PROFILES OF INFLATED SURFACES

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ABSTRACT. We study the shape of inflated surfaces introduced in [B1] and [P1]. More precisely, we analyze profiles of surfaces obtained by inflating a convex polyhedron, or more generally an almost everywhere flat surface, with a symmetry plane. We show that such profiles are in a one-parameter family of curves which we describe explicitly as the solutions of a certain differential equation.

1. INTRODUCTION

The classical isoperimetric problem says that the sphere is the unique surface of a given area which encloses the maximal volume (see e.g. [BZ1]). However, when the area is substituted with other functionals of the intrinsic metric, the uniqueness disappears and the problem becomes very difficult and sometimes intractable.

In this paper we study the geometry of *inflated surfaces*, defined as the surfaces Σ of maximal volume among all embedded surfaces homeomorphic to a sphere whose intrinsic metric is a submetric to the intrinsic metric of Σ. In other words, we require that for every surface Σ' homeomorphic to Σ if all geodesic distances between points in Σ' are less or equal to the geodesic distances between the corresponding points in Σ, then the volume of Σ' is less or equal to that of Σ. This notion was originally introduced by Bleecker [B1], and further studied in [P1].

Of course, every sphere is an inflated surface, since the smaller geodesic distances imply smaller surface area. However, not every convex surface is an inflated surface; in fact, very few of them are. For example, it was shown in [B2] that not all ellipsoids with distinct axes are inflated surfaces, and possibly none of them are. To see an example of this phenomenon, consider a nearly flat ellipsoid Σ, with two large equal axes and one very small one. Think of the surface of Σ as of non-stretchable closed balloon and blow some air into it. Although the resulting surface is non-convex, it is still isometric to Σ and encloses a larger volume, implying that Σ is not an inflated surface.

The idea behind inflated surfaces is to start with a given surface S and blow air into it until no longer possible. The resulting surface $\Sigma = \Sigma(S)$ will be the inflated surface, and will typically be non-convex and not everywhere smooth even if S was convex and smooth. Also, in the limit the geodesic distances can (and typically do) become smaller in Σ than in S, so the inflated surface Σ (perhaps, counterintuitively) has larger volume and smaller area than S. See sections 6, 7 and [P1] for many examples and conjectures on inflated surfaces.

Unfortunately, little is known about uniqueness and regularity of inflated surfaces, even in the most simple cases. We are motivated by [Pa] (see also [MO]), where the shape of a *mylar balloon* was computed, defined as the inflated surface of a doubly covered disc (glued along the boundary circle). In this case the symmetry can be utilized to obtain a complete profile by solving a one-dimensional variational problem. We consider a large class of almost everywhere flat convex surfaces with a plane symmetry. Examples include doubly covered regular polygons, cubes (or more general brick surfaces), other Platonic and Archimedean solids, etc. We then use tools from classical differential geometry and certain heuristic assumptions to compute the *profile* of the corresponding inflated surfaces, defined as the shape of the curve in the symmetry plane. We show that there is essentially one parametric family of such profiles, which are all solutions of a special third order differential equation. Given the variety of examples which include the mylar balloon, this result may again seem counterintuitive. This is the first result on the shape of general inflated surfaces.

The rest of this paper is structured as follows. In the next section we give formal definitions and state the main theorem. In Section 3 we present a heuristic argument on the metric of the inflated surfaces.[1] We use these results in Section 4 to write explicit equations on the curvature of inflated surfaces obtained by an inflation of a nearly flat surfaces. In Section 5 we use the plane symmetry to prove the main result. In the following section (Section 6) we discuss a number of special cases and make a special emphasis on the regularity assumptions. We conclude with final remarks in Section 7.

2. Definitions and main results

Let S be a closed compact surface embedded in $\mathbb{R}^3$ and homeomorphic to a sphere. Throughout the paper we assume our surfaces S are C^5 smooth everywhere except on a finite union of curves in S. We further discuss the regularity assumptions in Section 7.

We say that the surface $S' \subset \mathbb{R}^3$ is *submetric* to S, write $S' \preccurlyeq S$, if there exists a Lipschitz 1 homeomorphism $f : S \to S'$, i.e. such that the geodesic distance satisfies $|f(x)f(y)|_{S'} \le |xy|_S$ for all $x, y \in S$. In particular, if S' is isometric to S, then it is also submetric. Define

$$\upsilon(S) = \sup_{S' \preccurlyeq S} \mathrm{vol}(S'),$$

where here and throughout the paper by $\mathrm{vol}(S)$ we denote the volume enclosed by the surface S.

It is natural to assume, and has been explicitly conjectured in [P1], that when S is convex there is a unique (up to rigid motions) embedded surface $\Sigma = \Sigma(S)$ which attains the supremum: $\mathrm{vol}(\Sigma) = \upsilon(S)$. The surface Σ is the *inflated surface*, and we also refer to Σ as the *inflation of S*.

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[1] As we mention later on, this argument can be made rigorous under certain regularity assumptions.

From now on we consider only surfaces S which are convex and almost everywhere flat. The example include the surfaces ∂P of convex polytopes $P \subset \mathbb{R}^3$, doubly covered convex plane regions. Our goal is the description of $\Sigma = \Sigma(S)$, which we assume to be uniquely defined and satisfy the above regularity assumptions.

We are now ready to present the main result of this paper. Suppose S is symmetric with respect to a plane H, and let $C = \Sigma \cap H$ be the *profile* of Σ. We assume that Σ is C^5 smooth in the neighborhood of C. (if C contains finitely many non-smooth points, consider a portion of C between them). Let $k = k(t)$ be the geodesic curvature of C, considered as a curve in H, and parameterized by the length of C.

Theorem 2.1. *The curvature $k(t)$ of the profile C satisfies the following differential equation:*

$$(1) \qquad k(t)\,k'''(t) \,-\, k'(t)\,k''(t) \,+\, k^3(t)\,k'(t) \,=\, 0\,.$$

As a corollary, we conclude that non-constant solutions of $k(t)$ for which $k(0) = 0$ are given by the following integral formula:

$$\int_0^{k(t)} \frac{du}{\sqrt{(\mu - \lambda^2) + 4\lambda u^2 - u^4}} \;=\; \pm\frac{t}{2}\,,$$

where $\lambda, \mu \in \mathbb{R}$ are constants.

Note that the solutions are invariant under the change $k \to \lambda k$, $t \to t/\lambda$. This can be seen immediately from the invariance of equation (1) under the same change of variables. Therefore, there is only a one-parameter family of possible profiles, up to dilation. The examples and other applications of the theorem will be given in Section 6.

3. Basic description

Let $S \subset \mathbb{R}^3$ be an almost everywhere flat convex surface as in the previous section. For simplicity, the reader can always assume that S is the surface of a convex polytope, even though our results hold in greater generality. As before, we denote by Σ the inflation of S. We call g_0 and g the metrics on S and Σ, respectively. We also call J_0 and J the conformal structures of the metrics g_0 and g. By definition, g is obtained by contracting g_0 in some directions (at some points) and has maximal interior volume under this condition. Call Ω the subset of Σ where some direction is contracted. So Ω is an open subset of Σ.

Claim 3.1.   (1) *At each point $x \in \Omega$ one direction is not contracted. We call ξ a unit vector in the non-contracted direction, and σ the contraction factor in the direction orthogonal to ξ.*
   (2) *The integral curves of ξ are geodesics for g.*
   (3) *They are also geodesics for g_0.*
   (4) *At each point of Ω, $II(\xi, \xi) \geq 0$.*

Proof. For (1), note that if all directions are contracted it is possible to deform a little the surface Σ by making a little "bump", then g remains smaller than g_0 while the interior volume increases.

Part (2) follows from the fact that if integral curves of ξ are not geodesics in the neighborhood of some point x_0, then it is possible to deform Σ by adding a little "bump" while "correcting" the variation of the integral curves of ξ by deforming them towards their concave side on Σ. All those curves then keep the same length and g remains smaller than g_0 while the volume increases.

For (3), let γ be a segment of integral curve of ξ, so that by (2) it is geodesic for g. After replacing γ by a shorter segment if necessary, we can assume that γ is the unique minimizing segment between its endpoints. Let γ_0 be a minimizing segment for g_0 between the endpoints of γ. Denote by L_g the length for g, and by L_{g_0} the length for g_0. Then $L_g(\gamma) \leq L_g(\gamma_0)$ since γ is minimizing for g, while $L_{g_0}(\gamma_0) \leq L_{g_0}(\gamma)$ since γ_0 is minimizing for g_0.

On the other hand, from above, metric g is obtained from g_0 by contracting some directions, so that $L_g(\gamma_0) = L_{g_0}(\gamma_0)$. Putting the above inequalities together, we obtain:

$$L_g(\gamma) \leq L_g(\gamma_0) \leq L_{g_0}(\gamma_0) \leq L_{g_0}(\gamma).$$

But we also know that $L_g(\gamma) = L_{g_0}(\gamma)$, precisely because γ is an integral curve of ξ. It follows from here that the three inequalities above are actually equalities, so that $\gamma_0 = \gamma$. This finishes the proof of part (3).

For (4), if $\mathit{II}(\xi,\xi) < 0$, then it is possible to add a small "bump" increasing the volume and shortening all integral curves of ξ, a contradiction. $\qquad\square$

Remark 3.2. Let us note that if S and S' are the doubly covered polygons Q and Q', respectively, and $Q \subset Q'$, then the surface obtained by inflating S is not necessarily contained in the surface obtained by inflating S'. For example, take Q to be a square and Q' to be a very thin rectangle of length almost the diagonal of Q. On the other hand, the volumes of the inflated surfaces are monotonic in this case: $v(S) \leq v(S')$ (cf. [P1, §5.1]).

4. Equations

In this section we present the equations satisfied by the data describing g and Σ as a surface in $\mathbb{R}^3$. The basic hypothesis here is that Σ is the image of S by a contracting immersion (singular at the cone points of Σ_0 of course) which is a critical point of the volume among such contracting immersions.

Note first that the integral curves of ξ are not necessarily parallel on S; this apparently happens a lot in interesting situations but for the moment we stick to a more general setting. Consider such a line Δ_0, and another such line Δ very close to it. Let y be the function on Δ_0 defined as the distance between Δ_0 and Δ along the normal to Δ_0, and set

$$\rho = y'/y.$$

Then ρ makes sense as a limit $y \to 0$.

To simplify notations we use a prime for derivation along ξ, i.e., for any function f on Σ, $f' = df(\xi)$. We use a dot for the derivation along $J_0\xi$, i.e., $\dot{f} = df(J_0\xi)$.

Lemma 4.1. *The curvature of g is given by the following equation:*

$$K = -\sigma''/\sigma - 2\rho\sigma'/\sigma \, .$$

Proof. The distance between Δ_0 and Δ (for g now) is σy. Since both lines are geodesics, a basic argument on Jacobi fields on Riemannian surfaces gives that $K = -(y\sigma)''/(y\sigma)$, and the result follows from $y'' = 0$. $\qquad\square$

It is necessary to consider the second fundamental form of Σ, we write its coefficients in the basis $(\xi, J\xi)$ as $\begin{pmatrix} k_0 & \delta \\ \delta & k_1 \end{pmatrix}$.

Lemma 4.2. $K = k_0 k_1 - \delta^2$.

Proof. This is the *Gauss formula*. $\qquad\square$

Lemma 4.3. *The curvature k_0 is proportional to $y\sigma$ along integral curves of ξ. In other words,*

$$k_0'/k_0 = \sigma'/\sigma + y'/y \, .$$

Proof. Consider two bumps along an integral curve of ξ, in opposite directions so that the total length is not changed. The variation of length is proportional to k_0 and to the normal displacement, while the variation of volume is proportional to $y\sigma$ times the normal displacement. This implies the result. $\qquad\square$

Proposition 4.4. *Let ∇ be the Levi-Civita connection of g. Then $\nabla_\xi \xi = 0$, while $\nabla_{J\xi}\xi = (\sigma'/\sigma + \rho)J\xi$.*

Proof. The first point is a direct consequence of point (3) of Claim 3.1, since ξ is a unit vector field for g.

For the second point, let ∇^0 be the Levi-Civita connection of g_0. The definition of ρ shows that $\nabla^0_{J_0\xi}\xi = \rho J_0\xi$, while Claim 3.1 shows that $\nabla^0_\xi(J_0\xi) = 0$. Therefore

$$[\xi, J_0\xi] = \nabla^0_\xi(J_0\xi) - \nabla^0_{J_0\xi}\xi = -\rho J_0\xi \, .$$

By definition of σ, $g(J_0\xi, J_0\xi) = \sigma^2$, so that $J_0\xi = \sigma J\xi$, so that

$$[\xi, J\xi] = \left[\xi, \frac{1}{\sigma}J_0\xi\right] = \frac{1}{\sigma}[\xi, J_0\xi] + \left(\xi.\frac{1}{\sigma}\right)J_0\xi = -\frac{\rho}{\sigma}J_0\xi - \frac{\xi.\sigma}{\sigma^2}J_0\xi = -\rho J\xi - \frac{\sigma'}{\sigma}J\xi \, ,$$

and the result follows. $\qquad\square$

Lemma 4.5. *The* Codazzi equation *can be written as:*

$$\delta' - \dot{k}_0/\sigma + 2\delta(\sigma'/\sigma + \rho) = 0 \, , \text{ and}$$

$$k_1' - \dot{\delta}/\sigma + (k_1 - k_0)(\sigma'/\sigma + \rho) = 0 \, .$$

Proof. Let B be the shape operator of Σ, i.e., $I\!I(u,v) = I(Bu,v)$. Then the Codazzi equation is

$$(\nabla_u B)(v) - (\nabla_v B)(u) = 0 \,,$$

where ∇ is the Levi-Civita connection on Σ, and u, v are any two vector fields. Writing this for $\xi, J\xi$ we get:

$$\nabla_\xi(BJ\xi) - B(\nabla_\xi(J\xi)) - \nabla_{J\xi}(B\xi) + B(\nabla_{J\xi}\xi) = 0 \,.$$

Now expressing this in terms of the coefficients of $I\!I$, we obtain:

$$\nabla_\xi(\delta\xi + k_1 J\xi) - \nabla_{J\xi}(k_0\xi + \delta J\xi) + B((\sigma'/\sigma + \rho)J\xi) = 0 \,.$$

Therefore,

$$\delta'\xi + k_1' J\xi - dk_0(J\xi)\xi - k_0(\sigma'/\sigma+\rho)J\xi - d\delta(J\xi)J\xi + \delta(\sigma'/\sigma+\rho)\xi + (\sigma'/\sigma+\rho)(\delta\xi+k_1 J\xi) = 0 \,.$$

On the other hand, $\sigma J\xi = J_0\xi$, so $d\delta(J\xi) = \dot\delta/\sigma$, and similarly $dk_0(J\xi) = \dot k_0/\sigma$. Separating the terms in ξ and in $J\xi$ gives

$$\delta' - \dot k_0/\sigma + \delta(\sigma'/\sigma + \rho) + (\sigma'/\sigma + \rho)\delta = 0 \,.$$

We conclude:

$$k_1' - k_0(\sigma'/\sigma + \rho) - \dot\delta/\sigma + (\sigma'/\sigma + \rho)k_1 = 0 \,,$$

which implies the result. $\qquad\square$

5. Lines of symmetry

In this section we consider a special case as in the main theorem (Theorem 2.1), when S has a symmetry plane H. Consider an integral curve of $\xi \subset \Sigma \cap H$, which means that it is a line of symmetry of the inflated surface. For example, in a mylar balloon, all segments going through the center are such lines of symmetry. Other examples include the case when S is the doubly covered regular polygon or a rectangle. In each case plane H is orthogonal to the polygon and is a symmetry plane of both S and Σ.

We consider the case of a regular polygon for simplicity; as the reader will see the general case follows by the same argument. Let Q be a regular n-gon, made by gluing n copies of a triangle $T = (OAB)$. The copies of T are all glued so that their vertices O are glued together. Moreover the triangle is symmetric with respect to the line orthogonal to AB going through O. Denote by E be the midpoint of AB.

As before, we assume that there is a unique surface Σ which has maximal volume and is submetric to S. It follows from uniqueness that this surface has all the symmetries of Σ_0. So it sufficient to study the quantities describing the situation on T.

The equations above simplify somewhat when considered on an axis of symmetry, for instance on the segment OE of the triangle considered above. Then $\delta = 0$, and $\dot k_0 = \dot k_1 = 0$. Such lines are lines of curvature (integral lines of the curvature directions), we suppose that the corresponding principal curvature is k_0. In that case the three basic equations reduce to the following:

- for the Gauss equation,

$$(2) \qquad k_0 k_1 = -\frac{(\sigma y)''}{\sigma y},$$

- for the Codazzi equation,

$$(3) \qquad k_1' = \left(\frac{\sigma'}{\sigma} + \frac{y'}{y}\right)(k_0 - k_1),$$

- for the "conservation of curvature",

$$(4) \qquad \frac{k_0'}{k_0} = \frac{\sigma'}{\sigma} + \frac{y'}{y}.$$

One can use this last expression in the previous two to get

$$k_0 k_1 = -\frac{(\sigma y)''}{\sigma y} = -\left(\frac{(\sigma y)'}{\sigma y}\right)' - \left(\frac{(\sigma y)'}{\sigma y}\right)^2 = -\left(\frac{k_0'}{k_0}\right)' - \left(\frac{k_0'}{k_0}\right)^2 = -\frac{k_0''}{k_0},$$

$$k_0 k_1' = (k_0 - k_1) k_0'.$$

Replacing k_1 in those equations, we find after simple computations that k_0 satisfies equation (1), which we recall for convenience:

$$(5) \qquad k\,k''' - k'k'' + k^3 k' = 0.$$

As we mentioned in Section 3, this equation is invariant under the transformation $k \mapsto \lambda k$, $t \mapsto t/\lambda$, which makes sense since this homogeneity condition on k_2 corresponds to the invariance of the class of inflated surfaces under scaling.

The other key quantities describing the surface at the symmetry line can then be recovered from k. Setting $u = \sigma y$, we get:

$$\frac{u'}{u} = \frac{k'}{k}.$$

From here we see that u is proportional to k, while

$$k_1 = -\frac{k''}{k^2}.$$

Finally, let us mention that equation 5 can be solved implicitly in the following cases. The proof is straightforward.

Proposition 5.1. *The solutions of (5) vanishing at $t = 0$ are $k = 0$, and the functions defined implicitly by*

$$(6) \qquad \int_0^{k(t)} \frac{ds}{\sqrt{(\mu - \lambda^2) + 4\lambda s^2 - s^4}} = \pm\frac{t}{2},$$

with constants $\lambda, \mu \in \mathbb{R}$. The solutions of (5) with $k(0) = k'(0) = 0$ are $z = 0$, and the functions defined by

$$(7) \qquad \int_0^{k(t)} \frac{ds}{\sqrt{\lambda s^2 - s^4}} = \pm\frac{t}{2}.$$

Proof. Let k be a solution of (5), defined on an interval $I \subset \mathbb{R}$. Then

$$\left(k\,k'' - (k')^2 + \frac{k^2}{4} \right)' = 0 \ ,$$

so that

$$k\,k'' - (k')^2 + \frac{k^2}{4} = a$$

for some constant $a \in \mathbb{R}$. Let y be the inverse function of k (on a subinterval J of I where k is monotonous), then $k' \circ y = 1/y'$ and $y'^2 k'' \circ y + y'' k' \circ y = 0$, so that

$$-\frac{t\,y''(t)}{y'(t)^3} - \frac{1}{y'(t)^2} + \frac{t^4}{4} = a \ .$$

Set $u(t) = t\,y'(t)$, then, on the sub-interval where $u(t) \neq 0$,

$$\frac{u'(t)}{u(t)^3} = \left(\frac{t}{4} - \frac{a}{t^3} \right)$$

so that there exists $b \in \mathbb{R}$ such that

$$\frac{1}{u(t)^2} = \left(\frac{a}{t^2} - \frac{t^2}{4} \right) + b \ ,$$

for some $b \in \mathbb{R}$, and

$$u(t) = \pm \frac{2|t|}{\sqrt{4a - t^4 + 4bt^2}} \ .$$

Therefore,

$$y(t) = \pm \int^t \frac{2\,ds}{\sqrt{4a - s^4 + 4bs^2}} \ .$$

Since $y(k(r)) = r$ by definition of y, k satisfies the equation

$$\int_J^{k(r)} \frac{2\,ds}{\sqrt{4a - s^4 + 4bs^2}} = \pm \frac{r}{2} \ ,$$

which is equivalent to (6) by a simple change of the constants.

The solutions such that $k(0) = 0$ correspond to taking the primitive equal to 0 at 0, and those solutions have vanishing derivative at 0 if and only if the integrand goes to ∞ at $s = 0$, that is, if and only if $a = 0$. □

Remark 5.2. Taking into account the invariance under homotheties, the proposition implies that there is a one-parameter family of possible "profiles", which we call C_ν. For all the cases with planar symmetry obtained from a doubly covered convex figure, we have $k' = 0$ at the "equator". In addition to this, for the mylar balloon we have $k = 0$ at the "pole", which determines the curvature k uniquely. For the other examples, however, there is no reason to believe that $k = 0$ at the "pole" so k is completely really determined, we still need one more boundary condition.

6. EXAMPLES

6.1. **The mylar balloon.** Consider a *mylar balloon*, defined as above by gluing two copies of a disk. Both Theorem 2.1 and Proposition 5.1 can be applied in this case (see also Remark 5.2), and this determines the balloon profile in terms of curvature (see below). In fact, this is the only case when the profile was already computed [MO, Pa], but described in a different manner.

FIGURE 1. Pictures of two party balloons.

Paulsen [Pa] showed that the intersection with the upper right quadrant of the profile of the mylar balloon is the graph of the function $f : [0, a] \to \mathbb{R}$ given by

$$(8) \qquad f(x) = \int_x^a \frac{u^2 du}{\sqrt{a^4 - u^4}} \ ,$$

where a is the radius of the inflated balloon. This profile can be characterized by the following simple geometric property, which can be found (implicitly at least) in [MO]. We provide a direct proof here for the reader's convenience.

Proposition 6.1. *The profile of the mylar balloon is characterized by the fact that its curvature is a linear function of x: $k(x) = -2x/a^2$, where a is the radius of the balloon.*

Proof. Recall that the curvature of the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$k(x) = \frac{f''(x)}{\left(1 + f'(x)^2\right)^{3/2}} \ .$$

Here

$$f'(x) = -\frac{x^2}{\sqrt{a^4 - x^4}} \ ,$$

so that

$$\sqrt{1 + f'(x)^2} = \frac{a^2}{\sqrt{a^4 - x^4}} \ .$$

Similarly,

$$f''(x) = -\frac{2x(a^4 - x^4) + 2x^3 \cdot x^2}{\left(a^4 - x^4\right)^{3/2}} = \frac{-2xa^4}{\left(a^4 - x^4\right)^{3/2}} \ ,$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With this proposition in mind it is easy to check that the curvature of the profile of the mylar balloon is precisely the solution of (5) which vanishes at $t = 0$ – corresponding to the north pole of the balloon – and with vanishing derivative at the equator, corresponding to $x = a$. Note that we switch between a parameterization by the x coordinate, in (8) and in Proposition 6.1, to a parameterization by arclength, which is used in the description of the curvature as solution of (5). For this reason we will denote by $\bar{k}$ the curvature of the profile as a function of x, keeping the notation k for the curvature as a function of the arclength parameter t.

We will now check that the profile in (8) corresponds to the solution of (5) obtained by choosing $\lambda = 0, \mu = 16/a^4$ in (6). For this we use Proposition 6.1, which shows that

$$\int_0^{\overline{k}(x)} \frac{1}{\sqrt{16/a^4 - s^4}}\, ds = \int_0^{-2x/a^2} \frac{1}{\sqrt{16/a^4 - s^4}}\, ds = -\frac{1}{2} \int_0^x \frac{a^2}{\sqrt{a^4 - u^4}}\, du \ ,$$

where the last equality uses the change of variables $s = -2u/a^2$. Consider the arclength parameter t on the profile, as a function of x. Then

$$t(x) = \int_0^x \sqrt{1 + f'(x)^2}\, dx = \int_0^x \frac{a^2}{\sqrt{a^4 - u^4}}\, du \ .$$

As a consequence we obtain that

$$\int_0^{\overline{k}(x)} \frac{s^2}{\sqrt{16/a^4 - s^4}}\, ds = -\frac{t(x)}{2} \ ,$$

so that the profile in (8) corresponds to (6) for $\lambda = 0, \mu = 16/a^4$.

This profile has two prominent features.

- $k(0) = 0$, this clearly follows from (4), since $y'/y \to \infty$ at the north pole.
- $k'(t) = 0$ for t corresponding to $x = a$, that is, to the equator of the mylar balloon. This is clear since $x'(t) \to \infty$ as $x \to a$, while $\overline{k}'$ is bounded at that value of x because $\overline{k}$ is a linear function of x. This condition should apply to the profile of the mylar balloon for a clear symmetry reason.

Those two conditions characterize, up to dilation, the profile of the mylar balloon among solutions of (5).

6.2. **The square and rectangular pillow.** A rectangular pillow is the surface obtained by inflating the doubly covered rectangle. We call $v_\pm$ the centers of the two copies of the rectangle, and $v_i, 1 \leq i \leq 4$ its vertices. The rectangular pillow is thus the union of eight triangles, of vertices $v_\pm, v_i, v_{i+1}$. The square pillow is the special case of rectangular pillow for which the doubly covered rectangle which is inflated is a square. We make the following natural regularity hypothesis: *The square pillow is C^5 on the interior of each of the eight triangles.*

Clearly each of the triangles in the square pillow are congruent, by the general hypothesis at the beginning of the section. We consider one of them, say (v_+, v_1, v_2). Let w be the midpoint of (v_1, v_2). (v_+, w) is a line of symmetry of the triangle, therefore a line of curvature, and the corresponding principal curvature satisfies (5).

To have a better understanding of this profile of the square pillow, we can make a heuristically attractive hypothesis, which should be compared with experimental data: we suppose that the surface is C^3 at w. It then follows from symmetry that the derivative of k at w along (v_+, w) is zero.

Under this hypothesis, the principal curvature k along (v_+, w) is a solution of (5) with vanishing derivative at w. We conclude that the intersection of the rectangular pillow with a plane of symmetry containing v_-, v_+ but none of the v_i is the union of two copies of the profiles C_ν for some ν, obtained from equation (6) by a simple integration once only one additional parameter is known – for instance the curvature

of the profile at v_+. If this curvature were zero at v_+, it would imply that this curve is the same (up to dilation) as the profile of the mylar balloon.

Let us note that, to understand the more general case of rectangular pillows, it would be useful to determine the dependence of parameters λ, μ on the rectangle aspect ratio. The case of a square pillow is particularly attractive, and known as the *teabag problem* in recreational literature [K]. Let us also mention the simulations by Gammel [G] (see Figure 2), and physical experiments by Robin for the conjectured formula for the volume [R].

FIGURE 2. Computer simulation of the square pillow and the inflated cube.

6.3. **Doubles of polygons.** It is also possible to consider doubly covered regular n-gons (cf. Figure 1). We still call $v_\pm$ the centers of the two copies of the regular polygon, and $v_i, 1 \leq i \leq n$ their vertices (pairwise identified). The inflated double n-polygon is the copy of $2n$ triangles, each of which is $(v_\pm, v_i, v_{i+1})$. We make the same hypothesis as for the square pillow, and obtain a profile shape which again only depends on one parameter, for instance the curvature of the profile at $v_\pm$, as for the square pillow (which is obviously a special case).

6.4. **The inflated cube.** Let us start now with the surface $S = \partial P$ or a unit cube P. Using the same analysis, each face of the inflated cube is cut into four triangles and we can assume that each of them is C^5 smooth. Under this hypothesis we find that the intersection of the inflated cube with a plane of symmetry containing no vertex is the union of four arcs of C_ν, for some ν.

7. FINAL REMARKS

7.1. **Why is this a proper mathematical model?** There is a rather subtle point that needs to be made in favor of the relevance of the model studied here for the physical problem of understanding inflated surfaces.

It might appear at first sight that the model considered here, based on contracting embeddings of a surface with maximal volume, is quite different from what happens for true inflated surfaces. Indeed for those surfaces there is no contraction of the metric on the surface, but rather some "plaids" appear (see the wrinkles in Figure 1). One feature of those plaids is that they are necessarily along geodesics on the surface, and this seems to impose a constraint not present in the mathematical model. However, there is a very good fit between the mathematical models and the inflated surfaces as they are observed.

We believe that the resolution of this apparent paradox lies in parts (2) and (3) of Claim 3.1. These parts state precisely that the integral curves of the non-contracted directions are geodesics on the surface. In other terms, there is a mathematical constraint, coming from the maximality of the surface, which happens to coincide precisely with the physical constraints that the plaids have to be along lines. For this

reason the "mathematical" inflated surfaces are very close to the observed ones, in spite of apparently different constraints on their geometry.

7.2. **Future directions.** Perhaps the main open problem is to show that the inflated surfaces are well defined and uniquely determined (see [P1] for a complete statement). Presumably, this would imply the symmetry assumptions we made throughout the paper. Unfortunately, even in the case of the Mylar balloon or rectangular pillow this is completely open. It would also be important to prove the regularity conditions, in particular a formal proof of Claim 3.1.

Even under the uniqueness and regularity conditions, this paper goes only so far towards understanding of the inflated pillow shapes. Although we do not wish to suggest that in the case of rectangular pillows the shape of the surface can be expressed by means of classical functions, as in the case of the Mylar balloon [MO], it is perhaps possible that it is a solution of an elegant problem which completely defines it. For example, the linearity of the curvature as in Proposition 6.1 completely characterizes the Mylar balloon. It would be interesting if the shape of rectangular pillows has a similar characterization.

Finally, the *crimping density* and *crimping ratio* (the ratio of the area of inflated over non-inflated surface defined in [P1]) are interesting notions with potential applications to material science. Exploring them in the case of rectangular pillows would be of great interest.

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