NON-RIGIDITY OF SPHERICAL INVERSIVE DISTANCE CIRCLE PACKINGS

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Abstract. We give a counterexample of Bowers-Stephenson's conjecture in the spherical case: spherical inversive distance circle packings are not determined by their inversive distances.

1. Introduction

In this note we study inversive distance circle packing metrics on a surface F.

1.1. **Polyhedral surface.** Given a triangulated closed orientable surface F, a Euclidean (resp. spherical or hyperbolic) *polyhedral surface* is a map $l : E \to \mathbb{R}^+$, where E is the set of all edges of the triangulation, such that when e_1, e_2 and e_3 are the three edges of a triangle, then $l(e_1) + l(e_2) > l(e_3)$ (it is also required that $l(e_1) + l(e_2) + l(e_3) < 2\pi$ in the spherical case). From this l, there is a *polyhedral metric* in F such that the restriction of the metric to each triangle is isometric to a triangle in $\mathbb{E}^2$ (resp. $\mathbb{S}^2$ or $\mathbb{H}^2$) and the length of an edge e is given by $l(e)$. For instance, the boundary of a generic convex polyhedron in $\mathbb{E}^3$ (resp. $\mathbb{S}^3$ or $\mathbb{H}^3$) admits a natural polyhedral metric.

The *discrete curvature* k of a polyhedral surface is the map $k : V \to \mathbb{R}$, where V is the set of all vertices of the triangulation, and for a vertex $v \in V$, $k(v) = 2\pi - \sum_{i=1}^{m} \theta_i$, where θ_i are the angles at the vertex v.

1.2. **Inversive distance circle packings.** The notion of inversive distance circle packing was introduced by Bowers-Stephenson in [1], it is a generalization of Andreev and Thurston's circle packings on a surface, where two circles may intersect or not. We just give the definition of the spherical inversive distance circle packing, for Euclidean and hyperbolic cases, see [1] and [3] for more detailed discussions.

For two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ centered at v_1, v_2 of radii r_1 and r_2 in $\mathbb{S}^2$, so that v_1 and v_2 are of distance l apart, the *inversive distance* $I = I(\mathcal{C}_1, \mathcal{C}_2)$ between them is

$$I = \frac{\cos(l) - \cos(r_1)\cos(r_2)}{\sin(r_1)\sin(r_2)}. \tag{1.1}$$

When viewed B^3 as the Klein model of $\mathbb{H}^3$, the inversive distance is essentially the hyperbolic distance(or the intersection angle) between the two totally geodesic planes in $\mathbb{H}^3$ with $\mathcal{C}_i$ as their ideal boundaries. When those planes intersect, the

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inversive distance is the cos of their angle, and if they're disjoint, it is the cosh of their distance.

In a triangulated surface F, a *spherical inversive distance circle packing* is given as follows: fix a vector $I \in [-1,\infty)^E$, called the *inversive distance vector*. For any $r \in (0,\infty)^V$, called the *radius vector*, define the edge length by $l(e) = \sqrt{r(u)^2 + r(v)^2 + 2r(u)r(v)I(e)}$ for an edge e with u and v as its end points. If for any triangle with e_1, e_2 and e_3 as its three edges, we have $l(e_1)+l(e_2) > l(e_3)$ and $l(e_1) + l(e_2) + l(e_3) < 2\pi$, then the edge length function $l : E \to \mathbb{R}$ defines a spherical polyhedral metric on F, which is called the *spherical inversive distance circle packing metric* with inversive distance I.

The geometric meaning is that in F with this polyhedral metric, if we draw circles with radii r at the vertices V, then the inversive distance of two circles at the end points of an edge e is the given number $I(e)$.

It was conjectured by Bowers and Stephenson [1] that inversive distance circle packings have a global rigidity property: an inversive distance circle packing is determined by its combinatoric, inversive distance vector and discrete curvature at the vertices. Luo [3] proved Bowers-Stephenson's conjecture in the hyperbolic and Euclidean cases. In this note, we give a counterexample in the spherical case:

Theorem 2.4. *There is a triangulation of S^2 and two spherical inversive distance circle packings with the same inversive distance and discrete curvature, but they are not Möbius equivalent.*

The example we construct actually have zero discrete curvature at all vertices, so they are inversive distance circle patterns on the (non-singular) sphere.

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2. Proof of the theorem

The proof of our theorem uses a well-known infinitesimal flexible Euclidean polyhedron and the Pogorelov map which preserves the relative distances between two points in the configurations in different geometries. We first give a rapid preliminary.

Let $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$ be the symmetric 2-form in the Minkowski space $\mathbb{R}^4_1$, recall that the hyperbolic space is

$$\mathbb{H}^3 = \{x \in \mathbb{R}^4_1 |\ \|x\|^2 = -1, x_0 > 0\} \tag{2.1}$$

with the induced Riemannian metric on it, which is a hyperboloid in $\mathbb{R}^4_1$. The totally geodesic planes in $\mathbb{H}^3$ are the intersections between $\mathbb{H}^3$ and hyperplanes in $\mathbb{R}^4$ which pass through the origin.

Let B^3 be the unit ball in $\mathbb{R}^3$, then, there is a projective map $p_{\mathbb{H}} : \mathbb{H}^3 \to B^3$ given by $\rho((x_0, x_1, x_2, x_3)) = (x_1, x_2, x_3)/x_0$, which is a homeomorphism and which maps geodesic in $\mathbb{H}^3$ into geodesic in $\mathbb{R}^3$. This map is the projective model (Klein model) of the hyperbolic space.

A hyperideal hyperbolic polyhedron is the image of $p_{\mathbb{H}}^{-1} : Q \cap B^3 \to \mathbb{H}^3$, where Q is a Euclidean polyhedron in $\mathbb{R}^3$ such that all vertices of Q lie out of B^3 and all edges of Q intersect with B^3. For a point A in $\mathbb{R}^3 - \overline{B^3}$, consider the space $A^\perp$ of the points in $\mathbb{R}^4_1$ which are orthogonal to $p_{\mathbb{H}}^{-1}(A)$ in the symmetric 2-form. $p_{\mathbb{H}}^{-1}(A)$ is a hyperbolic plane in $\mathbb{H}^3$. Then take $A^* = p_{\mathbb{H}}(A^{\perp}) \cap B^3$, so it is a hyperbolic plane in the Klein model of the 3-dimensional hyperbolic space B^3. Thus its boundary is a round circle $\mathcal{C}_A$ in $\partial B^3 = S^2$. A^* is called the hyperbolic plane dual to A. By a simple argument, the planes dual to two vertices A and B of Q don't intersect. The length of an edge of a hyperideal hyperbolic polyhedron is defined as the distance between the dual planes.

The de Sitter space can be defined as

$$\mathbb{S}^3_1 = \{x \in \mathbb{R}^4_1 | \ \|x\|^2 = 1\} \tag{2.2}$$

with the induced Lorentzian metric on it, which is a one-sheeted hyperboloid in $\mathbb{R}^4_1$. The totally geodesic planes in $\mathbb{S}^3_1$ are the intersections of $\mathbb{S}^3_1$ with the hyperplanes in $\mathbb{R}^4$ which pass through the origin. Let

$$\mathbb{S}^3_{1,+} = \{x \in \mathbb{R}^4_1 | \ \|x\|^2 = 1, x_0 > 0\} \tag{2.3}$$

be the upper de Sitter space.

As for the hyperbolic space, there is a projective map $p : \mathbb{S}^3_{1,+} \to \mathbb{R}^3 - \overline{B^3}$ given by $\rho((x_0, x_1, x_2, x_3)) = (x_1, x_2, x_3)/x_0$, which is a homeomorphism and which maps geodesic in $\mathbb{S}^3_{1,+}$ into geodesic in $\mathbb{R}^3$.

In the projective model of $\mathbb{S}^3_{1,+}$, a geodesic maybe pass through B^3, and if it is the case, then the geodesic is *time-like*. If a geodesic does not pass through the closure of B^3, then this geodesic is *space-like*.

For more details on distances in the de Sitter space, see [5]: for two points x and y in $\mathbb{S}^3_{1,+}$, if the geodesic $[x, y]$ is a time-like geodesic, then the *distance* d between them is the negative number d such that $\cosh(d) = \langle x, y\rangle$; if the geodesic $[x, y]$ is a space-like geodesic, the *distance* d between them is the unique number in $i[0, \pi]$ such that $\cosh(d) = \langle x, y\rangle$.

There is a duality between points in the de Sitter space and oriented hyperplanes in the hyperbolic 3-space: consider the projective model of the upper de Sitter space $\mathbb{R}^3 - \overline{B^3}$, when A lies in $\mathbb{R}^3 - \overline{B^3}$, then the hyperplane A^* constructed above viewed as in hyperbolic 3-space is the dual of A.

When A and B are two points in $\mathbb{R}^3 - \overline{B^3}$, such that the Euclidean line L connecting A to B passes through B^3, then the de Sitter distance between A and B is essentially the hyperbolic distance between the two planes A^* and B^*: let l be the distance between A^* and B^* in the hyperbolic space, then $l = -d$. It is also essentially the inversive distance between the two circles $\mathcal{C}_A$ and $\mathcal{C}_B$, where $\mathcal{C}_A$ and $\mathcal{C}_B$ are the ideal boundaries of the planes $A^*$and B^* in $S^2 = \partial B_3$.

Lemma 2.1. *There is a Euclidean polyhedron Q such that*

1. *all vertices of Q lie out of B^3,*
2. *all edges of Q intersect with B^3,*
3. *in any neighborhood of Q, there are two Euclidean polyhedra Q_t and Q_{-t} which have the same combinatorics as Q and the same corresponding edge lengths.*

Proof. We first recall Schönhardt's twisted octahedron (see [8] and [2]): let ABC be an equilateral triangle in $\mathbb{R}^3$, and let L be a line that passes through the center of ABC and it is orthogonal to the plane of the triangle. Let $A^0B^0C^0$ be the image of ABC under a screw motion with axis L and rotation angle $\pi/2$. Consider a polyhedron Q bounded by triangles ABC, $A^0B^0C^0$, ABC^0, A^0BC, AB^0C,

A^0B^0C, AB^0C^0, and A^0BC^0. The polyhedron Q is combinatorially isomorphic to an octahedron, and has three edges with dihedral angles bigger than π: the edges AB^0, BC^0, and CA^0, see Figure 1.

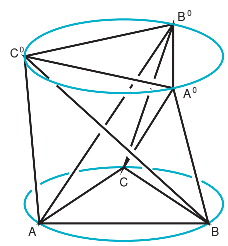

Figure 1. Schönhardt's twisted octahedron

This polyhedron Q is infinitesimal flexible (see [2]): there are vectors η_{A^0}, η_{B^0}, η_{C^0} in $\mathbb{R}^3$ such that the polyhedron Q_t with the vertices A, B, C, $A^0 + t\eta_{A^0}$, $B^0 + t\eta_{B^0}$, $C^0 + t\eta_{C^0}$, which is a small deformation of Q, is a non-trivial infinitesimally isometric deformation of Q, where η_{A^0} is a vector orthogonal to the plane A^0BC of norm 1 and pointing out from Q, and similarly for η_{B^0} and η_{C^0}. Then by a direct calculation(or see Lemma 4.1 of [2]), the pairs of corresponding edges of Q_t and Q_{-t} have the same lengths, for $0 < t$ small enough.

Let O be the center of the polyhedron Q, i.e, O lies in the line L and its distances to the planes ABC and $A^0B^0C^0$ are both equal to $h > 0$. Let a be the edge length of the equilateral triangle ABC, a simple calculation shows that conditions (1) and (2) of the lemma are equivalent to

(1) $h^2 + a^2/3 > 1$,
(2) $h^2 + a^2/12 < 1$,
(3) $a^2/3 < 1$.

So, we can assign $h = 1/2$ and a a little bigger than $3/2$, and the lemma follows. □

Pogorelov [4] has found remarkable maps from $\mathbb{S}^3 \times \mathbb{S}^3$ and $\mathbb{H}^3 \times \mathbb{H}^3$ to $\mathbb{R}^3 \times \mathbb{R}^3$, see also [5], [6], [7] for other forms of these maps and their infinitesimal versions. What we really need is the first four properties of the following proposition, we state it along with other properties for the sake of future reference.

Proposition 2.2. *There exists a map* $\Phi : \mathbb{S}^3_{1,+} \times \mathbb{S}^3_{1,+} \to \mathbb{R}^3 \times \mathbb{R}^3$ *such that:*

(1) *Φ is a homeomorphism from $\mathbb{S}^3_{1,+} \times \mathbb{S}^3_{1,+}$ to its image in $\mathbb{R}^3 \times \mathbb{R}^3$,*
(2) *the restriction of Φ to the diagonal $\triangle \subset \mathbb{S}^3_{1,+} \times \mathbb{S}^3_{1,+}$ is the projective map p(its image is in the diagonal $\triangle' \subset \mathbb{R}^3 \times \mathbb{R}^3$),*
(3) *let α be a time-orientation preserving isometry of $\mathbb{S}^3_1$, and $x \in \mathbb{S}^3_{1,+}$ with $\alpha(x) \in \mathbb{S}^3_{1,+}$, we have $\Phi(x, \alpha(x)) = (y, y')$ in $\mathbb{R}^3$, then there is a Euclidean isometry β such that for all x with $\alpha(x) \in \mathbb{S}^3_{1,+}$, we have $y' = \beta(y)$,*
(4) *if $[x, y]$ and $[x', y']$ are two time-like geodesics of the same length in $\mathbb{S}^3_{1,+}$, and if p_1, p_2 are the projections of $\mathbb{R}^3 \times \mathbb{R}^3$ on the two factors, then $[p_1 \circ \Phi \circ (x, x'), p_1 \circ \Phi \circ (y, y')]$ and $[p_2 \circ \Phi \circ (x, x'), p_2 \circ \Phi \circ (y, y')]$ are geodesics of the same length in $\mathbb{R}^3$,*

(5) *if* $g_1, g_2 : [0,1] \to \mathbb{S}^3_{1,+}$ *are space-like geodesic segments parametrized at the same speed, then* $p_1 \circ \Phi \circ (g_1, g_2)$ *and* $p_2 \circ \Phi \circ (g_1, g_2)$ *are geodesic segments parametrized at the same speed,*

(6) *there exists a point* $x^0 = p^{-1}(0) \in \mathbb{S}^3_{1,+}$ *such that, for each 2-plane* $\Pi \subset \mathbb{S}^3_{1,+}$ *containing* x^0*,*

$$\forall x \in \Pi,\ \forall y \in \mathbb{S}^3_{1,+},\ p_1 \circ \Phi(x, y) \in p(\Pi). \tag{2.4}$$

The proof of this proposition can be obtained by following those given by Pogorelov's book for the hyperbolic space or the sphere. More precisely, it is straightforward to adapt the proof of §3 Lemmas 1-4 and §4 Theorems 1-2 in Chapter V of [4].

Or from Section 6 of [9]: in Proposition 6.3 and 6.4 of [9], we should replace $f(a,b) = (a^2-b^2)^2 - 8(a^2+b^2-2)$(in hyperbolic case and $1 > a, b \geq 0$) to $g(a,b) = -(a^2-b^2)^2 + 8(a^2+b^2-2)$ (in de Sitter case and $1 < a, b$). Note that $g(a,b) = -(a^2-b^2)^2 + 8(a^2+b^2-2)$ is not always positive for $1 < a, b$, but this is true for $-4 < a^2-b^2 < 4$, so $\{(\xi,\eta) \in (\mathbb{R}^3-B^3)\times(\mathbb{R}^3-B^3) | -4 < |\xi|^2-|\eta|^2 < 4\} \subset Im(\Phi)$, which is an open neighborhood of the diagonal of $(\mathbb{R}^3-B^3)\times(\mathbb{R}^3-B^3) \subset \mathbb{R}^3\times\mathbb{R}^3$.

For the proof of (3) of Proposition 2.2, we just recall that for a time-orientation preserving isometry α of $\mathbb{S}^3_1$, in the matrix presentation $A_{4\times4}$ of it, the $(1,1)$-entry of A is positive, and then (3) of Proposition 2.2 follows from arguments similar to Proposition 6.5 of [9].

For the proof of (4) of Proposition 2.2, we need the transitivity of the time-orientation subgroup of $Iso(\mathbb{S}^3_1)$ on the space of time-like geodesic segments of a fixed length, which can be see from the duality between the de sitter space and the hyperbolic space. From this, we have an time-orientation isometry α, such that $\alpha([x,y]) = [x', y']$, and then (4) follows from (3). (5) is similar.

The map Φ in the Minkowski coordinate is given as follows: let $(x,y) \in \mathbb{S}^3_{1,+} \times \mathbb{S}^3_{1,+}$, where $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$, then

$$\Phi(x,y) = 2((x_1,x_2,x_3),(y_1,y_2,y_3))/(x_0+y_0) \in \mathbb{R}^3\times\mathbb{R}^3. \tag{2.5}$$

The converse $\Phi^{-1} : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{S}^3_{1,+}\times\mathbb{S}^3_{1,+}$ is given by

$$((\xi_1,\xi_2,\xi_3),(\eta_1,\eta_2,\eta_3)) \to (\rho((4-|\eta|^2+|\xi|^2,\xi_1,\xi_2,\xi_3)), \rho((4-|\xi|^2+|\eta|^2,\eta_1,\eta_2,\eta_3))), \tag{2.6}$$

where ρ is the linear normalization such that $\rho((4-|\eta|^2+|\xi|^2,\xi_1,\xi_2,\xi_3))$ and $\rho((4-|\xi|^2+|\eta|^2,\eta_1,\eta_2,\eta_3))$ lie in the hyperboloid $\mathbb{S}^3_{1,+}$.

Remark 2.3. The Pogorelov's maps in [4] and [9] are a little different, i.e, up to the multiple constant 2, we choose the one similar to [9], which is convenient for us.

Theorem 2.4. *There is a triangulation of* S^2 *and two spherical inversive distance circle packings with the same inversive distance and discrete curvature, but they are not Möbius equivalent.*

Proof. The triangulation of S^2 is given from the boundary of the Euclidean octahedron in Lemma 2.1. From Lemma 2.1, we have two Euclidean polyhedra which have the same edge lengths, but they are not congruent, say Q_t and Q_{-t} for a fixed $t > 0$ small enough, which are very near to Q. Denoted the vertices of Q by v^i and the corresponding vertices of $Q_t(Q_{-t})$ by $v^i_t(v^i_{-t})$. Note that $(v^i, v^i) \in Im(\Phi)$ by Proposition 2.2 (2), and from Proposition 2.2 (1), we can assume $(v^i_t, v^i_{-t}) \in Im(\Phi)$. Since $\Phi^{-1} \circ (v^i, v^i)$ give us a polyhedron in $\mathbb{S}^3_{1,+}$ such that each of the edges are time-like. Then we use Proposition 2.2 (4), $p_1 \circ \Phi^{-1} \circ (v^i_t, v^i_{-t})$ and $p_2 \circ \Phi^{-1} \circ (v^i_t, v^i_{-t})$

give us two polyhedra, say P_t and P_{-t}, in $\mathbb{S}^3_{1,+}$ such that each of the edges are time-like and which have the same corresponding edge lengths. For each vertex of P_t, we have a circle in S^2, which is the ideal boundary of the hyperbolic plane dual to the vertex. But recall that the de Sitter length here is essentially the inversive distance of the circles corresponding to two ideal vertices of the hyperideal hyperbolic polyhedra. So, we have two spherical inversive distance circle packing metrics, they induced the same standard spherical metric in $\mathbb{S}^2$, thus they have the same discrete curvature zero. These two spherical inversive distance circle packing are not Möbious equivalent can be seen also from the Pogorelov map.

□

Corollary 2.5. *There is a hyperideal polyhedron P such that each face of it is a triangle and in any neighborhood of P, there are two hyperideal polyhedra P_t and P_{-t} which have the same combinatorics and the corresponding edges of them have the same length.*

Proof. Now from Lemma 2.1, we have two Euclidean polyhedra Q_t and Q_{-t}, which have the same edge length, but they are not congruent. Then we use Proposition 2.2, we get two polyhedra in $\mathbb{S}^3_{1,+}$ such that each of the edges are time-like.

Such polyhedra in $\mathbb{S}^3_{1,+}$ can be viewed as hyperideal hyperbolic polyhedra, and the distance in the de Sitter geometry is just the distance of the circles corresponding to two ideal vertices of the hyperideal hyperbolic polyhedra, which is the edge length of the hyperideal hyperbolic polyhedra. □

Remark 2.6. Our polyhedra above are not convex, a similar phenomena appears in convex hyperbolic polyhedra, but, some of the faces are not triangle, see Theorem 2' of [6].

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