POTENTIAL MAPS, HARDY SPACES, AND TENT SPACES ON SPECIAL LIPSCHITZ DOMAINS

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Abstract: Suppose that Ω is the open region in $\mathbb{R}^n$ above a Lipschitz graph and let d denote the exterior derivative on $\mathbb{R}^n$. We construct a convolution operator T which preserves support in $\overline{\Omega}$, is smoothing of order 1 on the homogeneous function spaces, and is a potential map in the sense that dT is the identity on spaces of exact forms with support in $\overline{\Omega}$. Thus if f is exact and supported in $\overline{\Omega}$, then there is a potential u, given by $u = Tf$, of optimal regularity and supported in $\overline{\Omega}$, such that $du = f$. This has implications for the regularity in homogeneous function spaces of the de Rham complex on Ω with or without boundary conditions. The operator T is used to obtain an atomic characterisation of Hardy spaces H^p of exact forms with support in $\overline{\Omega}$ when $n/(n+1) < p \leq 1$. This is done via an atomic decomposition of functions in the tent spaces $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$ with support in a tent $T(\Omega)$ as a sum of atoms with support away from the boundary of Ω. This new decomposition of tent spaces is useful, even for scalar valued functions.

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1. Introduction

The study of potential maps on domains Ω of $\mathbb{R}^n$ has a rich history. Consider, for a moment, the following question. Suppose that a function f belongs to a Sobolev space $H_0^m(\Omega)$ where $m \geq 0$ and Ω is a bounded strongly Lipschitz domain, and suppose that $\int f = 0$. Is there a vector field u in $(H_0^{m+1}(\Omega))^n$ which satisfies $\operatorname{div} u = f$? The answer is yes, as was essentially proved by J. Nečas. Indeed, this follows by duality from [**13**, Chapter 3, Lemma 7.1]. An alternative proof was provided by M. E. Bogovskiĭ [**1**], [**2**], who, in particular, constructed an integral operator T which maps boundedly from the Sobolev space $W_0^{m,p}(\Omega)$ into $(W_0^{m+1,p}(\Omega))^n$ in the case when $m \geq 0$, $1 < p < \infty$, and Ω is starlike with respect to a ball. The potential u that solves the equation $\operatorname{div} u = f$ is then given by $u = Tf$, provided that $\int f = 0$. Since T gives the potential u which solves the equation and preserves support, we say that T is

a *potential map* for the domain Ω. Such a potential map is an important tool in the theory of equations of hydrodynamics. See the monograph [**7**] of G. P. Galdi and the papers mentioned below for further references to the extensive literature.

Subsequently M. Mitrea [**12**] (in 3 dimensions) and D. Mitrea, M. Mitrea and S. Monniaux [**11**] adapted Bogovskiĭ's operator to construct potential maps T_ℓ to solve the equation $du = f$, where d denotes the exterior derivative operator and where f is an exact ℓ-form with support in $\overline{\Omega}$ and components in a suitable Besov or Triebel-Lizorkin space. Using T_ℓ, they have thereby deduced sharp regularity estimates for important operators in the theory of hydrodynamics. As with Bogovskiĭ's operator, the mapping properties of T_ℓ ensure that there is no loss of regularity, and that support in $\overline{\Omega}$ is preserved. In the case when $n = 3$, the equation $du = f$ is equivalent to solving one of the equations

$$\mathbf{grad}\, u = f, \quad \mathbf{curl}\, u = f \quad \text{or} \quad \operatorname{div} u = f,$$

where f is interpreted either as a scalar or vector field depending on the value of ℓ.

More recently, M. Costabel and A. McIntosh [**6**] showed that the operators T_ℓ are pseudodifferential operators of order -1 and are therefore bounded in all the spaces of ℓ-forms with components in any one of the Besov or Triebel-Lizorkin spaces. For a domain starlike with respect to a ball, the special support properties of the operators imply regularity for the de Rham complex with full Dirichlet boundary conditions. Similar results hold for complexes without boundary conditions (using dual Poincaré-type operators). For bounded strongly Lipschitz domains, the same regularity results hold, and in addition the cohomology spaces can always be represented by $\mathscr{C}^\infty$ functions.

In the present paper, we turn our attention to unbounded special Lipschitz domains, that is, to domains in $\mathbb{R}^n$ that lie above the graph of a Lipschitz function, and consider the spaces $\dot{H}^s(\mathbb{R}^n, \Lambda)$ of forms with components in the homogeneous Sobolev space of degree s, where $s \in \mathbb{R}$. We construct an operator T with the following properties. First, T boundedly lifts forms in $\dot{H}^s(\mathbb{R}^n, \Lambda)$ to forms in $\dot{H}^{s+1}(\mathbb{R}^n, \Lambda)$. Second, T preserves support in $\overline{\Omega}$. Third, if $df = 0$ then a solution u of the equation $du = f$ is given by $u = Tf$. Hence the equation $du = f$ is solved on $\overline{\Omega}$ with optimal regularity, because if $f \in \dot{H}^s(\mathbb{R}^n, \Lambda^\ell)$, $df = 0$, and the support of f is in $\overline{\Omega}$, then $u \in \dot{H}^{s+1}(\mathbb{R}^n, \Lambda^{\ell-1})$, $du = f$, and the support of u is in $\overline{\Omega}$.

Our potential map T is a convolution integral operator involving kernels which are supported in a cone of $\mathbb{R}^n$ lying above its base at the

origin. This support property is ideally suited to boundary problems on a given special Lipschitz domain Ω, since such a cone (with appropriately chosen aperture) is contained within Ω when translated by points from Ω, and thus T preserves support in $\overline{\Omega}$. The bounded lifting property of T is valid not only when T acts on $\dot{H}^s(\mathbb{R}^n, \Lambda)$, but also when T acts on the space of forms whose components belong to any homogeneous Besov or Triebel-Lizorkin function space. These properties of T have immediate ramifications for the regularity of the de Rham complex in such spaces on $\overline{\Omega}$.

Our formula for T is motivated by a reproducing formula of D.-C. Chang, S. G. Krantz and E. M. Stein [3], which was also constructed to preserve support in a special Lipschitz domain.

In the second half of this paper, we apply this reproducing formula to show that Hardy spaces $H_d^p(\mathbb{R}^n, \Lambda)$ of exact forms can be characterised by exact atoms whenever $n/(n+1) < p \le 1$. Using the same method, we also show that the Hardy space $H_{z,d}^p(\overline{\Omega}, \Lambda)$ of exact forms with support in the closure $\overline{\Omega}$ of a special Lipschitz domain can be characterised using exact atoms supported in $\overline{\Omega}$. These results generalise the classical theorems of R. R. Coifman [4] and R. H. Latter [8] for $\mathbb{R}^n$, and the result of Chang, Krantz and Stein [3] for special Lipschitz domains, and have already been proved by Z. Lou and A. McIntosh [10], [9] for exact forms in the case when $p = 1$. In the three latter papers, the authors follow the method of R. R. Coifman, Y. Meyer and E. M. Stein [5] to show that each element u of the Hardy space can be written as a sum of atoms by first mapping u into a 'tent space' via an operator Q, then decomposing the image Qu as a sum of tent space atoms, before finally mapping the decomposition back into the Hardy space. In the case that u is supported in $\overline{\Omega}$, [3] and [9] then use reflection maps to express u as a sum of atoms whose supports lie in $\overline{\Omega}$. In our proof, we remove the need for reflection maps by proving a new result on the atomic decomposition of tent spaces, namely that if a function in a tent space itself has support in a tent $T_\beta(\Omega)$, then the supports of the atoms in its decomposition can be taken inside this tent, and away from the boundary of Ω.

This new result on tent spaces is quite powerful, even for scalar valued functions, as its use, along with the use of our reproducing formula, allows us to remain within the special Lipschitz domain the entire time. For this reason, we state here a simplified version of Theorem 9.4. For details, see Section 9. What is new is the requirement below that $5B_k \subset \Omega$.

Theorem 1.1. *Suppose that U belongs to the tent space $\mathcal{T}^1(\mathbb{R}^n \times \mathbb{R}^+)$, with $\mathrm{supp}(U)$ in a tent $T(\Omega)$ over an open set $\Omega \subset \mathbb{R}^n$. Then $U = \sum_k \lambda_k A_k$, where $\sum |\lambda_k| \lesssim \|U\|_{\mathcal{T}^1}$, $\mathrm{supp}(A_k) \subset B_k \times (0, 6r_k)$, for some ball B_k with radius r_k such that $5B_k \subset \Omega$, and $\mathrm{vol}(B_k) \int_0^\infty \|A_k(\cdot, t)\|_2^2 \frac{dt}{t} = 1$.*

The paper is organised as follows. In Section 2 we introduce notation and define the various spaces that we use, while in Section 3 we state reproducing formulae on special Lipschitz domains. In Section 4, the potential map T is defined, its properties are stated (see Theorem 4.1), and its utility for solving potential equations with boundary conditions on special Lipschitz domains is illustrated. Sections 5 and 6 are devoted to the proof of Theorem 4.1. Section 7 contains applications for potential maps on spaces without boundary conditions, by considering operators induced from T via quotient maps. In Sections 8 and 10, we apply the earlier part of the paper to prove our atomic characterisation of Hardy spaces of exact forms on $\mathbb{R}^n$ and on $\overline{\Omega}$. As noted already, Section 9 concerns the atomic definition of tent spaces, and is essentially independent of the rest of the paper.

Acknowledgments

2. Notation and definitions

Suppose that $\sigma > 0$ and x is a point of $\mathbb{R}^n$, where $n \geq 1$, $x = (x', x_n)$, $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Denote by $\Gamma_\sigma(x)$ and $\tilde{\Gamma}_\sigma(x)$ the cones of aperture σ and vertex x given by

$$\Gamma_\sigma(x) = \{y \in \mathbb{R}^n : \sigma(y_n - x_n) > |y' - x'|\}$$

and

$$\tilde{\Gamma}_\sigma(x) = \{y \in \mathbb{R}^n : \sigma(x_n - y_n) > |y' - x'|\}.$$

The cone $\Gamma_\sigma(x)$ lies above its vertex while $\tilde{\Gamma}_\sigma(x)$ lies beneath.

A subset Ω of $\mathbb{R}^n$ is said to be a *special Lipschitz domain* if

$$\Omega = \{x \in \mathbb{R}^n : x_n > \lambda(x')\},$$

where the function $\lambda \colon \mathbb{R}^{n-1} \to \mathbb{R}$ satisfies the Lipschitz condition

(2.1) $$|\lambda(x') - \lambda(y')| \leq A|x' - y'| \quad \forall \, x', y' \in \mathbb{R}^{n-1}$$

for some positive number A. The region that lies strictly beneath the graph of λ is denoted by Ω^-. Thus $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$. It follows immediately from the Lipschitz condition that when $0 < \sigma \leq A^{-1}$, the cone $\Gamma_\sigma(x)$ is contained entirely in Ω whenever $x \in \overline{\Omega}$, while $\check{\Gamma}_\sigma(x)$ lies entirely in Ω^- whenever $x \in \overline{\Omega^-}$.

Given a ball B of $\mathbb{R}^n$, let $r(B)$ denote its radius and $|B|$ its volume. Whenever $c > 0$, let cB denote the ball with the same centre as B and with radius $cr(B)$. If $r > 0$ and $z \in \mathbb{R}^n$ then $B_r(z)$ denotes the ball B of radius r and centre z.

Throughout, let $\mathscr{C}_0^\infty(\mathbb{R}^n)$ denote the space of smooth functions with compact support in $\mathbb{R}^n$. The Schwartz class of rapidly decreasing $\mathscr{C}^\infty$ functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$. Given f in $\mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\widehat{f}$ and inverse Fourier transform $\check{f}$ also belong to $\mathcal{S}(\mathbb{R}^n)$. (We use the convention $\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\langle x, \xi \rangle} f(x) \, dx$.) Let $\mathcal{Z}(\mathbb{R}^n)$ denote the set of functions φ in $\mathcal{S}(\mathbb{R}^n)$ possessing the property that $(\partial^\alpha \widehat{\varphi})(0) = 0$ for every multi-index α. Consider $\mathcal{Z}(\mathbb{R}^n)$ as a topological subspace of $\mathcal{S}(\mathbb{R}^n)$ and let $\mathcal{Z}'(\mathbb{R}^n)$ denote the topological dual of $\mathcal{Z}(\mathbb{R}^n)$. It is well-known (see, for example, [**16**, Section 5.1.2]) that $\mathcal{Z}'(\mathbb{R}^n)$ can be identified with the quotient space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where $\mathcal{S}'(\mathbb{R}^n)$ denotes the class of tempered distributions on $\mathbb{R}^n$ and $\mathcal{P}(\mathbb{R}^n)$ denotes the collection of all polynomials in $\mathbb{R}^n$. In this paper we use only the weak topology on $\mathcal{Z}'(\mathbb{R}^n)$. Hence we say that a sequence $(u_n)_{n=1}^\infty \subset \mathcal{Z}'(\mathbb{R}^n)$ converges in $\mathcal{Z}'(\mathbb{R}^n)$ to an element u of $\mathcal{Z}'(\mathbb{R}^n)$ if for each φ in $\mathcal{Z}(\mathbb{R}^n)$, $|u_n(\varphi) - u(\varphi)| \to 0$ as $n \to \infty$. The space $\mathcal{Z}'(\mathbb{R}^n)$ is complete with respect to this topology.

Many of the terms and definitions related to the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions have analogous formulations in $\mathcal{Z}'(\mathbb{R}^n)$. The Dirac delta distribution δ is defined as an element of $\mathcal{Z}'(\mathbb{R}^n)$ by the formula $\delta(\varphi) = \varphi(0)$ whenever $\varphi \in \mathcal{Z}(\mathbb{R}^n)$. Suppose that $u \in \mathcal{Z}'(\mathbb{R}^n)$. If Ω is an open set of $\mathbb{R}^n$ then we say that the support of u is contained in $\overline{\Omega}$, or that $\operatorname{supp} u \subset \overline{\Omega}$, if $u(\varphi) = 0$ for all φ in $\mathcal{Z}(\mathbb{R}^n)$ with support in $(\overline{\Omega})^c$. If $k \in \mathcal{S}(\mathbb{R}^n)$ then the convolution product $k * u$ is defined by

$$(k * u)(\varphi) = u(\tilde{k} * \varphi) \quad \forall \, \varphi \in \mathcal{Z}(\mathbb{R}^n),$$

where $\tilde{k}(x) = k(-x)$ whenever $x \in \mathbb{R}$. It is straightforward to show that $\tilde{k} * \varphi \in \mathcal{Z}(\mathbb{R}^n)$ and hence that $k * u \in \mathcal{Z}'(\mathbb{R}^n)$. Moreover

$$(2.2) \quad \operatorname{supp}(k*u) \subset \operatorname{supp} k + \operatorname{supp} u = \{x+y \in \mathbb{R}^n : x \in \operatorname{supp} k, \, y \in \operatorname{supp} u\}.$$

The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$ are subspaces of $\mathcal{Z}'(\mathbb{R}^n)$ defined in the following way. Fix a standard Littlewood-Paley dyadic system $(\theta_j)_{j\in\mathbb{Z}}$ of $\mathscr{C}^\infty$ functions on $\mathbb{R}^n$ with support in dyadic annuli centred at the origin (see [16, Chapter 5] for details). Given u in $\mathcal{Z}'(\mathbb{R}^n)$, define $\Delta_j u$ by $\Delta_j u(\varphi) = u((\theta_j\check{\varphi})^\frown) = (2\pi)^{-n/2}u(\widehat{\theta}_j * \varphi)$ for all φ in $\mathcal{Z}(\mathbb{R}^n)$. If $s \in \mathbb{R}$, $p > 0$ and $q > 0$ then $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$ are defined to be the spaces of all u in $\mathcal{Z}'(\mathbb{R}^n)$ with finite norms given by

$$\|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j u\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \qquad \text{and}$$

$$\|u\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = \left\|\left(\sum_{j\in\mathbb{Z}} 2^{jsq}|\Delta_j u(\cdot)|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}.$$

By suitable modification, one may also define homogeneous spaces when $p = \infty$ or $q = \infty$.

When $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q < \infty$, then $\mathcal{Z}(\mathbb{R}^n)$ is dense in $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and in $\dot{F}^s_{p,q}(\mathbb{R}^n)$ (see [16, Section 5.1.5]).

Both these classes include the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ as a special case, namely $\dot{H}^s(\mathbb{R}^n) = \dot{B}^s_{2,2}(\mathbb{R}^n) = \dot{F}^s_{2,2}(\mathbb{R}^n)$ whenever $s \in \mathbb{R}$. Moreover the Hardy spaces $H^p(\mathbb{R}^n)$, which are defined as subspaces of $\mathcal{S}'(\mathbb{R}^n)$, can be characterised by such norms because the natural projection from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{Z}'(\mathbb{R}^n)$ induces an isomorphism from $H^p(\mathbb{R}^n)$ to $\dot{F}^0_{p,2}(\mathbb{R}^n)$ when $0 < p < \infty$ (see [16, Section 5.2.4]).

To simplify notation, given any real number s, let $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ denote any one of the spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ (for fixed p and q satisfying $0 < p \leq \infty$, $0 < q \leq \infty$) or $\dot{F}^s_{p,q}(\mathbb{R}^n)$ (for fixed p and q satisfying $0 < p < \infty$, $0 < q \leq \infty$). Given a special Lipschitz domain Ω of $\mathbb{R}^n$, the spaces $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n)$ and $\dot{\mathscr{A}}^s(\Omega)$ are defined by

$$\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n) = \{u \in \dot{\mathscr{A}}^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Omega}\} \quad \text{and}$$

$$\dot{\mathscr{A}}^s(\Omega) = \dot{\mathscr{A}}^s(\mathbb{R}^n)/\dot{\mathscr{A}}^s_{\overline{\Omega^-}}(\mathbb{R}^n).$$

The quotient space $\dot{\mathscr{A}}^s(\Omega)$ can be interpreted both as a space of 'distributions' on $\mathbb{R}^n$ restricted to Ω, and as a space of 'distributions' on Ω. The subspace $\dot{\mathscr{A}}^s_{\bar{\Omega}}(\mathbb{R}^n)$ is a space of 'distributions' on $\mathbb{R}^n$.

If $f \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$ then f_t is given by the formula

$$f_t(x) = t^{-n} f(x/t) \quad \forall \, x \in \mathbb{R}^n.$$

It is easy to verify that the following formulae hold whenever $t > 0$, $1 \leq j \leq n$, and f and g belong to $\mathcal{S}(\mathbb{R}^n)$:

$$(\partial_j f)_t = t \partial_j (f_t), \quad (f * g)_t = f_t * g_t, \quad \text{and} \quad (f_t)^\frown(\xi) = \widehat{f}(t\xi).$$

Denote the full exterior algebra on $\mathbb{R}^n$ by $\Lambda := \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^n$, and the exterior product by $\wedge$. The interior product (or contraction) of $a = \sum_j a_j \, dx_j \in \Lambda^1$ with $u \in \Lambda^m$ is

$$a \lrcorner \, u := \sum_{k=1}^m (-1)^{k-1} a_{j_k} \, dx_{j_1} \wedge \cdots \wedge \widehat{dx_{j_k}} \wedge \cdots \wedge dx_{j_\ell} \in \Lambda^{m-1}$$

where the notation $\widehat{dx_{j_k}}$ means that the corresponding factor is to be omitted. (When $u \in \Lambda^0$ then $a \lrcorner \, u = 0$.) An identity that we shall need is

(2.3) $a \lrcorner \, (b \wedge u) + b \wedge (a \lrcorner \, u) = (a \cdot b)u \quad \forall \, a, b \in \Lambda^1$ and $u \in \Lambda$.

The space of differential forms with components in $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ is denoted by $\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda)$, and similarly for forms whose components lie in any of the test classes, functional classes, subspaces or quotient spaces mentioned above. The space of forms in $\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda)$ of order ℓ is denoted by $\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda^\ell)$. The topologies of these spaces are inherited in the obvious way. The exterior derivative d for forms acting on $\mathbb{R}^n$ is defined by $du = \Sigma_{j=1}^n e_j \wedge \partial_j u$, where $(e_j)_{j=1}^n$ denotes the standard basis for $\mathbb{R}^n$. It has the property $d^2 = 0$. By convention, $du = 0$ whenever u is an n-form. Note that d maps $\mathcal{Z}(\mathbb{R}^n, \Lambda)$ continuously into itself and is a well-defined map on $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$. (This is because ∂_j maps $\mathcal{Z}(\mathbb{R}^n)$ continuously into itself and hence by duality is a well-defined map on $\mathcal{Z}'(\mathbb{R}^n)$.) It also maps boundedly from $\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda)$ into $\dot{\mathscr{A}}^{s-1}(\mathbb{R}^n, \Lambda)$.

We defer definitions of Hardy spaces, tent spaces, and their atoms to Sections 8 and 9, except to note that the tent space $\mathcal{T}^1(\mathbb{R}^n \times \mathbb{R}^+)$ already used in Theorem 1.1 is the space of measurable functions $U \colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{C}$ with finite norm $\|U\|_{\mathcal{T}^1} = \int_{\mathbb{R}^n} \left(\iint_{|x-y| \leq t} |U(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} dx$.

3. Reproducing formulae on special Lipschitz domains

For the remainder of this paper, fix a Lipschitz constant A and a constant σ such that $A \geq 0$, $\sigma > 0$ and $\sigma A < 1$. Suppose that Ω is any special Lipschitz domain, defined by a function λ which satisfies the Lipschitz condition (2.1).

Before defining our potential map T on Ω, we investigate reproducing formulae for distributions with support in $\overline{\Omega}$, for which the functions in the integrand have support in Ω.

To do this, fix a function θ in $\mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \theta(x)\,dx = 1$ and

$$(3.1) \qquad \operatorname{supp}\theta \subset \{y \in \overline{\Gamma_\sigma(0)} : 1 \leq y_n,\ |y| \leq 2\},$$

and define the vector-valued $\mathscr{C}_0^\infty$ function Θ with the same support by

$$(3.2) \qquad \Theta(x) = \theta(x)\,x \quad \forall\, x \in \mathbb{R}^n.$$

Also, whenever $0 < a < b < \infty$, define the $\mathscr{C}_0^\infty(\mathbb{R}^n)$ function $\delta^{a,b}$ with support in $\overline{\Gamma_\sigma(0)}$ by

$$(3.3) \qquad \delta^{a,b} = \int_a^b (\nabla \cdot \Theta)_t \frac{dt}{t} = \int_a^b \nabla \cdot \Theta_t\,dt.$$

Proposition 3.1. *Whenever $0 < a < b < \infty$ we have the identity*

$$\delta^{a,b} = \theta_a - \theta_b.$$

Moreover, for each $u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$, $\delta^{a,b} * u \to u$ in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ as $a \to 0^+$ and $b \to \infty$. That is,

$$u = \int_0^\infty (\nabla \cdot \Theta)_t * u \frac{dt}{t} \quad \forall\, u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda),$$

where the improper integral converges in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$. Finally, if $\operatorname{supp}(u) \subset \overline{\Omega}$ then $\operatorname{supp}(\delta^{a,b} * u) \subset \overline{\Omega}$ whenever $0 < a < b < \infty$.

Proof: To verify $\delta^{a,b} = \theta_a - \theta_b$, it suffices to check that, for $0 < t < \infty$,

$$(3.4) \qquad \frac{d}{dt}\theta_t = -\frac{1}{t}(\nabla \cdot \Theta)_t.$$

A calculation using the product and chain rules shows that, for all $x \in \mathbb{R}^n$,

$$\frac{d}{dt}\theta_t(x) = \frac{d}{dt}\left(t^{-n}\theta\left(\tfrac{x}{t}\right)\right) = t^{-n}\frac{d}{dt}\left(\theta\left(\tfrac{x}{t}\right)\right) - nt^{-n-1}\theta\left(\tfrac{x}{t}\right)$$

$$= -\frac{1}{t}\left(\sum_{j=1}^{n} t^{-n}(\partial_j\theta)\left(\tfrac{x}{t}\right)\tfrac{x_j}{t} + nt^{-n}\theta\left(\tfrac{x}{t}\right)\right)$$

$$= -\frac{1}{t}\left(t^{-n}\sum_{j=1}^{n}(\partial_j\Theta_j)\left(\tfrac{x}{t}\right)\right)$$

$$= -\frac{1}{t}(\nabla \cdot \Theta)_t(x)$$

as claimed.

To prove that $\delta^{a,b} * u \to u$ for all $u \in \mathcal{Z}'(\mathbb{R}^n)$, we show that (i) $\theta_t * u \to u$ as $t \to 0^+$, and (ii) $\theta_t * u \to 0$ as $t \to \infty$.

To prove (i), it suffices to show that $\theta_t * \varphi \to \varphi$ in $\mathcal{Z}(\mathbb{R}^n)$ as $t \to 0^+$ whenever $\varphi \in \mathcal{Z}(\mathbb{R}^n)$. Suppose that $\varepsilon > 0$, $\varphi \in \mathcal{Z}(\mathbb{R}^n)$ and fix two multi-indices α and β. We need to show that

$$\sup_{x\in\mathbb{R}^n}\left|x^\beta\,\partial^\alpha(\theta_t * \varphi - \varphi)(x)\right| < \varepsilon$$

whenever t is sufficiently small. Since θ is compactly supported and φ and all its derivatives have rapid decay, the supremum over the set $\{x \in \mathbb{R}^n : |x| > R\}$ can be made arbitrarily small by taking R sufficiently large. Since θ has integral 1, the family $(\theta_t)_{0<t<1}$ is an approximate identity, and hence the supremum over the ball $\{x \in \mathbb{R}^n : |x| \le R\}$ can be made arbitrarily small by taking t sufficiently small. This completes the proof of (i).

We now prove (ii). Suppose that $\varepsilon > 0$, $\varphi \in \mathcal{Z}(\mathbb{R}^n)$, and fix two multi-indices α and β. Since the Fourier transform is continuous in the topology of $\mathcal{S}(\mathbb{R}^n)$, its suffices to show that

$$\sup_{\xi\in\mathbb{R}^n}\left|\xi^\alpha\partial^\beta(\widehat{\theta_t}\,\widehat{\varphi})(\xi)\right| < \varepsilon$$

whenever t is sufficiently large. This is not difficult to achieve using the fact that $\widehat{\theta_t}(\xi) = \widehat{\theta}(t\xi)$, along with the assumption that $\widehat{\varphi}$ and all its derivatives are zero at $0 \in \mathbb{R}^n$, as well as the rapid decay of the functions $\widehat{\theta}$, $\widehat{\varphi}$ and their derivatives at ∞.

Finally, note that if $\operatorname{supp}(u) \subset \overline{\Omega}$ then

$$\operatorname{supp}(\delta^{a,b} * u) \subset \overline{\Gamma_\sigma(0)} + \overline{\Omega} = \overline{\Omega} \quad \forall\, 0 < a < b.$$

This completes the proof. □

Remark 3.2. For those spaces $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ in which $\mathcal{Z}(\mathbb{R}^n)$ is dense (i.e. those spaces $\dot{\mathscr{A}}^s$ of the form $\dot{B}^s_{p,q}$ or $\dot{F}^s_{p,q}$ with $0 < p, q < \infty$), the convergence $\delta^{a,b} * u \to u$ also holds in $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$. To see this, use the convergence result in $\mathcal{Z}(\mathbb{R}^n, \Lambda)$, together with the uniform bound

$$\|\delta^{a,b} * u\|_{\dot{\mathscr{A}}^s} \leq C\|u\|_{\dot{\mathscr{A}}^s} \quad \forall\, 0 < a < b < \infty$$

where $C = C(\dot{\mathscr{A}}^s)$.

Our construction of the potential map T (see the beginning of the next section) was motivated by the reproducing formula used by Chang, Krantz and Stein in obtaining an atomic decomposition of functions in a Hardy space on a special Lipschitz domain. We indicate here the connection between our Proposition 3.1 and their reproducing formula [**3**, Lemma 3.4, Lemma 3.5].

Proposition 3.3. *Suppose that $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$ and*

(3.5) $$\operatorname{supp} \phi \subset \{y \in \overline{\Gamma_\sigma(0)} : \tfrac{1}{2} \leq y_n, |y| \leq 1\},$$

and define the vector-valued $\mathscr{C}_0^\infty$ function Ψ with the same support by

(3.6) $$\Psi(x) = 2\phi(x)\, x \quad \forall\, x \in \mathbb{R}^n.$$

When $0 < a < b < \infty$, define the $\mathscr{C}_0^\infty(\mathbb{R}^n)$ function $\underline{\delta}^{a,b}$ by

$$\underline{\delta}^{a,b} = \sum_{j=1}^n \int_a^b (\partial_j \phi)_t * (\Psi_j)_t \frac{dt}{t}.$$

Then, for each $u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$, $\underline{\delta}^{a,b} * u \to u$ in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ as $a \to 0^+$ and $b \to \infty$. Moreover, if $\operatorname{supp}(u) \subset \overline{\Omega}$ then $\operatorname{supp}(\underline{\delta}^{a,b} * u) \subset \overline{\Omega}$ whenever $0 < a < b < \infty$.

Proof: Suppose that ϕ satisfies the hypotheses of Proposition 3.3. So the function θ defined by $\theta := \phi * \phi$ satisfies the hypotheses of Proposition 3.1. Consequently $\delta^{a,b} * u \to u$ in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ for all $u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$, where

(3.7) $$\delta^{a,b} = \int_a^b (\nabla \cdot \Theta)_t \frac{dt}{t}$$

and $\Theta(x) = \theta(x)x$. We shall show that $\phi * \Psi = \Theta$, and hence that $\sum_{j=1}^n \partial_j \phi * \Psi_j = \nabla \cdot \Theta$, from which it follows by scaling and integrating that $\underline{\delta}^{a,b} = \delta^{a,b}$, thus proving the result.

What needs to be shown is that $(\phi * \Psi)(x) = (\phi * \phi)(x)x$ for all $x \in \mathbb{R}^n$. Indeed,

$$(\phi * \Psi)(x) = \int_{R^n} \phi(x - y) 2\phi(y) y \, dy$$

$$= \int_{\mathbb{R}^n} \phi(x - y) \phi(y) y \, dy + \int_{\mathbb{R}^n} \phi(y) \phi(x - y)(x - y) \, dy$$

$$= (\phi * \phi)(x) x,$$

thus proving the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.4. To be useful as a reproducing formula, the function ϕ can be chosen with some zero moments, in particular

$$(3.8) \qquad\qquad \int_{\mathbb{R}^n} \Psi(x) \, dx = 2 \int_{\mathbb{R}^n} \phi(x) \, x \, dx = 0.$$

Remark 3.5. For those spaces $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ in which $\mathcal{Z}(\mathbb{R}^n)$ is dense, the convergence $\underline{\delta}^{a,b} * u \to u$ also holds in $\dot{\mathscr{A}}^s_\Omega(\mathbb{R}^n, \Lambda)$.

4. The potential map for special Lipschitz domains

We now define the *potential map* T. Whenever $0 < a < b$, define the $\mathbb{R}^n$-valued kernel $K^{a,b}$ on $\mathbb{R}^n$ by

$$(4.1) \qquad\qquad K^{a,b}(x) = \int_a^b \Theta_t(x) \, dt \quad \forall \, x \in \mathbb{R}^n,$$

where Θ was defined at the start of Section 3. For each u in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$, define $T^{a,b}u$ and Tu by the formulae

$$(4.2) \qquad T^{a,b}u = K^{a,b} *_\lrcorner u \quad \text{and} \quad Tu = \lim_{a\to 0^+} \lim_{b\to\infty} K^{a,b} *_\lrcorner u,$$

where the limits are taken in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$. (If u is a 0-form then $Tu = 0$.) It is clear that $T^{a,b}u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$. It will be proved in Sections 5 and 6 that the limit in (4.2) exists in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$, thus defining $Tu \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$.

Note that if $u \in \mathcal{Z}(\mathbb{R}^n, \Lambda)$ then $T^{a,b}u \in \mathcal{Z}(\mathbb{R}^n, \Lambda)$ and $T^{a,b}u$ can be expressed by the formula

$$T^{a,b}u(x) = \int_a^b \int_{\mathbb{R}^n} \theta_t(x - y)(x - y) \lrcorner u(y) \, dy \frac{dt}{t}.$$

In this case we shall see that $Tu \in \mathcal{Z}(\mathbb{R}^n, \Lambda)$ and that $T^{a,b}u \to Tu$ in $\mathcal{Z}(\mathbb{R}^n, \Lambda)$.

We remark that in the case when $n = 1$, the domain Ω is a semi-open interval (α, ∞), $A = 0$ and $\Gamma_\sigma = (0, \infty)$. In this case, θ is a $\mathscr{C}_0^\infty(\mathbb{R})$

function with support in the interval $[1,2]$ such that $\int \theta(x)\,dx = 1$. We then obtain that

$$Tu(x) = \int_{-\infty}^{x} e_1 \lrcorner\, u(y)\,dy$$

whenever $u \in \mathcal{Z}(\mathbb{R}, \Lambda)$ and $x \in \mathbb{R}$. This is clearly the potential map for $d = \frac{d}{dx} e_1 \wedge$ which preserves support in (α, ∞).

The following theorem is the main result of this section.

Theorem 4.1. *Suppose that Ω is a special Lipschitz domain of $\mathbb{R}^n$ and that $s \in \mathbb{R}$. Then the operator T defined above has the following properties:*

(i) *the operator T maps from $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ to $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$;*

(ii) *for each space $\dot{\mathscr{A}}^s$ the operator T maps $\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda)$ to $\dot{\mathscr{A}}^{s+1}(\mathbb{R}^n, \Lambda)$, and there is a constant $c = c(\dot{\mathscr{A}}^s)$ such that*

$$\|Tu\|_{\dot{\mathscr{A}}^{s+1}(\mathbb{R}^n, \Lambda)} \le c\,\|u\|_{\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda)} \quad \forall\, u \in \dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda);$$

(iii) *$dTu + T\,du = u$ whenever $u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$;*

(iv) *if $u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$ and $\operatorname{supp}(u) \subset \overline{\Omega}$ then $\operatorname{supp}(Tu) \subset \overline{\Omega}$; and*

(v) *the operator T maps $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$ to $\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$, and for the same constant c of part (ii),*

$$\|Tu\|_{\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)} \le c\,\|u\|_{\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)} \quad \forall\, u \in \dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda).$$

Note that part (v) is an immediate consequence of parts (ii) and (iv) and the definition of $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$. We defer the proof of parts (i) to (iv) to Section 6.

Here we give an immediate application to the regularity of the exterior derivative on special Lipschitz domains.

Corollary 4.2. *Suppose that $s \in \mathbb{R}$ and that Ω is a special Lipschitz domain of $\mathbb{R}^n$. If $u \in \dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$ and $du = 0$ then there exists v in $\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$ such that $dv = u$. Moreover, there is a constant c independent of u such that*

$$\|v\|_{\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)} \le c\,\|u\|_{\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)}.$$

Consequently, the de Rham complex

$$0 \longrightarrow \dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^0) \xrightarrow{\ d\ } \dot{\mathscr{A}}^{s-1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^1) \xrightarrow{\ d\ } \dot{\mathscr{A}}^{s-2}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^2) \xrightarrow{\ d\ } \cdots$$

$$\cdots \xrightarrow{\ d\ } \dot{\mathscr{A}}^{s-n}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^n) \longrightarrow 0$$

is exact, and each space $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^k)$ *has a direct sum decomposition*

$$\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^k) = d\,\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{k-1}) \oplus T d\,\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^k)$$

with bounded projections dT *and* Td.

Proof of Corollary 4.2: If $du = 0$ then the anti-commutation relation in part (iii) of the theorem becomes $dTu = u$. So set $v = Tu$. It is straightforward to check that dT and Td are bounded projections and that $dT\,\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^k) = d\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^{k-1})$. $\qquad\square$

The important observation is that the projections are the same for all choices of the homogeneous Besov and Triebel-Lizorkin spaces $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^k)$, and consequently the spaces $d\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^k)$ of exact forms have the same interpolation properties as do the spaces $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^k)$.

We remark that the operator T as well as the constants in the estimates, depend on A and the choice of σ and Θ, but not on the precise domain Ω.

Remark 4.3. In the course of the proof, we show that the limit in formula (4.2) holds in $\mathcal{Z}(\mathbb{R}^n, \Lambda)$ when $u \in \mathcal{Z}(\mathbb{R}^n, \Lambda)$, and in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ when $u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda)$. It can also be shown that, for those spaces $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ in which $\mathcal{Z}(\mathbb{R}^n)$ is dense, the limit in formula (4.1) holds in $\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$ when $u \in \dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$. To see this, use the convergence result in $\mathcal{Z}(\mathbb{R}^n, \Lambda)$, together with the uniform bound

$$\|T^{a,b}u\|_{\dot{\mathscr{A}}^{s+1}} = \|K^{a,b} * u\|_{\dot{\mathscr{A}}^{s+1}} \le C\|u\|_{\dot{\mathscr{A}}^s} \quad \forall\, 0 < a < b < \infty$$

where $C = C(\dot{\mathscr{A}}^s)$. See Remark 5.4.

When $\Theta = \phi * \Psi$ as in Proposition 3.3, the operator T has the form

$$Tu = \lim_{a\to 0^+} \lim_{b\to\infty} \int_a^b \Theta_t * {\lrcorner}\, u \, dt$$

$$= \lim_{a\to 0^+} \lim_{b\to\infty} \int_a^b \phi_t * \Psi_t * {\lrcorner}\, u \, dt,$$

and, when $du = 0$,

(4.3) $$u = dTu = \lim_{a\to 0^+} \lim_{b\to\infty} \int_a^b (d\phi)_t * \wedge \Psi_t * {\lrcorner}\, u \frac{dt}{t}.$$

As before, convolution with ϕ_t and Ψ_t preserve support in the special Lipschitz domain Ω, and T is a bounded operator from $\dot{\mathscr{A}}^s_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$ to $\dot{\mathscr{A}}^{s+1}_{\overline{\Omega}}(\mathbb{R}^n, \Lambda)$ for every choice of $\dot{\mathscr{A}}^s$. Moreover, by Remarks 3.2

and 4.3, for those spaces $\mathscr{A}^s(\mathbb{R}^n)$ in which $\mathcal{Z}(\mathbb{R}^n)$ is dense, the limits exist in $\dot{\mathscr{A}}_{\overline{\Omega}}^{s+1}(\mathbb{R}^n, \Lambda)$ or $\dot{\mathscr{A}}_{\overline{\Omega}}^{s}(\mathbb{R}^n, \Lambda)$ when $u \in \dot{\mathscr{A}}_{\overline{\Omega}}^{s}(\mathbb{R}^n, \Lambda)$.

5. Tools used in proving Theorem 4.1

In this section we state and prove some rudimentary results about operators defined by Fourier multipliers. These will be used later in the proof of Theorem 4.1. We start with an homogeneous analogue of the Hörmander conditions.

Proposition 5.1. *Suppose that $m \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and $k \in \mathbb{R}$. Suppose also that, for each multi-index α, there exist c_α such that*

$$(5.1) \qquad |\partial^\alpha m(\xi)| \leq c_\alpha |\xi|^{-|\alpha|+k} \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then the operator S, given by

$$Su(\varphi) = u\big((m\check{\varphi})\widehat{}\big) \quad \forall\, u \in \mathcal{Z}'(\mathbb{R}^n) \quad \forall\, \varphi \in \mathcal{Z}(\mathbb{R}^n),$$

is well-defined on $\mathcal{Z}'(\mathbb{R}^n)$ and maps each space $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ into $\dot{\mathscr{A}}^{s-k}(\mathbb{R}^n)$ with

$$\|Su\|_{\dot{\mathscr{A}}^{s-k}(\mathbb{R}^n)} \leq c\,\|u\|_{\dot{\mathscr{A}}^{s}(\mathbb{R}^n)} \quad \forall\, u \in \dot{\mathscr{A}}^s(\mathbb{R}^n)$$

for some constant c (depending on k, c_α and the space $\dot{\mathscr{A}}^s$).

Proof: Assume the hypotheses of the proposition. To show that S is well-defined on $\mathcal{Z}'(\mathbb{R}^n)$, it suffices to verify that $(m\check{\varphi})\widehat{} \in \mathcal{Z}(\mathbb{R}^n)$ whenever $\varphi \in \mathcal{Z}(\mathbb{R}^n)$. Since m has at most polynomial growth at infinity, one need only show that $\lim_{\xi \to 0} \partial^\alpha(m\check{\varphi})(\xi) = 0$ for every multi-index α. But this follows from (5.1) and the fact that $\check{\varphi}(\xi)$ is $O(|\xi|^N)$ for every positive integer N.

To prove the bound, consider the Fourier multiplier m_k given by $m_k(\xi) = |\xi|^{-k} m(\xi)$. Clearly m_k has derivatives of all orders away from 0, and

$$\sup\left\{ |\xi|^{|\alpha|} |\partial^\alpha m_k(\xi)| : \xi \in \mathbb{R}^n \setminus \{0\},\ |\alpha| \leq N \right\}$$

is bounded for each positive integer N. So the operator Q_k, defined by

$$Q_k u(\varphi) = u\big((m_k \check{\varphi})\widehat{}\big) \quad \forall\, u \in \mathcal{Z}'(\mathbb{R}^n) \quad \forall\, \varphi \in \mathcal{Z}(\mathbb{R}^n)$$

is well-defined on $\mathcal{Z}'(\mathbb{R}^n)$ and is bounded on $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ by standard Fourier multiplier theory (see, for example, [**16**, Theorem 5.2.2]).

To complete the proof, let ι denote the $\mathbb{R}^n$-valued function on $\mathbb{R}^n$ given by $\iota(\xi) = \xi$ and $\dot{I}_k$ denote the lifting operator given by

$$\dot{I}_k v(\varphi) = v\big((|\iota|^k \check{\varphi})\widehat{}\big) \quad \forall\, v \in \mathcal{Z}'(\mathbb{R}^n) \quad \forall\, \varphi \in \mathcal{Z}(\mathbb{R}^n),$$

which maps $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ isomorphically onto $\dot{\mathscr{A}}^{s-k}(\mathbb{R}^n)$ (see [**16**, Theorem 5.2.3]). Then

$$Su(\varphi) = u\big((m\check{\varphi})^\frown\big) = u\big((|\iota|^k m_k \check{\varphi})^\frown\big) = \dot{I}_k Q_k u(\varphi),$$

or in other words, $S = \dot{I}_k Q_k$. This completes the proof. $\qquad\square$

Every homogeneous function m of degree $k \in \mathbb{R}$ (i.e., $m(\tau\xi) = \tau^k m(\xi)$, $\forall\, \xi \in \mathbb{R}^n \setminus \{0\}$, $\forall\, \tau > 0$) satisfies the hypotheses (with $c_\alpha = \sup\{|\partial^\alpha m(\omega)| : |\omega| = 1\}$), and therefore the conclusion, of the above proposition.

Let us turn our attention to bounds and convergence results for convolution operators.

Given a function ψ in $\mathscr{C}_0^\infty(\mathbb{R}^n)$, define the truncated kernel $k^{a,b}$ by

$$k^{a,b}(x) = \int_a^b \psi_t(x)\,dt \quad \forall\, x \in \mathbb{R}^n$$

whenever $0 < a < b$. Since $k^{a,b} \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, the operator $S^{a,b}$, given by

$$S^{a,b}u = k^{a,b} * u \quad \forall\, u \in \mathscr{Z}'(\mathbb{R}^n)$$

is well-defined on $\mathscr{Z}'(\mathbb{R}^n)$. Define the multiplier $m^{a,b}$ by

$$(5.2) \qquad\qquad m^{a,b} = (2\pi)^{n/2} \widehat{k^{a,b}},$$

so that $S^{a,b}$ can also be represented by

$$S^{a,b}u(\varphi) = u(\widetilde{k^{a,b}} * \varphi) = u\big((m^{a,b}\check{\varphi})^\frown\big) \quad \forall\, u \in \mathscr{Z}'(\mathbb{R}^n) \quad \forall\, \varphi \in \mathscr{Z}(\mathbb{R}^n).$$

Note that each component of $K^{a,b} * {}_\lrcorner u$, where $K^{a,b}$ is the kernel used to define $T^{a,b}$ in Section 4, is of the form $S^{a,b}u_I$, where $\psi(x) = \theta(x)x_j$ and u_I is a component of u corresponding to an index set I including j.

The following lemma will be used to show that the limit

$$\lim_{a\to 0^+} \lim_{b\to\infty} S^{a,b}u,$$

taken in the topology of $\mathscr{Z}'(\mathbb{R}^n)$, is well-defined whenever $u \in \mathscr{Z}'(\mathbb{R}^n)$.

Lemma 5.2. *Suppose that $(m^{a,b})_{0<a<b}$ is the net of Schwartz functions defined by* (5.2). *Then the function m, given by*

$$(5.3) \qquad m(\xi) = \lim_{a\to 0^+} \lim_{b\to\infty} m^{a,b}(\xi) \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\},$$

is well-defined. Moreover, $m \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and for any multi-index α,

$$(5.4) \qquad \partial^\alpha m(\xi) = \lim_{a\to 0^+} \lim_{b\to\infty} \partial^\alpha m^{a,b}(\xi) \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\},$$

where the convergence is uniform on annuli centred at the origin. The function m is homogeneous of degree -1 on $\mathbb{R}^n$.

For any multi-index α, there exists a constant c_α such that, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

(5.5) $$|\partial^\alpha m^{a,b}(\xi)| \leq c_\alpha |\xi|^{|\alpha|-1} \quad \forall \, 0 < a < b < \infty$$

and also $|\partial^\alpha m(\xi)| \leq c_\alpha |\xi|^{|\alpha|-1}$.

Proof: Fix a multi-index α. To show that m is well-defined and its derivatives are given by (5.4), it suffices to show that the net $(\partial^\alpha m^{a,b})_{a,b}$ is uniformly Cauchy on the annulus $\{\xi \in \mathbb{R}^n : r \leq |\xi| \leq R\}$ for some fixed numbers r and R satisfying $0 < r < R < \infty$. Henceforth, suppose that $r \leq |\xi| \leq R$.

To begin, note that

$$m^{a,b}(\xi) = \int_a^b \widehat{\psi}(t\xi) \, dt$$

whenever $0 < a < b$. Therefore

$$\left|(\partial^\alpha m^{a,b})(\xi)\right| = \left| \int_a^b t^{|\alpha|} (\partial^\alpha \widehat{\psi})(t\xi) \, dt \right|$$

(5.6) $$\leq |\xi|^{-|\alpha|} \int_a^b |t\xi|^{|\alpha|} |(\partial^\alpha \widehat{\psi})(t\xi)| \, dt$$

$$\leq c_\alpha |\xi|^{-|\alpha|} \int_a^b \min\{\tfrac{1}{4}, |t\xi|^{-2}\} \, dt$$

for some constant c_α, since $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$. Now

$$\left|(\partial^\alpha m^{b,b_0})(\xi)\right| \leq c_\alpha |\xi|^{-|\alpha|-2} \left(\frac{1}{b} - \frac{1}{b_0} \right) < \frac{c_\alpha}{r^{|\alpha|+2}b}$$

whenever $b_0 > b$, while

$$\left|(\partial^\alpha m^{a_0,a})(\xi)\right| \leq c_\alpha |\xi|^{-|\alpha|}(a - a_0) < c_\alpha r^{-|\alpha|}a$$

whenever $a_0 < a$. So if $0 < a_0 < a < b < b_0$ then

$$\left|(\partial^\alpha m^{a_0,b_0})(\xi) - (\partial^\alpha m^{a,b})(\xi)\right| < c_\alpha \left(r^{-|\alpha|-2}b^{-1} + r^{-|\alpha|}a \right),$$

which can be made as small as we like by taking b sufficiently large and a sufficiently close to 0.

It is straightforward to verify that $m^{a,b}(\lambda\xi) = \lambda^{-1}m^{\lambda a, \lambda b}(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^n$. On taking limits in a and b, we deduce the fact that m is homogeneous of degree -1.

Finally, we note that the bound (5.5) is an immediate consequence of (5.6), and the bound on $\partial^\alpha m$ follows by taking limits. $\square$

Using Proposition 5.1 and Lemma 5.2, we now define the operator S on $\mathcal{Z}'(\mathbb{R}^n)$ by

$$Su(\varphi) = u\big((m\check{\varphi})^\frown\big) \quad \forall\, u \in \mathcal{Z}'(\mathbb{R}^n) \quad \forall\, \varphi \in \mathcal{Z}(\mathbb{R}^n),$$

where m is the function given by (5.3). That is, S is the operator on $\mathcal{Z}'(\mathbb{R}^n)$ defined by the homogeneous Fourier multiplier m.

Proposition 5.3. *If $u \in \mathcal{Z}'(\mathbb{R}^n)$ then $S^{a,b}u$ converges to Su in $\mathcal{Z}'(\mathbb{R}^n)$ as $a \to 0^+$ and as $b \to \infty$.*

Proof: Suppose that $\varphi \in \mathcal{Z}(\mathbb{R}^n)$ and $u \in \mathcal{Z}'(\mathbb{R}^n)$. Since

$$S^{a,b}u(\varphi) = u\big((m^{a,b}\check{\varphi})^\frown\big) \quad \forall\, u \in \mathcal{Z}'(\mathbb{R}^n) \quad \forall\, \varphi \in \mathcal{Z}(\mathbb{R}^n),$$

it suffices to show that $m^{a,b}\check{\varphi} \to m\check{\varphi}$ in $\mathcal{S}(\mathbb{R}^n)$ as $a \to 0^+$ and $b \to \infty$. For then $(m^{a,b}\check{\varphi})^\frown \to (m\check{\varphi})^\frown$ in $\mathcal{S}(\mathbb{R}^n)$ and hence in $\mathcal{Z}(\mathbb{R}^n)$. It follows that

$$(S^{a,b}u - Su)(\varphi) = u\left(\big((m^{a,b} - m)\check{\varphi}\big)^\frown\right) \to 0$$

as $a \to 0^+$ and $b \to \infty$, which establishes the lemma.

Suppose that α and β are two multi-indices and $\epsilon > 0$. We need to show that there exist positive numbers a_0 and b_0 such that

$$\sup_{\xi \in \mathbb{R}^n} \left| \xi^\alpha \partial^\beta \big((m - m^{a,b})\check{\varphi}\big)(\xi) \right| < \epsilon$$

whenever $0 < a < a_0 < b_0 < b$. By expanding the left-hand side using the multidimensional version of Leibniz' rule, it suffices to show that there are positive numbers a_0 and b_0 such that

$$(5.7) \qquad \sup_{\xi \in \mathbb{R}^n} \sup_{|\gamma| \leq |\beta|} |\xi|^{|\alpha|} \left| \partial^\gamma(m - m^{a,b})(\xi) \right| \left| \partial^{\beta-\gamma}\check{\varphi}(\xi) \right| < \frac{\epsilon}{C_\beta}$$

whenever $0 < a < a_0 < b_0 < b$, where the constant C_β is the largest coefficient appearing the formula for Leibniz' rule.

By Lemma 5.2,

$$(5.8) \qquad |\partial^\gamma(m - m^{a,b})(\xi)| \leq 2c_\gamma |\xi|^{-|\gamma|-1} \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}$$

whenever $0 < a < b$. Now choose R in $(0, \infty)$ so large that

$$(5.9) \qquad \sup_{|\xi| > R} \sup_{|\gamma| \leq |\beta|} 2c_\gamma |\xi|^{|\alpha|-|\gamma|-1} |\partial^{\beta-\gamma}\check{\varphi}(\xi)| < \frac{\epsilon}{3C_\beta}.$$

This is possible since $\partial^{\beta-\gamma}\check{\varphi}$ is rapidly decreasing at infinity. Now choose r in $(0, \infty)$ so small that

$$(5.10) \qquad \sup_{0 < |\xi| < r} \sup_{|\gamma| \leq |\beta|} 2c_\gamma |\xi|^{|\alpha|-|\gamma|-1} |\partial^{\beta-\gamma}\check{\varphi}(\xi)| < \frac{\epsilon}{3C_\beta}.$$

This is possible since $\check{\varphi}$ and all its partial derivatives are 0 at the origin. By Lemma 5.2, there are positive numbers a_0 and b_0 such that

$$(5.11) \qquad \sup_{r \leq |\xi| \leq R} \sup_{|\gamma| \leq |\beta|} \left| \xi^\alpha \big(\partial^\gamma (m - m^{a,b}) \big)(\xi) \big(\partial^{\beta-\gamma} \check{\varphi} \big)(\xi) \right| < \frac{\epsilon}{3 C_\beta}$$

whenever $0 < a < a_0 < b_0 < b$. By combining estimates (5.8), (5.9), (5.10) and (5.11), it is easy to see that (5.7) holds whenever $0 < a < a_0 < b_0 < b$. This shows that $m^{a,b} \check{\varphi} \to m \check{\varphi}$ in $\mathcal{S}(\mathbb{R}^n)$ and completes the proof. $\qquad \square$

Remark 5.4. It is a consequence of (5.5) and Proposition 5.1 (with $k = -1$) that, in each space $\dot{\mathscr{A}}(\mathbb{R}^n)$, there is a uniform bound

$$\|S^{a,b} u\|_{\dot{\mathscr{A}}^{s+1}} = \|k^{a,b} * u\|_{\dot{\mathscr{A}}^{s+1}} \leq C \|u\|_{\dot{\mathscr{A}}^s} \quad \forall \, 0 < a < b < \infty$$

where $C = C(\dot{\mathscr{A}}^s)$. Thus, for those spaces $\dot{\mathscr{A}}^s(\mathbb{R}^n)$ in which $\mathcal{Z}(\mathbb{R}^n)$ is dense, $S^{a,b} u \to S u$ in $\dot{\mathscr{A}}^{s+1}(\mathbb{R}^n)$ as $a \to 0^+$ and as $b \to \infty$, whenever $u \in \dot{\mathscr{A}}^s(\mathbb{R}^n)$.

6. Proof of the main theorem on potential maps

We now use the tools presented in Section 5 to prove Theorem 4.1.

6.1. Mapping properties of T. Each component of T is a limit in $\mathcal{Z}'(\mathbb{R}^n)$ of convolution operators of the form $S^{a,b}$ and therefore has the same properties as S. In particular, Proposition 5.1 shows that T maps from $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ into $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ and boundedly lifts 'functions' of degree s in the homogeneous Besov and Triebel-Lizorkin spaces to 'functions' of degree $s + 1$. Thus we have proved parts (i) and (ii) of Theorem 4.1.

We also note that Remark 4.3 is a consequence of Remark 5.4.

6.2. Anticommutation relations. We now turn to the proof of Theorem 4.1(iii). Suppose that $0 < a < b$. Recall that the convolution operator $T^{a,b}$ is defined on $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$ by (4.2). Recall also that $\delta^{a,b}$ denotes the function in $\mathcal{Z}(\mathbb{R}^n)$ given by

$$\delta^{a,b} = \int_a^b (\nabla \cdot \Theta)_t \frac{dt}{t} = \int_a^b \nabla \cdot \Theta_t \, dt.$$

Fix u in $\mathcal{Z}'(\mathbb{R}^n, \Lambda)$. If we can show that

$$(6.1) \qquad\qquad d T^{a,b} u + T^{a,b} d u = \delta^{a,b} * u,$$

then, by taking limits as $a \to 0^+$ and $b \to \infty$ in $\mathcal{Z}'(\mathbb{R}^n)$ and applying Proposition 3.1 and Proposition 5.3, part (iii) of Theorem 4.1 will be proved.

Now

$$T^{a,b}u = \int_a^b \Theta_t * \lrcorner\, u\, dt,$$

so

$$dT^{a,b}u = \sum_{j=1}^n e_j \wedge \partial_j \int_a^b \Theta_t * \lrcorner\, u\, dt$$

$$= \sum_{j=1}^n \int_a^b e_j \wedge (\partial_j \Theta_t * \lrcorner\, u)\, dt$$

and

$$T^{a,b}du = \int_a^b \Theta_t * \lrcorner\, du\, dt$$

$$= \sum_{j=1}^n \int_a^b \partial_j \Theta_t * \lrcorner\, (e_j \wedge u)\, dt.$$

Therefore, using the identity (2.3), we obtain

$$dT^{a,b}u + T^{a,b}du = \sum_{j=1}^n \int_a^b \partial_j \Theta_t \cdot e_j * u\, dt$$

$$= \int_a^b \nabla \cdot \Theta_t * u\, dt$$

$$= \delta^{a,b} * u$$

as claimed. Hence we have shown equation (6.1), thus completing the proof of part (iii).

6.3. Support properties of T. To complete the proof of Theorem 4.1, it remains to show part (iv). Suppose $\operatorname{supp} u \subset \overline{\Omega}$. By (2.2) and (3.1),

$$\operatorname{supp}(\Theta_t * \lrcorner\, u) \subset \operatorname{supp}\Theta_t + \operatorname{supp} u$$

$$\subset \overline{\Gamma_\sigma(0)} + \overline{\Omega} = \overline{\Omega}$$

whenever $t \in (0, \infty)$. Hence $\operatorname{supp} T^{a,b}u \subset \overline{\Omega}$ whenever $0 < a < b$. By taking limits in $\mathscr{Z}'(\mathbb{R}^n)$ as $a \to 0^+$ and $b \to \infty$, one concludes that $\operatorname{supp}(Tu) \subset \overline{\Omega}$.

This completes the proof of Theorem 4.1.   □

7. Analogous results for complementary domains and restriction spaces

Suppose that Ω is a special Lipschitz domain of $\mathbb{R}^n$, and recall that Ω^- denotes the region strictly below the corresponding Lipschitz graph. Define an operator $\tilde{T}$ by the formula

$$\tilde{T}u = \lim_{a \to 0^+} \lim_{b \to \infty} \tilde{K}^{a,b} * \lrcorner\, u, \quad \forall\, u \in \mathcal{Z}'(\mathbb{R}^n, \Lambda),$$

where both limits are taken in $\mathcal{Z}'(\mathbb{R}^n)$ and $\tilde{K}^{a,b}(x) = K^{a,b}(-x)$ for all x in $\mathbb{R}^n$. Here $K^{a,b}$ is the truncated kernel given by (4.1). The analytic properties of $\tilde{T}$ are clearly the same as those of T. However, if $\operatorname{supp} u \subset \overline{\Omega^-}$ then $\operatorname{supp}(\tilde{T}u) \subset \overline{\Omega^-}$. Hence, in Theorem 4.1, one may replace T by $\tilde{T}$ throughout, and Ω by Ω^- in parts (iv) and (v), to obtain an analogous result for the complementary Lipschitz domain Ω^-.

We now draw some conclusions for the restriction space $\dot{\mathscr{A}}^s(\Omega, \Lambda)$. Given u in $\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda)$, let $[u]$ denote the equivalence class with representative u associated to the equivalence relation

$$v \sim w \iff v - w \in \dot{\mathscr{A}}^s_{\Omega^-}(\mathbb{R}^n, \Lambda).$$

By definition, $[u]$ belongs to $\dot{\mathscr{A}}^s(\Omega, \Lambda)$ and conversely every element of $\dot{\mathscr{A}}^s(\Omega, \Lambda)$ is of this form. Define an operator R by

$$R[u] = [\tilde{T}u] \quad \forall\, [u] \in \dot{\mathscr{A}}^s(\Omega, \Lambda).$$

Since $\tilde{T}$ maps boundedly from $\dot{\mathscr{A}}^s(\mathbb{R}^n, \Lambda)$ to $\dot{\mathscr{A}}^{s+1}(\mathbb{R}^n, \Lambda)$ and from $\dot{\mathscr{A}}^s_{\Omega^-}(\mathbb{R}^n, \Lambda)$ to $\dot{\mathscr{A}}^{s+1}_{\Omega^-}(\mathbb{R}^n, \Lambda)$, the operator R is well-defined and maps boundedly from $\dot{\mathscr{A}}^s(\Omega, \Lambda)$ to $\dot{\mathscr{A}}^{s+1}(\Omega, \Lambda)$.

Similarly, the exterior derivative d is defined as an operator on $\dot{\mathscr{A}}^s(\Omega, \Lambda)$ by

$$d[u] = [du] \quad \forall\, [u] \in \dot{\mathscr{A}}^s(\Omega, \Lambda),$$

and maps boundedly from $\dot{\mathscr{A}}^s(\Omega, \Lambda)$ into $\dot{\mathscr{A}}^{s-1}(\Omega, \Lambda)$. We thus obtain another variant of Theorem 4.1.

Proposition 7.1. *Suppose that Ω is a special Lipschitz domain of $\mathbb{R}^n$ and that $s \in \mathbb{R}$. Then the operator R defined above has the following properties:*

(i) *the operator R maps $\dot{\mathscr{A}}^s(\Omega, \Lambda)$ to $\dot{\mathscr{A}}^{s+1}(\Omega, \Lambda)$, and there is a constant c such that*

$$\|R[u]\|_{\dot{\mathscr{A}}^{s+1}(\Omega, \Lambda)} \le c\,\|[u]\|_{\dot{\mathscr{A}}^s(\Omega, \Lambda)} \quad \forall [u] \in \dot{\mathscr{A}}^s(\Omega, \Lambda);$$

(ii) *$dR[u] + Rd[u] = [u]$ whenever $[u] \in \dot{\mathscr{A}}^s(\Omega, \Lambda)$.*

One immediately obtains a regularity result for the exterior derivative on $\dot{\mathscr{A}}^s(\Omega, \Lambda)$.

Corollary 7.2. *Suppose that $s \in \mathbb{R}$ and Ω is a special Lipschitz domain. If $[u] \in \dot{\mathscr{A}}^s(\Omega, \Lambda)$ and $d[u] = 0$ then there exists $[v]$ in $\dot{\mathscr{A}}^{s+1}(\Omega, \Lambda)$ and a constant c independent of $[u]$ such that $d[v] = [u]$ and*

$$\|[v]\|_{\dot{\mathscr{A}}^{s+1}(\Omega, \Lambda)} \leq c \, \|[u]\|_{\dot{\mathscr{A}}^s(\Omega, \Lambda)}.$$

Consequently, the de Rham complex

$$0 \longrightarrow \dot{\mathscr{A}}^s(\Omega, \Lambda^0) \xrightarrow{\ d\ } \dot{\mathscr{A}}^{s-1}(\Omega, \Lambda^1) \xrightarrow{\ d\ } \dot{\mathscr{A}}^{s-2}(\Omega, \Lambda^2) \xrightarrow{\ d\ } \cdots$$

$$\cdots \xrightarrow{\ d\ } \dot{\mathscr{A}}^{s-n}(\Omega, \Lambda^n) \longrightarrow 0$$

is exact, and each space $\dot{\mathscr{A}}^s(\Omega, \Lambda^k)$ has a direct sum decomposition

$$\dot{\mathscr{A}}^s(\Omega, \Lambda^k) = d\dot{\mathscr{A}}^{s+1}(\Omega, \Lambda^{k-1}) \oplus Rd\dot{\mathscr{A}}^s(\Omega, \Lambda^k)$$

with bounded projections dR and Rd.

8. Atomic decomposition of Hardy spaces of exact forms on special Lipschitz domains

In this section we use the operator T of Theorem 4.1 and the reproducing formulae above, including the zero moment condition (3.8), to show that Hardy spaces of exact forms on special Lipschitz domains can be characterised by atomic decompositions.

In the following definitions of these spaces and their corresponding atoms, we at first allow Ω to be an arbitrary domain in $\mathbb{R}^n$, where $n \geq 1$.

Definition 8.1. Suppose that $1 \leq \ell \leq n$ and $n/(n+1) < p \leq 1$. Let $H_d^p(\mathbb{R}^n, \Lambda^\ell)$ denote the Hardy space of all ℓ-forms u in $H^p(\mathbb{R}^n, \Lambda)$ such that $u = dv$ for some $(\ell-1)$-form v in $\mathcal{S}'(\mathbb{R}^n, \Lambda^{\ell-1})$. Given a domain Ω in $\mathbb{R}^n$, we say that u is in $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$ if $u \in H_d^p(\mathbb{R}^n, \Lambda^\ell)$ and there exists v in $\mathcal{S}'(\mathbb{R}^n, \Lambda^{\ell-1})$ such that $u = dv$ and $\operatorname{supp} v \subset \overline{\Omega}$.

Remark 8.2. Definition 8.1 was first introduced in the papers [**10**] and [**9**] of Lou and McIntosh for the case when $p = 1$. When $n/(n+1) < p \leq 1$, the space $H_d^p(\mathbb{R}^n, \Lambda^n)$ is isomorphic to the classical Hardy space $H^p(\mathbb{R}^n)$, while $H_{z,d}^p(\overline{\Omega}, \Lambda^n)$ is isomorphic to the Hardy space $H_z^p(\Omega)$ of Chang, Krantz and Stein [**3**].

Following [**10**] and [**9**], we introduce atoms of Hardy spaces of exact forms.

Definition 8.3. Suppose that $1 \leq \ell \leq n$ and $n/(n+1) < p \leq 1$. We say that a is an $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atom if for some ball B in $\mathbb{R}^n$,

(a) there exists b in $L^2(\mathbb{R}^n, \Lambda^{\ell-1})$ such that $\operatorname{supp} b \subset \overline{B}$ and $a = db$, and

(b) $\|a\|_{L^2(\mathbb{R}^n, \Lambda)} \leq |B|^{1/2 - 1/p}$.

Note that if $n/(n+1) < p \leq 1$ and a is an $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atom, then each component of a is a classical $H^p(\mathbb{R}^n)$-atom.

Definition 8.4. Suppose that Ω is a domain of $\mathbb{R}^n$, $1 \leq \ell \leq n$ and $n/(n+1) < p \leq 1$. We say that a is an $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atom if for some ball B in $\mathbb{R}^n$,

(a) there exists b in $L^2(\mathbb{R}^n, \Lambda^{\ell-1})$ such that $\operatorname{supp} b \subset \overline{B}$ and $a = db$,

(b) $\|a\|_{L^2(\mathbb{R}^n, \Lambda)} \leq |B|^{1/2 - 1/p}$, and

(c) $4B \subset \Omega$.

Note that, following [9], the supports of $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atoms are away from the boundary of Ω, which is stronger than the classical definition of [3].

The following lemma gives an L^2 estimate for the function b of Definitions 8.3 and 8.4.

Lemma 8.5. *Suppose that* $1 \leq \ell \leq n$, $n/(n+1) < p \leq 1$ *and* a *is an* $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atom (respectively an $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atom). Then the $L^2(\mathbb{R}^n, \Lambda^{\ell-1})$ form b of Definition 8.3 (respectively Definition 8.4) can be chosen such that*

$$\|b\|_{L^2(\mathbb{R}^n, \Lambda)} \leq c_n r(B) |B|^{1/2 - 1/p}$$

where the constant c_n *depends only on* n.

Proof: Let $\mathscr{B}$ denote the collection of all balls in $\mathbb{R}^n$ and let $L_{\overline{B}}^2(\mathbb{R}^n, \Lambda^k)$ denote the space of k-forms with components in $L^2(\mathbb{R}^n)$ and support in the closure of a ball B. By applying the result of [12] or [6, Section 3] to a unit ball and then scaling, one obtains the following. There exists a constant c_n and a family of operators $\{T_\ell^B : 1 \leq \ell \leq n, B \in \mathscr{B}\}$ with the following properties:

(i) if $B \in \mathscr{B}$ and $1 \leq \ell \leq n$ then T_ℓ^B maps from $L_{\overline{B}}^2(\mathbb{R}^n, \Lambda^\ell)$ to $L_{\overline{B}}^2(\mathbb{R}^n, \Lambda^{\ell-1})$ and

$$\left\|T_\ell^B u\right\|_{L^2(\mathbb{R}^n, \Lambda)} \leq c_n r(B) \|u\|_{L^2(\mathbb{R}^n, \Lambda)} \quad \forall u \in L_{\overline{B}}^2(\mathbb{R}^n, \Lambda^\ell);$$

(ii) if $B \in \mathscr{B}$ and $1 \leq \ell < n$ then $dT_\ell^B u + T_{\ell+1}^B du = u$ for every $u \in L_{\overline{B}}^2(\mathbb{R}^n, \Lambda^\ell)$; and

(iii) if $B \in \mathscr{B}$ then there exists an n-form ϑ^B of $\mathscr{C}_0^\infty(\mathbb{R}^n, \Lambda^n)$ supported in $\frac{1}{2}B$ such that $dT_n^B u = u - (\int u)\vartheta^B$ for every $u \in L_{\overline{B}}^2(\mathbb{R}^n, \Lambda^n)$.

We return now to the proof of the lemma. Suppose that a is an $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atom, where $1 \leq \ell \leq n - 1$. Then there is a ball B and a form b' in $L^2(\mathbb{R}^n, \Lambda^{\ell-1})$ such that supp $b' \subset \overline{B}$, $a = db'$ and $\|a\|_2 \leq |B|^{1/p-1/2}$. Set $b = T_\ell^B a$, noting that supp $b \subset \overline{B}$. Moreover, by property (ii),

$$a = dT_\ell^B a + T_{\ell+1}^B da = db + T_{\ell+1}^B d^2 b' = db$$

and

$$\|b\|_{L^2(\mathbb{R}^n, \Lambda)} = \left\|T_\ell^B a\right\|_{L^2(\mathbb{R}^n, \Lambda)} \leq c_n r(B) \|a\|_{L^2(\mathbb{R}^n, \Lambda)} \leq c_n r(B)|B|^{1/2-1/p}.$$

This proves the lemma when $1 \leq \ell \leq n - 1$. The case when $\ell = n$ may be proved similarly, using property (iii) in place of property (ii). $\qquad \square$

Henceforth we suppose that Ω is a special Lipschitz domain with Lipschitz constant A. Recall that $H^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, $\dot{F}_{p,2}^0(\mathbb{R}^n) \subset \mathcal{Z}'(\mathbb{R}^n)$ and that the natural projection $\mathscr{J}$ from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{Z}'(\mathbb{R}^n)$ induces an isomorphism from $H^p(\mathbb{R}^n)$ to $\dot{F}_{p,2}^0(\mathbb{R}^n)$. In this way, the theory already developed in this paper can be applied, because, when $0 < p \leq 1$,

$$\mathscr{J} H^p(\mathbb{R}^n, \Lambda) = \dot{F}_{p,2}^0(\mathbb{R}^n, \Lambda);$$
$$\mathscr{J} H_d^p(\mathbb{R}^n, \Lambda) = d\dot{F}_{p,2}^1(\mathbb{R}^n, \Lambda); \quad \text{and}$$
$$\mathscr{J} H_{z,d}^p(\overline{\Omega}, \Lambda) = d\dot{F}_{p,2,\overline{\Omega}}^1(\mathbb{R}^n, \Lambda)$$

with equivalence of norms. The second and third identities follow from Theorem 4.1 and Corollary 4.2. In particular, it is a consequence of Corollary 4.2 that dT, correctly interpreted, is the identity on $H_{z,d}^p(\overline{\Omega}, \Lambda)$. Using the notation of Proposition 3.3, including the condition (3.8), we obtain a Calderón-type reproducing formula on this space, namely

$$(8.1) \quad \mathscr{J}u = dT\mathscr{J}u = \int_0^\infty d(\phi_t * \Psi_t * \lrcorner \, \mathscr{J}u)\, dt = \int_0^\infty (d\phi)_t * \wedge (\Psi_t * \lrcorner \, \mathscr{J}u)\frac{dt}{t}.$$

We note that this formula actually holds on the whole space $H_d^p(\mathbb{R}^n, \Lambda)$. By Remark 3.2 and the fact that $\mathcal{Z}(\mathbb{R}^n)$ is dense in $\dot{F}_{p,2}^0(\mathbb{R}^n)$, the implicit limits in these improper integrals exist in $H^p(\mathbb{R}^n, \Lambda)$ whenever $u \in H^p(\mathbb{R}^n, \Lambda)$.

The next two theorems, which characterise the spaces $H_d^p(\mathbb{R}^n, \Lambda)$ and $H_{z,d}^p(\overline{\Omega}, \Lambda)$ in terms of atoms, are the two main results of this section.

Theorem 8.6. *Suppose that $1 \leq \ell \leq n$ and $n/(n+1) < p \leq 1$. There exist constants c_p and c'_p with the following properties.*

(i) *If $(a_k)_{k=0}^\infty$ is a sequence of $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atoms and $(\lambda_k)_{k=0}^\infty$ belongs to $\ell^p(\mathbb{C})$ then the series*

$$\sum_{k=0}^\infty \lambda_k a_k$$

converges in $H^p(\mathbb{R}^n, \Lambda)$ to a form u in $H_d^p(\mathbb{R}^n, \Lambda^\ell)$, and

(8.2)
$$\|u\|_{H^p(\mathbb{R}^n, \Lambda)}^p \leq c_p \sum_{k=0}^\infty |\lambda_k|^p.$$

(ii) *Conversely, if $u \in H_d^p(\mathbb{R}^n, \Lambda^\ell)$ then there is a sequence $(a_k)_{k=0}^\infty$ of $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atoms and a sequence $(\lambda_k)_{k=0}^\infty$ in $\ell^p(\mathbb{C})$ such that*

$$u = \sum_{k=0}^\infty \lambda_k a_k,$$

where the sum converges in $H^p(\mathbb{R}^n, \Lambda)$, and

(8.3)
$$\sum_{k=0}^\infty |\lambda_k|^p \leq c'_p \|u\|_{H^p(\mathbb{R}^n, \Lambda)}^p.$$

Theorem 8.7. *Suppose that $1 \leq \ell \leq n$, $n/(n+1) < p \leq 1$ and Ω is a special Lipschitz domain in $\mathbb{R}^n$. Then there exist constants c_p and c'_p with the following properties.*

(i) *If $(a_k)_{k=0}^\infty$ is a sequence of $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atoms and $(\lambda_k)_{k=0}^\infty$ belongs to $\ell^p(\mathbb{C})$ then the series*

$$\sum_{k=0}^\infty \lambda_k a_k$$

converges in $H^p(\mathbb{R}^n, \Lambda)$ to a form u in $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$, and

$$\|u\|_{H^p(\mathbb{R}^n, \Lambda)}^p \leq c_p \sum_{k=0}^\infty |\lambda_k|^p.$$

(ii) *Conversely, if $u \in H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$ then there is a sequence $(a_k)_{k=0}^\infty$ of $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atoms and a sequence $(\lambda_k)_{k=0}^\infty$ in $\ell^p(\mathbb{C})$ such that*

$$u = \sum_{k=0}^\infty \lambda_k a_k,$$

where the sum converges in $H^p(\mathbb{R}^n, \Lambda)$, and

$$\sum_{k=0}^{\infty} |\lambda_k|^p \leq c_p' \|u\|_{H^p(\mathbb{R}^n, \Lambda)}^p .$$

The results of the preceding theorems are generalisations to exact forms of the classical atomic decompositions of [5] for $\mathbb{R}^n$ and [3] for special Lipschitz domains $\Omega \subset \mathbb{R}^n$. The generalisation to exact forms first appeared in [10] and [9] for the special case when $p = 1$. Apart from expanding the range of p, our contribution is a new proof using the reproducing formula (8.1), which is especially suited for application to special Lipschitz domains due to the support properties of ϕ. Consequently, our proof of Theorem 8.7 is shorter and more direct than the one given in [9], since we avoid using reflection maps and obtain more efficiently the desired support properties for $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atoms. As a by-product of our proof, we also obtain a special atomic decomposition for tent space functions supported in tents over Ω (see Theorem 9.4).

Before we can prove these characterisations, it is necessary to present a sequence of definitions and lemmata related to 'tents' over open sets, 'tent spaces' and tent space atoms. To help the reader contextualise what follows, we first offer a brief outline of the proof of part (ii) of each of the above theorems. Suppose that $u \in H_d^p(\mathbb{R}^n, \Lambda^\ell)$. Following the method developed in [5], we define an operator Q by

$$(Qu)(x,t) = \Psi_t * \lrcorner u(x) \quad \forall\, (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ = \mathbb{R}^n \times (0, \infty),$$

and show that Qu belongs to the tent space $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda^{\ell-1})$. Using the classical atomic decomposition for tent space functions, one may write $Qu = \sum_k \lambda_k A_k$, where each A_k is a $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda^{\ell-1})$-atom and the sequence (λ_k) belongs to ℓ^p. One then constructs a map π by

$$\pi U = \int_0^\infty (d\phi)_t * \wedge U(\cdot, t) \frac{dt}{t}$$

so that, by the reproducing formula (8.1),

$$u = dTu = \pi Qu = \sum_k \lambda_k \pi A_k$$

where each πA_k is an $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atom, thus obtaining the atomic decomposition for u.

The atomic decomposition for elements of $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$ will be proved along the same lines with the following variations. If $u \in H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$ then Qu is in fact supported in a 'tent' over Ω. So the tent space decomposition for Qu (presented in Theorem 9.4) gives tent space atoms A_k

with good support properties with respect to the domain Ω. It follows that $u = \pi Q u = \sum_k \lambda_k \pi A_k$, where each πA_k can be written as a finite sum of $H^p_{z,d}(\overline{\Omega}, \Lambda^\ell)$-atoms.

9. Atomic decomposition of tent spaces

The results in this section are independent of the previous sections, and we believe they are of interest in their own right.

The tent spaces were first introduced in the article [**5**] of Coifman, Meyer and Stein, and owe their name to the fact that, when $0 < p \leq 1$, their functions can be decomposed as a sum of atoms supported in tents over balls. Our aim is to show that if a function in such a tent space, itself has support in a tent $T_\beta(\Omega)$, then the supports of the atoms in its decomposition can be taken inside this tent, and away from the boundary of Ω.

We turn now to the relevant definitions. If $\beta > 0$ and $x \in \mathbb{R}^n$, let $\Gamma'_\beta(x)$ denote the cone in $\mathbb{R}^n \times \mathbb{R}^+$ with aperture β and vertex at x, namely

$$\Gamma'_\beta(x) = \{(y,t) \in \mathbb{R}^n \times \mathbb{R}^+ : |y - x| < \beta t\}.$$

If O is an open subset of $\mathbb{R}^n$, then the tent $T_\beta(O)$ over O with aperture β is defined by

$$T_\beta(O) = \{(y,t) \in \mathbb{R}^n \times \mathbb{R}^+ : d(y, O^c) \geq \beta t\}.$$

We follow the convention of writing $\Gamma'(x)$ for $\Gamma'_1(x)$ and $T(O)$ for $T_1(O)$. Given any measurable function U on $\mathbb{R}^n \times \mathbb{R}^+$, we define the Lusin area integral SU of U by the formula

$$(SU)(x) = \left(\iint_{\Gamma'(x)} |U(y,t)|^2 \frac{dy\, dt}{t^{n+1}} \right)^{1/2}.$$

Definition 9.1. Suppose that $p > 0$. The tent space $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$ is defined to be the set of all measurable functions U on $\mathbb{R}^n \times \mathbb{R}^+$ such that $\|U\|_{\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)}$ is finite, where

$$\|U\|_{\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)} = \|SU\|_{L^p(\mathbb{R}^n)}.$$

Definition 9.2. Suppose that $p > 0$. A measurable function A on $\mathbb{R}^n \times \mathbb{R}^+$ is said to be a $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$-atom if there exists a ball B in $\mathbb{R}^n$ such that $\operatorname{supp} A \subset T(B)$ and

$$\iint_{\mathbb{R}^n \times \mathbb{R}^+} |A(y,t)|^2 \, dy \frac{dt}{t} \leq |B|^{1-2/p}.$$

If $0 < p \leq 1$ then it is relatively straightforward to show that every $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$-atom A belongs to $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$ and that $\|A\|_{\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)} \leq 1$. Consequently, if $(\lambda_k)_{k \in \mathbb{N}} \in \ell^p(\mathbb{C})$ and $(A_k)_{k \in \mathbb{N}}$ is a sequence of $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$-atoms then $\sum_{k \in \mathbb{N}} \lambda_k A_k$ belongs to $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$. That the following converse is true is a deeper result due to Coifman, Meyer and Stein [**5**].

Theorem 9.3. *Suppose that* $0 < p \leq 1$. *There exists a constant* C *(depending only on* n *and* p*) with the following property: for all* U *in* $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$, *there exists a sequence* $(\lambda_k)_{k \in \mathbb{N}}$ *in* $\ell^p(\mathbb{C})$ *and a sequence* $(A_k)_{k \in \mathbb{N}}$ *of* $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$-*atoms such that*

$$U = \sum_{k \in \mathbb{N}} \lambda_k A_k$$

and

$$\sum_{k \in \mathbb{N}} |\lambda_k|^p \leq C \|U\|_{\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)}^p \, .$$

We introduce the following variant of the above theorem, where the tent space atoms are supported in Carleson boxes with good support properties with respect to an underlying domain Ω.

Theorem 9.4. *Suppose that* $0 < p \leq 1$ *and* $\beta > 0$. *There exist positive constants* C' *(depending only on* n, p *and* β*) and* c_β *(depending only on* β*), where* $0 < c_\beta < 1$, *satisfying the following property. If* $U \in \mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$ *and* $\operatorname{supp} U \subset T_\beta(\Omega)$ *for some proper open subset* Ω *of* $\mathbb{R}^n$, *then there exists a sequence* $(\lambda_k)_{k \in \mathbb{N}}$ *in* $\ell^p(\mathbb{C})$ *and a sequence* $(A_k)_{k \in \mathbb{N}}$ *of* $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$-*atoms, supported in corresponding tents* $(T(B_k))_{k \in \mathbb{N}}$, *such that*

(i) $U = \sum_{k \in \mathbb{N}} \lambda_k A_k$,

(ii) $\sum_{k \in \mathbb{N}} |\lambda_k|^p \leq C' \|U\|_{\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)}^p$,

(iii) $\operatorname{supp} A_k \subset c_\beta B_k \times (0, 6\beta^{-1} c_\beta \, r(B_k))$ *and* $5 c_\beta B_k \subset \Omega$ *whenever* $k \in \mathbb{N}$.

Proof: The proof is an adaptation of the proof of [**14**, Theorem 1.1], which in turn is based on the original ideas presented in [**5**]. Fix any number ν in the interval $(0, 1)$. For any k in $\mathbb{Z}$, let O_k denote the open subset of $\mathbb{R}^n$ given by

$$O_k = \{x \in \mathbb{R}^n : SU(x) > 2^k\}.$$

It can be shown that

(9.1) $$\operatorname{supp} U \subset \bigcup_{k \in \mathbb{Z}} T_\nu(O_k^*),$$

where each open set O_k^* is constructed using a corresponding set of global γ-density (see [**14**, pp. 128–130] for details). For each integer k, the Whitney lemma (see, e.g., [**14**, Lemma 2.2]) applied to the open set $O_k^* \cap \Omega$ gives a denumerable index set I^k, a sequence of balls $(B_j^k)_{j \in I^k}$ having radii $(r_j^k)_{j \in I^k}$ and centres $(x_j^k)_{j \in I^k}$, and a sequence $(\varphi_j^k)_{j \in I^k}$ of nonnegative functions on $\mathbb{R}^n$ with the following properties:

$$O_k^* \cap \Omega = \bigcup_{j \in I^k} B_j^k, \quad d(x_j^k, (O_k^* \cap \Omega)^c) = 10 r_j^k,$$

$$\operatorname{supp} \varphi_j^k \subset 2 B_j^k, \quad \sum_{j \in I^k} \varphi_j^k = \mathbf{1}_{O_k^* \cap \Omega},$$

and

(9.2) $$\tfrac{1}{4} B_i^k \cap \tfrac{1}{4} B_j^k = \emptyset \quad \text{if } i \neq j.$$

It can be shown that $O_{k+1}^* \subset O_k^*$ for all k (see [**14**, pp. 128, 130]). Therefore, for each (x, t) in $\mathbb{R}^n \times \mathbb{R}^+$,

$$\left(\mathbf{1}_{T_\nu(O_k^*)} - \mathbf{1}_{T_\nu(O_{k+1}^*)} \right)(x, t) \mathbf{1}_\Omega(x)$$
$$= \sum_{j \in I^k} \varphi_j^k(x) \left(\mathbf{1}_{T_\nu(O_k^*)} - \mathbf{1}_{T_\nu(O_{k+1}^*)} \right)(x, t) \mathbf{1}_\Omega(x)$$

and hence

$$U(x, t) = \sum_{k \in \mathbb{Z}} \sum_{j \in I^k} U(x, t) \varphi_j^k(x) \left(\mathbf{1}_{T_\nu(O_k^*)} - \mathbf{1}_{T_\nu(O_{k+1}^*)} \right)(x, t)$$

by (9.1). Define, for all integers k and all j in I^k,

$$\mu_j^k = \int_0^\infty \int_{\mathbb{R}^n} |U(y, t)|^2 \varphi_j^k(y)^2 \left(\mathbf{1}_{T_\nu(O_k^*)} - \mathbf{1}_{T_\nu(O_{k+1}^*)} \right)(y, t) \, dy \, \frac{dt}{t},$$

$$A_j^k(y, t) = U(y, t) \varphi_j^k(y) \left(\mathbf{1}_{T_\nu(O_k^*)} - \mathbf{1}_{T_\nu(O_{k+1}^*)} \right)(y, t) |B_j^k|^{1/2 - 1/p} (\mu_j^k)^{-1/2}$$

(unless $\mu_j^k = 0$, in which case we define $A_j^k = 0$) and

$$\lambda_j^k = |B_j^k|^{1/p - 1/2} (\mu_j^k)^{1/2}.$$

Then

$$U = \sum_{k \in \mathbb{Z}} \sum_{j \in I^k} \lambda_j^k A_j^k.$$

We claim that, up to a multiplicative constant, each A_j^k is a $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$-atom with the desired properties.

First we show that $\operatorname{supp} A_j^k \subset T(\tilde{B}_j^k)$, where $\tilde{B}_j^k = CB_j^k$ and $C = 2 + 12/\min\{\beta, \nu\}$. Let $(y, t) \in \operatorname{supp} A_j^k$, so that in particular $(y, t) \in T_\nu(O_k^*) \cap T_\beta(\Omega)$ and $y \in \operatorname{supp} \varphi_j^k$; that is,

$$d(y, (O_k^*)^c) \geq \nu t, \quad d(y, \Omega^c) \geq \beta t \quad \text{and} \quad |y - x_j^k| < 2r_j^k.$$

We aim to show that $d(y, (CB_j^k)^c) \geq t$ for then $\operatorname{supp} A_j^k \subset T(\tilde{B}_j^k)$. Suppose that $z \in (CB_j^k)^c$. Then

$$(9.3) \qquad |y - z| \geq |z - x_j^k| - |y - x_j^k| \geq (C - 2)r_j^k = \frac{12 r_j^k}{\min\{\beta, \nu\}}.$$

Also, $d(x_j^k, (O_k^* \cap \Omega)^c) = 10 r_j^k$. Suppose that $\varepsilon > 0$. There exists u in $(O_k^* \cap \Omega)^c$ such that $|x_j^k - u| < 10 r_j^k + \varepsilon$. So

$$\min\{\beta, \nu\} t \leq |y - u| \leq |y - x_j^k| + |x_j^k - u| < 12 r_j^k + \varepsilon.$$

Since this is true for every positive ε, it follows that $\min\{\beta, \nu\} t \leq 12 r_j^k$. Combining this with (9.3) gives $|y - z| \geq t$, and hence $d(y, (CB_j^k)^c) \geq t$ as required.

Second, the definition of A_j^k implies that

$$\iint |A_j^k(y, t)|^2 \, dy \frac{dt}{t} = |B_j^k|^{1-2/p} = C^{n(2/p-1)} |\tilde{B}_j^k|^{1-2/p},$$

and so up to the multiplicative constant $C^{n(2/p-1)}$, each A_j^k is a $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)$-atom.

Third, we prove that the A_j^k satisfy support properties as in part (iii) of the theorem. Now each A_j^k is supported in $T_\beta(\Omega) \cap (2B_j^k \times \mathbb{R}^+)$, where $5(2B_j^k) \subset \Omega$. So if $(y, t) \in T_\beta(\Omega) \cap (2B_j^k \times \mathbb{R}^+)$ then

$$\beta t \leq d(y, \Omega^c) \leq d(y, x_j^k) + d(x_j^k, \Omega^c) < 2r_j^k + 10 r_j^k$$

and hence $0 < t < 12\beta^{-1} r_j^k$. This shows that $\operatorname{supp} A_j^k \subset 2B_j^k \times (0, 12\beta^{-1} r_j^k)$. Defining the constant c_β by $c_\beta = 2/C$, it is now easy to see that

$$\operatorname{supp} A_j^k \subset T(\tilde{B}_j^k), \quad \operatorname{supp} A_j^k \subset c_\beta \tilde{B}_j^k \times (0, 6\beta^{-1} c_\beta \, r(\tilde{B}_j^k))$$

$$\text{and } 5c_\beta \tilde{B}_j^k \subset \Omega.$$

It remains to show that there exists a constant C', independent of Ω and U, such that

$$\sum_{k \in \mathbb{Z}} \sum_{j \in I^k} |\lambda_j^k|^p \leq C' \, \|U\|_{\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+)}.$$

The proof, which uses (9.2), proceeds exactly as in [**14**, pp. 132–133] and will not be reproduced here. Finally, relabel the balls $(\tilde{B}_j^k)$ as (B_k), $(C^{n(1/p-1/2)}\lambda_j^k)$ as (λ_k) and the functions $(C^{n(1/2-1/p)}A_j^k)$ as (A_k). This completes the proof of the theorem. □

Remark 9.5. A comparison of this proof with the proof of [**14**, Theorem 1.1] shows that Theorem 9.4 also holds when the underlying space $\mathbb{R}^n$ is replaced by any space X of homogeneous type that satisfies the assumptions of [**15**, §1.1.3].

Remark 9.6. Theorem 1.1 follows from the special case of Theorem 9.4 when $p = 1$ and $\beta = 1$, with $(c_\beta B_k)$, $(c_\beta^{n/2}\lambda_k)$ and $(c_\beta^{-n/2}A_k)$ renamed as (B_k), (λ_k) and (A_k).

10. Proofs of atomic decomposition of Hardy spaces

Our aim now is to use the reproducing formula (6.1) and the tent space decomposition in the previous section, to prove Theorems 8.6 and 8.7.

Recall that A and σ are fixed positive numbers such that $\sigma A < 1$. In order to apply the previous theorem, we need the following.

Lemma 10.1. *Suppose that $a > 0$, Ω is a special Lipschitz domain with Lipschitz constant A and that ψ is a $\mathscr{C}^\infty(\mathbb{R}^n)$ function supported in $\{y \in \overline{\Gamma_\sigma(0)} : y_n \geq a\}$. Suppose also that $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp} u \subset \overline{\Omega}$. Define Qu by*

$$Qu(x,t) = (\psi_t * u)(x) \quad \forall\, (x,t) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Then Qu is supported in $T_\beta(\Omega)$ where

$$(10.1) \qquad\qquad \beta = \frac{a(1-\sigma A)}{\sqrt{1+A^2}}.$$

Proof: What needs to be shown is that, under the stated hypotheses, $\operatorname{dist}(\operatorname{supp}(\psi_t * u), \Omega^c) \geq \beta t$. By (2.2), $\operatorname{supp}(\psi_t * u) \subset \overline{\Omega} + \{y \in \overline{\Gamma_\sigma(0)} : y_n \geq at\}$, so we need to show that if $x \in \overline{\Omega}$, $y \in \overline{\Gamma_\sigma(0)}$, $y_n \geq at$ and $z \in \Omega^c$, then $|x + y - z| \geq \beta t$. Let $w = z - x$, and note that by the assumption on Ω, $w_n \leq A|w'|$.

So the result is proved once we show that $|y - w| \geq \beta t$ whenever $-\infty < w_n \leq A|w'|$, $|y'| \leq \sigma y_n$ and $y_n \geq at$. We split into two cases.

Case (i): $w_n \leq \sigma A y_n$. Then

$$|y - w| \geq y_n - w_n \geq (1 - \sigma A)y_n \geq (1 - \sigma A)at > \beta t$$

by (10.1).

Case (ii): $w_n > \sigma A y_n$. Then $|w'| \geq \frac{1}{A}w_n > \sigma y_n \geq |y'|$. So

$$
\begin{aligned}
|y - w|^2 &= (y_n - w_n)^2 + |y' - w'|^2 \\
&> (y_n - w_n)^2 + (|w'| - |y'|)^2 \\
&> (y_n - w_n)^2 + (\tfrac{1}{A}w_n - \sigma y_n)^2 \\
&= \tfrac{1+A^2}{A^2}w_n^2 - \tfrac{2(\sigma+A)}{A}y_n w_n + (1 + \sigma^2)y_n^2 \\
&\geq \left(\tfrac{\beta}{a}\right)^2 y_n^2 \\
&\geq \beta^2 t^2,
\end{aligned}
$$

where we have minimised over w_n as usual way for quadratic expressions. □

We are now in a position to prove Theorems 8.6 and 8.7.

Proof: Suppose throughout that $n/(n+1) < p \leq 1$.

First we prove part (i) of Theorem 8.6. Suppose that $(a_k)_{k=0}^{\infty}$ is a sequence of $H_d^p(\mathbb{R}^n, \Lambda)$-atoms, $(\lambda_k)_{k=0}^{\infty}$ belongs to $\ell^p(\mathbb{C})$ and $a_k = db_k$. Since each component of a_k is a classical $H^p(\mathbb{R}^n)$-atom, the classical theory implies that there exist a constant c_p and u in $H^p(\mathbb{R}^n, \Lambda)$ such that (8.2) holds and

(10.2)
$$u = \sum_{k=0}^{\infty} \lambda_k a_k = \sum_{k=0}^{\infty} \lambda_k db_k,$$

where the sum converges in $H^p(\mathbb{R}^n, \Lambda)$.

Recall that $\mathscr{J}$ denotes the natural projection from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{Z}'(\mathbb{R}^n)$. By (10.2),

$$\sum_{k=0}^{M} \lambda_k \mathscr{J} a_k = d\left(\sum_{k=0}^{M} \lambda_k \mathscr{J} b_k\right) \to \mathscr{J} u \quad \text{in } \dot{F}_{p,2}^0(\mathbb{R}^n, \Lambda) \text{ as } M \to \infty,$$

and it follows from the continuity of d and Theorem 4.1 that $\mathscr{J} u \in d\dot{F}_{p,2}^1(\mathbb{R}^n, \Lambda) = \mathscr{J} H_d^p(\mathbb{R}^n, \Lambda)$. Hence $u \in H_d^p(\mathbb{R}^n, \Lambda)$. This completes the proof of Theorem 8.6(i).

Part (i) of Theorem 8.7 is proved along the same lines, noting that $d\dot{F}^1_{p,2,\overline{\Omega}}(\mathbb{R}^n, \Lambda) = \mathscr{J} H^p_d(\overline{\Omega}, \Lambda)$.

We now prove the converse statements in each of Theorems 8.6 and 8.7. Let ϕ denote the $\mathscr{C}^\infty_0(\mathbb{R}^n)$ function of Proposition 3.3, chosen such that

$$(10.3) \qquad \int_{\mathbb{R}^n} \Psi(x)\, dx = 0$$

where $\Psi(x) = 2\phi(x)x$. We remark also that $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$, and

$$(10.4) \qquad \int_{\mathbb{R}^n} (d\phi)(x)\, dx = 0$$

where $d\phi = \Sigma^n_{j=1} \partial_j \phi\, e_j$,

Given u in $\mathcal{S}'(\mathbb{R}^n, \Lambda)$, define Qu by

$$(Qu)(x,t) = \Psi_t * \lrcorner\, u(x)$$

whenever $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$. By the moment condition (10.3), it is well-known (see, for example, [**3**, p. 308]) that Q is bounded from $H^p(\mathbb{R}^n, \Lambda)$ to $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)$. Given U in $\mathcal{T}^2(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)$ with compact support in $\mathbb{R}^n \times \mathbb{R}^+$, define πU by the formula

$$\pi U = \int_0^\infty (d\phi)_t * \wedge\, U(\cdot, t) \frac{dt}{t}.$$

Again, by the moment condition (10.4), it is well known (see [**5**, Theorem 6]) that the operator π extends to a bounded linear operator from $\mathcal{T}^2(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)$ to $L^2(\mathbb{R}^n, \Lambda)$ and from $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)$ to $H^p(\mathbb{R}^n, \Lambda)$.

We focus now on the proof of Theorem 8.6(ii). Suppose that $u \in H^p_d(\mathbb{R}^n, \Lambda^\ell)$, where $1 \le \ell \le n$ and $n \ge 1$. Then Qu belongs to the tent space $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda^{\ell-1})$ and by Theorem 9.3, Qu has the atomic decomposition

$$Qu = \sum_{k \in \mathbb{Z}} \lambda_k A_k,$$

where each A_k is a $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda^{\ell-1})$-atom supported in a tent $T(B_k)$, the sum converges in $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)$ and

$$\sum_k |\lambda_k|^p < c''_p \|Qu\|^p_{\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)} \le c'_p \|u\|_{H^p(\mathbb{R}^n, \Lambda)}.$$

Define a_k by $a_k = \pi A_k$, so that

$$a_k(x) = \int_0^\infty \int_{\mathbb{R}^n} (d\phi)_t(x-y) \wedge A_k(y,t)\, dy \frac{dt}{t}$$

$$= d \int_0^\infty \int_{\mathbb{R}^n} \phi_t(x-y) A_k(y,t)\, dy\, dt$$

$$= db_k(x),$$

where

$$b_k = \int_0^\infty \phi_t * A_k(\cdot, t)\, dt.$$

We claim that each a_k is an $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atom, up to a multiplicative constant independent of k. First,

$$\|a_k\|_{L^2(\mathbb{R}^n, \Lambda)} = \|\pi A_k\|_{L^2(\mathbb{R}^n, \Lambda)} \le C \|A_k\|_{\mathcal{T}^2(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)} \le C|B_k|^{1/2 - 1/p},$$

where the constant C is independent of k. Second, we show that $b_k \in L^2(\mathbb{R}^n, \Lambda)$. Suppose that $B_k = B_{r_k}(z_k)$. Successive applications of the triangle, Cauchy-Schwarz and Young's inequalities yield

$$\|b_k\|_{L^2(\mathbb{R}^n, \Lambda)}^2 \le r_k \int_0^{r_k} \|\phi_t * A_k(\cdot, t)\|_{L^2(\mathbb{R}^n, \Lambda)}^2\, dt$$

$$\le r_k \int_0^{r_k} \|\phi_t\|_{L^1(\mathbb{R}^n)}^2 \|A_k(\cdot, t)\|_{L^2(\mathbb{R}^n, \Lambda)}^2\, dt$$

$$= C_1^2 r_k \int_0^{r_k} t \int_{\mathbb{R}^n} |A_k(y,t)|^2\, dy \frac{dt}{t}$$

$$\le C_1^2 r_k^2 |B_k|^{1-2/p},$$

where $C_1 = \|\phi\|_1$, and the final estimate follows from the fact that A_k is a $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda^\ell)$-atom. Third, we note that $\operatorname{supp} a_k \subset B_k$. Indeed, by (2.2) and (3.5),

$$\operatorname{supp} a_k = \operatorname{supp}(\pi A_k) \subset \bigcup_{0 \le t \le r_k} \left\{ \operatorname{supp} \phi_t + \left(1 - \frac{t}{r_k}\right) B_k \right\}$$

$$\subset \bigcup_{0 \le t \le r_k} \{tB_1(0) + (r_k - t)B_1(z_k)\}$$

$$= B_{r_k}(z_k)$$

$$= B_k.$$

Hence, up to a multiplicative constant, each a_k is an $H_d^p(\mathbb{R}^n, \Lambda^\ell)$-atom as claimed.

It remains to be shown that $u = \sum_k \lambda_k a_k$, where the sum converges in the topology of $H^p(\mathbb{R}^n, \Lambda)$. Since $\sum_k \lambda_k A_k$ converges in $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)$ and π is bounded from $\mathcal{T}^p(\mathbb{R}^n \times \mathbb{R}^+, \Lambda)$ to $H^p(\mathbb{R}^n, \Lambda)$, it follows that $\pi Q u = \sum_k \lambda_k a_k$, where the sum converges in the topology of $H^p(\mathbb{R}^n, \Lambda)$. But note by the definitions of Q and π that $\mathscr{J} \pi Q u = dT \mathscr{J} u$, where T is the operator given by (4.3). Recall from (8.1) that dT is the identity on $d\dot{F}_{p,2}^1(\mathbb{R}^n, \Lambda)$. Since $\mathscr{J} u \in d\dot{F}_{p,2}^1(\mathbb{R}^n, \Lambda)$, we have that

$$\mathscr{J} u = dT \mathscr{J} u = \mathscr{J} \pi Q u = \mathscr{J} \sum_k \lambda_k a_k$$

and hence that $u = \sum_k \lambda_k a_k$ as required. This, together with the bound already proved on $\sum |\lambda_k|^p$, completes the proof of Theorem 8.6(ii).

We turn now to prove Theorem 8.7(ii). Suppose that $u \in H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$. Using the same argument as above, $u = \pi Q u = \sum_k \lambda_k \pi A_k$ where the sum converges in H^p, $\pi A_k = db_k$, $\text{supp } A_k \subset T(B_k)$ and $\{\lambda_k\} \in \ell^p$. In this case, each πA_k is not (even up to a multiplicative constant) necessarily an $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atom because it may not satisfy the required support properties. We will instead show that each πA_k can be written as a finite sum of $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atoms.

Since $\text{supp } u \subset \overline{\Omega}$, we conclude by Lemma 10.1 that $\text{supp } Q u \subset T_\beta(\Omega)$, where β is given by (10.1) with $a = 1/2$. Theorem 9.4 gives the additional information that A_k can be chosen so that $\text{supp}(A_k) \subset cB_k \times (0, 6\beta^{-1} cr_k)$, and $5cB_k \subset \Omega$, where c is independent of k. Hence $\text{dist}(\overline{cB_k}, \Omega^c) \geq 4cr_k$. By (2.2) and (3.5),

$$\text{supp } a_k = \text{supp}(\pi A_k)$$

$$\subset \bigcup_{0 < t \leq 6\beta^{-1} cr_k} \left(\text{supp}(d\phi)_t + \overline{cB_k} \right)$$

$$\subset \overline{cB_k} + \{y \in \overline{\Gamma_\sigma(0)} : y_n \leq 6\beta^{-1} cr_k\} =: G_k.$$

Note that G_k is a compact subset of Ω and that $\text{dist}(G_k, \Omega^c) \geq 4cr_k$. This is because, if $x \in \overline{cB_k}$, $y \in \Gamma_\sigma(0)$ and $w \in \Omega^c$, then $w - y \in \Omega^c$, so $|(x + y) - w| = |x - (w - y)| \geq 4cr_k$. So we may cover G_k with finitely many balls $\{\frac{1}{2} B_k^j\}_{j=1}^M$ of radius $cr_k/2$ and centres z_k^j, where $z_k^j \in G_k$ and where (by scale and translation invariance) the integer M is independent of k. Let $\{\eta_k^j\}_{j=1}^M$ denote a smooth subordinate partition of unity with

the properties that $0 \leq \eta_k^j \leq 1$, supp $\eta_k^j \subset B_k^j$,

$$\sum_{j=1}^{M} \eta_k^j(x) = 1 \quad \forall\, x \in G_k$$

and $\|\nabla \eta_k^j\|_{L^\infty(\mathbb{R}^n)} \leq c' r_k^{-1}$ for some constant c' independent of j and k. For each k and j, define the function a_k^j and scalar μ_k^j by

$$a_k^j = (\mu_k^j)^{-1} d(\eta_k^j b_k) \quad \text{and} \quad \mu_k^j = |B_k^j|^{1/p-1/2} \|d(\eta_k^j b_k)\|_{L^2(\mathbb{R}^n, \Lambda)}.$$

Now each a_k^j is an $H_{z,d}^p(\overline{\Omega}, \Lambda^\ell)$-atom because $\eta_k^j b_k \in L^2(\mathbb{R}^n, \Lambda^{\ell-1})$, supp $a_k^j \subset B_k^j$ where $4B_k^j \subset \Omega$, and $\|a_k^j\|_{L^2(\mathbb{R}^n, \Lambda)} \leq |B_k^j|^{1/2-1/p}$.

Note now that

$$u = \sum_{k=0}^{\infty} \lambda_k \pi A_k = \sum_{k=0}^{\infty} \sum_{j=1}^{M} \lambda_k \mu_k^j a_k^j,$$

where the sum converges in H^p and $\sum_k |\lambda_k|^p < \infty$. To complete the proof of the theorem, it suffices to show that $\sup_{k,j} \mu_k^j \leq C'$ for some constant C'. This bound follows readily from the estimate

$$\|d(\eta_k^j b_k)\|_{L^2(\mathbb{R}^n, \Lambda)} = \left\| (d\eta_k^j) \wedge b_k + \eta_k^j\, db_k \right\|_{L^2(\mathbb{R}^n, \Lambda)}$$

$$\leq \|\nabla \eta_k^j\|_{L^\infty(\mathbb{R}^n)} \|b_k\|_{L^2(\mathbb{R}^n, \Lambda)} + \|\eta_k^j\|_{L^\infty(\mathbb{R}^n)} \|\pi A_k\|_{L^2(\mathbb{R}^n, \Lambda)}$$

$$\leq c' r_k^{-1} r_k |B_k|^{1/2-1/p} + C|B_k|^{1/2-1/p}$$

$$\leq (c' + C)|B_k|^{1/2-1/p}.$$

Thus we have completed the proof of Theorem 8.7. $\qquad\qquad\square$

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