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ALL SIMPLE GROUPS WITH ORDER FROM 1 MILLION TO 5 MILLION ARE EFFICIENT

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Abstract. There is much interest in finding short presentations for the finite simple groups. Indeed it has been suggested that all these groups are efficient in a technical sense. In previous papers we produced nice efficient presentations for all except one of the simple groups with order less than one million. Here we show that all simple groups with order between 1 million and 5 million are efficient by giving efficient presentations for all of them. Apart from some linear groups these results are all new. We also show that some covering groups and some larger simple groups are efficient. We make substantial use of systems for computational group theory and, in particular, of computer implementations of coset enumeration to find and verify our presentations.

1. Background

For presentations of groups there are various notions of short. These include length (addressed in [12] and [13]) and a technical notion: efficiency. Epstein studied geometric properties of groups in [10]. He used homological arguments to show that there is a lower bound on the minimal number of relations required to present a group. He called a group **efficient** if this lower bound could be achieved. A relevant, more formal, description of this notion appears in [3].

There is much interest in finding short presentations for (the nonabelian) finite simple groups and for their covering groups. We focus on the issue of efficiency. (Throughout this paper, simple group will refer to nonabelian simple group, since efficiency questions for cyclic groups have easy positive

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answers.) Recent work on short presentations for simple groups includes [2], [3], [4], [12], [13], [18] and [20]. Indeed, John Wilson says

> It seems reasonable to conjecture that the covering group of every finite simple group has a presentation with two generators and two relators.

If true, this implies that all finite simple groups are efficient.

We use Atlas notation [9] for the names of simple groups and we denote the covering group of the simple group G by $\widehat{G}$. In [3] and [4] we give nice efficient presentations for all except one, $S_4(4)$, of the simple groups with order less than one million. The efficiency question for one other covering group, $\widehat{U}_3(5)$, remains unresolved.

For simple groups with order between 1 million and 5 million the situation is rather different to that of smaller orders. For smaller orders most simple groups and their covering groups were already known to be efficient when we produced our nice efficient presentations. There we resolved six out of eight previously unknown cases among both the simple groups and their covers. For orders from 1 million to 5 million, apart from $L_2(p)$ (p prime) and $L_2(13^2)$, it was unknown whether the other seven simple groups are efficient. We prove that they are by giving efficient presentations for each of them. We also show that some larger groups are efficient. In some cases we prove that the covering groups are efficient, but we leave unresolved the efficiency question for covering groups of four simple groups with orders between 1 and 5 million.

This work was motivated in part by a request from Bill Kantor for an efficient presentation for A_{10}. In [13] it is shown that:

Theorem 1.1. [13, Theorem A.] *All nonabelian finite simple groups of Lie type, with the possible exception of the Ree groups ${}^2G_2(q)$, have presentations with* 2 *generators and at most* 80 *relations.*

All symmetric and alternating groups have presentations with 2 *generators and* 8 *relations.*

They state that a similar result holds for all finite simple groups, except perhaps the Ree groups, and also holds for many of the almost simple groups. They note that the bound 80 of Theorem A is not optimal and that much better bounds are obtained in various cases. Indeed this paper and our previous papers reveal some of these bounds via efficient presentations.

Aspects of their work [13, Examples 3.19] rely on using explicit efficient presentations for various "small" groups as building blocks. For example [13, Table 1], efficient presentations for A_{10} are used to construct 2-generator, 6-relator presentations for A_{47}. More generally, efficient presentations ([3] and [4]) for the universal central extension of A_m ($m \leq 9$) and for A_{10} (this paper) are used to reduce the number of relations in various presentations.

2. **Methodology**

We use three different methods to find efficient presentations. Method 1 involves a study of short presentations for perfect groups. In an extension of [16] we consider various 2-generator 3-relator presentations with longest relator of length up to 35. Method 2 considers presentations based on different generating pairs for the groups. In principle the method looks at all essentially different generating sets. The sizes of the groups considered here means that rather than trying to look at all generating sets we randomly choose generating sets. Method 3 involves a study of one-relator quotients of free products $C_m * C_n$ for coprime m and n. (By a one-relator quotient of a particular group we mean a presentation obtained by adding one extra relator to a presentation for the specified group.) The first two methods are described in detail in [3] while the third is described in [4]. There is some overlap between these methods: presentations constructed using Method 1 also arise from Method 3.

Each of our methods relies on investigations of search spaces of varying kinds. With increasing group size and presentation length the search spaces grow enormously. Under these circumstances we do not attempt exhaustive searches, but rather are satisfied when we have found at least one efficient presentation for each simple group. In part our new results are due to us being able to make deeper searches than previously undertaken.

We use GAP [11], MAGMA [1] and standalone programs to do our searches. We try to avoid listing presentations which are readily seen to be equivalent, using tools for culling equivalent presentations. Thus, we have a number of standalone programs which manipulate presentations in various ways to attempt to find better presentations. One of these, ACME, applies Andrews-Curtis moves and is described in [17]. The others investigate presentations obtained via simple canonical automorphisms (see [16]) and Whitehead automorphisms, combined with Andrews-Curtis moves.

We check that our presentations are correct by coset enumeration. We generally use the ACE enumerator [15], either as available in GAP or MAGMA, or as a standalone program for some more difficult cases.

As far as reliability of results is concerned we assert that all presentations given in this paper correctly define the groups. Each new presentation which appears has been verified by both GAP and MAGMA programs to present the specified group. Here we intentionally do not use ACE for the GAP check but rather use GAP's internal coset enumerator (which was entirely independently written), providing a strong level of confidence in the results.

We assess our presentations from a performance point of view by providing coset enumeration statistics. We generally measure coset enumeration performance by giving the total and maximum number of cosets used in a successful enumeration of the trivial subgroup using the `Hard` strategy of ACE, with the group generators given in alphabetical order. (We note that these enumerations are for performance measurement only and are by no means the best way to verify presentation correctness.)

We use two results from [3] and an extension each of which enable us to amalgamate relations in presentations to give presentations for associated groups with fewer relations. The proofs of these results are constructive, which allows us to build efficient presentations using them.

Theorem 2.1. *Let G be a finite simple group. Suppose that G, or some stem extension of G, can be presented as*

$$\{\ a, b \mid a^p = b^q = w(a,b) = 1\ \}.$$

Then the covering group of G, all stem extensions of G, and G itself, are efficient.

Corollary 2.2. *Let G be a finite simple group. Suppose that G, or some stem extension of G, can be presented as*

$$\{\ a, b \mid u(a, b)^p = v(a, b)^q = w(a, b) = 1\ \}.$$

Suppose also that $u(a, b)$ and $v(a, b)$ generate the free group on a and b. Then the covering group of G, all stem extensions of G, and G itself, are efficient.

A natural extension of Theorem 2.1 gives methods for amalgamating relations given a presentation for (a stem extension of) a simple group with more relations, such as

$$P = \{a, b \mid a^p = b^q = w_1(a, b) = \ldots = w_n(a, b) = 1\}.$$

We point out that our primary focus is on presentations which are efficient in terms of deficiency. This does not always coincide with best presentations for other purposes. For deficiency-zero groups, the deficiency-one presentation

$$\{a, b \mid a^p = b^q = w(a, b) = 1\}$$

is likely to be much more useful for practical computation than the efficient presentation produced by Theorem 2.1,

$$\{a, b \mid a^p b^{-q} = \widetilde{w}(a, b) = 1\},$$

where the derivation of $\widetilde{w}(a, b)$ from $w(a, b)$ is described in [3]. For example, these deficiency-one presentations are better for coset enumeration than the corresponding efficient presentations. A similar situation applies to presentations obtained from the extension of Theorem 2.1.

3. Orders 1 million to 5 million

In 1972 Sunday [19] gave efficient presentations for all $L_2(p)$ for prime $p \geq 5$. We give efficient presentations for all other simple groups with order between 10^6 and 5×10^6. (Only one of these, $L_2(13^2)$, was previously known to be efficient.) We remark that our methods give shorter presentations for individual groups than Sunday's generic presentation.

We adopt the convention of using upper-case letters to denote inverses in presentations so that, for example, $A = a^{-1}$. We give presentations by listing sets of relators (often only implicitly specifying the generators). For coset enumeration purposes the generators are always given in alphabetical order.

3.1. $\mathbf{S}_6(2)$**.** Using Method 2 we investigated 3395 random generating pairs for $S_6(2)$, taking about 50 cpu days. We obtained initial presentations with between 5 and 19 relators. We reduced 3 of these presentations to 3-relator presentations for preimages of $S_6(2)$ and 76 to 4-relator presentations. Most of these were proper preimages. However one of the 3-relator presentations does present the group, namely:

$$P_1 : b^{10}, BaBABAb^2AB^2, aBababa^2babAB^2.$$

In addition we are able to use the extension of Theorem 2.1 to provide other efficient presentations for $S_6(2)$ from among the 4-relator presentations. We give two examples. By amalgamating $(AB)^2$ and B^7 we obtain:

$$P_2 : abab^8, B^2aB^2aBA^2BaB^2aB^2a, AbaB^2aBAb^2a^2BAb^2A^2bAB^2a^2B^2a^2b^2$$

and by amalgamating b^4 and a^7 we obtain:

$$P_3 : b^4a^7, A^2Ba^3BA^3Ba^2b^2Ab^2A, AB^2A^2B^2A^3bA^2b^2a^2bA.$$

Presentation P_1 is much better for coset enumeration, requiring a maximum 1451520 cosets (that is the order of the group) and total 2353519 cosets. Presentations P_2 and P_3 use maximums 19251628 and 54263230, and totals 23949284 and 54356579, respectively.

Usually with presentations like these on a, b from Method 2 we leave them as produced by Magma to show how they appear. Our utilities for producing canonical presentations (see [16]) can often shorten them somewhat, but in no case led to significant alterations.

3.2. $\mathbf{A_{10}}$. Using Method 2 we investigated 7035 random generating pairs for A_{10}, taking about 100 cpu days. We obtained initial presentations with between 5 and 35 relators. We reduced 6 of these presentations to 3-relator presentations for preimages of A_{10}, five of which presented the group itself rather than the cover.

We tabulate those five presentations (in the order we found them) together with the maximum and total cosets used for enumerations over the trivial subgroup (index 1814400).

Table 1. Efficient presentations for A_{10}

Relators	Maximum	Total
$BA^2BA^3bAbA^3, a^2Ba^5B^3a^3, A^2baBaba^3baBabA^2B$	62863144	64956275
$babaBA^3babab^2a, A^4Ba^4abababab, Aba^2b^2aBA^2B^2a^2BaBab$	42591987	42606596
$AB^2AB^2Ab^2ab^2, b^2A^2BA^4ba^2b^5, A^2B^2A^2BA^2B^4ab^3$	12931389	13229582
$bA^4b^2Ab, BABAbABA^2Ba^2bAba^2, babA^2bABa^2BA^2bA^2Ba$	45828561	50469910
$aBa^2BA^2b^3ABa, aba^2ba^3baBABa, ba^2bABa^3BA^2ba^3$	34912095	35256724

3.3. $\mathbf{L_3(7)}$. Using Method 2 we investigated 442 random generating pairs for $L_3(7)$, taking about 2 cpu days. We obtained initial presentations with between 5 and 49 relators. We reduced one of these presentations (which started with 8 relators) to a 3-relator presentation for the group itself, which has order 1876896. The relators are:

$$ababA^2Babab, A^5ba^2ba^2ba^2b, aB^3abA^3BA^2b^3ab$$

and the enumeration over the trivial subgroup used maximum 35818192 and total 36370301 cosets.

3.4. $\mathbf{L}_2(2^7)$**.** The groups $L_2(p)$ for prime $p > 5$ are all presentable as one-relator quotients of the modular group, [5]. No such general result is known for any infinite family of groups $L_2(q)$ for proper prime powers q. However we observe that various individual $L_2(q)$ are so presentable, as may be seen in [6], [7] and [3].

Indeed $L_2(2^7)$ is a case in point. Using Method 1 we found two such defining sets of relators for this group:

$$(wZw)^2, (Wz)^3, w^2z^2w^2zwzwzwz^3wzwzwzw^2z^2 \text{ and}$$

$$(wZw)^2, (Wz)^3, wz^2wz^2w^4zw^2z^4w^2zw^4z^2wz^2.$$

This time adding the `Mendelsohn` strategy to ACE we found that to define the 2097024 cosets of the trivial subgroup requires maximums of 283082760 and 78097739 and totals of 285928760 and 80953562, respectively.

Replacing w by aB and z by ab we can obtain efficient presentations where the first two relators are a^2 and b^3, and the third relator is now twice as long. In many computations, shorter presentations generally perform better than longer ones, which is why we prefer the $\{w, z\}$ generating set for certain investigations. These and all other presentations on $\{w, z\}$ in this paper are symmetric in the sense described in [6]. We call this kind of presentation palindromic (defined in Subsection 4.1). They were found using Method 1 (and some of them were also found using Method 3).

The multiplier of $L_2(2^7)$ is trivial, so we need to reduce the number of relators to two. Using Corollary 2.2 we can construct efficient presentations for $L_2(2^7)$ by replacing the first two relators in these presentations for $L_2(2^7)$ with $wzWzwZ$ (without changing the other relator). Practical coset enumeration with these technically efficient presentations is about twice as hard compared with the three relator ones. We define the cosets of the trivial subgroup using maximums of 674416677 and 141785001 and totals of 767116814 and 158932585, respectively.

3.5. $\mathbf{L}_2(13^2)$**.** We readily found very many different presentations for $L_2(13^2)$ as a one-relator quotient of the modular group. We tabulate third relators (in order of nondecreasing length) which complement $(wZw)^2$ and $(Wz)^3$ to present the group, together with maximum and total cosets required for enumerations over the trivial subgroup, index 2413320. (The efficient presentation previously reported in [6] is the last one in our table.)

As is the case for all 3-relator groups in this paper on generators $\{w, z\}$ (and as already seen in the previous subsection) we can construct efficient presentations for their covering groups using Corollary 2.2. The enormous diversity of such presentations is discussed in [4, Subsection 4.2].

3.6. $\mathbf{U}_4(3)$**.** This is the third smallest of the simple groups with multiplier of rank 2. This means that efficient presentations on two generators have four relators. It also means that searches for efficient presentations are somewhat different from those for groups with lower multiplier rank. This is already witnessed in [3] with the groups $L_3(4)$ and $Sz(8)$, the smaller simple groups with multiplier of rank 2.

Using Method 2 we investigated 6549 random generating pairs for $U_4(3)$, taking about 136 cpu days. We obtained initial presentations with between 5 and 56 relators. We reduced 19 of these presentations

Table 2. Third relator for $L_2(13^2)$

Relator	Maximum	Total
$w^5z^3w^2z^8w^2z^3$	8872300	9029601
$w^5zwz^2w^{10}z^2wz$	2413320	3397688
$wzwzwz^2wzw^2z^2w^2z^2w^2zwz^2wzwz$	253717890	254196866
$wz^4w^3z^2w^2z^4w^2z^2w^3z^4$	49335198	50267677
$wzw^2zw^6z^6w^6zw^2z$	44896560	46570975
$w^2zw^3z^8w^3z^8w^3z$	7709166	8770251
$wz^2wzw^5zwz^2w^2z^2wzw^5zwz^2$	8136328	8449808
$wzw^6zwz^{12}wzw^6z$	108420272	112330062
$wz^2w^2zw^6zw^2z^2w^2zw^6zw^2z^2$	18110523	18260725
$wzwzwzwz^2w^2zwzwz^2wzwzw^2z^2wzwzwz$	95208904	96639511
$wz^2w^8z^{12}w^8z^2$	31366083	32010424
$w^2zwzw^5zw^3zw^5zw^3zw^5zwz$	23528008	23731554
$w^2zwzw^3zwz^2wzwzw^3zwzwz^2wzw^3zwz$	168593304	168977340
$w^5z^5w^6z^8w^6z^5$	3258522	4567814

to 4-relator presentations for preimages of $U_4(3)$, four of which presented the group itself rather than a stem extension.

We tabulate those four presentations (in the order we found them) together with the maximum and total cosets used for enumerations over the trivial subgroup (index 3265920).

Table 3. Efficient presentations for $U_4(3)$

Relators	Maximum	Total
$(Ba)^5, BA^2B^3a^2B^3A^2B, ABaB^2ABAbAbaB^3, babABaBa^2B^2AbA^2b$	147232399	152933805
$AB^2a^3B^2A^2, (BAB)^5, aB^2AB^2ABAB^2aB^2aB, b^2ABaBaBA^2Bababa^2BA$	11347637	13723387
$Ab^2aBAb^2ab^2, baB^3A^3baBab^2a, ABaB^3A^2bab^3Ab, (A^2b)^5$	4806024	6387193
$B^7, ab^2a^2BA^2B^3A^2B^2, BA^2b^2a^2Ba^2b^2a^2B^2a^2, Ab^2Ab^2aB^2Ab^2AbaB^3A$	41262105	42726475

Of these the third is best for coset enumeration. However we can do somewhat better using the extension of Theorem 2.1. Indeed, we can build efficient presentations in the following way.

Among 4-relator presentations for preimages of $U_4(3)$ we found:

$$a^5, b^7, Ab^3ABAB^3Ab, baBaBABAbab^3A$$

which are relators for an efficient presentation of the double cover. We can amalgamate the first two relators and adjust the last relator to get:

$$a^5b^7, Ab^3ABAB^3Ab, baBaBABAbaB^4A$$

which is an efficient presentation for the twelvefold cover. Alternatively by modifying the other relator we obtain relators (listed in order by length) for an efficient, but slightly longer, presentation of the sixfold cover:

$$a^5b^7, baBaBABAbab^3A, a^4b^3ABAB^3Ab.$$

It is easier to start with the smaller group to find the centre. In the sixfold cover we determine that $a^2Ba^2BaBA^2B^2AbABA^2B$ is central and has order 6, so by factoring it out we have relators

$$a^5b^7, baBaBABAbab^3A, a^4b^3ABAB^3Ab, a^2Ba^2BaBA^2B^2AbABA^2B$$

for an efficient presentation of $U_4(3)$ which uses a maximum of 3265920 (that is, the group order) and total 4902608 cosets to enumerate over the trivial subgroup.

In the twelvefold cover we find that $a^2baBab^2a^2bAbA^2bA^2b$ is central and has order 12, so by factoring it out we have relators

$$a^5b^7, Ab^3ABAB^3Ab, baBaBABAbaB^4A, a^2baBab^2a^2bAbA^2bA^2b$$

for a shorter efficient presentation of $U_4(3)$. It has similar coset enumeration performance, using a maximum of 3265920 and total 4771814 cosets.

3.7. $\mathbf{G}_2(3)$. It is surprisingly easy to find an efficient presentation for $G_2(3)$. Using random generating sets leads to presentations with many relations. So we focussed on (2, 3)-generating sets. The MAGMA `Relations` command when applied to an involution and a representative of conjugacy class 3C (that is, Atlas standard generators) regularly produced an almost efficient presentation with 4 relators:

$$a^2, B^3, (aB)^{13}, bababaBabaBaBabaBaBaBaBabababaBababaBabaBaBaBaba.$$

Coset enumeration over the trivial subgroup (index 4245696) is easy, using a maximum 4245696 and total 5090793 cosets.

Amalgamation of the first two relators to give a^2b^3 preserves the group, giving us an efficient presentation. The coset enumeration is about 8 times harder, requiring a maximum of 33202245 and total 35685039 cosets.

If instead we amalgamate the last two relators of the MAGMA presentation to give:

$$a^2, b^3, aBaBaBaBaBaBaBababaBaBabaBaBaBaBabababaBababaBabaBaBab$$

we get a deficiency one presentation for the cover $\widehat{G}_2(3)$ which has order 12737088. To enumerate the cosets of the trivial subgroup with the `Mendelsohn` strategy uses a maximum of 478295191 and total 479505174 cosets.

Now we can amalgamate the first two relators giving a^2b^3 and adjust (in one of very many different possible ways, as described in [3]) the exponents of the final relator to obtain:

$$aBaBaBaBaBaBababaBaBAbABABABABAbAbAbABAbAbABAbABABAb$$

which combine to give us an efficient presentation for the cover. We note that this presentation is **not** satisfied by the standard generators of the online Atlas [21] (for which no presentation is given).

Given the difficulty of the enumeration with the deficiency one presentation, we compare enumerations over the subgroup $\langle ab \rangle$. For the deficiency one presentation the index 326592 enumeration uses a maximum 15462319 and total 15539007 cosets while the enumeration with this given efficient presentation uses a maximum 79431266 and total 104749720.

3.8. $\mathbf{S}_4(5)$**.** We tabulate third relators (in order of nondecreasing length) which complement $(wZw)^2$ and $(Wz)^3$ to present $S_4(5)$, together with maximum and total cosets required for enumerations over the trivial subgroup, index 4680000. Note that all the coset enumerations are quite easy.

TABLE 4. Third relator for $S_4(5)$

Relator	Maximum	Total
$w^3z^2w^5z^4w^5z^2$	4680000	4704212
$w^2z^3w^3z^2wzw^2z^3w^2zwz^2w^3z^3$	4680000	4962393
$w^3zwzw^2zwz^2wzw^4zwz^2wzw^2zwz$	4680000	5342417
$w^3z^3w^3z^4w^4z^4w^4z^4w^3z^3$	4680000	4766748

We note that this group also turns up as a one-relator quotient of $C_2 * C_{13}$:

$$\{x^2, y^{13}, xyxy^3xy^2xy^{-3}xy^6\}.$$

The enumeration for this presentation is also easy, using a maximum of 4680000 and total 6201255 cosets.

4. Other groups

In this section we provide efficient presentations mainly for groups with order greater than 5 million. Of these, only one, $L_2(19^2)$, was previously known to be efficient.

4.1. **2-dimensional linear groups.** Previous work has shown that it is relatively easy to show that 2-dimensional linear groups are efficient. We tabulate (Table 5) some larger such groups, not previously known to be efficient, for which we have found efficient presentations as one-relator quotients of the modular group. We tabulate third relators (in order of nondecreasing length) which complement $(wZw)^2$ and $(Wz)^3$ to present the group, together with group order and maximum and total cosets required for enumerations over the trivial subgroup.

We note than an efficient presentation was previously reported for $L_2(19^2)$ in [6]. That presentation does not appear in our list since its relator length is outside the range we investigated here, whereas it has short syllable length which explains why it was found.

A linear word is a palindrome if it reads the same forwards and backwards. However group relators are best thought of as cyclic words. We say a presentation is **palindromic** if each of its relators when written around a circle can be read the same forwards and backwards from at least one place on the circle. The efficient presentations for $L_2(p)$ (for all odd primes p) given in [5] are palindromic, as are

Table 5. Third relator for some linear groups

Relator	Maximum	Total
Third relator for $L_2(3^5)$, order 7174332		
$wz^2w^2zw^{14}zw^2z^2$	72600049	73819479
$wzwz^2wz^3w^3zw^2zw^3z^3wz^2wz$	14758861	16910148
$wz^4wz^2wz^2wzw^2zw^2zw^2zwz^2wz^2wz^4$	76727537	77204593
$w^5zwz^2wzwz^2wzw^6zwz^2wzwz^2wz$	147759715	153131529
$wz^2wz^4w^2z^2w^4zw^2zw^4z^2w^2z^4wz^2$	20473674	26780217
Third relator for $L_2(17^2)$, order 12068640		
$wzw^2zwzwz^2w^6z^2wzwzw^2z$	21266263	24887892
$wzwz^4w^5z^4w^5z^4wz$	30017724	31397328
$w^3zwz^4w^3z^3w^4z^3w^3z^4wz$	120074333	125448989
$w^4z^3wz^5wzw^5zwz^5wz^3$	280550187	292430044
Third relator for $L_2(7^3)$, order 20176632		
$wz^2wzw^2z^2w^2z^2w^2z^2w^2z^2w^2zwz^2$	20176632	27577156
$wzwzwzwz^4wzwzw^2zwzwz^4wzwzwz$	61265317	66077373
$w^4z^4w^2z^2w^4z^5w^4z^2w^2z^4$	53383115	64531504
Third relator for $L_2(19^2)$, order 23522760		
$w^2zwz^2w^5z^3w^5z^2wz$	307943913	309894043
$w^2z^2w^2z^2wzwz^5wzwz^2w^2z^2$	23522760	29390662
$w^2z^3w^2z^2w^{11}z^2w^2z^3$	251881533	253137098
$w^2z^8w^2z^2w^3z^2w^2z^8$	78665641	81390063
$w^3z^3wzw^4zwz^4wzw^4zwz^3$	34893548	57037888
$w^5z^2wzw^2z^5w^6z^5w^2zwz^2$	161703657	174365181
$w^4z^2wzw^2z^2wzw^2z^5w^2zwz^2w^2zwz^2$	166227051	187655294
$wzw^2zwzwz^4wzwzw^4zwzwz^4wzwzw^2z$	37169063	53385139
$w^2z^3w^4z^2w^2z^4w^3z^4w^2z^2w^4z^3$	148086119	180629558
Third relator for $L_2(23^2)$, order 74017680		
$wzw^2zwzw^2z^2w^4z^2w^2zwzw^2z$	145105980	172170274
$wz^2wz^6w^8z^6wz^2$	113525791	139448007
$w^2zwz^3wzwzw^3zwz^3wzw^3zwzwz^3wz$	184168005	222297555
$w^2zwz^2w^2z^2w^3z^2w^2z^3w^2z^2w^3z^2w^2z^2wz$	239540807	258330801

all presentations on $\{w, z\}$ in this paper. We conjecture that all simple 2-dimensional linear groups have palindromic presentations.

We note that various other groups, including some very large direct products, can be presented as one relator quotients of the modular group with palindromic relators. A more general investigation of one relator quotients of the modular group, without insistence on palindromic relators, is in [8].

4.2. $\mathbf{L}_4(3)$. Using Method 2 we investigated 4106 random generating pairs for $L_4(3)$, taking about 136 cpu days. We obtained initial presentations with between 5 and 95 relators. We reduced 2 of these presentations to efficient presentations for $L_4(3)$:

$$BAB^2ABaBaBAB, AbabAbA^2b^2aBab^2, ABA^3BaBa^3BaBA^3BAB$$

which uses a maximum of 684365079 and total 718575037 cosets to enumerate the 6065280 cosets of the trivial subgroup, and:

$$BA^5B^3aB, B^{13}, ABa^6BAB^2ab$$

which uses a maximum of 57948651 and total 60171938 cosets.

4.3. $\mathbf{U}_3(8)$. Using Method 2 we investigated 3906 random generating pairs for $U_3(8)$, taking about 136 cpu days. We obtained initial presentations with between 5 and 98 relators. We reduced two of these presentations to efficient presentations for $U_3(8)$:

$$(ababa^2)^2, BAbaBA^3BabABA^2, bAB^2A^2BA^2B^2Ab^2a^2b$$

which uses a maximum of 188313099 and total 191400773 cosets to enumerate the 5515776 cosets of the trivial subgroup, and:

$$ABABa^2B^3A, BABABABab^3aBA, bAb^2AbAbABABAb^5ab^3abA$$

which uses a maximum of 20111706 and total 27583250 cosets.

4.4. $\mathbf{A}_{11}$. This group provides an interesting illustration of the effectiveness of the extension of Theorem 2.1 to presentations with more relations.

We applied each of our methods to investigate presentations for A_{11}. One variant of Method 2 restricts generating pairs to specific conjugacy classes. We looked at all essentially different generating pairs with orders (2,3), (2,4), (2,5) and (3,4) and their associated presentations. The (2,5) case took about 160 cpu hours to complete.

Of all of these, only in the (2,5) case did we find presentations with 6 initial relations, indeed three of them. The relator sets (as found) were:

$$a^2, B^5, (baBa)^4, (babaBaB^2ab)^2, (aB)^{11}, (bababaBaBaBa)^2;$$

$$a^2, B^5, (baB^2aBaB^2ab)^2, (baBab^2abaBaBab)^2, (Baba)^7, (aB)^{15}; \text{ and}$$

$$a^2, B^5, (aB)^5, (baBa)^6, (baBab^2aBaB^2aB^2ab)^2, (baBab^2aB^2aB^2aBab)^2.$$

Coset enumerations with them are relatively easy. In each case, enumerations over the trivial subgroup complete using a maximum of 19958400 cosets (that is, the group order) and totals 29347326, 25498310 and 21492223, respectively.

Each of these sets consists entirely of power relations, giving many options for amalgamation, leading to either the group itself or its cover. We started with the first relator set.

Replacing relators 3 and 4 by the amalgamation

$$baBabaBabaBabaBababaBaB^2ab^2abaBaB^2ab$$

preserves the group (and coset enumeration becomes about four times harder). Next, amalgamating original relators 5 and 6 to give $BaBaBaBaBaBaBaB^2aBaBababababaBaBaB$ again preserves the group (and the enumeration is not significantly harder).

Finally we amalgamate the first two relators to give a^2b^5 and we have a 3 relator presentation. Since it is perfect, it must be for A_{11} or its cover. Coset enumeration over the trivial subgroup confirms that it is the group itself. This coset enumeration is quite hard; with the `Mendelsohn` strategy it uses a maximum of 1490482794 and total 1727476818 cosets.

In a subsequent computation we found another efficient presentation for A_{11} which is much better for coset enumeration. This time using randomly selected generating sets we investigated 371 presentations in about 94 cpu days. We were able to reduce an initial 7 relator set on generators with orders 8 and 6 to a 4 relator set:

$$bABA^2Ba^2bAB^2a^2, (abAbaBa)^2, bA^2bA^4BabaBa^2, (BA)^9.$$

Again we amalgamate the power relators, this time giving

$$BABABABABABABABA^2baBa^2bAbaBa.$$

With these three relators we can enumerate the cosets of the trivial subgroup (again with the `Mendelsohn` strategy) using a maximum of 47094777 and total 68743286 cosets.

4.5. **$\mathbf{U}_3(5)$ revisited.** In [4] we pointed out that $U_3(5)$ (order 126000) was particulary difficult to handle with our methods. Indeed, we failed to find an efficient presentation for its cover, $\widehat{U}_3(5)$ (order 378000). This remains the case. However, by using the methods described here, we have found some better presentations and we list their canonical (see [16]) representatives.

TABLE 6. Better efficient presentations for $U_3(5)$

Relators	Length	Maximum	Total
$a^5, a^2bABabaBAb, ab^2Ab^2ab^2AB^2$	28	1031903	1240589
$a^7, abAb^2abAB^3, a^2baba^2b^2Ab^2$	30	223016	223400

The first of these presentations for $U_3(5)$ is shorter than our previous shortest. The second presentation has the same length as our previous shortest, but enumerates better than both that and the shorter one.

For the cover, $\widehat{U}_3(5)$, our previous shortest presentation has length 35. We have found many shorter ones and list canonical representatives of the five shortest that we have found.

Table 7. Shorter presentations for $\widehat{U}_3(5)$

Relators	Length	Maximum	Total
$a^7, abababaB^4, a^2BAbABa^2BAB$	30	4934202	5243558
$a^7, abAb^3abAB^2, a^2bAba^2B^2aB^2$	30	384634	610272
$a^8, a^2baBaba^2b^2, abAbabAbaB^2$	30	12243107	12950471
$a^8, ababAbabaB, a^2B^2abAbAbaB^2$	31	13595758	14002922
$(ab)^3, a^3bAbAbAbAb, a^2B^3abAbaB^3$	31	380511	589284

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