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# Quantum metrology from an information theory perspective

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**Abstract.** Questions about quantum limits on measurement precision were once viewed from the perspective of how to reduce or avoid the effects of quantum noise. With the advent of quantum information science came a paradigm shift to proving rigorous bounds on measurement precision. These bounds have been interpreted as saying, first, that the best achievable sensitivity scales as  $1/n$ , where  $n$  is the number of particles one has available for a measurement and, second, that the only way to achieve this Heisenberg-limited sensitivity is to use quantum entanglement. We review these results and show that using quadratic couplings of  $n$  particles to a parameter to be estimated, one can achieve sensitivities that scale as  $1/n^2$  if one uses entanglement, but even in the absence of any entanglement at any time during the measurement protocol, one can achieve a super-Heisenberg scaling of  $1/n^{3/2}$ .

**Keywords:** quantum metrology, nonlinear interferometry, Bose-Einstein condensate

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In quantum metrology, the description “Heisenberg-limited scaling” refers to the best possible scaling of the measurement uncertainty with the resources put into a measurement. The phrase arises not from Heisenberg uncertainty relations, but from uncertainty relations of the Mandelstam-Tamm type [1],  $\delta\gamma\langle\Delta^2K\rangle^{1/2} \geq \frac{1}{2}$ , in units with  $\hbar = 1$ . The uncertainty  $\delta\gamma$  in a parameter  $\gamma$  that, in part, determines the state of a quantum system is related to the standard deviation of the operator  $K$  that generates translations of the state along a path parameterized by  $\gamma$ . A sequence of logical and mathematical steps is needed to provide a rigorous connection between the problem of measurement uncertainty in quantum metrology and uncertainty relations of the Mandelstam-Tamm type. The pioneering work of Helstrom [2], Holevo [3], Braunstein, Caves and Milburn [4, 5] and others laid out and elucidated these steps.

The discussion in this Paper is restricted to single-parameter estimation. The first step in estimating the value of a parameter is to identify an elementary physical system that is sensitive to changes in the parameter. One or more of these elementary units makes up the measuring device or *probe*. We expect the measurement uncertainty to depend on the initial state of the quantum probe, its evolution, and the measurement made on the probe to extract information about the parameter. The *quantum Cramér-Rao bound* quantifies the idea that the optimal measurement uncertainty is inversely proportional to

the change in the state of the probe corresponding to small changes in the value of the parameter [2, 3, 4, 5]:

$$(\delta\gamma)^2 \geq \frac{1}{(ds_{\text{DO}}/d\gamma)^2} = \frac{1}{\mathfrak{I}(\gamma,t)} \geq \frac{1}{2\langle\Delta^2 K(\gamma,t)\rangle^{1/2}} \geq \frac{1}{\|K(\gamma,t)\|}. \quad (1)$$

Here  $ds_{\text{DO}}$  denotes a metric in the space of density operators of the probe,  $\mathfrak{I}(\gamma,t)$  is the quantum Fisher information and the Hermitian generator  $K(\gamma,t)$  is defined by  $\partial\rho/\partial\gamma = -i[K(\gamma,t), \rho(\gamma,t)]$ . The uncertainty in determining  $\gamma$  is quantified by the units-corrected, root-mean-square deviation of one’s estimate of the parameter,  $\gamma_{\text{est}}$ , from the true value  $\gamma$ :  $\delta\gamma = \langle(\gamma_{\text{est}}/|\partial\langle\gamma_{\text{est}}\rangle/\partial\gamma| - \gamma)^2\rangle^{1/2}$ . The last but one inequality in Eq. (1) is a rigorous statement of the Mandelstam-Tamm uncertainty relation. The last inequality in Eq. (1) is obtained by noting that the variance of a Hermitian operator is bounded from above by  $\langle\Delta^2 K\rangle \leq \|K\|^2/4$ , where  $\|\cdot\|$  denotes a particular semi-norm of a Hermitian operator, defined as the difference between its largest and smallest eigenvalues.

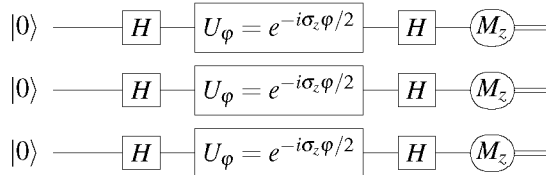
The number  $n$  of elementary units of the probe can be regarded as the most significant resource that goes into a measurement scheme. To see the dependence of the bound on  $\delta\gamma$  on  $n$  and to understand what “Heisenberg-limited scaling” means, we view quantum metrology from the perspective of quantum information theory using the language of quantum circuits in the next section.

## QUANTUM CIRCUITS FOR METROLOGY PROTOCOLS

We follow Giovannetti, Lloyd, and Maccone [6] in using quantum circuits to describe and analyze metrology protocols. From this perspective, we first look at a couple of well-known measurement schemes that were considered in [6]—Ramsey interferometry and cat-state interferometry—with the aim of generalizing these circuits to new protocols that were introduced in [7, 8]. In these initial discussions of Ramsey interferometry and cat-state interferometry, we assume that the probe units are qubits. The quantization axis is taken to be along the  $z$ -direction of a Bloch-sphere representation, with the basis states along this direction denoted as  $|0\rangle$  and  $|1\rangle$ .

### Ramsey and cat-state interferometry

A typical Ramsey interferometer, such as the one in [9], can be represented by the following quantum circuit.



In this measurement protocol, each of the  $n$  qubits that make up the probe evolves independently. Here and in the other quantum circuits depicted in this section, we

use  $n = 3$  probe units as an example. All the qubits are initialized in the state  $|0\rangle$ , which could represent the ground state in Ramsey interferometry using atoms. The Hadamard gate  $H$  puts each of the qubits in an equal superposition of the two basis states,  $(|0\rangle + |1\rangle)/\sqrt{2}$ . The parameter-dependent evolution of the quantum probe is generated by the Hamiltonian

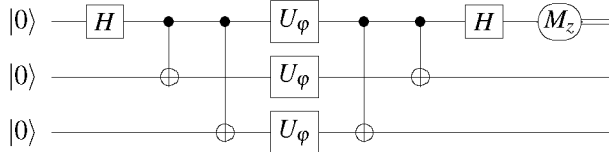
$$H_{\text{Ramsey}} = \gamma \sum_{j=1}^n \sigma_{z,j}/2 = \gamma J_z, \quad (2)$$

where  $\sigma_{z,j}$  denotes the  $\sigma_z$  operator acting on the  $j$ th probe qubit and  $J_z$  is the  $z$  component of the “total angular momentum” for all the qubits.

After evolution under this Hamiltonian for a time  $t$  and the remaining Hadamard gates, the state of the probe qubits just before the readout is  $\cos(\varphi/2)|0\rangle + \sin(\varphi/2)|1\rangle$ , where  $\varphi = \gamma t$ . Each qubit is measured along the  $z$ -direction. This leads to a measured signal  $\langle J_z \rangle \equiv \langle \sum_{j=1}^n \sigma_{z,j}/2 \rangle = n \cos \varphi/2$ , with variance  $\langle \Delta^2 J_z \rangle = n \langle \Delta^2 \sigma_z \rangle/4 = n \sin^2 \varphi/4$ . The uncertainty in the estimate of  $\gamma$  from the measured signal is  $\delta \gamma_{\text{Ramsey}} = \langle \Delta^2 J_z \rangle^{1/2} / |d \langle J_z \rangle / d \gamma| = 1/t \sqrt{n}$ .

For the Ramsey Hamiltonian (2), the generator of translations in  $\gamma$  is  $tJ_z$ . Thus, according to the quantum Cramér-Rao bound (1), the measurement uncertainty is bounded from below by  $\delta \gamma \geq 1/t \|J_z\| = 1/tn \|\sigma_z/2\| = 1/tn$ . The Ramsey interferometer described here does not achieve the best measurement uncertainty given by the quantum Cramér-Rao bound. The Hamiltonian  $H_{\text{Ramsey}}$  that governs the evolution of the probe qubits is fixed by the choice of physical systems that are the qubits. Given a choice of probe qubits, however, we still have the freedom to choose an optimal initial state for the probe and an optimal measurement of the qubits to minimize the measurement uncertainty. It turns out that the best possible scaling for the measurement uncertainty can be achieved if the probe is initialized in an entangled, “Schrödinger-cat” state.

The quantum circuit that employs an initial Schrödinger-cat state is the following.



The Hadamard gate on the first qubit, followed by the controlled-NOT gates to the remaining qubits, initializes the probe in the state  $|\text{cat}\rangle = (|0\dots 0\rangle + |1\dots 1\rangle)/\sqrt{2}$ . This state is often referred to as the cat state because when the number of qubits is large, it is a superposition of two macroscopically distinct states. The probe qubits evolve under the same Hamiltonian (2) as in Ramsey interferometry. The parameter-dependent evolution of the probe for a duration  $t$  changes the probe state to  $(e^{-in\varphi/2}|0\dots 0\rangle + e^{in\varphi/2}|1\dots 1\rangle)/\sqrt{2}$ .

After the parameter-dependent evolution, one way to handle the readout, depicted in the circuit above, is to subject the qubits to a sequence of gates that kick the phases picked up by the two components of the cat state into amplitudes on the first qubit, so that the state of the probe just before readout is  $[\cos(n\varphi/2)|0\rangle + \sin(n\varphi/2)|1\rangle] \otimes |0\rangle^{\otimes(n-1)}$ . The readout of the probe can then be performed by measuring the  $\sigma_z$  operator

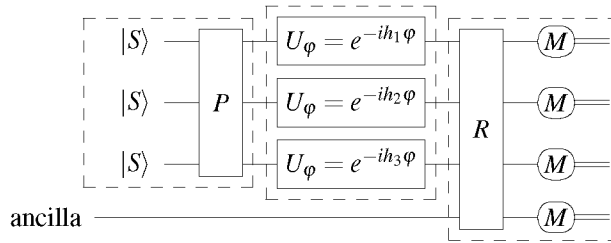
on the first qubit. This leads to a measured signal and variance given by  $\langle \sigma_{z,1} \rangle = \cos n\varphi$  and  $\langle \Delta^2 \sigma_{z,1} \rangle = \sin^2 n\varphi$ . The frequency of the  $\gamma$ -dependent fringe in cat-state interferometry is  $n$  times greater than the frequency of the signal in ordinary Ramsey interferometry. This leads to an enhanced sensitivity in the estimate of  $\gamma$  in cat-state interferometry:  $\delta\gamma_{\text{cat}} = \langle \Delta^2 \sigma_{z,1} \rangle^{1/2} / |d\langle \sigma_{z,1} \rangle / d\gamma| = 1/(tn)$

We can put our interferometry circuits in a general setting by considering the case in which the probe units are arbitrary systems and the Hamiltonian of the probe is of the form  $H_{\text{linear}} = \gamma h_{\text{linear}} = \gamma \sum_{j=1}^n h_j$ . Here the operators  $h_j$  denote identical couplings to the probe units; the use of independent couplings to the parameter is the source of our appellation ‘‘linear’’ for this Hamiltonian. The generator of translations in  $\gamma$  is  $K(\gamma, t) = t h_{\text{linear}}$ , so the quantum Cram er-Rao bound (1) on the uncertainty in a determination of  $\gamma$  takes the form

$$\delta\gamma \geq \frac{1}{t \|h_{\text{linear}}\|} = \frac{1}{tn(\Lambda - \lambda)} \quad (3)$$

where  $\Lambda$  and  $\lambda$  are the largest and smallest eigenvalues respectively of the single-unit operators  $h_j$ .

The quantum circuit for a measurement protocol of this sort is the following.



The dashed boxes highlight the three stages of this protocol: probe preparation, dynamics, and readout. All the probe units begin in a standard state  $|S\rangle$ . The arbitrary unitary operator  $P$  can then prepare any initial state as input to the dynamics. In the dynamics stage, the gates  $U_\varphi$  imprint information about the parameter on the probe. The final readout stage includes an arbitrary unitary interaction  $R$  among the probe units and with an arbitrary ancilla system. This unitary followed by measurements on each subsystem in a standard basis can be used to perform any quantum measurement. The quantum Cram er-Rao bound (3) applies to all circuits of the above form. Indeed, the bound actually applies to somewhat more general situations in which the unitary operator  $R$  is interleaved with the gate dynamics and the results of ancilla measurements are fed back onto the probe [7].

If the preparation unitary  $P$  is omitted from the circuit, making the input to the dynamics a product state, then the uncertainty in the generator of  $\gamma$  displacements is bounded by  $\langle \Delta^2 K \rangle^{1/2} \leq t \sqrt{n}(\Lambda - \lambda)/2$ . The resulting bound on measurement uncertainty, from Eq. (1), is  $\delta\gamma \geq 1/t \sqrt{n}(\Lambda - \lambda) \equiv \delta\gamma_{\text{QNL}}$ . This bound, a general form of that for Ramsey interferometry, is called the *quantum noise limit* or the *shot-noise limit*.

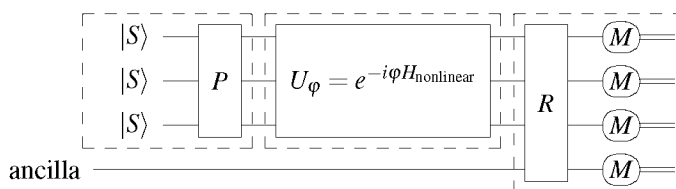
One can achieve the Cram er-Rao bound (3) by operating the circuit in a way that takes advantage of entangled input states. The preparation operator is chosen to take the

initial product of standard states to the state  $(|\Lambda, \dots, \Lambda\rangle + |\lambda, \dots, \lambda\rangle)/\sqrt{2}$ , where  $|\Lambda\rangle$  and  $|\lambda\rangle$  are the eigenstates of  $h$  corresponding to its largest and smallest eigenvalues. In the dynamics stage, this “cat-like” initial state is subject to a period of parameter-dependent evolution that changes it to  $(e^{-in\Lambda\varphi}|\Lambda, \dots, \Lambda\rangle + e^{in\lambda\varphi}|\lambda, \dots, \lambda\rangle)/\sqrt{2}$ . The readout process kicks back the differential phase shift into an amplitude information, which produces fringes with frequency proportional to  $n(\Lambda - \lambda)$ , thus achieving the optimal measurement uncertainty,  $\delta\gamma_{\text{HL}} = 1/in(\Lambda - \lambda)$ , of the Cramér-Rao bound (3). This optimal measurement uncertainty, a general form of that for cat-state interferometry, is often called the *Heisenberg limit*.

The general quantum metrology scheme considered in this subsection indicates that probe preparation buys an enhancement of  $1/\sqrt{n}$  over the case where the probe qubits are initialized in a product state. Readout has already been optimized to take advantage of this entangled input, so we conclude that when the parameter-dependent dynamics acts independently on the probe qubits, Heisenberg-limited scaling is indeed the  $1/n$  scaling. The one remaining way of exploring whether the  $1/n$  scaling can be improved is to consider more general dynamics [10, 11, 7, 12, 8, 13, 14].

## Heisenberg-limited metrology with nonlinear Hamiltonians

A generalized quantum metrology scheme in which the dynamics of the probe is generated by a Hamiltonian that includes all  $k$ -body couplings between the probe qubits was first considered in [7]. This nonlinear coupling Hamiltonian has the form  $H_{\text{nonlinear}} = \gamma h_{\text{nonlinear}} = \gamma(\sum_{j=1}^n h_j)^k = \gamma \sum_{j_1, \dots, j_k=1}^n h_{j_1} h_{j_2} \cdots h_{j_k}$ . The generator of translations in  $\gamma$  is  $K(\gamma, t) = t h_{\text{nonlinear}}$ , so the quantum Cramér-Rao bound for this dynamics is  $\delta\gamma \geq 1/tn^k(\Lambda_{\text{max}} - \Lambda_{\text{min}})$ , where  $\Lambda_{\text{max}}$  and  $\Lambda_{\text{min}}$  are functions of  $\Lambda$  and  $\lambda$ , the largest and smallest eigenvalues, respectively, of the single-unit operators  $h_j$ . For instance, if both  $\Lambda$  and  $\lambda$  are positive, then  $\Lambda_{\text{max}} = \Lambda^k$  and  $\Lambda_{\text{min}} = \lambda^k$  for all values of  $k$ . The other possible signs of  $\Lambda$  and  $\lambda$  are discussed in [8]; they all lead to a scaling  $1/n^k$ . The quantum circuit for metrology with nonlinear Hamiltonians has the following form.



The only difference between this circuit and the one for linear quantum metrology is in the dynamics stage, where the gate that imprints information about the parameter on the probe can involve multi-unit coupling to the probe units.

To achieve the  $1/n^k$  scaling made available by using a nonlinear Hamiltonian, the probe units have to be initialized in an entangled state that is very much like a cat state. Experimental limitations up till now have precluded making such cat-like states for large numbers of systems. To avoid this difficulty, Boixo *et al.* [8] analyzed the performance of quantum metrology protocols employing nonlinear Hamiltonians when the initial state

of the probe is a product state. In this case the optimal measurement uncertainty scales as  $\delta\gamma \sim 1/tn^{k-1/2}$ . The factors multiplying this scaling depend on the particular nonlinear coupling Hamiltonian [8]. It is noteworthy that the optimal  $1/tn^{k-1/2}$  sensitivity can be achieved using product measurements. The key point is that for a  $k$ -body coupling Hamiltonian, the use of a product-state input costs only a factor of  $\sim \sqrt{n}$  relative to the optimal possible sensitivity. The quantum noise limit and the Heisenberg limit of linear metrology are a special case of this  $\sqrt{n}$  loss of sensitivity when using input product states as opposed to an optimal entangled state.

With two-body couplings and an initial product state for the probe, a measurement uncertainty scaling as  $1/n^{3/2}$  is possible [15]. Since two-body couplings between all probe units is not an especially onerous requirement for a probe system, the prospect of improving upon the  $1/n$  Heisenberg scaling motivates us to investigate candidate systems for such metrology schemes. In a separate paper in this volume [16], we consider a Bose-Einstein condensate as such a candidate system, with the aim of developing a detailed, realistic, and viable proposal for an experiment that achieves better than  $1/n$  scaling for the measurement uncertainty in quantum single-parameter estimation.

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