Equivariant quantization of orbifolds

Norbert Poncin

Mathematics Research Unit
University of Luxembourg

Current Geometry, Levi-Civita Institute, Vietri, 2010

Outline

Equivariant quantization of

- vector spaces
- smooth manifolds
- foliated manifolds
- orbifolds
- supermanifolds

People:

A. Cap, M. Bordemann, C. Duval, H. Gargoubi, J. George, P. Lecomte, P. Mathonet, J.-P. Michel, V. Ovsienko, F. Radoux, J. Silhan, R. Wolak, ..., P

Geometric characterization of quantization

$$D \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(F))_{\mathrm{loc}} \overset{\mathrm{loc}}{\simeq} \mathrm{Hom}_{\mathbb{R}}(C^\infty(U, \bar{E}), C^\infty(U, \bar{F}))_{\mathrm{loc}}$$

$$f \in C^\infty(U), e \in \bar{E}, x \in U, \xi \in (\mathbb{R}^m)^*$$

$$\begin{aligned} D(fe) = & \sum_\alpha D_{\alpha,x}(e)\partial_{x^1}^{\alpha^1} \dots \partial_{x^m}^{\alpha^m} f \\ & \simeq \sum_\alpha D_{\alpha,x} \ (e \ )\xi_1^{\alpha^1} \dots \xi_m^{\alpha^m} \\ & = \sigma_{\mathrm{aff}}(D)(\xi; e) \end{aligned}$$

Differential operator $\overset{\mathrm{loc}}{\simeq}$ polynomial, total affine symbol

$$D \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(F))_{\mathrm{loc}} \overset{\mathrm{loc}}{\simeq}$$

$$D \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(F))_{\mathrm{loc}} \overset{\mathrm{loc}}{\simeq} \mathrm{Hom}_{\mathbb{R}}(C^\infty(U, \bar{E}), C^\infty(U, \bar{F}))_{\mathrm{loc}}$$

$f \in C^\infty(U), e \in \bar{E}, x \in U, \xi \in (\mathbb{R}^m)^*$

$$\begin{aligned} D(fe) = & \sum_\alpha D_{\alpha,x}(e) \partial_{x^1}^{\alpha^1} \dots \partial_{x^m}^{\alpha^m} f \\ & \simeq \sum_\alpha D_{\alpha,x}(e) \xi_1^{\alpha^1} \dots \xi_m^{\alpha^m} \\ & = \sigma_{\mathrm{aff}}(D)(\xi; e) \end{aligned}$$

Differential operator $\overset{\mathrm{loc}}{\simeq}$ polynomial, total affine symbol

$$D \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(F))_{\mathrm{loc}} \overset{\mathrm{loc}}{\simeq} \sigma_{\mathrm{aff}}(D) \in \Gamma(\mathcal{S}TU \otimes E^* \otimes F)$$

$$D \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(F))_{\mathrm{loc}} \overset{\mathrm{loc}}{\simeq} \mathrm{Hom}_{\mathbb{R}}(C^\infty(U, \bar{E}), C^\infty(U, \bar{F}))_{\mathrm{loc}}$$

$$f \in C^\infty(U), e \in \bar{E}, x \in U, \xi \in (\mathbb{R}^m)^*$$

$$\begin{aligned} D(fe) = & \sum_\alpha D_{\alpha,x}(e)\partial_{x^1}^{\alpha^1} \dots \partial_{x^m}^{\alpha^m} f \\ & \simeq \sum_\alpha D_{\alpha,x}(e)\xi_1^{\alpha^1} \dots \xi_m^{\alpha^m} \\ & = \sigma_{\mathrm{aff}}(D)(\xi; e) \end{aligned}$$

Differential operator $\overset{\mathrm{loc}}{\simeq}$ polynomial, total affine symbol

$$D \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(F))_{\mathrm{loc}} \overset{\mathrm{loc}}{\simeq} \sigma_{\mathrm{aff}}(D) \in \Gamma(\mathcal{S}TU \otimes E^* \otimes F)$$

Example:

Intertwining condition:

$$(L_X(TD) - T(L_X D))\,(\omega) = 0$$

Symbolic representation:

$$\begin{aligned}&(X.T)(\eta; D)(\xi;\omega) - \langle X,\eta\rangle\,((\tau_\zeta T)(\eta; D))\,(\xi;\omega)\\ &-\langle X,\xi\rangle\,(\tau_\zeta(T(\eta; D)))\,(\xi;\omega) + T(\eta+\zeta; X\tau_\zeta D)(\xi;\omega)\\ &-T(\eta; D)(\xi+\zeta;\zeta\wedge i_X\omega) + T(\eta+\zeta; D(\cdot+\zeta;\zeta\wedge i_X\cdot))(\xi;\omega) = 0\end{aligned}$$

Applications:

Flato-Lichnerowicz, De Wilde-Lecomte: cohomology of vector fields valued in differential forms

P: cohomology of the Nijenhuis-Richardson graded Lie algebra, nonexistence of universal classes

$M = \mathbb{R}^m$, $D \in \mathcal{D}(M)$, ϕ: coordinate change

$$
\begin{array}{ccc}
D(f) = \sum_\alpha D_{\alpha,x}\, \partial_x^\alpha f & \overset{\sigma_{\text{aff}}}{\leftrightarrow} & \sigma_{\text{aff}}(D)(\xi) = \sum_\alpha D_{\alpha,x}\, \xi^\alpha \\
\updownarrow \phi & & \updownarrow \phi \\
\ldots & \overset{\sigma_{\text{aff}}}{\leftrightarrow} & \ldots
\end{array}
$$

$M = \mathbb{R}^m$, $D \in \mathcal{D}(M)$, ϕ: coordinate change

$$
\begin{array}{ccc}
D(f) = \sum_\alpha D_{\alpha,x}\, \partial_x^\alpha f & \overset{\sigma_{\text{aff}}}{\leftrightarrow} & \sigma_{\text{aff}}(D)(\xi) = \sum_\alpha D_{\alpha,x}\, \xi^\alpha \\
\updownarrow \phi & & \updownarrow \phi \\
\dots & \overset{\sigma_{\text{aff}}}{\leftrightarrow} & \dots
\end{array}
$$

Non commutative, $\sigma_{\text{aff}}(D)$ not intrinsic, $\sigma(D)$ geometric meaning

$M = \mathbb{R}^m$, $D \in \mathcal{D}(M)$, ϕ: coordinate change

$$D(f) = \sum_\alpha D_{\alpha,x}\, \partial_x^\alpha f \quad \overset{\sigma_{\text{aff}}}{\leftrightarrow} \quad \sigma_{\text{aff}}(D)(\xi) = \sum_\alpha D_{\alpha,x}\, \xi^\alpha$$

$$\updownarrow \phi \qquad\qquad\qquad\qquad \updownarrow \phi$$

$$\ldots \qquad \overset{\sigma_{\text{aff}}}{\leftrightarrow} \qquad \ldots$$

Non commutative, $\sigma_{\text{aff}}(D)$ not intrinsic, $\sigma(D)$ geometric meaning

Vector space isomorphism:

$$\sigma_{\text{aff}}^{-1} : \text{Pol}(T^*M) = \Gamma(\mathcal{S}TM) =: \mathcal{S}(M) \to \mathcal{D}(M)$$

Nonequivariance (global version):

$$\exists \phi \in \text{Diff}(M) : \sigma_{\text{aff}}^{-1} \circ \phi \neq \phi \circ \sigma_{\text{aff}}^{-1}$$

Nonequivariance (local version):

$$\exists X \in \mathcal{X}(M) : \sigma_{\mathrm{aff}}^{-1} \circ L_X \neq \mathcal{L}_X \circ \sigma_{\mathrm{aff}}^{-1}$$

Nonequivariance (local version):

$$\exists X \in \mathcal{X}(M) : \sigma_{\mathrm{aff}}^{-1} \circ L_X \neq \mathcal{L}_X \circ \sigma_{\mathrm{aff}}^{-1}$$

Definition of an equivariant quantization (EQ) on a vector space [Lecomte, Ovsienko, 99]:

Nonequivariance (local version):

$$\exists X \in \mathcal{X}(M) : \sigma_{\mathrm{aff}}^{-1} \circ L_X \neq \mathcal{L}_X \circ \sigma_{\mathrm{aff}}^{-1}$$

Definition of an equivariant quantization (EQ) on a vector space [Lecomte, Ovsienko, 99]:

$$Q = \sigma_{\mathrm{tot}}^{-1} : \mathcal{S} \quad (M) = \Gamma(\mathcal{S}TM \quad) \overset{\mathrm{vs-isom}}{\to} \mathcal{D} \quad (M),$$

such that

$$\sigma_{\mathrm{tot}}^{-1} \circ L_X = \mathcal{L}_X \circ \sigma_{\mathrm{tot}}^{-1}, \forall X \in \quad \mathcal{X}(M)$$

and

$$\sigma \circ \sigma_{\mathrm{tot}}^{-1} |_{\mathcal{S}^k \ (M)} = \mathrm{id}_{\mathcal{S}^k \ (M)} \quad \text{(normalization)}$$

Nonequivariance (local version):

$$\exists X \in \mathcal{X}(M) : \sigma_{\mathrm{aff}}^{-1} \circ L_X \neq \mathcal{L}_X \circ \sigma_{\mathrm{aff}}^{-1}$$

Definition of an equivariant quantization (EQ) on a vector space [Lecomte, Ovsienko, 99]:

$$Q = \sigma_{\mathrm{tot}}^{-1} : \mathcal{S}_{\delta=\mu-\lambda}(M) = \Gamma(\mathcal{S}TM \otimes \Delta^{\delta} TM) \overset{\mathrm{vs-isom}}{\rightarrow} \mathcal{D}_{\lambda\mu}(M),$$

such that

$$\sigma_{\mathrm{tot}}^{-1} \circ L_X = \mathcal{L}_X \circ \sigma_{\mathrm{tot}}^{-1}, \forall X \in \mathcal{X}(M)$$

and

$$\sigma \circ \sigma_{\mathrm{tot}}^{-1} |_{\mathcal{S}^k{}_{\delta}(M)} = \mathrm{id}_{\mathcal{S}^k{}_{\delta}(M)} \quad \text{(normalization)}$$

Nonequivariance (local version):

$$\exists X \in \mathcal{X}(M) : \sigma_{\text{aff}}^{-1} \circ L_X \neq \mathcal{L}_X \circ \sigma_{\text{aff}}^{-1}$$

Definition of an equivariant quantization (EQ) on a vector space [Lecomte, Ovsienko, 99]:

$$Q = \sigma_{\text{tot}}^{-1} : \mathcal{S}_{\delta=\mu-\lambda}(M) = \Gamma(\mathcal{S}TM \otimes \Delta^{\delta} TM) \overset{\text{vs}-\text{isom}}{\rightarrow} \mathcal{D}_{\lambda\mu}(M),$$

such that

$$\sigma_{\text{tot}}^{-1} \circ L_X = \mathcal{L}_X \circ \sigma_{\text{tot}}^{-1}, \forall X \in \mathcal{X}(M)$$

and

$$\sigma \circ \sigma_{\text{tot}}^{-1} |_{\mathcal{S}^k{}_\delta(M)} = \text{id}_{\mathcal{S}^k{}_\delta(M)} \quad \text{(normalization)}$$

P. Lecomte, P. Mathonet, E. Tousset, 96: $\mathcal{D}_{\lambda\lambda}(M)$ and $\mathcal{D}_{\mu\mu}(M)$ not isomorphic as $\mathcal{X}(M)$-modules

Nonequivariance (local version):

$$\exists X \in \mathcal{X}(M) : \sigma_{\text{aff}}^{-1} \circ L_X \neq \mathcal{L}_X \circ \sigma_{\text{aff}}^{-1}$$

Definition of an $\mathfrak{g}$-equivariant quantization (EQ) on a vector space [Lecomte, Ovsienko, 99]:

$$Q = \sigma_{\text{tot}}^{-1} : \mathcal{S}_{\delta=\mu-\lambda}(M) = \Gamma(\mathcal{S}TM \otimes \Delta^{\delta} TM) \stackrel{\text{vs}-\text{isom}}{\to} \mathcal{D}_{\lambda\mu}(M),$$

such that

$$\sigma_{\text{tot}}^{-1} \circ L_X = \mathcal{L}_X \circ \sigma_{\text{tot}}^{-1}, \forall X \in \mathfrak{g} \subset \mathcal{X}(M)$$

and

$$\sigma \circ \sigma_{\text{tot}}^{-1} |_{\mathcal{S}^k{}_{\delta}(M)} = \text{id}_{\mathcal{S}^k{}_{\delta}(M)} \quad \text{(normalization)}$$

P. Lecomte, P. Mathonet, E. Tousset, 96: $\mathcal{D}_{\lambda\lambda}(M)$ and $\mathcal{D}_{\mu\mu}(M)$ not isomorphic as $\mathcal{X}(M)$-modules

Some motivations:

- Invariant star-products on T^*M obtained as pullbacks by

$$Q_\hbar(P) = \hbar^k Q(P), P \in \mathrm{Pol}^k(T^*M)$$

of the associative structure of the space of differential operators

- Classification of spaces of differential operators as modules over Lie algebras of vector fields

- Role of symmetries in relationship between classical and quantum systems – complete geometric characterization of quantization: a flat fixed G-structure on configuration space guarantees existence and uniqueness of a global $\mathfrak{g}$-equivariant quantization (where $\mathrm{Lie}(G) = \mathfrak{g}$)

Maximal Lie subalgebras $\mathfrak{g} \subset \mathcal{X}(M)$, $M = \mathbb{R}^m$:

Projective case: $G = \mathrm{PGL}(m+1, \mathbb{R})$, $\mathfrak{g} = \mathrm{sl}(m+1, \mathbb{R})$ can be embedded as maximal Lie subalgebra sl_{n+1} into $\mathcal{X}_*(M)$ (Lecomte, Ovsienko, 99)

Conformal case: $G = \mathrm{SO}(p+1, q+1)$ $(p + q = m)$, $\mathfrak{g} = \mathrm{o}(p+1, q+1)$ can be embedded as maximal Lie subalgebra $\mathrm{o}_{p+1,q+1}$ into $\mathcal{X}_*(M)$ (Duval, Lecomte, Ovsienko, 99)

Other cases: ... (Boniver, Mathonet, 01)

THE CASIMIR TECHNIQUE

Differential operators acting on tensor fields

- Projectively equivariant quantization for differential operators on differential forms [Boniver, Hansoul, Mathonet, P, 02]
- Efficiency of equivariant and standard affine symbol calculus as classification tools for modules of differential operators [P,04]
- Automorphisms and derivations of classical and quantum Poisson algebras [Grabowski, P, 04], [Grabowski, P, 05]

  Difference with automorphisms of the classical and the quantum Weyl algebra [Kanel, Kontsevich, 05]

$Q : (\mathcal{S}_{kp}, L_X) \to (\mathcal{D}_p^k \simeq \mathcal{S}_{kp}, \mathcal{L}_X)$: potential sl_{m+1}-EQ

Main observation:

$$Q \circ C = \mathcal{C} \circ Q$$

$$CP = \alpha P \Rightarrow \mathcal{C} QP = \alpha QP$$

Ideas:

- Diagonalization of C: $\mathcal{S}_p^k = \mathcal{A}_p^k \oplus \mathcal{B}_p^k$, eigenvalues α_p^k, β_p^k
- $$\mathcal{C} - C = N \underbrace{(\mathcal{L}_X - L_X)}_{\mathcal{S}_p^k \to \mathcal{S}_p^{k-1}} = \frac{1}{m+1} (\ \delta \ \mathrm{Div} \ \delta^* \quad + \delta^* \, \mathrm{Div} \ \delta)$$

$N : \mathcal{A}_p^k \ni P \to NP \in \mathcal{A}_p^{k-1}$. Set $QP = P + Q_1 P$ and try to define $Q_1 : \mathcal{A}_p^k \to \mathcal{A}_p^{k-1}$.

Since

$$\alpha_p^k P + \overbrace{\alpha_p^k Q_1 P}^{\in \mathcal{A}_p^{k-1}} = QCP = \mathcal{C}QP = \mathcal{C}(P + Q_1 P) =$$

$$(C + N)(P + Q_1 P) = \alpha_p^k P + \overbrace{\alpha_p^{k-1} Q_1 P + NP}^{\in \mathcal{A}_p^{k-1}} + \overbrace{NQ_1 P}^{\in \mathcal{A}_p^{k-2}},$$

$N : \mathcal{A}_p^k \ni P \to NP \in \mathcal{A}_p^{k-1}$. Set $QP = P + Q_1 P$ and try to define $Q_1 : \mathcal{A}_p^k \to \mathcal{A}_p^{k-1}$.

Since

$$\alpha_p^k P + \overbrace{\alpha_p^k Q_1 P}^{\in \mathcal{A}_p^{k-1}} = QCP = \mathcal{C}QP = \mathcal{C}(P + Q_1 P) =$$

$$(C + N)(P + Q_1 P) = \alpha_p^k P + \overbrace{\alpha_p^{k-1} Q_1 P + NP}^{\in \mathcal{A}_p^{k-1}} + \overbrace{NQ_1 P}^{\in \mathcal{A}_p^{k-2}},$$

- we get $Q_1 P = \frac{1}{\alpha_p^k - \alpha_p^{k-1}} \frac{1}{m+1} (\delta \, \mathrm{Div} \, \delta^* P)$
- and have to set $QP = P + \sum_{\ell=1}^k Q_\ell P$, $Q_\ell : \mathcal{A}_p^k \to \mathcal{A}_p^{k-\ell}$

$N : \mathcal{A}^k_p \ni P \to NP \in \mathcal{A}^{k-1}_p$. Set $QP = P + Q_1 P$ and try to define $Q_1 : \mathcal{A}^k_p \to \mathcal{A}^{k-1}_p$.

Since

$$\alpha^k_p P + \overbrace{\alpha^k_p Q_1 P}^{\in \mathcal{A}^{k-1}_p} = QCP = \mathcal{C}QP = \mathcal{C}(P + Q_1 P) =$$

$$(C + N)(P + Q_1 P) = \alpha^k_p P + \overbrace{\alpha^{k-1}_p Q_1 P + NP}^{\in \mathcal{A}^{k-1}_p} + \overbrace{NQ_1 P}^{\in \mathcal{A}^{k-2}_p},$$

- we get $Q_1 P = \frac{1}{\alpha^k_p - \alpha^{k-1}_p} \frac{1}{m+1} (\delta \, \mathrm{Div} \, \delta^* P)$
- and have to set $QP = P + \sum_{\ell=1}^k Q_\ell P$, $Q_\ell : \mathcal{A}^k_p \to \mathcal{A}^{k-\ell}_p$

$$Q\,|_{\mathcal{S}^k_p} = \mathrm{id} + \sum_{\ell=1}^k Q_\ell, \quad Q_\ell =$$

$$\left(\frac{1}{m+1}\right)^\ell \left(\left(\Pi_{1 \le j \le \ell} \frac{1}{\alpha^k_p - \alpha^{k-j}_p}\right) (\delta \, \mathrm{Div} \, \delta^*)^\ell + \left(\Pi_{1 \le j \le \ell} \frac{1}{\beta^k_p - \beta^{k-j}_p}\right) (\delta^* \, \mathrm{Div} \, \delta)^\ell\right)$$

EQUIVARIANT QUANTIZATION OF SMOOTH MANIFOLDS

Projective structure on a manifold *M*

Projective structure $\rightsquigarrow$ straight lines $\rightsquigarrow$ geodesics $\rightsquigarrow$ no canonical connection $\rightsquigarrow$ class of connections associated with the same geodesics

Torsion-free linear connections ∇, ∇' on M are projectively equivalent, i.e. define the same geometric geodesics, if and only if (H. Weyl)

$$\nabla'_X Y - \nabla_X Y = \omega(X)Y + \omega(Y)X \in \mathcal{X}(M),$$

where $X, Y \in \mathcal{X}(M), \omega \in \Omega^1(M)$

Projective structure on M = class $[\nabla]$ of projectively equivalent connections

Quantization associated with a connection (A. Lichnerowicz, star-products)

$D \in \mathcal{D}^k(\Gamma(E), \Gamma(F)) \leftrightarrow P \in \Gamma(\mathcal{S}^k TM \otimes E^* \otimes F)$

∇: covariant derivative of E

$\nabla^k : \Gamma(E) \ni f \to \nabla^k f \in \Gamma(\mathcal{S}^k T^*M \otimes E)$: iterated symmetrized

$Q_{\mathrm{aff}}(\nabla)P : \Gamma(E) \ni f \to (Q_{\mathrm{aff}}(\nabla)P)\, f = i_P(\nabla^k f) \in \Gamma(F)$

Towards natural and projectively invariant quantization

- There is no $Q : \mathcal{S} \to \mathcal{D}$ such that

$$Q \circ \phi^* = \phi^* \circ Q, \forall \phi \in \mathrm{Diff}(M)$$

i.e.

$$(Q(\phi^* P))(\phi^* f) = \phi^* ((QP)(f)), \forall \phi \in \mathrm{Diff}(M)$$

Towards natural and projectively invariant quantization

- There is no $Q : \mathcal{S} \to \mathcal{D}$ such that

$$Q \circ \phi^* = \phi^* \circ Q, \forall \phi \in \mathrm{Diff}(M)$$

i.e.

$$(Q(\phi^* P))\,(\phi^* f) = \phi^*\,((QP)(f))\,, \forall \phi \in \mathrm{Diff}(M)$$

- Is there $Q(\nabla) : \mathcal{S} \to \mathcal{D}$ such that

$$(Q(\phi^* \nabla)(\phi^* P))\,(\phi^* f) = \phi^*\,((Q(\nabla)P)(f))\,,$$

for all local diffeomorphisms ϕ?

Remarks

- Example of the gauge principle
- Q – problem: no solution
  $Q(\nabla)$ – problem: several solutions, standard ordering prescription, Weyl ordering prescription (half-densities)

Remarks

- Example of the gauge principle
- Q – problem: no solution
  $Q(\nabla)$ – problem: several solutions, standard ordering prescription, Weyl ordering prescription (half-densities)
- Group $\mathrm{PGL}(\mathrm{m}+1,\mathbb{R})$ does not preserve ∇, but the flows of $X \in \mathrm{sl}_{m+1} \subset \mathcal{X}_*(\mathbb{R}^m)$ preserve $[\nabla]$: the solution $Q = \sigma^{-1}_{\mathrm{sl}_{m+1}}$ of Lecomte and Ovsienko is intelligible,

Remarks

- Example of the gauge principle
- Q – problem: no solution
  $Q(\nabla)$ – problem: several solutions, standard ordering prescription, Weyl ordering prescription (half-densities)
- Group $\mathrm{PGL}(\mathrm{m}+1, \mathbb{R})$ does not preserve ∇, but the flows of $X \in \mathrm{sl}_{m+1} \subset \mathcal{X}_*(\mathbb{R}^m)$ preserve $[\nabla]$: the solution $Q = \sigma^{-1}_{\mathrm{sl}_{m+1}}$ of Lecomte and Ovsienko is intelligible, if $Q = Q[\nabla]$
- Additional requirement (uniqueness): $Q = Q[\nabla]$, i.e. Q is projectively invariant

Definition (Lecomte 01):

A natural and projectively invariant quantization is a family

$$Q_M : \mathcal{C}(M) \times \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda\mu}(M),$$

indexed by smooth manifolds M, s.th., for any $\nabla \in \mathcal{C}(M)$,

$$Q_M[\nabla] : \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda\mu}(M)$$

is a *vector space isomorphism* that verifies

1. the usual *normalization condition*
2. for any local diffeomorphism ϕ of M,

$$\left(Q_M[\phi^*\nabla](\phi^*P)\right)(\phi^*f) = \phi^*\left((Q_M[\nabla]P)(f)\right),$$

   $\forall P \in \mathcal{S}_\delta(M), \forall f \in \Gamma(\Delta^\lambda TM)$
3. $Q_M[\nabla]$ is independent of $\nabla \in [\nabla]$

Remarks:

- Generalization: If Q is a natural and projectively invariant quantization, then $Q_{\mathbb{R}^m}[\nabla_0]$ (∇_0: canonical flat connection) is an $\mathrm{sl}(m+1,\mathbb{R})$-equivariant quantization
- Functorial formulation: M. Bordemann wrote this definition in the language of natural bundles and operators (see I. Kolář, P. W. Michor, J. Slovăk)
- Existence results: M. Bordemann, 02; P. Mathonet, F. Radoux, 05; S. Hansoul, 06; A. Cap, J. Silhan, 09
- Techniques: Thomas-Whitehead connections, Cartan connections, tractor calculus

The Thomas-Whitehead technique

Example $M := S^m$

$g \in G := \mathrm{GL}(m+1, \mathbb{R})$

All g preserve the canonical connection of $\mathbb{R}^{m+1}$, the induced ϕ_g do usually not preserve the canonical LC-connection on S^m

$\tilde{M} := \mathbb{R}^{m+1} \backslash \{0\} \to S^m =: M$ is a bundle with typical fiber $\mathbb{R}_0^+$

Example $M := S^m$

$g \in G := \mathrm{GL}(m+1, \mathbb{R})$

All g preserve the canonical connection of $\mathbb{R}^{m+1}$, the induced ϕ_g do usually not preserve the canonical LC-connection on S^m

$\tilde{M} := \mathbb{R}^{m+1} \backslash \{0\} \to S^m =: M$ is a bundle with typical fiber $\mathbb{R}_0^+$

Lift the complex situation on M to the simpler situation on $\tilde{M}$:

$$Q[\nabla](P)(f)$$

Example $M := S^m$

$g \in G := \mathrm{GL}(m+1, \mathbb{R})$

All g preserve the canonical connection of $\mathbb{R}^{m+1}$, the induced ϕ_g do usually not preserve the canonical LC-connection on S^m

$\tilde{M} := \mathbb{R}^{m+1} \backslash \{0\} \to S^m =: M$ is a bundle with typical fiber $\mathbb{R}_0^+$

Lift the complex situation on M to the simpler situation on $\tilde{M}$:

- Construct natural lifts $\nabla \to \tilde{\nabla}$, $P \to \tilde{P}$, and $f \to \tilde{f}$

$$Q[\nabla](P)(f)$$

Example $M := S^m$

$g \in G := \mathrm{GL}(m+1, \mathbb{R})$

All g preserve the canonical connection of $\mathbb{R}^{m+1}$, the induced ϕ_g do usually not preserve the canonical LC-connection on S^m

$\tilde{M} := \mathbb{R}^{m+1} \setminus \{0\} \to S^m =: M$ is a bundle with typical fiber $\mathbb{R}_0^+$

Lift the complex situation on M to the simpler situation on $\tilde{M}$:

- Construct natural lifts $\nabla \to \tilde{\nabla}$, $P \to \tilde{P}$, and $f \to \tilde{f}$

$$Q[\nabla](P)(f) \qquad Q_{\mathrm{aff}}(\tilde{\nabla})(\tilde{P})(\tilde{f})$$

Example $M := S^m$

$g \in G := \mathrm{GL}(m+1, \mathbb{R})$

All g preserve the canonical connection of $\mathbb{R}^{m+1}$, the induced ϕ_g do usually not preserve the canonical LC-connection on S^m

$\tilde{M} := \mathbb{R}^{m+1} \backslash \{0\} \to S^m =: M$ is a bundle with typical fiber $\mathbb{R}_0^+$

Lift the complex situation on M to the simpler situation on $\tilde{M}$:

- Construct natural lifts $\nabla \to \tilde{\nabla}$, $P \to \tilde{P}$, and $f \to \tilde{f}$
- Define a quantization $Q[\nabla]$ by

$$\left(Q[\nabla](P)(f) \right)\tilde{} := Q_{\mathrm{aff}}(\tilde{\nabla})(\tilde{P})(\tilde{f})$$

Example $M := S^m$

$g \in G := \mathrm{GL}(m+1, \mathbb{R})$

All g preserve the canonical connection of $\mathbb{R}^{m+1}$, the induced ϕ_g do usually not preserve the canonical LC-connection on S^m

$\tilde{M} := \mathbb{R}^{m+1} \setminus \{0\} \to S^m =: M$ is a bundle with typical fiber $\mathbb{R}_0^+$

Lift the complex situation on M to the simpler situation on $\tilde{M}$:

- Construct natural lifts $\nabla \to \tilde{\nabla}$, $P \to \tilde{P}$, and $f \to \tilde{f}$
- Define a quantization $Q[\nabla]$ by

$$\left(Q[\nabla](P)(f) \right)\tilde{} := Q_{\mathrm{aff}}(\tilde{\nabla})(\tilde{P})(\tilde{f})$$

- Check if $\tilde{\nabla}$ only depends upon $[\nabla]$

The point is:

The point is:

- Standard affine quantization

$$Q_{\mathrm{aff}}(\nabla)(P)(f) = i_P\,(\nabla^{\cdot} f)$$

is natural, but not projectively invariant

The point is:

- Standard affine quantization

$$Q_{\mathrm{aff}}(\nabla)(P)(f) = i_P\,(\nabla^{\cdot} f)$$

  is natural, but not projectively invariant

- Naturality of all lifts and naturality of Q_{aff} entails naturality of Q:

$$(Q[\nabla](P)(f))\tilde{} = Q_{\mathrm{aff}}(\tilde{\nabla})(\tilde{P})(\tilde{f})$$

$$\begin{aligned}(Q[\phi_g^*\nabla](\phi_g^* P)(\phi_g^* f))\tilde{} &= Q_{\mathrm{aff}}(g^*\tilde{\nabla})(g^*\tilde{P})(g^*\tilde{f})\\ &= g^* Q_{\mathrm{aff}}(\tilde{\nabla})(\tilde{P})(\tilde{f})\\ &= g^*\,(Q[\nabla](P)(f))\tilde{}\\ &= (\phi_g^* Q[\nabla](P)(f))\tilde{}\end{aligned}$$

The point is:

- Standard affine quantization

$$Q_{\mathrm{aff}}(\nabla)(P)(f) = i_P\,(\nabla^{\cdot} f)$$

  is natural, but not projectively invariant

- Naturality of all lifts and naturality of Q_{aff} entails naturality of Q:

$$(Q[\nabla](P)(f))\tilde{} = Q_{\mathrm{aff}}(\tilde{\nabla})(\tilde{P})(\tilde{f})$$

$$\begin{aligned}(Q[\phi_g^*\nabla](\phi_g^* P)(\phi_g^* f))\tilde{} &= Q_{\mathrm{aff}}(g^*\tilde{\nabla})(g^*\tilde{P})(g^*\tilde{f}) \\ &= g^* Q_{\mathrm{aff}}(\tilde{\nabla})(\tilde{P})(\tilde{f}) \\ &= g^*\,(Q[\nabla](P)(f))\tilde{} \\ &= (\phi_g^* Q[\nabla](P)(f))\tilde{}\end{aligned}$$

- Projective invariance of $\tilde{\nabla}$ entails projective invariance of Q

Extension of the S^m-construction to an arbitrary M, $\dim M = m$

- $$\Delta^1 TM = P^1 M \times_{\mathrm{GL}(m,\mathbb{R})} \mathbb{R} \quad ,$$
  $$\mathrm{GL}(m,\mathbb{R}) \times \mathbb{R} \ni (g, r) \to g \cdot r = \mid \det g \mid^{-1} r \in \mathbb{R} \quad :$$
  rank 1 bundle of 1 – densities over M

Extension of the S^m-construction to an arbitrary M, $\dim M = m$

- $$\Delta^1 TM = P^1 M \times_{\mathrm{GL}(m,\mathbb{R})} \mathbb{R}_0^+,$$
  $\mathrm{GL}(m, \mathbb{R}) \times \mathbb{R}_0^+ \ni (g, r) \to g \cdot r = \mid \det g \mid^{-1} r \in \mathbb{R}_0^+$:

  rank 1 affine bundle of positive 1 – densities over M

Extension of the S^m-construction to an arbitrary M, $\dim M = m$

- $\tilde{M} := \Delta^1 TM = P^1 M \times_{\mathrm{GL}(m,\mathbb{R})} \mathbb{R}_0^+$,
  $\mathrm{GL}(m,\mathbb{R}) \times \mathbb{R}_0^+ \ni (g,r) \to g \cdot r = |\det g|^{-1} r \in \mathbb{R}_0^+$:
  rank 1 affine bundle of positive 1 – densities over M

Extension of the S^m-construction to an arbitrary M, $\dim M = m$

- $\tilde{M} := \Delta^1 TM = P^1 M \times_{\mathrm{GL}(m,\mathbb{R})} \mathbb{R}_0^+$,
  $\mathrm{GL}(m,\mathbb{R}) \times \mathbb{R}_0^+ \ni (g, r) \to g \cdot r = |\det g|^{-1} r \in \mathbb{R}_0^+$:
  rank 1 affine bundle of positive 1 – densities over M

- $\tilde{M} \times \mathbb{R}_0^+ \ni ([u, r], s) \to \rho_s[u, r] = [u, rs] \in \tilde{M}$: $\mathbb{R}_0^+$ – pb

Extension of the S^m-construction to an arbitrary M, $\dim M = m$

- $\tilde{M} := \Delta^1 TM = P^1 M \times_{\mathrm{GL}(m,\mathbb{R})} \mathbb{R}_0^+$,
  $\mathrm{GL}(m,\mathbb{R}) \times \mathbb{R}_0^+ \ni (g,r) \to g \cdot r = |\det g|^{-1} r \in \mathbb{R}_0^+$:
  rank 1 affine bundle of positive 1 – densities over M

- $\tilde{M} \times \mathbb{R}_0^+ \ni ([u,r],s) \to \rho_s[u,r] = [u,rs] \in \tilde{M}$: $\mathbb{R}_0^+$ – pb

  $\Delta^\lambda TM = \tilde{M} \times_{\mathbb{R}_0^+} \mathbb{R}$,
  $\mathbb{R}_0^+ \times \mathbb{R} \ni (s,t) \to s.t = s^\lambda t$:
  line bundle of λ-densities over M

Extension of the S^m-construction to an arbitrary M, $\dim M = m$

- $\tilde{M} := \Delta^1 TM = P^1 M \times_{\mathrm{GL}(m,\mathbb{R})} \mathbb{R}_0^+$,
  $\mathrm{GL}(m, \mathbb{R}) \times \mathbb{R}_0^+ \ni (g, r) \to g \cdot r = |\det g|^{-1} r \in \mathbb{R}_0^+$:
  rank 1 affine bundle of positive 1 – densities over M
- $\tilde{M} \times \mathbb{R}_0^+ \ni ([u, r], s) \to \rho_s[u, r] = [u, rs] \in \tilde{M}$: $\mathbb{R}_0^+$ – pb

  $\Delta^\lambda TM = \tilde{M} \times_{\mathbb{R}_0^+} \mathbb{R}$,
  $\mathbb{R}_0^+ \times \mathbb{R} \ni (s, t) \to s.t = s^\lambda t$:
  line bundle of λ-densities over M

  $f \in \Gamma(\Delta^\lambda TM) \leftrightarrow \tilde{f} \in C^\infty(\tilde{M})_{\mathbb{R}_0^+}$

- Lifts $\tilde{P}$ and $\tilde{\nabla}$ are technical

  Essential remark: Natural and projectively invariant lift $\tilde{\nabla}$ is a connection of some known type

- Lifts $\tilde{P}$ and $\tilde{\nabla}$ are technical

  Essential remark: Natural and projectively invariant lift $\tilde{\nabla}$ is a connection of some known type

Study of projective equivalence of connections

- Origins: Goes back to the twenties and thirties
- Objective: Associate a unique connection to each projective structure $[\nabla]$
- First answer: Thomas-Whitehead projective connection (T.Y. Thomas, J.H.C. Whitehead, O. Veblen)
- Second answer: Cartan projective connection (E. Cartan)

Thomas-Whitehead connections

- A *Thomas-Whitehead projective connection* is a torsion-free linear connection $\tilde{\nabla}$ over $\tilde{M}$, such that
  1. $\tilde{\nabla}\mathcal{E} = \frac{1}{m+1}\,\mathrm{id}$
  2. $\rho_{s*}\left(\tilde{\nabla}_X Y\right) = \tilde{\nabla}_{\rho_{s*}X}\,\rho_{s*}Y, \quad \forall X, Y \in \mathcal{X}(\tilde{M})$

- *Bordemann's connection* is a Thomas-Whitehead connection

- S. Hansoul *extended* the work of M. Bordemann from tensor densities to sections of arbitrary vector bundles associated with the principle bundle of linear frames

THE CARTAN TECHNIQUE

- $P^2M = \{\, j_0^2(\psi) \mid \psi : 0 \in U \subset \mathbb{R}^m \to M, \det \psi_{*0} \neq 0 \,\}$: 2nd order frame bundle

  $\mathcal{G}_m^2 = \{\, j_0^2(f) \mid f : 0 \in U \subset \mathbb{R}^m \to \mathbb{R}^m,\ f(0) = 0, \det f_{*0} \neq 0 \,\}$: structure group

- $G = \mathrm{PGL}(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$

  $H = G_{[e_{m+1}]} =$
  $\left\{ \begin{pmatrix} A & 0 \\ \alpha & a \end{pmatrix} : A \in \mathrm{GL}(m, \mathbb{R}), \alpha \in \mathbb{R}^{m*}, a \neq 0 \right\} / \mathbb{R}_0 \,\mathrm{id}$

  acts locally on $\mathbb{R}^m$ by $\mathbb{R}^m \supset U \ni Z \mapsto \frac{AZ}{\alpha Z + a} \in \mathbb{R}^m$: $H \subset \mathcal{G}_m^2$

- Proposition:

  Reductions $P(M,H)$ of P^2M to $H \subset \mathcal{G}_m^2$ are 1-to-1 with *projective structures* $[\nabla]$

  Theorem:

  To every projective structure $[\nabla] \simeq P(M,H)$ is associated a *unique normal Cartan connection* ω

- Definition:

  G: Lie group, H: closed subgroup, $\mathfrak{g}$,$\mathfrak{h}$: Lie algebras

  $P = P(M,H)$: PB s.th. $\dim M = \dim G/H$

  A *Cartan connection* on P is a 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ s.th.

  - $\mathfrak{r}_s^*\omega = \mathrm{Ad}(s^{-1})\omega$ ($\mathfrak{r}_s$ right action of $s \in H$)
  - $\omega(X^h) = h$ $(h \in \mathfrak{h})$
  - $\omega_u : T_uP \to \mathfrak{g}$ $(u \in P)$ is a vector space isomorphism (no horizontal SB)

$P = P(M, H)$ ($\leftrightsquigarrow \tilde{M}$): projective structure
ω ($\leftrightsquigarrow \tilde{\nabla}$): normal Cartan connection

$P = P(M, H)$ ($\leftrightsquigarrow \tilde{M}$): projective structure
ω ($\leftrightsquigarrow \tilde{\nabla}$): normal Cartan connection

- Lifts $\tilde{S}$ of symbols and $\tilde{f}$ of densities to objects on $P = P(M, H)$:

$$\mathcal{S}^k_\delta(M) = \Gamma(\mathcal{S}^k TM \otimes \Delta^\delta TM) = C^\infty(P^1 M, \mathcal{S}^k \mathbb{R}^m \otimes \Delta^\delta \mathbb{R}^m)_{\mathrm{GL}(m,\mathbb{R})}$$

$$\Gamma(\Delta^\lambda TM) = C^\infty(P^1 M, \Delta^\lambda \mathbb{R}^m)_{\mathrm{GL}(m,\mathbb{R})}$$

(V, ρ): representation of $\mathrm{GL}(m, \mathbb{R})$

$C^\infty(P^1 M, V)_{\mathrm{GL}(m,\mathbb{R})}$

$P = P(M, H)$ ($\leftrightsquigarrow \tilde{M}$): projective structure
ω ($\leftrightsquigarrow \tilde{\nabla}$): normal Cartan connection

- Lifts $\tilde{S}$ of symbols and $\tilde{f}$ of densities to objects on $P = P(M, H)$:

$$\mathcal{S}^k_\delta(M) = \Gamma(\mathcal{S}^k TM \otimes \Delta^\delta TM) = C^\infty(P^1 M, \mathcal{S}^k \mathbb{R}^m \otimes \Delta^\delta \mathbb{R}^m)_{\mathrm{GL}(m,\mathbb{R})}$$

$$\Gamma(\Delta^\lambda TM) = C^\infty(P^1 M, \Delta^\lambda \mathbb{R}^m)_{\mathrm{GL}(m,\mathbb{R})}$$

(V, ρ): representation of $\mathrm{GL}(m, \mathbb{R})$

$$C^\infty(P^1 M, V)_{\mathrm{GL}(m,\mathbb{R})}$$

$P = P(M, H)$ ($\leftrightsquigarrow \tilde{M}$): projective structure
ω ($\leftrightsquigarrow \tilde{\nabla}$): normal Cartan connection

- Lifts $\tilde{S}$ of symbols and $\tilde{f}$ of densities to objects on $P = P(M, H)$:

$$\mathcal{S}^k_\delta(M) = \Gamma(\mathcal{S}^k TM \otimes \Delta^\delta TM) = C^\infty(P^1 M, \mathcal{S}^k \mathbb{R}^m \otimes \Delta^\delta \mathbb{R}^m)_{\mathrm{GL}(m,\mathbb{R})}$$

$$\Gamma(\Delta^\lambda TM) = C^\infty(P^1 M, \Delta^\lambda \mathbb{R}^m)_{\mathrm{GL}(m,\mathbb{R})}$$

(V, ρ): representation of $\mathrm{GL}(m, \mathbb{R})$

$$C^\infty(P^1 M, V)_{\mathrm{GL}(m,\mathbb{R})} \simeq C^\infty(P, V)_H$$

- Idea:

$$\left(Q_M[\nabla](S)(f)\right)\tilde{} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^{\omega})^k \tilde{f}\right)$$

- Idea:

$$(Q_M[\nabla](S)(f))\tilde{} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k\tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

- Idea:

$$(Q_M[\nabla](S)(f))\tilde{} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^\infty(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^\omega g)(u)(v) =$$

- Idea:

$$(Q_M[\nabla](S)(f))\tilde{} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^\infty(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^\omega g)(u)(v) = \left(L \qquad\qquad g\right)(u) \in V$$

- Idea:

$$\widetilde{(Q_M[\nabla](S)(f))} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^\infty(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^\omega g)(u)(v) = \left(L_{\mathcal{X}(P) \ni X(\omega, v)} \qquad g\right)(u) \in V$$

- Idea:

$$\widetilde{(Q_M[\nabla](S)(f))} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^\infty(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^\omega g)(u)(v) = \left(L_{\mathcal{X}(P) \ni X(\omega, v)} \qquad g\right)(u) \in V$$

$$\omega_u \in \mathrm{Isom}(\mathrm{T_u P}, \mathfrak{g}),\ \mathfrak{g} = \mathrm{sl}(m+1, \mathbb{R}) \simeq \mathbb{R}^m \oplus \mathrm{gl}(m, \mathbb{R}) \oplus \mathbb{R}^{m*} \ni v$$

- Idea:

$$(Q_M[\nabla](S)(f))^{\tilde{}} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^\infty(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^\omega g)(u)(v) = \left(L_{\mathcal{X}(P) \ni X(\omega,v) = \omega^{-1} v}\, g\right)(u) \in V$$

$$\omega_u \in \mathrm{Isom}(\mathrm{T_u P}, \mathfrak{g}),\ \mathfrak{g} = \mathrm{sl}(m+1, \mathbb{R}) \simeq \mathbb{R}^m \oplus \mathrm{gl}(m, \mathbb{R}) \oplus \mathbb{R}^{m*} \ni v$$

- Idea:

$$\widetilde{(Q_M[\nabla](S)(f))} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^\infty(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^\omega g)(u)(v) = \left(L_{\mathcal{X}(P) \ni X(\omega, v) = \omega^{-1} v}\, g\right)(u) \in V$$

$$\omega_u \in \mathrm{Isom}(\mathrm{T_u P}, \mathfrak{g}),\ \mathfrak{g} = \mathrm{sl}(m+1, \mathbb{R}) \simeq \mathbb{R}^m \oplus \mathrm{gl}(m, \mathbb{R}) \oplus \mathbb{R}^{m*} \ni v$$

$(\nabla^\omega)^k : C^\infty(P, V) \to C^\infty(P, \mathcal{S}^k \mathbb{R}^{m*} \otimes V)$: iterated symmetrized

- Idea:

$$\left(Q_M[\nabla](S)(f)\right)^{\tilde{}} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^\omega : C^\infty(P, V) \to C^\infty(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^\infty(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^\omega g)(u)(v) = \left(L_{\mathcal{X}(P) \ni X(\omega, v) = \omega^{-1} v}\, g\right)(u) \in V$$

$$\omega_u \in \mathrm{Isom}(\mathrm{T_u P}, \mathfrak{g}),\ \mathfrak{g} = \mathrm{sl}(m+1, \mathbb{R}) \simeq \mathbb{R}^m \oplus \mathrm{gl}(m, \mathbb{R}) \oplus \mathbb{R}^{m*} \ni v$$

$(\nabla^\omega)^k : C^\infty(P, V) \to C^\infty(P, \mathcal{S}^k \mathbb{R}^{m*} \otimes V)$: iterated symmetrized

- Problem:

$$\tilde{f} \in C^\infty(P, \Delta^\lambda \mathbb{R}^m)_H,\ (\nabla^\omega)^k \tilde{f} \in C^\infty(P, \mathcal{S}^k \mathbb{R}^{m*} \otimes \Delta^\lambda \mathbb{R}^m)$$

$$\tilde{S} \in C^\infty(P, \mathcal{S}^k \mathbb{R}^m \otimes \Delta^{\delta = \mu - \lambda} \mathbb{R}^m)_H$$

$$i_{\tilde{S}}\left((\nabla^\omega)^k \tilde{f}\right) \in C^\infty(P, \Delta^\mu \mathbb{R}^m)$$

EXISTENCE OF EQ VIA CARTAN CONNECTIONS

- Idea:

$$(Q_M[\nabla](S)(f))^{\tilde{}} = Q_{\mathrm{aff}}(\omega)(\tilde{S})(\tilde{f}) = i_{\tilde{S}}\left((\nabla^{\omega})^k\tilde{f}\right)$$

- Derivative associated with ω:

$$\nabla^{\omega} : C^{\infty}(P, V) \to C^{\infty}(P, \mathbb{R}^{m*} \otimes V)$$

$$g \in C^{\infty}(P, V),\ u \in P,\ v \in \mathbb{R}^m$$

$$(\nabla^{\omega} g)(u)(v) = \left(L_{\mathcal{X}(P)\ni X(\omega,v)=\omega^{-1}v}\, g\right)(u) \in V$$

$$\omega_u \in \mathrm{Isom}(\mathrm{T_uP}, \mathfrak{g}),\ \mathfrak{g} = \mathrm{sl}(m+1, \mathbb{R}) \simeq \mathbb{R}^m \oplus \mathrm{gl}(m, \mathbb{R}) \oplus \mathbb{R}^{m*} \ni v$$

$(\nabla^{\omega})^k : C^{\infty}(P, V) \to C^{\infty}(P, \mathcal{S}^k\mathbb{R}^{m*} \otimes V)$: iterated symmetrized

- Problem:

$$\tilde{f} \in C^{\infty}(P, \Delta^{\lambda}\mathbb{R}^m)_H,\ (\nabla^{\omega})^k\tilde{f} \in C^{\infty}(P, \mathcal{S}^k\mathbb{R}^{m*} \otimes \Delta^{\lambda}\mathbb{R}^m)$$

$$\tilde{S} \in C^{\infty}(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^{\delta=\mu-\lambda}\mathbb{R}^m)_H$$

$i_{\tilde{S}}\left((\nabla^{\omega})^k\tilde{f}\right) \in C^{\infty}(P, \Delta^{\mu}\mathbb{R}^m)$: not H-equivariant

- Solution:

Add lower degree terms to $\tilde{S} \in C^{\infty}(P, \mathcal{S}^{k}\mathbb{R}^{m} \otimes \Delta^{\delta}\mathbb{R}^{m})_{H}$

- Solution:

Add lower degree terms to $\tilde{S} \in C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)_H$

$\mathrm{Div}^\omega : C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m) \to C^\infty(P, \mathcal{S}^{k-1}\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)$

- Solution:

Add lower degree terms to $\tilde{S} \in C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)_H$

$\mathrm{Div}^\omega : C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m) \to C^\infty(P, \mathcal{S}^{k-1}\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)$

e_ℓ: canonical basis of $\mathbb{R}^m$, ε^ℓ: dual basis in $\mathbb{R}^{m*}$

$\mathrm{div}\left(\sum_\ell X^\ell e_\ell\right) = \sum_j i_{\varepsilon^j}\partial_{x^j}\left(\sum_\ell X^\ell e_\ell\right)$

- Solution:

Add lower degree terms to $\tilde{S} \in C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)_H$

$\mathrm{Div}^\omega : C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m) \to C^\infty(P, \mathcal{S}^{k-1}\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)$

e_ℓ: canonical basis of $\mathbb{R}^m$, ε^ℓ: dual basis in $\mathbb{R}^{m*}$

$\mathrm{div}\left(\sum_\ell X^\ell e_\ell\right) = \sum_j i_{\varepsilon^j}\partial_{x^j}\left(\sum_\ell X^\ell e_\ell\right)$

$\mathrm{Div}^\omega \tilde{S} = \sum_j i_{\varepsilon^j}\nabla^\omega_{e_j}\tilde{S}$

- Solution:

Add lower degree terms to $\tilde{S} \in C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)_H$

$\mathrm{Div}^\omega : C^\infty(P, \mathcal{S}^k\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m) \to C^\infty(P, \mathcal{S}^{k-1}\mathbb{R}^m \otimes \Delta^\delta\mathbb{R}^m)$

e_ℓ: canonical basis of $\mathbb{R}^m$, ε^ℓ: dual basis in $\mathbb{R}^{m*}$

$\mathrm{div}\left(\sum_\ell X^\ell e_\ell\right) = \sum_j i_{\varepsilon^j}\partial_{x^j}\left(\sum_\ell X^\ell e_\ell\right)$

$\mathrm{Div}^\omega\,\tilde{S} = \sum_j i_{\varepsilon^j}\nabla^\omega_{e_j}\tilde{S}$

- Theorem:

For non critical δ,

$$(Q_M[\nabla](S)(f))\tilde{} = \sum_{\ell=0}^{k} c_{k\ell}\; i_{(\mathrm{Div}^\omega)^\ell\tilde{S}}\left((\nabla^\omega)^{k-\ell}\tilde{f}\right)$$

defines a natural projectively invariant quantization, if the $c_{k\ell}$ have some precise values

QUANTIZATION OF SINGULAR SPACES

E. Noether's theorem: Symmetries $\rightsquigarrow$ 1st integrals $\rightsquigarrow$ reduction of $(q_1, \ldots, p_n)$.

Reduced phase space: N/G, $N = \mu^{-1}\{0\}$ $\rightsquigarrow$ singular space: orbifold, stratified space...

Quantization:

$$
\begin{array}{ccccc}
 & S(N) & \xrightarrow{Q_N} & \mathcal{D}(N) & \\
R & \downarrow & & \downarrow & R \\
 & S(N/G)\ ? & \xrightarrow{Q_{N/G}\ ?} & \mathcal{D}(N/G)\ ? &
\end{array}
$$

Meta-principle: $[Q, R] = 0$

Problem: Construct $S(N/G)$..., and $Q_{N/G}$ s.th. $[Q, R] = 0$

K. Richardson:

G compact Lie group acting on N

$\mathcal{F}$ regular Riemannian foliation on compact (M, g):

$$N/G \simeq M/\bar{\mathcal{F}}$$

Problem:

Solve the $[Q, R]$-problem for $M/\bar{\mathcal{F}}$

Method:

Use foliated and adapted geometries on $(M, \mathcal{F})$ as desingularization of $M/\bar{\mathcal{F}}$

Foliation atlas on M:

$\phi_i : U_i \ni m \to (x, y) \in \mathbb{R}^p \times \mathbb{R}^q$, x: leaf, y: transverse

$\phi_{ji} := \phi_j \phi_i^{-1}$: verify gluing condition

$\mathcal{F}$: foliation

$N_m(M, \mathcal{F}) = T_m M / T_m \mathcal{F}$: normal bundle

Adapted and foliated geometric objects:

$X|_U = \mathcal{X}^a(x, y)\partial_{x^a} + \mathscr{X}^b(y)\partial_{y^b} \in \Gamma(TM)$: adapted vf

$[X]|_U = [\mathscr{X}^b(y)\partial_{y^b}] \in \Gamma(N(M, \mathcal{F}))$ is 'constant along the leaves': foliated vf

'Projections':

Adapted objects O_2 of $M \stackrel{p}{\to}$ foliated objects $O_1 \stackrel{p}{\to}$ singular objects O_0 of $M/\mathcal{F}$

Previous observation:

Adapted world in $M \stackrel{p^2=R}{\longrightarrow}$ singular world of $M/\mathcal{F}$

Projected constructions:

Construct Q_2 in M, Q_1, and Q_0 in $M/\mathcal{F}$ s.th.

$$Q_{i-1}[\nabla_{i-1}](S_{i-1})(f_{i-1}) = Q_{i-1}[p\nabla_i](pS_i)(pf_i) = p\,(Q_i[\nabla_i](S_i)(f_i))\,,$$

which then implies that

$$Q_0[R\nabla_2](RS_2)(Rf_2) = R\,(Q_2[\nabla_2](S_2)(f_2))\,,$$

i.e. that

$$[Q, R] = 0$$

Tedious constructions in the adapted and foliated worlds:

- Adapt the definitions of Cartan calculus
- Prove that the vital parts of the classical theorems used by the Cartan technique go through
- Extend the proof of existence of EQ
- Verify commutation of all constructions with the projections

Theorem (P, F. Radoux, R. Wolak, 09)

There exist adapted and foliated natural and projectively invariant quantizations that commute with the projection.

New ideas:

- Use foliated manifolds as desingularization of arbitrary orbifolds
- Use the foliated EQ to construct a singular EQ on orbifolds

Fixed points of a symmetry action generate singularities:

U_i: open ball around O in $\mathbb{R}^2$

$\Gamma_i = \{\mathrm{id}, \gamma_i, \gamma_i^2\}$: finite group of isometries

γ_i: rotation by angle $2\pi/3$ around O – fixed point O

$V_i = U_i/\Gamma_i$: cone – prototype of an orbifold

DEFINITION

An n-dimensional Riemannian orbifold V is a topological space with a cover V_i and charts (U_i, Γ_i, q_i), $q_i : U_i/\Gamma_i \xrightarrow{\sim} V_i$, (see figure) s. th. the chart changes $\varphi_{ji} : W_i \to W_j$, $q_j\varphi_{ji} = q_i$, are isometries.

ORBIFOLD GEOMETRIC OBJECTS

- No universal definitions of geometric objects on orbifolds exist
- Definitions of orbifold smooth maps, DO, symbols, vector fields, connections, differential forms, local isomorphisms... are needed
- Definitions must capture the nature of an orbifold
- Definitions must guarantee a 1-to-1 correspondence between orbifold and foliated geometric objects on the desingularization

Objective I:

For any Riemannian orbifold V construct a foliated smooth manifold $(\tilde{V}, \mathcal{F})$ s.th. $\tilde{V}/\mathcal{F} \simeq V$

Step 1:

(U_i, Γ_i, q_i): orbifold chart, Γ_i: finite group of isometries
$\tilde{U}_i(U_i, \pi_i, O(n))$: PB of orthonormal frames
γ_i: acts on U_i, γ_{i*}: acts on $\tilde{U}_i$
$\tilde{U}_i/\Gamma_i = \tilde{V}_i$: smooth manifold – action properly discontinuous (Γ_i finite) and free (figure)

Step 2:

φ_{ji}: chart-change isometries between U_is verify $\gamma_{ijk}\varphi_{ki} = \varphi_{kj}\varphi_{ji}$

$[\varphi_{ji*}]$: induced maps between $\tilde{V}_i$s verify $[\varphi_{ki*}] = [\varphi_{kj*}][\varphi_{ji*}]$

$\tilde{V}$: glued from $\tilde{V}_i$s is a smooth manifold

$O(n)$: right action on $\tilde{U}_i$s, $\tilde{V}_i$s, $\tilde{V}$

$\mathcal{F}$: regular foliation on $\tilde{V}$

V: Riemannian orbifold, $(\tilde{V}, \mathcal{F})$: foliated smooth resolution

$\mathcal{Q}(\mathcal{F})$: foliated EQ of $(\tilde{V}, \mathcal{F})$

$$Q_V[\nabla](S) := p_{\mathcal{D}}^{*-1}\left(\mathcal{Q}(\mathcal{F})[p_{\mathcal{C}}^{*}\nabla](p_{\mathcal{S}}^{*}S)\right)$$

THEOREM (P, F. RADOUX, R. WOLAK, 10)

There exists a natural and projectively invariant quantization of orbifolds.

Equivariant quantization of supermanifolds

Thank you!