken with $\{F_n : n \in \omega\}$ forces its models to be separated by prime, weak-filters that have the limited completeness property induced by the **C** predicate ($\{F_n : n \in \omega\}$ together demand that the **fin** predicate 'sees' all the sets it's supposed to, i.e. whenever a finite set of 'algebra' elements exists it defines a 'finite' set). The role of T_2^+ and T_3^+ is to ensure that in models of T^+ sufficiently many $\mathbb{S}$ sorted elements must be represented for '**C**-completeness' to guarantee separation by actual complete, prime, weak-filters, in a way that is made precise in the following lemma.

Lemma 5.2.9. *The class* $\{M^{\mathbb{A}} \restriction_{\mathscr{L}} : M \models T^+\}$ *of* $\mathscr{L}$*-reducts of models of* T^+ *is precisely the class of meet-completely representable posets.*

Proof. If P is a meet-completely representable poset we can use $(P, \wp(P), \in, finite)$ to model T^+ by Theorem 5.2.4 ($finite$ holds of a set X if and only if X is finite). Conversely if $A = M^{\mathbb{A}} \restriction_{\mathscr{L}}$ for some model M of T^+ then it is clearly a poset and by T_1^+ and $\{F_n : n \in \omega\}$ the interpretation of the $\in$ predicate can be used to naturally define a separating set K of prime, weak-filters of A. Let $S \in K$, let $T \subseteq S$ and suppose $x = \bigwedge T$ exists in A.

If $x = \bigwedge(\{y \in A : y > x\})$ then $T \subseteq \{y \in A : y > x\} \cap S \implies x = \bigwedge T \geq z$ for all lower bounds z of $\{y \in A : y > x\} \cap S$, moreover x *is* a lower bound for $\{y \in A : y > x\} \cap S$ so $x = \bigwedge(\{y \in A : y > x\} \cap S)$. By axioms T_2^+ and T_3^+, and the definitions of the set K and the predicate **C** we must have $x \in S$. If $T \not\subseteq \{y \in A : y > x\} \cap S$ then $x \in T$, and thus in $x \in S$ and we are done.

Alternatively suppose $x \neq \bigwedge(\{y \in A : y > x\})$ and $x \notin T$. Then $T \subseteq \{y \in A : y > x\}$. As x is clearly a lower bound for $\{y \in A : y > x\}$ if it is not the greatest lower bound there must be $z \in A$ with $z \leq y$ for all $y > x$ but $z \not\leq x$, but then if z would also be a lower bound for T, which would contradict the assumption that $x = \bigwedge T$. We conclude that S is complete, and in light of Theorem 5.2.4 that A is a meet-completely representable poset. □

We state the results of this section as a theorem.

Theorem 5.2.10. *The class of meet-completely representable posets is pseudoelementary.*

5.2.2 Failure of elementarity

We saw in Theorem 5.1.16 that **CRL** is not closed under elementary equivalence, and this result transfers into the poset and semilattice settings implying that the classes of completely representable posets and join/meet-semilattices are not elementary. Unlike in the distributive lattice case we *can* give a firm answer to the question of whether the class of posets admitting representation complete with respect to joins is elementary (likewise meets, by duality). The answer turns out to be negative, and we shall prove this here by constructing a counterexample to closure under ultraroots, thus obtaining our conclusion via an application of Theorem 2.3.21.

Construction of the counterexample

Let X_n be countable subsets of $(0,1] \subset \mathbb{R}$ for all $n \in \omega^+ = \{0,1,2,...,\omega\}$ with the following properties:

P1. for all $n < m \in \omega$, $X_n \cap X_m = \{1\}$,

P2. for all $n \in \omega^+$, for all $x \in (0,1]$, there is $y \in X_n$ with $y < x$,

P3. for all $n \in \omega$, $X_n \cap X_\omega = \emptyset$,

P4. for all $a < b \in \bigcup_{m\in\omega} X_m$, and for all $n \in \omega^+$, there is $c \in X_n$ with $a < c < b$.

It's not immediately obvious that such a collection exists, however, if $\{p_0, p_1, p_2, ...\}$ is an enumeration of the primes $\{2,3,5,...\}$ we can for example define $X_\omega = \mathbb{Q} \cap (0,1)$, and $X_n = \{r \in (0,1) : r = q.\sqrt{p_n} \text{ for some } q \in \mathbb{Q}\} \cup \{1\}$ for $n \in \omega$. It's easy to see P1, P2, and P3 hold for these choices, and P4 follows from Lemma 5.2.11 below.

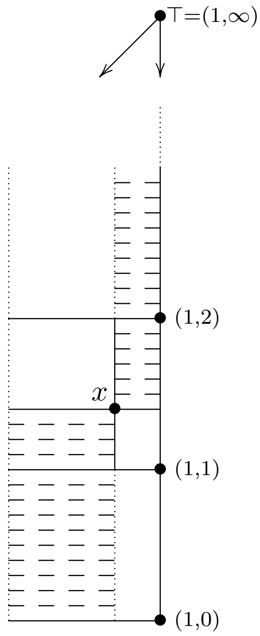
Lemma 5.2.11. *Let $r_1 < r_2 \in \mathbb{R}$, and let $s \in \mathbb{R}\setminus\{0\}$. Then there is $q \in \mathbb{Q}$ with $r_1 < qs < r_2$.*

Proof. Since $s \neq 0$ we have $r_1/s < r_2/s$, and by density of $\mathbb{Q}$ in $\mathbb{R}$ we have $r_1/s < q < r_2/s$, and thus $r_1 < qs, r_2$, for some $q \in \mathbb{Q}$. □

Let $Y = [0,\infty) \cap \mathbb{Q}$ and define a function $\Delta\colon Y \to \mathbb{N}$ by $\Delta(y) = n_y$, where $n_y \leq y < n_y + 1$. Define functions $\bar{\mathrm{x}}, \bar{\mathrm{y}}\colon (0,1] \times Y \to \mathbb{Q}$ by $\bar{\mathrm{x}}(x,y) = x$ and $\bar{\mathrm{y}}(x,y) = y$. Define P' to be the subset of $(0,1] \times Y$ composed of all pairs (x,y) with $x \in X_{\Delta(y)}$. Define an ordering on P' by $a \leq b \iff \bar{\mathrm{x}}(a) \leq \bar{\mathrm{x}}(b)$ and $\bar{\mathrm{y}}(a) \leq \bar{\mathrm{y}}(b)$. Note that whenever $\Delta(\bar{\mathrm{y}}(a)) = \Delta(\bar{\mathrm{y}}(b))$ we can define the meet and join of a and b as the infimum and supremum of $\{a,b\}$ respectively. When $\Delta(\bar{\mathrm{y}}(a)) \neq \Delta(\bar{\mathrm{y}}(b))$ the meet and join of a and b are defined if and only if either $a \leq b$ or $b \leq a$. Define P to be P' equipped with a top element $\top$ (see Figure 5.3). We extend our functions $\bar{\mathrm{x}}$ and $\bar{\mathrm{y}}$ to $P \to \mathbb{Q} \cup \{\infty\}$ by defining $\bar{\mathrm{x}}(1) = 1$, and $\bar{\mathrm{y}}(\top) = \infty$.

Proposition 5.2.12. *Let $p, q \in P$ with $\bar{\mathrm{y}}(p) = \bar{\mathrm{y}}(q)$ and $\bar{\mathrm{x}}(p) < \bar{\mathrm{x}}(q)$. Then there is no complete, prime weak-ideal containing p but not q.*

Proof. Suppose γ is a complete, prime weak-ideal containing p but not q, we claim that for each $n \geq \Delta(\bar{\mathrm{y}}(q)) \in \omega$ there is $q_n \in \gamma$ with $\Delta(\bar{\mathrm{y}}(q_n)) = n$. We proceed by induction: since the base



Here x is a point of P, the shaded area to the upper right of x marks the set of elements (other that $\top$) greater than x in P, while the shaded area to the bottom left of x marks those elements of P that are less than x.

Figure 5.3: P is not join-completely representable but we shall see it has an ultrapower that is.

case is trivial suppose the hypothesis holds for $n = k$ and suppose wlog that $\bar{y}(q_k) \geq \bar{y}(q)$. Let $q' = (1, \bar{y}(q_k))$. Then if $r \in P$ with $\Delta(\bar{y}(r)) = k$ and $\bar{x}(r) \leq \bar{x}(q_k)$ we have $q' \wedge r \leq q_k \in \gamma$ and thus $r \in \gamma$ by primality of γ (as $q' \geq q$, and so $q' \notin \gamma$).

Let $q_{k+1} = (x, (k+1))$ for some $x \in X_{k+1}$ with $0 < x < \bar{x}(q_k)$. Then q_{k+1} is the supremum of $\{r \in P : \Delta(\bar{y}(r)) = k \text{ and } \bar{x}(r) < x\} \subset \gamma$, and so $q_{k+1} \in \gamma$ by completeness as required.

Now, in view of the claim we have just proved γ must have $\top$ as its supremum, and so contains $\top$ and thus p, which is a contradiction. $\square$

Corollary 5.2.13. *P is not join-completely representable.*

Proof. This follows from Proposition 5.2.12 and the dual version of Theorem 5.2.4. $\square$

Now, let $U \subset \wp(\omega)$ be a non-principal ultrafilter and define $\prod_U P$ to be the ultraproduct of P over U. Define sets $u \in U$ to be *large*. We aim to prove that $\prod_U P$ is join-completely representable.

Let $[\alpha] \not\leq [\beta] \in \prod_U P$. Firstly, if $\{i \in \omega : \bar{y}(\beta(i)) < \bar{y}(\alpha(i)\} = u \in U$ we note that for each $i \in u$ we can choose $y_i \in Y$ with $\bar{y}(\beta(i)) < y_i < \bar{y}(\alpha(i))$, and $(1, y_i) > \beta(i)$ and

$(1, y_i) \not\geq \alpha(i)$. Define $[z] \in \prod_U P$ by $z(i) = (1, y_i)$ for all $i \in u$, and define $\gamma = [z]^\downarrow \subset \prod_U P$. For each $i \in u$ we know $z(i)$ is meet-prime, and by Theorem 2.3.19 this means $[z]$ will be meet-prime in $\prod_U P$, so γ is a complete, prime, weak-ideal with $[\beta] \in \gamma$ and $[\alpha] \notin \gamma$.

Suppose instead that $\{i \in \omega : \bar{\mathrm{x}}(\beta(i)) < \bar{\mathrm{x}}(\alpha(i))\} = u \in U$. We will use a family of complete, prime weak-ideals of P containing $\beta(i)$ but not $\alpha(i)$ for all $i \in u$ and use them to generate a complete, prime weak-ideal of $\prod_U P$ containing $[\beta]$ but not $[\alpha]$.

Definition 5.2.14 ($\bar{S}$). Given a family $S = \{S_i : i \in u\}$ of subsets of P indexed by some $u \in U$ we define $\bar{S} = \{[f] \in \prod_U P : \{i \in u : f(i) \in S_i\} \in U\}$.

Definition 5.2.15 (γ^k_G). Suppose $G = \{\gamma_i : i \in u\}$ is a family of prime weak-ideals of P indexed by $u \in U$, and let $k \colon \omega \to \omega$. Define $\gamma^k_G = \{[f] \in \bar{G} : \exists y \in \mathbb{Z}(\{i \in \omega : \Delta(\bar{\mathrm{y}}(f(i))) = k(i) + y)\} \in U)\}$.

Proposition 5.2.16. *Let $G = \{\gamma_i : i \in u\}$ be a family of prime weak-ideals of P where $\gamma_i = \{p \in P : \bar{\mathrm{x}}(p) < r_i\}$ for some $r_i \in X_\omega$. Let $\gamma = (\gamma^k_G)^\downarrow$ for some $k \colon \omega \to \omega$. Then γ is a complete, prime, weak-ideal of $\prod_U P$.*

Proof. Downward closure is automatic so suppose that $[f] \wedge [g]$ exists in $\prod_U P$ for some $[f], [g] \in \prod_U P$ and that $[f] \wedge [g] \in \gamma$. Then there is $[h] \in \gamma^k_G$ with $[h] \geq [f] \wedge [g]$. Since each $\gamma_i \in G$ is prime, applying Theorem 2.3.19 we assume wlog that $[f] \in \bar{G}$. Now, for each $i \in I$ we can only have $f(i) \wedge g(i)$ defined if either $f(i) \leq g(i)$ (or vice versa), or $\Delta(\bar{\mathrm{y}}(f(i))) = \Delta(\bar{\mathrm{y}}(g(i)))$, so we assume that $\{i \in I : \Delta(\bar{\mathrm{y}}(f(i))) = \Delta(\bar{\mathrm{y}}(g(i)))\} \in U$ as the alternative cases are trivial. Moreover, we can suppose that $\{i \in I : \bar{\mathrm{y}}(f(i)) < r_i$ and $\Delta(\bar{\mathrm{y}}(f(i))) \leq \Delta(\bar{\mathrm{y}}(h(i)))\} = u \in U$. For each $i \in u$ such that $\bar{\mathrm{y}}(f(i)) < r_i$ we can pick $x_i \in X_{\Delta(\bar{\mathrm{y}}(h(i))+1)}$ with $\bar{\mathrm{y}}(f(i)) < x_i < r_i$, and defining $[h']$ by $h'(i) = (x_i, \bar{\mathrm{y}}(h(i)) + 1)$ for each $i \in u$, and then $[h'] \in \gamma^k_G$, and $[f] \leq [h']$, so $[f] \in \gamma$ as required. This proof generalizes easily to arbitrary finite infima, so γ is prime.

We now prove completeness: Let $S \subseteq \gamma$ and let $[z]$ be an upper bound for S. We will show either $[z] \in \gamma$ or there is an upper bound $[z']$ for S with $[z'] \not\geq [z]$. If $[z] = [\top]$ (the top element of $\prod_U P$) define $[z']$ by $z'(i) = (1, k(i) + i)$. Then $[z']$ is an upper bound for the whole of γ^k_G that is strictly below $[\top]$ and we are done.

Suppose now that $[z] \neq [\top]$ and $[z] \notin \bar{G}$. Then $\{i \in \omega : z(i) > r_i\} = u_0 \in U$, so for each $i \in u_0$ we can choose $x_i \in X_{\Delta(\bar{\mathrm{y}}(z(i)))}$ so that $\bar{\mathrm{x}}(z(i)) > x_i > r_i$. Define $[z']$ by setting $z'(i) = (x_i, \bar{\mathrm{y}}(z(i)))$ for each $i \in u_0$. Then $[z'] > [s]$ for all $[s] \in S$, and $[z] \not\leq [z']$ as required.

Finally we suppose that $[z] \in \bar{G}$ and that $\{i \in \omega : z(i) \in \gamma_i\} = u_1 \in U$. If there is $m \in \omega$ such that $\{i \in u_1 : \Delta(\bar{\mathrm{y}}(z(i))) - k(i) < m\} = u_2 \in U$ then for each $i \in u_1 \cap u_2$ we can choose $p_i \in \gamma_i$ so that $p_i \geq z(i)$ and $\Delta(\bar{\mathrm{y}}(p_i)) = k(i) + m$, and thus defining $[z']$ by $z(i) = p_1$ for all $i \in u_1 \cap u_2$ we have $[z'] \in \gamma^k_G$ and $[z] \leq [z']$, so $[z] \in \gamma$ as required.

Alternatively, if $\{i \in u_1 : \Delta(\bar{\mathrm{y}}(z(i))) - k(i) > m\} \in U$ for all $m \in \omega$, then for each m define $v_m = \{i \in u_1 : \Delta(\bar{\mathrm{y}}(z(i)) - k(i) = m\}$, and for each $m > 0$ and each $i \in v_m$ define $p_i = (1, k(i) + m - 1)$. Let $p_i = \top$ for all $i \in v_0$ and define $[z']$ by $z'(i) = p_i$ for all $i \in u_1$. Then $[z']$ is an upper bound for γ^k_G, and for each $m > 0$ and $i \in v_m$ we have $\Delta(\bar{\mathrm{y}}(p_i)) = \Delta(k(i)+m-1) = k(i)+m-1 < k(i)+m = \Delta(\bar{\mathrm{y}}(z(i)))$, so $[z'] \not\geq [z]$ and we are done. □

Returning to $[\alpha]$ and $[\beta]$ recall that $\{i \in \omega : \bar{\mathrm{x}}(\beta(i)) < \bar{\mathrm{x}}(\alpha(i))\} = u \in U$ and that $\bar{\mathrm{y}}(\alpha(i)) = \bar{\mathrm{y}}(\beta(i))$ on a large set. For each $i \in u$ let $r_i \in X_\omega$ be such that $\bar{\mathrm{x}}(\beta(i)) < r_i < \bar{\mathrm{x}}(\alpha(i))$ and define $\gamma_i = \{p \in P : \bar{\mathrm{x}}(p) < r_i\}$. Let $G = \{\gamma_i : i \in u\}$, define $k\colon \omega \to \omega$ by $k(i) = \Delta(\bar{\mathrm{y}}(\beta(i))$ and let $\gamma = (\gamma^k_G)^{\downarrow}$. Then $[\beta] \in \gamma$, $[\alpha] \notin \gamma$ (it's not even in $\bar{G}$), and by Proposition 5.2.16 γ is a complete, prime weak-ideal. The dual to Theorem 5.2.4 and the discussion in this section yield the following result:

Proposition 5.2.17. *$\prod_U P$ is join-completely representable.*

Combining this with Corollary 5.2.13 and Theorem 2.3.21 we obtain the main result of this section.

Theorem 5.2.18. *The class of join-completely representable posets is not closed under ultraroots and thus is not elementary.*

In fact we can say more; firstly, by Corollary 5.2.7 and the fact that order duals commute with ultraproducts, we know that the class of meet-completely representable posets also fails to be closed under ultraroots. Secondly, it's easy to see the poset P is dually-separated by the set of its completely-prime, weak-ideals and so is meet-completely representable, thus, since $\prod_U P$ is join-completely representable by the discussion above, and meet-completely representable by closure of the class of meet-completely representable posets under ultraproducts (from pseudoelementarity), the class of posets with both join- and meet-complete representations cannot be closed under ultraroots either, and thus cannot be elementary.

5.3 The canonical extension and complete representability

Historically, complete representation is strongly associated with the canonical extension. Indeed, in the Boolean case, the construction of the canonical extension using ultrafilters as in [90] is a complete representation of itself when equipped with the identity map, so every canonical extension of a Boolean algebra must be completely representable. Less trivially, there is also a correspondence in the relation algeba case, which we state as Theorem 5.3.1 below.

Theorem 5.3.1 (Monk)**.** *A relation algebra is representable if and only if its canonical extension is completely representable.*

Proof. See e.g. [81, Theorem 3.36] □

The theorem above is a natural extension of the result in the Boolean case, as we know that every Boolean algebra is representable via Stone's theorem. Furthermore, we can pick out the class of Boolean algebras from **RA** as the subclass where $a\,;b = a \wedge b$ for all a, b, all elements are self converse, and $\mathbf{id} = 1$, so the result for Boolean algebras can be viewed in a roundabout way as a special case of Monk's result for relation algebras.

The situation for distributive lattices (without additional operations) is also straightforward. Here again we always have representability (via Theorem 3.2.5), and from Lemma 5.3.2 and Corollary 5.3.3 below we obtain that the canonical extension of a distributive lattice is always completely representable.

Lemma 5.3.2. *A complete lattice L is completely representable if and only if it is doubly algebraic (a complete lattice is algebraic if every element can be written as a join of compact elements, a complete lattice is doubly algebraic if both it and its order dual are algebraic).*

Proof. It is known, see e.g. [26], that a lattice L is doubly algebraic if and only if it is complete, completely distributive, and $J^\infty(L)$ and $M^\infty(L)$ are join- and meet-dense in L respectively. The result follows from Corollary 5.1.13. □

Corollary 5.3.3. *The canonical extension of any bounded distributive lattice is completely representable.*

Proof. The canonical extension L^σ of a bounded distributive lattice L must be doubly algebraic and complete (see e.g. [54, Theorem 2.5]). □

The situation for arbitrary posets is less clear cut, and representability of P does not guarantee complete representability of P^σ, because representability of P does not ensure distributivity of P^σ (see Example 5.3.6 below).

Theorem 5.3.4. *Given a poset P the following are equivalent*

1. *P^σ is distributive,*

2. *P^σ is completely representable,*

3. *P^σ is representable.*

Proof. By [39, Theorem 2.8] $J^\infty(P^\sigma)$ and $M^\infty(P^\sigma)$ are join- and meet-dense in P^σ respectively. If P^σ is distributive then $M_p(P^\sigma) = M(P^\sigma)$ and $J_p(P^\sigma) = J(P^\sigma)$. We aim to show that $J_p^\infty(P^\sigma) = J^\infty(P^\sigma)$ and appeal to Proposition 5.1.12(2).

Let $\{p\} \cup X \subseteq P^\sigma$. Then $p \wedge \bigvee X \geq \bigvee\{p \wedge x : x \in X\}$. Suppose $p \wedge \bigvee X \not\leq \bigvee\{p \wedge x : x \in X\}$. Then there is $y \in M_p(P^\sigma)$ with $y \geq \bigvee\{p \wedge x : x \in X\}$ and $y \not\geq p \wedge \bigvee X$. But $y \geq \bigvee\{p \wedge x : x \in X\} \implies y \geq p \wedge x$ for all $x \in X$, and thus by meet-primality we must have either $y \geq p$ or $y \geq \bigvee X$, which is a contradiction as we assumed $y \not\geq p \wedge \bigvee X$. So we must have $p \wedge \bigvee X = \bigvee\{p \wedge x : x \in X\}$, and so, if $p \in J^\infty(P^\sigma)$, then for all $X \subseteq P^\sigma$ we have

$$
\begin{aligned}
p \leq \bigvee X &\iff p \wedge \bigvee X = p \\
&\iff \bigvee\{p \wedge x : x \in X\} = p \\
&\iff p \vee x = p \text{ for some } x \in X \\
&\iff p \leq x \text{ for some } x \in X
\end{aligned}
$$

and thus $J^\infty(P^\sigma) = J_p^\infty(P^\sigma)$, and by Proposition 5.1.12(2) P^σ is completely representable. Clearly if P^σ is completely representable it must also be representable, and if it is representable it must be distributive, so we are done. □

Corollary 5.3.5. *P^σ is completely distributive if and only if it is distributive.*

Example 5.3.6. *P^σ need not be distributive.* Let P be the poset composed of four points $\{a, b, c, 0\}$ ordered by $0 \leq x$ for all $x \in \{a, b, c\}$. Then P is completely representable. However, the canonical extension P^σ of P is as in Figure 5.4 and isomorphic to the diamond lattice M_3, so is not distributive and thus cannot be representable.

From Example 5.3.6 we obtain the following easy result which we state as a theorem.

Theorem 5.3.7. *Neither the class of representable posets nor the class of representable meet-semilattices is canonical.*

A dual result also holds for join-semilattices.

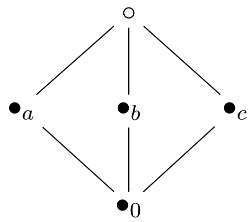


Figure 5.4: The canonical extension of a representable semilattice need not be distributive.

Chapter 6

Representing ordered domain algebras

Domain algebras provide a one-sorted formalism for automated reasoning about program and system verification [35, 36]. A thorough discussion of this is beyond the scope of this thesis, but the key idea is that algebra elements correspond to actions of some system on a set of states. As such, similar to relation algebras, the intended models are based on algebras of relations, but in this case domain and range operators are included in the signature, capturing the acceptable input states for an action and its possible output states respectively.

Single sortedness is achieved by embedding propositions into a set of actions as *tests* i.e. programs that succeed with no consequences if and only if their input is a particular state, and abort otherwise (see e.g. [94] for a discussion of tests in the setting of Kleene algebras). In particular, actions are modelled as a semiring, and a *test semiring* is a semiring S with an embedded Boolean algebra structure ($Test(S)$) representing the tests of underlying states. Domain operations map semiring elements to $Test(S)$, thus conferring first class status to neither states nor actions [33, 34]. Note that the full algebraic force of relation algebras is often not used, for example sometimes only the minimal signature of composition and domain is considered (e.g. [31]).

The algebraic behaviour of domain operations has been investigated in various contexts (see e.g. [34, 31, 32]), and as for many applications the intended models are relational there are representation problems similar to those for relation algebras. To address these questions we need a precise definition for the representable domain algebras $\mathbf{R}(;, \mathbf{dom})$.

6.1 Representable ordered domain algebras

Definition 6.1.1. The class $\mathbf{R}(;, \mathbf{dom})$ is defined as the isomorphs of $\mathcal{A} = (A, ;, \mathbf{dom})$ where $A \subseteq \wp(U \times U)$ for some base set U and

$$x\,;y = \{(u,v) \in U \times U : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w \in U\}$$
$$\mathbf{dom}(x) = \{(u,u) \in U \times U : (u,v) \in x \text{ for some } v \in U\}$$

for every $x, y \in A$.

The signature $(;, \mathbf{dom})$ can be expanded to larger signatures τ by including other operations. For instance, we can define

$$\mathbf{ran}(x) = \{(v,v) \in U \times U : (u,v) \in x \text{ for some } u \in U\}$$
$$x^{\smile} = \{(v,u) \in U \times U : (u,v) \in x\}$$
$$\mathbf{id} = \{(u,v) \in U \times U : u = v\}$$

and the corresponding representation classes $\mathbf{R}(\tau)$ analogously to the definition of $\mathbf{R}(;, \mathbf{dom})$, and thus analogously to **RRA** (see Section 3.4.1). We can also include bottom 0 and top 1 elements (interpreted as $\emptyset$ and $U \times U$, respectively) and the ordering $\subseteq$ to yield representable algebraic structures. Note that unlike in Section 3.3 we do *not* demand that a representation of an ODA interprets existing joins and meets as union and intersection, only that it interprets the domain algebra operations as operations on relations appropriately. In this regard the situation is similar to the situation for relation algebras, but without the concern for an underlying Boolean algebra structure. Indeed, representable domain algebras are a subreduct of representable relation algebras via the identities $\mathbf{dom}(x) = \mathbf{id} \wedge (x\,; x^{\smile})$ and $\mathbf{ran}(x) = \mathbf{id} \wedge (x^{\smile}\,; x)$.

A natural question is whether the class $\mathbf{R}(;, \mathbf{dom})$ of representable domain algebras of the minimal signature $(;, \mathbf{dom})$ is finitely axiomatisable, and it turns out the answer is no.

Theorem 6.1.2 ([82])**.** *Let τ be a similarity type such that $(;, \mathbf{dom}) \subseteq \tau \subseteq (;, \mathbf{dom}, \mathbf{ran}, 0, \mathbf{id})$. The class $\mathbf{R}(\tau)$ of representable τ-algebras is not finitely axiomatisable in first-order logic.*

Note that the above theorem does not apply to signatures where the ordering $\subseteq$ is definable (noting again that we do not demand that a representation preserve the partial lattice structure). In fact, Bredikhin [19] proved that the class

$$\mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, {}^{\smile}, \subseteq)$$

of representable algebraic structures is finitely axiomatisable. The relaxation of the demands on the representation of the partial lattice structure is important here, as it follows from [84, Theorem 2.3] that $\mathbf{R}(\cap, ;, \mathbf{dom}, \mathbf{ran}, {}^\smile)$ is not finitely axiomatizable, and if we had a finite axiomatization of the subclass of $\mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, {}^\smile, \subseteq)$ whose representations respected $\cap$ when it existed we could use it to finitely axiomatize $\mathbf{R}(\cap, ;, \mathbf{dom}, \mathbf{ran}, {}^\smile)$, which would be a contradiction.

Our aim is to provide an alternative, and slightly more general, proof that

$$\mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, {}^\smile, 0, \mathbf{id}, \subseteq)$$

is finitely axiomatisable. The advantage of our proof is that it uses a Cayley-type representation of abstract algebraic structures that also shows finite representability, i.e. that finite elements of $\mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, {}^\smile, 0, \mathbf{id}, \subseteq)$ can be represented on finite bases. We note in passing that if composition is not definable in τ then $\mathbf{R}(\tau)$ has the finite representation property, but it can be shown that every signature containing $(\cap, ;, \mathbf{id})$ or $(\cap, ;, {}^\smile)$ fails to have the finite representation property.

6.2 A representation theorem for ODAs

Let $\mathbf{Ax}$ denote the following formulas:

Partial order $\leq$ is reflexive, transitive and antisymmetric, with lower bound 0.

isotonicity and normality the operators ${}^\smile, ;, \mathbf{dom}, \mathbf{ran}$ are isotonic, e.g. $a \leq b \to a ; c \leq b ; c$ etc. and normal $0^\smile = 0 ; a = a ; 0 = \mathbf{dom}(0) = \mathbf{ran}(0) = 0$.

Involuted monoid ; is associative, $\mathbf{id}$ is left and right identity for ;, $\mathbf{id}^\smile = \mathbf{id}$ and ${}^\smile$ is an involution: $(a^\smile)^\smile = a$, $(a ; b)^\smile = b^\smile ; a^\smile$.

Domain/range axioms

(D1) $\mathbf{dom}(a) = (\mathbf{dom}(a))^\smile \leq \mathbf{id} = \mathbf{dom}(\mathbf{id})$

(D2) $\mathbf{dom}(a) \leq a ; a^\smile$

(D3) $\mathbf{dom}(a^\smile) = \mathbf{ran}(a)$

(D4) $\mathbf{dom}(\mathbf{dom}(a)) = \mathbf{dom}(a) = \mathbf{ran}(\mathbf{dom}(a))$

(D5) $\mathbf{dom}(a) ; a = a$

(D6) $\mathbf{dom}(a ; b) = \mathbf{dom}(a ; \mathbf{dom}(b))$

(D7) $\mathbf{dom}(\mathbf{dom}(a) ; \mathbf{dom}(b)) = \mathbf{dom}(a) ; \mathbf{dom}(b) = \mathbf{dom}(b) ; \mathbf{dom}(a)$

(D8) $\mathbf{dom}(\mathbf{dom}(a) ; b) = \mathbf{dom}(a) ; \mathbf{dom}(b)$

A consequence of axioms (D4) and (D5) is

(D9) $\mathbf{dom}(a)\,;\mathbf{dom}(a) = \mathbf{dom}(a)$

A model of these axioms is called an *ordered domain algebra*.

Another consequence of the ODA axioms is the following lemma, which we shall use later.

Lemma 6.2.1. *Let $\mathcal{B}$ be any ODA and let $b, c \in \mathcal{B}$. Then*

$$\mathbf{dom}(b\,;c)\,;b \geq b\,;\mathbf{dom}(c)$$

and

$$b\,;\mathbf{ran}(c\,;b) \geq \mathbf{ran}(c); b$$

Proof.

$$\begin{aligned}
\mathbf{dom}(b\,;c)\,;b &= \mathbf{dom}(b\,;\mathbf{dom}(c))\,;b && \text{by (D6)}\\
&\geq \mathbf{dom}(b\,;\mathbf{dom}(c))\,;b\,;\mathbf{dom}(c) && \text{(D1)}\\
&= b\,;\mathbf{dom}(c) && \text{(D5)}
\end{aligned}$$

The other part is similar. □

Each of the axioms (D1)–(D8) has a dual axiom, obtained by swapping domain and range and reversing the order of compositions, and we denote the dual axiom by a ∂ superscript, thus for example, $(D6)^\partial$ is $\mathbf{ran}(b\,;a) = \mathbf{ran}(\mathbf{ran}(b)\,;a)$. The dual axioms can be obtained from the axioms above, using the involution axioms and (D3).

Our main result is the following.

Theorem 6.2.2. *The class $\mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, \smile, 0, \mathbf{id}, \subseteq)$ is finitely axiomatisable:*

$$\mathcal{A} \in \mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, \smile, 0, \mathbf{id}, \subseteq) \textit{ iff } \mathcal{A} \models \mathbf{Ax}$$

and has the finite representation property.

We shall prove this in Section 6.4, using theory from Sections 4.2, 4.5.1, 4.5.4 and 6.3.

6.3 A completion process

We define a completion process for ODAs, though as we shall see later this process is quite badly behaved with respect to the ODA axioms. We base our approach on closure operators as in Section 4.5.1.

Definition 6.3.1 (Γ_D)**.** Given an ODA A with underlying poset P, define $\Gamma_D\colon\ P^{*\delta} \to P^{*\delta}$ by defining the closed sets of P^* to be those $X \in P^*$ such that $\{\mathbf{dom}(x)\,;y\,;\mathbf{ran}(z) : x, y, z \in X\}^\uparrow = X$.

Lemma 6.3.2. *Γ_D is a standard closure operator on $P^{*\delta}$.*

Proof. Routine. □

Lemma 6.3.3. *Given $X \in P^*$, if we define $X_0 = X$, and $X_{n+1} = \{\mathbf{dom}(x)\,;y\,;\mathbf{ran}(z) : x, y, z \in X_n\}^\uparrow$ for all $n \in \omega$, then $\Gamma_D(X) = \bigcup_\omega X_n$.*

Proof. It's easy to show that $X_n \subseteq X_{n+1}$ for all $n \in \omega$, so given $x, y, z \in \bigcup_\omega X_n$ there is $k \in \omega$ with $x, y, z \in X_k$. Thus $\mathbf{dom}(x)\,;y\,;\mathbf{ran}(z)^\uparrow \subseteq X_{k+1} \subseteq \bigcup_\omega X_n$. Clearly any closed set containing X must contain $\bigcup_\omega X_n$, so we must have $\Gamma_D(X) = \bigcup_\omega X_n$ as required. □

We can use the theory on lifting maps from Sections 4.2 and 4.5.4 lift the ODA operations to operations on the completion induced by Γ_D.

Definition 6.3.4 ($\Gamma_D[\mathcal{A}]$)**.** Given an ODA $\mathcal{A}$ with underlying poset P, we define $\Gamma_D[\mathcal{A}] = (\Gamma[P^*], \supseteq, f^\bullet_{\Gamma_D} : f \in \{;, \mathbf{dom}, \mathbf{ran}, {}^\smile, 0, \mathbf{id}\})$.

Henceforth we shall denote $f^\bullet_{\Gamma_D}$ by $f^\bullet$.

Lemma 6.3.5. *Given an ODA $\mathcal{A}$ with underlying poset P and the closure operator Γ_D. Then for all $f \in \{\mathbf{dom}, \mathbf{ran}, {}^\smile, 0, \mathbf{id}\}$, $f^\bullet(C_1, ..., C_n) = f[C_1 \times ... \times C_n]^\uparrow$.*

Proof. First note that $0^\bullet$ and $\mathbf{id}^\bullet$ are just $0^\uparrow$ and $\mathbf{id}^\uparrow$. For $\mathbf{dom}$ let $C \in \Gamma_D[P^*]$ and let $x, y, x \in C$. Then $\mathbf{dom}(\mathbf{dom}(x))\,;\mathbf{dom}(y)\,;\mathbf{ran}(\mathbf{dom}(z)) = \mathbf{dom}(x)\,;\mathbf{dom}(y)\,;\mathbf{dom}(z) = \mathbf{dom}(\mathbf{dom}(x)\,;y)\,;\mathbf{dom}(z)$ by ODA axioms (D4), (D7), and (D8). As C is Γ_D-closed we must have $\mathbf{dom}(x)\,;y \in C$, so we have something of form $\mathbf{dom}(x')\,;\mathbf{dom}(z)$ for $x', z \in C$. Another application of (D8) gives $\mathbf{dom}(x')\,;\mathbf{dom}(z) = \mathbf{dom}(\mathbf{dom}(x')\,;z)$, and thus as C is closed we have something of form $\mathbf{dom}(y')$ for $y' \in C$, which is in $\mathbf{dom}[C]$. The $\mathbf{ran}$ case is similar, and the $^\smile$ case follows from axiom (D3) and the fact that $^\smile$ is an involution. □

Notation 6.3.6. *Given $S \subseteq P$ we define $S^\smile = \{s^\smile : s \in S\}^\uparrow$, and we define $\mathbf{dom}(S)$ and $\mathbf{ran}(S)$ similarly. Given $S, T \subseteq P$ we define $S\,;T = \{s\,;t : s \in S$ and $t \in T\}^\uparrow$. By Lemma 6.3.5, when S is Γ_D-closed the unary operations on S defined in this way will coincide with their interpretation in $\Gamma_D[\mathcal{A}]$, e.g. $\mathbf{dom}(C) = \mathbf{dom}[C]^\uparrow = \Gamma_D(\mathbf{dom}[C]^\uparrow) = \mathbf{dom}^\bullet(C)$ for all Γ_D-closed sets C, though this is not the case for ;.*

We may wonder how close $\Gamma_D[\mathcal{A}]$ is to being an ODA. The answer is interesting, and to reach it we shall use the theory from the end of the last section. Most of the axioms (D1)-(D8) hold (Proposition 6.3.7), with the exceptions being (D2) and (D6) (Examples 6.3.11 and 6.3.12), the operations on $\Gamma_D[\mathcal{A}]$ remain isotone and normal, $\mathbf{id}^\bullet$ remains a left and right identity for composition and ${}^{\smile\bullet}$ is still an involution (Lemma 6.3.8). The dramatic deviation is that $;^\bullet$ is not necessarily associative (Example 6.3.13). The remainder of this section will be taken up with proving the claims in this paragraph.

Proposition 6.3.7. *Given ODA $\mathcal{A}$, axioms (D1), (D3), (D4), (D5), and (D7) hold in $\Gamma_D[\mathcal{A}]$.*

Proof. That $\Gamma_D[\mathcal{A}] \models \{(D1), (D3), (D4)\}$ follows easily from Corollary 4.5.45 and Lemma 6.3.5. Since $\mathbf{dom}^\bullet(C_1) ;^\bullet \mathbf{dom}^\bullet(C_2) = \Gamma_D(\{\mathbf{dom}(C_1) ; \mathbf{dom}(C_2))$ for all $C_1, C_2 \in \Gamma_D[\mathcal{A}]$, by Corollary 4.5.45 it is a necessary and sufficient condition for $\Gamma_D[\mathcal{A}] \models (D7)$ that $\Gamma_D(\mathbf{dom}(\mathbf{dom}(C_1); \mathbf{dom}(C_2)) = \mathbf{dom}[\Gamma_D(\mathbf{dom}(C_1); \mathbf{dom}(C_2))]^\uparrow$ for all $C_1, C_2 \in \Gamma_D[\mathcal{A}]$. We shall show that $\mathbf{dom}(C_1); \mathbf{dom}(C_2)$ is Γ_D-closed, as in that case the required equality follows from Lemma 6.3.5: Let $x_1, x_2, x_3 \in C_1$, and let $y_1, y_2, y_3 \in C_2$. Then

$$\begin{aligned} &\mathbf{dom}(\mathbf{dom}(x_1) ; \mathbf{dom}(y_1)) ; \mathbf{dom}(x_2) ; \mathbf{dom}(y_2) ; \mathbf{ran}(\mathbf{dom}(x_3) ; \mathbf{dom}(y_3)) \\ =\, &\mathbf{dom}(x_1) ; \mathbf{dom}(x_2) ; \mathbf{dom}(x_3) ; \mathbf{dom}(y_1) ; \mathbf{dom}(y_2) ; \mathbf{dom}(y_3) \end{aligned}$$

by axioms (D4) and (D7). Since $\mathbf{dom}[C_1]^\uparrow$ and $\mathbf{dom}[C_2]^\uparrow$ are closed by Lemma 6.3.5 it's easy to show that $\mathbf{dom}(x_1) ; \mathbf{dom}(x_2) ; \mathbf{dom}(x_3) \in \mathbf{dom}[C_1]^\uparrow$ and $\mathbf{dom}(y_1) ; \mathbf{dom}(y_2) ; \mathbf{dom}(y_3) \in \mathbf{dom}[C_2]^\uparrow$ and thus $\Gamma_D[\mathcal{A}] \models (D7)$ as required. That $\Gamma_D[\mathcal{A}] \models (D5)$ follows easily from Lemma 6.3.5. $\square$

Lemma 6.3.8. *For all $f \in \{;, \mathbf{dom}, \mathbf{ran}, {}^\smile, 0, \mathbf{id}\}$ the extension $f^\bullet$ is isotone and normal, moreover*

1. *$\mathbf{id}^\bullet$ is a left and right identity for $;^\bullet$, and*
2. *${}^{\smile\bullet}$ is an involution.*

Proof. Isotonicity of the operations is automatic from the lifting process, and normality follows from the fact that $0^\bullet = 0^\uparrow$. That $\mathbf{id}^\bullet$ is a left and right identity for $;^\bullet$ follows easily from the definition of $;^\bullet$ and the fact that $\mathbf{id}^\bullet = \mathbf{id}^\uparrow$. To see that $(a ; b)^\smile \leq b^\smile ; a^\smile$ holds in $\Gamma_D[\mathcal{A}]$ define $\phi = (a ; b)^\smile$ and $\psi = b^\smile ; a^\smile$. Then using Lemma 6.3.5 it's easy to see that $\psi^\bullet(C, D) = \Gamma_D(\psi[C \times D]^\uparrow)$ for all $C, D \in \Gamma_D[\mathcal{A}]$, and that $\phi^\bullet(C, D) = \Gamma_D(\phi[C \times D]^\uparrow)$ follows from Lemma 6.3.9 below. $\square$

Lemma 6.3.9. *For all $S \in P^*$, $\Gamma_D(S)^\smile = \Gamma_D(S^\smile)$.*

Proof. Since $S^\smile \subseteq \Gamma_D(S)^\smile$ and $\Gamma_D(S)^\smile$ is Γ_D-closed by Lemma 6.3.5, $\supseteq$ follows from properties of closure operators. Define $X_0 = S$ and X_n as in Lemma 6.3.3 for all $n \in \omega$. Then $X_0^\smile = S^\smile \subseteq \Gamma_D(S^\smile)$, and for all $k \in \omega$ and every $a \in X_k$ we have $a \geq b =$

$\mathbf{dom}(b_1)\,;b_2\,;\mathbf{ran}(b_3)$ for some $b_1, b_2, b_3 \in X_{k-1}$, so $b^{\smile} = \mathbf{dom}(b_3^{\smile})\,;b_2^{\smile}\,;\mathbf{ran}(b_1^{\smile})$ by involution and axioms (D1) and (D4), and so if $X_{k-1}^{\smile} \subseteq \Gamma_D(S)^{\smile} \implies X_k^{\smile} \subseteq \Gamma_D(S)^{\smile}$. Since $\Gamma_D(S)^{\smile} = \bigcup_{n\in\omega} X_n^{\smile}$ we are done. □

Lemma 6.3.10. *Let $X, Y \in \Gamma_D[\mathcal{A}]$. If $\mathbf{dom}(X) = \mathbf{dom}(Y)$ and $\mathbf{ran}(X) = \mathbf{ran}(Y)$ then $X \cup Y \in \Gamma_D[\mathcal{A}]$.*

Proof. This is straightforward. □

Example 6.3.11. *To show that (D2) can fail in $\Gamma_D[\mathcal{A}]$.* Let $\mathcal{A}$ be the full proper ODA over a base of four elements $\{a, b, c, d\}$. Define $x, y \in \mathcal{A}$ by $x = \{(a, b), (c, d)\}$, and $y = \{(a, d), (c, b)\}$. Then $\mathbf{dom}(x) = \mathbf{dom}(y)$ and $\mathbf{ran}(x) = \mathbf{ran}(y)$, and consequently $C = \{x, y\}^{\uparrow}$ is Γ_D-closed. We aim to show that $\Gamma_D(C\,;C^{\smile}) \nsubseteq \mathbf{dom}(C)$. Now, in particular $x\,;y^{\smile} \in \Gamma_D(C\,;C^{\smile})$, and $x\,;y^{\smile} = \{(a, c), (c, a)\}$, and $\mathbf{dom}(C) = \mathbf{dom}(x)^{\uparrow} = \mathbf{dom}(y)^{\uparrow} = \{(a, a), (c, c)\}^{\uparrow}$, so $x\,;y^{\smile} \notin \mathbf{dom}(C)$, and thus $\Gamma_D(C\,;C^{\smile}) \nsubseteq \mathbf{dom}(C)$, and $\Gamma_D[\mathcal{A}] \not\models (D2)$.

Example 6.3.12. *To show that (D6) can fail in $\Gamma_D[\mathcal{A}]$.* Let $\mathcal{A}$ be the full proper ODA over the two element base $\{a, b\}$. Define $x = \{(a, b), (b, a)\}$ and let $\mathbf{id} = \{(a, a), (b, b)\}$ be the identity as normal. Let $C = \{x, \mathbf{id}\}^{\uparrow}$. Then, as $\mathbf{dom}(x) = \mathbf{dom}(\mathbf{id})$ and $\mathbf{ran}(x) = \mathbf{ran}(\mathbf{id})$, C is Γ_D-closed. Define $D = \{(b, b)\}^{\uparrow}$. Then

$$\begin{aligned}
\mathbf{dom}(\Gamma_D(C\,;D)) &= \mathbf{dom}(\Gamma_D(\{x\,;\{(b,b)\}, \mathbf{id}\,;\{(b,b)\}\}^{\uparrow}))\\
&= \mathbf{dom}(\Gamma_D(\{\{(a,b),(b,a)\}\,;\{(b,b)\}, \{(b,b)\}\}^{\uparrow}))\\
&= \mathbf{dom}(\Gamma_D(\{\{(a,b)\}, \{(b,b)\}\}^{\uparrow}))\\
&= \mathbf{dom}(\emptyset^{\uparrow})\\
&= \emptyset^{\uparrow}
\end{aligned}$$

However, $\mathbf{dom}(C) = \{\{(a, a), (b, b)\}\}^{\uparrow} = \mathbf{id}$, and so $\mathbf{dom}(C)\,;D = D = \mathbf{dom}(D) \neq \emptyset^{\uparrow}$, and thus $\Gamma_D[\mathcal{A}] \not\models ((D6))$.

Example 6.3.13. *To show that associativity can fail in $\Gamma_D[\mathcal{A}]$.* Let $\mathcal{A}$ be the full proper ODA over a base of five elements $\{a, b, c, d, e\}$, let $x = \{(a, a)\}$, let $y = \{(a, b), (c, d)\}$, let $z = \{(a, d), (c, b)\}$, and let $u = \{(b, e), (d, e)\}$. Define $A = x^{\uparrow}$, $B = \{y, z\}^{\uparrow}$, and $C = u^{\uparrow}$. Then A and C are principal and hence Γ_D-closed, and $\mathbf{dom}(z) = \mathbf{dom}(y)$ and $\mathbf{ran}(z) = \mathbf{ran}(y)$ so C is also Γ_D-closed. Now, $\Gamma_D(A\,;B) = \Gamma_D(\{x\,;y, x\,;z\}^{\uparrow}) = \Gamma_D(\{\{(a, b)\}, \{(a, d)\}\}^{\uparrow}) = \emptyset^{\uparrow}$, as $\{a, b\}\,;\mathbf{ran}(\{a, d\}) = \emptyset$, so $\Gamma_D(\Gamma_D(A\,;B)\,;C) = \emptyset^{\uparrow}$. However, $B\,;C = \{y\,;u, z\,;u\}^{\uparrow} = \{(a, e), (c, e)\}^{\uparrow}$, which is principal and hence Γ_D-closed. Thus $\Gamma_D(A\,;\Gamma_D(B\,;C)) = \Gamma_D(A\,;B\,;C) = \Gamma_D(\{x\,;y\,;u, x\,;z\,;u\}^{\uparrow}) = \Gamma_D(\{(a, e)\}^{\uparrow}) = \{(a, e)\}^{\uparrow} \neq \emptyset^{\uparrow}$, and so $\Gamma_D(\Gamma_D(A\,;B)\,;C) \neq \Gamma_D(A\,;\Gamma_D(B\,;C))$.

6.4 Proving the representation theorem

We define a map h from $\mathcal{A}$ to a structure with base $\Gamma_D[\mathcal{A}]$ by setting

$$(X,Y) \in h(a) \iff X ;^{\bullet} a^{\uparrow} \subseteq Y \text{ and } Y ;^{\bullet} (a^{\smile})^{\uparrow} \subseteq X$$

We claim that h yields a representation of $\mathcal{A}$. First, some preparatory lemmas.

Lemma 6.4.1. *Let $\mathcal{A}$ be an ODA.*

1. *If $a \in \mathcal{A},\ X \in \Gamma_D[\mathcal{A}]$ and $\mathbf{dom}(a) \in \mathbf{ran}(X)$ then $\mathbf{ran}(X; a^{\uparrow}) \in \Gamma_D[\mathcal{A}]$.*
2. *If $a \in \mathcal{A},\ X \in \Gamma_D[\mathcal{A}],\ \delta \in \mathbf{dom}(\Gamma_D[\mathcal{A}])$ and $\mathbf{ran}(X) \supseteq \mathbf{dom}(a^{\uparrow}; \delta)$ then $X; a^{\uparrow}; \delta \in \Gamma_D[\mathcal{A}]$.*

Proof. For the first part, let $x_i \in X$ (for $i = 1, 2, 3$). We know that $\mathbf{ran}(x_i; a) \in \mathbf{ran}(X; a)$ and we are required to prove that

$$\mathbf{dom}(\mathbf{ran}(x_1; a)); \mathbf{ran}(x_2; a); \mathbf{ran}(\mathbf{ran}(x_3; a)) \in \mathbf{ran}(X; a)$$

Well,

$$\begin{aligned}
&\mathbf{dom}(\mathbf{ran}(x_1; a)); \mathbf{ran}(x_2; a); \mathbf{ran}(\mathbf{ran}(x_3; a)) \\
&= \mathbf{ran}(x_1; a); \mathbf{ran}(x_2; a); \mathbf{ran}(x_3; a) && \text{by (D4)} \\
&\geq \mathbf{ran}(\underbrace{x_1; \mathbf{ran}(x_2); \mathbf{ran}(x_3)}_{\in X}; a) && \text{by (D1), (D6)} \\
&\in \mathbf{ran}(X; a) && \text{since } X \in \Gamma_D[\mathcal{A}]
\end{aligned}$$

For the second part, let $x_i \in X$ and $d_i \in \delta$ (for $i = 1, 2, 3$), we are required to prove that

$$\mathbf{dom}(x_1; a; d_1); (x_2; a; d_2); \mathbf{ran}(x_3; a; d_3) \in X; a; \delta$$

For this,

$$\begin{aligned}
&\mathbf{dom}(x_1; a; d_1); (x_2; a; d_2); \mathbf{ran}(x_3; a; d_3) \\
&= \mathbf{dom}(x_1; \mathbf{dom}(a; d_1)); x_2; a; d_2; \mathbf{ran}(\mathbf{ran}(x_3; a); d_3) && \text{by (D6)} \\
&= \underbrace{\mathbf{dom}(x_1; \mathbf{dom}(a; d_1)); x_2}_{=x_2' \in X}; a; d_2; \mathbf{ran}(x_3; a); d_3 && \text{see (†) below} \\
&= x_2'; a; \mathbf{ran}(x_3; a); d_2; d_3 && \text{by (D7)} \\
&\geq x_2'; \mathbf{ran}(x_3); a; d_2; d_3 && \text{Lemma 6.2.1} \\
&\in X; a; \delta && \text{since } X \in \Gamma_D[\mathcal{A}],\ \delta \in \mathbf{dom}(\Gamma_D[\mathcal{A}])
\end{aligned}$$

(†) this follows from $(D7)$ and the facts that $X \in \Gamma_D[\mathcal{A}],\ \mathbf{ran}(X) \supseteq \mathbf{dom}(a; \delta)$. $\square$

Lemma 6.4.2. *h is 1-1.*

Proof. Let $a \not\leq b \in \mathcal{A}$. By isotonicity, (D5), (D2), and Lemma 6.3.5, $(\mathbf{dom}(a))^{\uparrow} ; a^{\uparrow} \subseteq a^{\uparrow}$ and $a^{\uparrow} ;(a^{\smile})^{\uparrow} \subseteq (\mathbf{dom}(a))^{\uparrow}$, so $((\mathbf{dom}(a))^{\uparrow}, a^{\uparrow}) \in h(a)$. Also, we cannot have $\mathbf{dom}(a) ; b \geq a$, by transitivity, isotonicity and (D1), since $a \not\leq b$. Thus $((\mathbf{dom}(a))^{\uparrow}, a^{\uparrow}) \notin h(b)$, and we are done. □

Lemma 6.4.3. $\{^{\smile}, 0, \mathbf{id}, \leq\}$ *are correctly represented*

Proof. $h(0) = \emptyset$, by normality and the partial order axioms, and $\leq$ is correctly represented by the partial order axioms and isotonicity. We have $h(\mathbf{id}) = \{(X, X) : X \in Cl(\mathcal{A})\}$ by the involuted monoid axioms, and $^{\smile}$ is correctly represented by the involution axioms. □

Lemma 6.4.4. *Let $a, b \in \mathcal{A}$, $X, Z \in \Gamma_D[\mathcal{A}]$ and suppose $X; a; b \subseteq Z$, and $Z; b^{\smile}; a^{\smile} \subseteq X$. Then the sets*

$$\begin{aligned} \alpha =& X ; a^{\uparrow} ; \mathbf{ran}(Z ;(b^{\smile})^{\uparrow}), \\ \beta =& Z ;(b^{\smile})^{\uparrow} ; \mathbf{ran}(X ; a^{\uparrow}), \textit{ and} \\ & \alpha \cup \beta \end{aligned}$$

are closed.

Proof. Consider $\alpha = X ; a^{\uparrow} ; \mathbf{ran}(Z ;(b^{\smile})^{\uparrow})$ first. If $z \in Z$ then

$$\begin{aligned} \mathbf{dom}(a; \mathbf{ran}(z; b^{\smile})) &= \mathbf{dom}(a; \mathbf{dom}(b; \mathbf{ran}(z))) && \text{by } (D3), (D6) \\ &= \mathbf{dom}(a; b; \mathbf{ran}(z)) && (D6) \\ &= \mathbf{ran}(\mathbf{ran}(z); b^{\smile}; a^{\smile}) && (D3) \\ &= \mathbf{ran}(z; b^{\smile}; a^{\smile}) && (D6) \\ &\in \mathbf{ran}(X) && Z; b^{\smile}; a^{\smile} \subseteq X \end{aligned}$$

hence $\mathbf{dom}(a^{\uparrow}; \mathbf{ran}(Z; b^{\smile\uparrow})) \subseteq \mathbf{ran}(X)$ and by Lemma 6.4.1(2) (with $\delta = \mathbf{ran}(Z ; b^{\smile\uparrow})$) α is closed. Similarly β is closed. Note that $\mathbf{dom}(\alpha) = \mathbf{dom}(\beta)$ and $\mathbf{ran}(\alpha) = \mathbf{ran}(\beta)$. By Lemma 6.3.10, $\alpha \cup \beta$ is also closed. □

Lemma 6.4.5. ; *is correctly represented.*

Proof. If $(X, Y) \in h(a)$ and $(Y, Z) \in h(b)$, then

$$\begin{aligned} & X ;^{\bullet} a^{\uparrow} \subseteq Y, \\ & Y ;^{\bullet} (a^{\smile})^{\uparrow} \subseteq X, \\ & Y ;^{\bullet} b^{\uparrow} \subseteq Z, \text{ and} \\ & Z ;^{\bullet} (b^{\smile})^{\uparrow} \subseteq Y. \end{aligned}$$

Hence $X\,;^{\bullet}(a\,;b)^{\uparrow} \subseteq Z$ and $Z\,;^{\bullet}((a\,;b)^{\smile})^{\uparrow} = Z\,;^{\bullet}(b^{\smile}\,;a^{\smile})^{\uparrow} \subseteq X$ by associativity and the involution axioms. So $(X, Z) \in h(a\,;b)$.

Conversely, assume that $(X, Z) \in h(a\,;b)$, i.e. that

$$X;^{\bullet}(a;b)^{\uparrow} \subseteq Z, \text{ and}$$
$$Z;^{\bullet}(b^{\smile};a^{\smile})^{\uparrow} \subseteq X.$$

Let $Y = \alpha \cup \beta = X;a^{\uparrow};\mathbf{ran}(Z;b^{\smile\uparrow}) \;\cup\; Z;b^{\smile\uparrow};\mathbf{ran}(X;a^{\uparrow})$. Then Y is closed by Lemma 6.4.4. We claim that $(X, Y) \in h(a)$, and $(Y, Z) \in h(b)$. To prove the claim we must show that $X;^{\bullet} a^{\uparrow} \subseteq Y$ and $Y;^{\bullet} a^{\smile\uparrow} \subseteq X$. For the first inclusion, we have $X;a \subseteq \alpha \subseteq Y$. For the other inclusion, let $y \in Y$, we have to prove that $y;^{\bullet} a^{\smile\uparrow} \in X$. Since $y \in Y = (X;a;\mathbf{ran}(Z;b^{\smile})) \cup (Z;b^{\smile};\mathbf{ran}(X;a))$ there are $x \in X,\ z \in Z$ and either $y \geq x;a;\mathbf{ran}(z;b^{\smile})$ or $y \geq z;b^{\smile};\mathbf{ran}(x;a)$. In the former case,

$$\begin{aligned}
y;a^{\smile} &\geq x;a;\mathbf{ran}(z;b^{\smile});a^{\smile} && \\
&\geq x;\mathbf{dom}(a;\mathbf{ran}(z;b^{\smile})) && \text{by (D2)}\\
&\geq x;\mathbf{dom}(a;b;z^{\smile}) && (D3),(D6)\\
&\in X && Z;b^{\smile};a^{\smile} \subseteq X, X \text{ closed}
\end{aligned}$$

while in the latter case

$$\begin{aligned}
y;a^{\smile} &= z;b^{\smile};\mathbf{ran}(x;a);a^{\smile} && \\
&\geq z;b^{\smile};\mathbf{dom}(a^{\smile};\mathbf{ran}(x));a^{\smile} && \text{by (D3), (D6)}\\
&\geq z;b^{\smile};a^{\smile};\mathbf{dom}(x^{\smile}) && \text{Lemma 6.2.1}\\
&\in X;\mathbf{ran}(X) = X && X \text{ is closed}
\end{aligned}$$

□

Lemma 6.4.6. $\mathbf{dom}$ *and* $\mathbf{ran}$ *are correctly represented.*

Proof. If $(X, Y) \in h(\mathbf{dom}(a))$, then $X\,;(\mathbf{dom}(a))^{\uparrow} \subseteq Y$. Since $\mathbf{dom}(a) \leq \mathbf{id}$ by (D1), we have that, for every $x \in X$, there is $y \in Y$ such that $x \geq x\,;\mathbf{dom}(a) \geq y$. Since Y is (upwards) closed, we get $X \subseteq Y$. Similarly, we get $Y \subseteq X$ by $Y\,;((\mathbf{dom}(a))^{\smile})^{\uparrow} \subseteq Y\,;(\mathbf{dom}(a))^{\uparrow} \subseteq X$ (using (D1)). Hence $X = Y$, i.e., $(X, X) \in h(\mathbf{dom}(a))$. Note also that $\mathbf{dom}(a) \in \mathbf{ran}(X)$, since $\mathbf{dom}(a) \in \mathbf{ran}(Y\,;(\mathbf{dom}(a)^{\uparrow}) \subseteq \mathbf{ran}(x)$.

Define the closed element $Z = X\,;a^{\uparrow}$. Then $(X, Z) \in h(a)$, since $X\,;a^{\uparrow} \subseteq Z$ by definition, and

$$X\,;a^{\uparrow}\,;(a^{\smile})^{\uparrow} \subseteq X\,;(\mathbf{dom}(a))^{\uparrow} \subseteq X$$

by (D2), and $\mathbf{dom}(a) \in \mathbf{ran}(X)$. Conversely, suppose $(X, Z) \in h(a)$ (for some Z). Then $X\,;a^{\uparrow} \subseteq Z$ and $Z\,;(a^{\smile})^{\uparrow} \subseteq X$. Since $Z\,;(a^{\smile})^{\uparrow} \subseteq X$, we have $\mathbf{dom}(a) = \mathbf{ran}(a^{\smile}) \in \mathbf{ran}(Z\,;(a^{\smile})^{\uparrow}) \subseteq \mathbf{ran}(X)$, whence $X\,;(\mathbf{dom}(a))^{\uparrow} \subseteq X$, i.e. $(X, X) \in h(\mathbf{dom}(a))$. So $\mathbf{dom}$ is correctly represented. Showing that $\mathbf{ran}$ is properly represented is similar. $\square$

We have shown that h yields a representation of $\mathcal{A}$, and clearly when $\mathcal{A}$ is finite the base of this representation is also finite. This concludes the proof of Theorem 6.2.2.

Chapter 7

Conclusions and further work

The themes of this thesis have been representation and completion for structures where a partial ordering is definable. Chapters 3 and 4 provided a short introduction to these areas, alongside some results claimed as original, while Chapter 5 narrowed the focus and extended the existing theory regarding complete representations. Chapter 6 gave an application of meet-completions to the representation theory of ordered domain algebras. In this chapter we recapitulate our main results, and discuss some natural questions arising from them alongside possible extensions of the theory. First, a summary of what has been achieved:

- We have given necessary and sufficient conditions for posets and lattices to admit complete representations as systems of sets where existing arbitrary joins and/or meets correspond to unions and/or intersections respectively. We have shown that the classes corresponding to completely and join/meet-completely representable lattices and posets are all pseudoelementary, and that the classes of posets and lattices with complete representations cannot be axiomatized in first order logic. We have shown also that the classes of posets with join or meet-complete representations cannot be axiomatized in first order logic (Chapter 5).

- We have seen under what circumstances poset meet-completions can preserve existing meets. In particular we have shown that when it is possible to define a meet-completion preserving a particular set of meets, then the set of such completions will form a topped lattice (that may not be bottomed), and we have shown that this lattice must be weakly upper semimodular, with stronger results holding for finite posets (Sections 4.5.2 and 4.5.3).

- We have made explicit a construction of Δ_1-completions using alternating meet- and join-completions, and shown that every Δ_1-completion can be obtained with such a process (Section 4.6.1).

- We have made explicit the role of a meet-completion in a proof that the class of representable ordered domain algebras is finitely axiomatizable and has the finite representation property (Chapter 6).

7.1 Further work

Perhaps the most obviously outstanding question is whether the classes of distributive lattices with meet/join-complete representations are elementary (see Conjecture 5.1.23), which due to their pseudoelementarity is equivalent to them being closed under ultraroots. In Section 5.1.3 we saw that if a counterexample to closure under ultraroots for the class **jCRL** (that is, a distributive lattice L and an ultrafilter U where $\prod_U L \in$ **jCRL** but $L \notin$ **jCRL**) exists then it must be rather complicated. In particular it must contain an infinitely dense subinterval (Proposition 5.1.22), and it must be $\vee(\bigwedge)$-distributive (Proposition 5.1.21).

Moreover to eliminate the possibility of distinguishing elements with principal filters such a lattice must display repeated branching behaviour, which in combination with the density requirement makes concrete examples difficult to construct.

The poset constructed in Section 5.2.2 is an example of the kind of substructure that must be present within a hypothetical counterexample, but attempts to freely generate a suitable lattice from this poset have been unsuccesful. Indeed, it can be shown that this particular poset cannot be used to generate a lattice counterxample in this way. Nevertheless it remains plausible that free generation could be used in combination with some other poset to provide a solution.

Attempts to rule out the possibility of a counterexample have also been unsuccessful. In the Boolean case the existence of complements is crucial for proof of elementarity, and it may be profitable to investigate the complete representability of classes of lattices with weaker forms of complementation.

Related to the discussion above, the semilattice cases remain open (note that there are two important cases here, as given a meet-semilattice we can either demand that arbitrary meets and finite existing joins are represented, or that finite meets and arbitrary existing joins are represented).

Regarding completions, we saw in Section 4.2.1 an approach to lifting maps between posets to maps between completions based on the manner in which the original poset densely sits in the completion. For canonical extensions and MacNeille completions questions such as functoriality, and preservation of identities have been addressed in the literature (see Sections 4.3 and 4.4 for references), and it would be interesting to see how these results fit into the more general setting of Δ_1-completions.

In this setting there may not be an obvious relationship between f^σ and f^π (Example 4.2.10), as to guarantee $f^\sigma \leq f^\pi$ requires a compactness property (Corollary 4.2.8), and there are Δ_1-completions that are not $(\mathcal{F}, \mathcal{I})$-compact for any choice of $\mathcal{F}$ and $\mathcal{I}$ [51, Example 5.2].

Futher to this, defining a Δ_1-completion to be an $(\mathcal{F}, \mathcal{I})$-dense completion, we saw that the 'natural' lifts f^σ and f^π are dependent on the choices for $\mathcal{F}$ and $\mathcal{I}$, and it is well known that the same basic structure can be obtained from different choices by varying the relation. Further study of what effect, if any, this has in practice could be instructive.

On the subject of ODAs, we have seen that $\mathbf{R}(;, \mathbf{dom}, \mathbf{ran}, ^\smile, 0, \mathbf{id}, \subseteq)$ is finitely axiomatizable, and it is known that $\mathbf{R}(\tau)$ is not *finitely* axiomatizable for any $(;, \mathbf{dom}) \subseteq \tau \subseteq (;, \mathbf{dom}, \mathbf{ran}, 0, \mathbf{id})$ (see Theorem 6.1.2). However, it is not currently known whether even the minimal class $\mathbf{R}(;, \mathbf{dom})$ is elementary.

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