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Polynomial Systems: a Lower Bound for the Weakened 16th Hilbert Problem

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In this paper we provide the greatest lower bound about the number of (non–infinitesimal) limit cycles surrounding a unique singular point for a planar polynomial differential system of arbitrary degree.

We prove that for m and n odd the maximum number $b_{m,n}$ of isolated zeros (taking into account their multiplicity) of the Abelian integral $I(h) = \int_{H(x,y)=h} y\bar{Q}(x,y)dx$, where $H(x,y) = \frac{1}{2}y^2 + \frac{1}{m+1}x^{m+1}$, and $\bar{Q}$ and arbitrary polynomial of degree at most $n-1$ is

$$\frac{(n+1)(n+3)}{8} - 1 \quad \text{if} \quad n \leq m\,, \qquad \frac{(m+1)(2n-m+3)}{8} - 1 \quad \text{if} \quad n \geq m\,.$$

Moreover, there are perturbations of the Hamiltonian system $\dot{x} = -\partial H/\partial y$, $\dot{y} = \partial H/\partial x$, such that the indicated maximum number $b_{m,n}$ of continuous families of limit cycles can be made to emerge from a corresponding number of arbitrarily prescribed periodic orbits within the period annulus of the center. Consequently,

$$b_{m,n} \leq N(m,n) \leq H_{\max\{m,n\}}\,.$$

This result provides the greatest lower bound about the number of (non–infinitesimal) limit cycles surrounding a unique singular point for a planar polynomial differential system of arbitrary degree $m = n$.

1. Introduction and the main result

We consider two–dimensional differential systems

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in the variables x and y. In his address to the International Congress of Mathematics in Paris in 1900, Hilbert raised the question of the number of limit cycles of these differential systems. It remains one of the most difficult open questions in the qualitative theory of planar polynomial differential systems.

Let H_m be the maximum possible number of limit cycles of (1) when P and Q are of degree at most m. The H_m are the *Hilbert numbers*, and it is still an open problem whether H_m is finite, even for the simplest case of quadratic polynomial differential systems ($m = 2$). Probably the best result in that direction has been the proof of *Dulac's Conjecture* by Il'yashenko [14] and Ecalle [9] using different methods. This result states that a given polynomial system cannot have infinitely many limit cycles. Note that this does not imply that the H_m are finite.

On the other hand there has been some success in finding lower bounds for H_m. Thus it is known that $H_2 \geq 4$ (see Shi [27]) and $H_3 \geq 11$ (see Li and Li [16]). Several authors have established that H_m grows at least as fast as m^2 with m. Thus, Il'yashenko [13] proved that

$$H_m \geq \frac{1}{2}\left(m^2 + m - 2\right);$$

Basarab–Horwath and Lloyd [2] shown that

$$H_m \geq \frac{1}{4}(m-1)(m+2);$$

Christopher and Lloyd [5] proved that

$$H_m \geq \frac{1}{2}(m+1)^2\left(\log_2(m+1) - 3\right) + 3m.$$

In these last three results the limit cycles occur in several nests, i.e., they are not surrounding a unique singular point.

Let $H(x, y)$ be a real polynomial of degree $m + 1$, and let $P(x, y)$ and $Q(x, y)$ be real polynomials of degree at most n. The problem of finding an upper bound $N(m, n, H, P, Q)$ for the number of isolated zeros of the Abelian integrals

$$I(h) = \int_{\Gamma_h} Q(x, y)dx - P(x, y)dy\,, \tag{2}$$

where Γ_h varies in the compact components of $H^{-1}(h)$ is called the *weakened 16th Hilbert problem*. It was posed by Arnold in [1].

The weakened 16th Hilbert problem is closely related to the problem of determinating an upper bound for the number of limit cycles of the perturbed Hamiltonian system

$$\dot{x} = -\frac{\partial H}{\partial y} + \varepsilon P(x, y)\,, \qquad \dot{y} = \frac{\partial H}{\partial x} + \varepsilon Q(x, y)\,, \tag{3}$$

where $0 < \varepsilon << 1$. The relationship between both problems comes from the following two facts:

(i) If $I(h^*) = 0$ and $I'(h^*) \neq 0$, then there exists a hyperbolic limit cycle L_{h^*} of system (3) such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$; and conversely, if there exists a hyperbolic limit cycle L_{h^*} of system (3) such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$, then $I(h^*) = 0$.

(ii) The total number of isolated zeros of (2) (taking into account their multiplicity) is an upper bound for the number of limit cycles of system (3) with $\varepsilon > 0$ tending to some periodic orbit Γ_h of system (3) with $\varepsilon = 0$ when $\varepsilon \to 0$.

Khovansky [15] and Varchenko [28] proved independently that $N(m, n, H, P, Q)$ is finite, but an explicit expression for $N(m, n, H, P, Q)$ is unknown. Many authors have contributed to estimate or to give upper bounds for the numbers $N(m, n, H, P, Q)$, usually they fix H and take arbitrary polynomials P and Q with n fixed or not. In this last case the upper bounds that they obtain are linear functions in n; see for instance Bogdanov [3] and [4], Petrov [24] and [25], Cushman and Sanders [6], Dumortier, Roussarie and Sotomayor [8], Drachman, van Gils and Zhang [7], Li and Rousseau [20], Gavrilov [10], Gavrilov and Horozov [11], Horozov and Iliev [12], Li, Llibre and Zhang [17] and [18], Li and Zhang [21], Novikov and Yakovenko [23], Zholadek [29], ...

Let $N(m, n)$ be the supremum of $N(m, n, H, P, Q)$ when H varies inside the class of all polynomials of degree at most $m+1$, and P and Q vary inside the class of all polynomials of degree at most n.

This paper is concerned with the rate of growth of $N(m, n)$, and since $N(m, n) \leq H_{\max\{m,n\}}$ we also provide a lower bound of $H_{\max\{m,n\}}$. Our result is the following.

For m odd let

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{m+1}x^{m+1}\,, \tag{4}$$

and let

$$P(x,y) \equiv 0\,, \qquad Q(x,y) = y\bar{Q}(x,y)\,, \tag{5}$$

be polynomials with degree of $\bar{Q}$ at most $n-1$. Then we consider the perturbed Hamiltonian system

$$\dot{x} = -y\,, \qquad \dot{y} = x^m + \varepsilon y \bar{Q}(x,y)\,. \tag{6}$$

Theorem. *For m and n odd the maximum number $b_{m,n}$ of isolated zeros (taking into account their multiplicity) of the Abelian integral (2) with H, P and Q given by (4) and (5) is*

$$\frac{(n+1)(n+3)}{8} - 1 \quad \text{if} \quad n \le m\,, \qquad \frac{(m+1)(2n-m+3)}{8} - 1 \quad \text{if} \quad n \ge m\,.$$

Moreover, there are perturbations of system (6) such that the indicated maximum number $b_{m,n}$ of continuous families of limit cycles can be made to emerge from a corresponding number of arbitrarily prescribed periodic orbits within the period annulus of the center. Consequently,

$$b_{m,n} \,\le\, N(m,n) \,\le\, H_{\max\{m,n\}}\,.$$

2. Proof of the theorem

The key point in the proof of this theorem is that, by using Green's Theorem, we will compute the Abelian integral through a double integral. These double integrals for system (6) are very easy to compute in comparison with the usual single Abelian integral. As far as we know this technique applied to Abelian integrals was used by first time in [19]. The proof of the theorem is given in the next section.

We write the function $\bar{Q}(x,y)$ of system (6) as follows

$$\bar{Q}(x,y) = \sum_{0 \le i+j \le n-1} a_{i,j} x^i y^j\,,$$

and a periodic orbit of the unperturbed system (6) for $\varepsilon = 0$ as

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{m+1}x^{m+1} = h\,.$$

By using Green's Theorem the Abelian integral (2) goes over to

$$
\begin{aligned}
I(h) &= \int\int_{H(x,y)\le h} \frac{\partial(y\bar{Q})}{\partial y} dxdy \\
&= \sum_{0\le i+j\le n-1} \int\int_{H(x,y)\le h} (j+1)a_{i,j}x^i y^j dxdy \\
&= \sum_{0\le 2i+2j\le n-1} (2j+1)a_{2i,2j} \int\int_{H(x,y)\le h} x^{2i}y^{2j} dxdy \\
&= \sum_{0\le 2i+2j\le n-1} 2a_{2i,2j} \int_{-\bar{x}}^{\bar{x}} x^{2i} \left(2h - \frac{2}{m+1}x^{m+1}\right)^{j+\frac{1}{2}} dx \\
&= \sum_{0\le 2i+2j\le n-1} C_{ij} h^{\alpha_{ij}}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{x} &= [(m+1)h]^{\frac{1}{m+1}} \,, \\
x &= [(m+1)h]^{\frac{1}{m+1}}\, y \,, \\
C_{ij} &= 2^{j+\frac{3}{2}} a_{2i,2j} (m+1)^{\frac{2i+1}{m+1}} \int_{-1}^{1} y^{2i}\left(1-y^{m+1}\right)^{j+\frac{1}{2}} dy \,, \\
\alpha_{ij} &= \frac{2i+1}{m+1} + j + \frac{1}{2} \,.
\end{aligned}
$$

We note that the number of α_{ij} that appear as exponents in the powers of h inside the last expression of the Abelian integral $I(h)$ is equal to

$$
\frac{(n+1)(n+3)}{8} \quad \text{if} \quad n \le m \,, \qquad \text{or to} \qquad \frac{(m+1)(2n-m+3)}{8} \quad \text{if} \quad n \ge m \,.
$$

Therefore the Theorem follows.

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