Efficient universal quantum computation with auxiliary Hilbert space

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We propose a scheme to construct the efficient universal quantum circuit for qubit systems with the assistance of possibly available auxiliary Hilbert spaces. An elementary two-ququart gate, termed the controlled-double-NOT gate, is proposed first in ququart (four-level) systems, and its physical implementation is illustrated in the four-dimensional Hilbert spaces built by the path and polarization states of photons. Then an efficient universal quantum circuit for ququart systems is constructed using the gate and the quantum Shannon decomposition method. By introducing auxiliary two-dimensional Hilbert spaces, the universal quantum circuit for qubit systems is finally achieved using the result obtained in ququart systems with the lowest complexity.

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Building a large-scale universal quantum computer (UQC) [1] capable of implementing arbitrary unitary operations is a major challenge in the field of quantum information science. The UQC can be built by physically implementing a universal quantum circuit [2–5], constructed by an ordered combination of elementary quantum gates which consist of one-qubit and two-qubit controlled-NOT (CNOT) gates. However, this is difficult to be realized in experiments when the complexity of quantum circuits (normally defined by the number of elementary two-qubit gates) is high. In fact, the complexity of the UQC grows exponentially with the number of qubits, and a large number of qubits is needed in practical applications. At present, the UQC has been achieved experimentally only in two-qubit systems with three elementary two-qubit gates [6]. Therefore, it is of crucial importance to find an efficient way to build a universal quantum circuit that has the lowest complexity.

In qubit systems, several matrix decomposition methods, such as the orthogonal-triangular decomposition (QR) [7], the cosine-sine decomposition (CSD) [8], and the quantum Shannon decomposition (QSD), also called n qubits decomposition (NQ) [9,10], have been introduced to construct a universal quantum circuit. Among them, the QSD method gives the best result where the complexity is $4^n \times 23/48 - 2^n \times 3/2 + 4/3$ for an n-qubit universal quantum circuit. Nonetheless, there is a gap toward the theoretical lower bound of $(4^n - 3n - 1)/4$ [11]. We note that all these results (including the lower bound) are obtained without the assistance of auxiliary dimensions or degrees of freedom (DOFs). This gives us the opportunity to further optimize the universal quantum circuit in qubit systems with the help of higher-level physical systems. An example can be found from Refs. [12,13], in which the three-qubit Toffoli gate and the general two-qubit controlled-unitary gate were significantly simplified by harnessing multilevel information carriers.

To simplify the UQC in qubit systems with higher-level physical systems, we may expand the qubits to qudits (d-level) by introducing auxiliary Hilbert spaces and then accomplish the general unitary operation of qubit systems by constructing the universal quantum circuits in qudit systems. For the UQC in qudit systems, there has also been significant progress to reduce the number of elementary two-qudit gates, such as the $\Gamma_2[Y_d]$ gate [14] and CINC gate [15,16], by using the spectral decomposition [14], QR [15,17], and CSD [18,19] methods. On the other hand, the auxiliary Hilbert spaces should have as few dimensions as possible from the viewpoint of physical implementation. Here, we expand the qubits to ququart (four-level) and propose a novel fundamental two-ququart gate, termed the controlled-double-NOT (CDNOT) gate. Based on the proposed gates and the QSD method, we construct the universal quantum circuit in ququart systems and then achieve the simplification of the UQC in qubit systems with the lowest complexity in the sense that the least-controlled operations are required both in ququart and qubit systems. For physical implementation, we pay special attention to the photonic systems. It is natural to use the path degree of freedom as auxiliary Hilbert spaces for polarized photon qubits, and thus qubits are expanded to ququarts (four-level). The lower complexity of UQC in qubit systems here means that the number of controlled operations between photons is significantly reduced.

The controlled-double-NOT gate in ququart systems. Consider a four-level generalization of the qubit, i.e., ququart, whose orthonormal basis is denoted by $|i\rangle$, $i = 0,1,2,3$. Let us define the single-ququart gate X, which acts σ_x on the two subspaces $\{|0\rangle,|2\rangle\}$ and $\{|1\rangle,|3\rangle\}$ of a ququart simultaneously by the matrix $X = \sigma_x \otimes I_2$, where below σ_x and σ_z are the Pauli matrices and I_l denotes the unit matrix with dimension l. For two ququarts, we introduce a new generalization of the CNOT gate, the controlled-double-NOT (CDNOT) gate, denoted by $\bigtriangledown(X)$,

$$\bigtriangledown(X)\,|ii\rangle_{AB} = \begin{cases} |i\rangle_A \otimes X|i\rangle_B & \text{if } |i\rangle_A = |2\rangle \text{ or } |3\rangle, \\ |ii\rangle_{AB} & \text{else,} \end{cases} \tag{1}$$

which means if the control ququart A is in states $|2\rangle$ or $|3\rangle$, then X acts on the target ququart B. Similarly, the controlled-double-phase-flip (CDPF) gate ($\bigtriangledown(P)$) and the general controlled-double-U gate ($\bigtriangledown(U)$) can be defined by replacing X with P and U in Eq. (1), respectively, where $P = \sigma_z \otimes I_2$ and U is an arbitrary unitary operation. The circuit representations for the CDNOT gate and CDPF gate are shown in Fig. 1.

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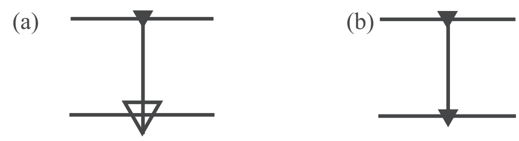

FIG. 1. (a) Circuit representation for the controlled-double-NOT gate $\bigtriangledown(X)$. The control (target) ququart is denoted by the closed (open) triangle. (b) Circuit representation for the controlled-double-phase-flip gate $\bigtriangledown(P)$.

Realization of the elementary ququart gates. We illustrate the realization of the ququart gates with optical two DOF systems. Let us build ququarts by the tensor product of the path (S,T) and polarization (H,V) of photons by defining the basis states as $|j\rangle = \{|SH\rangle, |SV\rangle, |TH\rangle, |TV\rangle\}$, where H (V) denotes the horizontal (vertical) polarization.

The arbitrary operation U of a single-photon ququart can be implemented in linear optical systems from the CSD decomposition [20], following the relation

$$U = \left(W_1^1 \oplus W_2^1\right) E \left(R_y^1 \oplus R_y^2\right) E \left(W_1^2 \oplus W_2^2\right), \qquad (2)$$

where W_l^g $(l = 1,2; g = 1,2)$ are unitary 2×2 matrices and $R_y^g(\theta_g) = \exp(i\sigma_y\theta_g)$; W_1^g, R_y^1 (W_2^g, R_y^2) act on the polarization state of the photon through polarization rotations in the path S (T); the operator E, defined as $E \equiv |0\rangle\langle 0| + |1\rangle\langle 2| + |2\rangle\langle 1| + |3\rangle\langle 3|$, exchanges the states between the two DOFs, and can be realized as in Fig. 2.

Because of the relation $X = \tilde{H}P\tilde{H}$, where $\tilde{H} = H \otimes I_2$ and H is the Hadamard gate, there exists the relation

$$\bigtriangledown(X) = (I_4 \otimes \tilde{H}) \bigtriangledown(P)(I_4 \otimes \tilde{H}). \qquad (3)$$

That means we only need to realize the CDPF gate $\bigtriangledown(P)$ and single ququart gate $\tilde{H}$ for implementing the CDNOT gate $\bigtriangledown(X)$. For two photons A, B with basis $|j\rangle_A|j\rangle_B$, the gate $\bigtriangledown(P)$ flips the phase of any polarization state of photon B in path T if the photon A is in path T and has no effect otherwise and can be realized by adapting the cavity-assisted setup proposed in Ref. [21], as illustrated in Fig. 3.

It is important to note that the difficulty for realizing the CDNOT gate is no more than the CNOT gate and CINC gate in term of the number of cavities, because only one cavity is required for one CDNOT gate.

QSD-based universal quantum circuit in ququart systems. In the following, we shall constructively prove the result: the CDNOT gate together with single-ququart operations can be used to form a new elementary quantum gate library for the four-level quantum computation.

Using the QSD method [9], we decompose the general n-ququart unitary operator U layer by layer. In each layer, one

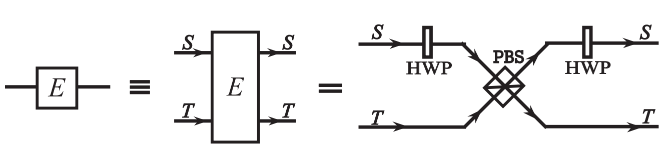

FIG. 2. Optical realization of the exchange operator E, where the PBS transmits photons in the $|H\rangle$ state while reflects photons in the $|V\rangle$ state. The half-wave plates $(\lambda/2)$ are set at 45° so as to convert the polarization of photon in path S from H (V) to V (H).

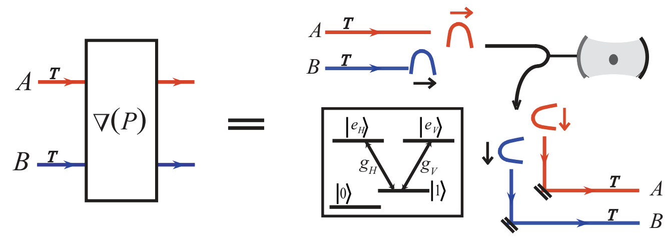

FIG. 3. (Color online) Schematic setup of implementing operation $\bigtriangledown(P)$ with the cavity-assisted interactions. The relevant energy level of the trapped atom [22] is shown in the inset. The atomic transitions $|1\rangle \to |e_L\rangle$ $(L = H,V)$ are resonantly coupled to the cavity modes a_L.

ququart will be separated out from other $n-1$ ququarts and the QSD method need to be used twice. The decomposition will be finished until only single-ququart operations are left. Take the first layer as an example: The quantum circuit is illustrated in Fig. 4(a) and the decomposition can be expressed as

$$\begin{aligned} U &= \left(A_0^1 \oplus B_0^1\right) Y_0^1 \left(A_1^1 \oplus B_1^1\right), \\ \left(A_q^1 \oplus B_q^1\right) &= \left(I_2 \otimes V_{2q}^1\right) Z_q^1 \left(I_2 \otimes V_{2q+1}^1\right), \\ V_p^1 &= \left(A_{2p}^2 \oplus B_{2p}^2\right) Y_p^2 \left(A_{2p+1}^2 \oplus B_{2p+1}^2\right), \\ \left(A_l^2 \oplus B_l^2\right) &= \left(I_2 \otimes W_{2l}\right) Z_l^2 \left(I_2 \otimes W_{2l+1}\right), \end{aligned} \qquad (4)$$

where $q = 0,1$; $p = 0,1,2,3$; $l = 2p+q$; and W_{2l+i} $(i = 0,1)$ are $(n-1)$-ququart operations. We then decompose the second layer where we take each W_{2l+i} as U and continue the recursion. The recursion will be finished until the encountered W are single-ququart operations. After the above decompositions, it can be seen that only the operators $Y_{p(g-1)}^g$ and $Z_{2p(g-1)+q}^g$, $g = 1,2$, are left besides single-ququart operations.

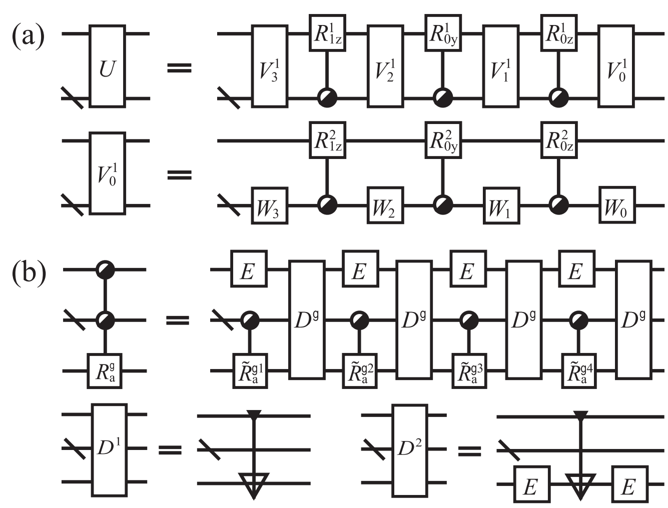

FIG. 4. (a) Recursive quantum circuits implementing the UQC in ququart systems, where the gates V_p^1 have the same decomposition form as gate V_0^1. The controlled gates R_{0y}^g (R_{qz}^g) with half-closed circle in control ququarts represent the uniformly controlled gate $F_n^{n-1}(R_y^g)$ $(F_n^{n-1}(R_z^g))$ corresponding to Y_0^g (Z_q^g) in Eq. (4). The line with the backslash symbol (\) represents more than one wire in the quantum circuit. (b) Recursive quantum circuits of the gate $F_n^{n-1}(R_{\mathbf{a}}^1)$, where some gates $\tilde{D}_n^g$ have been canceled each other out.

Here each matrix $Y^{g}_{p(g-1)}(Z^{g}_{2p(g-1)+q})$ represents a $(n-1)$-fold uniformly controlled gate $F_n^{n-1}(R_y^g)$ $(F_n^{n-1}(R_z^g))$ [8,9,19] with the first ququart as the target ququart. If we let the last ququart be the target ququart, the matrix of $F_n^{n-1}(R_{\mathbf{a}}^g)$ can be written as diag$[R_{\mathbf{a}}^g(\alpha_1,\beta_1),R_{\mathbf{a}}^g(\alpha_2,\beta_2),\ldots,R_{\mathbf{a}}^g(\alpha_{4^{n-1}},\beta_{4^{n-1}})]$, which represents 4^{n-1} different $(n-1)$-fold controlled rotations $R_{\mathbf{a}}^g(\alpha,\beta)$ of the target ququart, and $R_{\mathbf{a}}^1(\alpha,\beta)=E(R_{\mathbf{a}}(\alpha)\oplus R_{\mathbf{a}}(\beta))E$, $R_{\mathbf{a}}^2(\alpha,\beta)=R_{\mathbf{a}}(\alpha)\oplus R_{\mathbf{a}}(\beta)$, $R_{\mathbf{a}}(\phi)=\exp(i\mathbf{a}\cdot\boldsymbol{\sigma}\phi/2)$, $\mathbf{a}\cdot\boldsymbol{\sigma}=a_x\sigma_x+a_y\sigma_y+a_z\sigma_z$. For $R_{\mathbf{a}}^2(\alpha,\beta)$, $\alpha=\beta$.

Now we need to decompose the n-ququart uniformly controlled gate $F_n^{n-1}(R_{\mathbf{a}}^g)$ recursively, and one ququart is separated out each time. The recursion will be stopped until the single-ququart gate $F_1^0(R_{\mathbf{a}}^g)$ is encountered. The recursive quantum circuits are shown in Fig. 4(b), and the first layer of the decomposition is expressed as

$$\begin{aligned} F_n^{n-1}\left(R_{\mathbf{a}}^g\right) &= D_n^g\tilde{F}\left(R_{\mathbf{a}}^g\right)D_n^g\tilde{F}\left(R_{\mathbf{a}}^g\right), \\ \tilde{F}\left(R_{\mathbf{a}}^g\right) &= \tilde{D}_n^g\left[I_4\otimes F_{n-1}^{n-2}\left(R_{\mathbf{a}}^g\right)\right]\tilde{D}_n^g\left[I_4\otimes F_{n-1}^{n-2}\left(R_{\mathbf{a}}^g\right)\right], \end{aligned} \tag{5}$$

where $\tilde{D}_n^g=(E\otimes I_{4^{n-1}})D_n^g(E\otimes I_{4^{n-1}})$, $D_n^2=(I_{4^{n-1}}\otimes E)D_n^1(I_{4^{n-1}}\otimes E)$, and $D_n^1=I_{2\times4^{n-1}}\oplus(\bigoplus_{i=1}^{2\times4^{n-2}}X)$ is actually a CDNOT gate with the first (last) ququart as the control (target) ququart. Note that in the first decomposition in Eq. (5), $\tilde{F}(R_{\mathbf{a}}^g)=I_2\otimes(\bigoplus_{i=1}^{2\times4^{n-2}}R_{\mathbf{a}}^g(\tilde{\alpha}_i,\tilde{\beta}_i))$ and $(\tilde{\alpha}_i,\tilde{\beta}_i)$ can be solved by using the relations $R_{\mathbf{a}}^g(\tilde{\alpha}_1,\tilde{\beta}_1)R_{\mathbf{a}}^g(\tilde{\alpha}_2,\tilde{\beta}_2)=R_{\mathbf{a}}^g(\tilde{\alpha}_1+\tilde{\alpha}_2,\tilde{\beta}_1+\tilde{\beta}_2)$ and $XR_{\mathbf{a}}^1(\tilde{\alpha}_1,\tilde{\beta}_1)X=R_{\mathbf{a}}^1(-\tilde{\alpha}_1,-\tilde{\beta}_1)$. In each decomposition, the order of matrices can be reversed and some of the CDNOT gates may cancel each other out, i.e., $\tilde{D}_n^gD_n^g\tilde{D}_n^g=D_n^g$ and $D_n^g\tilde{D}_n^gD_n^g=\tilde{D}_n^g$.

Finally, we can see that the universal quantum circuit for realizing the general unitary operator U in ququart systems only contains an alternating sequence of CDNOT gates and one-ququart gates. From Fig. 4, we can get that the number of CDNOT gates needed to realize the gate $F_n^{n-1}(R_{\mathbf{a}})$ is 4^{n-1}, and the number of CDNOT gates required for the general $4^n\times4^n$ unitary operator U is

$$N_n^4=5\times(4^{2(n-1)}-4^{n-1}). \tag{6}$$

It can be seen from Table I that, compared with previous results [15,17,19] in ququart systems, the number of CDNOT gates required here is minimum.

Simplifying universal quantum circuit in qubit systems. We now propose a scheme for simplifying universal quantum circuit in qubit systems assisted with auxiliary DOF. We first consider a general unitary operator U acting on m qubits, represented by $|\psi\rangle\mapsto U|\psi\rangle$, where m is odd and we construct the quantum circuit in four steps, as shown in Fig. 5(a). First, we pair the qubits $A_1,A_2,\ldots,A_i,\ldots,A_m$ as $A_2A_3,A_4A_5,\ldots,A_{2j}A_{2j+1},\ldots,A_{m-1}A_m$. For one qubit of each pair, say when i is even, we use their auxiliary two-level DOF to construct ququarts by tensor producting the two DOFs.

TABLE I. Comparison of the numbers of elementary two-ququart gates needed to implement a general n-ququart unitary operation using the QR, CSD and our method. The upper two rows and the bold lower row denote the numbers of CINC and CDNOT gates, respectively.

n	2	3	4	6	7
QR	440	21 248	396 288	9.9×10^7	1.5×10^9
CSD	72	4464	40 824	2.2×10^7	3.6×10^8
Our method	**60**	**1200**	**20 160**	$\mathbf{5.2\times10^6}$	$\mathbf{8.4\times10^7}$

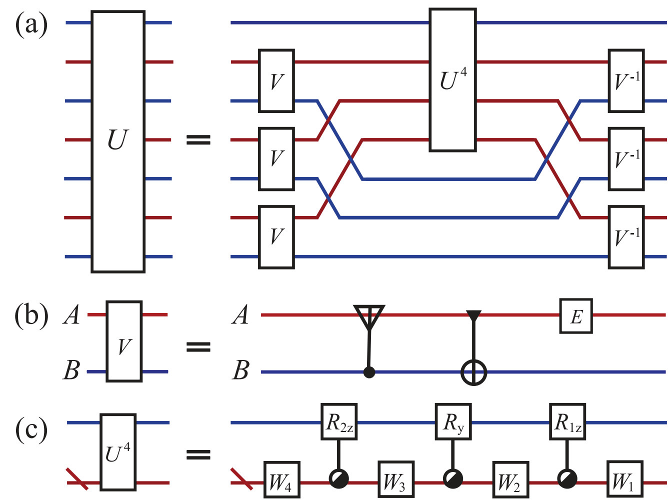

FIG. 5. (Color online) (a) Quantum circuit for a general seven-qubit unitary operation assisted with auxiliary DOF, where $U^4=U$. The ququarts (qubits) are denoted by the bold red lines (blue lines). (b) Quantum circuit for realizing the transferring operator V. The left controlled gate applies operation X to ququart A if qubit B is in state $|1\rangle$, and the right controlled gate flips the state of qubit B if ququart A is in state $|2\rangle$ or $|3\rangle$. (c) Quantum circuit for realizing U^4. The controlled gates between the qubit and ququart in (b) and (c) can be implemented as in ququart systems after expanding the state space of the qubit (red line) to four levels.

Then, by executing the transferring operator V, shown in Fig. 5(b), we transfer the states of each pair of qubits $A_{2j}A_{2j+1}$ to the ququarts A_{2j} with no change to the states, and let the state of A_{2j+1} to $|0\rangle$. That means the original state $|\psi\rangle$ is converted into $|\psi_1\rangle=|\psi\rangle_{A_1,A_2,A_4,\ldots,A_{m-1}}\otimes|00\ldots0\rangle_{A_3,A_5,\ldots,A_m}$.

Third, we need to realize U in the space of the qubit A_1 and ququarts $A_2,A_4,\ldots,A_{m-1}$. We decompose U with the QSD method and obtain four unitary matrices W_i $(i=1,2,3,4)$, as shown in Fig. 5(c), which are actually operators for ququart systems, so we can implement W_i by using the procedure of ququart systems. As a result, the state $|\psi_2\rangle=U|\psi\rangle\otimes|00\ldots0\rangle$ is yielded.

Finally, we achieve the operation U by performing the reverse process V^{-1} of transferring operator V to every pair of $A_{2j}A_{2j+1}$.

If m is even, the decomposition of U in the third step is not required, and we only need to implement it directly with the procedure of ququart systems.

Using Eq. (6), we can compute the total number of bipartite controlled gates N_m^2 in the universal quantum circuit of m-qubit systems, yielding

$$N_m^2=\begin{cases}(5/16)\times4^m-(5/4)\times2^m+2m & m\text{ even},\\(5/16)\times4^m-2^m+2(m-1) & m\text{ odd}.\end{cases} \tag{7}$$

From Table II, we can see that our method provides the most efficient universal quantum circuit in qubit systems.

TABLE II. Comparison of the numbers of elementary bipartite controlled gates needed to implement a general m-qubit unitary operation using different methods, among which only our results are derived with auxiliary Hilbert spaces.

m	2	3	4	5	6	7
QR [7]	4	64	536	4156	22 618	108 760
CSD [8]	8	48	224	960	3968	16 128
QSD [9,10]	3	20	100	444	1868	7660
Our method	**4**	**16**	**68**	**296**	**1212**	**5004**
Lower bounds [11]	3	14	61	252	1020	4091

Conclusion. The efficient universal quantum computation with the lowest complexity in qubit systems is achieved by expanding the qubits to ququarts with auxiliary Hilbert spaces and then accomplishing the general unitary operations with a universal quantum circuit constructed in ququart systems, in which a new elementary two-ququart gate CDNOT gate is proposed. In photonic systems, with the help of path degree of freedom, the polarized photon qubits are expanded to ququarts, and the lower complexity of UQC means that the number of controlled operations between photons is significantly reduced by changing controlled operations between photons to operations between different degrees of freedom (path and polarization) of photons, which is comparatively easy, and, more importantly, deterministic in contrast to probabilistic in controlled operations between photons. The UQC scheme proposed here paves the way to simplify the complicated quantum computation with assistance of auxiliary Hilbert spaces, e.g., some special algorithms.

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