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Analytical Guidance for Fitting Parsimonious Household-Portfolio Models to Data

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Sylwia Hubar, University of Exeter Business School, United Kingdom
Christos Koulovatianos, CREA, Université du Luxembourg
Jian Li, Goethe University Frankfurt, House of Finance, Germany

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For editorial correspondence, please contact: crea@uni.lu
University of Luxembourg
Faculty of Law, Economics and Finance
162A, avenue de la Faïencerie
L-1511 Luxembourg

UNIVERSITÉ DU LUXEMBOURG

Analytical Guidance for Fitting Parsimonious Household-Portfolio Models to Data ⋆

Sylwia Hubar[a], Christos Koulovatianos[b,c,*], Jian Li[d]

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[a] University of Exeter Business School, Rennes Drive, Exeter, EX4 4PU, United Kingdom. Email: sph209@exeter.ac.uk

[b] Department of Economics, University of Luxembourg, 162A avenue de la Faïencerie, Campus Limpertsberg, BRC 1.06E, L-1511, Luxembourg. Email: christos.koulovatianos@uni.lu, Tel.: +352-46-66-44-6356, Fax: +352-46-66-44-6341.

[c] Center for Financial Studies (CFS), Goethe University Frankfurt

[d] Department of Money and Macroeconomics, Goethe University Frankfurt, House of Finance, Grueneburgplatz 1, PF H32, D-60323 Frankfurt am Main, Germany. Email: Jian.Li@hof.uni-frankfurt.de

* Corresponding author.

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Analytical Guidance for Fitting Parsimonious Household-Portfolio Models to Data

Abstract

Saving rates and household investment in stocks and business equity are all increasing in income and wealth. Introducing subsistence consumption to a common-across-households Epstein-Zin-Weil utility function is up to a quantitative explanation, in the context of standardized parsimonious household-portfolio models with risky income. Closed forms in a simplified version of the model, with insurable labor-income risk and no liquidity constraints, reveal that if, (i) risky-asset returns are weakly correlated and, (ii) household resources are expected to grow over time, then poorer households can afford exiting subsistence concerns slowly by saving less and by taking less risk, while holding balanced portfolios.

Keywords: Epstein-Zin-Weil recursive preferences, subsistence consumption, household-portfolio shares, business equity, wealth inequality

JEL classification: G11, D91, D81, D14, D11, E21

1. Introduction

A look at Figure 1 reveals that the fractions of two key risky assets held by households, stocks and business equity, are increasing in income and wealth in the US. These monotonic relationships, together with the findings by Dynan et al. (2004), that the richer have higher saving rates, are a challenge to explain through standard portfolio-choice savings/consumption models. One reason that parsimonious explanations of such monotonic patterns are important, is the derivation of aggregate implications from micro data, in order to better understand taxation and regulation issues regarding financial markets in their entirety.[1]

This study has two goals. First, we want to jointly explain the monotonic patterns of multiple risky-asset portfolio shares and saving rates in the data using the simplest possible model with labor-income risk that adheres to the permanent-income hypothesis. Second, we want to shed light on the black box of household-portfolio analytics in such a model, by identifying clear modeling ingredients and mechanisms that lead to such monotonic portfolio shares and saving rates. In order to achieve these goals we step back and investigate a version of the multi-asset model with risky labor income that delivers analytical results: a version with insurable income risk and no liquidity constraints.[2]

How to calibrate household-portfolio models is another murky area in household-finance research. Our closed-form solutions combined with minimum-distance fitting to observed

[1] For a review paper about doing quantitative macroeconomics with heterogeneous households see Heathcote, Storesletten, and Violante (2009).

[2] The insurability of labor-income risk may not be an assumption far from reality, at least in the US economy. A recent study by Guvenen and Smith (2010) argues that the extent of uninsurability of lifetime income risk that researchers assume in calibrated macroeconomic models might be far too high. This is because part of the observed variation in labor incomes may be anticipated, giving the opportunity to insure against these fluctuations by using means such as precautionary savings and household investment in risky assets. Numerically simulated versions of our model with uninsurability of labor-income risk should not alter our qualitative conclusions. In addition, in light of the Guvenen and Smith (2010) study, most of our quantitative conclusions should not be sensitive to this insurability assumption. Finally, we stress the analogy between our paper and Wachter's (2002) contribution, who has assumed return predictability in a portfolio-choice model with mean-reverting returns in order to achieve insightful exact solutions.

portfolio shares serve as guides to understanding less obvious roles of specific parameter combinations in simulations. One set of such parameter combinations is entries of risky-asset covariance matrices, which are less easy to study due to lack of sufficiently long time series.

Before initiating this research we have been motivated and guided by recent advances in the literature. A recent study by Wachter and Yogo (2010) has made a breakthrough, as it provided reasonable fit of theoretical household portfolios shares to data. The key idea in Wachter and Yogo (2010) is that they distinguish between two categories of goods, basic goods and luxuries, so the rich invest more in risky assets because they are risking losses in mostly luxury consumption.[3] Similarly, Achury et al. (2012) introduced subsistence consumption into a simple Merton (1969, 1971) model with one type of goods, uncovering a similar mechanism to this of Wachter and Yogo (2010): the poor do not invest in risky assets because they are strongly averse to losing their subsistence consumption. Our study makes use of such building blocks, but pays attention to putting together as many pieces as possible analytically in order to study their interconnection.

First, we introduce subsistence consumption to a common-across-households Epstein-Zin-Weil utility function.[4] Second, we identify a condition that leads to closed-form results in models with many risky assets.[5] This condition is the labor-income-risk insurability restriction, that the squares of correlation coefficients between labor-income growth and risky-asset returns all sum up to unity. This parametric constraint does not imply unreasonable corre-

[3] This idea has been implicit in Browning and Crossley (2000).

[4] See Epstein and Zin (1989) and Weil (1989), and for the continuous-time version of recursive preferences that we use in this paper see Duffie ant Epstein, (1992a,b).

[5] Distinguishing between different types of risky assets poses a technical challenge. A model with multiple risky assets and income risk has several exogenous shocks already, a number of dimensions difficult to handle computationally, although, notably, Garlappi and Skoulakis (2010) have suggested a computational approach that is promising for overcoming such curse-of-dimensionality problems in numerical analysis. Moreover, the complexity of the model is likely to make simulation analysis a black box.

lation parametrizations and it retains the role of labor income as exogenous shifter of future expected resources through its long-term trend. So, one technical contribution of our paper is showing that closed-form solutions can be obtained despite that our preferences are recursive and include subsistence consumption. Through such analytical results we are able to examine the interplay between portfolio shares and the covariance matrix of risky assets.

Similarly to Achury et al. (2012), we find that poorer households do not take much risk because their consumption hovers around subsistence. For this reason, portfolio shares of the poor are smaller than these of the rich. Yet, that this mechanism can prevail in the setup of this paper, which has introduced a labor-income process and multi-asset portfolios is not a straightforward finding: exogenous shocks to labor income are an exogenous shifter of future resources that may affect savings and investment strategies.[6] Most importantly, our analysis reveals how portfolio shares of multiple assets may jointly increase with wealth in a balanced way: all that is needed is a weak correlation between the returns of these risky assets. Such a correlation of returns is difficult to uncover through market-return data, not only because of short time series.[7] Assets such as business equity embody unmeasured idiosyncratic risks due to additional frictions. Private business equity is not as easy to sell as common stocks of larger companies in capital markets. This trading friction of business equity implies additional idiosyncratic risks borne by business-equity holders. To the extent that private business-equity returns have a substantial idiosyncratic-risk component, this noise may be a reason to have weak correlation with stock returns, shedding light on why stock and business-equity holdings rise in wealth in such a balanced way in Figure 1.

Our analysis clearly explains why, in the presence of subsistence consumption, saving rates are also jointly increasing in wealth. If household resources are expected to grow over

[6] Achury et al. (2012) use a textbook-level Merton (1969, 1971) with no labor income and only one risky asset, an efficient-market portfolio.

[7] See, for example, Jin (2011) for a study that estimates business-equity risks from price realizations.

time, poorer households can afford to exit subsistence concerns slowly by saving less, which makes saving rates to increase in wealth. Yet, which number to assign as an individual's subsistence consumption is a question that arises naturally. Econometric studies such as these of Atkeson and Ogaki (1996) or Donaldson and Pendakur (2006) do not reject the existence of subsistence consumption levels. Yet, issues of econometric model specification affect the robustness of subsistence estimates, making them rather unpopular among applied-theory researchers. Here we rely on estimates from surveys regarding living-standards comparisons across households and we claim that an adult needs an annual amount of about 3,000 US dollars in order to just survive.[8] Among other roles in matching household-portfolio patterns, we show that such a subsistence estimate generates plausible intertemporal-elasticity-of-substitution (IES) variation across the rich and the poor.

2. Model

Our partial-equilibrium model is in continuous time. At any instant $t \in [0, \infty)$ a household receives a labor income stream, $y(t)$, that evolves according to the geometric process

$$\frac{dy(t)}{y(t)} = \mu_y dt + \sigma_y dz_y(t) \ , \tag{1}$$

with $\sigma_y > 0$, $\mu_y \geq 0$, with $z_y(t)$ being a Brownian motion, and for a given $y(0) = y_0 > 0$.[9] The household also possesses an initial stock of financial wealth, $a_0 \in \mathbb{R}$, and has the potential to invest this wealth in a risk-free asset with return r^f, and also in a set of $N \geq 1$ risky

[8] Our calibration in this paper refers to US dollars in year 2007. For the survey evidence see Koulovatianos et al. (2007, 2012) who use data in six countries derived using the survey method first suggested by Koulovatianos et al. (2005), and our discussion in the calibration section.

[9] Notice the equivalence between the continuous-time representation in (1) and its discrete-time permanent-income hypothesis counterpart in Carroll (1992, 1997). In particular, Carroll (1992, p. 65) uses a discrete-time stochastic framework in which income, Y_t, following his notation, is governed by $\ln(Y_t) = \ln(P_t) + \ln(V_t)$, $\ln(V_t) \sim N\left(0, \sigma_V^2\right)$, i.i.d. over time, with P_t denoting the permanent-labor-income component which obeys $\ln(P_{t+1}) = \ln(G) + \ln(P_t) + \ln(N_{t+1})$, and in which $\ln(N_t) \sim N\left(0, \sigma_N^2\right)$, i.i.d. over time. Combining these two equations leads to,

$$\ln(Y_{t+1}) - \ln(Y_t) = \ln(G) + \ln(\varepsilon_{t+1}) \ , \tag{2}$$

assets. The price of risky asset $i \in \{1, ..., N\}$, denoted by $p_i(t)$, is governed by the process

$$\frac{dp_i(t)}{p_i(t)} = R_i dt + \mathbf{e}_i \boldsymbol{\sigma} d\mathbf{z}^T(t) \ , \tag{4}$$

in which $\mathbf{z}(t) \equiv [z_1(t) \quad z_2(t) \quad \cdots \quad z_N(t)]$ is a row vector of Brownian motions with $z_i(t)$ being associated with asset $i \in \{1, ..., N\}$. The $N \times N$ matrix $\boldsymbol{\sigma}$ is derived from the decomposition of the covariance matrix, $\boldsymbol{\Sigma}$, which refers to risks of the N risky assets only. In particular, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$. Finally, $\mathbf{e}_i$ is a $1 \times N$ vector in which the value 1 is in position $i \in \{1, ..., N\}$, while all other elements are zero.

Labor income is correlated with risky asset $i \in \{1, ..., N\}$ through the correlation coefficient $\rho_{y,i}$. Specifically,

$$z_y(t) = \sqrt{1 - \rho_{y,1}^2 - ... - \rho_{y.N}^2} z_0(t) + \rho_{y,1} z_1(t) + ... + \rho_{y,N} z_N(t) \ , \tag{5}$$

in which $z_0(t)$ is also a Brownian motion. If $\rho_{y,1}^2 + ... + \rho_{y,N}^2 \neq 1$, then labor-income risk is uninsurable. If, instead, $\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$, then labor risk can be eliminated by trading financial assets. Numerical analysis of portfolio choice with multiple risky assets and labor-income risk is a demanding task.[10] In addition, solving complex models numerically may mask some of its key mechanics. So, in order to facilitate the derivation of analytical results for many risky assets without the need to resort to numerical analysis, we use the restriction $\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$.

in which $\ln(\varepsilon_{t+1}) = \ln(N_{t+1}) + \ln(V_{t+1}) - \ln(V_t)$. Given the assumption that $\ln(N_t)$ and $\ln(V_t)$ are independent, which is stated in Carroll(1992, p. 70), it follows that $\ln(\varepsilon_{t+1}) \sim N\left(0, \sigma_N^2 + 2\sigma_V^2\right)$, i.i.d. over time. After applying Itô's Lemma on (1) and stochastically integrating over a time interval $[t, t + \Delta t]$ for all $t \geq 0$ and any $\Delta t \geq 0$, we obtain,

$$\ln[y(t + \Delta t)] - \ln[y(t)] = \left(\mu_y - \frac{\sigma_y^2}{2}\right) \Delta t + \sigma_y [z_y(t + \Delta t) - z_y(t)] \ . \tag{3}$$

Setting $\Delta t = 1$, $\mu_y - \sigma_y^2/2 = \ln(G)$, and $\sigma_y^2 = \sigma_N^2 + 2\sigma_V^2$, makes equations (3) and (2) to coincide.

[10] Yet, our available toolkit for solving discrete-time dynamic portfolio choice problems with many assets and state variables has been recently advanced by Garlappi and Skoulakis (2010).

The evolution of assets is governed by the budget constraint,

$$da(t) = \left\{ \left\{ \boldsymbol{\phi}(t) \mathbf{R}^T + \left[1 - \boldsymbol{\phi}(t) \mathbf{1}^T \right] r^f \right\} a(t) + y(t) - c(t) \right\} dt + a(t) \boldsymbol{\phi}(t) \boldsymbol{\sigma} d\mathbf{z}^T(t) \ , \quad (6)$$

in which $\mathbf{R} = [R_1 \quad \cdots \quad R_N]$ is a row vector containing all mean asset returns and $\boldsymbol{\phi}(t) = [\phi_1(t) \quad \cdots \quad \phi_N(t)]$ is a row vector containing the chosen fraction of financial wealth invested in risky asset i, for all $i \in \{1, ..., N\}$ at any time $t \geq 0$ ($\mathbf{A}^T$ denotes the transpose of any matrix $\mathbf{A}$). We do not impose any short-selling restrictions on $\boldsymbol{\phi}(t)$.

The problem faced by a household is to maximize its lifetime expected utility subject to constraints (6) and (1). Our utility specification involves a small, yet influential step away from the continuous-time formulation and parameterization of recursive "Epstein-Zin" preferences, suggested by Duffie and Epstein (1992a,b). In particular, we use a subsistence-consumption level χ, defining expected utility as,

$$J(t) = E_t \left[\int_t^\infty f(c(\tau) \ , \ J(\tau)) \, d\tau \right] \ , \quad (7)$$

in which $f(c, J)$ is a normalized aggregator of continuation utility, J, and current consumption, c, with

$$f(c, J) \equiv \rho (1 - \gamma) \cdot J \cdot \frac{\left\{ \frac{c - \chi}{[(1-\gamma)J]^{\frac{1}{1-\gamma}}} \right\}^{1 - \frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} \ , \quad (8)$$

and in which $\chi \geq 0$ and $\rho, \eta, \gamma > 0$. In Appendix A we show an intuitive result for the case with $\chi > 0$: if $\gamma = 1/\eta$, then expected utility converges to the case of time-separable preferences with hyperbolic-absolute-risk-aversion (HARA) momentary utility.[11] If $\chi = 0$

[11]Specifically, in Appendix A we show that $f(c, J)\,|_{\gamma = 1/\eta}$ implies that continuation utility is

$$J(t) = \rho E_t \left\{ \int_t^\infty e^{-\rho(\tau - t)} \frac{[c(\tau) - \chi]^{1 - \frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} d\tau \right\} \ . \quad (9)$$

Notice that Koo (1998) has provided theoretical analysis to a model that is similar to ours but he has restricted his attention to the consant-relative-risk aversion utility function given by (9) after setting $\chi = 0$. Other notable analyses with time0separable preferences are Duffie et al. (1997) and Henderson (2005).

(standard formulation), then η denotes the household's elasticity of intertemporal substitution and γ is the coefficient of relative risk aversion. In Appendix A we show that the IES is equal to $\eta\left(1-\chi/c\right)$ no matter if $\gamma \neq 1/\eta$ or not. So, in case $\chi > 0$, which is central to our analysis, parameter η sets the upper bound of IES (recall that $c \geq \chi$) and plays the role of the IES only asymptotically, as $c \rightarrow \infty$.

2.1 Solution

In equilibrium, continuation utility, $J^*\left(t\right)$, is a value function depending on the household's assets and labor income, so $J^*\left(t\right) = V\left(a\left(t\right), y\left(t\right)\right)$ for all $t \geq 0$. With infinitely-lived households and constraints with time-invariant state-space representation, the optimization problem of the households falls in the category of stationary discounted dynamic programming. So, the time index is dropped from the Hamilton-Jacobi-Bellman equation (HJB) which is given by,

$$0 = \max_{c \geq \chi, \boldsymbol{\phi}} \left\{ f\left(c, V\left(a,y\right)\right) + \left\{\left[\boldsymbol{\phi}\mathbf{R}^T + \left(1-\boldsymbol{\phi}\mathbf{1}^T\right) r_f\right] a + y - c\right\} \cdot V_a\left(a,y\right) \right.$$
$$+ \frac{1}{2} a^2 \boldsymbol{\phi}\boldsymbol{\sigma}\boldsymbol{\sigma}^T\boldsymbol{\phi}^T \cdot V_{aa}\left(a,y\right) + \mu_y y \cdot V_y\left(a,y\right)$$
$$\left. + \frac{1}{2}\left(\sigma_y y\right)^2 \cdot V_{yy}\left(a,y\right) + \sigma_y a y \boldsymbol{\phi}\boldsymbol{\sigma}\boldsymbol{\rho}_y^T \cdot V_{ay}\left(a,y\right) \right\}, \qquad (10)$$

in which V_x denotes the first partial derivative with respect to variable $x \in \{a, y\}$, V_{xx} is the second partial derivative with respect to x, the notation for the cross-derivative is obvious, and $\boldsymbol{\rho}_y = \begin{bmatrix}\rho_{y,1} & \cdots & \rho_{y,N}\end{bmatrix}$ is a row vector containing all correlation coefficients between each of asset returns and the income process. Finally, r_f denotes the return of investment in the risk-free asset.

The first-order conditions of the problem expressed by (10) are,

$$f_c\left(c, V\left(a, y\right)\right) = V_a\left(a, y\right) \ , \tag{11}$$

$$\boldsymbol{\phi}^T = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\frac{V_a\left(a, y\right)}{-a \cdot V_{aa}\left(a, y\right)} - \sigma_y\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\frac{V_{ay}\left(a, y\right)}{V_{aa}\left(a, y\right)} \ . \tag{12}$$

We make two technical assumptions that enable us to secure interiority of solutions and analytical tractability. The rationale behind these assumptions becomes more obvious after the statement of Proposition 1, so we provide intuition at a later point.

Assumption 1 *Initial conditions are restricted so that,*

$$a_0 + \frac{y_0}{r_y} > \frac{\chi}{r_f} \ ,$$

with

$$r_y \equiv r_f - \mu_y + \sigma_y\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T$$

Assumption 2 *The parameter restriction,*

$$\frac{1}{\eta} > 1 - \frac{\rho}{r_f + \frac{\nu}{2\gamma}} \ ,$$

in which,

$$\nu \equiv \left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right) \ ,$$

holds.

Proposition 1 provides the formulas of the analytical solution to the model.

Proposition 1

If $\rho_{y,1}^2 + ... + \rho_{y,N}^2 = 1$*, short selling is allowed, and Assumptions 1 and 2 hold, the solution to the problem expressed by the HJB equation given by (10) is a decision rule for portfolio choice,*

$$\boldsymbol{\phi}^* = \boldsymbol{\Phi}(a, y) = \frac{1}{\gamma}\left(\mathbf{R} - r^f \mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1 - \frac{\frac{\chi}{r_f}}{a}\right)$$

$$+ \left[\frac{1}{\gamma}\left(\mathbf{R} - r_f \mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} - \sigma_y \boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}\right]\frac{\frac{y}{r_y}}{a}, \quad (13)$$

and a decision rule for consumption,

$$c^* = C(a, y) = \xi\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right) + \chi\ , \quad (14)$$

in which

$$\xi = \rho\eta + (1 - \eta) r_f - \frac{(\eta - 1)}{2\gamma}\nu\ , \quad (15)$$

while the value function is given by,

$$V(a, y) = \rho^{-\eta\frac{1-\gamma}{1-\eta}}\xi^{\frac{1-\gamma}{1-\eta}}\frac{\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right)^{1-\gamma}}{1 - \gamma}\ .$$

Proof See Appendix A. □

The term $y(t)/r_y$ is the present value of expected lifetime labor earnings at time $t \geq 0$.[12] So, the sum $(a + y/r_y)$ equals the present value of total expected lifetime resources. The term χ/r_f is the present value of lifetime subsistence needs which uses the risk-free rate as its discount factor.[13] In light of these observations, the term $(a + y/r_y - \chi/r_f)$ equals the discretionary expected lifetime resources.

[12]Since labor income is insurable, the effective discount factor, r_y, which is used to calculate the present value of expected lifetime labor earnings, involves three opportunity-cost ingredients. These ingredients are the risk-free rate, r_f, the trend of income, μ_y, and a term involving the excess returns and risks of other assets, $(\mathbf{R} - r_f\mathbf{1})\left(\boldsymbol{\sigma}^{-1}\right)^T$. In addition, $r_y = r_f - \mu_y + \sigma_y(\mathbf{R} - r_f\mathbf{1})\left(\boldsymbol{\sigma}^{-1}\right)^T \boldsymbol{\rho}_y^T$, takes into account the correlations of income with the risky assets, $\boldsymbol{\rho}_y$, and income volatility, σ_y. In particular, notice that $y(t) = y_0 \cdot e^{\mu_y t + \sigma_y z_y(t)}$ (see equation (1)) while equation (5) combined with the condition $\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T = 1$ gives $z_y(t) = \boldsymbol{\rho}_y \cdot \mathbf{z}^T(t)$.

[13]To see why the discount factor of lifetime subsistence needs is the risk-free rate alone, consider the special case of a household with minimum assets, $\underline{a}$, such that $\underline{a} + y/r_y = \chi/r_f$, i.e. total expected lifetime resources

The decision rule of consumption, (14), is an affine function of discretionary resources, $(a + y/r_y - \chi/r_f)$, with its gradient, ξ, influenced by risk aversion, which is driven by parameter γ. In particular, if parameter η is lower than 1 (i.e., $IES = \eta\,(1-\chi/c) < 1$), then a higher level of γ reduces the propensity to consume, ξ, creating incentives for precautionary savings. Yet, the impact of an increase in risk aversion on the saving rate is not unambiguous. Risk aversion affects the optimal portfolio composition and hence the expected asset income of a household. In the following section we elaborate on the characteristics of the saving rate.

2.2 Characterizing the saving rate

The saving rate is a function of (a, y), it is denoted by $s\,(a, y)$, and is given by

$$s\,(a,y) = 1 - \frac{C\,(a,y)}{I\,(a,y)}\ ,$$

in which $C\,(a, y)$ is given by (14) and $I\,(a, y)$ is a household's total income, subject to its optimal portfolio-choice vector dictated by the decision rule $\mathbf{\Phi}\,(a, y)$ in (13). After some algebra we obtain,

$$s\,(a,y) = \frac{\left[\eta\,(r_f-\rho) + \frac{\eta+1}{2}\frac{\nu}{\gamma}\right]\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right) - \mu_y \frac{y}{r_y}}{\left(\frac{\nu}{\gamma} + r_f\right)\left(a + \frac{y}{r_y} - \frac{\chi}{r_f}\right) + \chi - \mu_y \frac{y}{r_y}}\ . \tag{16}$$

Although equation (16) gives a closed form, it is still challenging to distinguish the dependence of the saving rate on total asset holdings, a, or on current income, y. One of the sources of complexity is the presence of subsistence consumption, χ. Yet, the introduction

equal subsistence needs (in slight violation of Assumption 1). In this special case, equation (13) implies that the household holds a portfolio of risky assets, $\boldsymbol{\phi}^* \cdot \underline{a} = -\sigma_y y/r_y \boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1}$ which enables it to perfectly insure against labor-income risk. In this way, the equilibrium consumption profile of such a household is $c^*\,(t) = \chi$ for all $t \geq 0$. So, the ability to insure against labor-income risk enables the household to avoid consumption fluctuations and to meet the condition $c\,(t) \geq \chi$ with equality at all times. Since this special household does not have any opportunity left for fluctuations in total income through its savings behavior (its total income is equal to χ for all $t \geq 0$), its intertemporal opportunity cost is determined solely by the risk-free rate r_f.

of χ in our model is crucial for our quantitative exploration, so we provide some step-by-step qualitative analysis of (16).

2.2.1 How subsistence consumption affects the saving rate

Households may save resources in order to be well above the level of lifetime subsistence needs, χ/r_f. A previous study indicates that the optimal transition of poorer households away from subsistence needs is slow, since the poor have lower saving rates (see Achury et al. (2012, Corollary 1, p. 113) which studies a model nested by our present framework for $\mu_y = 0$ and $\gamma = 1/\eta$).[14] With $\mu_y \neq 0$ the implied income trend affects incentives to save, since income growth exogenously shifts the resource constraint over time (unlike a which is endogenously determined). So, in order to examine how subsistence, χ, affects the dependence of the saving rates on a and y, below we distinguish between two cases, $\mu_y = 0$ and $\mu_y \neq 0$.

No expected income growth ($\mu_y = 0$) Equation (16) implies,

$$s\left(a,y\right)|_{\left(\chi>0,\mu_y=0\right)} = \frac{\eta\left(r_f-\rho\right)+\frac{\eta+1}{2}\frac{\nu}{\gamma}}{\frac{\nu}{\gamma}+r_f+\frac{\chi}{a+\frac{y}{r_y}-\frac{\chi}{r_f}}}\ , \tag{17}$$

which, in turn, implies a positive dependence of the saving rate on both wealth and income ($s_a\left(a,y\right)|_{\left(\chi>0,\mu_y=0\right)}$, $s_y\left(a,y\right)|_{\left(\chi>0,\mu_y=0\right)} > 0$, if and only if $s\left(a,y\right)|_{\left(\chi>0,\mu_y=0\right)} > 0$).[15] On the

14 Achury et al. (2012) study a Merton (1969, 1971) model with additively-separable HARA preferences and no labor income.

15 This monotonicity result is in accordance with findings in Achury et al. (2012, Proposition 3 and Corollary 1). The Achury et al. (2012) model can be nested in our analysis if we set $\mu_y = 0$. If $\mu_y = 0$, our assumption of full labor-income insurability ($\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T = 1$) makes labor income a trendless noise which can be fully absorbed by a and fully incorporated into future household asset holdings, a, which are endogenously accumulated. Yet, even within the special case of $\mu_y = 0$, Achury et al. (2012) study a more special case for us here, this with $\gamma = 1/\eta$. Equation (17) shows that, for $\mu_y = 0$, the monotonicity of the saving rate in Achury et al. (2012, Proposition 3 and Corollary 1) can be generalized for $\gamma \neq 1/\eta$.

contrary, setting $\chi = 0$, equation (16) implies,

$$s\left(a,y\right)|_{\left(\chi=0,\mu_y=0\right)} = \frac{\eta\left(r_f-\rho\right)+\frac{\eta+1}{2}\frac{\nu}{\gamma}}{\frac{\nu}{\gamma}+r_f}\ , \tag{18}$$

i.e., saving rates are the same across the rich and the poor. In order that the saving rate in both (17) and (18) be strictly positive, parameters should be such that the numerator in both formulas is strictly positive. In particular,[16]

$$s\left(a,y\right)|_{\left(\chi\geq 0,\mu_y=0\right)} > 0 \Leftrightarrow \frac{1}{\eta} > \frac{r_f+\frac{\nu}{2\gamma}-\rho}{r_f+\frac{\nu}{2\gamma}} > \frac{1}{\eta}\frac{-\frac{\nu}{2\gamma}}{r_f+\frac{\nu}{2\gamma}}\ . \tag{20}$$

In case $r_f + \nu/\left(2\gamma\right) > \rho$, Assumption 2 implies that the IES is smaller than an upper threshold determined by the rest of the model's parameters. This upper bound on the IES blocks the willingness to substitute consumption over time, preventing the possibility that households would seek corner solutions, and thus guaranteeing $c^*\left(t\right) > \chi$ for all $t \geq 0$.[17] The positive saving rate is also a result of a relatively low rate of time preference, ρ, which is another aspect taken care of by condition (20).

Non-zero expected income growth ($\mu_y \neq 0$) Focusing on the empirically plausible case of $\mu_y > 0$, equation (16) implies, after some algebra,

$$s\left(a,y\right) = 1 - \frac{\xi + \frac{\chi}{r_f}\frac{\eta\left(r_f-\rho\right)+(\eta-1)\frac{\nu}{2\gamma}}{a+\frac{y}{r_y}}}{\frac{\nu}{\gamma}+r_f-\frac{\chi}{r_f}\frac{\nu}{\gamma}\frac{1}{a+\frac{y}{r_y}}-\mu_y\frac{1}{1+r_y\frac{a}{y}}}\ . \tag{21}$$

[16]After some algebra, we find that $s\left(a,y\right)|_{\left(\chi\geq 0,\mu_y=0\right)} > 0$ if and only if,

$$1-\frac{\rho}{r_f+\frac{\nu}{2\gamma}} > \frac{1}{\eta}\frac{-\frac{\nu}{2\gamma}}{r_f+\frac{\nu}{2\gamma}}\ . \tag{19}$$

Combining (19) with Assumption 2 leads to (20).

[17]In this case of $r_f + \nu/\left(2\gamma\right) > \rho$, condition (20) is also automatically guaranteed, and a positive saving rate is guaranteed while $\mu_y = 0$.

Equation (21) is indicative of the importance of setting parameter $\chi > 0$. By setting $\chi = 0$, (21) implies,

$$s(a, y)\,|_{\chi=0} = 1 - \frac{\xi}{\frac{\nu}{\gamma} + r_f - \mu_y \frac{1}{1+r_y \frac{a}{y}}} \ ,$$

In this case of homothetic preferences it is easy to verify the monotonicity of $s_a(a, y)$ with respect to income, namely,

$$s_y(a, y)\,|_{\left(\chi=0,\mu_y>0\right)} < 0 \quad \text{if } a > 0 \ , \qquad s_y(a, y)\,|_{\left(\chi=0,\mu_y>0\right)} > 0 \quad \text{if } a < 0 \ . \tag{22}$$

The negative dependence of the saving rate on income when $a > 0$ in (22) reflects a dominant wealth effect on consumption. Since income grows exogenously at rate $\mu_y > 0$, higher future-consumption levels can be achieved without further sacrifices, i.e. without a higher saving rate. That both current and future consumption are normal goods corroborates this intuition. For indebted households ($a < 0$), an increase in labor income reduces the relative cost of servicing the current debt.

A ceteris-paribus increase in a implies an increase in the ratio a/y, which further implies a comparative disadvantage for the resource that grows without making sacrifices (i.e., y if $\mu_y > 0$). This comparative disadvantage is captured by the role of the ratio a/y in equation (21). From (16), after some algebra, we can verify that,

$$s_a(a, y) > 0 \Leftrightarrow \chi \frac{\eta\left(r_f - \rho\right) + \frac{\eta+1}{2}\frac{\nu}{\gamma}}{\frac{\nu}{\gamma} + r_f} + \mu_y \xi \frac{\frac{y}{r_y}}{\left(\frac{\nu}{\gamma} + r_f\right)^2} > 0 \ . \tag{23}$$

Noticing that $\eta\left(r_f - \rho\right) + (\eta + 1)\,\nu/\left(2\gamma\right) > 0$ is implied by condition (20), the equivalence given by (23) implies,

$$s_a(a, y)\,|_{\left(\chi\geq 0,\mu_y>0\right)} > 0 \ . \tag{24}$$

So, our conclusions regarding a saving rate which is increasing in a if $\chi > 0$, drawn by equation (17) above for the case of $\mu_y = 0$ are reconfirmed and strengthened. The positive dependence of the saving rate on a implied by (24) is a key takeout of our analytical investigation. Equation (21) makes clear that the sign of $s_y(a, y)|_{(\chi>0,\mu_y>0)}$ is ambiguous, necessitating calibration and numerical investigation. Finally, the less empirically plausible case of $\mu_y < 0$ implies ambiguous monotonicity of the saving rate with respect to a and y in most cases, which would require numerical verification.

Table 1 summarizes our findings under the parametric restriction given by (20).[18]

sign of $s_a(a, y)$

	$\mu_y = 0$	$\mu_y > 0$
$\chi = 0$	0	+
$\chi > 0$	+	+

How the saving rate depends on wealth

sign of $s_y(a, y)$

	$\mu_y = 0$	$\mu_y > 0$
$\chi = 0$	0	− if $a > 0$ and + if $a < 0$
$\chi > 0$	+	ambiguous

How the saving rate depends on income

Table 1 Dependence of the saving rate on wealth and income under condition (23)

2.3 Characterizing portfolio composition in the case of two risky assets

The most complicated analytical aspect of determining the dependence of portfolio shares, $\boldsymbol{\phi}$, on total asset holdings, a, and income, y, is the role played by the covariance matrix of risky assets. In the case of two risky assets ($N = 2$), the covariance matrix is,

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_s^2 & \rho_{s,b}\sigma_s\sigma_b \\ \rho_{s,b}\sigma_s\sigma_b & \sigma_b^2 \end{bmatrix},$$

[18] Recall that (20) is equivalent to having $s(a, y) > 0$ if $\mu_y = 0$.

in which σ_i is the standard deviation of asset $i \in \{s, b\}$, with subscript "s" denoting "stocks" and subscript "b" denoting "business equity", while $\rho_{i,j}$ denotes the correlation coefficient between two risky assets $i, j \in \{s, b\}$. The stochastic structure of the problem with $N = 2$ involves three volatility parameters, σ_s, σ_b, and σ_y, and two correlation coefficients, $\rho_{s,b}$ and $\rho_{y,s}$, since correlation $\rho_{y,b}$ can be deduced from the labor-risk-insurability constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$.

In Appendix A we show that the solution based on (13) for $N = 2$ is,[19]

$$\phi_s^* = \frac{1}{\gamma} \cdot \frac{1}{1-\rho_{s,b}^2} \cdot \frac{\frac{R_s - r_f}{\sigma_s} - \rho_{s,b}\frac{R_b - r_f}{\sigma_b}}{\sigma_s} \cdot \left(1 - \frac{\frac{\chi}{r_f}}{a}\right)$$
$$+ \left[\frac{1}{\gamma} \cdot \frac{1}{1-\rho_{s,b}^2} \cdot \frac{\frac{R_s - r_f}{\sigma_s} - \rho_{s,b}\frac{R_b - r_f}{\sigma_b}}{\sigma_s} - \sigma_y \left(\frac{\rho_{y,s}}{\sigma_s} - \frac{\sqrt{1-\rho_{y,s}^2} \cdot \rho_{s,b}}{\sigma_s \sqrt{1-\rho_{s,b}^2}}\right)\right] \frac{\frac{y}{r_y}}{a}, \tag{26}$$

and

$$\phi_b^* = \frac{1}{\gamma} \cdot \frac{1}{1-\rho_{s,b}^2} \cdot \frac{\frac{R_b - r_f}{\sigma_b} - \rho_{s,b}\frac{R_s - r_f}{\sigma_s}}{\sigma_b} \cdot \left(1 - \frac{\frac{\chi}{r_f}}{a}\right)$$
$$+ \left[\frac{1}{\gamma} \cdot \frac{1}{1-\rho_{s,b}^2} \cdot \frac{\frac{R_b - r_f}{\sigma_b} - \rho_{s,b}\frac{R_s - r_f}{\sigma_s}}{\sigma_b} - \sigma_y \cdot \frac{\sqrt{1-\rho_{y,s}^2}}{\sigma_b \sqrt{1-\rho_{s,b}^2}}\right] \frac{\frac{y}{r_y}}{a}. \tag{27}$$

The first observation about equations (26) and (27) is that parameter η, which is tightly linked with the IES does not affect the composition of portfolios. On the contrary, an increase in the relative-risk aversion coefficient γ influences the optimal portfolio share of each risky asset. In particular, the comparison between the ratio of the two Sharpe

[19] In Appendix A we also show that the magnitude of the discount factor used to calculate the present value of lifetime labor income equals,

$$r_y = r_f - \mu_y + \sigma_y \left[\frac{R_s - r_f}{\sigma_s} \cdot \left(\rho_{y,s} - \frac{\sqrt{1-\rho_{y,s}^2} \cdot \rho_{s,b}}{\sqrt{1-\rho_{s,b}^2}}\right) + \frac{R_b - r_f}{\sigma_b} \cdot \frac{\rho_{y,b}}{\sqrt{1-\rho_{s,b}^2}}\right]. \tag{25}$$

Equation (25) reveals that apart from r_f, μ_y, and σ_y, a linear relationship between the Sharpe ratios weighted by expressions involving the correlation coefficients $\rho_{y,s}$ and $\rho_{s,b}$ plays a key role in determining the magnitude of r_y which critically affects the level of lifetime labor income y/r_y.

ratios with the correlation coefficient between asset returns (i.e., how $\rho_{s,b}$ compares to $[(R_i - r_f)/\sigma_i]/[(R_j - r_f)/\sigma_j]$, $i, j \in \{s, b\}$ with $i \neq j$) determines whether an increase in γ leads to a decrease in both ϕ_s^* and ϕ_b^*, or in an increase in one of the two and in a reduction for the other.[20]

The dependence of ϕ_s^* and ϕ_b^*, on assets, a, and income, y, hinges upon a number of parameter combinations. If $\rho_{s,b} < [(R_i - r_f)/\sigma_i]/[(R_j - r_f)/\sigma_j]$, $i, j \in \{1, 2\}$ with $i \neq j$, then the first term of (26) and (27) contributes to making ϕ_s^* and ϕ_b^* increasing in a, as long as $\chi > 0$. So, the presence of subsistence consumption, $\chi > 0$ contributes to having portfolio shares of risky assets that are increasing in wealth, in accordance with what the data say. Nevertheless, the second term introduces a separate role for the income/wealth ratio y/a in generating portfolio shares which are increasing in wealth. This role of y/a depends on a more complicated relationship among parameters related to asset returns, their covariance matrix, and the correlation of risky asset with labor income shocks. Yet, equations (26) and (27) provide a useful pointer towards a successful calibration exercise for the $N = 2$ case: two key ingredients in order to match that portfolio-shares are increasing in wealth or income in the data are, (a) a positive level of subsistence consumption, $\chi > 0$, and (b) a low correlation coefficient between the two risky assets, especially one that guarantees $\rho_{s,b} < [(R_i - r_f)/\sigma_i]/[(R_j - r_f)/\sigma_j]$, $i, j \in \{s, b\}$ with $i \neq j$. The simulation exercise demonstrates the quantitative importance of these two key ingredients.

3. Calibration

3.1 Benchmark Calibration

Table 2 provides all calibrating parameters. Setting labor-income risk, σ_y, equal to 8.21%, is within the ballpark of a standard parametrization motivated by micro data (see, for

[20]Notice that since $\rho_{s,b} < 1$ it cannot be that an increase in γ causes both ϕ_s^* and ϕ_b^* to rise.

example, Gomes and Michaelides (2003 p. 736) for details). We also set the mean labor-income growth to 1.15%. Another standard parametrization is setting stock returns and their volatility close to their long-term values for R_s and σ_s of 7.56% and 21% (the corresponding values in Guvenen (2009) are 8% and 20%, while Gomes and Michaelides (2003) use 6% and 18%). Our calibration exercise worked better by giving the risk-free rate, r_f, the rather generous 3.56%, compared to the standard value close to 2% (see, for example, Gomes and Michaelides (2003) and Guvenen (2009)). While our implied equity premium is rather low (4%), it is not uncommon in the household-finance literature to consider such values. For example, an equity premium of 2.5% is within the range of values examined by Gomes and Michaelides (2003).

Preference Parameters

ρ	η	γ	χ
2.5%	0.08	4.78	2940[a]

Mean Returns

r_f	μ_y	R_s	R_b
3.56%	1.15%	7.56%	18%

Standard Deviations of Returns and Correlations

σ_y	σ_s	σ_b	ρ_{ys}	ρ_{yb}	ρ_{sb}
8.21%	21%	42.07%	48.93%	87.21%	1.74%

Table 2 Calibrating Parameters

[a]Annual subsistence cost per person in 2007 US Dollars.

Our preference parameters are close to the choices made by Achury et al. (2012), with the sole difference that the monthly subsistence consumption per person is USD 245 versus USD 230 in Achury et al. (2012). Nevertheless, the monthly amount of USD 245 is within the range of survey evidence about subsistence consumption reported by Koulovatianos et al. (2007, 2012), i.e. between USD 111 and 302.

After fixing the values of all parameters above, we performed a minimum-distance exercise within admissible ranges of all remaining parameter values, in order to best fit the model to the data.[21] The resulting benchmark calibration is given by Figure 2. While the share of business equity seems imperfectly matched, simulated patterns of portfolio shares are both increasing in income/wealth. The span of simulated business-equity shares for all income/wealth categories is satisfactorily close to the span indicated by the data, showing promise for future work. Notably, we have excellent data fit for the stockholding portfolio data. The minimum-distance exercise implied a number of parameters for business equity that best match the data. Most interesting and robust are the implications that the mean and standard deviation of business-equity returns, R_b and σ_b, are 18% and 42.07%. The value $R_b = 18\%$ is not far from the average estimates in Moskowitz and Vissing-Jorgensen (2002, Table 4, p. 756). Regarding our model's implication that $\sigma_b = 42.07\%$, Moskowitz and Vissing-Jorgensen (2002, p. 765) mention: "*[...] the annual standard deviation of the smallest decile of public firm returns is 41.1 percent. A portfolio of even smaller private firms is likely to be as volatile.*" It can be difficult to estimate idiosyncratic risks borne by a household. Unobservable limitations in outside options, such as frictions in relocating business if other family incomes could increase by relocating, imperfect insurance from theft, etc., may justify that a value for σ_b in the order of 40% may still be low.

Regarding the correlation between labor income shocks and stock returns, ρ_{ys}, Gomes and Michaelides (2003 p. 736) suggest an educated value of 30%, but try higher values, too. Our implied value for ρ_{ys} is 48.93%, which immediately implies that $\rho_{yb} = \sqrt{1-(48.93\%)^2} =$

[21]Income-tax calculations are based on data taken from the Federation of Tax Administrators at 444 N. Capital Street, Washington DC, projected from year 2003. See the Online Data Appendix for details on the tax raes and also Grant et al. (2010, Table 2).

87.21%, due to the parametric restriction $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, a key condition for obtaining closed-form solutions. The high correlation between business equity and family income may be plausible as a large fraction of households have family businesses and tend to employ family members or the owners themselves. Given equations (26) and (27) that we have derived above, we paid attention to the implied Sharpe ratios and concluded that an admissible and appropriate value for the crucial correlation between stock returns and business-equity returns, ρ_{sb}, is 1.74%.

In brief, our analysis suggests that data can be matched satisfactorily provided that business-equity returns are highly volatile and weakly correlated with stock returns. As idiosyncratic components of business-equity risk can be substantial, these implications seem plausible. Importantly, Figure 3 shows that the model replicates saving rates across the rich and the poor within the ranges suggested by Dynan et al. (2004).

3.2 Sensitivity Analysis

Figures 4 and 5 show the impact of changing the values of the subsistence parameter χ. A $12 - 18\%$ deviation above or below the benchmark value of USD 245 per month does not change fitted values of portfolio shares and saving rates substantially. Yet, a crucial exercise is to see the impact of discarding subsistence consumption from the model, through setting $\chi = 0$. Figures 4 and 5 show that portfolio shares and saving rates become almost flat across the rich and the poor. The U-shaped part that arises in Figure 5 is due to the cross-sectional pattern of the income-to-asset ratio, y/a, across the rich and the poor in the data, that is presented by Figure 6. That y/a has an impact on the saving rate and portfolio shares is obvious by equations (21), (26), and (27).

Another sensitivity-analysis exercise we preform focuses on changing the magnitude of the correlation coefficient between stock and business-equity returns, ρ_{sb}. Figures 7 and 8 show that slight changes in this correlation coefficients can have substantial impact on the portfolio shares and the saving rates. In other words, the benchmark value $\rho_{sb} = 1.74\%$ seems to be a robust implication of the model.

Finally we vary the elasticity of intertemporal substitution, η, and observe the impact of these changes on the saving rate. We remind that η affects the saving rate alone and not the household portfolio shares at all, as (21), (26), and (27) reveal. Unsurprisingly, small changes in η affect the saving rates substantially, but not its monotonicity property: the shifts in the saving-rate pattern are almost parallel. Finally, we emphasize that the implied saving-rate pattern under the constraint $\eta = 1/\gamma$ (time-separable preferences) is not quantitatively implausible. As the value of η leaves portfolio-shares unaffected, it is notable that the assumption of Epstein-Zin-Weil recursive preferences is not crucial for the goodness of fit of our exercise. Nevertheless, Epstein-Zin-Weil recursive preferences provide a valuable degree of freedom that may prove crucial for key extensions such as the introduction of liquidity constraints in a more descriptive and complicated version of the model.

4. Conclusion

We have introduced subsistence consumption to a simplified but standardized household-portfolio model with Epstein-Zin-Weil recursive preferences. We have analytically demonstrated the crucial role played by subsistence consumption in reconciling the model with stylized facts appearing the SCF data. In particular, we have shown how subsistence consumption can quantitatively match that saving rates and household-portfolio shares in stocks and business equity are altogether increasing in wealth. Both our analytical work and our simulations have uncovered an essential modeling ingredient which activates the mechanism through which subsistence consumption matches the data: private business-equity returns must have a substantial idiosyncratic-risk component; this component weakens the correlation between household stock-portfolio returns and business-equity returns from an individual household's perspective; in turn, this weak correlation makes both asset-holding shares to increase with wealth in a balanced way due to diversification incentives.

We have been able to obtain analytical results because our model was simplified, excluding liquidity constraints, while we also assumed insurable labor-income risk. A next step is to break each of these assumptions in order to pursue a parsimonious, yet versatile and robust household-portfolio theory that does not hinge upon heterogeneous-preference assumptions or behavioral approaches which are challenging to quantify. Simulation of such extended models can be challenging due to the curse of dimensionality. Advances such as this of Garlappi and Skoulakis (2010) can help in pursuing these extensions. Nevertheless, closed-form solutions can serve as valuable starting points when launching such numerical-simulation ventures. Moreover, our minimum-distance calibration and sensitivity analysis have revealed that calibrating multi-asset household-portfolio models is challenging. So, our present analysis can serve as a simulation guide in motivating parameters for more complicated models in future work. Altogether, we think our study points at a key take-out: introducing subsistence consumption to household-portfolio models seems promising for cracking central household-finance puzzles.

5. Appendix A

Proof that setting $\gamma = 1/\eta$ in equation (8) leads to time-separable preferences with HARA momentary utility

One can make a conjecture beforehand: we can use the transformation $\tilde{c} = c - \chi$; then $f(c, J) = \tilde{f}\left(\tilde{c}, \tilde{J}\right)$, in which $\tilde{f}\left(c, \tilde{J}\right)$ is the normalized aggregator function in the Epstein and Duffie (1992a,b) original formulation, with $\tilde{J}$ being its associated continuation utility (notice that $\tilde{f}\left(c, \tilde{J}\right) = f(c, J)|_{\chi=0}$); based on the identity $f(c, J) = \tilde{f}\left(\tilde{c}, \tilde{J}\right)$, one can use the well-known result that if $\gamma = 1/\eta$, then $\tilde{f}\left(c, \tilde{J}\right)|_{\gamma=1/\eta}$ implies that continuation utility is $\tilde{J}(t) = \rho \int_t^\infty e^{-\rho(s-t)} c(s)^{1-\gamma} / (1-\gamma)\, ds$; the conjecture to make is that $\tilde{f}\left(\tilde{c}, \tilde{J}\right)|_{\gamma=1/\eta}$ implies $\tilde{J}(t) = \rho \int_t^\infty e^{-\rho(s-t)} \tilde{c}(s)^{1-\gamma} / (1-\gamma)\, ds$, which is the desired result. Here we go through all steps of a formal proof in order to cross check whether the intuition and conjecture discussed above fail. In addition, throughout the proof, we use the expectations operator in order to cross check whether the result holds in the presence of parameter $\chi > 0$, when consumption is stochastic.

Equation (8) implies,

$$f(c, J)|_{\gamma=\frac{1}{\eta}} = \rho \frac{(c-\chi)^{1-\gamma}}{1-\gamma} - \rho J \ . \tag{28}$$

Let's use $J'(t)$ in order to denote the total derivative of J with respect to time evaluated at time t. Equation (7) implies that $E_t[J'(t)] = -E_t[f(c(t), J(t))]$, and after using equation (28) we obtain,

$$-E_t[J'(t)] = \rho E_t \left\{ \frac{[c(t)-\chi]^{1-\gamma}}{1-\gamma} \right\} - \rho E_t[J(t)] \ .$$

After multiplying both sides by $(1/\rho)\, e^{-\rho t}$ and after integrating the above equation with respect to time we obtain,

$$-\frac{1}{\rho} E_0 \left[\int_0^T e^{-\rho t} J'(t)\, dt \right] = E_0 \left\{ \int_0^T e^{-\rho t} \frac{[c(t)-\chi]^{1-\gamma}}{1-\gamma} dt \right\} - E_0 \left[\int_0^T e^{-\rho t} J(t)\, dt \right] \ , \tag{29}$$

for some $T \geq 0$. After applying integration by parts we obtain,

$$\int_0^T e^{-\rho t} J(t)\, dt = -\frac{1}{\rho}\left[e^{-\rho T} J(T) - J(0)\right] + \frac{1}{\rho}\int_0^T e^{-\rho t} J'(t)\, dt \ ,$$

and substituting this last expression into (29) results in,

$$J(0) = E_0 \left\{ \rho \int_0^T e^{-\rho t} \frac{\left[c(t) - \chi\right]^{1-\gamma}}{1-\gamma} dt \right\} + e^{-\rho T} E_T \left[J(T)\right] \ . \tag{30}$$

Since the choice of T was arbitrary, equation (30) should hold for all $T \geq 0$. The requirement of having a well-defined expected-utility function for all $T \geq 0$, i.e.,

$$-\infty < E_T \left[J(T)\right] < \infty \quad \text{ for all } T \geq 0 \ ,$$

implies that $\lim_{T\to\infty} e^{-\rho t} E_T \left[J(T)\right] = 0$. So, equation (30) implies,

$$J(T) = E_T \left\{ \rho \int_T^\infty e^{-\rho(t-T)} \frac{\left[c(t) - \chi\right]^{1-\gamma}}{1-\gamma} dt \right\} \ , \ \text{ for all } T \geq 0 \ , \tag{31}$$

which proves the statement that setting $\gamma = 1/\eta$ in equation (8) leads to time-separable preferences with HARA momentary utility. $\square$

Proof that the IES is equal to $\eta(1 - \chi/c)$

We can consider two distinct time instants, t and $t + \Delta t$, for any $t \geq 0$, and $\Delta t > 0$. Based on the definition of $J(t)$ given by (7), the IES at time t is,

$$IES(t) = -\lim_{\Delta t \to 0} \frac{d \ln \left[\frac{c(t+\Delta t)}{c(t)}\right]}{d \ln \left[\frac{\frac{\partial J(t)}{\partial c(t+\Delta t)}}{f_c(c(t), J(t))}\right]} \ . \tag{32}$$

With $\Delta t > 0$ it is,

$$J(t) = E_t \left[\int_t^{t+\Delta t_-} f(c(\tau), J(\tau))\, d\tau\right] + E_{t+\Delta t}\left[\int_{t+\Delta t}^\infty f(c(\tau), J(\tau))\, d\tau\right] \ ,$$

in which Δt_{-} is approaching Δt from below. Given the above equation and the definition of (7),

$$\lim_{\Delta t \to 0} \frac{\partial J(t)}{\partial c(t+\Delta t)} = \lim_{\Delta t \to 0} \left\{ E_t \left[\int_t^{t+\Delta t_{-}} f_J(c(\tau), J(\tau))\, d\tau \right] + 1 \right\} \cdot f_c(c(t+\Delta t), J(t+\Delta t)) \ , \tag{33}$$

in which the integral in the term $\lim_{\Delta t \to 0} \left\{ E_t \left[\int_t^{t+\Delta t_{-}} f_J(c(\tau), J(\tau))\, d\tau \right] + 1 \right\}$ is an acceptable approximation derived from $E_t \left[\int_t^{t+\Delta t_{-}} f_J(c(\tau), J(\tau)) \cdot \partial J(\tau) / \partial c(t+\Delta t)\, d\tau \right]$, given that $\Delta t \to 0$.

Combining (33) with (32) leads to,

$$IES(t) = \frac{-\lim_{\Delta t \to 0} d \ln \left[\frac{c(t+\Delta t)}{c(t)} \right]}{d \left\{ \lim_{\Delta t \to 0} \ln \left\{ E_t \left[\int_t^{t+\Delta t_{-}} f_J(c(\tau), J(\tau))\, d\tau \right] + 1 \right\} + \lim_{\Delta t \to 0} \ln \left[\frac{f_c(c(t+\Delta t), J(t+\Delta t))}{f_c(c(t), J(t))} \right] \right\}} \ . \tag{34}$$

Since $\lim_{\Delta t \to 0} \{\ln[x(t+\Delta t)] - \ln[x(t)]\} = [\dot{x}(t)/x(t)]\, dt$ (in which $\dot{x}(t) \equiv dx(t)/dt$), equation (34) can be expressed as,

$$IES(t) = \frac{-d \left[\frac{\dot{c}(t)}{c(t)} \right]}{d \left\{ \lim_{\Delta t \to 0} \frac{\ln \left\{ E_t \left[\int_t^{t+\Delta t_{-}} f_J(c(\tau), J(\tau)) d\tau \right] + 1 \right\}}{\Delta t} + \frac{d \ln[f_c(c(t), J(t))]}{dt} \right\}} \ . \tag{35}$$

The relationship between a discrete-time growth rate, g_d, with its continuous-time counterpart, g_c, is given by $g_c = \ln(1 + g_d)$. Since $\Delta t \to 0$ implies transition from a discrete-time approximation to continuous time, the term $\ln \left\{ E_t \left[\int_t^{t+\Delta t_{-}} f_J(c(\tau), J(\tau))\, d\tau \right] + 1 \right\} / \Delta t$ converges to $f_J(c(t), J(t))$, and can be substituted into (35) to give,

$$IES(t) = -\left\{ \frac{d \left\{ f_J(c(t), J(t)) + \frac{d \ln[f_c(c(t), J(t))]}{dt} \right\}}{d \left[\frac{\dot{c}(t)}{c(t)} \right]} \right\}^{-1} \ . \tag{36}$$

From (8) we obtain $f_c = \rho \left[(1-\gamma) J \right]^{(1/\eta - \gamma)/(1-\gamma)} \cdot (c - \chi)^{-1/\eta}$, which implies $d \ln(f_c)/dt = (1/\eta - \gamma)/(1-\gamma) \cdot \left(\dot{J}/J \right) - (1/\eta)\left[c/(c-\chi) \right] \cdot (\dot{c}/c)$ and becomes

$$\frac{d \ln[f_c(c(t), J(t))]}{dt} = -\frac{\frac{1}{\eta} - \gamma}{1-\gamma} \cdot \frac{f(c(t), J(t))}{J(t)} - \frac{1}{\eta} \cdot \frac{1}{1 - \frac{\chi}{c(t)}} \cdot \frac{\dot{c}(t)}{c(t)} \ , \tag{37}$$

after noticing that (7) gives $\dot{J}(t) = -\ f(c(t), J(t))$. After some algebra we obtain

$$f_J(c(t), J(t)) = \frac{\frac{1}{\eta} - \gamma}{1-\gamma} \cdot \frac{f(c(t), J(t))}{J(t)} \ . \tag{38}$$

After substituting (37) and (38) into (36) we arrive at,

$$IES(t) = \eta \left\{ \frac{d\left[\frac{1}{1-\frac{\chi}{c(t)}} \cdot \frac{\dot{c}(t)}{c(t)}\right]}{d\left[\frac{\dot{c}(t)}{c(t)}\right]} \right\}^{-1} \ . \tag{39}$$

Assuming that the current consumption level, $c(t)$, is constant, and focusing only on the impact of the change in the growth rate of consumption on the change in the growth rate of the marginal utility of consumption, equation (39) implies $IES(t) = \eta[1-\chi/c(t)]$ which proves the statement. $\square$

Proof of Proposition 1

We make a guess on the functional form of the value function, namely,

$$V(a, y) = b\frac{(a+\psi y-\omega)^{1-\gamma}}{1-\gamma} \ , \tag{40}$$

which implies,

$$V_a(a, y) = b(a+\psi y-\omega)^{-\gamma} \ , \tag{41}$$

and also

$$f_c(c, V(a, y)) = \rho b^{1-\frac{1-\frac{1}{\eta}}{1-\gamma}} (a+\psi y-\omega)^{\frac{1}{\eta}-\gamma} (c-\chi)^{-\frac{1}{\eta}} \ . \tag{42}$$

From (41), (42) and (11) it is,

$$c = \rho^{\eta} b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}} (a+\psi y-\omega) + \chi \ . \tag{43}$$

Similarly, calculating the appropriate partial derivatives and substituting them in (12), gives,

$$\boldsymbol{\phi}^T = \frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T - r_f\mathbf{1}^T\right)\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right) - \sigma_y\psi\frac{y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T \ . \tag{44}$$

Substituting (43), (40), (8), (44), and all derivatives stemming from (40) into the HJB given by (10) results in,

$$\begin{aligned}
\rho b\frac{(a+\psi y-\omega)^{1-\gamma}}{1-\frac{1}{\eta}} &= \frac{\rho^\eta b^{1-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1-\frac{1}{\eta}}(a+\psi y-\omega)^{1-\gamma}+ \\
&+b(a+\psi y-\omega)^{-\gamma}\left\{\frac{1}{\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right)(a+\psi y-\omega)-\right. \\
&\left.-\sigma_y\psi y\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right)+r_f a+y-\chi-\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}(a+\psi y-\omega)\right\}- \\
&-\frac{\gamma}{2}a^2 b(a+\psi y-\omega)^{-\gamma-1}\left\{\frac{1}{\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right)-\sigma_y\frac{\psi y}{a}\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right\}\times \\
&\times\boldsymbol{\sigma}\boldsymbol{\sigma}^T\left\{\frac{1}{\gamma}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right)\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right)-\sigma_y\frac{\psi y}{a}\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\right\}+ \\
&+\psi b(a+\psi y-\omega)^{-\gamma}\mu_y y-\frac{\gamma}{2}b\psi^2(\sigma_y y)^2(a+\psi y-\omega)^{-\gamma-1}-\gamma\sigma_y a y b\psi(a+\psi y-\omega)^{-\gamma-1}\times \\
&\times\left\{\frac{1}{\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(1+\psi\frac{y}{a}-\frac{\omega}{a}\right)-\sigma_y\frac{\psi y}{a}\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right\}\boldsymbol{\sigma}\boldsymbol{\rho}_y^T \ .
\end{aligned} \tag{45}$$

After some algebra, (45) leads to,

$$\begin{aligned}
\frac{\rho-\frac{1}{\eta}\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1-\frac{1}{\eta}}-\frac{1}{2\gamma}(\mathbf{R}-r_f\mathbf{1})\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right) &= r^f\frac{a+\frac{1+\psi\left[\mu_y-\sigma_y\left(\mathbf{R}-r_f\mathbf{1}\right)\left(\boldsymbol{\rho}\boldsymbol{\sigma}^{-1}\right)^T\right]}{r_f}y-\frac{\chi}{r_f}}{a+\psi y-\omega} \\
&+\frac{1}{2}\gamma\left(\frac{\sigma_y\psi y}{a+\psi y-\omega}\right)^2\left(\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T-1\right) \ .
\end{aligned} \tag{46}$$

Since we have assumed that $\rho_{y,1}^2+...+\rho_{y,N}^2=1$, $\boldsymbol{\rho}_y\boldsymbol{\rho}_y^T=1$, and the last term of the right-hand side in (46) vanishes. Moreover, we set

$$\omega = \chi/r_f \tag{47}$$

and

$$\psi = \frac{1+\psi\left[\mu_y - \sigma_y\left(\mathbf{R}-r_f\mathbf{1}\right)\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T\right]}{r_f}\ , \tag{48}$$

which gives

$$\psi = 1/r_y\ . \tag{49}$$

After substituting (48) into (46) we obtain

$$\frac{\rho - \frac{1}{\eta}\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}}}{1-\frac{1}{\eta}} - \frac{1}{2\gamma}\left(\mathbf{R}-r_f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}\left(\mathbf{R}^T-r_f\mathbf{1}^T\right) = r_f\ . \tag{50}$$

Solving (50) for $\rho^\eta b^{-\eta(1-1/\eta)/(1-\gamma)}$ gives,

$$\rho^\eta b^{-\eta\frac{1-\frac{1}{\eta}}{1-\gamma}} = \xi\ , \tag{51}$$

in which ξ is given by equation (15). Moreover, substituting (49) and (47) into (44) leads to (13). Substituting formulas (49) and (47) in (40) reveals that Assumption 1 is both necessary and sufficient in order that $V(a,y)$ be well-defined. From (15) the requirement that $\xi > 0$ is equivalent to the condition given by Assumption 2 in order to guarantee that, under Assumption 1 and equation (14), $c \geq \chi$ for all (a,y), completing the proof. □

Proof of equations (26), (27), and (25)

The decomposition of matrix $\boldsymbol{\Sigma}$ is

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T = \begin{bmatrix} \sigma_s & 0 \\ \rho_{s,b}\sigma_b & \sigma_b\sqrt{1-\rho_{s,b}^2} \end{bmatrix} \cdot \begin{bmatrix} \sigma_s & \rho_{s,b}\sigma_b \\ 0 & \sigma_b\sqrt{1-\rho_{s,b}^2} \end{bmatrix}\ , \tag{52}$$

with

$$\boldsymbol{\sigma}^{-1} = \frac{1}{\sigma_s\sigma_b\sqrt{1-\rho_{s,b}^2}}\begin{bmatrix} \sigma_b\sqrt{1-\rho_{s,b}^2} & 0 \\ -\rho_{s,b}\sigma_b & \sigma_s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ \frac{-\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{1}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}\ , \tag{53}$$

so,

$$\boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1} = \begin{bmatrix} \rho_{y,s} & \rho_{y,b} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sigma_s} & 0 \\ \frac{-\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{1}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}$$

or,

$$\boldsymbol{\rho}_y \boldsymbol{\sigma}^{-1} = \begin{bmatrix} \frac{\rho_{y,s}}{\sigma_s} - \frac{\rho_{y,b}\rho_{s,b}}{\sigma_s\sqrt{1-\rho_{s,b}^2}} & \frac{\rho_{y,b}}{\sigma_b\sqrt{1-\rho_{s,b}^2}} \end{bmatrix}. \tag{54}$$

Notice that since,

$$\boldsymbol{\Sigma}^{-1} = \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} = \frac{1}{\sigma_s^2\sigma_b^2\left(1-\rho_{s,b}^2\right)} \begin{bmatrix} \sigma_b^2 & -\rho_{s,b}\sigma_s\sigma_b \\ -\rho_{s,b}\sigma_s\sigma_b & \sigma_s^2 \end{bmatrix}, \tag{55}$$

after some algebra, the term $(1/\gamma)\left(\mathbf{R} - r^f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1}$ in (13) is,

$$\frac{1}{\gamma}\left(\mathbf{R} - r^f\mathbf{1}\right)\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)^{-1} = \frac{1}{\gamma}\cdot\frac{1}{1-\rho_{s,b}^2}\begin{bmatrix} \frac{\frac{R_s-r_f}{\sigma_s} - \rho_{s,b}\frac{R_b-r_f}{\sigma_b}}{\sigma_s} & \frac{\frac{R_b-r_f}{\sigma_b} - \rho_{s,b}\frac{R_s-r_f}{\sigma_s}}{\sigma_b} \end{bmatrix}. \tag{56}$$

After combining (56) and (54) with (13), and after imposing the constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, we obtain equations (26) and (27). Finally, since $r_y = r_f - \mu_y + \sigma_y\left(\mathbf{R} - r_f\mathbf{1}\right)\left(\boldsymbol{\rho}_y\boldsymbol{\sigma}^{-1}\right)^T$, after combining equation (54) with the constraint $\rho_{y,s}^2 + \rho_{y,b}^2 = 1$, we obtain equation (25). □

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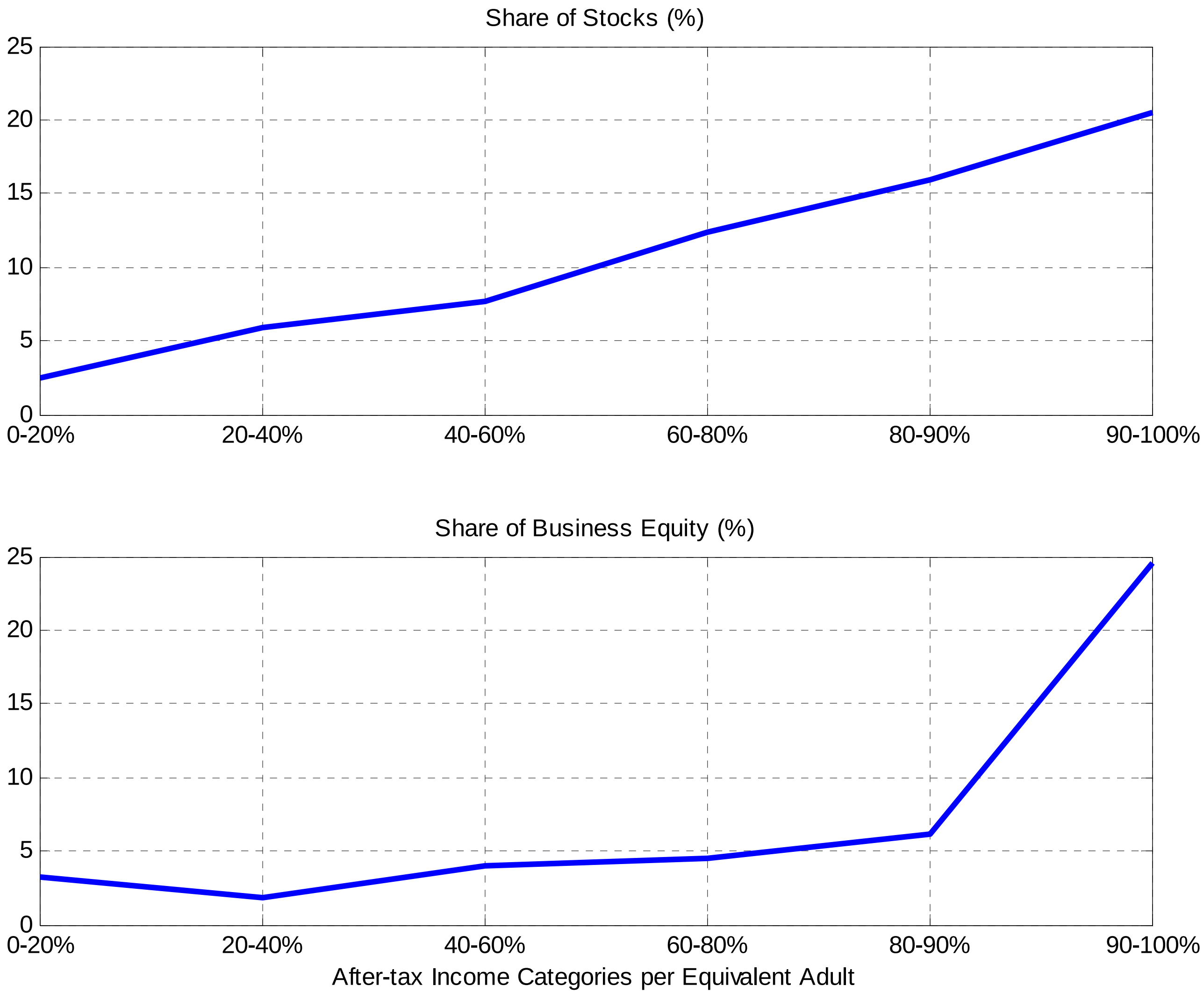

Figure 1 Stocks and business equity as fractions of total assets plotted against income categories (2007 USD).

Source: Survey of Consumer Finances (SCF) 2007. All income-wealth variables in Figure 1 are expressed in equivalent-adult terms in order to correct for household-size effects. The equivalence scale we used is $\sqrt{n}$, in which n is the number of household members. In addition, all variables refer to households with household heads of all ages. See the comparison between Tables 6 and 7 in the Online Appendix, which contrast the full-sample data of Figure 1 with those referring to a specific age with household heads aged between 25-59 years old. Demographic or life-cycle biases seem to play a rather mild role, so we have chosen to focus on information conveyed by the full sample throughout the paper.

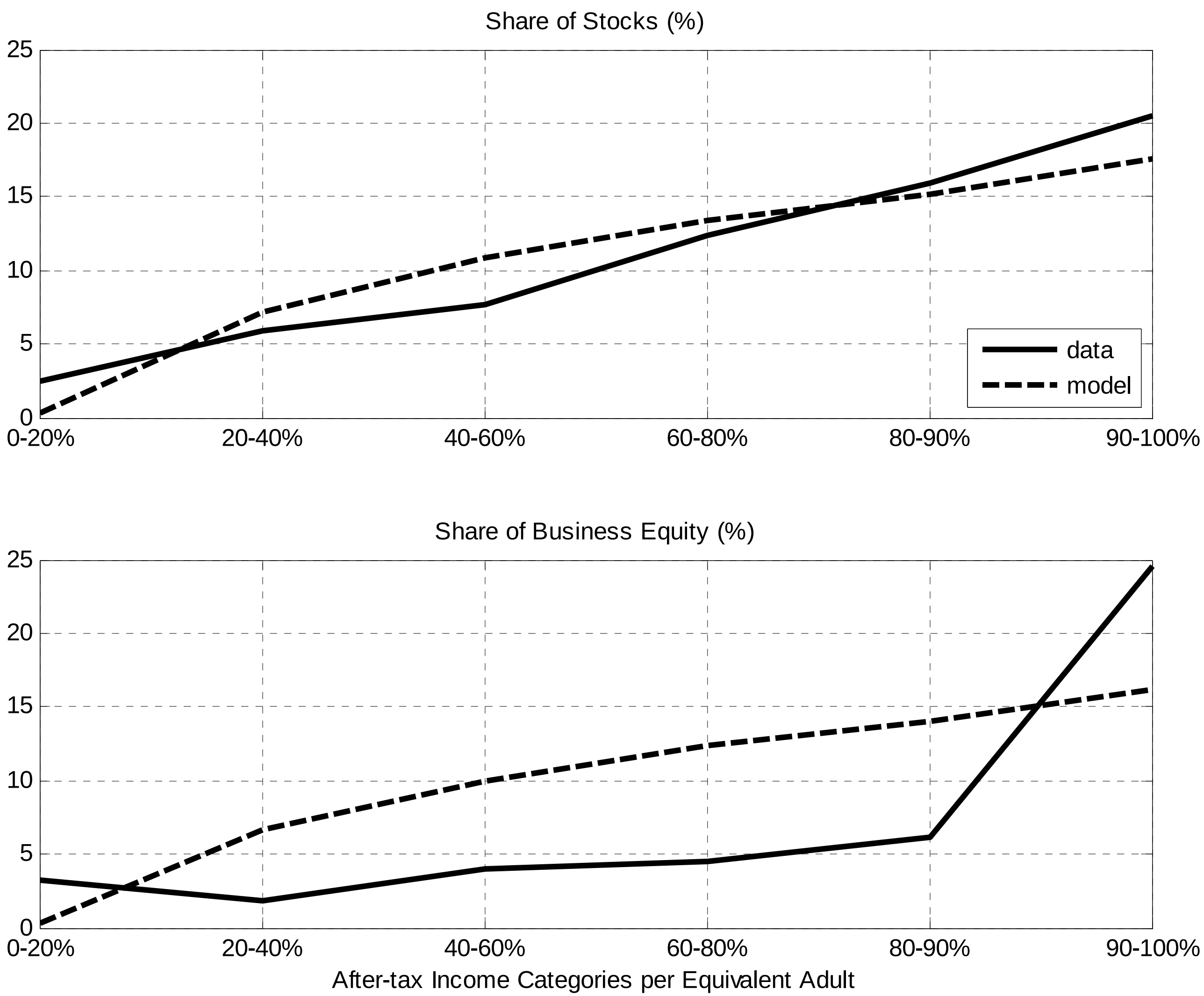

Figure 2 Benchmark calibration of $\mathbf{\Phi}(a, y)$. Income is in 2007 USD.

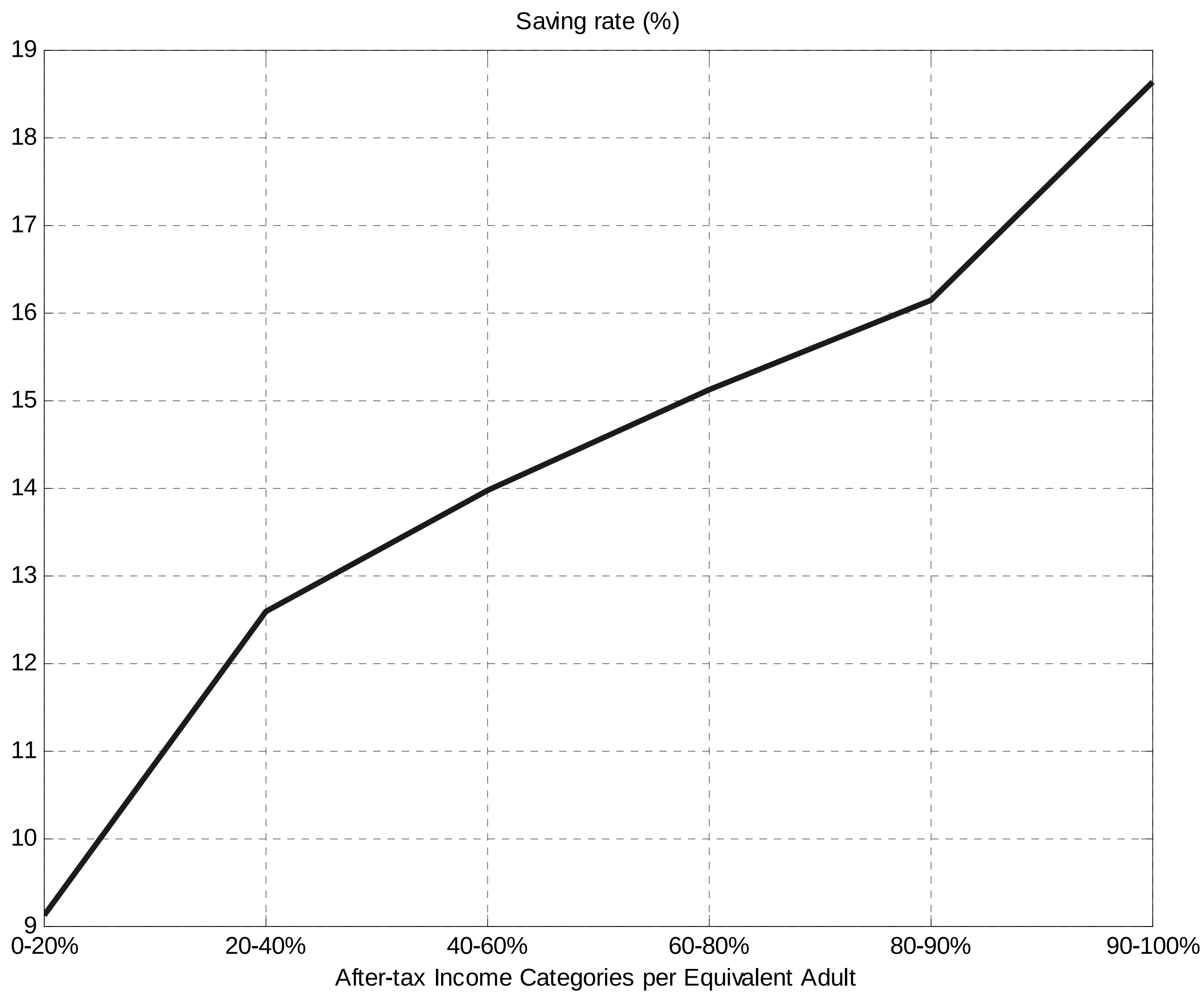

Figure 3 Benchmark calibration of the saving rate. Income is in 2007 USD.

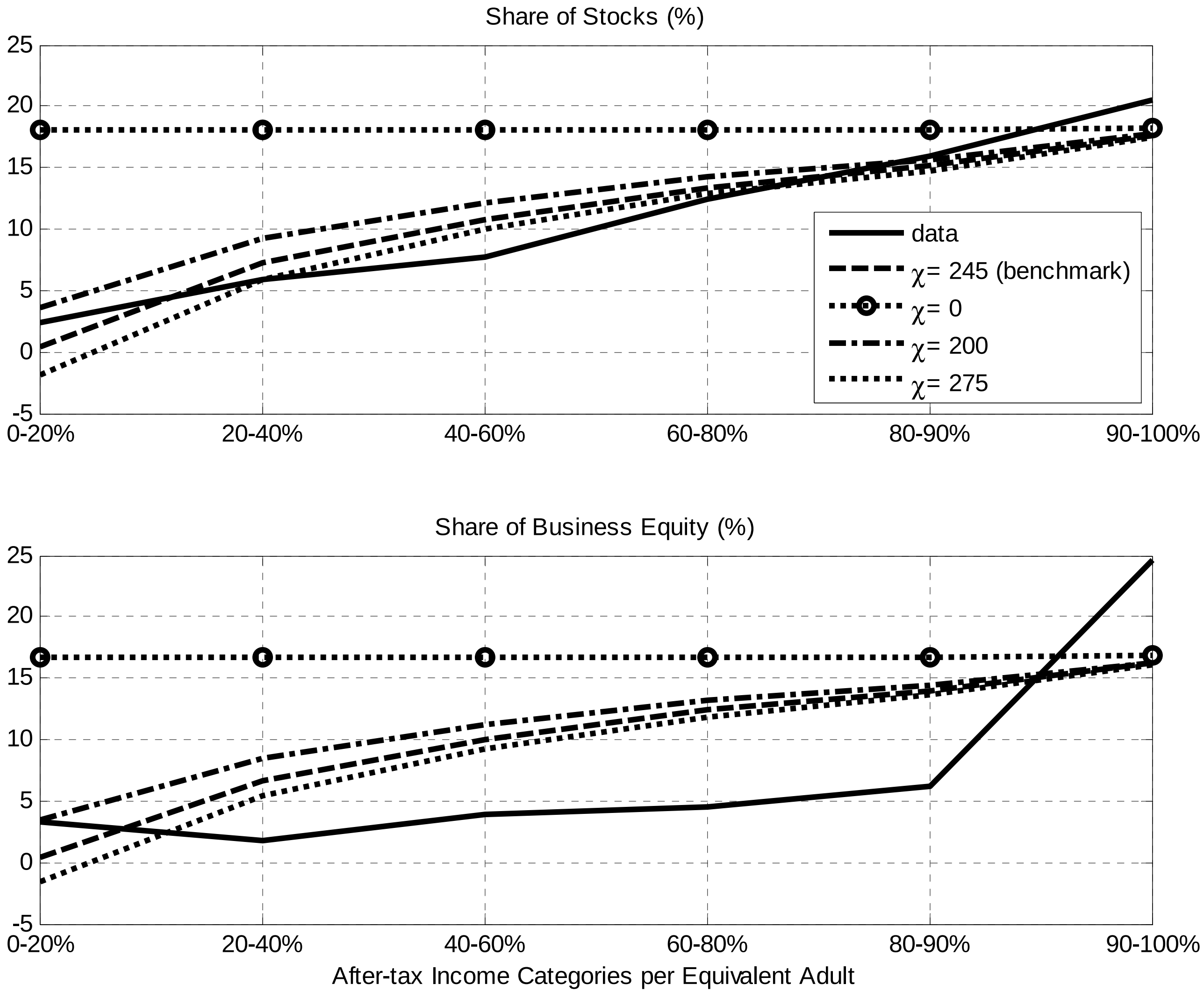

Figure 4 Sensitivity analysis of $\mathbf{\Phi}(a, y)$ by varying subsistence consumption.

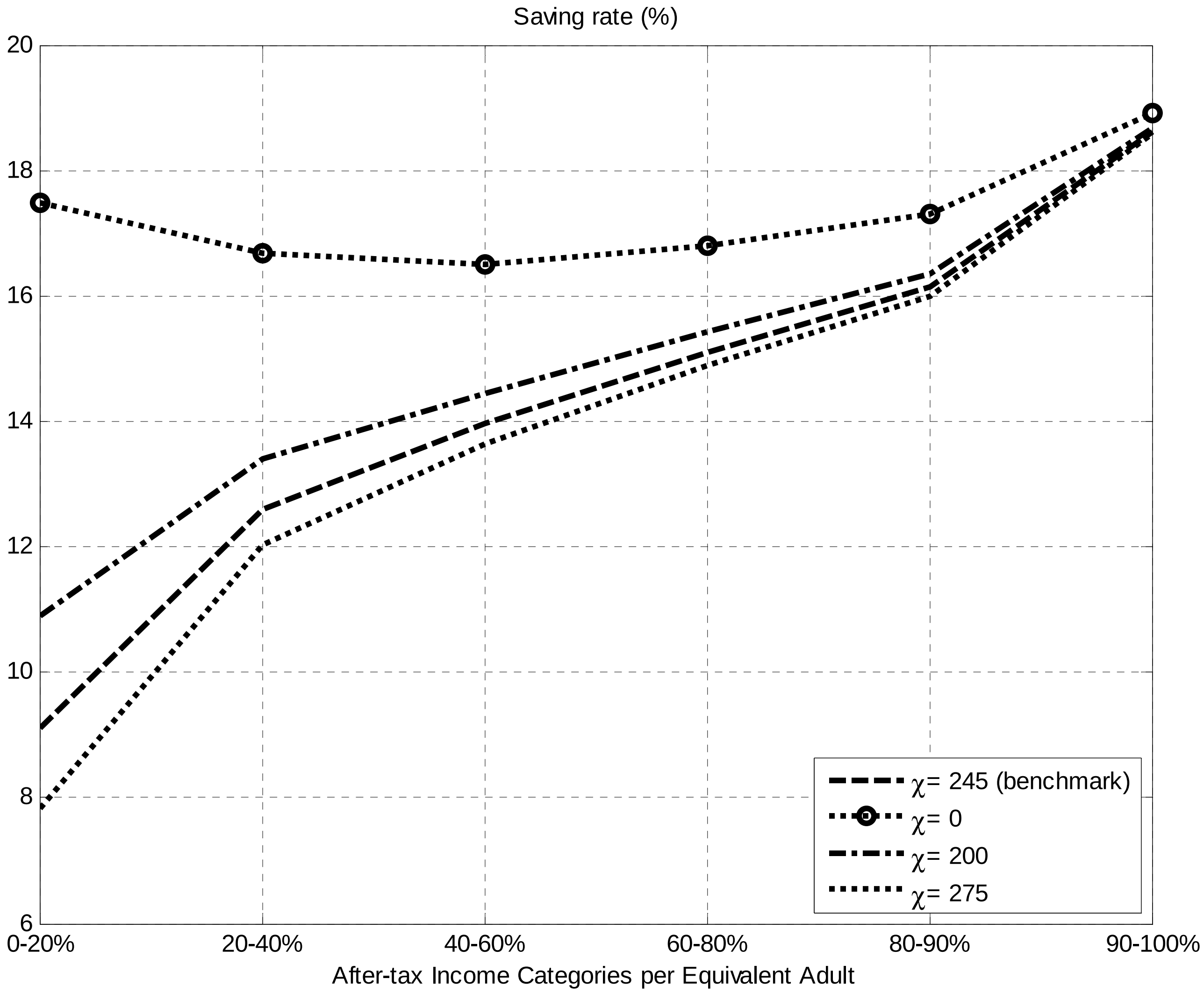

Figure 5 Sensitivity analysis of the saving rate (varying subsistence consumption).

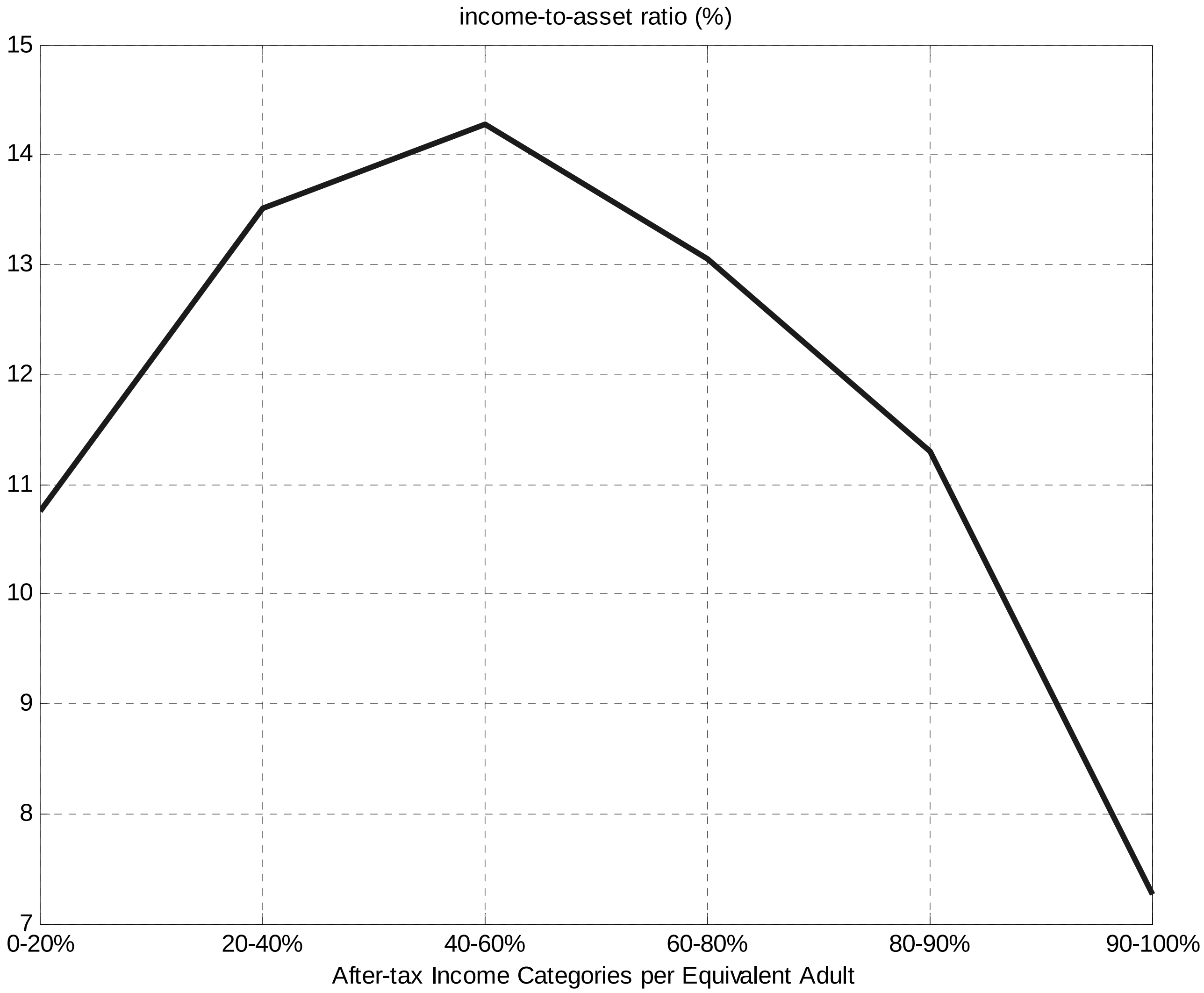

Figure 6 Income-to-asset ratio in the data. Both income and assets are per equivalent adult and in thousands of 2007 USD.

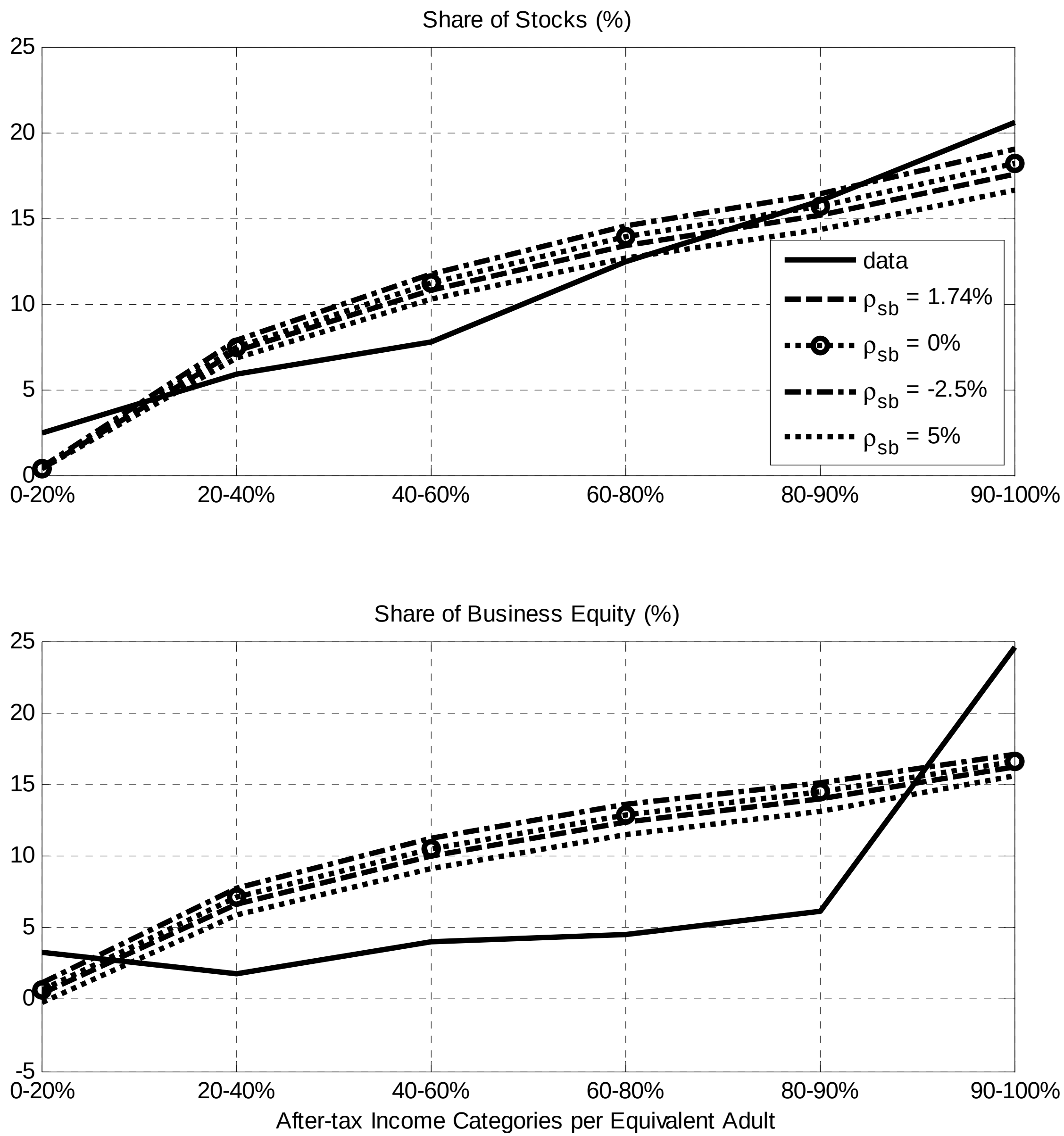

Figure 7 Sensitivity analysis of $\mathbf{\Phi}(a, y)$ by varying ρ_{sb}.

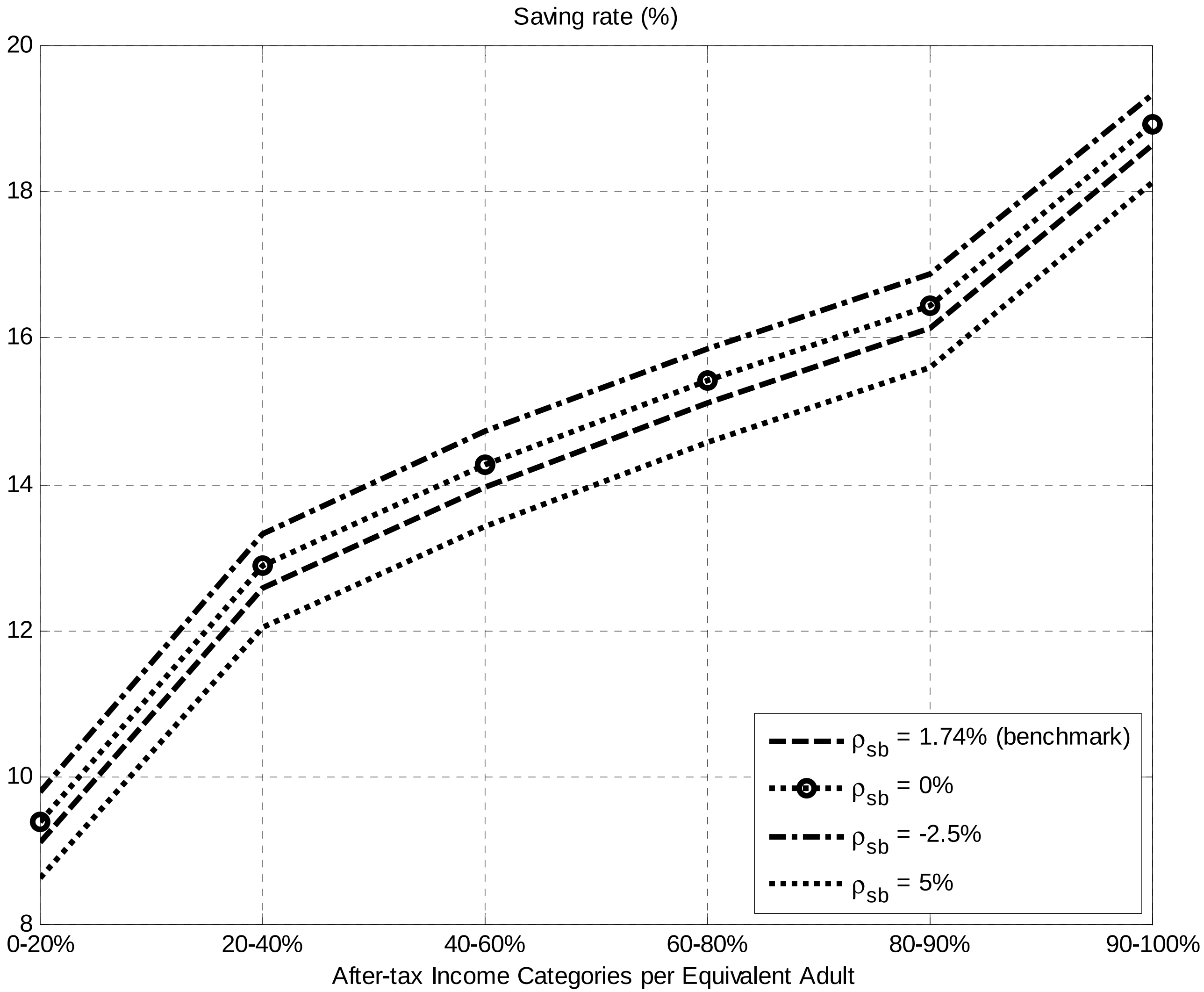

Figure 8 Sensitivity analysis of the saving rate by varying ρ_{sb}.

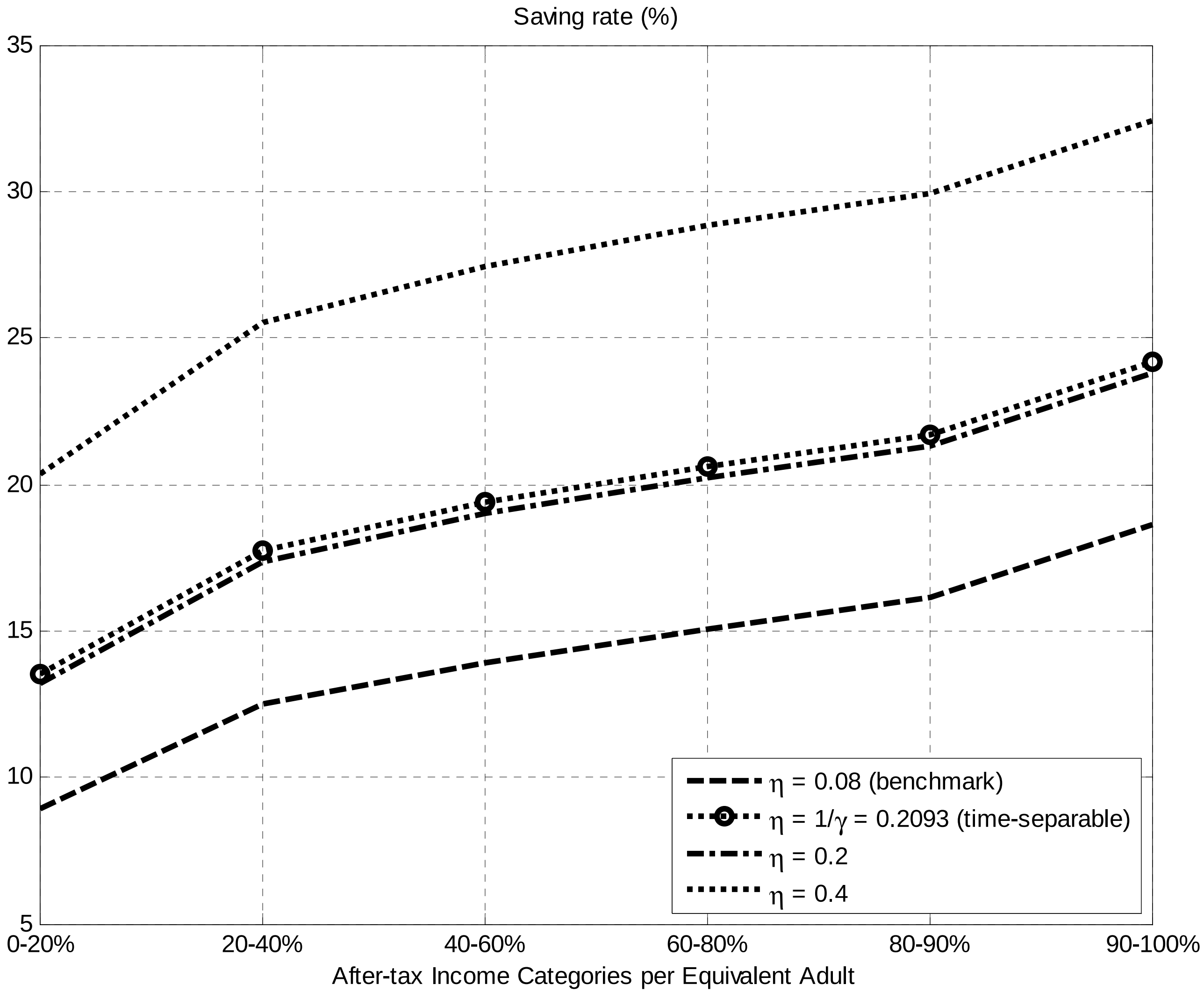

Figure 9 Sensitivity analysis of the saving rate by varying the IES.

Online Data Appendix
of

Analytical Guidance for Fitting Parsimonious Household-Portfolio Models to Data

Sylwia Hubar[a]

Christos Koulovatianos[b,c,*]

Jian Li[d]

July 2013

[a] Exeter School of Business, University of Exeter

[b] Department of Economics, University of Luxembourg

[c] Center for Financial Studies (CFS), Goethe University Frankfurt

[d] Goethe University Frankfurt

* Corresponding author: Department of Economics, University of Luxembourg, 162A avenue de la Faïencerie, Campus Limpertsberg, BRC 1.06E, L-1511, Luxembourg, Email: christos.koulovatianos@uni.lu, Tel.: +352-46-66-44-6356, Fax: +352-46-66-44-6341.

1 Description of Variables (Source: Survey of Consumer Finances (SCF) 2007)

1. `Stock Equity (Direct and Indirect Stockholding)`:

   (a) **Direct stockholding**
   - Publicly Traded Stocks.

   (b) **Stockholding through mutual funds**
   - Saving and Money Market Accounts.
   - Mutual Funds.
   - Annuities, Trusts and Managed Investment Accounts.

   (c) **Stockholding through Retirement Accounts**
   - IRA/KEOGH Accounts.
   - Past Pension Accounts.
   - Current Benefits and Future Benefits from Pensions.

2. `Business Equity`:
   - Actively Managed Business.
   - Non-Actively Managed Business.

3. `Total Assets`: Assets of all categories covered in the SCF 2007 database (stocks, business equity, bonds, saving and checking accounts, retirement accounts, life insurance, primary residence, and other residential real estate, nonresidential real estate, vehicles, artwork, jewelry, etc.).

4. `Total Income`: Income from all sources (salary, interest, dividend, compensations, transfers etc).

5. `Weight`: Weights are assigned in order to normalize the sample to representative-sampling standards (see the section "Analysis Weights" in the "Codebook for the 2007 Survey of Consumer Finances").[1]

6. `Income Percentiles`: Benchmark value from Bucks et al. (2009a, Table A.2, p. A53).

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[1] The "Codebook for the 2007 Survey of Consumer Finances" is downloadable from `http://federalreserve.gov/econresdata/scf/scf_2007documentation.htm`

7. `Equivalence Scales`: The equivalence scale is $\sqrt{n}$ in which n is the number of household members. This equivalence-scale measure approximates the standard OECD equivalance scales.

Table 1: **Income Percentiles**

Percentile	Total Labor Income
20	20,600
40	36,500
60	59,600
80	98,200
90	140,900

Notes: Full sample in 2007 USD. Data in the survey is in 2006 USD, which is adjusted according to the CPI-U table (U.S. Department of Labor Bureau of Labor Statistics, Consumer Price Index). The 2006-2007 average to average change is 2.84%.

2 Matching Data with Descriptive Statistics in the SCF 2007 Chartbook

To show that our database is constructed in a reliable way, we compare key statistics with those reported in the SCF2007 Chartbook. Our robustness checks are:

- **Matching median values of key variables in the SCF 2007 chartbook**:
  The reason for choosing medians instead of means in order to perform a robustness check is that median values capture more information regarding a variable's distribution. In addition, mean values can be substantially affected by outliers. Indeed, our database matches median values in the SCF2007 chartbook.

- **Matching median values of each income group in the SCF 2007 chartbook**:
  Our database generated should match the income benchmark in small differences by income quintile or decile, which is a more demanding task. Our results are listed in the following tables demonstrate that the matching is satisfactory.

Table 2: **Median Values of Key Variables**

Variables	SCF2007 Chartbook	Our Data
Total Asset	221.5	221.9
Total Income	47.3	46.5
Stock Equity	35.0	34.8
Business Equity	100.5	80.6

Notes: Full sample. Values in thousands of 2007 US dollars.

Table 3: **Median Values of Pre-Tax Family Income for All Families, Classified by Income Percentile**

Income Percentile	SCF2007 Chartbook	Our Data
Less than 20%	12.3	12.3
20%-39.9%	28.8	28.8
40%-59.9%	47.3	47.1
60%-79.9%	75.1	74.9
80%-89.9%	114.0	114.8
90%-100%	206.9	209.0

Notes: Full sample, in thousands of 2007 US dollars. Data in the survey are in 2006 US dollars. We adjusted them according to the CPI-U table (U.S. Department of Labor Bureau of Labor Statistics, Consumer Price Index). 2006-2007 Average to Average change is 2.84% .

Table 4: **Median Values of Total Assets for Families with Positive Asset Holdings, Classified by Income Percentile**

Income Percentile	SCF2007 Chartbook	Our Data
Less than 20%	23.5	26.1
20%-39.9%	84.9	90.1
40%-59.9%	183.5	182.2
60%-79.9%	343.1	345.6
80%-89.9%	567.5	561.2
90%-100%	1358.4	1355.5

Notes: Full sample, in thousands of 2007 US dollars.

Table 5: **Median Values of Different Asset Types for Families with Positive Asset Holdings by Income Percentile**

	SCF2007 Chartbook		Our Data	
Income Percentile	**Stocks**	**Business**	**Stocks**	**Business**
Less than 20%	6.5	50.0	7.3	43.6
20%-39.9%	8.4	19.5	10.0	20.4
40%-59.9%	17.7	30.8	17.4	33.8
60%-79.9%	34.2	55.1	33.8	41.6
80%-89.9%	62.0	72.1	59.1	70.6
90%-100%	219.6	379.5	223.8	381.0

Notes: Full sample, in thousands of 2007 US dollars.

3 Portfolio Shares of Risky Assets

Portfolio shares of risky assets are calculated by income groups. For each income group we have the formula,

$$SHARE_i = \frac{\sum_k \frac{\sum_n SHARE_{obs(n)}}{N}}{K} \ ,$$

in which n is the observation number, k is the imputation number and i is the risky-asset type. Final results are shown in the following tables. SCF weights are not shown in the above formula but have been included in the calculation. The comparison between Tables 6 and 7 justifies why we did not restrict the full sample into a particular age range such as household heads aged between 25-59 years old. Demographic or life-cycle biases seem to play a rather mild role, so we have chosen to utilize the entirety of the infomation provided by the SCF 2007 database in our calibration exercises.

Table 6: **Portfolio Share on Risky Assets by Income Percentile (Full Sample per Equivalent Adult)**

	Risky Assets (%)		General Information			Tax Information	
Income Percentile	**Stocks**	**Business**	**Total Income**	**Total Assets**	**Income/Asset (%)**	**Effective Marginal Tax Rate**	**After-tax Income**
Generated Data							
Less than 20%	2.44	3.24	9.03	85.52	10.56	-1.83%	9.19
20%-39.9%	5.84	1.84	19.42	139.82	13.89	2.78%	18.88
40%-59.9%	7.72	3.97	32.20	210.93	15.27	6.47%	30.11
60%-79.9%	12.44	4.51	49.84	327.22	15.23	14.28%	42.72
80%-89.9%	15.96	6.14	74.61	511.32	14.60	22.63%	57.73
90%-100%	20.53	24.55	252.12	2452.22	10.28	29.27%	178.33

Notes: Full sample, in thousands of 2007 US dollars.

Table 7: **Portfolio Share on Risky Assets by Income Percentile (Age group 25-59 per Equivalent Adult)**

	Risky Assets (%)		**General Information**			**Tax Information**	
Income Percentile	**Stocks**	**Business**	**Total Income**	**Total Assets**	**Income/Asset (%)**	**Effective Marginal Tax Rate**	**After-tax Income**
Generated Data							
Less than 20%	2.78	3.35	10.15	73.74	13.76	-1.83%	10.33
20%-39.9%	5.05	3.35	22.93	99.79	23.00	2.78%	22.30
40%-59.9%	8.32	3.73	37.60	188.48	19.95	6.47%	35.17
60%-79.9%	12.90	4.74	54.49	303.89	17.93	14.28%	46.71
80%-89.9%	14.89	7.26	78.69	475.10	16.57	22.63%	60.88
90%-100%	18.52	25.31	242.57	1908.87	12.71	29.27%	171.57

Notes: Age group 25-59, in thousands of 2007 US dollars.

References

[1] Bucks, Brian K., Arthur B. Kennickell, Traci L. Mach and Kevin B. Moore (2009a): "Changes in U.S. Family Finances from 2004 to 2007: Evidence from the Survey of Consumer Finances," Survey of Consumer Finances, Board of Governors of the Federal Reserve System.

[2] Bucks, Brian K., Arthur B. Kennickell, Traci L. Mach and Kevin B. Moore (2009b): "2007 SCF Chartbook," Survey of Consumer Finances, Board of Governors of the Federal Reserve System.