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Maximal surfaces and the universal Teichmüller space

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Abstract We show that any element of the universal Teichmüller space is realized by a unique minimal Lagrangian diffeomorphism from the hyperbolic plane to itself. The proof uses maximal surfaces in the 3-dimensional anti-de Sitter space. We show that, in AdS_{n+1}, any subset E of the boundary at infinity which is the boundary at infinity of a space-like hypersurface bounds a maximal space-like hypersurface. In AdS_3, if E is the graph of a quasi-symmetric homeomorphism, then this maximal surface is unique, and it has negative sectional curvature. As a by-product, we find a simple characterization of quasi-symmetric homeomorphisms of the circle in terms of 3-dimensional projective geometry.

1 Introduction

1.1 The universal Teichmüller space

We consider here the universal Teichmüller space $\mathcal{T}$, which can be defined as the space of quasi-symmetric homeomorphisms from $\mathbb{R}P^1$ to $\mathbb{R}P^1$ up to pro-

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jective transformations, see e.g. [18]. The quasi-symmetric homeomorphisms from $\mathbb{R}P^1$ to $\mathbb{R}P^1$ are precisely the homeomorphisms which are the boundary value of a quasi-conformal diffeomorphism from $\mathbb{H}^2$ to $\mathbb{H}^2$, so that the universal Teichmüller space $\mathcal{T}$ can be defined as the space of quasi-conformal diffeomorphisms from $\mathbb{H}^2$ to $\mathbb{H}^2$, up to composition with a hyperbolic isometry and up to the equivalence relation which identifies two quasi-conformal diffeomorphisms if they have the same boundary value.

It was conjectured by Schoen that any element in the universal Teichmüller space can be uniquely realized as a quasi-conformal harmonic diffeomorphism:

Conjecture 1.1 (Schoen [27]) *Let $\phi : \mathbb{R}P^1 \to \mathbb{R}P^1$ be a quasi-symmetric homeomorphism. There is a unique quasi-conformal harmonic diffeomorphism $\psi : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\partial\psi = \phi$.*

A number of partial results were obtained towards this conjecture, proving the uniqueness of ψ and its existence if ϕ is smooth enough, see [2, 24, 29] and the references there.

1.2 Minimal Lagrangian diffeomorphisms

Our first goal here is to prove an analog of Conjecture 1.1, with harmonic maps replaced by close relatives: minimal Lagrangian diffeomorphisms.

Definition 1.2 Let $\Phi : S \to S'$ be a diffeomorphism between two hyperbolic surfaces. Φ is *minimal Lagrangian* if it is area-preserving, and its graph is a minimal surface in $S \times S'$.

The relationship between harmonic maps and minimal Lagrangian maps is as follows.

Proposition 1.3

- *Let S_0 be a Riemann surface, and let $\psi : S_0 \to S$ be a quasi-conformal harmonic diffeomorphism from S_0 to a hyperbolic surface S. Let q be the Hopf differential of ψ. There is a unique harmonic diffeomorphism $\psi' : S_0 \to S'$ from S_0 to another hyperbolic surface S' with Hopf differential $-q$. Then $\psi' \circ \psi^{-1} : S \to S'$ is a minimal Lagrangian map.*
- *Conversely, let $\Phi : S \to S'$ be a minimal Lagrangian map between two (oriented) hyperbolic surfaces, and let S_0 be the graph of Φ, considered as a Riemann surface with the complex structure defined by its induced metric in $S \times S'$. Then the natural projections from S_0 to S and to S' are harmonic maps, and the sum of their Hopf differentials is zero.*

Thus minimal Lagrangian maps are a kind of "symmetric squares" of harmonic maps.

It is known that any diffeomorphism between two closed hyperbolic surfaces can be deformed to a unique harmonic diffeomorphism, see e.g. [20, 21]. In the same manner, it was proved by Schoen and by Labourie that any such diffeomorphism can be deformed to a unique minimal Lagrangian diffeomorphism [23, 27].

Our first result is an extension of this existence and uniqueness result to the universal Teichmüller space.

Theorem 1.4 *Let $\phi : \mathbb{R}P^1 \to \mathbb{R}P^1$ be a quasi-symmetric homeomorphism. There is a unique quasi-conformal minimal Lagrangian diffeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\partial\Phi = \phi$.*

The result of Schoen and Labourie on closed hyperbolic surfaces obviously follows from this. The proof of Theorem 1.4 can be found in Sect. 6. Note that partial results in this direction were obtained previously by Aiyama, Akutagawa and Wan [1], who proved the existence part of Theorem 1.4 when ϕ has small dilatation. Recently, Brendle has obtained results on the existence and uniqueness of minimal Lagrangian diffeomorphisms between two convex domains of the same, finite area in hyperbolic surfaces, see [14].

1.3 The anti-de Sitter space

The proof of Theorem 1.4 is essentially based on the geometry of maximal space-like surfaces in the anti-de Sitter (AdS) 3-dimensional space, as already in [1]. Recall that an embedded surface in a 3-dimensional Lorentzian manifold is *space-like* if its induced metric is Riemannian. It is *maximal* if its mean curvature vanishes, so that maximal space-like surfaces in Lorentzian manifolds are analogs of minimal surfaces in Riemannian manifolds.

This relationship between Teichmüller theory and 3-dimensional AdS geometry follows a pattern in some recent works (see [3, 11–13, 25] and also [1]), where results on Teichmüller theory were proved using 3-dimensional AdS geometry, although mostly in a somewhat different direction. The relationship between maximal surfaces in 3-dimensional AdS manifolds and minimal Lagrangian maps between closed hyperbolic surfaces was also used recently in [22].

The 3-dimensional AdS space can be considered as a Lorentzian analog of the 3-dimensional hyperbolic space. It can be defined as the quadric

$$AdS_3^* = \{x \in \mathbb{R}^{2,2} | \langle x, x\rangle = -1\},$$

where $\mathbb{R}^{2,2}$ is $\mathbb{R}^4$ endowed with bilinear symmetric form of signature $(2, 2)$. It is a geodesically complete Lorentz manifold of constant curvature -1.

Another way to define it is as the Lie group $SL(2,\mathbb{R})$, endowed with its bi-invariant Killing metric. More details are given in Sect. 2. The key point for us, basically discovered by Mess [3, 25] and used in the references mentioned above, is that space-like surfaces in AdS_3^* naturally give rise to area-preserving diffeomorphisms from the hyperbolic plane to itself. In this way, Theorem 1.4 is proved below to be equivalent to an existence and uniqueness statement for maximal space-like surfaces in AdS_3^*, and it is in this form that it is proved.

The anti-de Sitter space can of course be defined in higher dimensions. The existence part of the result on maximal surfaces is actually stated (and proved) below in the more general context of maximal hypersurfaces in AdS_{n+1}^*, see Theorem 1.6. The uniqueness part, however, is considered here only in AdS_3^* (and it needs hypotheses that are more interesting in dimension $2+1$), see Theorem 1.10.

1.4 Maximal surfaces and minimal Lagrangian diffeomorphisms

For closed hyperbolic surfaces, the existence of a minimal Lagrangian diffeomorphism is equivalent to the existence of a maximal space-like surface in a 3-dimensional globally hyperbolic AdS manifold, see [22]. This relation extends to maximal surfaces in AdS_3^* and the universal Teichmüller space as follows.

One way to consider the bridge between Teichmüller theory and AdS geometry is through the asymptotic boundary of AdS_3^*—denoted by $\partial_\infty AdS_3^*$—that, as for the hyperbolic space, furnishes a natural compactification of AdS_3^*. As in the hyperbolic case, a conformal Lorentzian structure is defined on $\partial_\infty AdS_3^*$. There is a natural projection $\partial_\infty AdS_3^*$ to $\mathbb{R}P^1 \times \mathbb{R}P^1$ that is a 2-to-1 covering (see Sect. 2.6 for details). The graph of any homeomorphism of $\mathbb{R}P^1$ lifts to a space-like closed curve in $\partial_\infty AdS_3^*$.

Proposition 1.5

- *Let $S \subset AdS_3^*$ be a maximal space-like graph with uniformly negative sectional curvature. Then there is a minimal Lagrangian diffeomorphism $\Phi_S : \mathbb{H}^2 \to \mathbb{H}^2$ associated to S, and the graph of $\partial\Phi_S : \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ is the projection of the boundary at infinity of S in $\partial_\infty AdS_3^*$.*
- *Conversely, to any quasi-conformal minimal Lagrangian diffeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ is associated a maximal surface S with uniformly negative sectional curvature and with boundary at infinity equal to the lifting of the graph of $\partial\Phi$ in $\partial_\infty AdS_3^*$.*

It is this proposition which provides the bridge between Theorem 1.4 and the existence and uniqueness of maximal surfaces in AdS_3^*.

1.5 Existence and regularity of maximal hypersurfaces in AdS_{n+1}

We can state an existence result for maximal hypersurfaces in the AdS space with fixed boundary values. The regularity conditions on the boundary values are quite weak, since we only demand that it bounds some space-like surface in AdS^*_{n+1}.

Theorem 1.6 *Let Γ be a closed acausal $C^{0,1}$ graph in $\partial_\infty AdS^*_{n+1}$ $(n \geq 2)$. If Γ does not contain light-like segments, then there is a maximal space-like hypersurface S_0 such that $\partial S_0 = \Gamma$.*

We provide in Sect. 4 a direct proof of this result, where the maximal surface is obtained as a limit of bigger and bigger maximal disks.

This existence result can be improved insofar as the regularity of the hypersurface is concerned. To state this improvement, we need a definition. Let Γ be a nowhere time-like graph in $\partial_\infty AdS^*_{n+1}$. Using the projective model of AdS^*_{n+1} which is also recalled in Sect. 2.5, we can consider the convex hull of Γ, it is a convex subset of AdS^*_{n+1} with boundary at infinity containing Γ, we use the notation $CH(\Gamma)$. We denote by $C(\Gamma)$ the intersection with AdS^*_{n+1} of $CH(\Gamma)$ (considered as a subset of projective space). The boundary of $C(\Gamma)$ is the disjoint union of two nowhere time-like hypersurfaces, which we call $\partial_+ C(\Gamma)$ and $\partial_- C(\Gamma)$.

Definition 1.7 The *width* of $C(\Gamma)$ (or by extension of Γ), denoted by $w(C(\Gamma))$ (resp. $w(\Gamma)$) is the supremum of the (time) distance between $\partial_- C(\Gamma)$ and $\partial_+ C(\Gamma)$.

It is proved below (Lemma 4.16) that $w(\Gamma)$ is always at most equal to $\pi/2$.

Theorem 1.8 *Suppose that $w(\partial_\infty S) < \pi/2$ in Theorem 1.6. Then S_0 can be taken to have bounded second fundamental form.*

The proof is also in Sect. 6.

1.6 The mean curvature flow

We also give in the Appendix another proof of Theorem 1.6. It is based on the mean curvature flow for hypersurfaces in the anti-de Sitter space.

Theorem 1.9 *Let $S \subset AdS_{n+1}$ be a space-like graph. There exists a long-time solution of the mean curvature flow with initial value S with fixed boundary at infinity, defined for all $t > 0$.*

This flow converges, as $t \to \infty$, to a maximal surface. When $w(\partial_\infty S) < \pi/2$, we also have bounds on the second fundamental form of the hypersurfaces occurring in the flow.

1.7 Uniqueness of maximal surfaces in AdS_3^*

We do not know whether maximal hypersurfaces with given boundary at infinity are unique in AdS_{n+1}^*. We can however state a result for surfaces in AdS_3^*, under a regularity assumption on the boundary at infinity.

Theorem 1.10 *Let S be a space-like graph in AdS_3^*. Suppose that the boundary at infinity of S is the graph of a quasi-symmetric homeomorphism from $\mathbb{R}P^1$ to $\mathbb{R}P^1$. Then there is a unique maximal surface in AdS_3^* with boundary at infinity $\partial_\infty S$ and with bounded second fundamental form, and it has negative sectional curvature.*

The proof, which can be found in Sect. 6, is based on the following proposition.

Proposition 1.11 *Let $S_0 \subset AdS_3$ be a maximal space-like graph with bounded principal curvatures. Then either it is flat, or its sectional curvature is uniformly negative (bounded from above by a negative constant).*

Those results should be compared to the existence and uniqueness of a maximal surface in a maximal globally hyperbolic AdS 3-dimensional manifold, see [4]. Theorem 1.6 applies to this case, with S_0 the lift of a closed surface in the globally hyperbolic manifold M. In this case the boundary at infinity of S is the limit set of M, which is the graph of a quasi-symmetric homeomorphism (see [3, 25]). Theorem 1.12 then shows that $w(\partial_\infty S) < \pi/2$, so that Theorem 1.10 also applies.

1.8 A characterization of quasi-symmetric homeomorphisms of the circle

Consider a homeomorphism $u : \mathbb{R}P^1 \to \mathbb{R}P^1$, let $\Gamma_u \subset \partial_\infty AdS_3$ be the lifting of the graph of u on $\partial_\infty AdS_3$.

Theorem 1.12 *$w(\Gamma_u)$ is at most $\pi/2$. It is strictly less than $\pi/2$ if and only if u is quasi-symmetric.*

The first part here is just Lemma 4.16, already mentioned above. The second part is proved in Sect. 6.1.

This statement can be considered in a purely projective way, because the fact that a point of $\partial_- C(\Gamma_u)$ is at distance strictly less than $\pi/2$ from $\partial_+ C(\Gamma_u)$

corresponds to a purely projective property, stated in terms of the duality between points and space-like planes in AdS_3, see Sect. 2.4. This duality is itself a projective notion, see Sect. 2.5.

The proof uses the considerations explained above on the properties of maximal surfaces bounded by Γ_u, it can be found in Sect. 6.1. It is based on Theorem 1.8 and to a partial converse, in dimension 3 only: if an acausal graph in $\partial_\infty AdS_3^*$ is the boundary of a maximal surface with bounded second fundamental form which is not a "horosphere" (as described in Sect. 5.2), then Γ is the graph of a quasi-symmetric homeomorphism from $\mathbb{R}P^1$ to $\mathbb{R}P^1$.

1.9 What follows

Section 2 contains a number of basic notions on the anti-de Sitter (AdS) space and some of it basic properties. It is included here for completeness, in the hope of making the paper reasonably self-contained for reader not yet familiar with AdS geometry. Section 3 similarly contains some basic facts (presumably less well-known) on space-like hypersurfaces in the AdS space.

Section 4 is perhaps the heart of the paper. After some preliminary statements on maximal space-like hypersurfaces in AdS, it contains both an existence theorem for maximal hypersurfaces with given boundary data at infinity, and a statement on the regularity of those hypersurfaces under a geometric condition on the boundary at infinity. This condition is later translated (for surfaces in the 3-dimensional AdS space) in terms of quasi-symmetric regularity of the data at infinity.

In Sect. 5 we further consider this regularity issue, with emphasis on surfaces in AdS_3^*, and we prove a uniqueness result for maximal surfaces with regular enough data at infinity. Finally we prove Theorem 1.4.

Appendix contains an alternative proof of the existence of a maximal hypersurface with given data at infinity, based on the mean curvature flow. This approach also yields some regularity results.

2 The anti de Sitter space

This section contains a number of basic statements on AdS geometry, which are necessary in the proof of the main results. Readers who are already familiar with AdS geometry will find little interest in it, we have however decided to include it to make the paper self-contained, hoping that it is useful for readers interested in Teichmüller theory but not yet in AdS geometry.

2.1 Definitions

We consider the hyperbolid model of the hyperbolic space: the hyperbolic space $\mathbb{H}^n$ is identified with the set of future-pointing unit time-like vectors in

$(n+1)$-dimensional Minkowski space $\mathbb{R}^{n,1}$. In this work, if it is not specified differently, we always use this identification. In particular points of $\mathbb{H}^n$ are identified with elements $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that $\sum_1^n x_i^2 - x_{n+1}^2 = -1$. We also fix the point $x^0 = (0, \ldots, 0, 1) \in \mathbb{H}^n$.

Let $\mathbb{R}^{n,2}$ be $\mathbb{R}^{n+2}$ equipped with the symmetric 2-form

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1} - x_{n+2} y_{n+2}.$$

The $(n+1)$-dimensional anti de Sitter space is the set

$$AdS^*_{n+1} = \{x \in \mathbb{R}^{n,2} | \langle x, x\rangle = -1\}.$$

The tangent space at a point $x \in AdS^*_{n+1}$ is the linear hyperplane orthogonal to x with respect to $\langle\cdot,\cdot\rangle$. The restriction of $\langle\cdot,\cdot\rangle$ to $T_x AdS^*_{n+1}$ is a Lorentzian scalar product.

Remark 2.1 With this definition of AdS^*_{n+1}, its isometry group is immediately seen to be $O(n, 2)$. In particular, this isometry group acts transitively on the points of AdS^*_{n+1}. More precisely, it acts simply transitively on the set of couples (x, e) where $x \in AdS^*_{n+1}$ and e is an orthonormal basis of $T_x AdS^*_{n+1}$. It is also clear (using the action of $O(n, 2)$ by isometries) that the geodesics in AdS^*_{n+1} are precisely the intersections of AdS^*_{n+1} with the linear planes in $\mathbb{R}^{n,2}$ containing 0.

There is a map

$$\Phi : \mathbb{H}^n \times \mathbb{R} \to AdS^*_{n+1}$$

defined by

$$\Phi((x_1, \ldots, x_{n+1}), t) = (x_1, x_2, \ldots, x_n, x_{n+1} \cos t, x_{n+1} \sin t). \tag{1}$$

Φ is a covering map, so topologically $AdS^*_{n+1} \cong \mathbb{H}^n \times S^1$. It will often be convenient to consider the universal cover AdS_{n+1} of AdS^*_{n+1}, that is $\mathbb{H}^n \times \mathbb{R}$, equipped with the pull-back of the metric on AdS^*_{n+1}.

It is easy to see that this metric at a point $((x_1, \ldots, x_{n+1}), t)$ takes the form

$$g_{\mathbb{H}} - x_{n+1}^2 dt^2. \tag{2}$$

If we consider the Poincaré model of $\mathbb{H}^n$, the metric can be written as

$$\frac{4}{(1-r^2)^2}(dy_1^2 + \cdots + dy_n^2) - \left(\frac{1+r^2}{1-r^2}\right)^2 dt^2, \tag{3}$$

where $r = \sqrt{y_1^2 + \cdots + y_n^2}$ and $y_1, \ldots, y_n$ are the Cartesian coordinates on the ball $\{y \in \mathbb{R}^n | r(y) < 1\}$.

By (2) we see that the time translations

$$(x, t) \to (x, t + a)$$

are isometries of AdS_{n+1}. The coordinate field $\frac{\partial}{\partial t}$ is a Killing vector field and the slices $\mathbb{H}^n \times \{t\}$ are totally geodesic.

We denote by $\bar{\nabla}$ the Levi-Civita connections of both AdS_{n+1} and $\mathbb{H}^n$. Since $\mathbb{H}^n \times \{t\}$ is totally geodesic, the restriction of $\bar{\nabla}$ on this slice coincides with its Levi-Civita connection.

We say that a vector $v \in T_{x,t}AdS_{n+1}$ is horizontal if it is tangent to the slice $\mathbb{H}^n \times \{t\}$. Analogously it is vertical if it is tangent to the line $\{x\} \times \mathbb{R}$.

The lapse function ϕ is defined by

$$\phi^2 = -\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle.$$

The gradient of t is a vertical vector at each point and it equal to

$$\bar{\nabla} t = -\frac{1}{\phi^2} \frac{\partial}{\partial t},$$

so its squared norm is equal to $-\frac{1}{\phi^2}$.

2.2 The asymptotic boundary and the causal structure

We denote by $\overline{AdS_{n+1}}$ the manifold with boundary $\overline{\mathbb{H}^n} \times \mathbb{R}$, where $\overline{\mathbb{H}}^n$ is the usual compactification of $\mathbb{H}^n$ (obtained for instance in the projective model of $\mathbb{H}^n$). Another way to consider $\overline{AdS_{n+1}}$ is as the universal cover of the compactification of AdS^*_{n+1} defined by adding the projectivization of the cone of vectors $x \in \mathbb{R}^{n,2}$ such that $\langle x, x \rangle = 0$.

Clearly AdS_{n+1} is the interior part of $\overline{AdS_{n+1}}$, whereas its boundary, $\partial \mathbb{H}^n \times \mathbb{R}$ is called the *asymptotic boundary* of AdS_{n+1} and is denoted by $\partial_\infty AdS_{n+1}$. The following statement is clear when considering the definition of AdS^*_{n+1} as a quadric.

Lemma 2.2 *Every isometry f of AdS_{n+1} extends to a homeomorphism of $\overline{AdS_{n+1}}$.*

The asymptotic boundary of a set $K \subset AdS_{n+1}$—denoted by $\partial_\infty K$—is the set of the accumulation points of K in $\partial_\infty AdS_{n+1}$. By (3) it is clear that the conformal structure on AdS_{n+1} extends to the boundary. This means that in the conformal class of the metric g there is a metric g^* that extends to the boundary. We can for instance put $g^* = \frac{1}{\phi^2} g$.

A vector v tangent at some point in $\partial_\infty AdS_{n+1}$ is time-like (light-like, space-like) if $g^*(v,v)<0$ $(=0, >0)$. Notice that the definition makes sense since the sign of $g^*(v,v)$ depends only on the conformal class of g^*.

Lemma 2.3 *Let $c:(-1,1)\to AdS_{n+1}$ be an inextensible time-like path. If the function t is bounded from above on c, there exists the limit $p_1 = \lim_{s\to 1} c(s)\in\partial_\infty AdS_{n+1}$.*

Proof The vertical component of $\dot c$ is

$$\dot c_V = \langle \dot c, \bar\nabla t\rangle \frac{\partial}{\partial t} = \dot t \frac{\partial}{\partial t}.$$

Since the norm of $\frac{\partial}{\partial t}$ for $g^*$is 1, we have $|\dot c_V|_{g^*} = \dot t$. On the other hand, the fact that c is time-like implies

$$|\dot c_H|_{g^*} \le |\dot c_V|_{g^*} = \dot t.$$

Since the function t is increasing along c, the bound on t along c implies that $\dot c_H$ is bounded in a neighbourhood of 1. It follows that the path c_H obtained by projecting c to $\mathbb{H}^n$, has finite length with respect to the metric $\frac{1}{\phi^2} g_{\mathbb{H}}$. This implies that there exists the limit $x_1 = \lim_{s\to 1} c_H(s)$. On the other hand, since t is increasing along c there exists the limit $t_1 = \lim_{s\to 1} t(c(s))$. The point $p_1 = (x_1, t_1)$ is the limit point of c. Since we assume that c is inextensible in AdS_{n+1}, $p_1 \in \partial_\infty AdS_{n+1}$. □

The point p_1 is an asymptotic end-point of c.

An inextensible path is without end-points if and only if the function t takes all the real values along c, or equivalently, if c does not admit any asymptotic end-point. Vertical lines are instances of inextensible paths without end-points.

2.3 Geodesics and geodesic hyperplanes in AdS_{n+1}

The next statement, which is classical, describes the geodesics in AdS^*_{n+1}, considered as a quadric in $\mathbb{R}^{n,2}$.

Lemma 2.4 (see [9]) *Geodesics in AdS^*_{n+1} are the intersection AdS^*_{n+1} with linear 2-planes in $\mathbb{R}^{n,2}$ containing 0. In particular, given a tangent vector v at some point $p\in AdS^*_{n+1}$, we have*

$$\exp_p(sv) = \begin{cases} \cos(s)p + \sin(s)v & \text{if } \langle v,v\rangle = -1;\\ p + sv & \text{if } \langle v,v\rangle = 0;\\ \cosh(s)p + \sinh(s)v & \text{if } \langle v,v\rangle = 1. \end{cases} \quad (4)$$

Remark 2.5 Totally geodesics k-planes in AdS^*_{n+1} are the intersection of AdS^*_{n+1} with $(k+1)$-linear planes of $\mathbb{R}^{n,2}$ containing 0.

Space-like and light-like geodesics are open simple curves. Homotopically, time-like geodesics are simple closed non-trivial curve. Moreover every complete time-like geodesic starting at p passes through $-p$ at time $(2k+1)\pi$ and at p at time $2k\pi$ for $k \in \mathbb{Z}$. Passing to the universal cover, we get the following statement.

Lemma 2.6 *Given a point $p = (x, t) \in AdS_{n+1}$ there is a discrete set $\{p_k | k \in \mathbb{Z}\}$ such that every time-like geodesic γ starting at p passes through p_k at time $t = k\pi$. Moreover, $p_{2k} = (x, t + 2k\pi)$ and $p_{2k+1} = (y, t + (2k+1)\pi)$ where y is some point in $\mathbb{H}^n$ independent of k.*

In what follows we will often use the points p_1 and p_{-1}. To simplify the notation we will denote these points by p_+ and p_-.

Time-like geodesics are time-like paths without end-points. On the other hand since space-like geodesics are conjugated to horizontal ones by some isometry, they have 2 asymptotic end-points. Using the projection Φ one can check that the path $c(s) = (x(s), \arccos(\frac{1}{\sqrt{1+s^2}}))$ where $x(s) = (s, 0, \dots, 0, \sqrt{1+s^2})$ is a light-like geodesic. Since c has two asymptotic end-points, the same property holds for every light-like geodesic.

Remark 2.7 Points in $\partial_\infty AdS_{n+1}$ related by a time-like arc in $\partial_\infty AdS_{n+1}$ are not joined by a geodesic arc in AdS_{n+1}. Indeed by the above description if a geodesic connects two points in the asymptotic boundary of AdS_{n+1} then it is either space-like or light-like (and in this case it is contained in the boundary).

Totally geodesic n-planes in AdS_{n+1} are distinguished by the restriction of the ambient metric on them. They can be time-like, space-like or light-like according as whether this restriction has Lorentzian, Euclidean or degenerate signature.

Space-like hyperplanes are conjugate by some isometry to horizontal planes. Time-like hyperplanes are conjugate by some isometry to the hyperplane $P_0 \times \mathbb{R}$, where P_0 is a totally geodesic hyperplane in $\mathbb{H}^n$. For light-like hyperplanes we will need a more precise description.

Lemma 2.8 *Let P be a light-like hyperplane. There are two points ζ_- and ζ_+ in $\partial_\infty AdS_{n+1}$ such that P is foliated by light-like geodesics with asymptotic end-points ζ_- and ζ_+. The foliation of P by light-like geodesics extends to a foliation of $\overline{P} \setminus \{\zeta_-, \zeta_+\}$ by light-like geodesics, where $\overline{P}$ denotes the closure of P in $\overline{AdS}_{n+1}$.*

Proof It is sufficient to prove the statement for a specific light-like plane. Consider the hypersurface $P_0 = \{(x,t) \in AdS_{n+1} | t = \arcsin(\frac{x_1}{x_{n+1}})\}$. Using the projection Φ one see that P_0 is a totally geodesic plane, indeed $\Phi(P_0)$ is a connected component of the intersection of AdS^*_{n+1} with the linear plane defined by the equation $y_1 - y_{n+2} = 0$.

We consider the natural parameterization $\sigma : \mathbb{H}^n \to P_0$ defined by $\sigma(x) = (x, \arcsin(\frac{x_1}{x_{n+1}}))$. Since the function $\frac{x_1}{x_{n+1}}$ extends to the boundary of $\mathbb{H}^n$, the map σ extends to $\overline{\mathbb{H}}^n$ and gives a parameterization of the closure $\overline{P}_0$ of P_0 in $\overline{AdS}_{n+1}$.

The level surfaces $H_a = \{\frac{x_1}{x_{n+1}} = a\}$ are totally geodesic hyperplanes orthogonal to the geodesic $c = \{x_2 = \cdots = x_n = 0\}$. Let N be the unit future-oriented vector field on $\mathbb{H}^n$ orthogonal to H_a for all a. A simple computation shows that

- for all a, $\sigma|_{H_a}$ is an isometric embedding;
- $\hat{N} = \sigma_*(N)$ is a light-like field;
- $\hat{N}$ is orthogonal to $\sigma(H_a)$.

It follows that P_0 is a light-like plane. The integral lines of $\hat{N}$ produce a foliation of P_0 by light-like geodesics. Notice that integral lines of $\hat{N}$ are the images of integral lines of the field N. By standard hyperbolic geometry, all these lines join the endpoints, say x_-, x_+, of the geodesic c. We conclude that light-like geodesics of P_0 join $\sigma(x_-)$ to $\sigma(x_+)$. Since the foliation of $\mathbb{H}^n$ by integral lines of N extends to a foliation of $\overline{\mathbb{H}}^n \setminus \{x_-, x_+\}$, the foliation given by $\hat{N}$ extends to a foliation of $\overline{P_0} \setminus \{\zeta_-, \zeta_+\}$. By continuity we conclude that the leaves of this foliation are light-like. □

For a light-like plane P the points ζ_- and ζ_+ are called respectively the past and the future end-points of the plane.

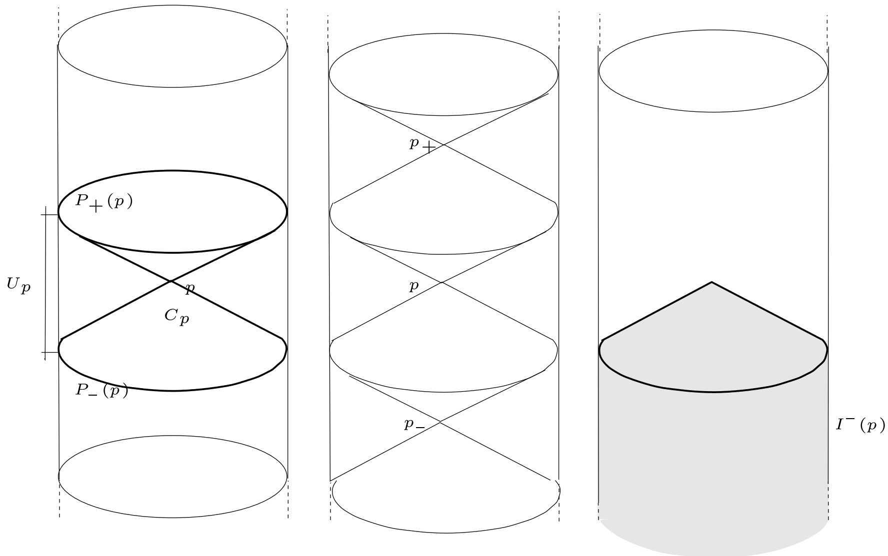
Space-like and light-like hyperplanes disconnect AdS_{n+1} in two connected components, that coincide with the past and the future of them. Their asymptotic boundary is a no-where time-like closed hypersurface of $\partial_\infty AdS_{n+1}$. On the other hand the asymptotic boundary of a time-like plane is the union of two inextensible time-like curves.

2.4 The causal structure of AdS_{n+1}

If $c : [0, 1] \to AdS_{n+1}$ is a time-like path, its length is defined in this way:

$$\ell(c) = \int_0^1 (-\langle \dot{c}(s), \dot{c}(s)\rangle)^{1/2} ds.$$

Given $p \in AdS_{n+1}$ we consider the set $P_-(p)$ (resp. $P_+(p)$) defined respectively as the set of points that can be joined to p through a past-directed (resp. future-directed) time-like geodesic of length $\pi/2$.



Remark 2.9 For a point $x \in AdS^*_{n+1}$ we can identify the set of unit time-like tangent vectors at x with the geodesic plane $P^*_x = x^\perp \cap AdS^*_{n+1}$ (where $x^\perp$ is the linear plane orthogonal to x). P^*_x has two connected components. Equation (4) shows that these components are the images of $P_+(p)$ and $P_-(p)$, where p is any preimage of x in AdS_{n+1}.

The following properties of $P_-(p)$ and $P_+(p)$ are a direct consequence of Remark 2.9.

Lemma 2.10 *The sets $P_-(p)$ and $P_+(p)$ are complete, space-like totally geodesic planes. Every time-like geodesic starting at p meets $P_-(p)$ and $P_+(p)$ orthogonally.*

Remark 2.11 For the point $p_0 = (x^0, 0)$, a direct computation (still using the projection Φ) shows that $P_-(p_0)$ and $P_+(p_0)$ are level curves of the time function t corresponding to values $-\pi/2$ and $\pi/2$ respectively.

The planes $P_-(p)$ and $P_+(p)$ are disjoint and bound an open precompact domain U_p in $\overline{AdS}_{n+1}$. For instance, for $p = (x^0, 0)$ we have $U_p = \{(x,t) \in \overline{AdS}_{n+1} | -\pi/2 < t < \pi/2\}$. By definition the interior of U_p (denoted by $\mathrm{int}(U_p)$) is the intersection of U_p with AdS_{n+1}. Notice that

$$\mathrm{int}(U_p) = I^+(P_-(p)) \cap I^-(P_+(p)).$$

Notice that $P_+(p_k) = P_-(p_{k+1})$ for every $k \in \mathbb{Z}$. In particular $U_{p_i} \cap U_{p_j} = \emptyset$ if $|i-j| > 1$ and $\overline{U_{p_i}} \cap \overline{U_{p_{i+1}}} = P_+(p_i)$.

Given $p \in AdS_{n+1}$ we denote by C_p the set of points joined to p through a time-like geodesic of length less than $\pi/2$.

Proposition 2.12

- *$C_p \subset U_p$.*
- *Space-like and light-like geodesics join p to points in $U_p \setminus C_p$, whereas time-like geodesics are contained in $\bigcup_{n\in\mathbb{Z}} C_{p_n}$.*
- *$I^+(p) \subset C_p \cup I^+(P_+(p)) = C_p \cup \bigcup_{k>0} U_{p_k}$.*
- *$\partial C_p \cap U_p$ is the light-like cone through p, whereas $\partial_\infty C_p$ is the union of the asymptotic boundary of $P_+(p)$ and the asymptotic boundary of $P_-(p)$.*

This proposition can be easily proved using the projection Φ and the explicit formula (4).

It is worth noticing that AdS_{n+1} is not geodesically convex. Indeed the set of points in AdS_{n+1} that can be joined to p by a geodesic is $\mathrm{int}(U_p) \cup \bigcup C_{p_k}$.

Corollary 2.13 *The set $I^-(p_+) \cap I^+(p_-)$ is the maximal star neighbourhood of p.*

Given $p \in AdS_{n+1}$ and $q \in I^+(p)$, the distance between them is defined as

$$\delta(p,q) = \sup\{\ell(c) | c \text{ time-like path joining } p \text{ to } q\}.$$

The next statement is true in a rather general context and can be proved by classical arguments.

Lemma 2.14 *If U is a star neighbourhood of p, then the distance from p*

$$\delta_p : U \cap I^+(p) \ni q \mapsto \delta(p,q) \in \mathbb{R}$$

is smooth. For $q \in U \cap I^+(p)$ the distance $\delta(p,q)$ is realized by the unique geodesic joining p to q contained in U.

Remark 2.15 The definition of the distance shows that for $q \in I^+(p) \cap U$ and $r \in I^+(q)$, the reverse of the triangle inequality holds

$$\delta(p,r) \geq \delta(p,q) + \delta(q,r). \tag{5}$$

2.5 The projective model

As noted in the proof of Lemma 2.6 the geodesics in AdS^*_{n+1} are obtained as the intersection of AdS^*_{n+1} with the linear planes of $\mathbb{R}^{n+2}$ containing 0.

For this reason the projection map

$$\pi : AdS_{n+1} \to \mathbb{R}P^{n+1}$$

is projective: it sends geodesics of AdS_{n+1} to projective segments. The image of this projective map is the interior of a quadric $Q \subset \mathbb{R}P^{n+1}$ of signature $(n-1, 1)$.

Notice for $p \in AdS_{n+1}$ the domain $\Phi(\mathrm{int}(U_p))$ is a connected component of $AdS^*_{n+1} \setminus P^*_{\Phi(p)}$. Thus the domain $\pi(\Phi(U_p))$ is contained in some affine chart of $\mathbb{R}P^{n+1}$.

In this way we construct a projective embedding

$$\pi^* : \mathrm{int}(U_p) \to \mathbb{R}^{n+1}.$$

The map π^* can be easily computed assuming $p = (x^0, 0)$. In this case $U_p = \{(x, t) | t \in (-\pi/2, \pi/2)\}$ so $\Phi(\mathrm{int}(U_p)) = \{(y_1, \dots, y_n, , y_{n+1}, y_{n+2}) \in AdS^*_{n+1} | y_{n+1} > 0\}$ and

$$\pi^*(x_1, \dots, x_{n+1}, t) = \left(\frac{x_1}{x_{n+1}\cos t}, \frac{x_2}{x_{n+1}\cos t}, \dots, \frac{x_n}{x_{n+1}\cos t}, \tan t \right) \tag{6}$$

for every $(x_1, \dots, x_n) \in \mathbb{H}^n$ and $t \in (-\pi/2, \pi/2)$.

Notice that the map extends continuously on U_p to a map, still denoted by π^*. From (6), the image $\pi^*(U_p)$ is the set

$$\left\{ (z_1, \dots, z_{n+1}) | \sum_{i=1}^{n} z_i^2 \leq z_{n+1}^2 + 1 \right\}. \tag{7}$$

In particular we deduce that every point $q \in U_p$ (even on the boundary) can be joined to p by a unique geodesic and that this geodesic continuously depend on q.

We have seen above how to associate to a point $p \in AdS_{n+1}$ two totally geodesic space-like hyperplanes $P_-(p)$ and $P_+(p)$. Both planes are sent by π to the intersection with $\pi(AdS^*_{n+1})$ of the same projective plane P, and P has a purely projective definition. Indeed the light-cone of p is tangent to Q along a circle C, and the image by π of the boundary at infinity of $P_-(p)$ is precisely C. One way to see this is by using the fact that in the projective model of AdS_{n+1} (as for the hyperbolic space) the distance between two points can be defined in terms of the Hilbert distance of the quadric Q, see e.g. [26].

This duality extends to a duality between totally geodesic (space-like) k-planes in $\pi(AdS_{n+1})$, with the dual of a k-plane P being a $(n-k)$-plane P^*. Then P^* can be defined as the intersection between the hyperplanes dual to the points of P, and conversely. Then P^* can be characterized as the set of points at distance $\pi/2$ from P along a time-like segment, and conversely.

2.6 The 3-dimensional AdS space

The general description of the n-dimensional anti-de Sitter space AdS^*_{n+1} above can be refined when $n = 2$, and AdS^*_3 has some quite specific properties.

One such specificity is that AdS^*_3 is none other than the Lie group $SL(2,\mathbb{R})$, with its Killing metric. This point of view, which is important in itself (see [3, 25]), will not be used explicitly here.

Another feature which is specific of AdS_3 is the fact that the boundary of $\pi(AdS_3)$ in $\mathbb{R}P^3$ is a quadric of signature $(1,1)$ which, as is well known, is foliated by two families of projective lines, which we will call $\mathcal{L}_l$ and $\mathcal{L}_r$ (l and r stand for "left" and "right" here). Those projective lines correspond precisely to the isotropic curves in the Lorentz-conformal structure on $\partial_\infty AdS_3$. Each line of one family intersects each line of the other family at exactly one point, this provides an identification of $\partial\pi(AdS^*_3)$ with $\mathbb{R}P^1 \times \mathbb{R}P^1$, with each copy of $\mathbb{R}P^1$ identified with one of the two families of lines foliating $\partial\pi(AdS^*_3)$.

This has interesting consequences, in particular those explained in Sect. 3.4. Another consequence is that the isometry group of AdS_3 can be naturally identified (up to finite index) with the product of two copies of $PSL(2,\mathbb{R})$. Indeed any isometry of AdS_3 in the connected component of the identity acts on the two families of lines foliating $\partial_\infty AdS_3$ by permuting those lines, and this action is projective on each family of lines. Conversely, any couple of elements of $PSL(2,\mathbb{R})$ can be obtained in this manner.

3 Space-like graphs in AdS_{n+1}

This section continues the description of the geometry of the AdS space, with emphasis on space-like surfaces. Readers already familiar with AdS geometry might not be very surprised by most of the results, but several notations and lemmas will be used in the next section.

3.1 Definitions

A smooth embedded hypersurface M in AdS_{n+1} is *space-like* if for every $x \in M$ the restriction of $\langle \cdot, \cdot \rangle$ on T_xM is positive definite. It turns out that a Riemannian structure is induced on every space-like hypersurface by the ambient metric.

We say that a space-like surface M in AdS_{n+1} is a *graph* if there is a function

$$u : \mathbb{H}^n \to \mathbb{R}$$

such that M coincides with the graph of u.

First let us check which functions correspond to space-like graphs. The function u induces a function on $\mathbb{H}^n \times \mathbb{R}$

$$\hat{u}(x,t) = u(x).$$

The gradient of $\hat{u}$ at a point (x,t) is the horizontal vector that projects to the gradient of u at x.

The graph of u, say $M = M_u$, is defined by the equation $\hat{u} - t = 0$. Thus the tangent space $T_{(x,u(x))}M = \ker(dt - d\hat{u})_{(x,u(x))}$. In particular the normal direction of M at $(x,u(x))$ is generated by the vector

$$\bar{\nu} = \bar{\nabla} t - \bar{\nabla}\hat{u} \tag{8}$$

whose norm is

$$|\bar{\nabla}\hat{u}|^2 - \frac{1}{\phi^2}.$$

Since $|\bar{\nabla}\hat{u}| = |\bar{\nabla} u|$ we deduce that M is space-like if and only if

$$1 - \phi^2|\bar{\nabla} u|^2 < 0, \tag{9}$$

and the future-pointing normal vector is

$$\nu = \frac{\phi}{\sqrt{1 - \phi^2|\bar{\nabla} u|^2}}(\bar{\nabla}\hat{u} - \bar{\nabla} t). \tag{10}$$

It is interesting to express (9) using the Poincaré model of hyperbolic space. In that case we have

$$\bar{\nabla} u = \frac{(1-r^2)^2}{4}\left(\frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_n}\right)$$

so

$$|\bar{\nabla} u|^2 = \frac{(1-r^2)}{4}\sum\left(\frac{\partial u}{\partial y_j}\right)^2$$

and condition (9) becomes

$$\sum_j \left(\frac{\partial u}{\partial y_j}\right)^2 < \frac{4}{(1+r^2)^2}. \tag{11}$$

In particular the function u is 2-Lipschitz with respect to the Euclidean distance of the ball.

Lemma 3.1 *Let $M = M_u$ be a smooth space-like graph in AdS_{n+1}. Then the function u extends to a continuous function*

$$\bar{u} : \bar{\mathbb{H}}^n \to \mathbb{R}.$$

In particular the closure of M in $\overline{AdS}_{n+1}$ is still a graph.

3.2 Acausal surfaces

A $C^{0,1}$ hypersurface M in AdS_{n+1} is said to be weakly space-like if for every $p \in M$ there is a neighbourhood U of p in AdS_{n+1} such that $U \setminus M$ is the disjoint union $I^+_U(M) \cup I^-_U(M)$. A neighbourhood satisfying the above property will be called a *good neighbourhood* of p.

It is not hard to see that a space-like surface is weakly space-like. On the other hand a C^1 weakly space-like surface is characterized by the property that no tangent plane is time-like.

A weakly space-like graph is a weakly space-like surface that is the graph of some function u. Weakly space-like graphs correspond to Lipschitz functions u such that the inequality

$$1 - \phi^2 |\bar{\nabla} u|^2 \leq 0$$

holds almost everywhere. As for space-like graphs it is still true that the closure of acausal graphs in $\overline{AdS}_{n+1}$ is a graph.

First we provide an intrinsic characterization of weakly space-like graphs.

Proposition 3.2 *Let M be a connected weakly space-like hypersurface. The following statements are equivalent*:

(1) *M is a weakly space-like graph*;
(2) *$AdS_{n+1} \setminus M$ is the union of 2 connected components*;
(3) *every inextensible time-like curve without end-points meets M exactly in one point.*

Proof The implication (1) $\Rightarrow$ (2) is clear.

Assume (3) holds. Then every vertical line meets M exactly in one point. This shows that the projection $\pi : M \to \mathbb{H}^n$ is one-to-one. Since M is a topological manifold, the Invariance of Domain Theorem implies that π is a homeomorphism. Thus M is a graph.

Finally suppose that (2) holds. We consider the equivalence relation on M such that $p \sim q$ if there are good neighbourhoods U and V of p and q respectively such that $I^+_U(p)$ and $I^+_V(q)$ are contained in the same component of $AdS_{n+1} \setminus M$. Equivalence classes are open. Since M is connected, all points are equivalent. We deduce that there is a component, say Ω_+, of $AdS_{n+1} \setminus M$

such that if $c = c(s)$ is a future-directed time-like path hitting M for $s = 0$, then there is $\epsilon > 0$ such that $c(s) \in \Omega_+$ for $0 < s < \epsilon$. In the same way, there is a component, say Ω_- such that $c(s) \in \Omega_-$ for $-\epsilon < s < 0$.

If U is a good neighbourhood of some point $p \in M$, then $U \subset \Omega_+ \cup M \cup \Omega_-$, so $\Omega_+ \cup \Omega_- \cup M$ is an open neighbourhood of M. Since the closure of every component of $AdS_{n+1} \setminus M$ contains points in M, by the assumption (2), Ω_+ and Ω_- are different components of $AdS_{n+1} \setminus M$ and $AdS_{n+1} = \Omega_- \cup M \cup \Omega_+$.

It follows that no future-directed time-like curve starting at a point of Ω_+ can end at some point of M. Since any future-directed time-like curve that starts on M intersects Ω_+, points of M are not related by time-like curves and $I^+(M) \subset \Omega_+$ and $I^-(M) \subset \Omega_-$.

In particular, given a point $p \in M$, the surface M is contained in U_p. It follows that the restriction of the time-function t on M is bounded in some interval $[a, b]$. Moreover Ω_+ contains the region $\{(x, t)|t > b\}$, instead Ω_- contains the region $\{(x, t)|t < a\}$.

Since the restriction of t on any inextensible time-like curve without end-points c takes all the values of the interval $(-\infty, +\infty)$ we have that c contains points of Ω_- and points of Ω_+. Thus it must intersect M. Since points of M are not related by time-like arcs, such intersection point is unique. □

Remark 3.3 Proposition 3.2 implies that space-like graphs are intrinsically described in terms of the geometry of AdS_{n+1}. In particular, if M is a space-like graph, and γ is an isometry of AdS_{n+1}, then $\gamma(M)$ is still a space-like graph.

Remark 3.4 Given a point $p \in AdS_{n+1}$ we have that $\partial I^+(p)$ is a weakly space-like graph. Indeed we can assume $p = (x^0, 0)$. In that case it turns out that $\partial I^+(p)$ is the graph of the function $\arccos(\frac{1}{x_{n+1}})$.

An important feature of weakly space-like graphs is that they are acausal as the following proposition states.

Proposition 3.5 *Let $M = M_u$ be a weakly space-like graph in AdS_{n+1}, and let $\overline{M}$ denote its closure in $\overline{AdS}_{n+1}$. Given $p \in M$, then, for every $q \in \overline{M}$, p and q are connected by a geodesic $[p, q]$ that is not time-like. Moreover, if this geodesic is light-like, then it is contained in M.*

Proof Proposition 3.2 implies that $M \cap I^+(p) = \emptyset$ and $M \cap I^-(p) = \emptyset$. In particular, $M \subset U_p$ that is a star-neighbourhood of p. It follows that any point q of $\overline{M}$ is connected to p by some geodesic that continuously depends on p. Since points of M cannot be connected to p by a time-like geodesic, the same holds for points in $\partial_\infty M$.

Finally, let us prove that if $[p,q]$ is light-like, then it is contained in M. Let $u_\pm : \mathbb{H}^n \to \mathbb{R}$ be such that $\partial I^\pm(p)$ is the graph of $u_\pm$. Let us set $p=(x_0,t_0)$ and $q=(x_1,t_1)$. Consider the geodesic arc of $\mathbb{H}^n$, say $x(s)$, starting from x_0 and ending at x_1 defined for $s\in[0,T]$ (T can be $+\infty$ if $x_1\in\partial\mathbb{H}^n$). Notice that the function of s defined by $u_+(s)=u_+(x(s))$ satisfies

$$\dot{u}_+ = \frac{1}{\phi(x(s))}, \qquad u_+(0)=t_0. \tag{12}$$

On the other hand the function $u(s)=u(x(s))$ satisfies

$$\dot{u} = \left\langle \bar{\nabla} u, \frac{dx}{ds}\right\rangle \le \frac{1}{\phi(x(s))}, \qquad u(0)=t_0. \tag{13}$$

Comparing (12) and (13) we deduce that

$$u(s)\le u_+(s),$$

and the equality holds at some s_0 if and only if $\dot{u}(s)=\frac{1}{\phi(x(s))}$ on the interval $[0,s_0]$, that is equivalent to say that the light-like segment joining $p=(x_0,t_0)$ to $q=(x(s_0),u(x(s_0)))$ is contained in M.

In an analogous way we show that $u_-(s)\le u(s)$. □

Remark 3.6 The hypothesis that M is a graph is essential in Proposition 3.5. It is not difficult to construct a space-like surface M containing points p,q that are related by a vertical segment.

For a weakly space-like surface M, a point $p\in M$ is *singular* if it is contained in the interior of some light-like segment contained in M. The singular set of M is the set of singular points. Analogously we define the singular set of the asymptotic boundary Σ of M. Notice that the singular set of Σ can be non-empty even if M does not contain singular points.

3.3 The domain of dependence of a space-like graph

Let M be a space-like graph in AdS_{n+1}, and let Σ denote its asymptotic boundary. We will suppose that M does not contain any singular point.

The domain of dependence of M is the set D of points $x\in AdS_{n+1}$ such that *every inextensible causal path through x intersects M*. It can be easily shown that this property is equivalent to requiring that $(I^+(x)\cup I^-(x))\cap M$ is precompact in AdS_{n+1}. There is an easy characterization of D in terms of Σ.

Lemma 3.7 *With the notations of Sect.* 2.3, *a point p lies in D if and only if Σ is contained in U_p.*

Proof Suppose that $p \in D$. Without loss of generality we can suppose that $p \in I^-(M)$. By the hypothesis, $I^+(p) \cap M$ is precompact in AdS_{n+1} (whereas $I^-(p) \cap M = \emptyset$). Thus there is a compact ball $B \subset \mathbb{H}^n$ such that $I^+(p) \cap M$ is contained in the cylinder above B. In particular, $M \setminus (B \times \mathbb{R})$ is contained in U_p. It follows that $\Sigma \subset \overline{U_p}$.

If some point x of Σ were contained in $\partial_\infty P_+(p)$ then the geodesic joining p to x would be light-like and would intersects M in some point q. Then by Proposition 3.5, the light-like geodesic segment joining q to x would be contained in M and this would contradict the hypothesis that M does not contain any singular point.

Let us consider now a point p such that $\Sigma \subset U_p$. Again we can suppose that $p \in I^-(M)$. By the assumption the asymptotic boundary of M and the asymptotic boundary of $I^+(p)$ are disjoint. It follows that $I^+(p) \cap M$ is precompact in AdS_{n+1}. ☐

Corollary 3.8 *Two space-like surfaces share the boundary at infinity if and only if their domains of dependence coincide.*

Proposition 3.9

- *The domain D is geodesically convex and its closure at infinity is precisely Σ.*
- *The boundary of D is the disjoint union of two weakly space-like graphs $\partial_\pm D = M_{u_\pm}$ whose boundary at infinity is Σ.*
- *Every point $p \in \partial D$ is joined to Σ by a light-like ray.*

To prove this proposition we need a technical lemma of AdS geometry.

Lemma 3.10 *Given two points $p, q \in AdS_{n+1}$ connected along a geodesic segment $[p, q]$ and given any point r lying on such a segment, we have that*

$$U_p \cap U_q \subset U_r.$$

Proof Let u_p (resp. v_p) be the real function on $\mathbb{H}^n$ such that $P_+(p)$ (resp. $P_-(p)$) is the graph of u_p (resp. v_p). Analogously define u_q, v_q, u_r, v_r.

We have that

$$U_p = \{(x,t) | v_p(x) < t < u_p(x)\}, \qquad U_q = \{(x,t) | v_q(x) < t < u_q(x)\},$$
$$U_r = \{(x,t) | v_r(x) < t < u_r(x)\}.$$

In particular, $U_p \cap U_q = \{(x,t) | \max\{v_p(x), v_q(x)\} < t < \min\{u_p(x), u_q(x)\}\}$. Then, the statement turns out to be equivalent to the inequalities

$$v_r \leq \max\{v_p, v_q\}, \qquad \min\{u_p, u_q\} \leq u_r.$$

If the segment $[p, q]$ is time-like, then, up to isometry, we can suppose that $p = (x^0, 0)$, $q = (x^0, a)$, $r = (x^0, b)$ with $0 \le b \le a$. In this case we have $u_p(x) = \pi/2$, $u_q(x) = a + \pi/2$, $u_r(x) = b + \pi/2$ so the statement easily follows.

Suppose now that the geodesic $[p, q]$ is space-like. Up to isometry, we can suppose that $p = (x_p, 0)$, $q = (x_q, 0)$, $r = (x_r, 0)$ where x_p, x_q, x_r are the following points in (the hyperboloid model of) $\mathbb{H}^n$:

$$x_p = (-\sinh\epsilon, 0, \dots, 0, \cosh\epsilon), \qquad x_q = (\sinh\eta, 0, \dots, 0, \cosh\eta),$$
$$x_r = (0, \dots, 0, 1),$$

where η and ϵ are respectively the distance from p and q to r.

The corresponding points $p^*, q^*, r^* \in AdS^*_{n+1}$ are

$$p^* = (-\sinh\epsilon, 0, \dots, 0, \cosh\epsilon, 0), \qquad q^* = (\sinh\eta, 0, \dots, 0, \cosh\eta, 0),$$
$$r^* = (0, \dots, 0, 1, 0).$$

By Remark 2.9, $\Phi(P_+(p))$ is a component of the intersection of AdS^*_{n+1} with the hyperplane defined by the equation

$$-y_1 \sinh\epsilon - y_{n+1} \cosh\epsilon = 0.$$

In particular, pulling-back this equation, we deduce that the set $P_+(p)$ is a component of the set

$$\{((x_1, \dots, x_{n+1}), t) \in \mathbb{H}^n \times \mathbb{R} \mid -x_1 \sinh(\epsilon) - x_{n+1} \cos t \cosh(\epsilon) = 0\}.$$

Since the function t takes value in $(0, \pi)$ on $P_+(p)$ we deduce that

$$u_p(x_1, \dots, x_{n+1}) = \arccos\left(-\frac{x_1 \sinh\epsilon}{x_{n+1}\cosh\epsilon}\right).$$

Analogously, we derive

$$u_r(x_1, \dots, x_{n+1}) = \pi/2, \qquad u_q(x_1, \dots, x_{n+1}) = \arccos\left(\frac{x_1 \sinh\eta}{x_{n+1}\cosh\eta}\right).$$

Notice that $u_p \le \pi/2$ if $x_1 \le 0$, whereas $u_q \le \pi/2$ if $x_1 \ge 0$. It follows that $\min\{u_p, u_q\} \le u_r$.

Since $v_p = -u_p$, $v_q = -u_q$ and $v_r = -u_r$, we deduce that $\max\{v_p, v_q\} \ge v_r$.

When $[p, q]$ is light-like, the computation is completely analogous. $\square$

Remark 3.11 From the proof of the lemma we have that $P_+(p)$ and $P_+(q)$ are disjoint in $\overline{AdS_{n+1}}$ if p and q are joined by a time-like segment, while they meet along a $(n-1)$-dimensional geodesic plane if p and q are connected by a space-like geodesic. Finally in the light-like case, they meet at the asymptotic end-points of the geodesic through p and q.

Proof of Proposition 3.9 Let p be a point contained in D and consider the nearest conjugate points $p_\pm$ to p as defined in Sect. 2.3. First we show that D is contained in the star neighbourhood $I^-(p_+) \cap I^+(p_-)$ of p. Let $q \notin I^-(p_+)$. If $q \in \overline{I^+(p_+)}$ then $I^-(p_+) \subset I^-(q)$. Since Σ is contained in the asymptotic boundary of the past of $P_+(p) = P_-(p_+)$ that in turn coincides with the asymptotic boundary of $I^-(p_+)$, we see that $\Sigma \subset \partial_\infty I^-(q)$, so that $\Sigma \cap U_q = \emptyset$. Suppose now that q is related to p_+ by a space-like geodesic. Remark 3.11 shows that $\partial_\infty P_-(p_+) \cap \partial_\infty P_-(q)$ contains a point (ξ, t). Since Σ is a graph on $\partial \mathbb{H}^n$, there is a point in Σ of the form (ξ, t') and since $\Sigma \subset I^-(P_-(p_+))$ we get $t' < t$. It follows that (ξ, t') is not contained in U_q. Eventually we obtain that $q \notin D$. The same argument shows that any point in D must be contained in $I^+(p_-)$ so D is contained in $I^-(p_+) \cap I^+(p_-)$.

We deduce from this that given two points $p, q \in D$, the geodesic segment $[p, q]$ joining them exists and does not contain any point conjugate to p. Given a point $r \in [p, q]$ the region U_r contains $U_p \cap U_q$, so that U_r contains Σ. By Lemma 3.7 it follows that $r \in D$. This shows that D is convex.

Clearly Σ is contained in the boundary of D. On the other hand, given any other point $q \in \partial_\infty AdS_{n+1}$, the vertical line through q meets Σ at a point q'. By Remark 2.7, there is no geodesic arc in AdS_{n+1} joining q to q'. Since D is convex, q' cannot lie on D. In particular, the asymptotic boundary of D coincides with Σ.

To prove that the boundary of D has two components, we notice that every time-like geodesic, say c, through a point $p \in M$ must intersect ∂D in two points which are contained in the future and in the past of M respectively. Indeed, since D is contained in some compact region of $\overline{AdS_{n+1}}$, it turns out that $c \cap D$ is precompact without asymptotic points. By the convexity of D, we have that $c \cap D$ is a compact segment and clearly there is an end-point in the future of M and another end-point in the past of M.

Let us define $\partial_\pm D = \partial D \cap I^\pm(M)$. The previous argument proves that no time-like geodesic can join points of $\partial_+ D$. Since D is convex, points of $\partial_+ D$ are joined by light-like or space-like geodesic arcs. In particular $\partial_+ D$ is an acausal set. By general results (see e.g. [8]) it is a weakly space-like surface (in particular it is a $C^{0,1}$-embedded surface).

In addition, every inextensible time-like path without endpoints must intersect $\partial_+ D$ at some point. By Proposition 3.2 we deduce that $\partial_+ D$ is a weakly space-like graph.

To conclude we have to prove that points in ∂D are connected to Σ by some light-like ray. By the characterization of D given by Lemma 3.7, we have that ∂D is the set of points p such that $\Sigma \subset \overline{U_p}$ and $\Sigma \cap \partial_\infty(P_-(p) \cup P_+(p)) \neq \emptyset$. Take a point y in this intersection. By the convexity of D, the segment c joining x to y (that is light-like) is contained in $\overline{D}$. Points on c are joined to $y \in \Sigma$ by a light-like geodesic, so they cannot be contained in D. In particular $c \subset \partial D$. □

Remark 3.12 Since time-like arcs in D do not contain conjugate points, their length is less than π. In particular, the length of any time-like geodesic segment joining a point of $\partial_- D$ and a point of $\partial_+ D$ is less than π. If there exists a point $q_+ \in \partial_+ D$ and $q_- \in \partial_- D$ such that $\delta(q_-, q_+) = \pi$, then we have $P_-(q_+) = P_+(q_-) = P$ and $\overline{U_{q_+}} \cap \overline{U_{q_-}} = P$. Since Σ is contained in $\overline{U_{q_+}} \cap \overline{U_{q_-}}$, we conclude that $\Sigma = \partial_\infty P$. In this case $D = I^-(q_+) \cap I^+(q_-)$.

Remark 3.13 The closure of D in $\overline{AdS}_{n+1}$ is compact.

Lemma 3.14 *For every $p \in D$ the intersection $\overline{I^+(p)} \cap \overline{D}$ is compact in AdS_{n+1}.*

Proof Since the closure of D in $\overline{AdS}_{n+1}$ is compact, it is sufficient to show that no point in $\partial_\infty AdS_{n+1}$ is an accumulation point for $\overline{D} \cap \overline{I^+(p)}$. However the set of boundary accumulation points of $\overline{I^+(p)}$ is disjoint from U_p, whereas the set of boundary accumulation points for D is Σ, that is contained in U_p. □

Lemma 3.15 *There is a point $p \in D$ such that $D \subset U_p$.*

Proof We first assume there are points $q_+ \in \partial_+ D$ and $q_- \in \partial_- D$ such that $\delta(q_-, q_+) = \pi$. By Remark 3.12, we deduce that $D = I^-(q_+) \cap I^+(q_-)$ and any point on the plane $P_-(q_+) = P_+(q_-)$ satisfies the statement.

Now we consider the case where $\delta(q, q') < \pi$ for $q \in \partial_- D$ and $q' \in \partial_+ D$. We define two functions on D

$$\tau_+(p) = \sup_{q \in D \cap I^+(p)} \delta(p, q), \qquad \tau_-(p) = \sup_{q \in D \cap I^-(p)} \delta(q, p)$$

that are Lipschitz-continuous (see [9]). By Lemma 3.14, for $p \in D$ there is $q_+(p) \in \overline{D}$ such that $\tau_+(p) = \delta(p, q_+(p))$ and analogously there is a point $q_-(p)$ such that $\tau_-(p) = \delta(q_-(p), p)$. Clearly $q_+(p) \in \partial_+ D$ and $q_-(p) \in \partial_- D$.

Notice that by the reverse of triangle inequality we have $\tau_+(p) + \tau_-(p) \leq \delta(q_-(p), q_+(p)) < \pi$. In particular the open sets $\Omega_- = \{\tau_- < \pi/2\}$ and

$\Omega_+ = \{\tau_+ < \pi/2\}$ cover D. Since they are not empty, it follows that there exists a point p such that $\tau_-(p) < \pi/2$ and $\tau_+(p) < \pi/2$, so $D \subset U_p$. □

3.4 From space-like graphs in AdS_3 to diffeomorphisms of $\mathbb{H}^2$

There is a relation between some space-like surfaces in AdS_3^* (satisfying some specific properties) and diffeomorphisms from $\mathbb{H}^2$ to $\mathbb{H}^2$. More specifically, there is a one-to-one relation between maximal graphs in AdS_3^* with negative sectional curvature and minimal Lagrangian diffeomorphisms from the hyperbolic disk to itself. The quasi-conformal minimal Lagrangian diffeomorphisms correspond precisely to the maximal graphs with uniformly negative sectional curvature.

This relation, which is well-known (see [1]), is at the heart of the proof of Theorem 1.4, so we outline its construction and its main properties here, referring to [3, 5, 11, 22, 25] for more details.

Let $S \subset AdS_3$ be a space-like graph. Let I be its induced metric, B its shape (or Weingarten) operator, and let E be the identity map from TS to TS at each point. Denote by J the complex structure of I on S. We can then define two metrics μ_l, μ_r as:

$$\mu_l = I((E + JB)\cdot, (E + JB)\cdot), \qquad \mu_r = I((E - JB)\cdot, (E - JB)\cdot).$$

It is then not difficult to show that both μ_l and μ_r are hyperbolic metrics (see [5, 22])—the reason for this being that $E \pm JB$ satisfies the Codazzi equation, $d^\nabla(E \pm JB) = 0$ on S, and that $\det(E \pm JB) = 1 + \det(B)$ is equal to minus the sectional curvature of the induced metric I on S, which by the Gauss equation in AdS_3 is equal to $-1 - \det(B)$.

However μ_l and μ_r are not necessarily smooth metrics, they might have singularities when $E \pm JB$ is singular, that is—by the determinant computation just mentioned—when $1 + \det(B) = 0$. This means that μ_l and μ_r are smooth hyperbolic metrics whenever the induced metric on S has negative sectional curvature.

There is a nice geometric interpretation of metrics μ_l and μ_r that is based on a specific feature of AdS_3. Every leaf of the left (right) foliation of $\partial_\infty AdS_3$ meets the boundary of any space-like planes exactly at one point. Consider a fixed totally geodesic plane P_0. Given any other plane P there are two natural identifications $\Phi_{P,l}, \Phi_{P,r} : \partial_\infty P \to \partial_\infty P_0$ obtained by following each of the families of lines $\mathcal{L}_l, \mathcal{L}_r$.

By means of the projective model, it can be easily seen that maps $\Phi_{P,l}$ and $\Phi_{P,r}$ extend uniquely to isometries of AdS_3—still denoted by $\Phi_{P,l}$, $\Phi_{P,r}$—sending P to P_0 (see [13, 25] for details).

It is also not difficult to check that replacing P_0 by another geodesic plane does not change $\Phi_{P,l}$ and $\Phi_{P,r}$ up to left composition by some isometry of AdS_3 preserving respectively $\mathcal{L}_l$ and $\mathcal{L}_r$.

Now given any space-like surface S we can define two maps $\Phi_l, \Phi_r : S \to P_0$ as

$$\Phi_l(x) = \Phi_{P(x),l}(x), \qquad \Phi_r(x) = \Phi_{P(x),r}(x),$$

where $P(x)$ is the geodesic plane tangent to S at x. Still in this case, replacing P_0 does not change Φ_l and Φ_r, up to left composition with some isometry of AdS_3 that preserves respectively $\mathcal{L}_l$ and $\mathcal{L}_r$.

The following is a basic remark, see e.g. [22] for a proof—it can actually be checked by a direct computation, by choosing P_0 as the tangent plane at the point x.

Lemma 3.16 *The pull-backs by Φ_l (resp. Φ_r) of the hyperbolic metric on P_0 is precisely the metric μ_l (resp. μ_r).*

A consequence is that Φ_l and Φ_r are non-singular when μ_l, μ_r are non-degenerate metrics, and we have seen that this is the case when $\det(B) \neq -1$. We are therefore lead to consider surfaces with negative sectional curvature (the Gauss formula indicates that the sectional curvature of S is $K = -1 - \det(B)$).

Lemma 3.16, which is a local statement, can be improved, under the condition that S is a space-like maximal graph with negative curvature. Here we call π_l (resp. π_r) the map from $\partial_\infty AdS_3$ to P_0 sending a point $x \in \partial_\infty AdS_3$ to the intersection with P_0 of the line of $\mathcal{L}_l$ (resp. $\mathcal{L}_r$) containing x.

Proposition 3.17 *Suppose that S is a maximal space-like graph with sectional curvature bounded from above by some negative constant. Then Φ_l (resp. Φ_r) is a global diffeomorphism from S to P_0. Φ_l (resp. Φ_r) extends continuously to the closure of S in $\overline{AdS_3}$, and its boundary value is the restriction of π_l (resp. π_r) to $\partial_\infty S$.*

The difficult part to prove is the extension result. We need the following technical lemma that gives a condition for the extension. Unfortunately this lemma does not apply directly to S, but to the surface S^+ of points whose distance from S is $\pi/4$. We then factorize the map Φ_l as the composition of the corresponding map $\Phi_l^+ : S^+ \to P_0$ and a diffeomorphism $\sigma : S \to S_+$ that is given by the normal evolution and that is the identity on the boundary.

Lemma 3.18 *Let S be a space-like surface in AdS_3 with negative curvature whose boundary curve Γ does not contain singular points (that is, $\partial_\infty S$ does not contain any light-like segment). Consider the maps $\Phi_l, \Phi_r : S \to P_0$ described above. Suppose that there is no sequence of points x_n on S such that the totally geodesic planes P_n tangent to S at x_n converge to a light-like plane P whose past end-point and future end-point are not on Γ.*

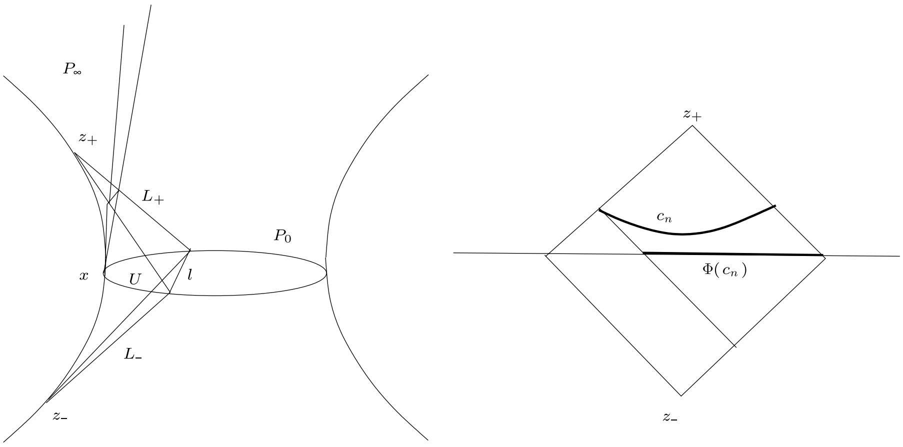


Fig. 1 The rhombus in the proof of Lemma 3.18

Then for any sequence of points $x_n \in S$ converging to $x \in \partial_\infty S$ we have that $\Phi_l(x_n) \to \pi_l(x)$ (resp. $\Phi_r(x_n) \to \pi_r(x)$) in $\overline{P_0}$

Proof We prove that for any sequence $x_n \to x \in \partial_\infty S$ there is a subsequence such that $\Phi_l(x_{n_k})$ converges to $\pi_l(x)$. Indeed, up to passing to a subsequence we can suppose that the totally geodesic plane P_n tangent to S at x_n converges to a plane P_∞. Since x is the limit of points on P_n, it belongs to $\partial_\infty P_\infty$.

We distinguish two cases

(1) P_∞ is space-like;
(2) P_∞ is light-like.

First we deal with the first case. We have that $\Phi_l(x_n) = \Phi_{\underline{P_n},l}(x_n)$. Since $P_n \to P_\infty$ it can be checked that $\Phi_{P_n,l} \to \Phi_{P,l}$ uniformly on $\overline{AdS_3}$ (see [13]). So we have

$$\Phi_l(x_n) \to \Phi_{P_\infty,l}(x) = \pi_l(x).$$

Consider now the case where P_∞ is light-like. By the assumption either the past or the future end-point of P_∞ is contained in $\Gamma = \partial_\infty S$. Since points on Γ are not joined by light-like segments, the intersection between Γ and P_∞ is only this point. Since $x \in \Gamma \cap P_\infty$, we conclude that x is either the past endpoint or the future end-point of P. Up to reversing the time-orientation we can suppose that x is the past end-point of P_∞.

Up to some isometry of AdS_3 preserving the leaves of $\mathcal{L}_l$ we can suppose that $x \in P_0$ so it is sufficient to prove that $\Phi_l(x_n) \to x$.

Consider any geodesic l on P_0 and let U be the half-plane bounded by l containing the point x. We will show that for n large enough $\Phi_l(x_n) \in U$.

The four leaves of $\mathcal{L}_l$ and $\mathcal{L}_r$ passing through the end-points of l bound a rhombus R in $\partial_\infty AdS_3$ containing x in its interior (see Fig. 1). The end-points of l are two opposite vertices of R and there are two other opposite vertices z_- and z_+ such that z_- is the past end-point of both edges adjacent to it and z_+ is the future end-point of both edges adjacent to it.

Since x is the past endpoint of P_∞, this plane intersects the frontier of R in two points, one for each edge with vertex z_+. In particular also $P_n \cap R$ is for n large enough an arc c_n joining two points on the edges adjacent to z_+.

Let L_- be the light-like plane whose past end-point is z_- and L_+ be the light-like plane whose future end-point is z_+. Notice that $V = I^-(L_+) \cap I^+(L_-)$ is a neighbourhood of x in $\overline{AdS_3}$ and the asymptotic boundary of V is exactly R. In particular, for n large enough, $x_n \in V$.

The boundary of L_+ is the union of the two past-directed light-like rays starting from z_+ and L_- is the union of two future-directed light-like rays starting from z_+.

It turns out that $H_n = P_n \cap I^-(L_+)$ is the half-plane on P_n that is the convex hull of c_n. Since c_n is contained in the future of $\partial_\infty L_-$ we have that $H_n \subset I^+(L_-)$. And we conclude that

$$P_n \cap V = H_n.$$

Since for n large enough $x_n \in P_n \cap V$, we have that

$$\Phi_l(x_n) = \Phi_{P_n,l}(x_n) \in \Phi_{P_n,l}(H_n).$$

Now $\Phi_{P_n,l}(H_n)$ is the half-plane of P_0 whose asymptotic boundary is $\pi_l(c_n)$. Notice that $\pi_l(c_n)$ is contained in $\partial_\infty U$ so we have $\Phi_l(x_n) \in \Phi_{P_n,l}(H_n) \subset U$. □

Remark 3.19 If S is a future-convex graph and its boundary does not contain singular points then the condition required in Lemma 3.18 is satisfied. Indeed totally geodesic planes tangent to S are support planes so if we take a sequence of such planes P_n that converges to some light-like plane P_∞, we have that P_∞ cannot intersects S transversely. In particular S is contained in the past of P_∞. This implies that either the boundary of S is disjoint from the boundary of P_∞ or that the past end-point of P_∞ is contained in the boundary of S.

Now if the tangency points x_n of P_n with S converge to some asymptotic point x, clearly $x \in S \cap P_\infty$. Thus, in this case we have that the past end-point of P_∞ is contained in the boundary of S. Since the boundary of S does not contain light-like segments, the point x must coincide with the past end-point of P_∞.

Lemma 3.20 *Let S be a maximal space-like graph with sectional curvature bounded from above by some negative constant. The asymptotic boundary of S does not contain any light-like segment.*

The proof is based on some simple preliminary claims.

Claim 3.21 *Let $S \subset AdS_3$ be a space-like graph with principal curvatures in $(-1, 1)$. Then the equidistant surfaces S_r at (oriented) time-like distance r from S, for all $r \in [-\pi/4, \pi/4]$, are smooth, space-like graphs. If the principal curvatures of S are in $(-1+\epsilon, 1-\epsilon)$, then, for r close enough to $\pi/4$, S_r is past-convex, and S_{-r} is future-convex.*

Proof If $(S_r)_{r\in I}$ is a non-singular foliation of a neighborhood of S by space-like surfaces at constant distance r from S, then the shape operator B_r of S_r satisfies a Riccati type equation relative to r:

$$\frac{dB_r}{dr} = B_r^2 - I,$$

where I is the identity. It follows that the principal curvatures of S evolve as $\tan(r - r_0)$, where r_0 is chosen so that $\tan(r_0)$ is the principal curvature of S at the corresponding point and in the corresponding direction.

Suppose now that S has principal curvatures $k \in (-1+\epsilon, 1-\epsilon)$ at each point, for some $\epsilon > 0$. This implies that, at each point and in each principal direction, $r_0 \in (-\pi/4+\alpha, \pi/4-\alpha)$, where $\alpha > 0$ is another constant. As a consequence, the equidistant foliation (S_r) is well-defined for $r \in [-\pi/4, \pi/4]$, and moreover for $\alpha' < \alpha$ the surfaces $S_{\pi/4-\alpha'}$ and $S_{-\pi/4+\alpha'}$ are smooth and respectively strictly concave and strictly convex, so that the domain

$$\Omega = \bigcup_{r\in[-\pi/4+\alpha', \pi/4-\alpha']} S_r$$

is convex with smooth boundary, with principal curvatures bounded from below by a strictly positive constant. $\square$

Applying Lemma 3.15 to the domain of dependence of S, we deduce that S embeds in the projective model of AdS_3. In particular we can consider its convex hull K, that is the minimal convex set containing S. The width of S—denoted by $w(S)$—is defined as the width of its convex hull, that is supremum of the length of timelike geodesics contained in K.

Corollary 3.22 *Let S be a space-like maximal surface, with sectional curvature bounded from above by a negative constant. Then $w(S) < \pi/2$.*

Proof This follows from the claim because the convex hull of S is contained in Ω, and $w(\Omega) \le \pi/2 - 2\alpha < \pi/2$. $\square$

Claim 3.23 *Suppose that there is a light-like segment in $\partial_\infty S$. Then $w(S) = \pi/2$.*

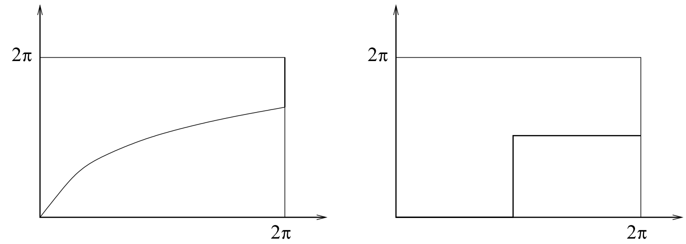


Fig. 2 Deforming a graph to the standard 2-step graph

Proof We consider the surface S embedded in the projective model of the Anti-de Sitter space. The boundary at infinity of S is the graph of a map $u : \mathbb{R}P^1 \to \mathbb{R}P^1$. If $\partial_\infty S$ contains a light-like segment then u is not continuous, and its graph has a "jump", as in the left-hand side of Fig. 2. Composing u on the left with a sequence of projective transformations, we can make its graph as close as wanted (in the Hausdorff topology) from the standard 2-step graph shown on the right-hand side of Fig. 2. (This is achieved by composing u on the right with a sequence of powers of a projective transformation having as attracting fixed point the point where the "jump" occurs.) We call Γ_0 this 2-step graph, considered as a subset of $\partial\pi(AdS_3)$ (here π is the map in the projective model of AdS_3).

Now Γ_0, as a subset of $\partial\pi(AdS_3)$, is composed of four light-like segments. It has four vertices, and it is not difficult to check that the lines Δ and Δ^* connecting the two pairs of opposite points are two dual space-like lines in $\pi(AdS_3)$. In particular, if $CH(\Gamma_0)$ denotes the convex hull of Γ_0, then $w(CH(\Gamma_0)) = \pi/2$ (a more detailed analysis of this situation can be found in [9, Sect. 7.3.3]).

Since $\partial_\infty S$ can be made arbitrarily close to Γ_0 by applying AdS isometries (corresponding to composing u on the left and on the right with projective transformations of $\mathbb{R}P^1$), it follows that $w(S) = \pi/2$. □

Proof of Lemma 3.20 The statement follows directly from Corollary 3.22 and Claim 3.23. □

Let us come back to Proposition 3.17.

Proof of Proposition 3.17 We consider again the surface S_+ of points in the future of S at distance $\pi/4$ from S. We have seen that S_+ is smooth and past-convex. Moreover a diffeomorphism $\sigma : S \to S_+$ is uniquely determined so that the Lorentzian distance between x and $\sigma(x)$ is exactly $\pi/4$.

Since the distance between points on S_+ and points on S is bounded, they share the same boundary. Moreover, since the boundary of S does not contain light-like segments, it can easily seen that the map σ extends to the identity at the boundary.

We claim that the map Φ_l can be factorized as the composition of σ and Φ_l^+, where $\Phi_l^+ : S_+ \to P_0$ is the map constructed in the same way as Φ_l. The claim and Remark 3.19 imply that Φ_l extends to the boundary.

Let us prove the claim. Given any point $x \in S$, we have to check that $\Phi_l(x) = \Phi_l^+(\sigma(x))$. Up to isometry we can suppose that:

- P_0 is the plane tangent to S at x,
- $x = (x^0, 0)$ and P_0 is the horizontal plane.

With this assumption clearly $\Phi_l(x) = x$.

Since the segment joining x to $\sigma(x)$ is orthogonal to both S and S^+, it follows that $\sigma(x) = (x^0, \pi/4)$ and the plane P_+ tangent to S_+ at $\sigma(x)$ is the horizontal plane.

In this case the map $\Phi_{P_+,l}$ can be explicitly computed. In particular it is given by $\Phi_{P_+,l}(y,t) = (R(y), t - \pi/4)$ where $R \in Isom(\mathbb{H}^2)$ is a rotation of angle $\pi/4$ around x^0. It easily follows that $\Phi_l^+(\sigma(x)) = \Phi_{P_+,l}(\sigma(x)) = x$, and this proves the claim.

Notice that the map Φ_l and Φ_r turn to be proper maps. On the other hand, under the hypothesis that S has negative sectional curvature, Φ_l and Φ_r are local diffeomorphisms from S to P_0, so that, by the Dependence of Domain Theorem, they are global diffeomorphism from S to P_0. □

Definition 3.24 Suppose that S has negative sectional curvature. We call $\Phi_S : \Phi_l^{-1} \circ \Phi_r : \mathbb{H}^2 \to \mathbb{H}^2$. Φ_S is a global diffeomorphism, well-defined up to composition by a hyperbolic isometry.

By construction the differential of Φ_S is given at each point by $(E + JB)^{-1}(E - JB)$. It follows that, as long as the principal curvatures of S are in $[-1+\epsilon, 1-\epsilon]$ for some $\epsilon > 0$, the diffeomorphism Φ_S is quasi-conformal (and conversely).

Lemma 3.25 *The map Φ_S extends to a homeomorphism from $\overline{\mathbb{H}^2}$ to $\overline{\mathbb{H}^2}$, and the graph of $\partial\Phi_S : \mathbb{R}P^1 \to \mathbb{R}P^1$ in (the image by π) of AdS_3 is the boundary at infinity of S in $\partial_\infty AdS_3$.*

Proof The extension of Φ_S to the boundary is a direct consequence of its definition and of the extension to the boundary of Φ_l and Φ_r. It is then clear that the graph of $\partial\Phi_S$ is equal to $\partial_\infty S$, since the restrictions of π_l and π_r to $\partial_\infty S$ are equal to the boundary values of Φ_l and Φ_r. □

We have now proved the first two points in Proposition 1.5. To prove the third point it is necessary to construct, given a quasi-conformal minimal Lagrangian diffeomorphism $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$, a maximal space-like S such that $\Phi = \Phi_S$. One way to do this is through the identification of $\mathbb{H}^2 \times \mathbb{H}^2$ with the space of time-like geodesics in AdS_3 (see [5]). We rather use here local arguments (as in [22]).

Let $\Phi : \mathbb{H}^2 \to \mathbb{H}^2$ be a minimal Lagrangian diffeomorphism. Call ρ_l and ρ_r the hyperbolic metrics on the two copies of $\mathbb{H}^2$ (this underlines the relationship with the construction in the previous paragraphs). The fact that Φ is minimal Lagrangian is equivalent (see [23]) to the fact that

$$\Phi^* \rho_r = \rho_l(b\cdot, b\cdot),$$

where b is self-adjoint (for ρ_l), positive, of determinant 1, and satisfies the equation

$$d^{\nabla^l} b = 0,$$

where ∇^l is the Levi-Civita connection of ρ_l and $d^{\nabla^l} b$ is defined (see [10]) as

$$(d^{\nabla^l} b)(x, y) = \nabla^l_x (by) - \nabla^l_y (bx) - b([x, y]).$$

We can then define a metric I on S by

$$4I = \rho_l((E + b)\cdot, (E + b)\cdot). \tag{14}$$

Since b is non-singular and has positive eigenvalues, I is a metric on $\mathbb{H}^2$. Since $d^{\nabla^l} b = 0$ we also have $d^{\nabla^l}(E + b) = 0$, it follows from standard arguments (see e.g. [22]) that the Levi-Civita connection of I is

$$\nabla_x y = (E + b)^{-1} \nabla^l_x ((E + b)y),$$

and therefore that the curvature K of I is equal to

$$K = \frac{K_l}{\det((E + b)/2)} = -\frac{4}{\det(E + b)} = -\frac{4}{2 + \mathrm{tr}(b)}.$$

Let J be the complex structure of I, we now define $B : T\mathbb{H}^2 \to T\mathbb{H}^2$ as follows:

$$JB = (E + b)^{-1}(E - b). \tag{15}$$

Then JB has some remarkable properties.

(1) $d^\nabla JB = 0$. This follows from a direct computation, because $d^{\nabla^l}(E - b) = 0$. Since J is parallel for ∇, it follows that $d^\nabla B = 0$.
(2) JB is self-adjoint for I, because $E - b$ is self-adjoint for ρ_l. It follows that B is traceless.
(3) JB is traceless—this follows from a direct computation in a basis where b is diagonal, using the fact that $\det(b) = 1$. It follows that B is self-adjoint.

(4) $\det(JB) = \frac{\det(E-b)}{\det(E+b)} = \frac{2-\mathrm{tr}(b)}{2+\mathrm{tr}(b)}$. It follows that $K = -1 - \det(B)$.

In other terms, setting $\mathbb{I} = I(B\cdot, \cdot)$, we see that $\mathbb{I}$ satisfies the Gauss and Codazzi equation relative to I. It follows that there exists a (unique) isometric embedding of $(\mathbb{H}^2, I)$ in AdS_3 with second fundamental form $\mathbb{I}$ (and shape operator B).

Equation (15) then shows that $E + JB = 2(2+b)^{-1}$, so that $\mu_l = \rho_l$, and a direct computation shows also that $\mu_r = \rho_r$. If Φ is quasi-conformal then b is bounded, so that the sectional curvature of S is uniformly negative. The first part of this section shows that the graph of $\partial\Phi$ in $\mathbb{R}P^1 \times \mathbb{R}P^1 \simeq \partial_\infty AdS_3$ is equal to the boundary at infinity of S, and this finishes the proof of Proposition 1.5.

4 The existence and regularity of maximal graphs

Given a smooth space-like surface M in AdS_{n+1} we consider the future-oriented normal vector field ν.

The gradient function with respect to the field $T = -\phi\bar{\nabla}t$ is

$$v_M = -\langle \nu, T\rangle.$$

It measures the angle between the hypersurface M and the horizontal slice. Notice that $v_M(x) \geq 1$ for every $x \in M$. If M is the graph of a function u then

$$v_M = \frac{1}{\sqrt{1-\phi^2|\bar{\nabla}u|^2}}.$$

In that case the normal field ν is equal to $\nu = \phi v_M(\nabla u - \nabla t)$.

The shape operator of M is the linear operator of TM defined by

$$B(v) = \bar{\nabla}_v \nu$$

whereas the second fundamental form is defined by $\mathbb{I}(v, w) = \langle v, B(w)\rangle$. The mean curvature, denoted by H, is the trace of B. A space-like surface M is *maximal* if its mean curvature vanishes.

In [6] a general formula for the mean curvature of a space-like graph is given. If M is the space-like graph of a function u we have

$$H = \frac{1}{v_M}\big(\mathrm{div}_M(\phi\,\mathrm{grad}_M u) + \mathrm{div}_M T\big), \tag{16}$$

where div_M is the operator on M defined

$$\mathrm{div}_M X = \sum \langle e_i, \bar{\nabla}_{e_i} X\rangle, \qquad X \in \Gamma(TAdS_{n+1})$$

where e_i is any orthonormal basis.

4.1 Maximal hypersurfaces and convex subsets

We concentrate here on convexity properties of maximal hypersurfaces in AdS_{n+1}.

Lemma 4.1 *Let M be a compact maximal graph. Suppose that there exists a space-like plane P such that ∂M is contained in $I^-(P)$. Then M is contained in $I^-(P)$.*

Proof Suppose by contradiction that a point p_0 of M lies in the future of P. Without loss of generality we can suppose that P is the horizontal plane $\{t = 0\}$ and $p_0 = (x^0, a)$ with $a > 0$. Since M is contained in $I^+((p_0)_-) \cap I^-((p_0)_+)$, by our assumption on the boundary we have that $0 < a < \pi$ and ∂M is contained in the region of points with $-\pi < t < 0$.

Consider the function $u : AdS_{n+1} \to \mathbb{R}$ defined at the point $p = (x, t)$ as

$$u(p) = x_{n+1} \sin(t).$$

By our assumption,

$$u(p) < 0 \quad \text{for every } p \in \partial M. \tag{17}$$

We compute now Δu, where Δ is the Beltrami-Laplace operator of M. Notice that u is the pull-back of the function u^* defined on AdS^*_{n+1} as

$$u^*(y) = \langle y, e\rangle,$$

where $e = (0, \ldots, 0, -1)$. Thus we can suppose that M is immersed in AdS^*_{n+1} and compute Δu^*. Notice that the gradient of u^* is the orthogonal projection of e on M, that is,

$$\nabla u(y) = e + \langle e, y\rangle y + \langle e, \nu^*\rangle \nu^* = e + uy + \langle e, \nu^*\rangle \nu^*,$$

where ν^* is the normal field of M in AdS^*_{n+1}. Since for $v \in T_y M$, $\nabla_v(\nabla u)$ is the tangential part of $\bar{\nabla}_v(\nabla u)$ (where $\bar{\nabla}$ is the standard connection in $\mathbb{R}^{2,2}$) we have

$$\nabla_v(\nabla u^*) = u^* v + \langle e, \nu^*\rangle B(v).$$

Taking the trace we get $\Delta u^* = nu^* + \langle e, \nu^*\rangle H = nu^*$, where the last equality holds since M is maximal. Eventually we have

$$\Delta u = nu.$$

In particular if the maximum of the function u is achieved at some interior point of M, then it must be negative. Since $u(p_0) > 0$ we get a contradiction. □

Definition 4.2 A convex slab of AdS_{n+1} is a convex domain in AdS_{n+1} whose boundary is the union of two acausal graphs.

Let K be a convex slab and M_v and M_u be its boundary components that are graph respectively of functions $v, u : \mathbb{H}^n \to \mathbb{R}$ and suppose that $v < u$. The domain K is

$$\{(x, t)|u(x) \le t \le v(x)\}.$$

The component M_v (resp. M_u) is called the past (resp. future) boundary of K. Notice that the future boundary is past-convex: this means that points of M_v are related by a space-like geodesic that lies in the past of M_v. Analogously M_u is future convex. Since points of a convex slab K can be connected by geodesics, Remark 2.7 implies that the asymptotic boundary of K can intersect each vertical line in $\partial_\infty AdS_{n+1}$ in at most one point. So we have

Corollary 4.3 *If K is a convex slab then its boundary components share the same asymptotic boundary.*

Remark 4.4 Let u and v be two space-like functions defined on $\mathbb{H}^n$ such that M_u is past convex, M_v is future convex and $v(x) < u(x)$. Corollary 4.3 implies that in general the domain $\Omega = \{(x, t)|v(x) < t < u(x)\}$ is not convex. On the other hand it is not difficult to see that if the functions u and v coincide on $\partial\mathbb{H}^n$, then Ω is a convex slab.

Remark 4.5 Let K be a convex slab and D be the domain of dependence of its asymptotic boundary. Then K is contained in $\bar{D}$.

An important property of convex slabs is that a maximal surface whose boundary is contained in a convex slab is completely contained in the slab.

Proposition 4.6 *Let Ω be a convex slab. If M is a compact maximal surface such that ∂M is contained in Ω. Then M is contained in Ω.*

Proposition 4.6 is a direct consequence of Lemma 4.1 and the following lemma.

Lemma 4.7 *Let Ω be a convex slab and let S_-, S_+ denote respectively its past and future boundary. For every $p \in S_-$ (resp. $p \in S_+$) there is a space-like geodesic plane P_p passing through p such that $\Omega \subset I^+(P_p)$ (resp. $\Omega \subset I^-(P_p)$). Moreover we have*

$$\Omega = \bigcap_{p \in S_-} I^+(P_p) \cap \bigcap_{p \in S_+} I^-(P_p).$$

Proof Since Ω is contained in the domain of dependence D of its asymptotic boundary, there is a point p such that $\Omega \subset U_p$. Up to isometry we can suppose that $p = (x^0, 0)$ and consider the projective map

$$\pi^* : U_p \to \mathbb{R}^{n+1}$$

constructed in Sect. 2.5. Since π^* is a projective map, the set $\pi^*(\Omega)$ is convex in $\mathbb{R}^{n+1}$.

Given a point $q \in S_+$ the point $q^* = \pi^*(q)$ lies on the boundary of $\pi^*(\Omega)$, so there is a support plane P^* passing through it. We can consider the plane in U_p equal to $P_q = (\pi^*)^{-1}(P^*)$. This plane passes through q and does not meet the interior of Ω. Since any time-like arc passing through q meets the interior of Ω, the plane P_q is not time-like. In particular, P disconnects AdS_{n+1} in two components that are the future and the past of P_q. Since $q \in S_+$, it turns out that $\Omega \subset I^-(P_q)$. Analogously for $q \in S_-$ we find a plane P_q such that $\Omega \subset I^+(P_q)$.

In particular the inclusion

$$\Omega \subset \bigcap_{p \in S_-} I^+(P_p) \cap \bigcap_{p \in S_+} I^-(P_p)$$

is proved. Now take a point $q \notin \Omega$. Consider a time-like geodesic arc contained in $AdS_{n+1} \setminus \Omega$ such that q is an end-point and the other end-point, say p, lies on $\partial\Omega$. Without loss of generality we can assume $p \in S_+$. In that case it turns out that $q \in I^+(P_p)$, so the reverse inclusion is also proved. □

Lemma 4.8 *Let Σ be a space-like graph in $\partial_\infty AdS_{n+1}$. There is a convex slab $K(\Sigma)$, called the convex hull of Σ, such that*:

- *The asymptotic boundary of $K(\Sigma)$ is Σ.*
- *Every convex slab with boundary Σ contains $K(\Sigma)$.*

Proof Let D be the domain of dependence of Σ and take $p \in D$. Consider the image Σ^* of Σ through the projective map

$$\pi^* : U_p \to \mathbb{R}^{n+1}.$$

Clearly Σ^* is contained in the image, say D^*, of D. In particular the convex hull in $\mathbb{R}^{n+1}$ of Σ^*, say K, is contained in D^*.

We denote by $K(\Sigma)$ the convex set $(\pi^*)^{-1}(K)$. It is clear that Σ is contained in the asymptotic boundary of $K(\Sigma)$. By Corollary 4.3, Σ coincides with the asymptotic boundary of $K(\Sigma)$.

Clearly no support plane of $K(\Sigma)$ can be time-like. Indeed time-like planes disconnect the asymptotic boundary of $K(\Sigma)$. This implies that the boundary of $K(\Sigma)$ in AdS_{n+1} is locally achronal. Moreover it has two components, and each of them disconnects AdS_{n+1} in two components. It follows easily that $K(\Sigma)$ is a convex slab. □

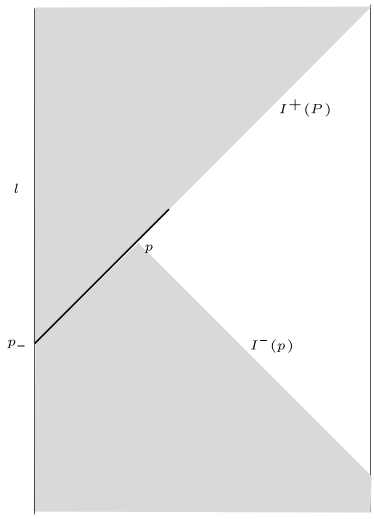
Remark 4.9 The same proof shows that: for a space-like graph M in AdS_{n+1}, there is convex slab, say $K(M)$, such that

- $K(M)$ contains M.
- If K is a convex slab containing M, then $K(M) \subset K$.

The slab $K(M)$ is called the convex hull of M.

Clearly if D is the domain of dependence of Σ we have $K(\Sigma) \subset \overline{D}$. The following statement is an important technical point for what follows. Recall that singular points of Σ are points contained in some light-like segment contained in Σ.

Lemma 4.10 *If Σ is space-like graph in $\partial_\infty AdS_{n+1}$ without singular points, then the boundary components of $K = K(\Sigma)$ do not contain singular points. Moreover, in this case, no point of K is contained in ∂D.*



Proof Suppose that a light-like segment c is contained in $\partial_+ K$. Take a support plane P of $\partial_+ K$ at some point of c. Clearly P is light-like and contains c. For every $p \in c$ notice that

$$I^+(P) \cap \partial_+ K = \emptyset, \qquad \Sigma \subset \overline{U}_p. \tag{18}$$

Let p_- be the past end-point of the light-like geodesic through p contained in P. Let l be the vertical line through p_-. Since Σ is a graph, it must intersect

l at some point. Notice that one component of $l \setminus \{p\}$ is contained in $I^+(P)$ whereas the other component is contained in $I^-(p)$. This remark and (18) show that Σ must intersect l at p_-, that is, $p_- \in \Sigma$.

By a classical theorem on convex sets in Euclidean space (still using the projective map π^* as in Lemma 4.7), $P \cap K(\Sigma)$ is the convex hull of $P \cap \Sigma$. Thus there is another point $q \in P \cap \Sigma$. By Lemma 2.8, we conclude that p_- and q are connected by a light-like segment and this contradicts the assumption that Σ does not contain any singular point.

Eventually, segments joining points of $\partial_+ K(\Sigma)$ to Σ are space-like. By Proposition 3.9 we conclude that no point of $\partial_+ K(\Sigma)$ is contained in ∂D. □

4.2 Existence of entire maximal graph with given boundary condition

Let Σ be a space-like graph in $\partial_\infty AdS_{n+1}$ without singular points. In this section we prove the main theorem on the existence of a maximal graph with given asymptotic boundary.

Theorem 4.11 *There is a maximal graph M in AdS_{n+1} whose boundary at infinity coincides with Σ.*

Let us consider the following notation that we will use through this section:

- D is the domain of dependence of Σ;
- K is the convex hull of Σ;
- S is the future boundary of K;
- B_r is the ball in $\mathbb{H}^n$ centered at x^0 of radius r;
- S_r is the intersection of S with the cylinder $B_r \times \mathbb{R}$.

In [7] (Theorem 4.1) it is shown that there is a maximal surface M_r such that $\partial M_r = \partial S_r$. Moreover M_r is homotopic to S_r (rel. ∂S_r) in the sense that there exists a family of space-like embeddings

$$h_s : S_r \to AdS_{n+1}$$

such that

(1) $h_0 = \mathrm{Id}$, $h_1(S_r) = M_r$;
(2) $h_s(x) = x$ for $x \in \partial S_r$ and $s \in [0, 1]$;
(3) the map $s \mapsto h_s(x)$ is a vertical path for every $x \in S_r$.

It easily follows that M_r is the graph of some function defined on B_r. Putting the previous results together we obtain the following lemma.

Lemma 4.12 *For every $r > 0$, there is a maximal surface M_r such that $\partial M_r = \partial S_r$. Moreover, the surface M_r is a graph of a function u_r defined on B_r and is contained in K.*

The basic idea of the proof of Theorem 4.11 is to construct a sequence $r_k \to +\infty$ such that u_{r_k} converges C^2 on compact subset of $\mathbb{H}^n$. The proof is based on an a-priori gradient estimate, that is a particular case of an estimate proved by Bartnik [7]. Given a point $p \in AdS_{n+1}$ and $\epsilon > 0$ we denote by $I^+_\epsilon(p)$ the set of points in the future of p whose distance from p is at least ϵ.

Lemma 4.13 *Let $p \in AdS_{n+1}$ and $\epsilon > 0$, and let $H \subset I^-(p_+)$ be a compact domain (where p_+ is defined in Sect. 2.3). There is a constant $C = C(p, \epsilon, H)$ such that, for every maximal graph M that verifies the following conditions:*

- $\partial M \cap I^+(p) = \emptyset$,
- *$M \cap I^+(p)$ is contained in H,*

we have that

$$\sup_{M \cap I^+_\epsilon(p)} v_M < C$$

where v_M is the gradient function of M.

Proof Let us consider the time-function

$$\tau(x) = \delta(x, p) - (\epsilon/2)$$

where $\delta(x, p)$ is the Lorentzian distance between x and p. This function is smooth on the domain $\mathcal{V} = H \cap I^+(p)$.

Notice that by the assumption on M, the region $M \cap \mathcal{V}$ contains the region of M where $\tau \geq 0$ and $M \cap I^+_\epsilon(p)$ is contained in $\mathcal{V}$.

We can apply Theorem 3.1 of [7] and conclude that

$$\sup_{M \cap I^+_\epsilon(p)} v_M < C$$

where C depends on the C^2-norms of t and τ and on the C^0 norm of Ric, taken on the domain $\mathcal{V}_{\tau \geq 0}$ with respect to a reference Riemannian metric. □

We can prove now Theorem 4.11.

Proof of Theorem 4.11 For every point $p \in D \cap I^-(\partial_- K)$ we choose $\epsilon = \epsilon(p)$ such that the family $\{I^+_{\epsilon(p)}(p) \cap K\}_{p \in D \cap I^-(\partial_- K)}$ is an open covering of K.

Given a number R, the intersection $(B_R \times \mathbb{R}) \cap K$ is compact, so there is a finite numbers of points $p_1, \ldots, p_{k_0} \in D \cap I^-(\partial_- K)$ such that putting $\epsilon_k = \epsilon(p_k)$ we have

$$(B_R \times \mathbb{R}) \cap K \subset \bigcup_1^{k_0} I^+_{\epsilon_k}(p_k).$$

For all $k \in \{1, \dots, k_0\}$, $p_k \in D$, so that the intersection $\overline{I^+(p_k) \cap D}$ is compact. Moreover, $D \subset I^-((p_k)_+)$. It follows that the set $H_k = \overline{I^+(p_k)} \cap K$ is compact and contained in $I^-((p_k)_+)$.

By Lemma 4.13, there is a constant C_k, such that

$$\sup_{M \cap I^+_{\epsilon_k}(p_k)} v_M < C_k$$

for every maximal surface M that satisfies the following requirements:

- $\partial M \cap I^+(p_k) = \emptyset$;
- $M \cap I^+(p_k)$ is contained in H_k.

By the compactness of $I^+(p_k) \cap D$, there is $r_0 > 0$ such that

$$I^+(p_k) \cap D \subset B_{r_0} \times \mathbb{R}$$

for $k = 1, \dots, k_0$.

Let $\{M_r\}$ be the family of maximal surfaces constructed in Lemma 4.12. Then $M_r \subset K$. Moreover there exists $r_0 > 0$ such that, for $r > r_0$, $\partial M_r \cap I^+(p_k) = \emptyset$ for $k = 1, \dots, k_0$.

It follows that $\sup_{M_r \cap I^+_{\epsilon_k}(p_k)} v_{M_r} \le C_k$ for $k = 1, \dots, k_0$. Since $M_r \cap (B_R \times \mathbb{R}) \subset \bigcup_k I^+_{\epsilon_k}(p_k)$ we conclude that

$$\sup_{M_r \cap (B_R \times \mathbb{R})} v_{M_r} \le \max\{C_1, \dots, C_{k_0}\} \tag{19}$$

for every $r > r_0$.

Eventually we deduce that for every R there is a constant $C(R)$ such that the gradient function of v_{M_r} is bounded by $C(R)$ for r sufficiently big.

Take now any divergent sequence r_i. Let u_i be the function defined on B_{r_i} such that $M_{r_i} = M_{u_i}$. By comparing (16) with estimate (19), we see that the restriction of u_i on B_R is solution of a uniformly elliptic quasi-linear operator on B_R, with bounded coefficients.

Since $|u_i|$ and $|\bar{\nabla} u_i|$ are uniformly bounded on B_R, by elliptic regularity theory (see e.g. [19]) the norms of u_i in $C^{2,\alpha}(B_{R-1})$ are uniformly bounded. It follows that the family u_i is precompact in $C^2(B_{R-1})$.

By a diagonal process we extract a subsequence u_{i_h} converging to a function u_∞ defined on $\mathbb{H}^n$ in such a way that the convergence is C^2 on compact sets. Since the u_{i_h} are uniformly space-like, so is u_∞. Moreover, since it is the C^2 limit of solutions of (16), it is still a solution.

As a consequence, the graph of u, say M, is a maximal graph. Since M is a limit of surfaces contained in K, it is contained in K. In particular the asymptotic boundary of M is contained in Σ, and so it coincides with Σ. $\square$

4.3 Regularity of maximal hypersurfaces

We will now show that if the distance between K and the past boundary of D is strictly positive, then any maximal surface contained in K has bounded second fundamental form.

Theorem 4.14 *Suppose that there exists $\epsilon > 0$ such that, for every $y \in \partial_- K$, there exists a point $x \in \partial_- D$ such that $\delta(x, y) \geq \epsilon$. Then there exists a constant $C > 0$, depending on ϵ, such that the second fundamental form of any maximal graph contained in K is bounded by C.*

To prove this theorem we will need the following relation between the boundaries of D and K. The first part of the lemma will be used in the proof of Theorem 4.14, while the second part will be necessary below.

Lemma 4.15 *Let $\Sigma \subset \partial_\infty AdS_{n+1}$ be space-like graph, let $K = K(\Sigma)$ be its convex hull, and let $D = D(\Sigma)$ be its domain of dependence. Then:*

(1) *For all $q \in K$ and $p \in \partial_- D \cap I^-(q)$ we have that $\delta(p, q) \leq \pi/2$.*
(2) *For all $q \in \partial_+ K$ there exists $p \in \partial_- D \cap I^-(q)$ such that $\delta(p, q) = \pi/2$.*

The proof of the first point in dimension $2 + 1$ can be found in [9]. That argument actually applies in every dimension. For the sake of completeness we sketch the argument here.

Proof Since $p \in \partial_- D$, Σ is contained in $\overline{U}_p$ and $\Sigma \cap (P_+(p) \cup P_-(p)) \neq \emptyset$.

Notice that the plane $P_+(p)$ does not disconnect Σ, so, it is a support plane for K. In particular $K \subset \overline{I^-(P_+(p))}$. This implies that the distance of every point of $K \cap I^+(p)$ from p is bounded by $\pi/2$, and proves the first point. Moreover, since $P_+(p)$ is a support plane of K, its intersection with $\partial_+ K$ is non-empty. But for any point $q \in P_+(p)$ we have $\delta(p, q) = \pi/2$, and this proves the second point. □

As a consequence we find a bound on the width of the boundary at infinity of a space-like graph in AdS_{n+1}. This estimate is improved for $n = 2$ when the boundary at infinity is the graph of a quasi-symmetric homeomorphism, see Theorem 1.12.

Lemma 4.16 *Let $M \subset AdS_{n+1}$ be a space-like graph. Then $w(\partial_\infty M) \leq \pi/2$.*

We can now prove Theorem 4.14.

Proof of Theorem 4.14 We consider $q_0 = (x^0, 0)$ and consider the horizontal plane P_0 passing though $(x^0, \pi/2 - \epsilon/2)$, and define $H_0 = \overline{I^+(q_0) \cap I^-(P_0)}$.

From Lemma 4.13, we find a constant C (depending on ϵ) such that

$$\sup_{N\cap I^+_{\epsilon/3}(q_0)} v_N < C$$

for every maximal surface N such that

(1) $\partial N \cap I^+(q_0) = \emptyset$,
(2) $N \cap I^+(q_0) \subset H_0$.

Moreover, by applying the elliptic regularity theory as in the proof of Theorem 4.11, we see that there is another constant, still denoted by C, such that

$$\sup_{N\cap I^+_{\epsilon/2}(q_0)} |A|^2 < C$$

for the same class of maximal surfaces.

Now consider a point p on the maximal surface M. By the assumption there is a point $p_0 \in \partial_- D$ such that $\delta(p, p_0) > \epsilon$. We can fix a point q on the segment $[p_0, p]$ such that $\delta(p,q) > \epsilon/2$. Since $I^+(q) \cap K$ is compact, there is a point $r \in \partial_+ K$ that maximizes the distance from q. Lemma 4.15 and the reverse triangle inequality imply that $\bar{s} := \delta(q,r) < \pi/2 - \epsilon/2$. Moreover the plane passing through r and orthogonal to the segment $[q,r]$ is a support plane P for K (that is $K \subset \overline{I^-(P)}$).

Now consider an isometry γ of AdS_{n+1} such that $\gamma(q) = (x^0, 0)$ and $\gamma(r) = (x^0, \bar{s})$. We have that $\gamma(P)$ is the horizontal plane through $(x^0, \bar{s})$. Since $\bar{s} < \pi/2 - \epsilon/2$, $\gamma(P) \subset I^-(P_0)$. Thus, $\gamma(K) \subset I^-(P_0)$, and $\gamma(M) \cap I^+(q_0) \subset H_0$.

In particular $\gamma(M)$ satisfies the conditions (1), (2) above and we conclude that

$$\sup_{\gamma(M)\cap I^+_{\epsilon/2}(q_0)} |\tilde{A}|^2 < C$$

where $\tilde{A}$ denotes the second fundamental form of $\gamma(M)$. Since $\gamma(p) \in I^+_{\epsilon/2}(q_0)$ we conclude that

$$|A|^2(p) = |\tilde{A}|^2(\gamma(p)) < C$$

where the constant C is independent of the point p. □

Corollary 4.17 *Suppose that $w(K) < \pi/2$. Then there exists $C > 0$ such that any maximal space-like graph in K has second fundamental form bounded by C.*

Proof Let $\epsilon = \pi/2 - w(K)$, so that $\epsilon > 0$. Let $y \in \partial_- K$. Consider a point $z \in \partial_+ K \cap I^+(y)$ for which $\delta(y,z)$ is maximal. Then $\delta(y,z) \le w(K)$ by definition of w.

Let now Δ be the past-oriented time-like geodesic ray starting from z and containing y, and let x be its intersection with $\partial_- D$. By the definition of z, the space-like plane orthogonal to Δ at z is a support plane of K (otherwise z would not maximize $\delta(y,\cdot)$ on $\partial_+ K$).

This shows that z is also a critical point of $\delta(x,\cdot)$ on $\partial_+ K$ and, since K is convex, it is a maximum of this function on $\partial_+ K$. Therefore $\delta(x,z) = \pi/2$ by the second point of Lemma 4.15. Therefore $\delta(x,y) \ge \epsilon$. So we can apply Theorem 4.14, which yields the result. □

5 Uniqueness of maximal surfaces in AdS_3

We consider in this section the uniqueness of maximal graphs with given boundary at infinity and bounded second fundamental form in AdS_3. The argument has two parts. The first is to show that those surfaces have negative sectional curvature. The second part is to show that the existence of such a negatively curved maximal space-like graph forbids the existence of any other maximal graph with the same boundary. Both parts use a version "at infinity" of the maximum principle, for which a compactness argument is needed. For the first part we need a simple compactness statement on sequences of maximal surfaces.

5.1 A compactness result for sequences of maximal hypersurfaces

The following statement will allow us below to use the maximum principle "at infinity".

Lemma 5.1 *Choose $C > 0$, a point $x_0 \in AdS_{n+1}$, and a future-oriented unit time-like vector $n_0 \in T_{x_0}AdS_{n+1}$. There exists $r_0 > 0$ as follows. Let P_0 be the space-like hyperplane orthogonal to n_0 at x_0, let D_0 be the disk of radius r_0 centered at x_0 in P_0, and let $(S_n)_{n\in\mathbb{N}}$ be a sequence of maximal space-like graphs containing x_0 and orthogonal to n_0, with second fundamental form bounded by C. After extracting a sub-sequence, the restrictions of the S_n to the cylinder above D_0 converge C^∞ to a maximal space-like disk with boundary contained in the cylinder over ∂D_0.*

The proof given here applies with a few modifications to the more general context of maximal (resp. minimal) immersions of hypersurfaces in any Lorentzian (resp. Riemannian) manifold with bounded geometry, we state the lemma in AdS_{n+1} for simplicity.

Proof For all n, the surface S_n is the graph of a function f_n over P_n. The bound on the second fundamental form of S_n, along with the fact that the S_n are orthogonal to n_0, indicates that, for some $r > 0$, the derivative of f_n is bounded on the disk of center x_0 and radius r, more precisely there exists $\epsilon > 0$ such that

$$\phi \|\nabla f_n\| < 1 - \epsilon$$

on this disk of center x_0 and radius r.

This, along with the bound on the second fundamental form of S_n (again) shows that the Hessian of f_n is bounded by a constant depending on r (for r small enough). Thus we can extract from $(f_n)_{n\in\mathbb{N}}$ a subsequence which is $C^{1,1}$ converging to a function f_∞ on the disk of center x_0 and radius r. Moreover the gradient of f_∞ is uniformly bounded, so that the graph of f_∞ is a disk which is uniformly space-like.

By definition the f_n are solutions of (16), which just translates analytically the fact that their graphs are maximal surfaces. Since f_∞ is a $C^{1,1}$-limit of the f_n, it is itself a weak solution of (16). Since (16) is quasi-linear, it then follows from elliptic regularity that f_∞ is C^∞, and that (f_n) is C^∞-converging to f_∞ (see [19]). This means that the restriction of the S_n to the cylinder above the disk of radius r_0 in P_0, for some $r_0 > 0$ (depending only on C) converge to a limit which is a maximal surface, the graph of f_∞ over the disk of radius r_0. □

5.2 Maximal surfaces with bounded second fundamental form

The first proposition of this section is the following, its proof is based on Lemma 5.1. Note that from this point on we will often consider space-like graphs in the projective model of AdS_3.

Proposition 5.2 *Let S be a complete maximal surface in AdS_3. Suppose that the norm of the fundamental form of S is bounded. Then S either has negative sectional curvature, or S is flat. If the supremum of the sectional curvature of S is* 0, *then* $w(\partial_\infty S) = \pi/2$.

The completeness mentioned here is with respect to the induced metric on S. The proof uses two preliminary statements. The first is taken from [22], where it can be found in the proof of Lemma 3.11, p. 214. Note that the sign of the Laplacian used here is defined so that Δ is negative as an operator acting on L^2.

Lemma 5.3 *Let Σ be a maximal space-like surface in a* 3*-dimensional AdS manifold. Let B be its shape operator, and let* $\chi = \log(-\det(B))/4$. *Then χ satisfies the equation*

$$\Delta \chi = e^{4\chi} - 1.$$

As a consequence, we can apply the maximum principle to χ, it shows that χ cannot have a positive local maximum. This can be translated into a statement on K, using the Gauss formula, which shows that $K = -1 + e^{4\chi}$.

Lemma 5.4 *Suppose that K has a local maximum at a point where it is non-negative. Then $K = 0$ at that point, and on the whole surface S, so that S is flat (in the intrinsic sense).*

We need another elementary statement, characterizing the maximal surfaces with flat induced metric in AdS_3. We include the proof for the reader's convenience.

Lemma 5.5 *Let Σ be a space-like maximal surface in AdS_3, with zero sectional curvature. Then Σ is a subset of a "horosphere", that is, its principal curvatures are -1 and 1, and its lines of curvature form two orthogonal foliations by parallel lines. If Σ is a space-like graph, then its boundary at infinity is the union of four light-like segments in $\partial_\infty AdS_3$.*

Proof Since Σ is maximal, its principal curvatures are at each point two opposite numbers, k and $-k$. The Gauss formula asserts that the sectional curvature of Σ is $K = -1 + k^2$, so $k = 1$. Let (e_1, e_2) be an orthonormal frame of unit principal vectors on Σ_0, and let $\mathbb{I}$ be the second fundamental form of Σ. The Codazzi equation can be written as follows, at any point $m \in \Sigma$, for any vector field x on Σ such that $\nabla x = 0$ at m:

$$I((d^\nabla B)(e_1, e_2), x) = e_1.\mathbb{I}(e_2, x) - e_2.\mathbb{I}(e_1, x) - \mathbb{I}([e_1, e_2], x) = 0.$$

If ω is the connection form for the frame (e_1, e_2), a simple computation (using that ∇x vanishes at m) shows that, at m,

$$e_1.\mathbb{I}(e_2, x) - e_2.\mathbb{I}(e_1, x) = \mathbb{I}(\omega(e_1)e_1 + \omega(e_2)e_2, x) = -\mathbb{I}([e_1, e_2], x).$$

Since $\mathbb{I}$ is non-degenerate, it follows that $\omega(e_1)e_1 + \omega(e_2)e_2 = 0$.

Therefore e_1 and e_2 are both parallel vector fields, and the first part of the statement follows.

There is a simple way to describe such a horosphere. Consider a space-like line Δ in AdS_3, and the set Σ_0 of endpoints of the future-oriented time-like segments of length $\pi/4$ starting from Δ. An explicit computation (as in the proof of Proposition 5.2 below) shows that Σ_0 is precisely a horosphere as described above. The action of the isometry group of AdS_3 shows that there exists a unique surface of this type passing through each point x of AdS_3, with fixed (time-like) normal and fixed principal direction at x for the principal curvature $+1$, so any maximal graph with zero sectional curvature is of this type.

Let Δ^* be the line dual to Δ, that is, the set of endpoints of future-oriented time-like segments of length $\pi/2$ starting from Δ (see Sect. 2.5). Now let $\partial\Sigma_0$ be the boundary at infinity of Σ_0. Considering the projective model of AdS_3 shows that $\partial\Sigma_0$ contains the endpoints at infinity Δ_- and Δ_+ of Δ, and also the endpoints at infinity of Δ^*_+ and Δ^*_- of Δ^*. Since $\partial\Sigma_0$ is a nowhere time-like curve in $\partial_\infty AdS_3$, it is necessarily made of the four segments from Δ_+ to Δ^*_+, from Δ^*_+ to Δ_-, from Δ_- to Δ^*_-, and from Δ^*_- to Δ_+, which are all light-like. This proves the last part of the lemma. □

Proof of Proposition 5.2 Since S has bounded second fundamental form, its sectional curvature K is bounded, we call K_S the upper bound of K on S. Lemma 5.4 already shows that if this upper bound is attained on S, then it is non-positive, and if it is equal to 0 then S is flat. We will use Lemma 5.1 to extend this argument to the case where the upper bound K_S is not attained.

Consider a sequence $(s_n)_{n\in\mathbb{N}}$ of points in S such that $K_S - 1/n < K(s_n) < K_S$, and apply to S a sequence of isometries $(\phi_n)_{n\in\mathbb{N}}$ which sends s_n to a fixed point x_0 and the oriented unit normal vector to S at s_n to a fixed vector n_0. Since S has bounded second fundamental form, Lemma 5.1 shows that we can extract from the sequence $(\phi_n(S))_{n\in\mathbb{N}}$ a subsequence which converges, in the neighborhood of x_0, to a maximal space-like graph S_0. By construction the curvature of S_0 has a local maximum at x_0, and this local maximum is equal to K_S. Lemma 5.4 therefore shows that $K_S \le 0$.

Suppose now that $K_S = 0$. Then the sequence $\phi_n(S)$ converges, in a neighborhood of x_0, to a "horosphere" Σ_0, as described in Lemma 5.5. Lemma 5.1 shows that the convergence is C^∞ in compact subsets of AdS_3. Let E_n be the boundary at infinity of $\phi_n(S)$. Since $\phi_n(S)$ is space-like, E_n is a nowhere time-like curve in $\partial_\infty AdS_3$. By construction, $E_n = (\rho_{l,n}, \rho_{r,n})E$, where $E = \partial_\infty S$, $(\rho_{l,n})$ and $(\rho_{r,n})$ are two sequences of elements of $PSL_2(\mathbb{R})$, and, for all $n \in \mathbb{N}$, $(\rho_{l,n}, \rho_{r,n})$ is considered as an isometry acting on AdS_3 through the natural identification (see Sect. 2.6 or [3, 25]).

By Lemma 3.1 (more precisely the fact that space-like hypersurfaces in AdS_{n+1} are the graphs of 2-Lipschitz functions), since $\phi_n(S)$ converges on compact subsets of AdS_3 to Σ_0, E_n converges to the boundary at infinity of Σ_0, which we call E_0. In particular, using the notations in the proof of Lemma 5.5, for each $n \in \mathbb{N}$ there are four points $x_n^+, x_n^-, x_n^{+*}, x_n^{-*} \in E_n$ which can be chosen so that $x_n^+ \to \Delta_+$, $x_n^- \to \Delta_-$, $x_n^{+*} \to \Delta^*_+$ and $x_n^{-*} \to \Delta^*_-$.

Therefore, for n large enough, there are points y_n, z_n which are arbitrarily close to Δ and to Δ^* respectively, with (y_n) and (z_n) converging to limits respectively in Δ and to Δ^*. The distance between the limits is $\pi/2$, so that the distance between y_n and z_n goes to $\pi/2$ as $n \to \infty$, this shows that $w(K) = \pi/2$. □

5.3 Quasi-symmetric homeomorphisms and the width

There is another important relation which is valid only in AdS_3, as stated in the next proposition.

Proposition 5.6 *Let E be a weakly space-like graph in $\partial_\infty AdS_3$ (that is, E is a weakly space-like curve). Let K be the convex hull of E. Suppose that $w(K) = \pi/2$. Then E is not the graph of a quasi-symmetric homeomorphism from $\mathbb{R}P^1$ to $\mathbb{R}P^1$ (see e.g. [28, Sect. 2.1]).*

Proof We suppose that $w(K) = \pi/2$, it follows that there exist two sequences of points (x_n) in $\partial_- K$ and (y_n) in $\partial_+ K$ such that $\delta(x_n, y_n) \to \pi/2$. We can suppose (replacing x_n and y_n by points in the same face of ∂K if necessary) that x_n is contained in a space-like geodesic $\Delta_n \subset \partial_- K$, and that y_n is contained in a space-like geodesic $\Delta'_n \subset \partial_+ K$.

We can find a sequence (ϕ_n) of isometries of AdS_3 such that $\phi_n(x_n) \to x$, $\phi_n(y_n) \to y$, with $\delta(x, y) = \pi/2$. Moreover, $\phi_n(K)$ is the convex hull $\phi_n(E)$. Since the $\phi_n(K)$ are convex, they converge (perhaps after extracting a subsequence) in the Hausdorff topology to a limit K_0, which is the convex hull of $E_0 = \lim \phi_n(E)$. Moreover, extracting a subsequence again if necessary, we can suppose that $\phi_n(\Delta_n) \to \Delta$ and that $\phi_n(\Delta'_n) \to \Delta'$. Since $x \in \Delta$, $y \in \Delta'$, and $\delta(x, y) = \pi/2$, $\Delta' = \Delta^*$, otherwise the width of K_0 would have to be strictly larger than $\pi/2$, contradicting Lemma 4.16.

Then E_0 contains the endpoints Δ_-, Δ_+ of Δ, and the endpoints Δ^*_-, Δ^*_+ of Δ^*. Since E is weakly space-like, so is E_0, so it is the union of four light-like segments joining those four points.

Since E_0 is composed of four light-like segments (with endpoints Δ_+, Δ^*_+, Δ_- and Δ^*_-) there are points u, v and u', v' in $\mathbb{R}P^1$, with $u \neq v$ and $u' \neq v'$, such that, in the identification of $\partial_\infty AdS_3$ with $\mathbb{R}P^1 \times \mathbb{R}P^1$, $\Delta_+ = (u, u')$, $\Delta^*_+ = (u, v')$, $\Delta_- = (v, v')$, and $\Delta^*_- = (v, u')$.

So E_0 is the graph of the function $f_0 : \mathbb{R}P^1 \to \mathbb{R}P^1$ sending (u, v) to v' and (v, u) to u'. After composing on the right and on the left with projective transformations, we can suppose that it is the graph of the function $f_0 : \mathbb{R}P^1 \to \mathbb{R}P^1$ sending $(0, 2)$ to 0 and $(2, \infty] \cup [-\infty, 0)$ to 1.

Consider the points $-3, -1, 1, \infty \in \mathbb{R}P^1$. A direct computation shows that their cross-ratio is $[-3, -1; 1, \infty] = 2$, while the cross-ratio of their images by f_0 is $[0, 0; 1, 1] = 1$.

It follows that there are 4-tuples of points on $\phi_n(E)$ whose projection by π_l are 4-tuples of points with cross-ratio arbitrarily to 2 and whose projection by π_r are 4-tuples of points with cross-ratio arbitrarily close to 1. This means precisely, by definition of a quasi-symmetric homeomorphism, that E is not the graph of a quasi-symmetric homeomorphism. $\square$

5.4 Uniqueness of negatively curved maximal surfaces

We now turn to the second proposition of this section, the fact that maximal space-like graphs with negative sectional curvature are uniquely determined, among all maximal space-like graphs, by their boundary at infinity.

Proposition 5.7 *Let S be a maximal graph in AdS_3, with sectional curvature bounded from above by a negative constant. Then S is unique among complete maximal graphs with given boundary curve at infinity and bounded second fundamental form.*

We first state a preliminary lemma (see also Lemma 4.8).

Lemma 5.8 *Let $u : \mathbb{R}P^1 \to \mathbb{R}P^1$ be a homeomorphism, and let $E_u \subset \mathbb{R}P^1 \times \mathbb{R}P^1 \simeq \partial\pi(AdS_3)$ be its graph. Let $C(E_u)$ be the convex hull of E_u. Then any maximal surface in AdS_3 with boundary at infinity E_u is contained in $C(E_u)$.*

Proof Let $S \subset AdS_3$ be a maximal surface, with boundary at infinity E_u. The image of S in the projective model of AdS_3 is a saddle surface, that is, a surface which has opposite principal curvatures at each point. A characterization of saddle surfaces (see [15, Sect. 6.5.1]) is that, for any relatively compact subset $G \subset S$, then G is contained in the convex hull of ∂G. This property, applied to an exhaustion of the image of S in the projective model by compact subsets, is precisely what we need. □

Proof of Proposition 5.7 We consider the domain Ω introduced in the proof of Claim 3.21, as the set of points at time-like distance at most $\pi/4$ from S. Claim 3.21 shows that Ω is convex, with smooth,space-like boundary.

Consider now another maximal graph $S' \subset AdS_3$, complete, with the same boundary at infinity as S, and with bounded second fundamental form. By construction the boundary of Ω is equal to E. Since Ω is convex, it contains the convex hull of E and therefore, by Lemma 5.8, it contains S'. Let r_1 be the supremum over S' of the distance to S. The argument above shows that $r_1 \in [0, \pi/4 - \alpha)$, and the maximum principle shows that, if $r_1 > 0$, then it cannot be attained at an interior point of S', since then S' would have to be tangent from the interior of S_{r_1}, which would contradict the maximality of S'.

Since S' is complete, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points in S' such that $d(x_n, S) \to r_1$ and that the norm of the differential at x_n of the restriction to S' of the distance to S goes to zero as $n \to \infty$ (this is a very weak form of a lemma appearing e.g. in [30]).

Consider a sequence of isometries $(\phi_n)_{n\in\mathbb{N}}$ chosen such that $\phi_n(x_n)$ is equal to a fixed point x_0, and that the normal to $\phi_n(S')$ at $\phi_n(x_n)$ is a fixed vector n_0. Lemma 5.1 shows that, after extracting a sub-sequence, $(\phi_n(S'))_{n\in\mathbb{N}}$

converges in a neighborhood of x_0 to a smooth, maximal surface S'_∞. Moreover, since the differential at x_n of the distance to S goes to zero, the images by ϕ_n of S also converge to a limit S_∞, in a neighborhood of its intersection with the normal to S'_∞ at x_0.

We can now apply the maximum principle to the distance to S'_∞ as a maximal surface in the foliation by the surfaces equidistant to S_∞, and obtain a contradiction if $r_1 > 0$. So $r_1 = 0$, and $S' = S$. □

Together with Proposition 5.2 and Proposition 5.6, Proposition 5.7 leads directly to a simple consequence.

Corollary 5.9 *Let S be a maximal graph in AdS_3, with bounded second fundamental form. Suppose that the boundary at infinity of S is the graph E of a quasi-symmetric homeomorphism from $\mathbb{R}P^1$ to $\mathbb{R}P^1$. Then S is the unique maximal surface with boundary at infinity E and bounded second fundamental form.*

6 Proof of the main results

6.1 A characterization of quasi-symmetric homeomorphisms

We now prove Theorem 1.12. Let $u : \mathbb{R}P^1 \to \mathbb{R}P^1$ be a homeomorphism, and let E_u be its graph. We already know, from Lemma 4.16, that $w(E_u) \leq \pi/2$. Moreover Proposition 5.6 shows that if u is quasi-symmetric, then $w(E_u) < \pi/2$.

Suppose conversely that $w(E_u) < \pi/2$. We can apply Theorem 4.11 to E_u, and obtain a maximal graph M in AdS_3 with boundary at infinity equal to E_u. Corollary 4.17 shows that M has bounded second fundamental form.

Proposition 5.2 then shows that M has sectional curvature bounded from above by a negative constant. Therefore we obtain through Proposition 1.5 a minimal Lagrangian quasi-conformal diffeomorphism ϕ with boundary value equal to u. Since ϕ is quasi-conformal, u is quasi-symmetric, as claimed.

6.2 Theorems 1.4 and 1.10

Theorem 1.4 clearly follows, through Proposition 1.5, from Theorem 1.10, so we now concentrate on this last statement.

Proof of Theorem 1.10 Let $E = \partial_\infty S \subset \partial_\infty AdS_3$, and let M be the maximal graph with boundary at infinity E which is provided by Theorem 4.11. Since E is the graph of a quasi-symmetric homeomorphism, Proposition 5.6 shows that $w(E) < \pi/2$.

The argument in the previous paragraph then shows that E is the boundary at infinity of a maximal graph M in AdS_3, which has bounded second fundamental form by Theorem 4.14. Then Proposition 5.2 shows that M has sectional curvature bounded from above by a negative constant. Proposition 5.7 can therefore be used to obtain that M is unique among maximal graphs with boundary at infinity E and bounded second fundamental form. □

Appendix: Mean curvature flow for space-like graphs

In this section we prove a longtime existence solution for the mean curvature flow of space-like graphs in AdS_{n+1}. The proof is based on Ecker's estimates [16], that are the parabolic analogous of Bartnik's estimates we have used in Lemma 4.13. This argument provides an alternate proof of the existence and regularity of maximal surfaces with given asymptotic boundary already proved in Sect. 4.

We recall that a mean curvature flow of a space-like surface is a family of space-like embeddings $\sigma_s : M \to AdS_{n+1}$ such that

$$\frac{\partial \sigma}{\partial s}(x, s) = H(x, s)\nu(x, s) \tag{20}$$

where $H(x, s)$ and $\nu(x, s)$ are respectively the mean curvature and the normal vector of the surface $M_s = \sigma_s(S)$ at point $\sigma_s(x)$.

We also consider the case where M is compact with boundary. In that case we always consider the Dirichlet condition

$$\sigma_s(x) = \sigma_0(x) \quad \text{for all } x \in \partial M. \tag{21}$$

Lemma A.1 *Let $(M_s)_{s\in[0,s_0]}$ be a family of space-like surfaces moving by mean curvature flow. If M_0 is a graph of a function u_0 defined on some domain Ω of $\mathbb{H}^n$ with smooth boundary, then so is M_s for every $s \in [0, s_0]$.*

Moreover, if $u_s : \Omega \to \mathbb{R}$ is the function defining M_s then

$$\frac{\partial u}{\partial s} = \phi^{-1} v^{-1} H \tag{22}$$

where v is the gradient function on M_s.

Proof Since M_s is homotopic to M_0 through a family of space-like surfaces with fixed boundary, then M_s is contained in the domain of dependence of M_0 that, in turn, is contained in $\Omega \times \mathbb{R}$.

Moreover, M_s disconnects $\Omega \times \mathbb{R}$ in two regions. The same argument as in Proposition 3.2 shows that M_s is a graph on Ω of a function u_s.

The evolution equation of u_s is computed in [17]. □

Remark A.2

(1) Notice that $\frac{\partial(t\circ\sigma)}{\partial s} = \phi^{-1}vH$, that is different from (22). The reason is that the curve $\sigma(x,\cdot)$ at some point s is tangential to the normal of M_s, so in general it is not a vertical line. This implies that the function u_s agrees with $t|_{M_s}$ only up some tangential diffeomorphism of M_s.

(2) Equation (22) is equivalent, up to tangential diffeomorphisms, to (20). This means that if $(u_s)_{s\in[0,s_0]}$ is a solution of (22), there is a time-dependent field X_s on Ω such that the map $\sigma : \Omega\times[0,s_0]\to AdS^{n+1}$ defined by

$$\sigma(x,s) = (\psi_s(x), u_s(\psi_s(p)))$$

is a solution of (20), where ψ_s is the flow of X_s.

Proposition A.3 ([16]) *Let M_0 be a space-like $\mathrm{C}^{0,1}$ compact graph in AdS_{n+1}. Then there is a smooth solution of (20) for $s\in(0,+\infty)$ such that*

- $\partial M_s = \partial M_0$ *for every* s;
- $M_s \to M_0$ *in the Hausdorff topology as* $s\to 0$;
- $M_s\to M_\infty$ *in the* C^∞*-topology as* $s\to+\infty$, *where* M_∞ *is the unique maximal space-like surface with the property that* $\partial M_\infty = \partial M_0$;
- *if* H_s *denotes the mean curvature on* M_s *we have*

$$H_s^2(x) \le \frac{n}{2}\frac{1}{s}. \tag{23}$$

A.1 Mean curvature flow and convex subsets

To show the convergence of the mean curvature flow, we need to remark that, under suitable hypothesis, it does not leave convex subsets of AdS_{n+1}.

Lemma A.4 *Let M_s be a compact solution of (20). Suppose that there exists a space-like plane P such that M_0 is contained in $I^-(P)$ and $\partial M_0\subset I^-(P)$. Then M_s is contained in $I^-(P)$ for every $s>0$.*

Proof Without loss of generality we can suppose that P is the horizontal plane. We consider the function $u : AdS^{n+1}\to\mathbb{R}$ defined, as in the proof of Lemma 4.1, by $u(x,t) = x_{n+1}\sin t$.

By our assumption

$$\begin{aligned} u(p) &\le 0 \quad \text{for every } p\in M_0,\\ u(p) &< 0 \quad \text{for every } p\in\partial M_s. \end{aligned} \tag{24}$$

On the other hand the computation in Lemma 4.1 shows that

$$\left(\frac{d}{ds} - \Delta\right)u = -nu$$

where Δ is the Laplace-Beltrami operator on M_s.

In particular if the maximum of the function u is achieved at some interior point of M_s we have

$$\frac{du_{\max}}{ds} \leq n u_{\max}.$$

By (24), we deduce that $u_{\max}(s) < 0$ for every $s > 0$. In particular M_s is contained in the region $\{(x,t) | 0 < t < \pi\}$ for every $s > 0$. □

Lemma A.4 and Lemma 4.7 imply the following property.

Proposition A.5 *If M_s be a compact solution of (20) such that M_0 is contained in the closure of some convex slab Ω, and ∂M_0 is contained in Ω, then M_s is contained in Ω for every $s > 0$.*

Let $M = \Gamma_u$ be a weakly space-like graph and Σ be its asymptotic boundary. We will assume that neither M nor Σ contains any singular point. Finally we denote by D the domain of dependence of M and by K its convex hull, introduced in Remark 4.9. The same argument as in Lemma 4.10 shows that $\overline{K} \cap \partial D = \emptyset$.

For every $r > 0$ let u^r be the restriction of u on B_r (that is the ball in $\mathbb{H}^n$ of center at x^0 and radius r). We consider the mean curvature flow with Dirichlet condition of the compact graph of u^r, that is, a map

$$\sigma^r : \overline{B}_r \times (0, +\infty) \to AdS_{n+1}$$

that verifies (20) and satisfies

- $\sigma^r(x, 0) = (x, u(x))$ for every $x \in B_r$;
- $\sigma^r(x, s) = (x, u(x))$ for every $x \in \partial B_r$.

Let us denote by M^r_s the image of B_r through the map $\sigma(\cdot, s)$.

By Lemma A.1 and Proposition A.3 there is a family of space-like functions

$$u^r_s : \overline{B}_r \to \mathbb{R}$$

such that M^r_s is the graph of u^r_s and the family (u^r_s) satisfies (22).

Proposition A.6 *For every $R > 0$, $\eta > 0$ there is $\bar{r} > 0$ and constants $C, C_0, C_1, \ldots$ such that for every $r > \bar{r}$ and every $s > \eta$ we have*

$$\sup_{M^r_s \cap B_R \times \mathbb{R}} v < C$$
$$\sup_{M^r_s \cap B_R \times \mathbb{R}} |\nabla^m A|^2 < C_m \quad \textit{for } m = 0, \ldots.$$

Proof The scheme of the proof is the same as for Theorem 4.14. In particular we use the notations introduced there.

We choose points $p_1, \dots, p_{k_0} \in D \cap I^-(\partial_- K)$ and numbers $\epsilon_1, \dots, \epsilon_k$ such that

$$(B_R \times \mathbb{R}) \cap K \subset \bigcup_1^{k_0} I^+_{\epsilon_k}(p_k).$$

On $I^+_{\epsilon_k}(p_k)$ we consider the time function $\tau_k = \tau_{p_k} - \epsilon_k$ where τ_{p_k} denote the Lorentzian distance from p_k and is a time function on $I^+(p_k)$. Notice that τ_k is smooth on the domain $\mathcal{V} = I^+(p_k) \cap I^-((p_k)_+)$. Moreover $\overline{K \cap I^+_{\epsilon_k/2}(p_k)}$ is a compact domain in $\mathcal{V}$.

Since M^r_s is contained in K for every r and s, we deduce that there exists r_0 such that for $r \geq r_0$ and $k = 1, \dots, k_0$

$$\partial M^r_s \cap I^+(p_k) = \emptyset$$

and $M_r \cap \{\tau_k \geq 0\} = M^r_s \cap \overline{I^+_{\epsilon_k/2}(p_k)}$ is compact.

Thus we are in the hypothesis of Theorem 2.1 of [16], there is a constant a_k

$$\sup_{M^r_s \cap I^+_{\epsilon_k}(p_k)} v_{M^r_s} \leq a_k\left(1 + \frac{1}{s}\right) \tag{25}$$

where a_k depends on the C^2 norm of τ_k and t and the C^0 norm of *Ric* taken on the domain $\overline{K \cap I^+_{\epsilon_k/2}(p_k)}$ with respect to a reference Riemannian metric.

In particular for $s > \eta$ we have

$$\sup_{M^r_s \cap I^+_{\epsilon_k}} v_{M^r_s} \leq a_k\left(1 + \frac{1}{\eta}\right). \tag{26}$$

By Theorem 2.2 of [16] we also have that for every $m = 0, 1, \dots$ there are constants $a_{k,m}$ such that

$$\sup_{M^R_s \cap I^+_{\epsilon_k}(p_k)} |\nabla^m A|^2 \leq a_{k,m}.$$

In particular, the constants $C = \sup\{a_1, \dots a_{k_0}\}$, $C_m = \sup\{a_{1,m}, \dots, a_{k,m}\}$ satisfy the statement. □

Theorem A.7 *There is a family of space-like functions*

$$\bar{u}_s : \mathbb{H}^n \to \mathbb{R}$$

for $s \in (0, +\infty)$ *that verifies* (22) *such that*

- *$\bar{u}_s \to u$ as $s \to 0$ in the compact open topology.*
- *$\{\bar{u}_s\}_{s>1}$ is a relatively compact family in $C^\infty(\mathbb{H}^n)$.*
- *the graph M_s of $\bar{u}_s$ is contained in K for every $s > 0$.*
- *the mean curvature of M_s satisfies $H_s(x)^2 < \frac{n}{2s}$.*

Proof For any $R > 0$ and $\epsilon > 0$ we consider the restriction of u^r on $B_R \times [-\epsilon, +\infty)$. Proposition A.6 implies that such restrictions form a pre-compact family in $C^\infty(B_R \times [-\epsilon, +\infty))$.

By a diagonal process, we can construct a sequence $r_n \to +\infty$ such that (u^{r_n}) converges to $\bar{u}$ in the C^∞-topology on compact subsets of $\mathbb{H}^n \times (0, +\infty)$. Notice that by construction $(\bar{u}_s)_{s>1}$ is precompact in $C^\infty(\mathbb{H}^n)$.

By the uniform estimate on the gradient function of u^r_s on B_R we get that the graph M_s of $\bar{u}_s$ is space-like. Clearly $\bar{u}_s$ verifies (22). Since (23) holds for every u^r_s, we get that $H(\bar{u}_s)^2 < \frac{n}{2s}$.

Analogously, passing to the limit in the inclusion $M^r_s \subset K$, we get that M_s is contained in K.

Comparing (22) with (23), it follows that

$$|u^r_s(x) - u(x)| \leq \sqrt{ns}.$$

Taking the limit for $r \to +\infty$ we get

$$|\bar{u}_s(x) - u(x)| \leq \sqrt{ns}$$

which shows that $\bar{u}_s \to u$ in the compact open topology. □

Remark A.8 Taking the limit of M_{s_k} for a suitable sequence $s_k \to +\infty$ we obtain a maximal surface contained in D. Thus Theorem A.7 furnishes another proof of Theorem 4.11.

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