EXTENDING FUNCTIONS TO NATURAL EXTENSIONS

GEORGES HANSOUL AND BRUNO TEHEUX

ABSTRACT. We investigate the problem of extending maps between algebras of a finitely generated prevariety to their natural extensions. As for canonical extension of lattice-based algebras, a new topology has to be introduced in order to be able to define an algebra inside its natural extension.

Under the assumption that there is a structure that yields a logarithmic duality for the prevariety, this topology is used to define the natural extension of a map. This extension turns out to be a multivalued map and we investigate its properties related to continuity, composition and smoothness. We also prove that our approach completely subsume the lattice-based one.

In the meanwhile, we characterize the natural extension of Boolean products.

1. INTRODUCTION

The tool of canonical extension has a long standing history. It was introduced (and ignored for a long subsequent period) in the two seminal papers [16] and [17] in which the authors defined the perfect extension of a Boolean algebra with operators. The development of canonical (perfect) extension for modal algebras can be considered as one of the key elements of the success of the algebraic approach for normal modal logics. Indeed, varieties of modal algebras closed under canonical extension are algebraic counterparts of canonical logics, which form a class of KRIPKE complete logics. One of the most elegant success of this theory is given in [15] in which SAHLQVIST completeness results are obtained in a completely algebraic way.

This successful approach lead some authors to consider generalizations of canonical extension to wider classes of algebras: bounded distributive lattices with operators in [9], bounded distributive lattices with monotone maps in [10], bounded distributive lattice expansions in [11], bounded lattice expansions in [8]. These tools were also recently applied to shed light on varieties of lattice-based algebras (see [5, 12]). The canonical extension of a lattice-based algebra $\mathbf{A}$ is built in two steps: first build the canonical extension of the lattice reduct of $\mathbf{A}$, then consider non-lattice operations of

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$\mathbf{A}$ as functions on the lattice reduct of $\mathbf{A}$ and use the tools provided by the theory of canonical extension to define extension(s) of these operations.

For a while, canonical extension was considered as a tool that is available even though a topological duality was not known. However, it has been recently noted that canonical extension may actually *lead* to topological dualities in at least two different ways. In [2], canonical extension is used to bring structure to the spectrum of an algebra $\mathbf{A}$ in order to shape the bidual of $\mathbf{A}$ into an isomorphic copy of $\mathbf{A}$. In [7], the aim is to restore symmetry in the construction of natural dualities by swapping topology to the algebraic side. This approach leads to a duality between topological algebras and non-topological structures.

Hence, canonical extension appears as a companion tool to topological dualities: Stone duality for Boolean algebras and Priestley duality for bounded distributive lattices. The theory of Natural Dualities provides a general framework for the development of such Stone like dualities for prevarieties of algebras. Hence, the question of designing a tool that generalizes canonical extension in this setting naturally arises. There are at least two main steps in the development of such a generalization. Step 1 is to define the natural extension of an algebra. Step 2 is to define natural extension of functions between algebras.

Step 1 was initiated in [4] in which the *natural extension* of an algebra is introduced and generalizes canonical extension to non lattice-based algebras. There are several (often equivalent) ways to define the natural extension of an algebra $\mathbf{A}$. A first one is to define it as the closure of $\mathbf{A}$ in the topological algebra of the dual of the non-topological dual of $\mathbf{A}$. A second approach is to define it as the profinite extension of $\mathbf{A}$. The first approach connects natural extension to the history of canonical extension and its applications in algebraic logic (specially in modal logic). The second one could be considered as an impulse towards developments and applications of natural extension for broader classes of algebras (*i.e* algebras not arising from logic). In this view, the question of determining in general what should be a canonical extension and what should it be used for is very intriguing. In cite [4], the authors prove that the two mentioned definitions coincide in some general setting.

In this paper, we propose to initiate step 2, that is, to provide a way to extend functions between algebras to their natural extensions. Formally, the problem is the following. Consider a prevariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ in which natural extension of algebras can be computed (thanks to step 1) and denote by $\mathbf{A}^\delta$ the natural extension of $\mathbf{A} \in \mathcal{A}$. Let $u : \mathbf{A} \to \mathbf{B}$ be a map for some $\mathbf{A}, \mathbf{B} \in \mathcal{A}$. How to extend u to a map $u^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$?

We propose a solution that generalizes the corresponding historical construction for canonical extension of maps between bounded distributive lattices. The latter construction is topological in nature. As for the distributive lattice case, there is a natural "intrinsic" topology ι that equips any $\mathbf{A} \in \mathcal{A}$.

It is the topology induced by the product of the discrete topology on $\mathbf{M}$ when $\mathbf{A}$ is considered as a subalgebra of $\mathbf{M}^{\mathcal{A}(\mathbf{A},\mathbf{M})}$. We introduce a new topology, denoted by δ, in order to be able to topologically define $\mathbf{A}$ in $\langle \mathbf{A}^\delta, \delta\rangle$. It turns out that $\mathbf{A}$ is not always dense in $\langle \mathbf{A}^\delta, \delta\rangle$ but it is so if the $\underset{\sim}{M}$ generates a logarithmic duality. We use this topology to extend maps $u : \mathbf{A} \to \mathbf{B}$. The resulting construction is a relation $\widetilde{u} \subseteq \mathbf{A}^\delta \times \mathbf{B}^\delta$. Interestingly, it can be considered as a map defined on $\mathbf{A}^\delta$ and valued in the set $\Gamma(\mathbf{B}^\delta_\iota)$ of closed subsets of $\langle \mathbf{B}^\delta, \iota\rangle$ with good continuity properties.

The paper is organized as follows. In the next section, we recall the definition of the natural extension of an algebra in a residualy finite prevariety. We also introduce the new topology δ and give its very first properties. To extend non fundamental operations on an algebra, it is important to know how natural extension behaves on finite products of algebras. We prove that under some general conditions, the canonical extension of the product is the product of the canonical extension of the factors. This result is then generalized to Boolean products.

The third section is the core of the paper. We work under the assumption that there is a structure $\underset{\sim}{M}$ that yields a duality for $\mathcal{A}$ and that is injective in the dual category. Moreover, we assume that $\mathbf{A}$ is dense in $\langle \mathbf{A}^\delta, \delta\rangle$ for any $\mathbf{A} \in \mathcal{A}$, which is the case if the duality is logarithmic. For any $u : \mathbf{A} \to \mathbf{B}$, we provide an extension $\widetilde{u} \subseteq \mathbf{A}^\delta \times \mathbf{B}^\delta$.

In the fourth section, we introduce two properties of functions $u : \mathbf{A} \to \mathbf{B}$, namely *smoothness* and *strongness*. A function u is smooth if $\widetilde{u}(x)$ is a singleton for any $x \in \mathbf{A}^\delta$. Strongness is a continuity property that is shared by extension of homomorphisms. If u is a strong function, then the $\widetilde{u} : \mathbf{A}^\delta \to \Gamma(\mathbf{B}^\delta_\iota)$ can be lifted to a map $\overline{u} : \Gamma(\mathbf{A}^\delta_\iota) \to \Gamma(\mathbf{B}^\delta_\iota)$. These two properties are used to study preservation of composition of functions through canonical extension. We end the fourth section by providing a concrete illustration of our constructions in the variety of median algebras (a non lattice-based variety).

In the lattice-based case, canonical extension provides with two single-valued extensions u^σ and u^π of a map $u : \mathbf{A} \to \mathbf{B}$. Our approach provides with one multiple-valued extension $\widetilde{u}$. In the fifth section of the paper, we prove that u^σ and u^π can be recovered from $\widetilde{u}$, but not conversely. This constitutes the definitive confirmation that our approach completely subsume the lattice-based one.

2. A topology for extension of maps

2.1. **Natural extension of algebras.** In [4], the authors define a generalization of canonical extension to classes of non necessarily lattice-based algebras. They provide such an extension, called *the natural extension* for *any residually finite algebra*, *i.e.*, for any algebra that belongs to a prevariety $\mathbb{ISP}(\mathcal{M})$ where $\mathcal{M}$ is a set of finite algebras of the same type. Such a prevariety is called an *internally residually finite* prevariety, in short, an *IFR-prevarierty.* The authors give two different ways to characterize the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathbf{A}$ in an IFR-prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$:

(1) $n_{\mathcal{A}}(\mathbf{A})$ can be obtained as a closed subspace of a topological algebra obtained through Hom functors,
(2) this construction is proved to be equivalent ([4], Theorem 3.6) to the profinite extension of $\mathbf{A}$ in $\mathcal{A}$.

Moreover, if $\mathcal{M}$ is a *finite* set of finite algebras and if $\underset{\sim}{M}$ is a multi-sorted structure that yields a natural duality for $\mathcal{A}$,

(3) $n_{\mathcal{A}}(\mathbf{A})$ is the algebra of the 'non topological' morphisms (*i.e.* structure preserving maps) from the dual of $\mathbf{A}$ to $\underset{\sim}{M}$.

Our main results are developed in this setting of a dualisable prevariety. Moreover, to avoid cumbersome notations, we prefer to state our results in the one-sorted case (where $\mathcal{M}$ contains only one finite algebra). The latter restriction is more a matter of convenience than a technical restriction: we claim that our developments admit the obvious generalization to the multi-sorted case.

Hence, in what follows, unless stated otherwise, we denote by $\mathbf{M}$ a finite algebra and by $\mathcal{A}$ the IFR-prevariety $\mathbb{ISP}(\mathbf{M})$. We use $\underset{\sim}{M}$ to denote a topological structure

$$\underset{\sim}{M} = \langle M, G \cup H \cup R, \iota \rangle$$

where ι is the discrete topology on M and G, H and R are respectively a set (possibly empty) of algebraic operations, algebraic partial operations (with nonempty domain) and algebraic (nonempty) relations on $\mathbf{M}$. We use $\mathcal{X}$ to denote the topological prevariety $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{M})$, *i.e.*, the class of topological structures that are isomorphic to a closed substructure of a nonempty power of $\underset{\sim}{M}$. For any $X, Y \in \mathcal{X}$ we denote by $\mathcal{X}(X, Y)$ the set of structure preserving continuous maps $f : X \to Y$. We use X_* to denote $\mathcal{X}(X, \underset{\sim}{M})$.

For any $\mathbf{A} \in \mathcal{A}$, we denote by $\mathbf{A}^*$ the set $\mathcal{A}(\mathbf{A}, \mathbf{M})$ of the homomorphims from $\mathbf{A}$ to $\mathbf{M}$. Recall (see Preduality Theorem 5.2 in [3] for instance) that $\mathbf{A}^* \in \mathcal{X}$ if $\mathbf{A}^*$ inherits the structure and the topology from $\underset{\sim}{M}^A$. Moreover, the evaluation map

$$e_{\mathbf{A}} : \mathbf{A} \to (\mathbf{A}^*)_* : a \mapsto e_{\mathbf{A}^*}(a) : \phi \mapsto \phi(a)$$

is an embedding. Similarly, for any $X \in \mathcal{X}$ the map

$$\epsilon_X : X \to (X_*)^* : \phi \mapsto \epsilon_X(\phi) : x \mapsto x(\phi)$$

is an embedding.

We use characterization (1) as the definition of the natural extension of an element of $\mathcal{A}$. In order to precise this definition, we need to introduce further notation.

We denote by $\mathbf{M}_\iota$ the topological algebra obtained by equipping $\mathbf{M}$ with the discrete topology. For any set X we use $\mathbf{M}_\iota^X$ to denote the power algebra $\mathbf{M}^X$ equipped with the product topology induced by ι on M. We denote by $\mathcal{A}_\iota$ the category of topological algebras that are isomorphic to a closed subalgebra of a nonempty power of $\mathbf{M}_\iota$ with continuous homomorphisms as arrows. For any topological structure X and any topological algebra $\mathbf{A}$, we denote respectively by $X^\flat$ and $\mathbf{A}^\flat$ the structure obtained from X and the algebra obtained from $\mathbf{A}$ by forgetting the topology. We denote by $\mathcal{X}^\flat$ the category whose objects are the $X^\flat$ where $X \in \mathcal{X}$ with structure preserving maps as arrows. By abuse of notation, we write $\mathcal{X}^\flat(X, Y)$ instead of $\mathcal{X}^\flat(X^\flat, Y^\flat)$.

Definition 2.1 ([4])**.** Let $\mathbf{A} \in \mathcal{A}$. The *natural extension* of $\mathbf{A}$, in notation $\mathbf{A}^\delta$, is the topological closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\mathbf{M}_\iota^{\mathbf{A}^*}$. The algebra $\mathbf{A}^\delta$ is turned into an element of $\mathcal{A}_\iota$ - and we denote it by $\mathbf{A}_\iota^\delta$ when we want to stress this fact - by equipping it with the topology induced by $\mathbf{M}_\iota^{\mathbf{A}^*}$.

Note that $\mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M})$ is easily seen to be a closed subalgebra of $\mathbf{M}_\iota^{\mathbf{A}^*}$. Hence, since $e_{\mathbf{A}}(\mathbf{A}) \subseteq \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M})$, we could have equivalently computed the closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\langle \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}), \iota\rangle$ (where ι denotes the induced topology) instead of $\mathbf{M}^{\mathbf{A}^*}$ in Definition 2.1. We shall use this fact without further notice throughout the paper.

Let us briefly comment notation. We use ι to denote the discrete topology on $\mathbf{M}$ because it coincides with the interval topology when $\mathbf{M}$ is equipped with a total order. For the natural extension of an algebra, several notations could have been adopted: the notation $n_{\mathcal{A}}(\mathbf{A})$ introduced in [4], the profinite extension notation $\mathrm{pro}_{\mathcal{A}}(\mathbf{A})$ or $\mathbf{A}^\delta$. We have chosen the latter because it fits with the historical notation of the canonical extension for bounded distributive lattices that is well established among the community.

Recall the two following results (Theorem 3.6 and Theorem 4.3 of [4]).

Proposition 2.2. *Assume that $A \in \mathcal{A}$.*

(1) *The definition of $\mathbf{A}_\iota^\delta$ is independent of the structure $G \cup H \cup R$ used to defined $\underset{\sim}{M}$ and of the algebra $\mathbf{M}$ used to define $\mathcal{A}$.*
(2) *If in addition $\underset{\sim}{M}$ yields a duality for $\mathcal{A}$ then $\mathbf{A}_\iota^\delta$ is isomorphic to $\langle \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}), \iota\rangle$.*

Thanks to the second result of the previous Proposition, in what follows, we identify $\langle \mathbf{A}^\delta, \iota\rangle$ with $\langle \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}), \iota\rangle$ when $\underset{\sim}{M}$ yields a duality for $\mathcal{A}$.

2.2. **The topology δ for natural extensions.** In the lattice-based setting, it is well known that the topology ι that naturally equips the canonical extension of an algebra $\mathbf{A}$ can be enriched into a finer topology in which $\mathbf{A}$ is definable as the algebra of isolated points. Authors have reserved various notations for this topology: Gehrke and Jónsson denote it by σ in [11] and Gehrke and Vosmaer denote it by δ in [12]. Since σ is usually used to denote Scott topologies, we shall prefer the notation δ.

If $X, Y \in \mathcal{X}$ we denote by $\mathcal{X}_p(X, Y)$ the set of partial morphisms from X to Y, *i.e.*, the set of maps $f : \mathrm{dom}(f) \to Y$ where $\mathrm{dom}(f)$ is a closed substructure of X and where $f \in \mathcal{X}(\mathrm{dom}(f), Y)$.

Definition 2.3. Assume that $\mathbf{A} \in \mathcal{A}$. For any $f \in \mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{M})$, we define

$$O_f = \{x \mid x \supseteq f\}.$$

We denote by $\Delta_{\mathbf{A}}$, or simply Δ the family

$$\Delta = \{O_f \mid f \in \mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{M})\}.$$

The topology $\delta(\mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}))$, or simply δ, is the topology on $\mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M})$ generated by Δ.

It is clear that Definition 2.3 will be essentially used when $\underset{\sim}{M}$ yields a duality for $\mathcal{A}$ since in this case $\mathbf{A}^\delta = \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M})$. The following result is another reason to stress the importance of the dualisable setting.

Proposition 2.4. *Assume that $A \in \mathcal{A}$.*

1. *The element of $\mathcal{X}(\mathbf{A}^*, \underset{\sim}{M})$ are isolated points of $\langle \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}), \delta \rangle$.*
2. *If $\underset{\sim}{M}$ is injective in $\mathcal{X}$ and if Δ is a base of δ then $\mathcal{X}(\mathbf{A}^*, \underset{\sim}{M})$ is dense in $\langle \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}), \delta \rangle$.*
3. *If $\underset{\sim}{M}$ is injective in $\mathcal{X}$ and yields a duality for $\mathcal{A}$ and if Δ is a base of δ then $e_{\mathbf{A}}(\mathbf{A})$ is a discrete dense subset of $\langle \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}), \delta \rangle$.*

Proof. (1) If $x \in \mathcal{X}(\mathbf{A}^*, \underset{\sim}{M})$ then $O_x = \{x\}$ is a base open of δ.

(2) Let F be a closed substructure of $\mathbf{A}^*$ and $f \in \mathcal{X}(F, \underset{\sim}{M})$. By injectivity of $\underset{\sim}{M}$, the map f extends to an $x \in \mathcal{X}(\mathbf{A}^*, \underset{\sim}{M})$. By construction $x \in O_f$, which gives the desired result.

(3) We know by (1) and (2) that the isolated points of $\langle \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}), \delta \rangle$ are exactly the elements of $e_{\mathbf{A}}(\mathbf{A})$. We conclude the proof by the fact that $\mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}) = \mathbf{A}^\delta$. □

Hence, it appears that $\mathbf{A}$ is topologically definable in $\mathbf{A}^\delta$ when Δ is a base of δ. The following result precises some rather general setting under which this happens. Recall that a strong duality is said to be *logarithmic* if (finite) coproducts in the dual category (these always exist since they are dual to products) are realized as the direct union, that is, disjoint union with constants amalgamated (see section 6.3 in [3]).

Proposition 2.5. *If $\underset{\sim}{M}$ yields a logarithmic duality for $\mathcal{A}$ then Δ is a base of δ.*

Proof. Let $\mathbf{A} \in \mathcal{A}$ and $f, g \in \mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{M})$. First, we note that $\mathrm{dom}(f) \cup \mathrm{dom}(g)$ is a substructure of $\mathbf{A}^*$. Indeed, if i_f and i_g denote the inclusion maps $i_f : \mathrm{dom}(f) \to \mathbf{A}^*$ and $i_g : \mathrm{dom}(g) \to \mathbf{A}^*$ and if s_f and s_g denote respectively the canonical embeddings of $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ into $\mathrm{dom}(f) \amalg \mathrm{dom}(g)$ (which is direct union with constants amalgamated), there is a map $i : \mathrm{dom}(f) \amalg \mathrm{dom}(g) \to \mathbf{A}^*$ such that $i \circ s_f = i_f$ and $i \circ s_g = i_g$. Then $\mathrm{Im}(i) = \mathrm{dom}(f) \cup \mathrm{dom}(g)$ is a substructure of $\mathbf{A}^*$, even in the presence of partial operations in the language of $\underset{\sim}{M}$

Now, if $f \cup g$ is not a map (*i.e.*, if f and g do not coincide on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$) then $O_f \cap O_g = \varnothing$. If $f \cup g$ is a map but does not belong to $\mathcal{X}^\flat(\mathrm{dom}(f) \cup \mathrm{dom}(g), \underset{\sim}{M})$ then $O_f \cap O_g$ is also empty

If $f \cup g \in \mathcal{X}^\flat(\mathrm{dom}(f) \cup \mathrm{dom}(g), \underset{\sim}{M})$, then it also belongs to $\mathcal{X}(\mathrm{dom}(f) \cup \mathrm{dom}(g), \underset{\sim}{M})$ since f and g are continuous on $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ respectively. In that case, it follows that $O_f \cap O_g = O_{f \cup g}$. $\square$

Priestley duality for bounded distributive lattices is a logarithmic one. We prove that the topology δ of Definition 2.3 coincides with the topology usually defined on canonical extensions of bounded distributive lattices in order to extend maps. This topology is denoted by σ in [11] and by δ in [12]. To make things clear, we denote it by δ'. Recall that if we view $\mathbf{A}^\delta$ has the lattice of decreasing subsets of the Priestley dual $\mathbf{A}^*$ of $\mathbf{A}$, then δ' is the topology which has for basis the sets $[F, O]$ where F and O are respectively a closed element of $\mathbf{A}^\delta$ (*i.e.*, a closed decreasing subset of $\mathbf{A}^*$ of $\mathbf{A}$) and an open element of $\mathbf{A}^\delta$ (*i.e.*, an open decreasing subset of $\mathbf{A}^*$).

Proposition 2.6. *If $\mathcal{A}$ is the variety of bounded distributive lattices and $\underset{\sim}{M} = \langle \{0,1\}, \leq \rangle$, then $\delta(\mathbf{A}^\delta) = \delta'(\mathbf{A}^\delta)$ for any $\mathbf{A} \in \mathcal{A}$.*

Proof. First, we prove that $\delta' \subseteq \delta$. Let F and O be respectively a closed and an open element of $\mathbf{A}^\delta$ with $F \subseteq O$. Then $G = F \cup -O$ is a closed substructure of $\mathbf{A}^*$. Let $f : G \to \underset{\sim}{M}$ be the map defined by $f^{-1}(0) = F$. Then $f \in \mathcal{X}(G, \underset{\sim}{2})$ and $[F, O] = O_f$.

Conversely, let $f \in \mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{M})$. Then $f^{-1}(0)$ is a decreasing clopen subset of $\mathrm{dom}(f)$. Hence, it is a closed subspace of $\mathbf{A}^*$ and $F = f^{-1}(0) \downarrow$ is a decreasing closed subspace of $\mathbf{A}^*$. Similarly, $F' = f^{-1}(1) \uparrow$ is an increasing closed subspace of $\mathbf{A}^*$. It follows that

$$\begin{aligned} x \in O_f &\Leftrightarrow f^{-1}(0) \subseteq x \text{ and } -x \supseteq f^{-1}(1) \\ &\Leftrightarrow F \subseteq x \text{ and } x \subseteq -F'. \end{aligned}$$

We conclude that $O_f = [F, -F'] \in \delta'$. $\square$

Remark 2.7. Note that it is not always possible to compare δ and ι. We can for instance state that $\iota \subseteq \delta$ if any finite subset of $\mathbf{A}^*$ generates a finite substructure in $\mathbf{A}^*$. This happens for instance when $\underset{\sim}{M}$ is a purely relational structure.

An important feature of canonical extension in the lattice-based setting is that it commutes with finite products. This property is fundamental since it allows to consider extensions of n-ary functions with $n \geq 2$ which are n-ary functions. It turns out that this phenomenon generalizes to natural extensions of dualisable prevarieties under rather mild assumptions.

Theorem 2.8. *Assume that $\mathbf{M}$ is of finite type, that $\underset{\sim}{M}$ yields a full duality for $\mathcal{A}$ and let $A, B \in \mathcal{A}$. Then $(\mathbf{A} \times \mathbf{B})^\delta \simeq \mathbf{A}^\delta \times \mathbf{B}^\delta$ if and only if $(\mathbf{A}^* \amalg \mathbf{B}^*)^\flat \simeq (\mathbf{A}^*)^\flat \amalg (\mathbf{B}^*)^\flat$.*

This holds in particular if $\underset{\sim}{M}$ generates a logarithmic duality and in that case, there is an isomorphism between $(\mathbf{A}\times\mathbf{B})^\delta$ and $\mathbf{A}^\delta\times\mathbf{B}^\delta$ which is also a ι- and δ-homeomorphism.

Proof. Assume that $(\mathbf{A}^*\amalg\mathbf{B}^*)^\flat \simeq (\mathbf{A}^*)^\flat \amalg (\mathbf{B}^*)^\flat$. It then follows successively that

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})^\delta &\simeq \mathcal{X}^\flat((\mathbf{A}\times\mathbf{B})^{*\flat}, \underset{\sim}{M}^\flat) \\
&\simeq \mathcal{X}^\flat((\mathbf{A}^*\amalg\mathbf{B}^*)^\flat, \underset{\sim}{M}^\flat) \qquad (2.1)\\
&\simeq \mathcal{X}^\flat((\mathbf{A}^*)^\flat\amalg(\mathbf{B}^*)^\flat, \underset{\sim}{M}^\flat) \qquad (2.2)\\
&\simeq \mathcal{X}^\flat((\mathbf{A}^*)^\flat, \underset{\sim}{M}^\flat)\times\mathcal{X}^\flat((\mathbf{B}^*)^\flat, \underset{\sim}{M}^\flat) \qquad (2.3)\\
&\simeq \mathbf{A}^\delta\times\mathbf{B}^\delta,
\end{aligned}$$

where (2.1) is obtained because full dualities turn products to co-products, (2.3) follows from the fact that $(\mathcal{A}_\iota(\cdot,\mathbf{M}_\iota), \mathcal{X}^\flat(\cdot,\underset{\sim}{M}^\flat), e, \epsilon)$ is a dual adjunction between $\mathcal{A}_\iota$ and $\mathbb{ISP}(\underset{\sim}{M}^\flat)$ and (2.2) holds by assumption. Moreover, if $\underset{\sim}{M}$ yields a logarithmic duality for $\mathcal{A}$, the isomorphism given by the previous piece of argument is easily seen to be a ι and δ-homeomorphism.

Conversely, if $(\mathbf{A}\times\mathbf{B})^\delta \simeq \mathbf{A}^\delta\times\mathbf{B}^\delta$, it follows successively that

$$\begin{aligned}
(\mathbf{A}^*\amalg\mathbf{B}^*)^\flat &\simeq ((\mathbf{A}\times\mathbf{B})^*)^\flat \qquad (2.4)\\
&\simeq \mathcal{A}_\iota((\mathbf{A}\times\mathbf{B})^\delta, \mathbf{M}_\iota) \qquad (2.5)\\
&\simeq \mathcal{A}_\iota(\mathbf{A}^\delta\times\mathbf{B}^\delta, \mathbf{M}_\iota) \qquad (2.6)\\
&\simeq \mathcal{A}_\iota(\mathbf{A}^\delta, \mathbf{M}_\iota)\amalg\mathcal{A}_\iota(\mathbf{B}^\delta, \mathbf{M}_\iota) \qquad (2.7)\\
&\simeq (\mathbf{A}^*)^\flat\amalg(\mathbf{B}^*)^\flat \qquad (2.8)
\end{aligned}$$

where (2.5), (2.7) and (2.8) follows from Theorem 2.4 in [7] that states that $(\mathcal{A}_\iota(\cdot,\mathbf{M}_\iota), \mathcal{X}^\flat(\cdot,\underset{\sim}{M}^\flat), e, \epsilon)$ is a dual equivalence between $\mathcal{A}_\iota$ and $\mathbb{ISP}(\underset{\sim}{M}^\flat)$ and (2.6) holds by assumption. □

Remark 2.9. There is an other way to obtain the sufficiency of the condition in the preceding result when $\underset{\sim}{M}$ yields a full duality for which coproducts in the dual category are realized by direct unions (logarithmic ones for instance). Indeed, it is easily checked that under these conditions, for any $\mathbf{A},\mathbf{B}\in\mathcal{A}$, the natural embedding $\mathrm{Con}(\mathbf{A})\times\mathrm{Con}(\mathbf{B})\hookrightarrow\mathrm{Con}(\mathbf{A}\times\mathbf{B})$ turns out to be an isomorphism. It follows that $\delta_{\mathbf{A}\times\mathbf{B}}\simeq\delta_\mathbf{A}\times\delta_\mathbf{B}$ (where $\delta_\mathbf{A}$ denotes the directed set of finite index congruences of $\mathbf{A}$) and that $\mathrm{pro}_\mathcal{A}(\mathbf{A}\times\mathbf{B})\simeq\mathrm{pro}_\mathcal{A}(\mathbf{A})\times\mathrm{pro}_\mathcal{A}(\mathbf{B})$.

Generalizing Theorem 2.8 to Boolean products (such generalizations are known for canonical extension of bounded distributive lattice expansions, see [11] for instance) depends on the possibility to express emptyness in the dual space in terms of formulas in the algebra, as seen in the next result. Recall the following notation: if $a\in\mathbf{A}$ and $m\in\mathbf{M}$ we denote by $[a:m]$ the set $\{\psi\in\mathcal{A}(\mathbf{A},\mathbf{M})\mid\psi(a)=m\}$. The family $\{[a:m]\mid a\in\mathbf{A}, m\in\mathbf{M}\}$ is a base of clopen subsets of $\mathbf{A}^*$.

The following theorem generalizes the developments in [13] about Boolean products of bounded distributive lattices.

Theorem 2.10. *Assume that* $\underset{\sim}{M}$ *yields a logarithmic duality for* $\mathcal{A}$ *and that* $\mathbf{M}$ *is of finite type. Let* $\mathbf{A}$ *be a Boolean product of the family* $(\mathbf{A}_i)_{i\in I}$ *of algebras of* $\mathcal{A}$*. If for every* $n \in \mathbb{N}$ *and every* $m_1, \ldots, m_n \in M$ *there is an open formula* $\phi(x_1, \ldots, x_n)$ *in the language of* $\mathbf{M}$ *such that for every* $i \in I$ *and every* $a_1, \ldots, a_n \in \mathbf{A}_i$*, it holds*

$$\mathbf{A}_i^* = \bigcup_{\lambda=1}^{n} [a_\lambda, m_\lambda] \quad \Leftrightarrow \quad \mathbf{A}_i \models \phi(a_1, \ldots, a_n),$$

then $\mathbf{A}^\delta$ *is* $\mathcal{A}_\iota$*-isomorphic to* $\prod_{i\in I} \mathbf{A}_i^\delta$*.*

Proof. Let $f : \mathbf{A} \hookrightarrow \prod_{i\in I} \mathbf{A}_i$ be a Boolean representation of the family $(\mathbf{A}_i)_{i\in I}$ of algebras of $\mathcal{A}$. For any $i \in I$ we denote by ρ_i the embedding $(\pi_i)^* : A_i^* \hookrightarrow \amalg\{\mathbf{A}_i^* \mid i \in I\}$ where π_i denotes the projection map from $\prod_{i\in I} A_i$ onto its i-th factor $\mathbf{A}_i$, *i.e.*, ρ_i is the map defined by $\rho_i(\psi) = \psi \circ \pi_i$. Let X denote the set $\bigcup\{\rho_i(\mathbf{A}_i^*) \mid i \in I\}$. Since $\underset{\sim}{M}$ yields a logarithmic duality for $\mathcal{A}$, it is easily seen that $\bigcup\{\rho_i(\mathbf{A}_i^*) \mid i \in J\}$ is isomorphic to $\amalg\{\mathbf{A}_i^* \mid i \in J\}$ for any finite subset J of I. It follows that X is a (not necessarily closed) substructure of $\amalg\{A_i^* \mid i \in I\}$ (such a verification involves only finitely many terms $\rho_i(\mathbf{A}_i^*)$). In particular, X can be seen as

$$X = \amalg\{(A_i^*)^\flat \mid i \in I\}. \tag{2.9}$$

We are going to prove that we can equip X with a Boolean topology to obtain a topological structure that is isomorphic to A^* and that is embeddable into $\amalg\{\mathbf{A}_i^* \mid i \in I\}$.

We define the topology τ on X as the topology generated by the sets

$$[a : m] = \bigcup\{[\pi_i(f(a)) : m] \mid i \in I\}, \quad a \in \mathbf{A}, m \in \mathbf{M}.$$

The topology τ is clearly finer than the topology induced on X by $\amalg\{\mathbf{A}_i^* \mid i \in I\}$. Let us show that $\langle X, \tau\rangle$ is Boolean. It suffices to prove that it is compact. Assume that $X = \bigcup\{[a_\lambda : m_\lambda] \mid \lambda \in L\}$ for some $a_\lambda \in \mathbf{A}$ and $m_\lambda \in \mathbf{M}$. For any $i \in I$, the family $\{[\pi_i(f(a_\lambda)) : m_\lambda] \mid \lambda \in L\}$ is an open covering of $\rho_i(\mathbf{A}_i^*)$ and there is a finite subset L_i of L such that

$$\rho_i(A_i^*) = \bigcup\{[\pi_i(f(a_\lambda)) : m_\lambda] \mid \lambda \in L_i\}. \tag{2.10}$$

By hypothesis, for any $i \in I$ there is an open formula formula ϕ_{in} with n variables (where n denotes $|L_i|$) such that identity (2.10) is equivalent to

$$\mathbf{A}_i \models \phi_{in}((\pi_i(a_\lambda))_{\lambda\in L_i}). \tag{2.11}$$

Now, for any $i \in I$ let Ω_i be the set defined by

$$\Omega_i = \{j \in I \mid \mathbf{A}_j \models \phi_{in}((\pi_j(f(a_\lambda)))_{\lambda\in L_i})\}.$$

The family $\{\Omega_i \mid i \in I\}$ is an open covering of I. By compactness, there is a finite subset J of I such that

$$I = \bigcup \{\Omega_j \mid j \in J\}. \tag{2.12}$$

By combining (2.11) and (2.12), we obtain,

$$X = \bigcup \{ \bigcup \{[a_\lambda : m_\lambda] \mid \lambda \in L_j\} \mid j \in J\},$$

which is a finite open covering of X extracted from $\{[a_\lambda : m_\lambda] \mid \lambda \in L\}$.

Let us denote by g the restriction of f^* to X. Hence, for any $\rho_i(\psi) \in \rho_i(\mathbf{A}_i)$, we have $g(\rho_i(\psi)) = \psi \circ \pi_i \circ f$. We aim to prove that g is an $\mathcal{X}$-isomorphism between $\langle X, \tau\rangle$ and $\mathbf{A}^*$.

First we prove that g is a $\mathcal{X}^\flat$-embedding. We have to prove that if r represents an n-ary relation or the graph of a (partial) operation in the language of $\underset{\sim}{M}$ and if $\psi_1, \ldots, \psi_n \in X$, we have the following equivalence

$$(\psi_1, \ldots, \psi_n) \in r^X \Leftrightarrow (g(\psi_1), \ldots, g(\psi_n)) \in r^{\mathbf{A}^*}. \tag{2.13}$$

Let J be a finite subset of I such that $\{\psi_1, \ldots, \psi_n\} \subseteq \bigcup\{\rho_j(\mathbf{A}_j^*) \mid j \in J\}$. Let us denote by Y the latter set. We have already noted that Y, considered as a substructure of $\amalg\{A_i^* \mid i \in I\}$ is isomorphic to $\amalg\{A_j^* \mid j \in J\}$. Since $f : A \hookrightarrow \prod_{i\in I} A_i$ is a Boolean representation of A, the map $f_J : A \to \prod_{j \in J} A_j : a \mapsto (\pi_j(a))_{j\in J}$ is onto. Hence, the dual map $f_J^* : Y \to A^*$ is an embedding and is clearly equal to the restriction of g to Y. Then, it follows successively

$$\begin{aligned} (\psi_1, \ldots, \psi_n) \in r^X &\Leftrightarrow (\psi_1, \ldots, \psi_n) \in r^Y \\ &\Leftrightarrow (f_J^*(\psi_1), \ldots, f_J^*(\psi_n)) \in r^{\mathbf{A}^*} \\ &\Leftrightarrow (g(\psi_1), \ldots, g(\psi_n)) \in r^{\mathbf{A}^*}, \end{aligned}$$

which establishes equivalence (2.13), as required.

Finally, since g is the restriction to X of a continuous map, it is a continuous map for the induced topology on X. From the fact that τ is finer than the induced topology we eventually conclude that $g : \langle X, \tau \rangle \to A^*$ is an $\mathcal{X}$-embedding. We deduce that $\langle X, \tau\rangle \in \mathcal{X}$.

For the last part of the proof, we show that the evaluation map

$$h : \mathbf{A} \to \mathcal{X}(X, \underset{\sim}{M}) : a \mapsto h(a) : \rho_i(\psi) \mapsto \psi(\pi_i(f(a)))$$

is an isomorphism. It is clearly an homomorphism. Moreover, if $a, b \in \mathbf{A}$ and $a \neq b$ then there is an $i \in I$ such that $\pi_i(f(a)) \neq \pi_i(f(b))$, *i.e.* such that $e_{\mathbf{A}_i}(\pi_i(f(a))) \neq e_{\mathbf{A}_i}(\pi_i(f(b)))$. Let $\psi \in \mathbf{A}_i^*$ with $e_{\mathbf{A}_i}(\pi_i(f(a)))(\psi) \neq e_{\mathbf{A}_i}(\pi_i(f(b)))(\psi)$. It means that $\psi(\pi_i(f(a))) \neq \psi(\pi_i(f(b)))$ which proves that h is one-to-one. Moreover, since $h^* = g$ and since g is an embedding, we deduce that h is onto and so, an isomorphism.

Hence, it follows successively that

$$\mathbf{A}^\delta \simeq \mathcal{X}^\flat(\mathbf{A}^*, \underset{\sim}{M}) \simeq \mathcal{X}^\flat(X, \underset{\sim}{M}) \simeq \mathcal{X}^\flat(\amalg_{i\in I}(\mathbf{A}_i^*)^\flat, \underset{\sim}{M}),$$

where we have used (2.9) to obtain the latter isomorphism. Then, we obtain

$$\mathcal{X}^\flat(\amalg_{i\in I}(\mathbf{A}_i^*)^\flat, \underset{\sim}{M}) \simeq \prod_{i\in I}\mathcal{X}^\flat(\mathbf{A}_i^*, \underset{\sim}{M}) \simeq \prod_{i\in I}\mathbf{A}_i^\delta$$

where the first isomorphism is obtained by partnership duality (Theorem 2.4 in [7]) and is an $\mathcal{A}_\iota$-isomorphism. □

For an illustration of the previous result, see Example 4.10.

3. Natural extension of functions

In view of Proposition 2.4, we adopt the following working assumption for the remainder of the paper.

Assumption 3.1 (Working Assumption)**.** *The structure $\underset{\sim}{M}$ yields a duality for $\mathcal{A}$, $\underset{\sim}{M}$ is injective in $\mathcal{X}$ and Δ is a basis of the topology δ.*

Recall that Proposition 2.5 states that if $\underset{\sim}{M}$ yields a logarithmic duality for $\mathcal{A}$ then Assumption 3.1 is satisfied. As noted in [3], many known strong dualities are logarithmic and hence, satisfy Assumption 3.1.

We consider two algebras $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and a map $u : \mathbf{A} \to \mathbf{B}$. We aim to define an extension $u^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$ of u which is 'as close' to u as possible. As we shall see, there are in general several candidates for such an extension. Unlike the lattice-based case, none of these candidates can 'naturally' (*i.e.* in a reasonably continuous way) be picked out.

We write $F \Subset X$ if F is a finite subset of X. If $x \in X$ and if τ is a topology on X, we denote by τ_x the set of open τ-neighborhoods of x.

Definition 3.2. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. For any $V \in \delta(\mathbf{A}^\delta)$ and $F \Subset \mathbf{B}^*$, let us define

$$u(V, F) = \{e_{\mathbf{B}}(u(a))\restriction_F \mid e_{\mathbf{A}}(a) \in V\} \subseteq M^F,$$

and for any $x \in \mathbf{A}^\delta$, define

$$u(x, F) = \bigcap\{u(V, F) \mid V \in \delta_x\}$$

(the latter intersection exists and is nonempty since $\{u(V, F) \mid V \in \delta_x\}$ is a lower-directed family of nonempty finite sets). Let us also denote by $V(u, x, F)$, or simply by $V(x, F)$ the open δ-neighborhood of x defined by

$$V(x, F) = \bigcup\{W \in \delta_x \mid u(W, F) = u(x, F)\},$$

that is, the greatest open δ-neighborhood of x that realizes $u(x, F)$.

Fact 3.3. *For any $V \in \delta_x$,*

(1) *$u(x, F) \subseteq u(V, F)$,*
(2) *$u(x, F) = u(V, F)$ if and only if $V \subseteq V(x, F)$.*

Now, given $x \in \mathbf{A}^\delta$, we are going to let F run through the finite subsets of $\mathbf{B}^*$. To compare the $u(x, F)$, we need a copy of each of these that does not depends on F. We build such a copy by considering $u(x, F)$ as a trace in M^F of a subset of $M^{\mathbf{B}^*}$. This construction is precisely described in the following definition.

Definition 3.4. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. For any $x \in \mathbf{A}^\delta$ and $F \Subset \mathbf{B}^*$, define

$$\widetilde{u}(x, F) = \{y \in M^{\mathbf{B}^*} \mid Y \restriction_F \in u(x, F)\},$$

and finally, define

$$\widetilde{u}(x) = \bigcap\{\widetilde{u}(x, F) \mid F \Subset \mathbf{B}^*\}.$$

We gather some properties of the preceding constructions in the following technical Lemma.

Lemma 3.5. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$, let $u : \mathbf{A} \to \mathbf{B}$, let $F, G \Subset \mathbf{B}^*$ and let $x, z \in \mathbf{A}^\delta$. Then,*

1. *$\widetilde{u}(x, F)$ is a nonempty closed subset of $\mathbf{M}_\iota^{\mathbf{B}^*}$,*
2. *if $z \in V(x, F)$ then $V(z, F) \subseteq V(x, F)$ and $\widetilde{u}(z, F) \subseteq \widetilde{u}(x, F)$,*
3. *if $F \Subset G$ then $V(x, G) \subseteq V(x, F)$, $\widetilde{u}(x, G) \subseteq \widetilde{u}(x, F)$ and $\widetilde{u}(x, G) \restriction_F = u(x, F)$,*
4. *$\widetilde{u}(x)$ is an nonempty closed subset of $\mathbf{B}_\iota^\delta$,*
5. *$\widetilde{u}(x) \restriction_F = u(x, F)$,*
6. *$\widetilde{u}(x) = \bigcap_{F \Subset \mathbf{B}^*} \overline{u(V(x, F) \cap \mathbf{A})}$ where the closure is computed in $\mathbf{B}_\iota^\delta$.*

Proof. (1) is clear.

(2) If $z \in V(x, F)$, then $V(x, F) \in \delta_z$. Since $u(x, F) = u(V(x, F), F)$, it follows that

$$u(z, F) \subseteq u(x, F), \tag{3.1}$$

and so that $\widetilde{u}(z, F) \subseteq \widetilde{u}(x, F)$. Then, let W denote the δ-open set $V(x, F) \cup V(z, F)$. We obtain successively

$$\begin{aligned} u(W, F) &= u(V(x, F), F) \cup u(V(z, F), F) \\ &= u(x, F) \cup u(z, F) \\ &= u(x, F) \end{aligned}$$

where the latter equality is obtained thanks to (3.1). It means that $W \subseteq V(x, F)$, hence that $V(z, F) \subseteq V(x, F)$.

(3) For $F \Subset \mathbf{B}^*$ and $V \in \delta_x$, let $\widetilde{u}(V, F) = \{y \in \mathbf{B}^* \mid Y \restriction_F \in u(V, F)\}$. Clearly, $F \Subset G \Subset \mathbf{B}^*$ implies that $\widetilde{u}(V, G) \subseteq \widetilde{u}(V, F)$, and therefore that $\widetilde{u}(x, G) \subseteq \widetilde{u}(x, F)$. Hence, for any $y \in \mathbf{B}^*$,

$$y \restriction_G \in u(x, G) \Rightarrow y \restriction_F \in u(x, F). \tag{3.2}$$

In particular, we obtain that

$$\{e_{\mathbf{B}}(u(a)) \mid a \in V(x, G)\} \subseteq \{e_{\mathbf{B}}(u(a)) \mid a \in V(x, F)\},$$

which proves that $V(x,G) \subseteq V(x,F)$. Now, it follows by definition and (3.2) that $\widetilde{u}(x,G)\restriction_F = u(x,G)$.

(4) Since $\widetilde{u}(x)$ is the intersection of a lower-directed family of nonempty closed subsets of the compact space $\mathbf{M}_\iota^{\mathbf{B}^*}$, it is a nonempty closed subset of $\mathbf{M}_\iota^{\mathbf{B}^*}$. It remains to prove that $\widetilde{u}(x) \subseteq \mathbf{B}^\delta$.

Let $y \in \widetilde{u}(x)$. The family formed by the

$$\Omega_F = \{z \in \mathbf{M}_\iota^{\mathbf{B}^*} \mid \forall \phi \in F \ \big(z(\phi) = y(\phi)\big)\}, \quad F \Subset \mathbf{B}^*,$$

is a basis of neighborhoods of y in $\mathbf{M}_\iota^{\mathbf{B}^*}$. Now, for any $F \Subset \mathbf{B}^*$, we have $y \in \widetilde{u}(x) \subseteq u(x,F)$ and hence, there is an $e_{\mathbf{A}}(a) \in V(x,F)$ such that $y\restriction_F = e_{\mathbf{B}}(u(a))\restriction_F$. It means that Ω_F meets $e_{\mathbf{B}}(\mathbf{B})$ and so that $y \in \mathbf{B}^\delta$ which is by definition the closure of $e_{\mathbf{B}}(\mathbf{B})$ in $\mathbf{M}_\iota^{\mathbf{B}^*}$.

(5) Clearly, $u(x,F) = \widetilde{u}(x,F)\restriction_F \supseteq \widetilde{u}(x)\restriction_F$. Let us prove the converse inclusion and let $\alpha \in u(x,F)$. Denote by Ω the clopen subset of $\mathbf{B}_\iota^\delta$ defined by $\Omega = \{y \in \mathbf{B}^\delta \mid y\restriction_F = \alpha\}$. For any G such that $F \Subset G \Subset \mathbf{B}^*$, we know by (3) that $\widetilde{u}(x,G)\restriction_F = u(x,F)$. Hence, there is a $y \in \widetilde{u}(x,G)$ such that $y\restriction_F = \alpha$. We have proved that $\Omega \cap \widetilde{u}(x,G) \neq \varnothing$. By compactness of $\mathbf{B}_\iota^\delta$, it follows that there is a z in $\bigcap_{G \Supset F} \Omega \cap \widetilde{u}(x,G) \subseteq \widetilde{u}(x)$. Hence, $\alpha \in \widetilde{u}(x)\restriction_F$.

(6) First, let $F \Subset \mathbf{B}^*$ and $x \in \mathbf{A}^\delta$. We have to prove that $\widetilde{u}(x) \subseteq \overline{u(V(x,F) \cap \mathbf{A})}$. Let $y \in \widetilde{u}(x)$, $G \Subset \mathbf{B}^*$ and prove that $\{z \in \mathbf{B}^\delta \mid z\restriction_G = y\restriction_G\}$ meets $u(V(x,F) \cap \mathbf{A})$. We may assume that $F \subseteq G$. Then, it follows that

$$y\restriction_G \in \widetilde{u}(x)\restriction_G = u(V(x,G),F) = u(V(x,G) \cap A)\restriction_G \subseteq u(V(x,F) \cap A)\restriction_G,$$

where the inclusion is obtained by (3). Hence, there is an $e_{\mathbf{A}}(a) \in e_{\mathbf{A}}(A) \cap V(x,F)$ such that $y\restriction_G = u(e_{\mathbf{A}}(a))\restriction_G$.

The converse inclusion follows directly from the fact that $u(V(x,F) \cap \mathbf{A}) \subseteq \widetilde{u}(x,F)$. ☐

The set $\widetilde{u}(x)$ is clearly a good candidate for a reservoir of potential values for an extension u^δ at x. However, there is no immediate way to uniformly pick out an element $u^\delta(x)$ in $\widetilde{u}(x)$. Nevertheless, as we shall see in the next section, the following theorem allows to recover the well-known continuity properties of the upper and lower extensions in the lattice-based case.

Let us recall that if $\langle X, \tau\rangle$ is a compact Hausdorff space, the *co-*Scott *topology* $\sigma\downarrow$ is defined on the set $\Gamma(X)$ of the closed subsets of X as the topology that has the sets

$$\Box U = \{F \in \Gamma(X) \mid F \subseteq U\}, \quad U \in \tau,$$

as basis.

It should be noted that the topology most commonly considered on $\Gamma(X)$ is the Vietoris *topology* λ which is the join of the co-Scott topology $\sigma\downarrow$ and the *upper topology* $\iota\uparrow$ which has the sets

$$\Diamond U = \{F \in \Gamma(X) \mid F \cap U \neq \varnothing\}, \quad U \in \tau,$$

as basis. It is well-known that the VIETORIS topology is the LAWSON topology on the complete lattice $\Gamma(X)$ ordered by reverse inclusion. Equipped with VIETORIS topology, $\Gamma(X)$ is a compact HAUSDORFF space.

Theorem 3.6. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. The map $\widetilde{u} : \mathbf{A}^\delta \to \Gamma(\mathbf{B}^\delta_\iota) : x \mapsto \widetilde{u}(x)$ is $(\delta, \sigma\downarrow)$-continuous and satisfies $\widetilde{u}(e_{\mathbf{A}}(a)) = \{e_{\mathbf{B}}(u(a))\}$ for every $a \in \mathbf{A}$.*

Proof. Let $F \Subset \mathbf{B}^*$ and $\alpha \in \mathbf{M}^F$ and denote by U the basic ι-open $U = \{Y \in \mathbf{B}^\delta \mid Y\restriction_F = \alpha\}$. Let $x \in \widetilde{u}^{-1}(\Box U)$. We claim that the δ-neighborhood $V(x, F)$ of x is a subset of $\widetilde{u}^{-1}(\Box U)$. Indeed, for any $z \in V(x, F)$, we obtain successively,

$$\widetilde{u}(z)\restriction_F = u(z, F) = \widetilde{u}(z, F)\restriction_F \subseteq \widetilde{u}(x, F)\restriction_F = u(x, F) = \widetilde{u}(x)\restriction_F = \alpha,$$

where we have applied item (5) and (2) of Lemma 3.5. We have proved the continuity of $\widetilde{u}$.

Now, thanks to item (3) of Proposition 2.4, we know that $e_{\mathbf{A}}(a)$ is a δ-isolated point of $\mathbf{A}^\delta$. Hence, for any $F \Subset \mathbf{B}^*$, we have $u(e_{\mathbf{A}}(a), F) = \{e_{\mathbf{B}}(u(a))\}$ and finally $\widetilde{u}(x) = \{e_{\mathbf{B}(u(a))}\}$. □

Definition 3.7. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. In view of the preceding result, we say that a map $u' : \mathbf{A}^\delta \to \Gamma(\mathbf{B}^\delta_\iota)$ is a Γ-*extension* of u if $u'(e_{\mathbf{A}}(a)) = \{e_{\mathbf{B}}(u(a))\}$ for every $a \in A$.

As shown in the next result, the map $\widetilde{u}$ is the smallest (for inclusion order) of the $(\delta, \sigma\downarrow)$-continuous Γ-extensions of u.

Theorem 3.8. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. If $u' : \mathbf{A}^\delta \to \Gamma(\mathbf{B}^\delta_\iota)$ is a $(\delta, \sigma\downarrow)$-continuous Γ-extension of u then $\widetilde{u}(x) \subseteq u'(x)$ for any $x \in \mathbf{A}^\delta$.*

Proof. We localize, that is, we first consider the case where $\mathbf{B}$ is finite, and hence, where $\mathbf{B} = \mathbf{B}^\delta$. Assume that there are some $x \in \mathbf{A}^\delta$ and some $b \in \mathbf{B}$ such that $b \in \widetilde{u}(x)$ while $b \notin u'(x)$. Note that, since $\mathbf{B}$ is finite,

$$\widetilde{u}(x) = \widetilde{u}(x)\restriction_B = u(x, B) = \bigcap\{\{e_{\mathbf{B}}(u(a)) \mid e_{\mathbf{A}}(a) \in V\} \mid V \in \delta_x\}.$$

Hence, since $b \in \widetilde{u}(x)$, for any $V \in \delta_x$, there is an $e_{\mathbf{A}}(a) \in V$ such that $b = e_{\mathbf{B}}(u(a))$. Applied to $V = \{z \in \mathbf{A}^\delta \mid b \notin u'(z)\} = u'^{-1}(\Box(\mathbf{B} \setminus \{b\}))$, we obtain an $a \in \mathbf{A}$ such that $b \notin u'(e_{\mathbf{A}}(a)) = \{u(e_{\mathbf{A}}(a))\} = \{e_{\mathbf{B}}(u(a))\}$ while $b = e_{\mathbf{B}}(u(a))$, a contradiction.

Let us now prove the general case, that is, without assuming that $\mathbf{B}$ is finite. For any $F \Subset \mathbf{B}^*$ let us denote by u_F the map $u_F : e_{\mathbf{A}}(\mathbf{A}) \to \mathbf{M}^F : e_{\mathbf{A}}(a) \mapsto e_{\mathbf{B}}(u(a))\restriction_F$ and by u'_F the map $u'_F : \mathbf{A}^\delta \to \Gamma(\mathbf{M}^F_\iota) : x \mapsto u'(x)\restriction_F$. Then u'_F is a $(\delta, \sigma\downarrow)$-continuous Γ-extension of u_F. By the previous discussion, if $x \in \mathbf{A}^\delta$ we deduce that $\widetilde{u}_F(x) \subseteq u'_F(x)$ for any $F \Subset \mathbf{B}^*$, that is, that the elements of $\widetilde{u}(x)$ are locally in $u'(x)$. We conclude that $\widetilde{u}(x) \subseteq u'(x)$ since $u'(x)$ is a closed subset of $\mathbf{B}^\delta_\iota$. □

There is an evident lack of symmetry in Theorem 3.6 and Theorem 3.8, at least if, as these theorems suggest, we consider $\widetilde{u}$ as a map $\widetilde{u} : \mathbf{A}^\delta \to \Gamma(\mathbf{B}^\delta_\iota)$ rather than as a relation $\widetilde{u} \subseteq \mathbf{A}^\delta \times \mathbf{B}^\delta$. Indeed, $\mathbf{A}^\delta$ is a topological algebra while $\Gamma(\mathbf{B}^\delta_\iota)$ is just a topological space. To somehow restore symmetry, we observe that $\Gamma(\mathbf{B}^\delta_\iota)$ can be considered as a topological algebra as well. Although this fact should be considered as folklore, we include a proof for the sake of completeness.

Definition 3.9. Let $\langle \mathbf{X}_\tau, (f_\mu)_{\mu\in\Lambda}\rangle$ be a Hausdorff compact topological algebra. The algebra $\langle \Gamma(X_\tau), (g_\mu)_{\mu\in\Lambda}\rangle$ over the language Λ is defined on the set of closed elements of X_τ by setting

$$g_\lambda(F_1, \dots, F_{n_\mu}) = f_\lambda(F_1 \times \dots \times F_{n_\mu})$$

(this is a closed subset of X_τ since $F_1 \times \dots \times F_{n_\mu}$ is compact) for any $\mu \in \Lambda$ of arity n_μ.

Proposition 3.10. *If $\langle X_\tau, (f_\mu)_{\mu\in\Lambda}\rangle$ is a compact* Hausdorff *(resp. Boolean) topological algebra, then so are $\langle \Gamma(X_\tau)_\lambda, (g_\mu)_{\mu\in\Lambda}\rangle$ and $\langle \Gamma(X_\tau)_{\sigma\downarrow}, (g_\mu)_{\mu\in\Lambda}\rangle$.*

Proof. Let $\mu \in \Lambda$ with arity n and f and g be the operations associated to μ on $\mathbf{X}_\tau$ and $\Gamma(X_\tau)$ respectively. We prove that g is co-Scott and upper-continuous.

Let $U \in \tau$ and $(F_1, \dots, F_n) \in g^{-1}(\Box U)$. This means that $F_1 \times \dots \times F_n \subseteq f^{-1}(U)$, which belongs to τ. Hence, there are some $U_1, \dots, U_n \in \tau$ such that

$$F_1 \times \dots \times F_n \subseteq U_1 \times \dots \times U_n \subseteq f^{-1}(U).$$

Thus, the set $\Box U_1 \times \dots \times \Box U_n$ is a neighborhood of $(F_1, \dots, F_n)$ in $\Gamma(X)^n_{\sigma\downarrow}$. Moreover, it is a subset of $g^{-1}(\Box U)$ since if $F'_1, \dots, F'_n$ are closed subsets of X_τ with $F'_1 \subseteq U_1, \dots, F'_n \subseteq U_n$ then $g(F'_1, \dots, F'_n) \subseteq f(U_1 \times \dots \times U_n) \subseteq U$.

Now, let $U \in \tau$ and $(F_1, \dots, F_n) \in g^{-1}(\Diamond U)$. Hence, let $k_1 \in F_1$, $\dots$, $k_n \in F_n$ such that $f(k_1, \dots, k_n) \in U$. By continuity of f, there are some elements $U_1, \dots, U_n$ of τ with

$$(k_1, \dots, k_n) \in U_1 \times \dots \times U_n \subseteq f^{-1}(U).$$

We deduce that $\Diamond U_1 \times \dots \times \Diamond U_n$ is a neighborhood of $(F_1, \dots, F_n)$ which is included in $g^{-1}(U)$. Indeed, if $F'_1, \dots, F'_n \in \Gamma(X_\tau)$ and $k'_1 \in F'_1 \cap U_1$, $\dots$, $k'_n \in F'_n \cap U_n$ then $f(k'_1, \dots, k'_n) \in g(F'_1, \dots, F'_n) \cap U$. $\Box$

4. Strongness, smothness and function composition

Theorem 3.6 provides with an extension of a map $u : \mathbf{A} \to \mathbf{B}$ into a relation $\widetilde{u} \subseteq \mathbf{A}^\delta \times \mathbf{B}^\delta$. Nevertheless, in some cases, the relation $\widetilde{u}$ can be considered as a map. This gives rise to the following natural definition.

Definition 4.1. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$. A map $u : \mathbf{A} \to \mathbf{B}$ is *smooth* if $\widetilde{u}(x)$ is a one-element set for every $x \in \mathbf{A}^\delta$. If u is smooth, we denote by u^δ the map $u^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$ defined by $u^\delta(x) \in \widetilde{u}(x)$.

Examples of smooth maps are given by the elements of $\mathbf{A}^*$ (this follows from the δ-density of $e_{\mathbf{A}}(\mathbf{A})$ in $\mathbf{A}^\delta$). Other examples are provided in Proposition 4.4 below.

If $u : \mathbf{A} \to \mathbf{B}$ is not smooth, one could be tempted to define a map $u^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$ by picking out an element $u^\delta(x)$ in $\widetilde{u}(x)$ for any $x \in \mathbf{A}^\delta$. The next result states that it is not possible to do so in a reasonably continuous way. We give a direct proof of this statement although it could be considered as a consequence of Theorem 3.8.

Proposition 4.2. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. If u is not smooth, then there is no (δ, ι)-continuous extension $u^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$ of u such that $u^\delta(x) \in \widetilde{u}(x)$ for any $x \in \mathbf{A}^\delta$.*

Proof. Assume by way of contradiction that such a u^δ exists. Let $x \in \mathbf{A}^\delta$ and $y \in \widetilde{u}(x)$ such that $y \neq u^\delta(x)$. Let ϕ be an element of $\mathbf{B}^*$ such that $y(\phi) \neq u^\delta(x)(\phi)$. The set $W = \{z \in \mathbf{B}^\delta \mid z(\phi) = u^\delta(x)(\phi)\}$ is an open subset of $\mathbf{B}^\delta_\iota$. By continuity of u^δ, the set $V = (u^\delta)^{-1}(W)$ is a δ-neighborhood of x. It follows that

$$y(\phi) \in \widetilde{u}(x) \restriction_{\{\phi\}} = u(x, \phi) \subseteq u(V, \{\phi\}),$$

where the equality is item (5) of Lemma 3.5. By definition of $\widetilde{u}(x)$, it means that there is an $e_{\mathbf{A}}(a)$ in V such that

$$y(\phi) = e_{\mathbf{A}}(u(a))(\phi) = \phi(u(a)) = u^\delta(a)(\phi).$$

Since $e_{\mathbf{A}}(a) \in V$, we get $u^\delta(e_{\mathbf{A}}(a)) \in W$ and hence $u^\delta(e_{\mathbf{A}}(a))(\phi) = u^\delta(x)(\phi)$. We conclude that $y(\phi) = u^\delta(x)(\phi)$, a contradiction. □

If u is a smooth map, continuity properties of $\widetilde{u}$ (Theorem 3.6 and Theorem 3.8) can be translated into continuity properties of u^δ in the following way.

Proposition 4.3. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$.*

(1) *If u is smooth then $u^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$ is a (δ, ι)-continuous extension of u.*
(2) *If u admits a (δ, ι)-continuous extension $u' : \mathbf{A}^\delta \to \mathbf{B}^\delta$ then u is smooth and $u^\delta = u'$.*

As in the lattice-based case, we have the two following classical examples of smooth functions.

Proposition 4.4. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$.*

(1) *If $u \in \mathcal{A}(\mathbf{A}, \mathbf{B})$ then u is smooth.*
(2) *If $f^{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}$ is a fundamental operation (*i.e.* the corresponding operation of an element of the type of $\mathbf{M}$) and if $\iota \subseteq \delta$ then f is smooth.*

Proof. (1) It follows directly from the definition of $\widetilde{u}$ that $\widetilde{u}(x)$ contains only one element, namely $u^\delta(x) : \phi \mapsto x(\phi \circ u)$.

(2) The interpretation $f^{\mathbf{A}^\delta}$ of f on $\mathbf{A}^\delta$ is a (ι, ι)-continuous extension of $f^{\mathbf{A}}$. □

Hence, any homomorphism is smooth. So, its natural (map) extension is (δ, ι)-continuous. Actually, they it is easily seen to be (ι, ι)-continuous. This fact motivates the following definition.

Definition 4.5. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$. A map $u : \mathbf{A} \to \mathbf{B}$ is *strong* if the map $\widetilde{u} : \mathbf{A}^\delta \to \Gamma(\mathbf{B}^\delta_\iota)$ is $(\iota, \sigma\downarrow)$-continuous.

For a strong map $u : \mathbf{A} \to \mathbf{B}$, the relation $\widetilde{u}$ can be lifted into a map on $\Gamma(\mathbf{A}^\delta_\iota)$ in a fairly good way, as seen in the next result.

Proposition 4.6. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. If u is strong, then the map $\bar{u} : \Gamma(\mathbf{A}^\delta_\iota) \to \Gamma(\mathbf{B}^\delta_\iota) : F \mapsto \widetilde{u}(F)$ is $\sigma\downarrow$-continuous.*

Proof. First, we prove by an easy compactness argument that if $F \in \Gamma(\mathbf{A}^\delta_\iota)$ then $\widetilde{u}(F) \in \Gamma(\mathbf{B}^\delta_\iota)$. Indeed, let $y \in \mathbf{B}^\delta_\iota \setminus \widetilde{u}(F)$. For any $x \in F$ let V_x and W_x be two disjoint ι-neighborhoods of $\widetilde{u}(x)$ and y respectively. By continuity, the family $\{\widetilde{u}^{-1}(\Box V_x) \mid x \in F\}$ forms an open covering of the compact F. If $\{\widetilde{u}^{-1}(\Box V_{x_i}) \mid 0 \leq i \leq n\}$ is a finite subcovering of F and if $W = \bigcap_{i=0}^n W_{x_i}$ then W is an open neighborhood of y that does not meet $\widetilde{u}(F)$.

Now, we prove that the map $\bar{u}$ is $\sigma\downarrow$-continuous. Assume that $F \in \bar{u}^{-1}(\Box U)$ for some open set U of $\mathbf{B}^\delta_\iota$. This means that $\widetilde{u}(F) \subseteq \Box U$, so that F is a subset of $\widetilde{u}^{-1}(\Box U)$, which is an open set in $\mathbf{A}^\delta_\iota$ by assumption. It follows that $F \in \Box\widetilde{u}^{-1}(\Box U)$ while $\bar{u}(\Box\widetilde{u}^{-1}(\Box U)) \subseteq \Box U$ as required. $\Box$

Another important feature of strongness is that it can be seen as a step to obtain the preservation of composition of functions through their extensions.

Proposition 4.7. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}$, $u : \mathbf{A} \to \mathbf{B}$ and $v : \mathbf{B} \to \mathbf{C}$. If v is strong then $\widetilde{vu}(x) \subseteq \widetilde{v}(\widetilde{u}(x)) = \bar{v}(\widetilde{u}(x))$ for any $x \in \mathbf{A}$.*

Proof. By Theorem 3.6 and Proposition 4.6, we obtain that $\bar{v}\widetilde{u} : \mathbf{A}^\delta \to \Gamma(\mathbf{C}^\delta_\iota)$ is a $(\delta, \sigma\downarrow)$-continuous map and the conclusion follows from Theorem 3.8. $\Box$

Corollary 4.8. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}$, $u : \mathbf{A} \to \mathbf{B}$ and $v : \mathbf{B} \to \mathbf{C}$. If u is smooth and if v is strong and smooth, then vu is smooth and $(vu)^\delta = v^\delta u^\delta$.*

A similar conclusion can be achieved under somehow different assumptions.

Proposition 4.9. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}$, $u : \mathbf{A} \to \mathbf{B}$ and $v \in \mathcal{A}(\mathbf{B}, \mathbf{C})$. Then $\widetilde{vu}(x) = \widetilde{v}(\widetilde{u}(x))$ for every $x \in \mathbf{A}^\delta$.*

Proof. We have already mentioned that homomorphisms are strong. Hence, we know by Proposition 4.7 that $\widetilde{vu}(x) \subseteq \widetilde{v}(\widetilde{u}(x))$.

Let us prove the converse inclusion. First, we prove the result when $\mathbf{C} = \mathbf{C}^\delta$ is finite. We identify $\mathbf{A}$ with $e_{\mathbf{A}}(\mathbf{A})$, $\mathbf{B}$ with $e_{\mathbf{B}}(\mathbf{B})$ and $\mathbf{C}$ with $e_{\mathbf{C}}(\mathbf{C})$. Then

$$v : e_{\mathbf{B}}(\mathbf{B}) \to e_{\mathbf{C}}(\mathbf{C}) : e_{\mathbf{B}}(b) \mapsto e_{\mathbf{C}}(v(b)) : \psi \mapsto e_{\mathbf{B}}(b)(\psi \circ v),$$

that is v, is equal to the map $\pi_F : e_{\mathbf{B}}(\mathbf{B}) \to \mathbf{M}^F : x \mapsto x\restriction_F$ where $F = \{\psi \circ v \mid \psi \in \mathbf{C}^*\}$ is a finite subset of $\mathbf{B}^*$. Thus, we obtain that

$$\widetilde{v} = \widetilde{\pi_F} : \mathbf{B}^\delta \to \mathbf{M}^F : x \mapsto \{x\restriction_F\}.$$

It follows that

$$\widetilde{v}(\widetilde{u}(x)) = \widetilde{u}(x)\restriction_F = u(x, F) = u(V(u, x, F))\restriction_F,$$

where we use notation introduced in Definition 3.2. Then, according to Fact 3.3, we have

$$u(V(u, x, F))\restriction_F = u(V(u, x, F) \cap V(\pi_F u, x, F))\restriction_F,$$

where the latter is equal to

$$u(V(\pi_F u, x, F))\restriction_F = \pi_F(u(V(\pi_F u, x, F))) = \widetilde{\pi_F u}(x) = \widetilde{vu}(x).$$

Consider now the general case where $\mathbf{C}$ is not supposed to be finite. Let $F \Subset \mathbf{C}^*$ and denote by v_F the map $\pi_F \circ v : e_{\mathbf{B}}(\mathbf{B}) \to \mathbf{M}^F$. By applying the previous argument to v_F, we obtain

$$\widetilde{vu}(x)\restriction_F = \widetilde{v_F u}(x) = \widetilde{\pi_F}\widetilde{v}\widetilde{u}(x) = \widetilde{v}\widetilde{u}(x)\restriction_F,$$

which concludes the proof. $\square$

Let us give some concrete illustrations of the previous constructions. For this, let us consider (for the end of the section) that $\mathcal{A}$ is the variety of median algebras, that is, $\mathcal{A} = \mathbb{ISP}(\mathbf{2})$ where $\mathbf{2} = \langle\{0, 1\}, (\cdot, \cdot, \cdot)\rangle$ is the algebra with a single ternary operation $(\cdot, \cdot, \cdot)$ defined as the majority function on $\{0, 1\}$. For our purpose, this variety is interesting because (i) it is a non lattice-based variety and (ii) it has a majority term and hence is strongly dualisable. Moreover, the considered duality is logarithmic and, as we shall recall soon, finitely generated substructures are finite. Hence, $\iota(\mathbf{A}^\delta) \subseteq \delta(\mathbf{A}^\delta)$ for any $\mathbf{A} \in \mathcal{A}$.

It has been shown (see [14, 18] and [3]) that the topological structure

$$\underset{\sim}{2} = \langle\{0, 1\}, 0, 1, \leq, {}^\bullet, \iota\rangle$$

with two constants 0 and 1, the natural order $\leq$ and the unary operation ${}^\bullet$ defined by $x^\bullet = (x + 1) \bmod 2$, yields a strong duality for $\mathcal{A}$. Members of $\mathcal{X} = \mathbb{IS}_c\mathbb{P}(\underset{\sim}{2})$ are called *strongly complemented* Priestley *spaces.* A topological structure $X = \langle X, 0, 1, \leq, {}^\bullet, \iota\rangle$ is a member of $\mathcal{X}$ provided that $\langle X, \leq, \iota\rangle$ is a Priestley space with bounds 0 and 1 and ${}^\bullet$ is a continuous order reversing homeomorphism that interchanges 0 and 1 and that satisfies $\phi \leq \phi^\bullet \Rightarrow \phi = 0$ and $\phi^{\bullet\bullet} = \phi$.

There is an equivalent spectrum-based formulation of this duality that is comfortable to adopt to ease computations. A subset ϕ of a median algebra $\mathbf{A}$ is a *prime ideal of* $\mathbf{A}$ if for every $x, y, z \in \mathbf{A}$, the element (x, y, z) belongs to ϕ if and only if at least one of the sets $\{x, y\}$, $\{x, z\}$, $\{z, y\}$ is a subset of ϕ. A subset x of a structure $X \in \mathcal{X}$ is a *disjoint ideal* of X if it is a decreasing set which is disjoint from $x^\bullet$. If in addition x is a clopen subset of X, then

x is called a *continuous disjoint ideal.* A *(continuous) maximal disjoint ideal* of X is a (continuous) ideal that contains ϕ or $\phi^\bullet$ for any $\phi \in X$.

With these definitions, $\mathbf{A}^*$ is isomorphic (by the map $\phi \mapsto \phi^{-1}(0)$) to the prime spectrum of $\mathbf{A}$ (*i.e.*, the set of prime ideals of $\mathbf{A}$) with inclusion order, $\varnothing$ and $\mathbf{A}$ as bottom and top element respectively, set complementation for map $^\bullet$ and ZARIKSI topology. Conversely if $X \in \mathcal{X}$ the dual of X is isomorphic to the maximal continuous disjoint spectrum of X (*i.e.*, the set of continuous maximal disjoint ideals of X) equipped with the operation $(\cdot,\cdot,\cdot)$ inherited from the median operation defined on the powerset of X by

$$(x,y,z) = (x \cap y) \cup (x \cap z) \cup (y \cap z). \tag{4.1}$$

Moreover, if $\mathbf{A} \in \mathcal{A}$ then $\mathbf{A}^\delta$ is isomorphic to the maximal disjoint spectrum of $\mathbf{A}^*$ (*i.e.*, the set of the maximal disjoint ideals of $\mathbf{A}^*$ with median operation defined in (4.1)).

Example 4.10. As an illustration of Theorem 2.10, we prove that *the natural extension of a Boolean representation* $\mathbf{A} \hookrightarrow \mathbf{2}^X$ *of the two element median algebra* $\mathbf{2}$ *is equal to the full product* $\mathbf{2}^X$ *of its factors.*

The dual of $\mathbf{2}$ is represented in Figure 1. Observe that for any non-empty finite subsets I and J, any $a \in \mathbf{2}^I$ and $b \in \mathbf{2}^J$, equality

$$\mathbf{2}^* = \bigcup_{i\in I}[a_i : 1] \cup \bigcup_{j \in J}[b_j : 0]$$

holds if and only if $\bigcap_{j\in J}[b_j : 1] \subseteq \bigcup_{i\in I}[a_i : 1]$, or equivalently if the following condition is satisfied in $\mathbf{2}$ (for some $j_0 \in J$),

$$\bigwedge_{j\in J}(b_j = b_{j_0}) \Rightarrow \bigvee_{i \in I}(a_i = b_{j_0}).$$

The latter formula is also equivalent to

$$\bigvee_{k,l\in J;\, i\in I}\big((a_i, b_k, b_l) = a_i\big).$$

The desired result then follows by application of Theorem 2.10.

From the identity $\mathbf{A}^\delta = \mathbf{2}^X$, we deduce that $\mathbf{A}^\delta$ is a ternary Boolean algebra, *i.e.*, that for any $x \in \mathbf{A}^\delta$ there is an $x^c \in \mathbf{A}^\delta$ such that $\mathbf{A}^\delta$ satisfies

$$(x, z, x^c) = z. \tag{4.2}$$

Indeed, it suffices to define x^c by

$$x^c(\phi) = (1 + x(\phi) \bmod 2), \quad \phi \in \mathbf{A}^*.$$

Since the operation $\cdot^c$ is defined pointwise, we obtain that x^c is continuous if x is continuous. It follows that the restriction of $\cdot^c$ on $\mathbf{A}$ is valued in $\mathbf{A}$ and that $\mathbf{A}$ satisfies equation (4.2) and hence is a ternary Boolean algebra. Equivalently it means that for any $a \in \mathbf{A}$ the algebra $\mathbf{B}_{(\mathbf{A},a)} = \langle A, \wedge, \vee, \cdot^c, 0, 1\rangle$ defined by $b \wedge d = (a,b,d)$, $b \vee d = (b,d,a^c)$, $0 = a$ and $1 = a^c$ is a Boolean algebra.

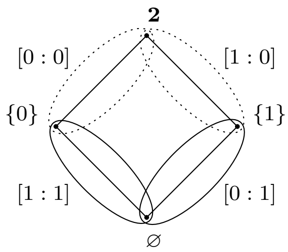

FIGURE 1. Dual of median algebra **2**

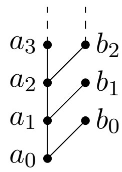

FIGURE 2. Graph of median algebra **A**

Since it is clear that the $\{(\cdot,\cdot,\cdot)\}$-reduct of a Boolean algebra is a Boolean product of the median algebra **2**, we actually have obtained the following result.

Proposition 4.11. *The class of the $\{(\cdot,\cdot,\cdot)\}$-reducts of the ternary Boolean algebras (considered as algebras over the language $\{(\cdot,\cdot,\cdot),\cdot^c\}$) coincide with the class of the Boolean representations of the median algebra* **2**.

This result is rather surprising since it is not clear how one should construct directly an operation $\cdot^c$ on a Boolean power of the median algebra **2** by using only properties of Boolean products (*i.e.*, the patchwork property and the fact that equalizers are clopen).

Example 4.12. Let us consider the undirected simple graph $G = (A, E)$ represented in Figure 2, where $A = \bigcup\{\{a_i, b_i\} \mid i \in \omega\}$ and $E = \{\{a_i, b_i\} \mid i \in \omega\}$. Being a tree, this graph is a median graph. Hence, for any $a, b, c \in A$, there is a unique vertex (a, b, c) that belongs to shortest paths between any two of a, b, c. The algebra $\mathbf{A} = \langle A, (\cdot,\cdot,\cdot)\rangle$ is a median algebra (see [1]). The median operation is easy to compute: for any $j, k, l \in \omega$

$$\begin{aligned} (a_j, a_k, a_l) &= a_{(j,k,l)} \qquad & (a_j, a_k, b_l) &= a_{(j,k,l)} \\ (a_j, b_k, b_l) &= a_{(j,k,l)} \qquad & (b_j, b_k, b_l) &= a_{(j,k,l)}, \end{aligned}$$

where (j, k, l) denotes the median element of $j, k, l \in \omega$.

In order to describe the elements of $\mathbf{A}^*$, it is helpful to consider G as the graph of a poset whose covering relation is

$$\bigcup\{\{(a_i, a_{i+1}), (a_i, b_i)\} \mid i \in \omega\}.$$

Clearly, the elements of $\mathbf{A}^*$ are the following

$$A_i = a_i\!\uparrow, \quad A_i^\bullet = V \setminus a_i\!\uparrow, \quad B_i = \{b_i\}, \quad B_i^\bullet = V \setminus \{b_i\}, \qquad i \in \omega.$$

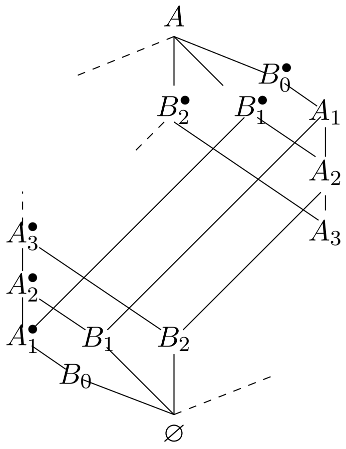

FIGURE 3. The dual $\mathbf{A}^*$ of $\mathbf{A}$.

Hence, the dual of $\mathbf{A}$ is depicted in Figure 3.

The elements of the bidual of $\mathbf{A}$ are easily computed:

$$e_{\mathbf{A}}(b_n) = A_{n+1}\!\downarrow \cup A_n^\bullet\!\downarrow = B_n^\bullet\!\downarrow, \qquad n \in \omega,$$

$$e_{\mathbf{A}}(a_n) = A_{n+1}\!\downarrow \cup A_n^\bullet\!\downarrow \cup \{B_n\}, \qquad n \in \omega.$$

Then $\mathbf{A}^\delta \setminus e_{\mathbf{A}}(\mathbf{A}) = \{\infty\}$ where

$$\infty = \bigcup\{\{A_n^\bullet, B_n\} \mid n \in \omega\}.$$

A simple computation shows that, up to identification of $\mathbf{A}$ to $e_{\mathbf{A}}(\mathbf{A})$

$$(\infty, a_m, b_n) = (\infty, a_m, a_n) = (\infty, b_m, b_n) = a_{m\vee n}, \qquad m, n \in \omega.$$

We have already noted that $\iota(\mathbf{A}^\delta) \leq \delta(\mathbf{A}^\delta)$ since finitely generated substructures of $\mathbf{A}^*$ are finite. Hence, for any $C \in \mathbf{A}^*$, the subasis clopen subsets $\{x \mid C \in x\}$ and $\{x \mid C \notin x\}$ are respectively equal to O_f and O_g where $f = \{C\}$ and $g = \{C^\bullet\}$ correspond to morphisms defined on the closed substructure $\{C, C^\bullet\}$ of $\mathbf{A}^*$.

Now, let $u : \mathbf{A} \to \mathbf{2}$ be the map defined by $u(b_i) = 1$ and $u(a_i) = 0$ for any $i \in \omega$. Clearly, the map u is not an homomorphism. Let us denote by u' the extension of u on $\mathbf{A}^\delta$ that satisfies $u'(\infty) = 0$. We prove that u' is (δ, ι)-continuous which implies that u is smooth. We have to prove that $u'^{-1}(0) = \{\infty, a_0, a_1, \ldots\}$ is a δ-open subset of $\mathbf{A}^\delta$. Consider $F = \{\varnothing, b_0, b_1, b_2, \ldots\} = \bigcap_{i\in\omega} e_{\mathbf{A}}(a_i)$. It follows from the continuity of $^\bullet$ that $F \cup F^\bullet$ is a closed substructure of $\mathbf{A}^*$. Hence, the map $f : F \cup F^\bullet \to 2$ defined by $f(x) = 0$ if and only if $x \in F$ is a partial morphism. It is easily seen that $\infty \in O_f \subseteq u'^{-1}(0)$.

The variety of median algebras has other interesting properties related to natural extensions. We delay the study of these properties in a forthcoming paper.

5. Local extensions of functions

As it should now appears clearly, the general framework of this paper has a huge intersection with the historically important developments of canonical extension for lattice-based algebras. Our Working Assumption 3.1 (that we continue to adopt in this section) is indeed satisfied in the case of the variety $\mathcal{A}$ of bounded distributive lattices and Priestley duality. Nevertheless, it should be noted that the traditional solution adopted to extend a map between two distributive lattices to their canonical extension seems of a completely different nature than the solution presented in this paper. Indeed, instead of one extension that is a multiple-valued map, the canonical extension provides with two extensions: the lower and the upper extensions that are single-valued functions.

In this section, we want to adapt our constructions to parallel the historical ones and prove that our approach subsumes the distributive lattice one.

We need to introduce two new tools to develop this parallelism. The first one is a natural tool that we call *localization*. The second one is as artificial in our general setting as it is natural in the distributive lattice one: we need to equip $\underset{\sim}{M}$ with a total order.

Definition 5.1. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. For $\phi \in \mathbf{B}^*$, the localization of u at ϕ, denoted by u_ϕ, is the map $u_\phi = \phi \circ u : A \to \underset{\sim}{M} : a \mapsto \phi(u(a))$.

A property $\mathcal{P}$ of maps is said to be *local* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$, it holds

$$u \models \mathcal{P} \Leftrightarrow \forall \phi \in \mathbf{B}^* u_\phi \models \mathcal{P}.$$

Proposition 5.2. *The property 'being smooth' is a local property.*

Proof. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. If u is smooth then Corollary 4.8 ensures that u_ϕ is smooth for any $\phi \in \mathbf{B}^*$. Conversely, assume that $u \circ \phi$ is smooth for any $\phi \in \mathbf{B}^*$. It follows by Proposition 4.9 that for any $x \in \mathbf{A}^\delta$, $\widetilde{u_\phi}(x) = \widetilde{\phi}(\widetilde{u}(x))$. Hence, if $y, z \in \widetilde{u}(x)$, we obtain $y(\phi) = \phi^\delta(y) \in \widetilde{u_\phi}(x)$. The same conclusion holds for z, which proves that $y(\phi) = z(\phi)$ for any $\phi \in \mathbf{B}^*$. $\square$

Let us now introduce the second tool and its associated topologies.

Notation 5.3. We denote by $\leq$ a total order on M (fixed once for all). As usual, we denote by $\iota\uparrow$ and $\iota\downarrow$ the topologies formed by the increasing subsets of M and the decreasing subsets of M respectively. Therefore, for any $\mathbf{A} \in \mathcal{A}$, its dual $\mathbf{A}^* \subseteq M^A$ is equipped with two topologies $\iota\uparrow$, $\iota\downarrow$ and a partial order induced respectively by the product of the corresponding topologies and the product order on $\underset{\sim}{M}^A$. We denote by $\mathbf{A}^+$ the complete sublattice of $M^{\mathbf{A}^*}$ consisting of all the order preserving maps from $\mathbf{A}^*$ to M.

We have the following easy comparison between $\mathbf{A}^+$ and $\mathbf{A}^\delta$.

Lemma 5.4. *Let* $\mathbf{A} \in \mathcal{A}$.

(1) $\mathbf{A}^+$ *is a closed subset of* $\mathbf{M}_\iota^{\mathbf{A}^*}$ *and* $\mathbf{A}^\delta \subseteq \mathbf{A}^+$.
(2) *If* $\leq$ *is algebraic on* $\mathbf{M}$ *then* $\mathbf{A}^+$ *is a closed subalgebra of* $\mathbf{M}_\iota^{\mathbf{A}^*}$.

Proof. The only part that is not a consequence of the Pre-duality Theorem (Theorem 5.2 in [3]) is the inclusion $\mathbf{A}^\delta \subseteq \mathbf{A}^+$. Hence, assume that $x \in \mathbf{A}^\delta$ and that $\phi \leq \psi \in \mathbf{A}^*$. According to our General Assumption, the element x is ι-locally in $e_{\mathbf{A}}(\mathbf{A})$. In particular, there is an $a \in \mathbf{A}$ such that $x(\phi) = \phi(a)$ and $x(\psi) = \psi(a)$. We conclude that $x(\phi) \leq x(\psi)$ since $\phi \leq \psi$. $\square$

We mimic the construction of the upper and lower extension of a map between to distributive lattices.

Definition 5.5. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u : \mathbf{A} \to \mathbf{B}$. We define the maps $u^\Delta, u^\nabla : \mathbf{A}^\delta \to \mathbf{M}^{\mathbf{B}^*}$ as follows: for any $x \in \mathbf{A}^\delta$ and any $\phi \in \mathbf{B}^*$, we set

$$u^\nabla(x)(\phi) = \bigwedge \widetilde{u_\phi}(x), \qquad u^\Delta(x)(\phi) = \bigvee \widetilde{u_\phi}(x).$$

We call u^Δ and u^∇ respectively *the upper* and *the lower extension of* u.

As noted in the preceding definition, we can not guarantee that the maps u^Δ and u^∇ are valued in $\mathbf{B}^\delta$. It is possible to overcome this problem for algebras that are locally semilattices.

Definition 5.6. An algebra $\mathbf{A} \in \mathcal{A}$ is a *local* $\wedge$*-semilattice* (with respect to the total order $\leq$ we have added on M) if for every $b, c \in \mathbf{A}$ and every $F \Subset \mathbf{A}^*$,

$$(e_{\mathbf{A}}(b) \wedge e_{\mathbf{A}}(c)) \restriction_F \in e_{\mathbf{A}}(\mathbf{A}) \restriction_F .$$

Local $\vee$*-semilattices* are defined dually. A *local lattice* is an algebra of $\mathcal{A}$ that is both a local $\wedge$-semilattice and a local $\vee$-semilattice.

Theorem 5.7. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ *and* $u : \mathbf{A} \to \mathbf{B}$.

(1) *The maps* u^∇ *and* u^Δ *are extensions of* u *valued in* $\mathbf{B}^+$ *that are respectively* $(\delta, \iota\uparrow)$ *and* $(\delta, \iota\downarrow)$*-continuous.*
(2) *For any* $x \in \mathbf{A}^\delta$, *we have* $u^\nabla(x) = \bigwedge \widetilde{u}(x)$ *and* $u^\Delta(x) = \bigvee \widetilde{u}(x)$.
(3) *If* $\mathbf{B}$ *is a local lattice, then* u^∇ *and* u^Δ *are valued in* $\mathbf{B}^\delta$.

Proof. We treat the case of u^∇. The case of u^Δ may be obtained by duality.

(1) We already know that u^∇ is an extension of u. Let us prove that u^∇ is valued in $\mathbf{B}^+$. Let $x \in \mathbf{A}^\delta$ and $\phi \leq \psi \in \mathbf{B}^*$. Denote by V the δ-neighborhood of x defined as the intersection of $V(x, \{\phi\})$ and $V(x, \{\psi\})$ that are defined in Definition 3.2. Then, according to Fact 3.3, we have

$$\widetilde{u_\rho}(x) = \{e_{\mathbf{B}}(u(a))(\rho) \mid a \in V\}, \quad \rho \in \{\phi, \psi\}.$$

Hence, there is an $a \in A$ such that

$$u^\nabla(x)(\phi) = \bigwedge \widetilde{u_\phi}(x) = e_{\mathbf{B}}(u(a))(\phi) \leq e_{\mathbf{B}}(u(a))(\psi) = \bigwedge \widetilde{u_\phi}(x) = u^\nabla(x)(\psi),$$

which proves that u^∇ is order-preserving.

We now prove that u^∇ is $(\delta, \iota\uparrow)$-continuous. Let $\omega = \{y \in \mathbf{B}^+ \mid y\restriction_F \geq m\}$ be a basic $\iota\uparrow$-open set for some $F \Subset \mathbf{B}^*$ and some $m \in M^F$. If U_ϕ denotes the open set $m(\phi)\uparrow$ of M_ι, it is easily checked that $u^{\nabla -1}(\omega) = \bigcap\{\widetilde{u_\phi}^{-1}(\Box U_\phi) \mid \phi \in F\}$. The result then follows from the $(\delta, \sigma\downarrow)$-continuity (Theorem 3.6) of $\widetilde{u_\phi}$ for every $\phi \in F$.

(2) Since $\wedge$ is computed pointwise in $\mathbf{B}^+$, equation $u^\nabla(x) = \bigwedge \widetilde{u}(x)$ can be checked locally and is equivalent to

$$u^\nabla(x)(\phi) = \bigwedge \widetilde{u_\phi}(x) \quad \forall \phi \in \mathbf{B}^*.$$

But this is precisely the definition of u^∇.

(3) First, observe that $\mathbf{B}^\delta$ is a lattice: if $x, y \in \mathbf{B}^\delta$ then $x \wedge y$ and $x \vee y$ are locally in $e_{\mathbf{B}}(\mathbf{B})$ since $\mathbf{B}$ is a local lattice. Moreover, it is clearly a topological one. Since it is compact and HAUSDORFF, it is complete (see Proposition 2.1 in [6] for instance). Hence, since $\widetilde{u}(x) \subseteq \mathbf{B}^\delta$, we deduce that $u^\nabla(x) = \bigwedge \widetilde{u}(x)$ is an element of $\mathbf{B}^\delta$ as required. □

Theorem 5.7 shows that the maps u^∇ and u^Δ are determined by $\widetilde{u}$. We observe that the converse is not true, even in the lattice-based case, as shown in the following example.

Example 5.8. Let $\mathbf{L}$ be the lattice of the finite subsets of ω together with ω with inclusion order. The PRIESTLEY dual $\mathbf{L}^* = \omega \cup \{\infty\}$ is the one point ALEXANDROFF compactification of the antichain ω, with ∞ as a top element. Hence, $\mathbf{L}^\delta = 2^\omega \cup \{\top\}$ is the power set of $\mathbf{L}^* \setminus \{\infty\}$ with a top element $\top = \mathbf{L}^*$.

(1) We easily build functions $u : \mathbf{L} \to \mathbf{2}$ that are smooth without being homomorphisms. Indeed let u be the non trivial permutation of $\mathbf{2}$ and $\phi \in \mathbf{L}^*$. Then $u \circ \phi : \mathbf{L} \to \mathbf{2}$ is a smooth function that does not belong to $\mathbf{L}^*$. Other examples are given by the maps $u_A : \mathbf{L} \to \mathbf{2}$ (for $A \subseteq \omega$) that are defined by $u_A(x) = 0$ if and only if $x \subseteq A$. If A is infinite and co-infinite then u_A is smooth but not strong.
(2) The function $u : \mathbf{L} :\to \mathbf{2}$ defined by $u(X) = |X| \bmod 2$ if $X \neq \omega$ and $u(\omega) = 1$ is not smooth. Indeed, if X is an infinite proper subset of ω then $\widetilde{u}(X) = \{0, 1\} = [u^\nabla(x), u^\Delta(x)]$.
(3) The function $u : \mathbf{L} \to \mathbf{2}^2$ defined by

$$u(X) = (|X| \bmod 2, (|X| + 1) \bmod 2), \quad X \neq \omega,$$

$$u(\omega) = (1, 1),$$

is not smooth. Moreover, contrary to example (2), the set $\widetilde{u}(x)$ is not determined by $u^\nabla(x)$ and $u^\Delta(x)$. Indeed, if X is an infinite proper subset of ω then $\widetilde{u}(X) = \{(0, 1), (1, 0)\}$ while $u^\nabla(x) = (0, 0)$ and $u^\Delta(x) = (1, 1)$.
(4) For $k \leq 2$ let $u_k : \mathbf{A} \to \mathbf{L}$ be the function defined by $u_k(X) = (1 + |X| \bmod k).X$ for any $X \neq \omega$ and $u(\omega) = \omega$. Then u is not smooth. Indeed, if X is a proper infinite subset of ω then $\widetilde{u}(X) = \{X, 2.X, \ldots, k.X\}$. Moreover, the behavior of the u_k $(k \geq 2)$ with

respect to composition is not optimal since $\widetilde{u_l \circ u_k} = \widetilde{u_l} \circ \widetilde{u_k}$ if and only if l and k are coprime.

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Département de Mathématiques, Université de Liège, Grande Traverse, 12, 4000 Liège, Belgium

E-mail address: `g.hansoul@ulg.ac.be`

Mathematics Research Unit, FSTC, University of Luxembourg, 6, Rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg

E-mail address: `bruno.teheux@uni.lu`