Characterization of some aggregation functions stable for positive linear transformations

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Abstract

This paper deals with the characterization of some classes of aggregation functions often used in multicriteria decision making problems. The common properties involved in these characterizations are "increasing monotonicity" and "stability for positive linear transformations". Additional algebraic properties related to associativity allow to completely specify the functions.

Keywords: aggregation functions; interval scale; invariance; algebraic properties of aggregation operators; multiple criteria decision making.

1 Introduction

Synthesizing judgments is an important part of multiple criteria decision making methods. The most typical situation concerns individuals who form quantifiable judgments about a measure of an object (weight, length, area, height, volume, importance or other attributes, for instance in the framework of a hierarchy)(see [3,4]) or quantifiable judgments on pairs of alternatives along each criterion. In the latter case, the judgments are very often expressed with the help of fuzzy preference relations (see [8,9]).

In order to reach a consensus (overall opinion) on these jugdments, classical aggregation functions have been proposed: arithmetic means, geometric means, root-power means and many others. Of course, given such an aggregation function, we can ask for a motivation of its use, i.e. for natural, reasonable assumptions which lead to this function. Conversely, we can specify some assumptions (called axioms or properties) and determine all the aggregation functions satisfying these. This is the topic with which we deal here.

This paper aims at describing the family of all aggregation functions fulfilling three specific properties. The first two are increasing monotonicity and stability for the same transformations of interval scales in the sense of the theory of measurement (see [15]), i.e. stability for positive linear transformations (we refer to the corresponding functional equation in [5,6] where the arithmetic mean is characterized). The third property is chosen among well-known algebraic properties such as associativity, decomposability and bisymmetry. See Section 3 for details.

We make a distinction between aggregation functions having a fixed number of arguments (aggregation m-functions) and aggregation functions defined for all number of arguments (aggregation operators or aggregators). Section 4 is devoted to characterizations of aggregation

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m-functions whereas Section 5 presents characterizations of aggregators. For space limitation, no proof of the results will be given.

2 Basic definitions

We first want to clarify somewhat the difference between an aggregation m-function and an aggregator. In this paper, $\mathbb{N}^*$ denotes the set of strictly positive integers and $\mathbb{R}$ the set of real numbers. Moreover, we assume that the information to be aggregated consists of numbers belonging to the interval $[0,1]$ as required in most applications. In Section 6, we show that this assumption can be weakened.

Definition 1 *Let $m \in \mathbb{N}^*$. An* aggregation m-function $M^{(m)}$ *defined on* $[0,1]$ *is a real valued function of m arguments:*

$$M^{(m)} : [0,1]^m \to \mathbb{R} : (x_1, \ldots, x_m) \to M^{(m)}(x_1, \ldots, x_m).$$

Definition 2 *An* aggregation operator (or aggregator) M *defined on* $[0,1]$ *is a sequence* $(M^{(m)})_{m\in\mathbb{N}^*}$ *of aggregation m-functions $M^{(m)}$ defined on* $[0,1]$.

Obviously, an aggregator M defined on $[0,1]$ can be viewed as a function defined for any number of arguments:

$$M : \bigcup_{m=1}^{\infty} [0,1]^m \to \mathbb{R} : (x_1, \ldots, x_m) \to M(x_1, \ldots, x_m).$$

In order to avoid heavy notations, we introduce the following terminology. It will be used all along this paper.

- $\mathbb{I} := [0,1]$
- For all $m \in \mathbb{N}^*, N_m := \{1, \ldots, m\}$
- For all $m \in \mathbb{N}^*$ and all $x \in \mathbb{I}, m \cdot x := \underbrace{x, \ldots, x}_{m}$
- Given a vector $(x_1, \ldots, x_m) \in \mathbb{I}^m$, let $x_{(1)}, \ldots, x_{(m)}$ denote the elements of this vector sorted in increasing order: that is, $x_{(1)} \leq \ldots \leq x_{(m)}$.

3 Aggregation properties

As mentioned in the introduction, if we want to obtain a reasonable or satisfactory aggregation, any aggregation m-function should not by used. In order to evacuate the "undesirable" m-functions, we can adopt an axiomatic approach and impose that these m-functions fulfil some selected properties. Such properties can be divided in three categories: natural properties, stability properties and algebraic properties.

3.1 Natural properties

Definition 3 *The aggregation m-function $M^{(m)}$ defined on $\mathbb{I}$ is*

- symmetric (Sy) *if $M^{(m)}$ is a symmetric function on $\mathbb{I}^m$, i.e. if, for all permutations σ of N_m and all $(x_1,\ldots,x_m)\in\mathbb{I}^m$, we have*

$$M^{(m)}(x_1,\ldots,x_m)=M^{(m)}(x_{\sigma(1)}\ldots,x_{\sigma(m)}).$$

- increasing (In) *if $M^{(m)}$ is increasing in each argument, i.e. if, for all $i\in N_m$ and all $x_1,\ldots,x_m,x'_i\in\mathbb{I}$, we have*

$$x_i<x'_i\Rightarrow M^{(m)}(x_1,\ldots,x_i,\ldots,x_m)\leq M^{(m)}(x_1,\ldots,x'_i,\ldots,x_m).$$

- compensative (Comp) *if, for all $(x_1,\ldots,x_m)\in\mathbb{I}^m$,*

$$\min(x_1,\ldots,x_m)\leq M^{(m)}(x_1,\ldots,x_m)\leq\max(x_1,\ldots,x_m).$$

- idempotent (I) *if, for all $x\in\mathbb{I}$,*

$$M^{(m)}(m\cdot x)=x.$$

It should be noted that any compensative aggregation m-function defined on $\mathbb{I}$ necessarily takes its values in $\mathbb{I}$. Moreover, we have the following result which can easily be checked:

Proposition 1 *For every aggregation m-function $M^{(m)}$ defined on $\mathbb{I}$,*

$$\begin{array}{rl}(i) & (Comp)\Rightarrow(I)\\ (ii) & (In,I)\Rightarrow(Comp)\end{array}$$

The properties mentioned in Definition 3 can be adapted to aggregators as follows.

Definition 4 *The aggregator M defined on $\mathbb{I}$ fulfils (Sy) (resp. (In), (Comp), (I)) if, for all $m\in\mathbb{N}^*$, the aggregation m-function $M^{(m)}$ fulfils (Sy) (resp. (In), (Comp), (I)).*

3.2 Stability properties

Definition 5 *The aggregation m-function $M^{(m)}$ defined on $\mathbb{I}$ is*

- stable for the admissible similarities (SSI) *if*

$$M^{(m)}(rx_1,\ldots,rx_m)=rM^{(m)}(x_1,\ldots,x_m)$$

for all $(x_1,\ldots,x_m)\in\mathbb{I}^m$ and all $r>0$ such that $rx_i\in\mathbb{I}$ for all $i\in N_m$.

- stable for the admissible translations (STR) *if*

$$M^{(m)}(x_1+t,\ldots,x_m+t)=M^{(m)}(x_1,\ldots,x_m)+t$$

for all $(x_1,\ldots,x_m)\in\mathbb{I}^m$ and all $t\in\mathbb{R}$ such that $x_i+t\in\mathbb{I}$ for all $i\in N_m$.

- stable for the admissible positive linear transformations (SPL) *if*

$$M^{(m)}(rx_1+t,\ldots,rx_m+t)=rM^{(m)}(x_1,\ldots,x_m)+t$$

for all $(x_1,\ldots,x_m)\in\mathbb{I}^m$ *and all* $r>0, t\in\mathbb{R}$ *such that* $rx_i+t\in\mathbb{I}$ *for all* $i\in N_m$.

- stable for the standard negation N (SSN) *if*

$$M^{(m)}(1-x_1,\ldots,1-x_m)=1-M^{(m)}(x_1,\ldots,x_m)$$

for all $(x_1,\ldots,x_m)\in\mathbb{I}^m$.

The use of stability properties supposes that the values to be aggregated are given according to some scale type as defined by Roberts [15]. Note that some characterization theorems were obtained by Nagumo [14] for (SSI) and (STR) and by Silvert [16] for (SSN) (see also [9: pp.117-126] and [13]). The next result gives some relations between stability properties.

Proposition 2 *For all aggregation m-function* $M^{(m)}$ *defined on* $\mathbb{I}$, *we have*

$$\begin{array}{rl}
(i) & \text{(SSI, STR)} \Leftrightarrow \text{(SPL)}\\
(ii) & \text{(SSI)} \Rightarrow M^{(m)}(m\cdot 0)=0\\
(iii) & \text{(SPL)} \Rightarrow \text{(I)}\\
(iv) & \text{(SSI, SSN)} \Rightarrow \text{(SPL)}
\end{array}$$

It clearly turns out, by the previous proposition, that the condition "$r>0$" in the statement of (SSI) or (SPL) can be replaced by "$r\geq 0$" without any effect.

The next proposition characterizes the aggregation m-functions $M^{(m)}$ defined on $\mathbb{I}$ and satisfying (SPL). A similar characterization was obtained by Aczél and Roberts [5: p.220 (Case 5b)] in the case of aggregation m-functions $M^{(m)}$ defined on $\mathbb{R}$.

Proposition 3 *An aggregation m-function* $M^{(m)}$ *defined on* $\mathbb{I}$ *fulfils (SPL) if and only if there exists an aggregation m-function* $F^{(m)}$ *defined on* $\mathbb{I}$ *such that, for all* $(x_1,\ldots,x_m)\in\mathbb{I}^m$, *we have*

$$M^{(m)}(x_1,\ldots,x_m)=\begin{cases} x \text{ if } (x_1,\ldots,x_m)=(x,\ldots,x),\\ (x_{(m)}-x_{(1)})F^{(m)}\left[\frac{x_1-x_{(1)}}{x_{(m)}-x_{(1)}},\ldots,\frac{x_m-x_{(1)}}{x_{(m)}-x_{(1)}}\right]+x_{(1)} \text{ otherwise.}\end{cases}$$

Definition 6 *The aggregator M defined on* $\mathbb{I}$ *fulfils (SSI) (resp. (STR), (SPL), (SSN)) if, for all* $m\in\mathbb{N}^*$, *the aggregation m-function* $M^{(m)}$ *fulfils (SSI) (resp. (STR), (SPL), (SSN)).*

This paper mostly concentrates on the characterization of aggregation functions and operators satisfying properties (In) and (SPL), as well as some additional properties such as (Sy), (SSN), or algebraic properties to be introduced next.

3.3 Algebraic properties

Definition 7 *The aggregation m-function* $M^{(m)}$ *defined on* $\mathbb{I}$ *is*

- associative (A) *if* $m=2$ *and*

$$M^{(2)}(M^{(2)}(x_1,x_2),x_3)=M^{(2)}(x_1,M^{(2)}(x_2,x_3))$$

for all $(x_1,x_2,x_3)\in\mathbb{I}^3$.

- autodistributive (AD) *if* $m = 2$ *and*

$$\begin{aligned} M^{(2)}(x_1, M^{(2)}(x_2, x_3)) &= M^{(2)}(M^{(2)}(x_1, x_2), M^{(2)}(x_1, x_3)), \\ M^{(2)}(M^{(2)}(x_1, x_2), x_3) &= M^{(2)}(M^{(2)}(x_1, x_3), M^{(2)}(x_2, x_3)) \end{aligned}$$

  for all $(x_1, x_2, x_3) \in \mathbb{I}^3$.

- bisymmetric (B) *if* $m \geq 2$ *and*

$$\begin{aligned} & M^{(m)}(M^{(m)}(x_{11}, \ldots, x_{1m}), \ldots, M^{(m)}(x_{m1}, \ldots, x_{mm})) \\ = \; & M^{(m)}(M^{(m)}(x_{11}, \ldots, x_{m1}), \ldots, M^{(m)}(x_{1m}, \ldots, x_{mm})) \end{aligned}$$

  for all square matrices

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm} \end{pmatrix} \in \mathbb{I}^{m \times m}.$$

Note that those definitions make sense only if $M^{(m)}$ *takes its values in* $\mathbb{I}$.

The properties mentioned in Definition 7 were investigated by several authors. For a list of references see [2]. Ling [11] has specifically investigated the associative property (A). Bisymmetry (B) has been used by Aczél [1] and Fodor and Marichal [7] to characterize certain mean values. This property expresses that aggregation can be performed first on the rows, then on the columns of any square matrix, or conversely. The next proposition presents an immediate link between (B) and (AD).

Proposition 4 *For every aggregation* 2*-function* $M^{(2)}$ *defined on* $\mathbb{I}$*, we have (I, B)*$\Rightarrow$*(AD).*

The next algebraic properties concerns aggregators.

Definition 8 *The aggregator* M *defined on* $\mathbb{I}$ *is*

- associative (A) *if each subset of consecutive elements from* $(x_1, \ldots, x_m)$ *can be substitued by the partial aggregation of this subset without changing the global aggregation, i.e. formally, if* $M^{(1)}(x) = x \; \forall x \in \mathbb{I}$ *and if, for all* $m \in \mathbb{N}^*$*, all* $(x_1, \ldots, x_m) \in \mathbb{I}^m$ *and all* $0 \leq j < k \leq m$*, we have*

$$\begin{aligned} & M^{(m)}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k, x_{k+1}, \ldots, x_m) \\ = \; & M^{(m-k+j+1)}(x_1, \ldots, x_j, M^{(k-j)}(x_{j+1}, \ldots, x_k), x_{k+1}, \ldots, x_m). \end{aligned}$$

- decomposable (D) *if each element of any subset of consecutive elements from* $(x_1, \ldots, x_m)$ *can be substitued by the partial aggregation of this subset without changing the global aggregation, i.e. formally, if* $M^{(1)}(x) = x \; \forall x \in \mathbb{I}$ *and if, for all* $m \in \mathbb{N}^*$*, all* $(x_1, \ldots, x_m) \in \mathbb{I}^m$ *and all* $0 \leq j < k \leq m$*, we have*

$$\begin{aligned} & M^{(m)}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_k, x_{k+1}, \ldots, x_m) \\ = \; & M^{(m)}(x_1, \ldots, x_j, (k-j) \cdot M^{(k-j)}(x_{j+1}, \ldots, x_k), x_{k+1}, \ldots, x_m). \end{aligned}$$

- strongly decomposable (SD) *if each element of any subset of elements from* $(x_1, \ldots, x_m)$ *can be substitued by the partial aggregation of this subset without changing the global aggregation, i.e. formally, if* $M^{(1)}(x) = x \ \forall x \in \mathbb{I}$ *and if, for all* $m \in \mathbb{N}^*$*, all* $(x_1, \ldots, x_m) \in \mathbb{I}^m$ *and all* $N = \{i_1, \ldots, i_p\} \subseteq N_m$*, with* $i_1 < \ldots < i_p$*, we have*

$$M^{(m)}(x_1, \ldots, x_m) = M^{(m)}(x'_1, \ldots, x'_m)$$

where, for all $i \in N_m$*,*

$$x'_i = \begin{cases} x_i & \text{if } i \notin N, \\ M^{(p)}(x_{i_1}, \ldots, x_{i_p}) & \text{otherwise.} \end{cases}$$

- strongly bisymmetric (SB) *if* $M^{(1)}(x) = x \ \forall x \in \mathbb{I}$ *and if, for all* $m, p \in \mathbb{N}^*$*,*

$$\begin{aligned} & M^{(p)}(M^{(m)}(x_{11}, \ldots, x_{1m}), \ldots, M^{(m)}(x_{p1}, \ldots, x_{pm})) \\ = \; & M^{(m)}(M^{(p)}(x_{11}, \ldots, x_{p1}), \ldots, M^{(p)}(x_{1m}, \ldots, x_{pm})) \end{aligned}$$

for all matrices

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pm} \end{pmatrix} \in \mathbb{I}^{p \times m}.$$

Note that those definitions make sense only if, for all $m \in \mathbb{N}^*$*,* $M^{(m)}$ *takes its values in* $\mathbb{I}$*.*

Associativity (A) is a well-known algebraic property which allows to omit "parentheses" in an aggregation of at least three elements (see e.g. [2]). Observe that, if the aggregator M is associative, then the 2-function $M^{(2)}$ is associative (just set $m = 3$ in Definition 8). Of course, associativity can be viewed as an iterative property since it allows to define completely any aggregator M only from its 2-function $M^{(2)}$.

Decomposability is a property introduced by Kolmogoroff [10] and Nagumo [14] in the case of symmetric aggregators (Sy)(see also [7]). In the nonsymmetric case, we generalize this property in two ways: decomposability (D) and strong decomposability (SD). Of course, under (Sy), these two properties are identical. We also introduce the property of strong bisymmetry (SB) as a generalization of (B).

The next proposition points out a link between (A) and (D).

Proposition 5 *For every aggregator* M *defined on* $\mathbb{I}$*, we have (I, A)⇒(D).*

4 Characterization of some aggregation m-functions

This section is devoted to aggregation m-functions which are increasing and stable for positive linear transformations. The next definition introduces some of them (all are defined on $\mathbb{I}$).

Definition 9 *Let* $m \in \mathbb{N}^*$.

- *For any weight vector* $\omega^{(m)} = (\omega_1^{(m)}, \ldots, \omega_m^{(m)}) \in \mathbb{I}^m$ *such that*

$$\sum_{i=1}^{m} \omega_i^{(m)} = 1,$$

the weighted arithmetic mean *m-function* $WAM^{(m)}_{\omega^{(m)}}$ *and the* ordered weighted averaging *m-function* $OWA^{(m)}_{\omega^{(m)}}$ *associated to* $\omega^{(m)}$*, are respectively defined by*

$$\begin{aligned} WAM^{(m)}_{\omega^{(m)}}(x_1,\ldots,x_m) &= \sum_{i=1}^{m} \omega_i^{(m)} x_i \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m, \\ OWA^{(m)}_{\omega^{(m)}}(x_1,\ldots,x_m) &= \sum_{i=1}^{m} \omega_i^{(m)} x_{(i)} \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m. \end{aligned}$$

- *The* arithmetic mean *m-function* $AM^{(m)}$ *is defined by*

$$AM^{(m)}(x_1,\ldots,x_m) = \frac{1}{m}\sum_{i=1}^{m} x_i \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m.$$

- *For any* $i \in N_m$*, the* projection *m-function* $P_i^{(m)}$ *associated to the* i*-th argument is defined by*

$$P_i^{(m)}(x_1,\ldots,x_m) = x_i \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m.$$

- *The* minimum *m-function* $MIN^{(m)}$ *and the* maximum *m-function* $MAX^{(m)}$ *are respectively defined by*

$$\begin{aligned} MIN^{(m)}(x_1,\ldots,x_m) &= \min_{i\in N_m} x_i \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m, \\ MAX^{(m)}(x_1,\ldots,x_m) &= \max_{i\in N_m} x_i \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m. \end{aligned}$$

- *For any nonempty subset* $N^{(m)} \subseteq N_m$*, the* partial minimum *m-function* $MIN^{(m)}_{N^{(m)}}$ *and the* partial maximum *m-function* $MAX^{(m)}_{N^{(m)}}$ *associated to* $N^{(m)}$*, are respectively defined by*

$$\begin{aligned} MIN^{(m)}_{N^{(m)}}(x_1,\ldots,x_m) &= \min_{i\in N^{(m)}} x_i \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m, \\ MAX^{(m)}_{N^{(m)}}(x_1,\ldots,x_m) &= \max_{i\in N^{(m)}} x_i \quad \forall (x_1,\ldots,x_m) \in \mathbb{I}^m. \end{aligned}$$

The following result shows that any aggregation 2-function $M^{(2)}$ fulfilling (In, SPL) is completely defined by the values $M^{(2)}(0,1)$ and $M^{(2)}(1,0)$.

Proposition 6 *The aggregation 2-function* $M^{(2)}$ *defined on* $\mathbb{I}$ *fulfils (In, SPL) if and only if, for all* $(x_1,x_2) \in \mathbb{I}^2$*, we have*

$$M^{(2)}(x_1,x_2) = \begin{cases} (1-\theta)x_1 + \theta x_2 & \text{if } x_1 \le x_2, \\ \overline{\theta}x_1 + (1-\overline{\theta})x_2 & \text{if } x_1 \ge x_2, \end{cases}$$

with $\theta, \overline{\theta} \in \mathbb{I}$*. Moreover, we have* $\theta = M^{(2)}(0,1)$ *and* $\overline{\theta} = M^{(2)}(1,0)$*.*

Some particular examples according to the values of θ and $\overline{\theta}$ can be found in Table 1. Moreover, the next corollary trivially follows.

$(\theta, \overline{\theta})$	$M^{(2)}$
$(\theta, \overline{\theta}) = (0,0)$	$\mathrm{MIN}^{(2)}$
$(\theta, \overline{\theta}) = (1,1)$	$\mathrm{MAX}^{(2)}$
$(\theta, \overline{\theta}) = (0,1)$	$\mathrm{P}_1^{(2)}$
$(\theta, \overline{\theta}) = (1,0)$	$\mathrm{P}_2^{(2)}$
$\theta + \overline{\theta} = 1$	$\mathrm{WAM}_{(1-\theta,\theta)}^{(2)}$
$\theta = \overline{\theta}$	$\mathrm{OWA}_{(1-\theta,\theta)}^{(2)}$

Table 1: Some examples

Corollary 1 *The aggregation 2-function $M^{(2)}$ defined on $\mathbb{I}$ fulfils (Sy, In, SPL) if and only if there exists $\omega^{(2)} \in \mathbb{I}^2$ such that*

$$M^{(2)} = OWA_{\omega^{(2)}}^{(2)}.$$

Note that a complete characterization of the $\mathrm{OWA}_{\omega^{(m)}}^{(m)}$ m-functions can be found in [12] (see also [9: p.133]).

The next theorems describe the families of aggregation m-functions fulfilling (In, SPL, A), (In, SPL, AD) and (In, SPL, B) respectively.

Theorem 1 *The aggregation 2-function $M^{(2)}$ defined on $\mathbb{I}$ fulfils (In, SPL, A) if and only if*

$$M^{(2)} \in \{MIN^{(2)},\ MAX^{(2)},\ P_1^{(2)},\ P_2^{(2)}\}.$$

Theorem 2 *Let $M^{(2)}$ be any aggregation 2-function defined on $\mathbb{I}$. Then the following three assertions are equivalent:*

$$\begin{array}{rl} (i) & M^{(2)} \text{ fulfils (In, SPL, AD)}, \\ (ii) & M^{(2)} \text{ fulfils (In, SPL, B)}, \\ (iii) & M^{(2)} \in \{MIN^{(2)},\ MAX^{(2)}\} \cup \{WAM_{\omega^{(2)}}^{(2)} | \omega^{(2)} \in \mathbb{I}^2\}. \end{array}$$

Theorem 3 *Let $m \in \mathbb{N}^*, m \geq 2$. The aggregation m-function $M^{(m)}$ defined on $\mathbb{I}$ fulfils (In, SPL, B) if and only if*

$$M^{(m)} \in \{MIN_{N^{(m)}}^{(m)},\ MAX_{N^{(m)}}^{(m)} | N^{(m)} \subseteq N_m\} \cup \{WAM_{\omega^{(m)}}^{(m)} | \omega^{(m)} \in \mathbb{I}^m\}.$$

To summarize this section, we present a table containing the characterizations obtained above (Table 2). The table also contains some corollaries which can be checked easily (recall that (SSI, SSN) implies (SPL)). Among them, we can find a complete characterization of the weighted arithmetic mean m-functions. Note that another characterization of this family can be found in [2: pp.234-239] (see also [5]). Moreover, by introducing new properties, we can obtain other corollaries. For instance, $\mathrm{AM}^{(m)}$ alone could be characterized using a property of strict increasing monotonicity.

5 Characterization of some aggregators

This section is devoted to aggregators which are increasing and stable for positive linear transformations. The next definition introduces some of them.

In, SPL, A ($m = 2$)	Sy, In, SPL, A ($m = 2$)	In, SSI, SSN, A ($m = 2$)
$\text{MIN}^{(2)}$, $\text{MAX}^{(2)}$, $\text{P}_1^{(2)}$, $\text{P}_2^{(2)}$.	$\text{MIN}^{(2)}$, $\text{MAX}^{(2)}$.	$\text{P}_1^{(2)}$, $\text{P}_2^{(2)}$.
In, SPL, AD ($m = 2$)	**Sy, In, SPL, AD ($m = 2$)**	**In, SSI, SSN, AD ($m = 2$)**
$\text{MIN}^{(2)}$, $\text{MAX}^{(2)}$, $\{\text{WAM}^{(2)}_{\omega^{(2)}} \| \omega^{(2)} \in \mathbb{I}^2\}$.	$\text{MIN}^{(2)}$, $\text{MAX}^{(2)}$, $\text{AM}^{(2)}$.	$\{\text{WAM}^{(2)}_{\omega^{(2)}} \| \omega^{(2)} \in \mathbb{I}^2\}$.
In, SPL, B ($m \geq 2$)	**Sy, In, SPL, B ($m \geq 2$)**	**In, SSI, SSN, B ($m \geq 2$)**
$\{\text{MIN}^{(m)}_{N^{(m)}}, \text{MAX}^{(m)}_{N^{(m)}} \| N^{(m)} \subseteq N_m\}$, $\{\text{WAM}^{(m)}_{\omega^{(m)}} \| \omega^{(m)} \in \mathbb{I}^m\}$.	$\text{MIN}^{(m)}$, $\text{MAX}^{(m)}$, $\text{AM}^{(m)}$.	$\{\text{WAM}^{(m)}_{\omega^{(m)}} \| \omega^{(m)} \in \mathbb{I}^m\}$.

Table 2: Results from Section 4

Definition 10

- *For any sequence $\omega = (\omega^{(m)})_{m\in\mathbb{N}^*}$ of weight vectors $\omega^{(m)} \in \mathbb{I}^m$, such that*

$$\sum_{i=1}^{m} \omega_i^{(m)} = 1 \ \forall m \in \mathbb{N}^*,$$

  the weighted arithmetic mean *aggregator WAM_ω defined on $\mathbb{I}$ and associated to ω is the aggregator $(WAM^{(m)}_{\omega^{(m)}})_{m\in\mathbb{N}^*}$.*

- *For any $\theta \in \mathbb{I}$, the* decomposable weighted arithmetic mean *aggregator $DWAM_\theta$ defined on $\mathbb{I}$ and associated to θ is the aggregator $(WAM^{(m)}_{\omega^{(m)}})_{m\in\mathbb{N}^*}$ where, for all $i \in N_m$,*

$$\omega_i^{(m)} = \frac{(1-\theta)^{m-i}\theta^{i-1}}{\sum_{j=1}^{m}(1-\theta)^{m-j}\theta^{j-1}}.$$

- *The* arithmetic mean *aggregator AM defined on $\mathbb{I}$ is the aggregator $(AM^{(m)})_{m\in\mathbb{N}^*}$.*

- *The* first projection *aggregator FP and the* last projection *aggregator LP, both defined on $\mathbb{I}$, are the aggregators $(P_1^{(m)})_{m\in\mathbb{N}^*}$ and $(P_m^{(m)})_{m\in\mathbb{N}^*}$. Observe that $FP=DWAM_0$ and $LP=DWAM_1$.*

- *The* minimum *aggregator MIN and the* maximum *aggregator MAX, both defined on $\mathbb{I}$, are the aggregators $(MIN^{(m)})_{m\in\mathbb{N}^*}$ and $(MAX^{(m)})_{m\in\mathbb{N}^*}$.*

- *For any sequence $N = (N^{(m)})_{m\in\mathbb{N}^*}$ of nonempty subsets $N^{(m)} \subseteq N_m$, the* partial minimum *aggregator MIN_N and the* partial maximum *aggregator MAX_N, both defined on $\mathbb{I}$ and associated to N, are the aggregators $(MIN^{(m)}_{N^{(m)}})_{m\in\mathbb{N}^*}$ and $(MAX^{(m)}_{N^{(m)}})_{m\in\mathbb{N}^*}$.*

The next theorems describe the families of aggregators fulfilling (In, SPL, A), (In, SPL, D), (In, SPL, SD) and (In, SPL, SB) respectively.

Theorem 4 *The aggregator M defined on $\mathbb{I}$ fulfils (In, SPL, A) if and only if*

$$M \in \{MIN,\ MAX,\ FP,\ LP\}.$$

In, SPL, A	Sy, In, SPL, A	In, SSI, SSN, A
MIN, MAX, FP, LP.	MIN, MAX.	FP, LP.
In, SPL, D	Sy, In, SPL, D	In, SSI, SSN, D
MIN, MAX, $\{\text{DWAM}_\theta \vert \theta \in \mathbb{I}\}$.	MIN, MAX, AM.	$\{\text{DWAM}_\theta \vert \theta \in \mathbb{I}\}$.
In, SPL, SD	Sy, In, SPL, SD	In, SSI, SSN, SD
MIN, MAX, FP, LP, AM.	MIN, MAX, AM.	FP, LP, AM.
In, SPL, SB	Sy, In, SPL, SB	In, SSI, SSN, SB
$\{\text{MIN}_N,\text{MAX}_N \vert N = (N^{(m)} \subseteq N_m)_{m\in\mathbb{N}^*}\}$, $\{\text{WAM}_\omega \vert \omega = (\omega^{(m)} \in \mathbb{I}^m)_{m\in\mathbb{N}^*}\}$.	MIN, MAX, AM.	$\{\text{WAM}_\omega \vert \omega = (\omega^{(m)} \in \mathbb{I}^m)_{m\in\mathbb{N}^*}\}$.

Table 3: Results from Section 5

Theorem 5 *(i) The aggregator M defined on $\mathbb{I}$ fulfils (In, SPL, D) if and only if*

$$M \in \{MIN,\ MAX\} \cup \{DWAM_\theta | \theta \in \mathbb{I}\}.$$

(ii) The aggregator M defined on $\mathbb{I}$ fulfils (In, SPL, SD) if and only if

$$M \in \{MIN,\ MAX,\ FP,\ LP,\ AM\}.$$

Theorem 6 *The aggregator M defined on $\mathbb{I}$ fulfils (In, SPL, SB) if and only if*

$$M \in \{MIN_N,\ MAX_N | N = (N^{(m)} \subseteq N_m)_{m\in\mathbb{N}^*}\} \cup \{WAM_\omega | \omega = (\omega^{(m)} \in \mathbb{I}^m)_{m\in\mathbb{N}^*}\}.$$

To summarize this section, we present a table containing the above characterizations (Table 3). The table also contains some corollaries which can be checked easily. In particular, we obtain a complete characterization of the weighted arithmetic mean aggregators. Moreover, AM could be isolated using a property of strict increasing monotonicity.

6 Aggregation functions defined on real intervals containing $\mathbb{I}$

Let Ω be any interval such that $\mathbb{I} \subseteq \Omega \subseteq \mathbb{R}$. Obviously, all the definitions and properties introduced earlier can be defined on Ω rather than $\mathbb{I}$. In this section, we show that all the results obtained so far can be adapted to aggregation functions defined on Ω.

The next two results establish a link between some aggregation functions defined on Ω and their restrictions to $\mathbb{I}$.

Proposition 7 *Let $P\in$ {Sy, In, Comp, I, SSI, STR, SPL, SSN}, $Q\in$ {A, AD, B}, $R\in$ {A, D, SD, SB}. Then*

1. *Let $m \in \mathbb{N}^*$. If the aggregation m-function $F^{(m)}$ defined on Ω fulfils (P) (resp. (Comp, Q)) then its restriction $M^{(m)}$ to $\mathbb{I}$ fulfils (P) (resp. (Comp, Q)).*

2. *If the aggregator F defined on Ω fulfils (P) (resp. (Comp, R)) then its restriction M to $\mathbb{I}$ fulfils (P) (resp. (Comp, R)).*

Proposition 8 *Any aggregation m-function $F^{(m)}$ ($m \in \mathbb{N}^*$) defined on Ω and fulfilling (SPL) is completely defined by its restriction to $\mathbb{I}$. The same holds true for any aggregator F.*

Proposition 7 and Proposition 8 allow to obtain characterizations for aggregation functions defined on Ω. For instance, we have the following:

Corollary 2 *Let $m \in \mathbb{N}^*, m \geq 2$. The aggregation m-function $F^{(m)}$ defined on $\mathbb{R}$ fulfils (Sy, In, SPL, B) if and only if*

$$F^{(m)} \in \{\ MIN^{(m)},\ MAX^{(m)},\ AM^{(m)}\}.$$

7 Conclusion

We have characterized some aggregation m-functions and some aggregators which can be useful in multicriteria decision making procedures. The results contribute to the theory of MCDM and can help the decision maker in choosing a particular family of functions on the basis of some properties expected in advance from an aggregation function.

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