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ELENA SAFULINA

Parallel and semiparallel space-like submanifolds of low dimension in pseudo-Euclidean space

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Parallel and semiparallel space-like submanifolds of low dimension in pseudo-Euclidean space

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Introduction

Let $N_s^n(c)$ be a space form of constant curvature c. If $s = 0$ (or $s = n$) it is Riemannian and if $0 < s < n$, it is pseudo-Riemannian [1]. A submanifold M^m in $N_s^n(c)$ is called *semiparallel* if $\overline{R}(X,Y)h = 0$ (this is the integrability condition of the system $\overline{\nabla}h = 0$ which characterizes a parallel submanifold). Here $\overline{R}$ is the curvature operator of the van der Waerden-Bortolotti connection $\overline{\nabla} = \nabla \oplus \nabla^{\perp}$ and h is the second fundamental form.

Parallel submanifolds in the case $s = 0$, $c = 0$ are classified by Ferus [2]. His results have been extended to the case $s = 0$, $c \neq 0$ by Takeuchi [3], Backes and Reckziegel [4], and to the case of pseudo-Riemannian space form $N_s^n(c)$, $s > 0$ by Blomstrom [5] and Naitoh [6]. Some special classes of parallel submanifolds in E_1^n and E_2^n are described by Magid [7].

Semiparallel submanifolds M^m in $N_s^n(c)$ by $s = 0$ have been classified and described in the following cases: surfaces ($m = 2$) if $c = 0$ by Deprez [8]; surfaces ($m = 2$) if $c > 0$ by Mercuri and Asperti [9], [10]); three-dimensional submanifolds, two-codimensional submanifolds (i.e. $m = n - 2$) and hypersurfaces ($m = n - 1$) if $c = 0$ by Lumiste [11], [13], Lumiste and Riives [12], Deprez [14], respectively; submanifolds M^m with flat normal connection $\nabla^{\perp}$ if $c = 0$ by Lumiste [15] and if $c > 0$ by Dillen and Nölker [16]. It is shown (Lumiste, [17]) that every semiparallel submanifold is a second order envelope of corresponding parallel submanifolds. A survey on parallel and semiparallel submanifolds with their generalizations in a Euclidean space is given by Lumiste in [18] and [19]. His results on semiparallel time-like surfaces in a Lorentzian spacetime form are published in [20].

Käesoleva väitekirja uurimisobjektideks on paralleelsed ja semiparalleelsed ruumisarnased madalamõõtmelised (M^1, M^2 ja M^3) alammuutkonnad pseudoeukleidilises ruumis E_s^n. Using the Cartan moving frame method and the Cartan exterior differentiation. Alammuutkondade geomeetriliste omaduste kirjeldamiseks .

In **Chapter 1** of the thesis the classification of the subspaces of a pseudo-Euclidean space E_s^n depending on the Euclidean metric is obtained. The definitions of the index and the defect of a semi-pseudo-Euclidean subspace $E_{l,d}^k$ are given. The normal vector space of the submanifolds M^m at a point $x \in M^m$ is introduced in *Section*

1.1 The moving frame adapted to the space-like submanifold $M^m \subset E^n_s$, which have a positive definite inner metric is presented. The curvature 2-forms of Levi-Civita connection, the normal connection and van der Waerden-Bortolotti connection are constructed. In *Section 1.2* the form of the structure of a space-like submanifold M^m in pseudo-Euclidean space E^n_s is determined. The equation of the isotropic cone C_x of this space is obtained. The definitions of the timelike and the spacelike directions belonging to the different domains of intersection the normal vector space with the cone C_x are given. In the last part of this Section the example of the submanifold M^2 in pseudo-Euclidean space E^4_1 with three different possible mutual locations of the cone C_x and the normal space is considered. The definition of the principal normal subspace of submanifolds M^m at a point $x \in M^m$ is introduced in *Section 1.3*, where the cases of regular, singular non-vanishing and completely vanishing metrics are defined in more details.

The **Chapter 2** is devoted to the general aspects of parallel and semiparallel space-like submanifolds. In *Section 2.1* it is proved that a parallel space-like submanifold M^m in a pseudo-Euclidean space E^n_s with a principal normal subspace of completely vanishing metric is a submanifold with m families of parabola generators, some of them can degenerate into a straight line. The equation which represents the parallel space-like submanifold M^m is constructed. In addition, here it is proved that every space-like submanifold M^m in the pseudo-Euclidean space E^n_s having the principal normal subspace of completely vanishing metric is semiparallel. At the end of *Section 2.2* it is obtained that semiparallel space-like submanifold M^m in space $E^{m+2n_1}_{0,n_1}$ with the principal normal space of dimension n_1 and completely vanishing metric has a flat normal connection. *Section 2.3* deals with some definitions of terms, which are connected with the second order of enveloping. The criterion of semiparallelism for submanifold M^m in the space $N^n_s(c)$ is derived. The concept of Veronese submanifolds and the results of research work on the question of the existence of second order envelopes of Veronese submanifolds (see also Lumiste [18], [21], [22], [23] and Riives [24]) are introduced in *Section 2.4*. Segre submanifolds and their second order envelopes are presented and characterized in *Section 2.5* by using the results of Lumiste [25], [26] and [27].

The **Chapter 3** deals with the classification and description of the semiparallel space-like curves and surfaces in pseudo-Euclidean spaces E^n_s. These results are published by the author in [31]. At the beginning of this Chapter the concepts of the reducible and irreducible submanifolds are introduced. All parallel surfaces are determined. Their 2nd order envelopes are found in the main part of this Chapter. The classification of semiparallel space-like surfaces is given in *Section 3.1*. It is proved that there exists an open and dense part U of M^2 such that the connected components of U are of the following types:
(i) open parts of totally umbilical M^2 (in particular, of totally geodesic M^2) in the space E^n_s;

(ii) surfaces with flat $\overline{\nabla}$;
(iii) isotropic surfaces with nonflat $\nabla^{\perp}$ satisfying $\| H \| = 3K$, where K is the Gaussian curvature and H is the mean curvature vector (Theorem 3.1).
The classification of the parallel lines and surfaces with flat Waerden-Bortolotti connection $\overline{\nabla}$ is derived in *Section 3.2*. It is proved that a parallel space-like M^1 in the space E^n_s is either a straight line, or a circle (it can be a either real, or imaginary radius), or a parabola (Proposition 3.1). The parallel space-like surfaces M^2 with flat $\overline{\nabla}$ in the space E^n_s are classified by Proposition 3.2 and Theorem 3.2. The problem of the existence of a nontrivial 2nd order envelope of parallel surfaces M^2 is raised in *Section 3.3*. It is proved that there exists a nontrivial 2nd order envelope with some arbitrariness (Proposition 3.3). The concept of maximal submanifold is introduced in *Section 3.4*. It is established that non-trivial maximal semiparallel space-like surfaces exist in the space E^n_s with $s > 0$ and their geometrical description is given. The classification of the maximal semiparallel space-like surface M^2 in the space E^n_s (with $s > 0$), which is not totally geodesic, is investigated.

The **Chapter 4** is devoted to the normally flat semiparallel space-like submanidols M^3. In this case the curvature 2-forms of the normal connection $\nabla^{\perp}$ is zero. The concept of a flat normal connection of a space-like submanifold M^m in a pseudo-Euclidean space E^n_s is introduced. The result of Proposition 2.2 are applied for normally flat parallel space-like submanifolds M^3. It is proved that a normally flat parallel space-like M^m in a space E^n_s with principal normal subspace of completely vanishing metric is a submanifold in space $E^{3+n_1}_{0,n_1}$ with three families of parabola generators (Proposition 4.1). The *Section 4.1* deals with the principal curvature vectors. The regular, singular non-vanishing and completely vanishing metric is derived for case of the 3-dimensional principal normal subspace. The case of one-dimensional principal normal subspace is discussed in *Section 4.2*. A normally flat semiparallel space-like submanifold M^3 in a space E^n_s with N_xM^3 of dimension 1 and regular metric are investigated in Proposition 4.3. The case of the completely vanishing metric is examined in Proposition 4.4. The two-dimensional principal normal subspace and its metric (regular, singular non-vanishing and completely vanishing) are derived in *Section 4.3* (Proposition 4.5, Proposition 4.6). In *Section 4.4* the case with the 3-dimensional principal normal subspace of a either regular, or singular non-vanishing, or completely vanishing metric is considered (Proposition 4.7).

The aim of the **Chapter 5** is to investigate all possibilities for the principal normal subspace of normally non-flat semiparallel space-like submanifolds M^3 and to classify the normally non-flat parallel space-like M^3. The geometrical structure of semiparallel submanifolds as a second envelopes of the corresponding parallel submanifolds needs complementary investigations. The case of six-dimensional principal normal subspace is discussed in *Section 5.1*. It is proved that a normally non-flat semiparallel space-like M^3 in a space E^n_s with $\dim N_xM^3 = 6$ is either a Veronese submanifold, or a submanifold with three families of parabola generators, or a second order envelope of the such submanifolds. In [11] Lumiste has shown that in Euclidean space E^n (i.e. for the case $s = 0$) there exists no semiparallel submanifold M^3 with $\dim N_xM^3 = 5$. In the case of the space E^n_s with $s > 0$ the situation is different (*Section 5.2*). If the considered space-like M^3 is a semiparallel space-like submanifold in a pseudo-Euclidean space E^n_s with $\dim N_xM^3 = 5$, then the metric of the principal normal subspace N_xM^3 vanishes completely (Proposition 5.2). A normally non-flat semiparallel space-like M^3 in E^n_s with $\dim N_xM^3 = 5$ is either a submanifold with 3 families of parabola generators, which can be represented by reduced equation (Proposition 5.3), or a second order envelope of submanifolds from this Proposition. In *Section 5.3* the subcases for case of four-dimensional principal normal subspace are considered. It is proved that in different cases the metric of the principal normal subspace can be either regular, or singular non-vanishing, or completely vanishing (Proposition 5.4, Proposition 5.5). The case of three-dimensional principal normal subspace is discussed in *Section 5.4*. The metric of the principal normal subspace generated by pseudo-Euclidean space can be either regular, or singular non-vanishing (with non–isotropic mean curvature vector H, Proposition 5.8), or completely vanishing. The case of two-dimensional principal normal subspace is investigated in *Section 5.5*. As distinct from Euclidean space the semiparallel submanifolds M^3 in a space E^n_s with $\dim N_xM^3 = 2$ can have not only a flat normal connection but also a non-flat normal connection. In particular, if the submanifold M^3 is a semiparallel space-like submanifold in E^n_s with $\dim N_xM^3 = 2$, then the normal connection is non-flat only in case of completely vanishing metric of the principal normal subspace (Proposition 5.11).

The results, presented this thesis, have been published in [31] and [32]. The author has introduced these results at the following international conferences: "VIII International Conference devoted to the memory of academician M. Kravchuk (1892-1942)" (Kyiv, 2000), "IX Oporto meeting on geometry, topology and physics" (Porto, 2000), "Differential geometry Valencia 2001, An International meeting on the occasion of the 60^{th} birthday of prof. A. M. Naveira" (Valencia, 2001), "Ukrainian Mathematical Congress - UMC2001" (Kyiv, 2001), "8th International Conference on Differential Geometry and Its Applications" (Opava, 2001), "International Conference on Geometry and Topology" (Cherkassy, 2002), "XVII International Fall Workshop on Geometry and Physics" (Castro Urdiales, 2008), "Algebra, Geometry and Mathematical Physics: 4^{th} Baltic-Nordic Workshop" (Tartu, 2008).

Chapter 1

Preliminaries

Let E_s^n be a $n-$dimensional pseudo-Euclidean space of index s with coordinate $(x_1, \dots, x_n)$ and a metric $\sum_{r=1}^{n-s} x_r^2 - \sum_{q=n-s+1}^{n} x_q^2$. Subspaces of pseudo-Euclidean space E_s^n can carry various metrics: a positive- or negative-definite Euclidean metric, a pseudo-Euclidean (occasionally also called a *semi-Euclidean*, see [33]) or a degenerate metric. Depending on the metric a subspace of E_s^n is either a pseudo-Euclidean E_s^k subspace, $k < n$ (Euclidean if $s = 0$), or a semi-pseudo-Euclidean $E_{l,d}^k$ subspace which has an orthogonal frame consisting of l vectors of imaginary length, d vectors of zero length, and $k - l - d$ vectors of real length. Here the integers l and d are called the *index* and the *defect* of subspace $E_{l,d}^k$, accordingly. If $l = 0$ then a semi-pseudo-Euclidean subspace $E_{l,d}^k$ is a semi-Euclidean $E_{0,d}^k$.

1.1 Moving frame adapted to the space-like submanifolds

Let $\{x, e_I\}$, $(I = 1, 2, ..., n)$ be the moving frame in E_s^n, i.e. a free element of the frame bundle in E_s^n.

At a point $x \in M^m$ the *tangent vector space* $T_x M^m$ is a vector subspace of $T_x[E_s^n]$ and has an orthogonal compliment $T_x^{\perp} M^m$ in the latter, which is a $(n-m)-$dimensional vector space, called the *normal vector space* of the submanifolds M^m at x.

The moving frame is said to be adapted to a space–like submanifold $M^m \in E_s^n$, if to take , $e_i \in T_x M^m$, $e_\alpha \in T_x^{\perp} M^m$, where $i, j = 1, \dots, m;\quad \alpha, \beta = m+1, ..., n$. Denoting scalar product of the frame vectors e_I and e_J, as usually, $\langle e_I, e_J \rangle = g_{IJ}$, one has $g_{i\alpha} = 0$ and it can be taken $g_{ij} = \delta_{ij}$; moreover let denote $\langle e_\alpha, e_\alpha \rangle = \varepsilon_\alpha$ and $\langle e_\alpha, e_\beta \rangle = g_{\alpha\beta},\ \alpha \neq \beta$. In the formulae

$$dx = e_I \omega^I, \quad de_I = e_J \omega_I^J, \tag{1.1}$$

$$d\omega^I = \omega^J \wedge \omega_J^I, \quad d\omega_J^I = \omega_J^K \wedge \omega_K^I \tag{1.2}$$

(where the point x is identified with its radius–vector) there hold $\omega^\alpha = 0$ and

$$\omega_i^j = -\omega_j^i, \tag{1.3}$$
$$g_{\alpha\beta}\omega_i^\beta + \omega_\alpha^i = 0, \tag{1.4}$$
$$dg_{\alpha\beta} = g_{\gamma\beta}\omega_\alpha^\gamma + g_{\alpha\gamma}\omega_\beta^\gamma. \tag{1.5}$$

The equations $\omega^\alpha = 0$ lead to

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ij}^\alpha. \tag{1.6}$$

Let h_{ijk}^α denote the covariant derivative of h_{ij}^α defined by

$$\overline{\nabla} h_{ij}^\alpha (\equiv dh_{ij}^\alpha - h_{kj}^\alpha \omega_i^k - h_{ik}^\alpha \omega_j^k + h_{ij}^\beta \omega_\beta^\alpha) = h_{ijk}^\alpha \omega^k, \quad h_{ijk}^\alpha = h_{ikj}^\alpha. \tag{1.7}$$

The relation

$$\overline{\nabla} h_{ijk}^\alpha \wedge \omega^k = \overline{\Omega} \circ h_{ij}^\alpha, \tag{1.8}$$

where

$$\overline{\Omega} \circ h_{ij}^\alpha = -h_{kj}^\alpha \Omega_i^k - h_{ik}^\alpha \Omega_j^k + h_{ij}^\beta \Omega_\beta^\alpha, \tag{1.9}$$

can be obtained from the previous by exterior differentiation. In formulae (1.9)

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = -g_{\alpha\beta}\omega_i^\alpha \wedge \omega_j^\beta, \tag{1.10}$$
$$\Omega_\beta^\alpha = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha = -\sum_i g_{\alpha\gamma}\omega_i^\gamma \wedge \omega_i^\beta \tag{1.11}$$

are the curvature 2-forms of the Levi-Civita connection ∇ and the normal connection $\nabla^\perp$, respectively. Together they represent the curvature 2-forms of the van der Waerden-Bortolotti connection $\overline{\nabla}$.

Remark here, that $\Omega_i^j = -\Omega_j^i$ and that exterior differentiation leads from (1.5) to the following relations $g_{\gamma\beta}\Omega_\alpha^\gamma + g_{\alpha\gamma}\Omega_\beta^\gamma = 0$.

1.2 The isotropic cones of pseudo-Euclidean space

The structure of a space-like submanifold M^m in pseudo-Euclidean space E_s^n is determined by the form

$$g = g_{ij}\omega^i\omega^j, \qquad i, j = 1, \ldots, m, \tag{1.12}$$

and the isotropic cones C_x of this space are defined by the equation

$$g = g_{rq}\omega^r\omega^q = 0, \qquad r, q = 1, \ldots, n, \tag{1.13}$$

whose left-hand side is a nondegenerate quadratic form. Thus the normal vector space $T_x^\perp M^m$ can have a real intersection with the cone C_x and is divided by it into two domains – internal and external. Directions belonging to the first domain are called *timelike*, and directions belonging to the second domain are called *spacelike* (see Figure 1).

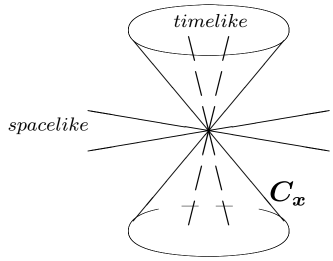

Figure 1

Therefore, the normal space $T_x^\perp M^m$ can have different signatures that depend on the numbers p and s, and on the mutual location of this normal space $T_x^\perp M^m$ and isotropic cone C_x.

Let us consider, for example, the pseudo-Euclidean space E_1^4. Three different possible mutual locations of the cone C_x and the normal space $T_x^\perp M^2$ to the submanifold $M^2 \subset E_1^4$ are presented in Figure 2, 3, and 4.

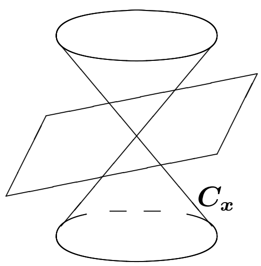

Figure 2

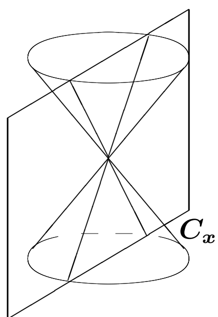

Figure 3

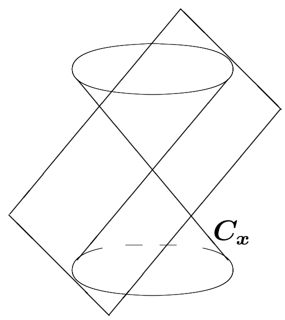

Figure 4

In the first case (see Figure 2) the normal space $T_x^\perp M^2$ to a spacelike M^2 contains only spacelike directions, which located outside of the cone C_x. In the second case (see Figure 3) the normal space $T_x^\perp M^2$ to a spacelike M^2 contains both spacelike and timelike directions. Finally, in the third case (see Figure 4), where the normal space $T_x^\perp M^2$ is tangent to the isotropic cone C_x, the directions are spacelike and *isotropic* (or *lightlike*).

1.3 The principal (first) normal subspace

For investigation of semiparallel submanifolds M^m the vector subspace $N_xM^m = \mathrm{span}\{h_{ij}\}$ at an arbitrary fixed point $x \in M^m$, where $h_{ij} = h^\alpha_{ij} e_\alpha$ $i, j = 1, \ldots, m$ are components of h, is important, called the *principal normal subspace* of submanifolds M^m at x. As a subspace of a pseudo-Euclidean space it can have either regular, or singular non-vanishing, or completely vanishing metric.

Let us consider such metric possibilities in more details, denoting dimension of the principal vector subspace as n_1. The frame vectors belonging to the normal space $T^\perp_x M^m = N_xM^m \oplus N^\perp_x M^m$ can be taken so that

$$e_a \in N_xM^m, \quad e_\xi \in N^\perp_x M^m, \tag{1.14}$$

where $a \in \{m+1, \ldots, m+n_1\}$, $\xi \in \{m+n_1+1, \ldots, n\}$.

The case of regular metric. Here, in general, metric is indefinite because there are k frame vectors with real length, $2l$ with zero length and $n_1 - k - 2l$ with imaginary length, i.e.

$$g_{ab} = \begin{array}{c} \left(\begin{array}{ccc} \| E \| & 0 & 0 \\ 0 & \left\| \begin{array}{cc} 0 & E \\ E & 0 \end{array} \right\| & 0 \\ 0 & 0 & \| -E \| \end{array} \right) \end{array} \begin{array}{l} \} \quad k \\ \left.\vphantom{\begin{array}{c} 0 \\ E \end{array}}\right\} \quad 2l \\ \} \quad n_1 - k - 2l, \end{array} \tag{1.15}$$

where E is a unit matrix. If $l = 0$ and $k = n_1$, then metric is positively definite, if $l = k = 0$, then it is negatively definite.

The case of singular non-vanishing metric. Here one has n_1 frame vectors where p vectors have real length, q vectors have imaginary length and r vectors have zero length (here $r = n_1 - p - q$). The next r frame vectors can be taken so that

$$g_{a'b'} = \left(\begin{array}{ccc} \| E \| & 0 & 0 \\ 0 & \| -E \| & 0 \\ 0 & 0 & \left\| \begin{array}{cc} 0 & E \\ E & 0 \end{array} \right\| \end{array} \right) \begin{array}{l} \} \quad p \\ \} \quad q \\ \left.\vphantom{\begin{array}{c} 0 \\ E \end{array}}\right\} \quad 2r, \end{array} \tag{1.16}$$

where $a', b' = m+1, \ldots, m+n_1+r$.

The case of completely vanishing metric. In this case all n_1 frame vectors e_a belonging to N_xM^m have zero scalar squares and their pairwise scalar products are zero, too. Now the next n_1 frame vectors $e_{\overline{a}}$ $(\overline{a} = a + n_1)$ can be taken as in the previous case (1.16) supposing $p = q = 0$ and $r = n_1$, i.e.

$$g_{a'b'} = \left(\begin{array}{cc} 0 & E \\ E & 0 \end{array} \right) \quad \begin{array}{l} \} \quad n_1 \\ \} \quad n_1. \end{array} \tag{1.17}$$

Chapter 2

General aspects on parallel and semiparallel submanifolds

2.1 The parallelity condition

Due to definition a parallel submanifolds M^m have parallel second fundamental form, i.e. $\overline{\nabla} h = 0$. Thus from (1.7) parallel condition is (see [18])

$$dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta = 0. \tag{2.1}$$

The result of [18] can be extended to parallel space-like submanifolds M^m in E^n_s.

Proposition 2.1. *A parallel space-like M^m in E^n_s with constant dimension of the principal normal space lies in either $E^{m+n_1}_s$ $0 \le s \le n_1$, or $E^{m+n_1}_{r,k+1}$, $0 \le r \le n_1 - k - 1$, $0 \le k \le n_1 - 1$, or $E^{m+n_1}_{0,n_1}$.*

Proof. In this case one has

$$h_{ij} = e_a h^a_{ij}, \quad h^\xi_{ij} = 0 \tag{2.2}$$

and the parallelity condition (2.1) now leads to $h^a_{ij}\omega^\xi_a = 0$, i.e. $\omega^\xi_a = 0$, because the matrix of the coefficients has the rank n_1. Now in (1.1) one has

$$\begin{aligned}
dx &= e_i\omega^i,\\
de_i &= e_j\omega^j_i + e_a h^a_{ij}\omega^j,\\
de_a &= -g_{ab}\sum_{i=1}^m e_i h^b_{ij}\omega^i + e_b\omega^b_a.
\end{aligned}$$

It means that $\operatorname{span}\{x, e_i, e_a\}$ is invariant along space-like M^m. With respect to the metric of N_xM^m there are the following possibilities.

If the metric of the principal normal subspace is regular, then due to (1.15) it can be concluded that parallel space-like submanifolds lie in $E^{m+n_1}_s$, $0 \leq s \leq n_1$.

The space-like parallel submanifold with principal normal subspace of singular non-vanishing metric (1.16) lies in semi-pseudo-Euclidean space $E^{m+n_1}_{r,k+1}$, $0 \leq r \leq n_1 - k - 1$, $0 \leq k \leq n_1 - 1$.

At last, if the metric of the principal normal subspace is completely vanishing, then the considered space-like parallel M^m lies in semi-Euclidean space $E^{m+n_1}_{0,n_1}$ and transformation $\frac{\sqrt{2}}{2}e'_a = e_a + e_{\overline{a}}$, $\frac{\sqrt{2}}{2}e'_{\overline{a}} = e_a - e_{\overline{a}}$ gives the frame for the pseudo-Euclidean space E^n_s with $s \geq n_1$, $n \geq 2n_1 + m$. $\square$

Corollary 2.1. *Let M^m be a space-like submanifold in pseudo-Euclidean space E^n_s and its normal space has completely vanishing metric. Then denoting* $\mathrm{dim} N_x M^m = n_1$ *one has $n \geq 2n_1 + m$, $s \geq n_1$.*

Proof. In general case one has $\mathrm{dim} T_x M^m + \mathrm{dim} T^{\perp}_x M^m = n$. The dimension of the tangent space is m. Thus

$$\mathrm{dim} T^{\perp}_x M^m = n - m. \tag{2.3}$$

On the other hand the normal space is pseudo-Euclidean and there exists isotropic cone with n_1-dimensional flat generators. In [34] it was shown that such cone lies in pseudo-Euclidean space with $s \geq n_1$ and $\mathrm{dim} T^{\perp}_x M^m \geq 2n_1$. Together with (2.3) it leads to $n \geq 2n_1 + m$. $\square$

Proposition 2.2. *A parallel space-like M^m in pseudo-Euclidean space E^n_s with principal normal subspace of completely vanishing metric is either a submanifold in $E^{m+n_1}_{0,n_1}$ with m families of parabola generators (some of them can degenerate into a straight line) and can be represented by the equation*

$$x = \tfrac{1}{2}h_{ii}(u^i)^2 + h_{ij}u^i u^j + h_{0i}u^i, \tag{2.4}$$

$(i, j = 1, \dots, m;\ i \neq j)$ all coefficients here are some constant vectors, or an open part of such a submanifold. In case where $n_1 < \frac{1}{2}m(m+1)$ there are some linear relations between vectors in (2.4).

Proof. Due to [18] for parallel M^m one has

$$\nabla h_{ij} = -\sum_{k=1}^{m} e_k \langle h_{ij}, h_{kl} \rangle \omega^l$$

and so $\nabla h_{ij} = 0$, if $N_x M^m$ has completely vanishing metric. Due to (1.10) then $\Omega^j_i = 0$, i.e. M^m is locally Euclidean. Thus every point $x \in M^m$ has a neighborhood U, on which there is a parallel field of tangent orthogonal frames. For this field $de_i = 0$, so $\omega^j_i = 0$ and $\nabla h_{ij} = 0$ reduces to $dh_{ij} = 0$, but $d\omega^i = \omega^j \wedge \omega^i_j$ reduces to

$d\omega^i = 0$. Hence on this U it can be made $\omega^i = du^i$, h_{ij} =const. Now the derivation formulae are

$$dx = e_i\omega^i, \quad de_i = h_{ij}\omega^j, \quad dh_{ij} = 0 \tag{2.5}$$

and for the principal and the second derivatives of x one has

$$x_{u^i} = e_i, \quad x_{u^iu^i} = h_{ii}, \quad x_{u^iu^j} = h_{ij}$$

whereas all third derivatives are zero. Thus the geodesic lines are parabolas and the considered parallel space-like submanifold M^m can be represented by the equation (2.4). □

2.2 The semiparallelity condition

The semiparallelity condition $\overline{\nabla} h_{ijk} \wedge \omega^k = 0$ with the help of the Cartan's lemma leads to

$$\overline{\nabla} h_{ijk}(\equiv dh^\alpha_{ijk} - h^\alpha_{ljk}\omega^l_i - h^\alpha_{ikl}\omega^l_j - h_{ijl}\omega^l_k + h^\beta_{ijk}\omega^\alpha_\beta) = h^\alpha_{ijkl}\omega^l, \tag{2.6}$$

where

$$h^\alpha_{ijkl} = h^\alpha_{ijlk} \tag{2.7}$$

Using (1.9) the semiparallelity condition can be represented in the following way

$$h^\alpha_{kj}\Omega^k_i + h^\alpha_{ik}\Omega^k_j - h^\beta_{ij}\Omega^\alpha_\beta = 0\ . \tag{2.8}$$

Denoting $H_{ij,kl} = g_{\alpha\beta}h^\alpha_{ij}h^\beta_{kl}$ and using (2.2), one concludes that the semiparallelity condition (2.8) is equivalent to

$$\sum_k (h_{kj}H_{i[p,q]k} + h_{ik}H_{j[p,q]k} - H_{ij,k[p}h_{q]k}) = 0. \tag{2.9}$$

Proposition 2.3. *Every space-like submanifold M^m in pseudo-Euclidean space E^n_s, with the principal normal subspace of completely vanishing metric is semiparallel.*

Proof. Taking $e_a \in N_xM^m$, $e_{\overline{a}}, e_\xi \in N^\perp_x M^m$ one has $h^{\overline{a}}_{ij} = h^\xi_{ij} = 0$ ($\xi = m + 2n_1 + 1, ..., n$), i.e.

$$\omega^{\overline{a}}_i = \omega^\xi_i = 0. \tag{2.10}$$

Due to the completely vanishing metric together with (1.5) one has in (1.10) and (1.11) that $\Omega^j_i = \Omega^b_a = 0$. Thus the semparallelity condition (2.8) transforms into zero identity. □

Denoting $\widetilde{\Omega}^a_b = d\omega^a_b - \omega^c_b \wedge \omega^a_c$ and $NM^m = \cup_{x\in M^m} N_xM^m$, one has the normal vector bundle $NM^m \to M^m$ with fibre N_xM^m at every $x \in M^m$. The fact that 2-forms $\widetilde{\Omega}^a_b$ are semi-basic, shows that there is connection in this bundle with connection forms ω^a_b (Evtushik, [35], Ch. II, par.1). This connection will be denoted by ∇^N and called the *first normal connection.*

Proposition 2.4. *Semiparallel space-like submanifold M^m in $E^{m+2n_1}_{0,n_1}$ with the normal space of completely vanishing metric has flat normal connection ∇^N.*

Proof. In general 2-forms $\widetilde{\Omega}^a_b$ can be written in the following way

$$\widetilde{\Omega}^a_b = \omega^i_b \wedge \omega^a_i + \omega^{\bar{c}}_b \wedge \omega^a_{\bar{c}} + \omega^\xi_b \wedge \omega^a_\xi.$$

If submanifold M^m lies in $E^{m+2n_1}_{0,n_1}$, then all 1-forms with indexes ξ are absent. Using frame from previous Proposition one has that 1-forms ω^i_b are zero due to (1.4) and (2.10). It remains to show that all 1-forms $\omega^{\bar{a}}_b = 0$.

Since the dimension of the principal normal space is n_1, then among vectors h_{ij} there are n_1 linearly independent. Let denote them $h_{k_s l_s}$, $s = 1, \dots, n_1$. Thus the frame vectors $e_{m+1}, \dots, e_{m+n_1}$ can be taken so that

$$e_{m+1} = h_{k_1 l_1},\ e_{m+2} = h_{k_2 l_2}, \dots, e_{m+n_1} = h_{k_{n_1} l_{n_1}}, \tag{2.11}$$

and all others vectors of the second fundamental form can be written

$$h_{pq} = \chi_1 e_{m+1} + \chi_2 e_{m+2} + \dots + \chi_{n_1} e_{m+n_1} = \sum_{s=1}^{n_1} \chi_s h_{k_s l_s}.$$

Now for semiparallel not parallel submanifolds one has $\omega^{\bar{a}}_b = h^{\bar{a}}_{k_{b-m} l_{b-m} p}\ \omega^p$. In [18] (Theorem 19.1) it is obtained $\langle h_{ijk}, h_{kl} \rangle = 0$. In our case it gives for linearly independent vectors $h_{k_s l_s}$ that

$$\langle h_{k_{a-m} l_{a-m}}, h_{k_{b-m} l_{b-m} p} \rangle = 0, \tag{2.12}$$

where $h_{k_{b-m} l_{b-m} p} = h^a_{k_{b-m} l_{b-m} p} e_a + h^{\bar{a}}_{k_{b-m} l_{b-m} p} e_{\bar{a}} + h^\xi_{k_{b-m} l_{b-m} p} e_\xi$. Due to the frame choosing (2.12) can be rewritten in the following way

$$\langle e_a,\ h^{\bar{b}}_{k_{b-m} l_{b-m} p} e_{\bar{b}} \rangle = 0,$$

i.e. $h^{\bar{a}}_{k_{b-m} l_{b-m} p} = 0$ and $\omega^{\bar{a}}_b = 0$. Thus 2-forms $\widetilde{\Omega}^a_b$ are zero and the considered semiparallel submanifold has flat normal connection ∇^N. □

2.3 Semiparallel submanifolds as a second order envelope of parallel ones

Let us start with some definitions of terms, which are connected with the second order of enveloping.

Two paths λ and $\widetilde{\lambda}$ in E^n_s are said to have the first order tangency at their common point x_0, corresponding to $t = 0$, if their tangent vectors X at x_0 coincide. They are said to have the second order tangency at x_0, if in addition, their curvature vectors $h(X^0, X^0)$ at x_0 coincide; here λ and $\widetilde{\lambda}$ are considered as an 1-dimensional submanifolds, and X^0 is their common unit tangent vector at x_0.

Two submanifolds M^m and $\widetilde{M}^m$ with a common point x_0 in E^n_s are said to have the v *order tangency* at x_0, if for every path λ through x_0 in M^m there is a path $\widetilde{\lambda}$ through the same x_0 in $\widetilde{M}^m$, which has the v order tangency with λ at x_0.

It is obvious that first order tangency means that the tangent m-planes of these submanifolds at x_0 coincide. In case of second order tangency of two (pseudo-)Riemannian submanifolds M^m and $\widetilde{M}^m$ in $N^n_s(c)$ at their common point x_0 it is necessary that their fundamental triplets at x_0 coincide.

Let a submanifold M^m in $N^n_s(c)$ have for its every point x a submanifold $\widetilde{M}^m$ in $N^n_s(c)$, which has the second order tangency with M^m at x. Then M^m is said to be the *second order envelope* of the family of such submanifolds $\widetilde{M}^m$.

Theorem.[Lumiste, [17]] *A submanifold M^m in $N^n_s(c)$ is semiparallel if and only if it is a second order envelope of parallel submanifolds.*

2.4 Veronese submanifolds and their second order envelopes

In [18] for parallel submanifolds M^m in a connected complete Riemannian manifold with constant curvature $N^n(c)$ with maximal possible dimension of the principal normal subspace $n_1 = \frac{1}{2}m(m+1)$ it is obtained that it is intrinsically a Riemannian manifold of constant curvature $K > 0$, immersed into an $(n-1)$-dimensional sphere $S^{n-1}(2K(m+1)m^{-1})$, $n = \frac{1}{2}m(m+3)$, as a minimal submanifold. If this M^m is connected and complete then all its inner motions are induced by the isometries of this sphere.

A such submanifolds M^m is called the *Veronese submanifold* and denoted as V^m; it lies, at least locally, in $E_s^{\frac{1}{2}m(m+3)}$, where $s = 0$, or $s = \frac{1}{2}m(m+1)$.

A semiparallel submanifolds M^m with $m \geq 3$ in $N^n(c)$ with $n = \frac{1}{2}m(m+3)$, whose principal normal subspace at arbitrary point has the maximal possible dimension $\frac{1}{2}m(m+1)$ is parallel (see [18]).

Existence of semiparallel submanifolds M^m as a 2nd order envelopes of a such parallel submanifolds in $N^n(c)$, $n > \frac{1}{2}m(m+3)$ is affirmatively shown by Lumiste in [22] and in [23]. In case of $V^2 \subset E^6$ is shown (Riives, [24]) that there are exist semiparallel, but not parallel 2nd order envelope of a family V^2. The question on existence of 2nd order envelopes of two-dimensional Veronese submanifolds in E^6_s, (s is either 0, or 3, or 4) and in E^7_s, (s is either 0, or 3, or 4, or 5) with some arbitrariness is solved by Lumiste in [21].

2.5 Segre submanifolds and their second order envelopes

Among parallel submanifolds M^m in E^n (see [18]) there exists a class of submanifolds $M^{p_1+p_2}$ in a sphere $S^{p_1p_2+p_1+p_2}(a^2)$, generated by p_1- and p_2-dimensional great spheres of the latter, totally orthogonal at each point of $M^{p_1+p_2}$. A submanifolds of a such class is called the *Segre submanifolds*.

Taking the moving orthogonal frame with origin x so that the basic vectors e_π, $\pi \in \{0, 1, \dots, p_1\}$ are tangent to the generating great p_1-sphere, $e_{\overline{\pi}}$, $\overline{\pi} \in \{0, 1, \dots, p_2\}$ are tangent to the generating great p_2-sphere, e_{m+1} is opposite to the normalized radius-vector x of the point x and e_ξ are the remaining normal vectors, $\xi \in \{m+2, \dots, m+p_1p_2\}$ one has the system, which determines the Segre submanifolds in the following way:

$$\begin{aligned}
dx &= e_\pi \omega^\pi + e_{\overline{\pi}} \omega^{\overline{\pi}},\\
de_\pi &= e_\sigma \omega^\sigma_\pi + a e_{m+1} \omega^\pi + a e_{\pi\overline{\pi}} \omega^{\overline{\pi}},\\
de_{\overline{\pi}} &= e_{\overline{\sigma}} \omega^{\overline{\sigma}}_{\overline{\pi}} + a e_{m+1} \omega^{\overline{\pi}} + a e_{\pi\overline{\pi}} \omega^\pi,\\
de_{m+1} &= -a(e_\pi \omega^\pi + e_{\overline{\pi}} \omega^{\overline{\pi}}),\\
de_{\pi\overline{\pi}} &= -a(e_\pi \omega^{\overline{\pi}} + e_{\overline{\pi}} \omega^\pi) + e_{\sigma\overline{\pi}} \omega^\sigma_\pi + e_{\pi\overline{\sigma}} \omega^{\overline{\sigma}}_{\overline{\pi}},
\end{aligned}$$

where $\sigma \in \{0, 1, \dots, p_1\}$, $\overline{\sigma} \in \{0, 1, \dots, p_2\}$ and the point c with radius-vector

$$x = a^{-1} e_{m+1},$$

is a center of the Segre submanifold.

The second order envelope of the Segre submanifolds $M^{p_1+p_2}$ in E^n with a given dimension $m = p_1 + p_2$, $p_1 \geq p_2$, and arbitrary centers and radii $r = a^{-1}$ is

1) by $p_1 = p_2 = 1$ a surface M^2 with flat $\overline{\nabla}$, the two principal curvature vectors of which have at every point the same length $\sqrt{2}a$,

2) by $p_1 > 1$, $p_2 = 1$ a submanifold M^m in $E^{2m} \subset E^n$, generated by an 1-parametric family of the concentric p_1-dimensional spheres, the orthogonal trajectories of which are the congruent logarithmic spirals (or circles in the limit case) with the common pole in the center of the family of spheres,

3) by $p_1 > 1$, $p_2 > 1$ a single $M^{p_1+p_2}$.

More information can be found in Lumiste articles [25] and [26].

Chapter 3

Semiparallel and parallel space-like surfaces

A submanifold M^m in E^k_s is said to be a *product of submanifolds* M^{m_q} in $E^{k_q}_{s_q}$ $(q = 1, \dots, r)$ if (i) $M^m = M^{m_1} \times \dots \times M^{m_r}$, (ii) $E^k_s = E^{k_1}_{s_1} \times \dots \times E^{k_r}_{s_r}$, where in the right hand side every two different components are totally orthogonal.

By relaxing the last requirement in (ii) (i.e. every two components $E^{k_q}_{s_q}$ are not necessary to be mutually orthogonal) one obtains a *translation submanifold* M^m of submanifolds M^{m_q} in $E^{k_q}_{s_q}$.

If a submanifold M^m in E^n_s is decomposable into a translation submanifold then such a M^m is said to be *reducible*, otherwise *irreducible*.

To classify a translation submanifolds one needs to study corresponding irreducible submanifolds. Therefore the classification of the latter is important to describe all such submanifolds. Here is natural to start with the low-dimensional cases.

3.1 Classification of semiparallel space-like surfaces

For the dimension $m = 1$ there is easy to see that every curve is semiparallel, namely has flat $\overline{\nabla}$. To obtain the classification result of $m = 2$ one needs some preparations.

If $m = 2$ then the tangent part $\{e_1, e_2\}$ of the adapted frame can be transformed according to

$$e_1^{'} = e_1 \cos\phi + e_2 \sin\phi, \qquad e_2^{'} = -e_1 \sin\phi + e_2 \cos\phi. \tag{3.1}$$

Then

$$\omega_{1'}^{2'} = \omega_1^2 + d\phi, \qquad \omega^1 = \omega^{1'} \cos\phi - \omega^{2'} \sin\phi\ , \qquad \omega^2 = \omega^{1'} \sin\phi + \omega^{2'} \cos\phi \tag{3.2}$$

and for $h = h_{ij}\omega^i\omega^j$ one obtains

$$h'_{11} = \tfrac{1}{2}(h_{11} + h_{22}) + \tfrac{1}{2}(h_{11} - h_{22})\cos 2\phi + h_{12}\sin 2\phi \ ,$$

$$h'_{12} = \tfrac{1}{2}(h_{22} - h_{11})\sin 2\phi + h_{12}\cos 2\phi \ ,$$

$$h'_{22} = \tfrac{1}{2}(h_{11} + h_{22}) + \tfrac{1}{2}(h_{22} - h_{11})\cos 2\phi - h_{12}\sin 2\phi \ .$$

Therefore span$\{h_{11}, h_{22}, h_{12}\}$ is an invariant vector subspace of $T_x^{\perp}M^2$ at an arbitrary fixed point $x \in M^2$. Is is the principal normal subspace of M^2 at x, denoted by N_xM^2. Let us denote $\frac{1}{2}(h_{11} - h_{22}) = A$, $h_{12} = B$, and $\frac{1}{2}(h_{11} + h_{22}) = H$; then $A' = A\cos 2\phi + B\sin 2\phi$, $B' = -A\sin 2\phi + B\cos 2\phi$, $H' = H$. It is seen that H is an invariant vector, called the *mean curvature vector*, and that span$\{A, B\}$ is an invariant vector subspace at x, denoted by I_xM^2; the latter is the plane of the *normal curvature indicatrix* determined as $\{y : y - x = h_{ij}X^iX^j, X \in T_xM^2, \| X \| = 1\}$. Since

$$\langle A', B' \rangle = \langle A, B \rangle \cos 4\phi + \tfrac{1}{2}(B^2 - A^2)\sin 4\phi,$$

there exists ϕ_0 such that $\langle A', B' \rangle = 0$. So it can be made $\langle A, B \rangle = 0$. If now to take $\phi = \frac{\pi}{4}$, then $A' = B$, $B' = -A$, and $\langle A', B' \rangle = 0$, so that the roles of A and B can be interchanged, if this is not obstructed by the metric.

Theorem 3.1. *Let M^2 be a semiparallel space-like surface in E^n_s. There exists an open and dense part U of M^2 such that the connected components of U are of the following types:*

(i) open parts of totally umbilical M^2 (in particular, of totally geodesic M^2) in E^n_s;

(ii) surfaces with flat $\overline{\nabla}$;

(iii) isotropic surfaces with nonflat $\nabla^{\perp}$ satisfying $\| H \|^2 = 3K$, where K is the Gaussian curvature and H is the mean curvature vector.

Proof. The indicatrix of normal curvature is generally an ellipse whose plane has direction span$\{A, B\}$ and goes through the endpoint of the vector H, with initial point placed at x; it could also be a degenerate form of such an ellipse (a line segment or a point).

Let us start proof with the case where dim$I_xM^2 = 0$, i.e. the indicatrix of normal curvature degenerates into a point and one has $A = B = 0$. If here dim$N_xM^2 = 1$, then $H \neq 0$ and e_3 can be taken so that $H = \delta e_3$ and the components of the second fundamental form h_{ij} can be written as follows: $h_{11} = h_{22} = \delta e_3$, $h_{12} = 0$. Thus M^2 is totally umbilic. For the case dim$N_xM^2 = 0$ one has $\delta = 0$ and the considered surface is totally geodesic. This leads to case (i) of the Theorem 3.1.

In case dim$I_xM^2 = 1$ one has that the indicatrix of normal curvature degenerates into a line segment. In this case the vectors A and B are collinear and at least one of them is nonzero. Since A and B can be interchangeable, it is always assumed that $A \neq 0$. Then the frame vector e_3 can be taken so that

$$A = ae_3,\ a > 0,\ B = be_3.$$

Let $\dim N_x M^2 = 2$. The next frame vector e_4 can be taken so that $H = \delta e_3 + \sigma e_4$. If $\dim N_x M^2 = 1$, one has $\sigma = 0$. The Pfaff system (1.6) can be written as

$$\omega_1^3 = (\delta + a)\omega^1 + b\omega^2,\ \ \omega_1^4 = \sigma\omega^1,\ \ \omega_1^\xi = 0, \tag{3.3}$$
$$\omega_2^3 = b\omega^1 + (\delta - a)\omega^2,\ \ \omega_2^4 = \sigma\omega^2,\ \ \omega_2^\xi = 0, \tag{3.4}$$
$$\tag{3.5}$$

where $\xi = 5, ..., n$. Hence

$$\Omega_1^2 = -\Omega_2^1 = [\varepsilon_3 a^2 + \varepsilon_3 b^2 - H^2]\omega^1 \wedge \omega^2, \tag{3.6}$$

where $H^2 = \varepsilon_3\delta^2 + \varepsilon_4\sigma^2 + 2g_{34}\delta\sigma$, and all Ω_α^β are zero.

With respect to the metric in subspaces $I_x M^2$ and $N_x M^2$ there are the following possibilities.

If the metric of $I_x M^2$ is regular, then $\varepsilon_3 = \pm 1$, and $b = 0$. Thus the vector e_4 can be taken so that either

$$\varepsilon_4 \neq 0,\ \ g_{34} = 0, \tag{3.7}$$

or

$$\varepsilon_4 = 0,\ g_{34} = 0, \tag{3.8}$$

and, moreover, in the last case the vector e_5 can be taken so that $\varepsilon_5 = 0,\ g_{45} = 1$. So (3.7) and (3.8) mean that the metric of $N_x M^2$ is regular or singular non-vanishing, respectively.

If the metric of $I_x M^2$ is vanishing, then $\varepsilon_3 = 0$ and semiparallelity condition leads to $b \neq 0$ or $b = 0$). Now the metric of $N_x M^2$ is either regular, or singular non-vanishing, or vanishing. This means that the vector e_4 can be taken so that either

$$\varepsilon_4 = 0,\ g_{34} = 1, \tag{3.9}$$

or

$$\varepsilon_4 \neq 0,\ g_{34} = 0, \tag{3.10}$$

and, moreover, in (3.10) the vector e_5 can be taken so that $\varepsilon_5 = 0,\ g_{35} = 1$, or

$$\varepsilon_4 = 0,\ \ g_{34} = 0. \tag{3.11}$$

In the last case the frame vectors $e_5,\ e_6$ can be chosen so that $\varepsilon_5 = \varepsilon_6 = 0$, $g_{35} = g_{46} = 1,\ g_{36} = g_{45} = g_{56} = 0,\ n \geq 6$.

For all these cases the semiparallelity condition (2.8) reduces to

$$b\Omega_1^2 = 0,\ a\Omega_1^2 = 0.$$

Since $a > 0$, one has $\Omega_1^2 = 0$. This result together with $\Omega_\alpha^\beta = 0$ gives that $\overline{\nabla}$ is flat, i.e. leads to case (ii) of the Theorem.

At last, the non-degenerate indicatrix $\{y : y - x = H + A\cos 2\psi + B\sin 2\psi\}$ is an ellipse. In this case the mutually orthogonal vectors A and B are noncollinear. The orthogonal frame vectors e_3 and e_4 (i.e. with $g_{34} = 0$) in I_xM^2 can be taken so that

$$A = ae_3,\ B = be_4,\ a \geq b > 0.$$

The dimension of the subspace N_xM^2 is either 3 or 2.

Let $\dim N_xM^2 = 3$, thus the next frame vector e_5 can be taken so that $H = \delta e_3 + \sigma e_4 + \tau e_5$. Here the components h_{ij} can be written as follows:

$$h_{11} = (\delta + a)e_3 + \sigma e_4,\ h_{22} = (\delta - a)e_3 + \sigma e_4,\ h_{12} = be_4.$$

Thus M^2 is determined by the Pfaff system

$$\omega_1^3 = (\delta + a)\omega^1,\ \ \omega_1^4 = \sigma\omega^1 + b\omega^2,\ \ \omega_1^5 = \tau\omega^1,\ \ \omega_1^\xi = 0,$$

$$\omega_2^3 = (\delta - a)\omega^2,\ \ \omega_2^4 = b\omega^1 + \sigma\omega^2,\ \ \omega_2^5 = \tau\omega^2,\ \ \omega_2^\xi = 0,$$

where $\xi = 6, ..., n$. Hence the curvature 2-forms in (1.10) are

$$\Omega_1^1 = \Omega_2^2 = 0,\ \Omega_1^2 = -\Omega_2^1 = (\varepsilon_3 a^2 + \varepsilon_4 b^2 - H^2)\omega^1 \wedge \omega^2, \tag{3.12}$$

where

$$H^2 = \varepsilon_3\delta^2 + \varepsilon_4\sigma^2 + \varepsilon_5\tau^2 + 2g_{35}\delta\tau + 2g_{45}\sigma\tau,$$

and in (1.11)

$$\Omega_3^4 = -2\varepsilon_3 ab\omega^1 \wedge \omega^2,\ \Omega_4^3 = 2\varepsilon_4 ab\omega^1 \wedge \omega^2, \tag{3.13}$$
$$\Omega_5^3 = 2g_{54}ab\omega^1 \wedge \omega^2,\ \Omega_5^4 = -2g_{35}\omega^1 \wedge \omega^2, \tag{3.14}$$

all others Ω_α^β are zero. Thus the semiparallelity condition (2.8) transforms into

$$\begin{array}{ll} ab(\varepsilon_4\sigma + g_{45}\tau) = 0, & b(2\varepsilon_3 a^2 + \varepsilon_4 b^2 - H^2 + \varepsilon_3 a\delta + g_{35}a\tau) = 0, \\ a(\varepsilon_3 a^2 + 2\varepsilon_4 b^2 - H^2) = 0, & b(2\varepsilon_3 a^2 + \varepsilon_4 b^2 - H^2 - \varepsilon_3 a\delta - g_{35}a\tau) = 0. \end{array}$$

The consideration of this system gives, due to $ab\tau \neq 0$, that $\varepsilon_3 = \varepsilon_4 = \varepsilon_5$, having the values 1, -1 or 0.

In the last case the metric of N_xM^2 vanishes completely and the frame vectors can be taken so that

$$\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0, \;\; \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 0, \;\; g_{36} = g_{47} = g_{58} = 1, \;\; n \geq 8. \tag{3.15}$$

It can be obtained by the appropriate choice of remaining frame vectors so that all others $g_{\alpha\beta}, \; \alpha \neq \beta$, are zero.

If $\dim N_xM^2 = 2$, then $\tau = 0$ and the semiparallelity condition leads to $\varepsilon_3 = \varepsilon_4 = 0$. Here the vectors $e_5, \; e_6$ can be taken so that

$$\varepsilon_5 = \varepsilon_6 = 0, \; g_{35} = g_{46} = 1, \; g_{56} = 0, \; n \geq 6. \tag{3.16}$$

Note that in the cases of this section the choice of all others e_α depends on the value of s in E^n_s.

Let us consider the cases (3.15), (3.16) in more detail, denoting $\dim N_xM^2 = n_1, \; (n_1 = 3, 2)$ and $a, b = \{3, ..., n_1 + 2\}, \; \overline{a}, \overline{b} = \{n_1 + 3, ..., 2n_1 + 2\}$, in (1.3)–(1.5) one has

$$\omega_i^{a+n_1} + \omega_a^i = 0, \;\; \omega_i^{\overline{a}-n_1} + \omega_{\overline{a}}^i = 0, \tag{3.17}$$
$$\omega_a^a + \omega_{\overline{a}}^{\overline{a}} = 0, \;\; \omega_a^{\overline{a}} = \omega_{\overline{a}}^a = 0, \tag{3.18}$$
$$\omega_a^{\overline{b}} + \omega_b^{\overline{a}} = 0, \;\; \omega_{\overline{a}}^b + \omega_{\overline{b}}^a = 0, \;\; \omega_a^b + \omega_{\overline{b}}^{\overline{a}} = 0. \tag{3.19}$$

The substitution from (3.17)–(3.19) into (1.10), (1.11) gives that $\Omega_i^j = \Omega_\alpha^\beta = 0$, i.e. $\overline{\nabla}$ is flat. This leads to case *(ii)* of the Theorem 3.1.

In the case where the first normal subspace has a regular metric (i.e. $\varepsilon_3 = \varepsilon_4 = \varepsilon_5$, equalling to 1 or -1), the semiparallelity condition leads to $\delta = \sigma = 0, \; a = b$, $\tau = a\sqrt{3}$. It follows that the normal curvature vector has a constant scalar square at every point $x \in M^2$, i.e. the considered surface is an isotropic surface and $H^2 = 3K$. It gives case *(iii)* of the Theorem 3.1. □

3.2 Classification of the parallel lines and surfaces

Proposition 3.1. *A parallel space-like M^1 in E^n_s is either a straight line, or a circle (it can be either real, or imaginary radius), or a parabola.*

Proof. For the principal normal of the curve M^1 in pseudo-Euclidean spaces there are three possibilities: it can be space-like, time-like, or light-like. In the first two cases one has $\varepsilon_2 = \pm 1$. The parallel curve can be treated like in [18]. It is a straight line or a circle; the latter can be of either real, or imaginary radius. If the principal normal is light-like, then $\varepsilon_2 = 0$ and the next frame vector e_3 can be taken so that $\varepsilon_3 = 0, \; g_{23} = 1$. Thus the Bartels-Frenet formulae can be written as

$$dx = e_1 ds, \; de_1 = k_1 e_2 ds, \; de_2 = -d\ln k_1 e_2 ds.$$

The parallelity condition leads to k_1 =const, thus $de_2 = 0$. After integration it gives $x = \frac{1}{2}cs^2 + c_1 s + c_2$, where all coefficients are constant vectors. Therefore the parallel curve of this case is a parabola. □

As it is noted in Introduction, for the geometric description of the surfaces of Theorem 3.1 more detailed classification and characterization of the surfaces of type (ii) are needed. First, the same must be done for the corresponding parallel surfaces.

Proposition 3.2. *Let M^2 be a space-like parallel surface in E^n_s with flat $\overline{\nabla}$, which lies essentially in an affine subspace of the E^n_s. Such an M^2 is either*

(ii_1) a translation surface of two parallel curves, or

(ii_2) a surface in E^4_1 on its isotropic cone C^3, with a fixed vertex; the mean curvature vector of this surface is isotropic and $T^{\perp}_x M^2$ goes through the generator of the cone, or

(ii_3) a surface in $E^3_{0,1}$, $E^4_{0,2}$ or $E^5_{0,3}$ with two families of parabola generators (one of them can degenerate into a family of straight lines). This surface can be represented by the equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + c_1 u + c_2 v$, where all coefficients are some constant vectors and the first three of them are isotropic (no matter whether this degeneration occurs or not).

Proof. For the full classification of parallel surfaces (ii) with flat $\overline{\nabla}$ there must be considered the frame possibilities (3.7)–(3.11), (3.15), (3.16).

Let, at first, the frame vectors be taken as shown in (3.7). If here $\sigma \neq 0$, then from the parallelity condition one has $\omega^4_3 = \omega^2_1 = \omega^\xi_3 = \omega^\xi_4 = da = d\sigma = d\delta = 0$. Thus the derivation formulae are

$$\begin{aligned}
dx &= e_1\omega^1 + e_2\omega^2,\\
de_1 &= [(\delta + a)e_3 + \sigma e_4]\omega^1,\\
de_2 &= [(\delta - a)e_3 + \sigma e_4]\omega^2,\\
de_3 &= -\varepsilon_3[(\delta + a)e_1\omega^1 + (\delta - a)e_2\omega^2],\\
de_4 &= -\varepsilon_4[\sigma e_1\omega^1 + \sigma e_2\omega^2].
\end{aligned}$$

If $\varepsilon_3 = \varepsilon_4$, then the considered parallel surface lies in E^4_s, s is 0 or 2 (it depends on the signature of the metric). Since $d\omega^1 = 0$, $d\omega^2 = 0$, at least locally $\omega^1 = du$, $\omega^2 = dv$. The geodesic lines v =const and u =const are circles. Hence the parallel surface is a translation surface of two circles on the totally orthogonal E^2_s, s is 0 or 1. If $\varepsilon_3 \neq \varepsilon_4$, the considered parallel surface is a translation surface of two lines; one of them is a circle with real radius on E^2 and the other is a circle with imaginary radius on E^2_1. Because of assumption $\sigma = 0$, from (3.6) one has that $a^2 = \delta^2$ and the vector e_3 can be directed so that $a = \delta$, thus one of the geodesic lines degenerates into a straight line.

In the case where $N_x M^2$ has the metric (3.8), the equality (3.6) leads to $a^2 = \delta^2$.

Then the vector e_3 can be taken so that $a = \delta$ and the parallelity condition gives $\omega_1^2 = \omega_4^3 = \omega_3^4 = \omega_3^\xi = \omega_4^\xi = 0$, $\omega_4^4 = -\frac{d\sigma}{\sigma}$, a =const and the derivation formulae can be written as $dx = e_1\omega^1 + e_2\omega^2$, $de_1 = (2ae_3 + \sigma e_4)\omega^1$, $de_2 = \sigma e_4\omega^2$, $de_3 = -2\varepsilon_3 a e_1\omega^1$, $d(\sigma e_4) = 0$. So the considered surface lies in $E^4_{0,1}$ (or in $E^4_{1,1}$) if $\varepsilon_3 = 1$ (or $\varepsilon_3 = -1$, respectively), which is spanned by the point x and mutually orthogonal vectors e_1, e_2, e_3, σe_4. Investigation of geodesic lines gives that parallel M^2 is a translation surface of circles and parabolas. On supposition $\sigma = 0$, it is easy to see that geometry of the corresponding parallel surface is the same as in the previous case on analogous supposition.

In the case where I_xM^2 has a vanishing metric and the frame is described by the equalities (3.9), one has $\Omega_1^2 = 0$; then $d\omega_1^2 = 0$, i.e. $\omega_1^2 = d\psi$. Thus the formulae (3.1)–(3.2) lead to $\omega_{1'}^{2'} = 0$ and h_{ij} can be written as

$$h'_{11} = (\delta + a')e_3 + \sigma e_4,\ h'_{22} = (\delta - a')e_3 + \sigma e_4,\ h'_{12} = b'e_3,$$

where $a' = a\cos2\psi + b\sin2\psi$ and $b' = -a\sin2\psi + b\cos2\psi$. Thus

$$A = a'e_3,\ B = b'e_3,\ H = \delta e_3 + \sigma e_4.$$

The parallelity condition leads to $\omega_3^\xi = \omega_4^\xi = 0$, $\omega_3^3 = -\frac{da'}{a'} = -\frac{db'}{b'} = -\frac{d\delta}{\delta} = \frac{d\sigma}{\sigma}$, i.e. $b' = k_1a'$, $\delta = k_2a'$, $\sigma = \frac{k_3}{a'}$, where k_1, k_2, k_3 are some constants. Moreover, from (3.6) and the semiparallelity condition one has $\delta\sigma = 0$ and either

$$\begin{aligned}
&1)\ \delta = 0,\ \sigma \neq 0, (k_2 = 0), \text{or}\\
&2)\ \delta \neq 0,\ \sigma = 0, (k_3 = 0), \text{or}\\
&3)\ \delta = \sigma = 0, (k_2 = k_3 = 0).
\end{aligned}$$

Since $d\omega^{1'} = 0$, $d\omega^{2'} = 0$, at least locally $\omega^{1'} = du$, $\omega^{2'} = dv$, and the derivation formulae in subcase 1) by $b' \neq 0$ can be written so that

$$\begin{aligned}
&dx = e'_1du + e'_2dv,\\
&de'_1 = (a'e_3 + \frac{k_3}{a'}e_4)du + k_1a'e_3dv,\\
&de'_2 = k_1a'e_3du + (-a'e_3 + \frac{k_3}{a'}e_4)dv,\\
&d(a'e_3) = -k_3dx,\\
&d(\frac{k_3}{a'}e_4) = -k_3[(e'_1 + k_1e'_2)du + (k_1e'_1 - e'_2)dv].
\end{aligned}$$

The considered surface M^2 lies in E^4_1, spanned by the point x and vectors e'_1, e'_2, $a'e_3$, $\frac{k_3}{a'}e_4$. The point $z \in E^4_1$ with the radius vector $z = x + \frac{1}{k_3}a'e_3$ is fixed for the surface since $dz = 0$, thus there is an isotropic cone C^3 with a vertex at point z. The surface lies on the cone, its normal plane $T_x^{\perp}M^2$ goes through the generator of the cone (collinear to e_3) and the mean curvature vector is isotropic (noncollinear to e_3).

On supposition $b'=0$, the derivation formulae for the considered parallel M^2 from subcase 1) can be written as

$$\begin{aligned}
&dx = e_1' du + e_2' dv,\\
&de_1' = (a' e_3 + \frac{k_3}{a'} e_4) du,\\
&de_2' = (-a' e_3 + \frac{k_3}{a'} e_4) dv,\\
&d(a' e_3 + \frac{k_3}{a'} e_4) = -2k_3 e_1' du,\\
&d(-a' e_3 + \frac{k_3}{a'} e_4) = 2k_3 e_2' dv.
\end{aligned}$$

Since $(a'e_3+\frac{k_3}{a'}e_4)^2 = 2k_3$ and $(-a'e_3+\frac{k_3}{a'}e_4)^2 = -2k_3$, this surface lies in E_1^4, spanned by the point x and mutually orthogonal vectors $e_1',\ e_2',\ a'e_3 + \frac{k_3}{a'}e_4,\ -a'e_3 + \frac{k_3}{a'}e_4$. Investigation of its geodesics gives that the considered parallel surface is a translation surface of two plane lines of constant curvature.

In subcase 2), when $k_3 = 0$ on supposition $k_1 \neq 0$ (i.e. $b' \neq 0$), in the derivation formulae one has

$$\begin{aligned}
&dx = e_1' du + e_2' dv,\\
&de_1' = [(k_2+1)du + k_1 dv] a' e_3,\\
&de_2' = [k_1 du + (k_2-1)dv] a' e_3,\\
&d(a' e_3) = 0.
\end{aligned}$$

The considered surface lies in $E_{0,1}^3 \subset E_1^4$, spanned by the point x and mutually orthogonal vectors $e_1',\ e_2',\ a'e_3$. Denoting the partial derivatives of x by x_u, x_v, etc., one has

$$\begin{aligned}
&x_u = e_1,\ x_v = e_2,\\
&x_{uu} = (k_2+1)a'e_3,\ x_{uv} = k_1 a' e_3,\ x_{vv} = (k_2-1)a'e_3,\\
&x_{uuu} = x_{uuv} = x_{vvu} = x_{vvv} = 0.
\end{aligned}$$

Since for this case $(k_2+1)a'e_3 = h_{11}$, $(k_2-1)a'e_3 = h_{22}$, $k_1 a' e_3 = h_{12}$, then parallel M^2 can be represented by the equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + c_1 u + c_2 v$, where all coefficients are some constant vectors; the absolute term can be made zero, if to exchange the initial point. It is seen that the geodesic lines on this parallel surface are parabolas (one of them can degenerate into a straight line).

On supposition $k_1 = 0$, the considered surface lies in $E_{0,1}^3 \subset E_1^4$ and is a translation surface of either two parabolas, or of a parabola and a straight line.

In subcase 3) with $\delta = \sigma = 0$ the derivation formulae can be written so that

$$\begin{aligned} &dx = e_1' du + e_2' dv, \\ &de_1' = a' e_3 du + k_1 a' e_3 dv, \\ &de_2' = k_1 a' e_3 du - a' e_3 dv, \\ &d(a' e_3) = 0. \end{aligned}$$

Thus M^2 lies in $E^3_{0,1}$ and due to $h_{22} = -h_{11}$ either is determined by the equation $x = \frac{1}{2} h_{11}((u)^2 - (v)^2) + h_{12} uv + c_1 u + c_2 v$, where all coefficients are some constant vectors, and thus has two families of parabola generators, or is a hyperbolic paraboloid.

If the frame vectors are taken as shown in (3.10), then geometry of the corresponding parallel surface is the same as in the previous case, subcase 2).

At last, if (3.11) holds, then the parallelity condition implies $\omega_3^3 = -\frac{db'}{b'} = -\frac{da'}{a'} = -\frac{d\delta}{\delta}$ (i.e. $b' = k_1 a'$, $\delta = k_2 a'$, where k_1, k_2 are some constants), $\omega_4^4 = -\frac{d\sigma}{\sigma}$ and $\omega_3^4 = \omega_4^3 = \omega_3^\xi = \omega_4^\xi = 0$.

Since $d\omega^1 = d\omega^2 = 0$, then at least locally $\omega^1 = du$, $\omega^2 = dv$ and the derivation formulae can be written so that

$$\begin{aligned} &dx = e_1' du + e_2' dv, \\ &de_1' = [(k_2 + 1) a' e_3 + \sigma e_4] du + k_1 a' e_3 dv, \\ &de_2' = k_1 a' e_3 du + [(k_2 - 1) a' e_3 + \sigma e_4] dv, \\ &d(a' e_3) = 0, \quad d(\sigma e_4) = 0. \end{aligned}$$

On supposition $\sigma = 0$, the considered surface lies in $E^3_{0,1}$ spanned by the point x and mutually orthogonal vectors e_1', e_2', $a' e_3$ and geometry of this M^2 coincides with geometry in the case with (3.10), subcase 2).

If $\sigma \neq 0$ and $b' \neq 0$, then M^2 lies in $E^4_{0,2}$ spanned by the point x and mutually orthogonal vectors e_1', e_2', $a' e_3$, σe_4 and is determined by the equation $x = \frac{1}{2} h_{11}(u)^2 + \frac{1}{2} h_{22}(v)^2 + h_{12} uv + c_1 u + c_2 v$, where $h_{11} = (k_2 + 1) a' e_3 + \sigma e_4$, $h_{12} = k_1 a' e_3$, $h_{22} = (k_2 - 1) a' e_3 + \sigma e_4$ (all coefficients are constant vectors), and has two families of parabola generators. At last, if $\sigma \neq 0$, but $b' = 0$, then the considered parallel surface lies in $E^4_{0,2}$ and is a translation surface of two geodesic lines, each of which is a parabola.

It remains to consider surfaces for which the metric in $N_x M^2$ vanishes completely and is described either by (3.15) or (3.16).

In the first case the subspace N_xM^2 has a maximal dimension. Then the frame can be adapted to the considered surface so that $A = e_3,\ B = e_4,\ H = e_5$, i.e. $a = b = \tau = 1$ and $\delta = \sigma = 0$. Thus the Pfaff system can be written as follows:

$$\omega_1^3 = \omega^1,\ \omega_1^4 = \omega^2,\ \omega_1^5 = \omega^1,\ \omega_1^{\overline{a}} = \omega_1^{\xi} = 0, \tag{3.20}$$
$$\omega_2^3 = \omega^2,\ \omega_2^4 = \omega^1,\ \omega_2^5 = \omega^2,\ \omega_2^{\overline{a}} = \omega_2^{\xi} = 0. \tag{3.21}$$

Here $\overline{a} = 6, 7, 8;\ \xi = 9, ..., n$, and substitution into the parallelity condition leads to $\omega_3^3 = \omega_4^4 = \omega_5^5 = \omega_3^5 = \omega_4^5 = \omega_5^3 = \omega_5^4 = 0,\ 2\omega_1^2 = \omega_3^4 = -\omega_3^4$. Due to $e_3 + e_5 = h_{11},\ \ e_4 = h_{12},\ \ e_5 - e_3 = h_{22}$, the derivation formulae can be written as

$$\begin{aligned}
dx &= e_1\omega^1 + e_2\omega^2,\\
de_1 &= \omega_1^2 e_2 + h_{11}\omega^1 + h_{12}\omega^2,\\
de_2 &= -\omega_1^2 e_1 + h_{12}\omega^1 + h_{22}\omega^2,\\
dh_{11} &= \omega_1^2 h_{12},\\
dh_{22} &= -\omega_1^2 h_{12},\\
dh_{12} &= -\omega_1^2(h_{11} - h_{22}).
\end{aligned}$$

Since $\Omega_1^2 = 0$, due to (1.10) here $d\omega_1^2 = 0$, i.e. $\omega_1^2 = d\psi$. Using transformation formulae (3.1)–(3.2), one has $\omega_{1'}^{2'} = 0,\ d\omega^{1'} = 0,\ d\omega^{2'} = 0$. The last two equalities imply, at least locally, that $\omega^{1'} = du,\ \ \omega^{2'} = dv$, thus

$$\begin{aligned}
dx &= e_1' du + e_2' dv,\\
de_1' &= h_{11}' du + h_{12}' dv,\\
de_2' &= h_{12}' du + h_{22}' dv,\\
dh_{11}' &= dh_{12}' = dh_{22}' = 0.
\end{aligned}$$

So the considered parallel space-like M^2 lies in $E^5_{0,3}$ spanned by the point x and mutually orthogonal vectors $e_1,\ e_2,\ h_{11}',\ h_{22}',\ h_{12}'$, the last three of which are light-like, two others space-like. This surface can be represented by the equation $x = \frac{1}{2}h_{11}'(u)^2 + \frac{1}{2}h_{22}'(v)^2 + h_{12}'uv + h_{01}'u + h_{02}'v$, where all coefficients are some constant vectors. It is seen that the geodesic lines u =const and v =const on this parallel surface are parabolas.

For the parallel space-like surface M^2, with $\dim N_xM^2 = 2$, i.e. when the frame vectors are taken as (3.16) and the frame can be adapted to M^2 so that $A = e_3$, $B = e_4$, i.e. $a = b = 1$. Moreover, if the mean curvature vector H is nonzero, then A and B can be taken so that $A \parallel H$ and thus $H = \delta e_3$. Now the Pfaff system can be written as

$$\omega_1^3 = (\delta + 1)\omega^1,\ \ \omega_1^4 = \omega^2,\ \ \omega_1^{\overline{a}} = \omega_1^{\xi} = 0, \tag{3.22}$$
$$\omega_2^3 = (\delta - 1)\omega^2,\ \ \omega_2^4 = \omega^1,\ \ \omega_2^{\overline{a}} = \omega_2^{\xi} = 0. \tag{3.23}$$

Here $\overline{a} = 5, 6;\ \xi = 7, ..., n$ and from the parallelity condition (2.1) one has $\omega_3^3 = \omega_4^4 = \omega_4^5 = \omega_3^6 = \omega_1^2 = \omega_3^4 = \omega_4^3 = d\delta = 0$. Thus the derivation formulae

$$
\begin{aligned}
& dx = e_1\omega^1 + e_2\omega^2, \\
& de_1 = (\delta+1)e_3\omega^1 + e_4\omega^2, \\
& de_2 = e_4\omega^1 + (\delta-1)e_3\omega^2, \\
& de_3 = de_4 = 0
\end{aligned}
\tag{3.24}
$$

give that the considered surface lies in $E_{0,2}^4$ spanned by the point x and mutually orthogonal vectors e_1, e_2, e_3, e_4.

Since $h_{11} = (\delta+1)e_3,\ h_{22} = (\delta-1)e_3,\ h_{12} = e_4$, the considered surface can be represented by the equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + c_1u + c_2v$. Here $\omega^1 = du,\ \omega^2 = dv$; the u- and v-lines are geodesics of this surface and if $\delta^2 - 1 \neq 0$, then they are parabolas, but if $\delta^2 - 1 = 0$, then one of them degenerates into a straight line. □

The surfaces (i) are already parallel. For the surface (iii) the parallelity condition implies a =const, $\omega_1^2 - \omega_3^4 = \omega_3^\xi = \omega_4^\xi = 0$, where $\xi = 5, \dots, n$. Therefore for such a surface

$$
\begin{aligned}
& dx = e_1\omega^1 + e_2\omega^2, \\
& de_1 = e_2\omega_1^2 + a(e_3\omega^1 + e_4\omega^2 + \sqrt{3}e_5\omega^1), \\
& de_2 = -e_1\omega_1^2 + a(e_4\omega^1 - e_3\omega^2 + \sqrt{3}e_5\omega^2), \\
& de_3 = -\varepsilon_3 a(e_1\omega^1 - e_2\omega^2) + 2e_4\omega_1^2, \\
& de_4 = -\varepsilon_4 a(e_1\omega^2 + e_2\omega^1) - 2e_3\omega_1^2, \\
& de_5 = -\varepsilon_5 a\sqrt{3}dx.
\end{aligned}
$$

Hence the considered surfaces M^2 lies in a space E_s^5 ($s = 0$ or $s = 3$) spanned by the point x and the vectors e_1, e_2, e_3, e_4 and e_5. Moreover, since $\mathrm{d}(x + (a\sqrt{3})^{-1}e_5) = 0$, the point with radius vector $z = x + (a\sqrt{3})^{-1}e_5$ is a fixed point. A such M^2 is a Veronese surface V^2 from Section 2.4. Thus the parallel space-like surfaces M^2 in E_s^n are classified by the following

Theorem 3.2. *Let M^2 be a space-like parallel surface in E_s^n. Then it is either*
(i') a totally geodesic or totally umbilic surface, or
(ii') a surface with flat $\overline{\nabla}$ from Proposition 3.2, or
(iii') a Veronese surface V^2, or its open part.

3.3 Existence of semiparallel surfaces

Now there arises the problem of the existence of a nontrivial 2nd order envelope of parallel surfaces from (i)–(iii) of the Theorem 3.1.

It is known that parallel surfaces of (i) of the Theorem have only trivial 2nd order envelopes (i.e. they are umbilic-like in the sense of [18]).

For case (iii) it is established in [21] that in E^6_s, (s is 0, 3, or 4) there exists the most general semiparallel surface M^2 with some arbitrariness and it is the 2nd order envelope of a 2-parameter family of mutually non-congruent Veronese orbits. Moreover, these results can be used in E^6_s by $s = 1$, if to take e_α so that $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1$, $\varepsilon_6 = -1$, $\varepsilon_7 = ... = \varepsilon_n = 1$. Thus the difference from [21] will be in (1.4), where $\omega^6_K = -\varepsilon_6 \omega^K_6$, $K = 1, ..., 5$ but it does not influence the final result.

In case (ii) Proposition 3.2 can be used. In the latter for subcase (ii_1), when the parallel surface M^2 is a translation surface, the existence of the nontrivial 2nd order envelope of these surfaces is obvious. Thus it remains to consider subcases (ii_2) and (ii_3) of Proposition 3.2.

Proposition 3.3. *Let M^2 be a parallel surface of subcase (ii_2) or (ii_3). They possess nontrivial 2nd order envelopes with some arbitrariness.*

Proof. Without a loss of generality only the frame possibilities (3.9) with $b \neq 0$, (3.15) and (3.16) can be considered.

In the first of them, taking into account that $\omega^4_3 = \omega^3_4 = 0$ and $\omega^3_3 = -\omega^4_4$, the Pfaff system (3.3), (3.4) after exterior differentiation gives

$$\begin{aligned}
&\big(d(\delta + a) + (\delta + a)\omega^3_3 - 2b\omega^2_1\big) \wedge \omega^1 + (db + b\omega^3_3 + 2a\omega^2_1) \wedge \omega^2 = 0,\\
&(db + b\omega^3_3 + 2a\omega^2_1) \wedge \omega^1 + \big(d(\delta - a) + (\delta - a)\omega^3_3 + 2b\omega^2_1\big) \wedge \omega^2 = 0,\\
&\big(d\sigma - \sigma\omega^3_3\big) \wedge \omega^1 = 0, \qquad \big((\delta + a)\omega^\xi_3 + \sigma\omega^\xi_4\big) \wedge \omega^1 + b\omega^\xi_3 \wedge \omega^2 = 0,\\
&\big(d\sigma - \sigma\omega^3_3\big) \wedge \omega^2 = 0, \qquad b\omega^\xi_3 \wedge \omega^1 + \big((\delta - a)\omega^\xi_3 + \sigma\omega^\xi_4\big) \wedge \omega^2 = 0.
\end{aligned}$$

Now let 1) $\sigma \neq 0, \delta = 0$. It is easy to see that then $d\sigma = \sigma\omega^3_3$. Since $b \neq 0$, the basis of secondary forms consists of da, db, $2\omega^2_1$, ω^3_3, ω^ξ_3, ω^ξ_4 and the ranks of the polar systems $s_1 = 2+2(n-4)$ and $s_2 = 2$. Thus the Cartan's number is $Q = 6+2(n-4)$. On the other hand, due to the Cartan's lemma the number of the independent coefficients is $6+2(n-4)$. Thus the Cartan's criterion is satisfied and this Pfaff system is compatible and determines the considered M^2 for subcase (ii_2) with arbitrariness of two real holomorphic functions of two variables.

Let 2) $\delta \neq 0$, $\sigma = 0$. If here $\delta^2 - a^2 \neq 0$, then due to the Cartan's lemma for the first two equalities one has 8 independent coefficients. Consideration of the two lasts equalities gives $\omega^\xi_3 = r^\xi_1\omega^1 + r^\xi_2\omega^2$, whereas $r_1 = r^\xi_1 e_\xi$ and $r_2 = r^\xi_2 e_\xi$ are either both zero or linearly dependent vectors. The common number of independent coefficients is either 8 or $8 + (n-4)$, respectively. In both cases $Q = N$ and $s_2 = 3$.

If $\delta^2 - a^2 = 0$ (for example $\delta = a$), then $r_1^\xi = r_2^\xi = 0$, $N = Q = 6$, where $s_1 = 2$, $s_2 = 2$.

Thus the semiparallel surface of subcase (ii_3) in $E^3_{0,1}$ exists either with arbitrariness of three real holomorphic functions of two variables (it has two families of parabola generators), or with arbitrariness of two real holomorphic functions of two variables (this occurs when parabola degenerates into a straight line).

At last, 3) $\delta = \sigma = 0$, then due to the Cartan's lemma one has $\omega_3^\xi = 0$ and the number of independent coefficients is 2; since the basis of secondary forms consists of $da + a\omega_3^3 - 2b\omega_1^2$, $db + b\omega_3^3 + 2a\omega_1^2$ and it is easy to see that $s_1 = 2$, $s_2 = 0$; here $Q = N = 2$ and the considered surface for this case exists with arbitrariness of two real functions of one variable.

In the case (3.15) the Pfaff system (3.20), (3.21) gives by exterior differentiation

$$\begin{aligned}
(\omega_3^3 + \omega_5^3) \wedge \omega^1 + (2\omega_1^2 + \omega_4^3) \wedge \omega^2 = 0, &\qquad \omega_5^6 \wedge \omega^1 + \omega_4^6 \wedge \omega^2 = 0,\\
(2\omega_1^2 + \omega_4^3) \wedge \omega^1 + (\omega_5^3 - \omega_3^3) \wedge \omega^2 = 0, &\qquad \omega_4^6 \wedge \omega^1 + \omega_5^6 \wedge \omega^2 = 0,\\
(2\omega_1^2 - \omega_3^4 - \omega_5^4) \wedge \omega^1 - \omega_4^4 \wedge \omega^2 = 0, &\qquad (\omega_3^7 + \omega_5^7) \wedge \omega^1 = 0,\\
\omega_4^4 \wedge \omega^1 + (2\omega_1^2 - \omega_3^4 + \omega_5^4) \wedge \omega^2 = 0, &\qquad (\omega_5^7 - \omega_3^7) \wedge \omega^2 = 0,\\
(\omega_3^5 + \omega_5^5) \wedge \omega^1 + \omega_4^5 \wedge \omega^2 = 0, &\qquad \omega_3^8 \wedge \omega^1 + \omega_4^8 \wedge \omega^2 = 0,\\
\omega_4^5 \wedge \omega^1 + (\omega_5^5 - \omega_3^5) \wedge \omega^2 = 0, &\qquad \omega_4^8 \wedge \omega^1 - \omega_3^8 \wedge \omega^2 = 0,
\end{aligned}$$

and $(\omega_3^\xi + \omega_5^\xi) \wedge \omega^1 + \omega_4^\xi \wedge \omega^2 = 0$, $\omega_4^\xi \wedge \omega^1 + (\omega_5^\xi - \omega_3^\xi) \wedge \omega^2 = 0$, where $\xi = 9, ..., n$. Using here the Cartan's lemma and also the relations (3.17)–(3.19) one has $\omega_3^8 = \omega_3^7 = \omega_4^8 = \omega_4^6 = \omega_5^7 = \omega_5^6 = 0$. The common number of coefficients in the right sides is $N = 14 + 4(n-8) = 4n - 18$. On the other hand, first, the basis of the secondary forms consists of ω_3^3, $2\omega_1^2 + \omega_4^3$, ω_5^3, $2\omega_1^2 - \omega_3^4$, ω_4^4, ω_5^4, ω_3^5, ω_4^5, ω_5^5, ω_3^ξ, ω_4^ξ, ω_5^ξ; second, the ranks of the polar systems are: $s_1 = 6 + 2(n-8)$ and $s_1 + s_2$, where $s_2 = n - 4$, thus the Cartan's number $Q = s_1 + 2s_2 = 4n - 18$. Hence the Cartan's criterion is satisfied and the semiparallel surface of subcase (ii_3) in $E^5_{0,3}$ exists with arbitrariness of $n - 4$ real holomorphic functions of two variables.

For the case (3.16) the first eight equations of the Pfaff system (3.22), (3.23) lead by the exterior differentiation to

$$\begin{aligned}
\Big(d\delta + (\delta+1)\omega_3^3 + \sigma\omega_4^3\Big) \wedge \omega^1 + \Big(2\omega_1^2 + \omega_4^3\Big) \wedge \omega^2 = 0, &\qquad \omega_4^5 \wedge (\sigma\omega^1 + \omega^2) = 0,\\
\Big(2\omega_1^2 + \omega_4^4\Big) \wedge \omega^1 + \Big(d\delta + (\delta-1)\omega_3^3 + \sigma\omega_4^3\Big) \wedge \omega^2 = 0, &\qquad \omega_4^5 \wedge (\omega^1 + \sigma\omega^2) = 0,\\
\Big(d\sigma - 2\omega_1^2 + (\delta+1)\omega_3^4 + \sigma\omega_4^4\Big) \wedge \omega^1 + \omega_4^4 \wedge \omega^2 = 0, &\qquad (\delta+1)\omega_3^6 \wedge \omega^1 = 0,\\
\omega_4^4 \wedge \omega^1 + \Big(d\sigma + 2\omega_1^2 + (\delta-1)\omega_3^4 + \sigma\omega_4^4\Big) \wedge \omega^2 = 0, &\qquad (\delta-1)\omega_3^6 \wedge \omega^2 = 0.
\end{aligned}$$

This system together with (3.17)–(3.19) gives $\omega_4^5 = \omega_3^6 = 0$. After exterior differentiation the equations $\omega_i^\xi = 0,\ \xi = 7, ..., n$, give

$$[(\delta+1)\omega_3^\xi + \sigma\omega_4^\xi] \wedge \omega^1 + \omega_4^\xi \wedge \omega^2 = 0,\ \omega_4^\xi \wedge \omega^1 + [(\delta-1)\omega_3^\xi + \sigma\omega_4^\xi] \wedge \omega^2 = 0.$$

Now the basis of secondary forms consists of $2\omega_1^2$, ω_3^3, $d\delta$, ω_4^3, ω_3^4, ω_4^4, ω_3^ξ, ω_4^ξ.

Let $\delta^2 - 1 \neq 0$; then $s_1 = 4+2(n-6)$ and $s_2 = 6+2(n-6)-4-2(n-6) = 2$, the Cartan's number is $8+2(n-6)$ and it is equal to the number of independent coefficients.

In the case $\delta^2 - 1 = 0$ (for example, $\delta = 1$) there are $s_1 = 4 + 2(n-6)$, $s_2 = 1$ and $Q = N = 6 + 2(n-6)$.

The Cartan's criterion is satisfied and the semiparallel surface of subcase (ii_3) in $E^4_{0,2}$ exists either with arbitrariness of two real holomorphic functions of two variables, or with arbitrariness of one real holomorphic function of two variables (depending on the occurrence of degeneration). □

3.4 Maximal semiparallel space-like sufaces

The space-like submanifold M^m in E^n_s is said to be *maximal submanifold* (or just maximal), if the mean curvature vector H is identically zero. In fact, according to the theory of minimal submanifolds in E^n, it is known that every minimal semiparallel submanifold is totally geodesic (see [8] and [18]). Hence the class of all such submanifolds are very small. In E^n_1 there exist minimal semiparallel time-like surfaces (strings), which are not totally geodesic (see [20]). It can be shown that among surfaces of type (ii) in E^n_s with $s > 0$ there do exist maximal semiparallel space-like surfaces not totally geodesic.

Proposition 3.4. *In E^n_s with $s > 0$ a maximal semiparallel space-like surface M^2, which is not totally geodesic, has flat $\overline{\nabla}$ and is either*

1) a surface in $E^3_{0,1}$ or in $E^4_{0,2}$, which has two families of parabola generators and can be represented by the equation $x = \frac{1}{2}h_{11}((u)^2 - (v)^2) + h_{12}uv + c_1u + c_2v$, where all coefficients are some constant vectors, moreover, the first two of them are isotropic, or

2) a hyperbolic paraboloid in $E^3_{0,1}$, or

3) a 2nd order envelope of a family, consisting of the surfaces of one of the previous classes in E^n_s.

Proof. Due to Proposition 3.2 here the cases when $\dim I_xM^2$ is either 1 (subcases (3.8)–(3.11)) or 2 (subcase (3.16)) are to be considered with the additional condition $H = 0$ (i.e. $\delta = \sigma = 0$).

Let $\dim I_x M^2 = 1$; then using the result of Proposition 3.3 a such semiparallel space-like surface exists with arbitrariness of two real functions of one variable.

Geometrically a such M^2 lies in $E^3_{0,1}$ and either is determined by the equation $x = \frac{1}{2}h_{11}((u)^2 - (v)^2) + h_{12}uv + c_1u + c_2v$, where all coefficients are some constant vectors, and thus has two families of parabola generators, or is a hyperbolic paraboloid.

For the case $\dim I_x M^2 = 2$, the maximal M^2 occurs in (3.16). The Pfaff system now transforms into

$$\omega_1^3 = \omega_2^4 = \omega^1,\ \omega_2^3 = -\omega_1^4 = -\omega^2,\ \omega_1^\xi = \omega_2^\xi = 0,\ \xi = 5, ..., n.$$

After exterior differentiation it gives

$$\begin{aligned} \omega_3^3 \wedge \omega^1 + \left(2\omega_1^2 + \omega_4^3\right) \wedge \omega^2 = 0, &\qquad \left(2\omega_1^2 + \omega_4^3\right) \wedge \omega^1 - \omega_3^3 \wedge \omega^2 = 0, \\ -\left(2\omega_1^2 - \omega_3^4\right) \wedge \omega^1 + \omega_4^4 \wedge \omega^2 = 0, &\qquad \omega_4^4 \wedge \omega^1 + \left(2\omega_1^2 - \omega_3^4\right) \wedge \omega^2 = 0, \\ \omega_3^\xi \wedge \omega^1 = 0, &\qquad \omega_3^\xi \wedge \omega^2 = 0. \end{aligned}$$

Due to the Cartan's lemma all ω_3^ξ are zero; the others equalities give that $N = 4 + 2 = 6$; the basis in the left sides consists of $2\omega_1^2,\ \omega_3^3,\ \omega_4^3,\ \omega_3^4,\ \omega_4^4$ and the ranks of the polar systems $s_1 = 4,\ s_2 = 1$. So the Cartan's number is equal to the number of independent coefficients and the Cartan's criterion is satisfied. The extended Pfaff system determines M^2 with arbitrariness of one real holomorphic function of two variables. For this surface $dx = e_1\omega^1 + e_2\omega^2,\ de_1 = e_3\omega^1 + e_4\omega^2,\ de_2 = e_4\omega^1 - e_3\omega^2,\ de_3 = de_4 = 0$, thus the considered maximal M^2 lies in $E^4_{0,2}$ and can be represented by the equation $x = \frac{1}{2}h_{11}((u)^2 - (v)^2) + h_{12}uv + c_1u + c_2v$, where all coefficients are constant vectors. This M^2 has two families of parabola generators. $\square$

Chapter 4

Normally flat semiparallel space-like submanifolds M^3

The normal connection $\nabla^\perp$ of a space-like submanifold M^m in E^n_s is said to be flat if $\Omega^\beta_\alpha = 0$. Then the matrices $\parallel h^\alpha_{ij} \parallel$ and $\parallel h^\beta_{ij} \parallel$ commute due to (1.11), and are diagonalizable simultaneously by choosing a suitable orthonormal frame in T_xM^m. In this frame which is called the *principal frame*, one has

$$h_{ij} = k_i \delta_{ij}; \tag{4.1}$$

its basic directions are called the *principal directions* and the normal vectors $k_i = \kappa^\alpha_i e_\alpha$ are called the *principal curvature vectors* of the M^m with flat $\nabla^\perp$ in E^n_s. The several principal curvature vectors k_{i_1} and k_{i_2} corresponding to the same vector e_α, i.e. $k_{i_1} = \kappa_{i_1} e_\alpha$, $k_{i_2} = \kappa_{i_2} e_\alpha$, is called *non-simple principal curvature vectors*.

Now parallelity condition (2.1) for normally flat submanifolds transforms into

$$d\kappa_i + \kappa_i \omega^\alpha_\alpha = 0, \quad (\kappa_i - \kappa_j)\omega^j_i = 0, \quad \omega^\beta_\alpha = \omega^\xi_\alpha = 0, (\alpha \neq \beta). \tag{4.2}$$

The result of Proposition 2.2 can be used for normally flat parallel space-like submanifolds M^3. It gives

Proposition 4.1. *A normally flat parallel space-like M^m in a space E^n_s with principal normal subspace of completely vanishing metric is either*

1) a submanifold in $E^{3+n_1}_{0,n_1}$ with three families of parabola generators (one or two of them can degenerate into a straight line) and can be represented by the equation

$$x = \tfrac{1}{2}\Big(k_1(u^1)^2 + k_2(u^2)^2 + k_3(u^3)^2\Big) + k_{01}u^1 + k_{02}u^2 + k_{03}u^3, \tag{4.3}$$

where k_i are the principal curvature vectors and k_{0i} are some constant vectors, or

2) an open part of such a submanifold.

In case where $n_1 < \frac{1}{2}m(m+1)$ there are some linear relations between vectors k_i in (4.3).

Due to principal frame (4.1) for curvature 2-forms of the Levi-Civita connection ∇ in (1.10) one has

$$\Omega_i^j = -\langle k_i, k_j \rangle \omega^i \wedge \omega^j \tag{4.4}$$

and semiparallelity condition (2.8) is equivalent with

$$(k_i - k_j)\langle k_i, k_j \rangle = 0, \qquad \text{(see [18])}. \tag{4.5}$$

4.1 The principal curvature vectors

In case with $\dim N_x M^3 = 0$ the semiparallel M^3 is a 3-plane E^3 in E_s^n, so in the future work will be considered M^3 with $\dim N_x M^3 \geq 1$, i.e. the following possibilities for the principal curvature vectors.

The case $dim N_x M^3 = 1$. Here the principal curvature vectors can be taken so that

$$k_1 = \kappa_1 e_4, \quad k_2 = \kappa_2 e_4, \quad k_3 = \kappa_3 e_4.$$

With respect to the metric one has either regular metric where all vectors k_i have non-zero scalar square, i.e. for the frame vector e_4 one has that ε_4 is either 1 or -1 and

$$\varepsilon_\alpha = 0,\ g_{4\alpha} = g_{\alpha\beta} = 0,\ \alpha, \beta = 5, \dots, n, \tag{4.6}$$

or a completely vanishing metric, where all curvature vectors k_i are isotropic, i.e. for the frame vector e_4 one has that $\varepsilon_4 = 0$; thus the next frame vector e_5 can be taken so that

$$\varepsilon_5 = 0, \quad g_{45} = 1, \quad \text{and} \quad g_{\alpha\beta} = 0, \quad \alpha, \beta = 6, \dots, n. \tag{4.7}$$

The case $dim N_x M^3 = 2$. Now, without the loss of generality, the principal curvature vectors can be taken so that

$$k_1 = \kappa_1 e_4, \quad k_2 = \kappa_2 e_5, \quad k_3 = \kappa_3 e_5.$$

In this case the metric of $N_x M^3$ is either regular, or singular non-vanishing, or completely vanishing. If the metric of $N_x M^3$ is regular, then the both principal curvature vectors k_1 and k_2 have non-zero scalar square. Thus the frame vectors e_4, e_5 can be taken so that their scalar squares ε_4, ε_5 either

$$\text{have values 1, or } -1 \text{ and all others } g_{\alpha\beta} = 0,\ \alpha, \beta = 6, \dots, n. \tag{4.8}$$

If the metric is singular non-vanishing, then one from mutually orthogonal vectors k_1, k_2 have zero square. Let, at first, k_2 is isotropic. Then one obtains

$$\varepsilon_5 = 0 \text{ and } \varepsilon_4 \neq 0 \tag{4.9}$$

and the next vector e_6 can be taken so that $\varepsilon_6 = 0$, moreover $g_{56} = 1$, and it is orthogonal to $e_1, \ldots, e_5$.

In case where k_1 is isotropic for the frame vectors e_4, e_5 one has

$$\varepsilon_4 = 0 \text{ and } \varepsilon_5 \neq 0 \tag{4.10}$$

and the next vector e_6 can be taken so that $\varepsilon_6 = 0$, moreover $g_{46} = 1$, and it is orthogonal to $e_1, \ldots, e_5$.

At last, in case of completely vanishing metric one has that the both vectors k_1, k_2 are isotropic. Thus the frame vectors e_4, e_5 have zero scalar squares and their scalar product is zero, too, i.e. $\varepsilon_4 = \varepsilon_5 = g_{45} = 0$. Now the next vectors e_6, e_7 can be taken so that

$$\varepsilon_6 = \varepsilon_7 = 0, \; g_{46} = g_{57} = 1, \; g_{\alpha\beta} = 0, \; \alpha, \beta = 8, \ldots, n. \tag{4.11}$$

The case $dim N_x M^3 = 3$. In this case

$$k_1 = \kappa_1 e_4, \quad k_2 = \kappa_2 e_5, \quad k_3 = \kappa_3 e_6.$$

and the principal normal subspace has either regular, or singular non-vanishing, or completely vanishing metric.

If metric is regular, then it is sufficient to consider the case where all linearly independent k_i have non-zero scalar square, i.e. for the frame vectors e_4, e_5, e_6 one has that

$$\varepsilon_4, \varepsilon_5, \varepsilon_6 \text{ have values 1 or } -1 \text{ and all } g_{\alpha\beta} = 0, \quad \alpha, \beta = 6, \ldots, n. \tag{4.12}$$

In case of singular non-vanishing metric one or two of mutually orthogonal vectors k_1, k_2, k_3 have zero square. Let us assume that one of them is isotropic. Without loss of generality it can be taken that $(k_1)^2 = 0$, then orthogonal to k_1 vectors in E^n_s are in a $n-1$-dimensional subspace which contains also k_1 (the tangent subspace of the isotropic cone). If in this subspace there are two non-zero mutually orthogonal non-isotropic vectors k_2 and k_3, then the dimension of this subspace must be ≥ 3, therefore $n - 3 \geq 4$, $s \geq 1$ and one obtains

$$\varepsilon_4 = 0, \text{ and there can be madden for } \varepsilon_5, \; \varepsilon_6, \text{ values 1 or} -1. \tag{4.13}$$

The next frame vectors e_7 can be taken so that $\varepsilon_7 = 0$, moreover $g_{47} = 1$ and it is orthogonal to $e_1, \ldots, e_6$.

The similar reasoning in case of singular non-vanishing metric with two isotropic vectors leads to $(k_1)^2 = (k_2)^2 = 0$. In the tangent subspace of the isotropic cone there is one non-zero and non-isotropic vector k_3, i.e.

$$\varepsilon_4 = \varepsilon_5 = 0, \text{ and there can be madden for } \; \varepsilon_6, \text{ values 1 or} -1. \tag{4.14}$$

The next frame vectors e_7, e_8 can be taken so that $\varepsilon_7 = \varepsilon_8 = 0$, moreover $g_{47} = g_{58} = 1$ and it is orthogonal to $e_1, \dots, e_6$.

At last, in case of completely vanishing metric all mutually orthogonal vectors k_i are isotropic. Thus for the frame vectors e_a $(a = 4, 5, 6)$ and the next frame vectors $e_{\overline{a}}$ $(\overline{a} = a + 3)$ one has

$$\varepsilon_a = 0, \ \varepsilon_{\overline{a}} = 0, \ \text{ and } \ g_{\overline{a}a} = 1. \tag{4.15}$$

Remark. In description of regular metric for 2- and 3-dimensional N_xM^3 the following result is used

Proposition 4.2. *Let M^m be a normally flat semiparallel space-like submanifold in E^n_s, whose principal normal subspace N_xM^m has regular metric. Thus in* (1.15) *one has $l = 0$.*

Proof. Let us suppose that the principal normal subspace N_xM^m is r- dimensional, $1 \le r \le m$ and linearly independent vectors $e_{m+1}, \dots, e_{m+r}$, are such as it is described in (1.15), i.e. at least $2l$ of them are such that

$$\langle e_q, e_q\rangle = \langle e_{\widetilde{q}}, e_{\widetilde{q}}\rangle = 0, \quad \langle e_q, e_{\widetilde{q}}\rangle = 1$$

for every value $q \in \{m + k + 1, \dots m + k + l\}$ and $\widetilde{q} = q + l$. Now one has

$$\Omega^{j_{\widetilde{q}}}_{i_q} = -\kappa_{i_q}\kappa_{j_{\widetilde{q}}}\omega^{i_q} \wedge \omega^{j_{\widetilde{q}}},$$

the other curvature 2-forms are zero. The semiparallelity condition transforms into

$$\kappa^2_{i_q}\kappa_{j_{\widetilde{q}}} = 0, \quad \kappa_{i_q}\kappa^2_{j_{\widetilde{q}}} = 0, \quad \forall q,$$

it means that among vectors $k_{(1)}, \dots, k_{(r)}$ at least l vectors must be zero, which contradict with their linear independence. □

4.2 The case of one-dimensional principal normal subspace

In this case the Pfaff system is following

$$\omega^4_1 = \kappa_1\omega^1, \ \omega^4_2 = \kappa_2\omega^2, \ \omega^4_3 = \kappa_3\omega^3, \ \omega^\xi_i = 0, \ \xi = 5, \dots, n \tag{4.16}$$

where either

$$\kappa_1\kappa_2\kappa_3 \neq 0, \ \text{or} \tag{4.17}$$

$$\kappa_1\kappa_2 \neq 0, \ \kappa_3 = 0, \ \text{or} \tag{4.18}$$

$$\kappa_1 \neq 0, \ \kappa_2 = \kappa_3 = 0. \tag{4.19}$$

Proposition 4.3. *A normally flat semiparallel space-like submanifold M^3 in E^n_s with N_xM^3 of dimension 1 and regular metric is a part of either*

1) a sphere S^3_τ in $E^4_\tau \subset E^n_s$, ($\tau \in \{0,1\}$), or

2) a product $S^3_\tau \times E^1$ in $E^4_\tau \subset E^n_s$, ($\tau \in \{0,1\}$), or

3) a product $S^1_\tau \times E^2$ in $E^4_\tau \subset E^n$, ($\tau \in \{0,1\}$), or

4) a second order envelope of submanifolds with (4.18)-(4.19).

Proof. Let start with (4.17), then the exterior differentiation of the Pfaff system (4.16) leads to $d\ln\kappa_q \wedge \omega^q = 0$, $\omega_4^\xi \wedge \omega^q = 0$, $q = 1,2,3$ and due to the Cartan's lemma one obtains $d\ln\kappa_q = A_q\omega^q$ and $\omega_4^\xi = 0$. The semiparallel condition (4.5) leads to $k_1 = k_2 = k_3 = k = \kappa e_4$. It means that coefficients $A_q = 0$ and the considered semiparallel submanifold is a parallel one with

$$\begin{aligned}
dx &= e_1\omega^1 + e_2\omega^2 + e_3\omega^3,\\
de_1 &= \omega_1^3 e_2 + \omega_1^3 e_3 + k\omega^1,\\
de_2 &= -\omega_1^2 e_1 + \omega_2^3 e_3 + k\omega^2,\\
de_3 &= -\omega_1^3 e_1 - \omega_2^3 e_2 + k\omega^3,\\
dk &= -\varepsilon_4\kappa^2 dx.
\end{aligned}$$

Geometrically it is either a sphere S^3 (in case of a positively definite metric), or a sphere S^3_1 (in case of a negatively definite metric).

In case (4.18) the exterior differentiation of (4.16) with $\kappa_3 = 0$ leads to

$$d\ln\kappa_q \wedge \omega^q + \omega_q^3 \wedge \omega^3 = 0, \quad \omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2 = 0, \quad \omega_4^\xi \wedge \omega^q = 0, \quad q = 1,2.$$

The semiparallelity condition gives $\kappa_1 = \kappa_2 = \kappa$. The latter together with the Cartan's lemma leads to

$$d\ln\kappa = A\omega^3, \quad \omega_1^3 = A\omega^1, \quad \omega_2^3 = A\omega^2, \quad \omega_4^\xi = 0. \tag{4.20}$$

Applying here the same procedure one obtains

$$dA = A^2\omega^3. \tag{4.21}$$

The exterior differentiation of $d\omega_1^2 = -(A^2 + \varepsilon_4\kappa^2)\omega^1 \wedge \omega^2$ together with obtained formulae (4.20), (4.21) and structure formulae (1.4) leads to

$$A(A^2 + \varepsilon_4\kappa^2) = 0, \tag{4.22}$$

i.e. in case of the positively definite metric (4.6) one has $A = 0$. Hence the considered submanifold is a parallel one. The condition (4.5) gives that $k_1 = k_2 = k = \kappa e_4$ and due to the parallelity condition (4.2) one has $d\kappa = 0$, $\omega_1^3 = \omega_2^3 = 0$, $\omega_4^\alpha = \omega_4^\xi = 0$. There holds $dx = e_i\omega^i$, and

$$de_1 = e_2\omega_1^2 + k\omega^1, \;\; de_2 = -e_1\omega_1^2 + k\omega^2, \;\; de_3 = 0, \;\; dk = -\varepsilon_4\kappa^2(e_1\omega^1 + e_2\omega^2).$$

Thus geometrically it is a product of a straight line and a sphere $S^2(r)$. In case of the regular negatively definite metric from (4.22) there are occur two possibilities: either $A = 0$ or $A = \kappa$. In the first of them semiparallel submanifold is parallel one and geometrically is a product of a straight line and a sphere $S_1^2(r)$. In the second case ($A = \kappa$) the prolonged system is totally integrable. Thus semiparallel submanifold exists and geometrically is 2nd order envelope of the corresponding parallel submanifolds.

At last, for subcase (4.19) the exterior differentiation of the Pfaff system (4.16) with $\kappa_2 = \kappa_3 = 0$ leads to

$$d\ln\kappa_1 \wedge \omega^1 + \omega_1^2 \wedge \omega^2 + \omega_1^3 \wedge \omega^3 = 0, \quad \omega_1^2 \wedge \omega^1 = 0, \quad \omega_1^3 \wedge \omega^1 = 0, \quad \omega_4^\xi \wedge \omega^1 = 0.$$

Using the Cartan's lemma one has

$$d\ln\kappa_1 = A_1\omega^1 + A_2\omega^2 + A_3\omega^3, \quad \omega_1^2 = A_2\omega^1, \quad \omega_1^3 = A_3\omega^1, \quad \omega_4^\xi = X^\xi\omega^1.$$

Here the basis of the secondary forms consists of $d\ln\kappa_1$, ω_1^2, ω_1^3, ω_4^ξ and the rank of the polar matrices are $s_1 = 3 + (n-4) = n-1$ and $s_2 = 0$. The Cartan's number is $Q = n-1$. On the other hand the number of independent coefficients N is equal to $n-1$, so on $N = Q$ and the Cartan's criterion is satisfied. The considered semiparallel submanifold exists with arbitrariness of $n-1$ holomorphic functions of one real argument.

For the corresponding parallel submanifold one has $d\kappa_1 = 0$, $\omega_1^2 = \omega_1^3 = 0$, $\omega_4^\alpha = \omega_4^\xi = 0$ and there holds

$$dx = e_i\omega^i, \;\; de_1 = k\omega^1, \;\; de_2 = de_3 = 0, \;\; dk = -\varepsilon_4\kappa^2 e_1\omega^1.$$

It means that the considered space-like M^3 is a product of a circle S_τ^1 ($\tau = \{0, 1\}$ on depending of the sign of a metric) and a plane E^2. □

Proposition 4.4. *A normally flat semiparallel space-like submanifold M^3 in E_s^n with N_xM^3 of dimension 1 and completely vanishing metric is either*

1) a parallel submanifold M^3 from Proposition 4.1, or

2) a second order envelope of such submanifolds with some arbitrariness.

Proof. In case (4.17) with the completely vanishing metric (4.7) the exterior differentiation of the Pfaff system (4.16) gives $(d\ln\kappa_q + \omega_4^4) \wedge \omega^q = 0$, $\omega_4^\xi \wedge \omega^q = 0$, $q = 1, 2, 3$. Thus the basis of the secondary forms consists of $d\ln\kappa_q + \omega_4^4$ and the rank of the polar matrices are $s_1 = 3$ and $s_2 = 0$. The Cartan's number is $Q = 3$. On the other hand due to the Cartan's lemma one obtains $(d\ln\kappa_q + \omega_4^4) = A_q\omega^q$ and $\omega_4^\xi = 0$ and the number of independent coefficients N is equal to 3, so on $N = Q$ and the Cartan's criterion is satisfied. The considered semiparallel submanifold is either a translation submanifold of three families of parabola generators in $E_{0,1}^4$ from

Proposition 4.1, or a 2nd order envelope of such parallel submanifolds with arbitrariness of 3 holomorphic functions of one real argument.

For the case (4.18) with $\kappa_3 = 0$ the exterior differentiation of the Pfaff system leads to

$$(d\ln\kappa_q + \omega_4^4)\wedge\omega^q + \omega_q^3\wedge\omega^3 = 0, \quad \omega_1^3\wedge\omega^1 + \omega_2^3\wedge\omega^2 = 0, \quad \omega_4^\xi\wedge\omega^q = 0,$$

where $q = 1, 2$ and here applying the Cartan's lemma one has

$$\begin{aligned} d\ln\kappa_1 + \omega_4^4 &= A\omega^1 + B\omega^3, \qquad \omega_1^3 = B\omega^1,\\ d\ln\kappa_2 + \omega_4^4 &= C\omega^2 + D\omega^3, \qquad \omega_2^3 = D\omega^2. \end{aligned}$$

Now the basis of the secondary forms consists of $d\ln\kappa_1 + \omega_4^4$, $d\ln\kappa_2 + \omega_4^4$, ω_1^3, ω_2^3 and the rank of the polar matrices are $s_1 = 4$ and $s_2 = 0$. The Cartan's number is $Q = 4$. On the other hand the number of independent coefficients N is equal to 4, so on $N = Q$ and the Cartan's criterion is satisfied. The considered semiparallel is either a translation submanifold two parabolas and a straight line from Proposition 4.1, or a 2nd order envelope of a such parallel submanifolds with arbitrariness of 4 holomorphic functions of one real argument.

In the last case (4.19) with $\kappa_2 = \kappa_3 = 0$ the exterior differentiation leads to

$$\begin{aligned} &(d\ln\kappa_1 + \omega_4^4)\wedge\omega^1 + \omega_1^2\wedge\omega^2 + \omega_1^3\wedge\omega^3 = 0,\\ &\omega_1^2\wedge\omega^1 = 0, \quad \omega_1^3\wedge\omega^1 = 0, \quad \omega_4^\xi\wedge\omega^1 = 0. \end{aligned}$$

Using the Cartan's lemma one has

$$d\ln\kappa + \omega_4^4 = A_1\omega^1 + A_2\omega^2 + A_3\omega^3, \quad \omega_1^2 = A_2\omega^1, \quad \omega_1^3 = A_3\omega^1, \quad \omega_4^\xi = X^\xi\omega^1.$$

Here the basis of the secondary forms consists of $d\ln\kappa_1 + \omega_4^4$, ω_1^2, ω_1^3, ω_4^ξ and the rank of the polar matrices are $s_1 = 3 + (n-4) = n-1$ and $s_2 = 0$. The Cartan's number is $Q = n-1$. On the other hand the number of independent coefficients N is equal to $n-1$, so on $N = Q$ and the Cartan's criterion is satisfied. The considered semiparallel submanifold is either a translation submanifold of two straight lines and a parabola from Proposition 4.1, or a 2nd order envelope of a such parallel submanifolds with arbitrariness of $n-1$ holomorphic functions of one real argument. □

4.3 The case of two-dimensional principal normal subspace

Now the principal normal subspace is two-dimensional

$$\begin{aligned} &k_1 = \kappa_1 e_4, \quad k_2 = \kappa_2 e_5, \quad k_3 = \kappa_3 e_5, \quad \text{where either}\\ &\kappa_1\kappa_2\kappa_3 \neq 0, \quad \text{or} && (4.23)\\ &\kappa_1\kappa_2 \neq 0,\ \kappa_3 = 0. && (4.24) \end{aligned}$$

and its metric is either regular (4.8), or singular non-vanishing (4.9) and (4.10), or completely vanishing (4.11).

Proposition 4.5. *A normally flat semiparallel space-like submanifold M^3 in E^n_s with N_xM^3 of dimension 2 on supposition (4.23) is either*

1) a product $S^2_\tau \times S^1_\sigma$, $\tau, \sigma \in \{0,1\}$ in $E^5_{\tau+\sigma} \subset E^n_s$, or

2) a translation submanifold of a circle S^1_σ and two parabolas in $E^5_{\sigma,1} \subset E^n_s$, or

3) a translation submanifold in $E^5_{0,2}$ with three families of parabola generators and can be represented by the equation

$$x = \tfrac{1}{2}\Big(k_1(u^1)^2 + k_2(u^2)^2 + k_3(u^3)^2\Big) + k_{01}u^1 + k_{02}u^2 + k_{03}u^3,$$

here k_i are the principal curvature vectors and h_{0i} are some constant vectors, or

4) a second order envelope of a submanifolds above with some arbitrariness.

Proof. The Pfaff system for the case (4.23) is

$$\omega^4_1 = \kappa_1\omega^1,\ \omega^5_2 = \kappa_2\omega^2,\ \omega^5_3 = \kappa_3\omega^3,\ \omega^4_2 = \omega^4_3 = \omega^5_1 = \omega^\xi_i = 0, \tag{4.25}$$

where $\xi = 6, \dots, n$. Its exterior differentiation leads to

$$\begin{aligned}
&(d\ln\kappa_1 + \omega^4_4)\wedge\omega^1 + \omega^2_1\wedge\omega^2 + \omega^3_1\wedge\omega^3 = 0,\quad \kappa_1\omega^2_1\wedge\omega^1 + \kappa_2\omega^4_5\wedge\omega^2 = 0,\\
&\kappa_1\omega^5_4\wedge\omega^1 - \kappa_2\omega^2_1\wedge\omega^2 - \kappa_3\omega^3_1\wedge\omega^3 = 0,\qquad \kappa_1\omega^3_1\wedge\omega^1 + \kappa_3\omega^4_5\wedge\omega^3 = 0,\\
&\kappa_2\omega^2_1\wedge\omega^1 - (d\kappa_2 + \kappa_2\omega^5_5)\wedge\omega^2 - (\kappa_2 - \kappa_3)\omega^3_2\wedge\omega^3 = 0,\\
&\kappa_3\omega^3_1\wedge\omega^1 - (\kappa_2 - \kappa_3)\omega^3_2\wedge\omega^2 - (d\kappa_3 + \kappa_3\omega^5_5)\wedge\omega^3 = 0,\\
&\kappa_1\omega^\xi_4\wedge\omega^1 = 0,\quad \kappa_2\omega^\xi_5\wedge\omega^2 = 0,\quad \kappa_3\omega^\xi_5\wedge\omega^3 = 0.
\end{aligned}$$

Using the Cartan's lemma for the equations above one has

$$\begin{aligned}
&d\ln\kappa_1 + \omega^4_4 = A\omega^1 + B\omega^2 + C\omega^3, && (\kappa_2 - \kappa_3)\omega^3_2 = M\omega^2 + P\omega^3,\\
&d\kappa_2 + \kappa_2\omega^5_5 = -\kappa_2 D\omega^1 + L\omega^2 + M\omega^3, && \omega^2_1 = B\omega^1 + D\omega^2,\\
&d\kappa_3 + \kappa_3\omega^5_5 = -\tfrac{(\kappa_3)^2}{\kappa_2}D\omega^1 + P\omega^2 + Q\omega^3, && \omega^3_1 = C\omega^1 + \tfrac{\kappa_3}{\kappa_2}D\omega^3,\\
&\kappa_1\omega^5_4 = R\omega^1 - \kappa_2 B\omega^2 - \kappa_3 C\omega^3, && \kappa_2\omega^4_5 = \kappa_1 D\omega^1,\\
&\omega^\xi_4 = X^\xi\omega^1,\quad \omega^\xi_5 = 0.
\end{aligned} \tag{4.26}$$

In case of the regular metric (4.8) one has $\omega^4_4 = \omega^5_5 = 0$ and $-\varepsilon\omega^4_5 = \omega^5_4$, (here $\varepsilon = 1$, if $\varepsilon_4 = \varepsilon_5$ and $\varepsilon = -1$, if $\varepsilon_4 = -\varepsilon_5$). Moreover, the semiparallelity condition (4.5) leads to $\kappa_2 = \kappa_3$. Thus in (4.26) one has

$$B = C = M = P = L = Q = 0,\quad R = -\varepsilon\frac{(\kappa_1)^2}{\kappa_2}D,$$

i.e. the system (4.26) transforms into

$$d\ln\kappa_1 = A\omega^1,\ \ d\ln\kappa_2 = -D\omega^1,\ \ \omega_1^2 = D\omega^2,\ \ \omega_1^3 = D\omega^3,\ \ \kappa_2\omega_5^4 = \kappa_1 D\omega^1.$$

Applying here exterior differentiation and Cartan's lemma ones more one obtains $dD = -D^2\omega^1$. Using this result in differential prolongation of $d\omega_2^3 = -(\varepsilon_5\kappa_2^2 + D^2)\omega^2\wedge\omega^3$ one obtains $D(\varepsilon_5(\kappa_2)^2 + D^2) = 0$. The latter in case $\varepsilon_5 = 1$ leads immediately to $D = 0$, i.e. one has $Q = N = 1 + (n-5) = n-4$ and the semiparallel space-like submanifolds exist with arbitrariness of $n-4$ functions of one real argument. If ε_5 has value -1, then either $D = 0$, or $D = \pm\kappa_2$. In both these cases the prolonged system is totally integrable and the considered normally flat semiparallel M^3 is a 2nd order envelope of corresponding parallel submanifolds.

For the corresponding parallel submanifold M^3 the condition (4.5) leads to $\kappa_2 = \kappa_3$ and in the derivation formulae one has

$$\begin{aligned}&dx = e_i\omega^i,\ \ de_1 = k_1\omega^1\ \ de_2 = \omega_2^3 e_3 + k_2\omega^2,\ \ de_3 = -\omega_2^3 e_2 + k_2\omega^3,\\ &dk_1 = -\varepsilon_4(\kappa_1)^2 e_1\omega^1,\ \ dk_2 = -\varepsilon_5(\kappa_2)^2(e_2\omega^2 + e_3\omega^3).\end{aligned}$$

Thus the geodesic lines are a sphere S_τ^2, $\tau\in\{0,1\}$, and a circle S_σ^1, $\sigma\in\{0,1\}$ and, in general, this parallel M^3 lies in $E_{\tau+\sigma}^5$.

In case of the singular non-vanishing metric (4.9) due to (1.5) one has $\omega_4^4 = \omega_5^6 = 0$, $\omega_4^6 = -\omega_5^4$ and together with (4.26) it leads to $Q = N = 8 + (n-6) = n+2$, i.e. the considered semiparallel space-like submanifold exists with arbitrariness of $n+2$ functions of one real argument.

For the corresponding parallel submanifolds one has

$$d\ln\kappa_1 = d\kappa_2 + \kappa_2\omega_5^5 = \omega_5^4 = \omega_4^5 = \omega_1^2 = \omega_1^3 = \omega_4^\xi = \omega_5^\xi = 0.$$

The derivation formulae can be written as

$$\begin{aligned}&dx = e_i\omega^i,\ \ de_1 = k_1\omega^1\ \ de_2 = k_2\omega^2,\ \ de_3 = k_3\omega^3,\\ &dk_1 = -\varepsilon_4(\kappa_1)^2 e_1\omega^1,\ \ dk_2 = dk_3 = 0.\end{aligned}$$

and the considered parallel submanifold M^3 is a translation submanifold of a circle S_σ^1 and two parabolas in $E_{\sigma,1}^5$.

The case of the singular non-vanishing metric (4.10) in (1.5) gives $\omega_5^5 = \omega_4^6 = 0$, $\omega_5^6 = -\omega_4^5$ and the semiparallelity condition (2.8) gives $\kappa_2 = \kappa_3$. Thus (4.26) transforms into

$$\begin{aligned}&d\ln\kappa_1 + \omega_4^4 = A\omega^1,\quad d\ln\kappa_2 = -D\omega^1,\quad \kappa_2\omega_5^4 = \kappa_1 D\omega^1,\\ &\omega_1^2 = D\omega^2,\quad \omega_1^3 = D\omega^3,\quad \omega_4^\xi = X^\xi\omega^1,\quad \omega_5^\xi = 0,\ \xi = 6,\ldots,n.\end{aligned}$$

The same way as it is done in the present proof for the regular metric (4.8) one has $D(\varepsilon_5(\kappa_2)^2 + D^2) = 0$. The latter in case $\varepsilon_5 = 1$ leads immediately to $D = 0$, i.e. one has $Q = N = 1 + (n-6) = n-5$ and the considered semiparallel space-like submanifold exists with arbitrariness of $n-5$ functions of one real argument. If ε_5 has value -1, then either $D = 0$, or $D = \pm\kappa_2$. In both these cases the prolonged system is totally integrable and the considered normally flat semiparallel M^3 is a 2nd order envelope of the corresponding parallel submanifolds.

For the corresponding parallel M^3 the equations

$$d\ln\kappa_1 + \omega_4^4 = d\ln\kappa_2 = \omega_1^2 = \omega_1^3 = \omega_5^4 = \omega_4^\xi = \omega_5^\xi = 0$$

are to be added. The extended system is completely integrable and yields

$$\begin{aligned} &dx = e_i\omega^i, \;\; de_1 = k_1\omega^1 \;\; de_2 = \omega_2^3 e_3 + k_2\omega^2, \;\; de_3 = -\omega_2^3 e_2 + k_3\omega^3, \\ &dk_1 = -\varepsilon_4(\kappa_1)^2 e_1\omega^1, \;\; dk_2 = dk_3 = 0. \end{aligned}$$

Thus, in general, this parallel submanifold lies in $E^5_{\sigma,1}$ and geometrically is a translation submanifold of a sphere S^2_τ and a parabola.

In case of the completely vanishing metric (4.11) the geometry of the normally flat parallel space-like submanifold is known from Proposition 4.1, it is a translation submanifold in $E^5_{0,2}$ with three families of parabola generators and can be represented by the equation

$$x = \tfrac{1}{2}\Big(k_1(u^1)^2 + k_2(u^2)^2 + k_3(u^3)^2\Big) + k_{01}u^1 + k_{02}u^2 + k_{03}u^3.$$

It remains to investigate the existence of a second order envelope of such submanifolds.

Now the frame can be adapted to the space-like submanifold M^3 so that $\kappa_1 = \kappa_2 = \kappa_3 = 1$. Thus in (4.26) one has $M = P = L = Q = 0$. Moreover, taking there in the two last equalities $\xi = 6$ and $\xi = 7$ due to the metric (4.11) and the structure formulae (1.5) one has $\omega_4^6 = \omega_5^6 = \omega_4^7 = \omega_5^7 = 0$ and (4.26) transforms into

$$\begin{aligned} &\omega_4^4 = A\omega^1 + B\omega^2 + C\omega^3, \quad \omega_5^5 = -D\omega^1, \quad \omega_4^5 = R\omega^1 - B\omega^2 - C\omega^3, \\ &\omega_1^2 = B\omega^1 + D\omega^2, \quad \omega_1^3 = C\omega^1 + D\omega^3, \quad \omega_4^\xi = X^\xi\omega^1, \quad \omega_5^\xi = 0, \end{aligned}$$

where $\xi = 8, \ldots, n$. The basis of the secondary forms consists of ω_4^4, ω_5^5, ω_1^2, ω_1^3, ω_4^5, ω_4^ξ. Now the ranks of the polar matrices are $s_1 = 5 + (n-7)$ and $s_2 = 0$. The Cartan's number is $Q = N = 5 + (n-7) = n-2$ and the semiparallel space-like submanifold exists with arbitrariness of $n-2$ functions of one real argument. □

Proposition 4.6. *A normally flat semiparallel space-like submanifold M^3 in E^n_s with N_xM^3 of dimension 2 on supposition (4.24) is either*

1) a product $S^1_\tau \times S^1_\sigma \times E^1$, $\tau, \sigma \in \{0,1\}$ in $E^5_{\tau+\sigma} \subset E^n_s$, or

2) a translation submanifold of $S^1_\tau \times E^1$ and a parabola in $E^5_{\tau,1} \subset E^n_s$, or

3) a translation submanifold in $E^5_{0,2}$ with two families of parabola generators and a straight line; it can be represented by the equation

$$x = \tfrac{1}{2}\Big(k_1(u^1)^2 + k_2(u^2)^2\Big) + k_{01}u^1 + k_{02}u^2 + k_{03}u^3,$$

here k_i are the principal curvature vectors and k_{0i} are some constant vectors, or

4) a second order envelope of a submanifolds above with some arbitrariness.

Proof. The exterior differentiation of the Pfaff system (4.25) on supposition $\kappa_3 = 0$ gives

$$\begin{aligned}
&(d\ln\kappa_1 + \omega_4^4) \wedge \omega^1 + \omega_1^2 \wedge \omega^2 + \omega_1^3 \wedge \omega^3 = 0,\\
&\omega_1^2 \wedge \omega^1 - (d\ln\kappa_2 + \omega_5^5) \wedge \omega^2 - \omega_2^3 \wedge \omega^3 = 0,\\
&\kappa_1\omega_4^5 \wedge \omega^1 - \kappa_2\omega_1^2 \wedge \omega^2 = 0, \quad \kappa_2\omega_2^3 \wedge \omega^2 = 0,\\
&\kappa_1\omega_1^2 \wedge \omega^1 + \kappa_2\omega_5^4 \wedge \omega^2 = 0, \quad \kappa_1\omega_1^3 \wedge \omega^1 = 0,\\
&\kappa_1\omega_4^\xi \wedge \omega^1 = 0, \quad \kappa_2\omega_5^\xi \wedge \omega^2 = 0.
\end{aligned} \tag{4.27}$$

Using the Cartan's lemma one has

$$\begin{aligned}
&d\ln\kappa_1 + \omega_4^4 = A\omega^1 + B\omega^2 + C\omega^3, \qquad \kappa_2\omega_5^4 = \kappa_1 D\omega^1 + S\omega^2,\\
&d\ln\kappa_2 + \omega_5^5 = -D\omega^1 + L\omega^2 + M\omega^3, \quad \kappa_1\omega_4^5 = R\omega^1 - \kappa_2 B\omega^2,\\
&\omega_1^2 = B\omega^1 + D\omega^2, \quad \omega_1^3 = C\omega^1, \quad \omega_2^3 = M\omega^2,
\end{aligned} \tag{4.28}$$

and $\omega_4^\xi = X^\xi\omega^1$, $\omega_5^\xi = Y^\xi\omega^2$, where $\xi = 6, \dots, n$.
In case of regular metric (4.8) from the structure formulae (1.5) one has $\omega_4^5 = -\omega_5^4$. Thus the rank of the polar system $s_1 = 6 + 2(n-6) = 2n-6$. On the other hand relations betweens D and R, S and B give that the number of independent coefficients is the same $N = 6 + 2(n-6) = 2n-6$ and the considered semiparallel space-like submanifold exists with arbitrariness of $2n-5$ functions of one real argument.

For the corresponding parallel submanifold one has $dx = e_1\omega^1 + e_2\omega^2 + e_3\omega^3$,

$$\begin{aligned}
&de_1 = k_1\omega^1, \quad de_2 = k_2\omega^2, \quad de_3 = 0,\\
&dk_1 = -\varepsilon_4(\kappa_1)^2 e_1\omega^1, \quad dk_2 = -\varepsilon_5(\kappa_2)^2 e_2\omega^2,
\end{aligned}$$

i.e. parallel submanifold M^3 is a product $S^1_\tau \times S^1_\sigma, \times E^1$ in $E^5_{\tau+\sigma}$.

The same way as in previous Proposition 4.5 the singular non-vanishing metric (4.9) give $\omega_4^4 = \omega_5^6 = 0$, $\omega_4^6 = -\omega_5^4$, thus in (4.28) one has $S = 0$. It means that the Cartan's criterion is satisfied with $Q = N = 7 + 2(n-6) = 2n-5$ and semiparallel space-like submanifold exists with arbitrariness of $2n-5$ functions of one real argument and is a 2nd order envelope of parallel M^3 with $dx = e_1\omega^1 + e_2\omega^2 + e_3\omega^3$,

$$de_1 = k_1\omega^1, \quad de_2 = k_2\omega^2, \quad de_3 = 0, \quad dk_1 = -\varepsilon_4(\kappa_1)^2 e_1\omega^1, \quad dk_2 = 0,$$

i.e. a translation submanifold of $S^1_\tau \times E^1$ and a parabola in $E^5_{\tau,1}$.

The singular non-vanishing metric (4.10) together with (1.5) gives $\omega^5_5 = \omega^6_4 = 0$, $\omega^6_5 = -\omega^5_4$ and in (4.28) one has $R = 0$. It means that the Cartan's criterion is satisfied with $Q = N = 7 + 2(n-6) = 2n-5$ and semiparallel space-like submanifold exists with arbitrariness of $2n-5$ functions of one real argument. Here the geometry of the corresponding parallel submanifold is the same as in previous case with (4.9).

In case of the completely vanishing metric (4.11) the formulae for $\xi = 6, 7$ and leads to $\omega^6_4 = \omega^6_5 = \omega^7_4 = \omega^7_5 = 0$. Thus the basis of the secondary forms consists of $d\ln\kappa_1 + \omega^4_4$, $d\ln\kappa_2 + \omega^5_5$, ω^2_1, ω^3_1, ω^3_2, ω^4_5, ω^5_4, ω^ξ_4, ω^ξ_5, where $\xi = 8, \dots, n$ and the ranks of the polar systems are $s_1 = 6 + 2(n-8) = 2(n-5)$ and $s_2 = 1$, i.e. the Cartan's number is $Q = 2(n-4)$ and the number of independent coefficients N is the same. Thus the considered semiparallel space-like submanifold exists with arbitrariness of one holomorphic function of two real arguments. The description of the corresponding parallel submanfold is done in Proposition 4.1. Remark, that one of the geodesic lines in this case degenerate into a straight line. □

4.4 The case of three-dimensional principal normal subspace

In this last Section the 3-dimensional principal normal subspace has the either regular (4.12), or singular non-vanishing (4.13) and (4.14), or completely vanishing metric (4.15).

Proposition 4.7. *A normally flat semiparallel space-like submanifold M^3 in E^n_s with N_xM^3 of dimension 3 is either*

1) a product $S^1_\tau \times S^1_\sigma \times S^1_\varrho$, $\tau, \sigma, \varrho \in \{0, 1\}$ in $E^6_{\tau+\sigma+\varrho} \subset E^n_s$, or

2) a translation submanifold of $S^1_\tau \times S^1_\sigma$, $\tau, \sigma \in \{0, 1\}$ and a parabola in $E^6_{\tau+\sigma} \subset E^n_s$, or

3) a translation submanifold of two parabolas and a circle S^1_τ in $E^6_{\tau,2}$, or

4) a translation submanifold in $E^6_{0,3}$ with three families of parabola generators; it can be represented by the equation

$$x = \tfrac{1}{2}\Big(k_1(u^1)^2 + k_2(u^2)^2 + k_3(u^3)^2\Big) + k_{01}u^1 + k_{02}u^2 + k_{03}u^3,$$

here k_i are the principal curvature vectors and k_{0i} are some constant vectors, or

5) a second order envelope of a submanifolds above with some arbitrariness.

Proof. In this case the Pfaff system is

$$\omega^4_1 = \kappa_1\omega^1,\ \omega^5_2 = \kappa_2\omega^2,\ \omega^6_3 = \kappa_3\omega^3, \quad \omega^4_2 = \omega^4_3 = \omega^5_1 = \omega^5_3 = \omega^6_1 = \omega^6_2 = 0,$$

and $\omega_i^\xi = 0$, where $\xi = 7, \dots, n$ and its exterior differentiation leads to

$$
\begin{array}{ll}
(d\ln\kappa_1 + \omega_4^4)\wedge\omega^1 + \omega_1^2\wedge\omega^2 + \omega_1^3\wedge\omega^3 = 0, & \omega_4^\xi\wedge\omega^1 = 0,\\
\omega_1^2\wedge\omega^1 + (d\ln\kappa_2 + \omega_5^5)\wedge\omega^2 + \omega_2^3\wedge\omega^3 = 0, & \omega_5^\xi\wedge\omega^2 = 0,\\
\omega_1^3\wedge\omega^1 + \omega_2^3\wedge\omega^2 + (d\ln\kappa_3 + \omega_6^6)\wedge\omega^3 = 0, & \omega_6^\xi\wedge\omega^3 = 0,\\
\\
\kappa_1\omega_1^2\wedge\omega^1 + \kappa_2\omega_5^4\wedge\omega^2 = 0, & \kappa_1\omega_4^5\wedge\omega^1 - \kappa_2\omega_1^2\wedge\omega^2 = 0,\\
\kappa_1\omega_1^3\wedge\omega^1 + \kappa_3\omega_6^4\wedge\omega^3 = 0, & \kappa_1\omega_4^6\wedge\omega^1 - \kappa_3\omega_1^3\wedge\omega^3 = 0,\\
\kappa_2\omega_2^3\wedge\omega^2 + \kappa_3\omega_6^5\wedge\omega^3 = 0, & \kappa_2\omega_5^6\wedge\omega^2 - \kappa_3\omega_2^3\wedge\omega^3 = 0.
\end{array}
\tag{4.29}
$$

Due to the Cartan's lemma one has

$$
\begin{array}{lll}
d\ln\kappa_1 + \omega_4^4 = A\omega^1 + B\omega^2 + C\omega^3, & \omega_1^2 = B\omega^1 + D\omega^2, & \\
d\ln\kappa_2 + \omega_5^5 = D\omega^1 + K\omega^2 + L\omega^3, & \omega_1^3 = C\omega^1 + F\omega^3, & \\
d\ln\kappa_3 + \omega_6^6 = F\omega^1 + M\omega^2 + P\omega^3, & \omega_2^3 = L\omega^2 + M\omega^3, & \\
\\
\kappa_2\omega_5^4 = \kappa_1 D\omega^1 + R\omega^2 & \kappa_1\omega_4^5 = U\omega^1 - \kappa_2 B\omega^2, & \omega_4^\xi = X^\xi\omega^1,\\
\kappa_3\omega_6^4 = \kappa_1 F\omega^1 + S\omega^3 & \kappa_1\omega_4^6 = V\omega^1 - \kappa_3 C\omega^3, & \omega_5^\xi = Y^\xi\omega^2,\\
\kappa_3\omega_6^5 = \kappa_2 M\omega^2 + T\omega^3 & \kappa_2\omega_5^6 = W\omega^2 - \kappa_3 L\omega^3, & \omega_6^\xi = Z^\xi\omega^3.
\end{array}
\tag{4.30}
$$

In case of the regular metric (4.12) one has $\omega_4^4 = \omega_5^5 = \omega_6^6 = 0$, $\varepsilon_5\omega_4^5 = -\varepsilon_4\omega_5^4$, $\varepsilon_6\omega_4^6 = -\varepsilon_4\omega_6^4$, $\varepsilon_6\omega_5^6 = -\varepsilon_5\omega_6^5$. Thus the basis of the secondary forms consists of $\ln\kappa_1$, $\ln\kappa_2$, $\ln\kappa_3$, ω_1^2, ω_1^3, ω_2^3, ω_5^4, ω_6^4, ω_5^6, ω_4^ξ, ω_5^ξ, ω_6^ξ, the rank of the polar matrices are $s_1 = 9 + 3(n-6) = 3n-9$ and $s_2 = 0$, i.e. the Cartan's number is $Q = 3n-9$. On the other hand in (4.30) one has six independent relations, thus the number of all independent coefficients is the same $N = 3n-9$ and the considering semiparallel submanifold M^3 in this case is a 2nd order envelope of the corresponding parallel submanifolds with arbitrariness of $3n-9$ holomorphic functions of one real argument.

For the corresponding parallel submanifold one has that the coefficients on the right side (4.30), as components of h_{ijk}, are zero. The extended system consisting from the previous equations and

$$d\ln\kappa_1 = d\ln\kappa_2 = d\ln\kappa_3 = \omega_i^j = \omega_5^4 = \omega_6^4 = \omega_6^5 = \omega_4^\xi = \omega_5^\xi = \omega_6^\xi = 0$$

is completely integrable and there holds

$$dx = e_i\omega^i, \quad de_i = k_i\omega^i, \quad dk_i = \varepsilon_{3+i}(\kappa_i)^2 e_\omega^i,$$

where κ_i=const, i.e parallel submanifold is a product $S_\tau^1 \times S_\sigma^1 \times S_\varrho^1$, $\tau, \sigma, \varrho \in \{0,1\}$ in $E^6_{\tau+\sigma+\varrho}$.

Let us continue with the singular non-vanishing metric of one isotropic vector (4.13). Here $\omega_5^5 = \omega_6^6 = 0$, $\varepsilon_6\omega_5^6 = -\varepsilon_5\omega_6^5$, $\omega_4^5 = -\omega_5^7$, $\omega_4^6 = -\omega_6^7$, $\omega_4^7 = 0$ and the basis of the secondary forms consists of $d\ln\kappa_1 + \omega_4^4$, $d\ln\kappa_2$, $d\ln\kappa_3$, ω_1^2, ω_1^3,

ω_2^3, ω_4^5, ω_4^6, ω_5^4, ω_6^4, ω_6^5, ω_4^ξ, ω_5^ξ, ω_6^ξ, $\xi = 8, \dots, n$. Now the ranks of the polar system is $s_1 = 11 + 3(n-7) = 3n - 10$. The number of independent coefficient is $11 + 3(n-7) = 3n - 10$. The Cartan's criterion is satisfied and the considered semiparallel M^3 exists with arbitrariness of $3n - 10$ holomorphic functions of one real argument.

For the corresponding parallel submanifold the equations

$$d\ln\kappa_1 + \omega_4^4 = d\ln\kappa_2 = d\ln\kappa_3 = \omega_1^2 = \omega_1^3 = \omega_2^3 = \omega_4^5 = \omega_4^6 = \omega_5^4 = \omega_6^4 = \omega_6^5 = 0$$

and $\omega_4^\xi = \omega_5^\xi = \omega_6^\xi = 0$ are to be added. The extended system is completely integrable and yields

$$\begin{aligned} &dx = e_i\omega^i, \quad de_i = k_i\omega^i, \\ &dk_1 = 0, \quad dk_2 = -\varepsilon_5(\kappa_2)^2 e_2\omega^2, \quad dk_3 = -\varepsilon_5(\kappa_3)^2 e_3\omega^3. \end{aligned}$$

Thus the considered parallel M^3 is a translation submanifold of $S^1_\tau \times S^1_\sigma$ and a parabola, which lies in $E^6_{\tau+\sigma}$.

The next step is a singular non-vanishing metric with two isotropic vectors (4.14). In this case the system (4.29) for $\xi = 7, 8$ together with (1.5) and the Cartan's lemma leads to $\omega_4^7 = \omega_5^7 = \omega_4^8 = \omega_5^8 = 0$. Since due to the metric $\omega_6^6 = 0$, $\omega_6^7 = -\omega_4^6$, $\omega_6^8 = -\omega_5^6$, then the basis of the secondary forms consists of $d\ln\kappa_1 + \omega_4^4$, $d\ln\kappa_2 + \omega_5^5$, $d\ln\kappa_3$, ω_i^j, ω_5^4, ω_6^4, ω_4^5, ω_6^5, ω_4^6, ω_5^6, ω_4^ξ, ω_5^ξ, ω_6^ξ, $\xi = 9, \dots, n$ and the ranks of the polar matrices are $s_1 = 11 + 3(n-8)$ and $s_2 = 1$. The Cartan's number $Q = 13 + 3(n-8)$. The relations (4.30) give $13 + 3(n-8)$ independent coefficients. So the considered M^3 exists with arbitrariness of one holomorphic function of two real arguments.

Here for the corresponding parallel submanifold one has that the extended system is completely integrable and yields

$$dx = e_i\omega^i, \quad de_i = k_i\omega^i, \quad dk_1 = dk_2 = 0, \quad dk_3 = -\varepsilon_6(\kappa_3)^2 e_3\omega^3,$$

i.e. geometrically a such parallel M^3 is a translation submanifold of two parabolas and a circle S^1_τ in $E^6_{\tau,2}$.

In case of the completely vanishing metrics (4.15) the system (4.29) for $\xi = 7, 8, 9$ together with (1.5) and the Cartan's lemma leads to $\omega_4^7 = \omega_5^7 = \omega_6^7 = \omega_4^8 = \omega_5^8 = \omega_6^8 = \omega_4^9 = \omega_5^9 = \omega_6^9 = 0$. Since the frame was adapted to the submanifold so that $\kappa_i = 1$, then the basis of the secondary forms consists of ω_{3+i}^{3+i}, ω_i^j, ω_5^4, ω_6^4, ω_4^5, ω_6^5, ω_4^6, ω_5^6, ω_{3+i}^ξ, $\xi = 10, \dots, n$ and the Cartan's number Q is equal $s_1 + 2s_2$, where s_i, $(i = 1, 2)$ are ranks of polar matrices and $s_1 = 9 + 3(n-9)$, $s_2 = 3$. It gives $N = 15 + 3(n-9)$ independent coefficients and the Cartan's criterion is satisfied; considered semiparallel submanifold exists as a 2nd order envelope of corresponding parallel submanifolds with arbitrariness of 3 holomorphic functions of two arguments.

The corresponding parallel submanifold due to Proposition 4.1 is a translation submanifold in $E^6_{0,3}$ with three families of parabola generators; it can be represented by the equation

$$x = \frac{1}{2}\Big(k_1(u^1)^2 + k_2(u^2)^2 + k_3(u^3)^2\Big) + k_{01}u^1 + k_{02}u^2 + k_{03}u^3,$$

here k_i are the principal curvature vectors and k_{0i} are some constant vectors. $\square$

Chapter 5

Normally non-flat semiparallel space-like submanifolds M^3

In this part all possibilities for the principal normal subspace of normally non-flat semiparallel space-like submanifolds M^3 are investigated and the normally non-flat parallel space-like submanifolds M^3 are classified. The geometrical structure of semiparallel submanifolds as a second envelopes of the corresponding parallel submanifolds needs further investigation.

5.1 The case of six-dimensional principal normal subspace

Let us start with $\dim N_xM^3 = 6$, then all six vectors h_{11}, h_{22}, h_{33}, h_{12}, h_{13} and h_{23} are linearly independent and (2.9) leads to

$$H_{11,22} = H_{11,33} = H_{22,33} = 2K, \quad H_{12,12} = H_{13,13} = H_{23,23} = K,$$
$$H_{11,11} = H_{22,22} = H_{33,33} = 4K, \quad H_{aa,ab} = H_{aa,bc} = H_{ab,2ac} = 0,$$

for every three distinct value a, b and c $(a, b, c = 1, 2, 3)$. The matrix

$$\begin{pmatrix} 4K & 2K & 2K & 0 & 0 & 0 \\ 2K & 4K & 2K & 0 & 0 & 0 \\ 2K & 2K & 4K & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & 0 & K \end{pmatrix}$$

with main minors $4K$, $12K^2$, $32K^3$, $32K^4$, $32K^5$, $32K^6$ gives that if $K > 0$ then the metric of the principal normal subspace is regular positively definite; if $K < 0$, then the metric of N_xM^3 is regular negatively definite; at last, if $K = 0$, then the metric is completely vanishing.

Proposition 5.1. *A normally non-flat parallel space-like M^3 in E^n_s with $dim N_x M^3 = 6$ in $E^9_{0,6}$ is either*

1) a Veronese submanifold $V^3 \subset E^9_s$, ($s = 0$ or $s = 6$), or

2) a submanifold in $E^9_{0,6}$ with three families of parabola generators; it can be represented by the equation

$$x = \tfrac{1}{2}\Big(h_{11}(u^1)^2 + h_{22}(u^2)^2 + h_{33}(u^3)^2\Big) +$$

$$+h_{12}u^1u^2 + h_{13}u^1u^3 + h_{23}u^2u^3 + h_{01}u^1 + h_{02}u^2 + h_{03}u^3,$$

where h_{0i} are some constant vectors.

Proof. It is known that if the metric of the principal normal subspace has the maximal dimension and is regular ($K > 0$ and $K < 0$), then the corresponding semiparallel M^3 is either a Veronese submanifold V^3 in a 8−sphere, i.e. an orbit of a 6−parametric Lie subgroup of rotation of E^9_s, ($s = 0$ or $s = 6$) around the center of this 8−sphere or a second order envelope of a family of congruent Veronese V^3 (see Section 2.4). The second assertion follows from Proposition 2.2. □

A normally non-flat semiparallel space-like M^3 with $\dim N_x M^3 = 6$ in E^n_s is a second order envelope of submanifolds from this Proposition.

5.2 The case of five-dimensional principal normal subspace

In [11] it is shown that in Euclidean space E^n (i.e. for the case $s = 0$) there no exist semiparallel submanifold M^3 with $\dim N_x M^3 = 5$. In the case of E^n_s with $s > 0$ the situation is different. There holds

Proposition 5.2. *Let M^3 be a semiparallel space-like submanifold in pseudo-Euclidean space E^n_s with $dim N_x M^3 = 5$. Then the metric of the principal normal subspace $N_x M^3$ vanishes completely.*

Proof. If the dimension of $N_x M^3$ is 5, then one has at the point $x \in M^3$ a linear dependence between vectors h_{ij}, and there exist six coefficients ϑ^{ij} so that

$$h_{ij}\vartheta^{ij} = 0, \quad \sum(\vartheta^{ij})^2 \neq 0.$$

Here h_{ij} are components of a vector valued symmetric tensor field, hence ϑ^{ij} are components of a symmetric tensor field to a multiplier. Now the vectors e_1, e_2 and e_3 can be taken at the point $x \in M^3$ so that this dependency transforms into

$$h_{11}\vartheta^{11} + h_{22}\vartheta^{22} + h_{33}\vartheta^{33} = 0,$$

which leads (after renumbering, if needed) to

$$h_{33} = \tau_1 h_{11} + \tau_2 h_{22}. \tag{5.1}$$

The five vectors h_{11}, h_{22}, h_{12}, h_{13} and h_{23} must be here linearly independent. Let us now investigate the semiparallelity condition (2.9) denoting it by $[i,j;k,l]$ and using (5.1). All coefficients by linearly independent vectors in (2.9) must be equal to zero. If to take [1,1;1,2] and [2,2;2,1] the coefficients by h_{12} give as a result all $H_{i,j;k,l}$ are zero. □

Using Proposition 2.2 and (5.1) can be formulated

Proposition 5.3. *A normally non-flat parallel space-like M^3 in $E^8_{0,5}$ with $dim N_x M^3 = 5$ is a submanifold with 3 families of parabola generators, which can be represented by the equation*

$$x = \frac{1}{2}\Big(h_{11}\big((u^1)^2 + \tau_1 (u^3)^2\big) + h_{22}\big((u^2)^2 + \tau_2 (u^3)^2\big)\Big) +$$
$$+ h_{12} u^1 u^2 + h_{13} u^1 u^3 + h_{23} u^2 u^3 + h_{01} u^1 + h_{02} u^2 + h_{03} u^3,$$

where h_{0i} are some constant vectors.

A normally non-flat semiparallel space-like M^3 with $\dim N_x M^3 = 5$ in E^n_s is a second order envelope of submanifolds from this Proposition.

5.3 The case of four-dimensional principal normal subspace

In this case the same way as in [11] can be obtained the following possibilities for linear dependence between vectors h_{ij}:

$$(A) \quad h_{33} = \mu h_{11}, \quad h_{23} = \nu_1 h_{11} + \nu_2 h_{22} + \nu_3 h_{12} + \nu_4 h_{13}, \tag{5.2}$$
$h_{11}, h_{22}, h_{12}, h_{13}$ are linearly independent,

$$(B) \quad h_{33} = \mu h_{11}, \quad h_{22} = \nu h_{11}, \tag{5.3}$$
$h_{11}, h_{12}, h_{13}, h_{23}$ are linearly independent,

$$(C) \quad h_{33} = \mu h_{11}, \quad h_{13} = \nu_1 h_{11} + \nu_2 h_{22}, \tag{5.4}$$
$h_{11}, h_{22}, h_{12}, h_{23}$ are linearly independent,

$$(D) \quad h_{33} = 0, \quad h_{12} = \nu_1 h_{11} + \nu_2 h_{22}, \tag{5.5}$$
$h_{11}, h_{22}, h_{13}, h_{23}$ are linearly independent,

$$(E) \quad h_{33} = h_{11} = 0, \tag{5.6}$$
$h_{22}, h_{12}, h_{13}, h_{23}$ are linearly independent.

Remark, that the first two cases must be considered with their limit cases when $\mu = 0$.

Proposition 5.4. *Let M^3 be a semiparallel space-like submanifold in E^n_s with $dim N_x M^3 = 4$. In the case (A) the metric of the first normal subspace can be either regular, or singular non-vanishing, or completely vanishing.*

Proof. From (2.9) on supposition (5.2) one has the following relations:

$$\begin{aligned}
&[1,1;1,2] && \text{by } h_{11}: && H_{11,12} - \nu_1 H_{11,13} = 0,\\
& && \text{by } h_{22}: && H_{11,12} + \nu_2 H_{11,13} = 0,\\
&[2,2;1,2] && \text{by } h_{11}: && H_{22,12} + \nu_1\big(2H_{12,23} - 3H_{22,13}\big) = 0,\\
& && \text{by } h_{22}: && H_{22,12} - \nu_2\big(2H_{12,23} - 3H_{22,13}\big) = 0,\\
&[1,2;1,2] && \text{by } h_{11}: && 2H_{12,12} - H_{11,22} + \nu_1\big(H_{11,23} - 2H_{12,13}\big) = 0,\\
& && \text{by } h_{22}: && 2H_{12,12} - H_{11,22} - \nu_2\big(H_{11,23} - 2H_{12,13}\big) = 0,\\
&[3,3;1,2] && \text{by } h_{11}: && \mu H_{11,12} + \nu_1\big(2H_{22,13} - 2H_{12,23} - \mu H_{11,13}\big) = 0,\\
& && \text{by } h_{22}: && \mu H_{11,12} - \nu_2\big(2H_{22,13} - 2H_{12,23} - \mu H_{11,13}\big) = 0.
\end{aligned} \tag{5.7}$$

The analysis of this system on supposition $\nu_1 + \nu_2 \neq 0$ leads to

$$\begin{aligned}
&H_{11,12} = H_{22,12} = H_{11,13} = H_{22,13} = H_{12,23} = 0,\\
&2H_{12,12} - H_{11,22} = H_{11,23} - 2H_{12,13} = 0.
\end{aligned} \tag{5.8}$$

These results together with

$$\begin{aligned}
&[1,1;1,2] && \text{by } h_{13}: && 3H_{11,23} - 2H_{12,13} - \nu_4 H_{11,13} = 0,\\
&[2,2;1,2] && \text{by } h_{13}: && H_{22,23} - \nu_4\big(2H_{12,23} - 3H_{22,13}\big) = 0,
\end{aligned}$$

show that $H_{12,13} = 0$ and $H_{11,23} = H_{22,23} = 0$. The latter with (5.2) leads to

$$\nu_1 H_{11,11} + \nu_2 H_{11,22} = 0, \quad \nu_1 H_{11,22} + \nu_2 H_{22,22} = 0. \tag{5.9}$$

Moreover,

$$\begin{aligned}
&[1,1;1,2] && \text{by } h_{12}: && H_{22,12} + \nu_1\big(2H_{12,23} - 3H_{22,13}\big) = 0,\\
&[2,2;1,2] && \text{by } h_{12}: && 2H_{12,12} - H_{11,22} + \nu_1\big(H_{11,23} - 2H_{12,13}\big) = 0,\\
&[1,3;1,2] && \text{by } h_{11}: && H_{12,13} + (1-\mu)\big(H_{12,13} - H_{11,23}\big) +\\
& && && \qquad + \nu_1(H_{11,22} - H_{12,12} - H_{13,13}) = 0,
\end{aligned}$$

give

$$H_{12,12} = H_{13,13} = K, \quad H_{11,22} = 2K, \quad H_{11,11} = H_{22,22} = 4K. \tag{5.10}$$

Thus (5.9) together with supposed above $\nu_1 + \nu_2 \neq 0$ give $K = 0$ and the metric of the first normal subspace is completely vanishing.

Let now $\nu_1 = -\nu_2 = \nu \neq 0$. Then the system (5.7) transforms into

$$\begin{aligned}
&H_{11,12} - \nu H_{11,13} = 0, \quad H_{22,12} + \nu\big(2H_{12,23} - 3H_{22,13}\big) = 0,\\
&2H_{12,12} - H_{11,22} + \nu\big(H_{11,23} - 2H_{12,13}\big) = 0,\\
&\mu H_{11,12} + \nu\big(2H_{22,13} - 2H_{12,23} - \mu H_{11,13}\big) = 0,
\end{aligned}$$

and together with equalities

$$\begin{array}{lll}
[1,1;1,3] & \text{by } h_{11}: & (1-\mu)H_{11,13} - H_{11,12} = 0,\\
 & \text{by } h_{22}: & \nu H_{11,12}\big) = 0,\\
[2,2;1,3] & \text{by } h_{11}: & (1-\mu)H_{22,13} + \nu\big(2\mu H_{11,12} - 2H_{13,23} - H_{22,12}\big) = 0,\\
 & \text{by } h_{22}: & \nu\big(2\mu H_{11,12} - 2H_{13,23} - H_{22,12}\big) = 0,\\
[1,2;1,3] & \text{by } h_{11}: & (2-\mu)H_{12,13} - H_{11,23} + \nu\big(\mu H_{11,11} - H_{12,12} - H_{13,13}\big) = 0,\\
 & \text{by } h_{22}: & H_{11,23} - H_{12,13} - \nu\big(\mu H_{11,11} - H_{12,12} - H_{13,13}\big) = 0,
\end{array}$$

$$\begin{array}{lll}
[2,3;1,3] & \text{by } h_{11}: & H_{13,23} + \mu\big(\mu H_{11,12} - 2H_{13,23}\big) - \nu H_{12,23} = 0,\\
 & \text{by } h_{22}: & H_{13,23} - \mu H_{11,12} + \nu H_{12,23} = 0,\\
[1,3;1,2] & \text{by } h_{11}: & H_{12,13} + (1-\mu)\big(H_{12,13} - H_{11,23}\big) +\\
 & & \qquad\qquad +\nu\big(H_{11,22} - H_{12,12} - H_{13,13}\big) = 0,\\
 & \text{by } h_{22}: & H_{12,13} + \nu\big(H_{11,22} - H_{12,12} - H_{13,13}\big) = 0,\\
[2,3;1,2] & \text{by } h_{11}: & H_{12,23} + \mu\big(H_{12,23} - H_{22,13}\big) - H_{13,23} = 0,\\
 & \text{by } h_{22}: & H_{22,13} - 2H_{12,23} - \nu H_{13,23} = 0.
\end{array}$$

on supposition $\mu \neq 1$ gives

$$H_{11,12} = H_{11,13} = H_{22,12} = H_{22,13} = H_{12,13} = H_{11,23} = H_{13,23} = 0.$$

Remark, that the relations (5.10) act here and supposition $K \neq 0$ contradicts to the assumption $\mu \neq 0$. If $K = 0$, then the straightford computation gives a completely vanishing metric.

The next step is $\mu = 1$ and $K \neq 0$. Now $\nu \neq 0$ leads to a contradiction, thus $\nu = 0$. Further, $\nu_4 = 0$ and $\nu_3^2 - 1 = 0$. Here the case $\nu_3 = 1$ can be reduced to the case $\nu_3 = -1$, taking $-e_1$ instead of e_1, and so the basic linear dependencies become: $h_{33} = h_{11}$, $h_{23} = h_{12}$. But the latter can be simplified by frame transformation

$$e_1^* = e_2, \quad e_2^* = \frac{1}{\sqrt{2}}(-e_1 + e_3), \quad e_3^* = \frac{1}{\sqrt{2}}(e_1 + e_3)$$

to $h_{13}^* = h_{23}^* = 0$.

Let * be omitted further, so that $h_{13} = h_{23} = 0$ and h_{11}, h_{22}, h_{33}, h_{12} are linearly independent. Now the consequences from the semiparallelity condition (2.9) reduce to

$$H_{11,12} = H_{22,12} = H_{33,12} = H_{11,33} = H_{22,33} = 0,$$
$$H_{11,11} = H_{22,22} = 2H_{11,22} = 4H_{12,12} = K.$$

Due to the arbitrariness of $H_{33,33}$ the metric of the first normal subspace can be either regular ($H_{33,33} \neq 0$), or singular non-vanishing ($H_{33,33} = 0$). □

Proposition 5.5. *Let M^3 be a semiparallel space-like submanifold in E^n_s with $dim N_x M^3 = 4$. In the cases (B)-(E) the metric of the first normal subspace can be completely vanishing, only.*

Proof. Let start with the case (B). Here the equation $[1,1;1,2]$ and $[2,2;1,2]$ give by h_{12}:

$$3H_{11,22} - 2H_{12,12} - H_{11,11} = 0, \quad 3H_{11,22} - 2H_{12,12} - H_{22,22} = 0, \tag{5.11}$$

thus $H_{11,11} = H_{22,22}$. If $H_{11,11} \neq 0$, then due to (5.3) and $[1,1;2,3]$ by h_{23}: $H_{11,22} - H_{11,33} = 0$ one has $\nu^2 = 1$ and $\mu = \nu$.

Let $\nu = \mu = 1$. Thus $[1,3;1,2]$ by h_{23}: $H_{11,22} - H_{12,12} - H_{13,13} = 0$ together with $H_{11,22} = H_{12,12}$ shows that $H_{13,13} = 0$. The latter due to $[1,1;1,3]$ by h_{13}: $3H_{11,33} - 2H_{13,13} - H_{11,11} = 0$ contradicts with $H_{11,11} \neq 0$.

Supposing $\nu = \mu = -1$ from (5.11) one has $H_{12,12} + 2H_{11,11} = 0$, on the other hand $[1,2;1,2]$ by h_{11} leads to $2H_{12,12} - H_{11,22} = 0$, i.e. here $H_{11,11} \neq 0$ is impossible, too.

Now obtained that $H_{11,11} = 0$. Thus $H_{13,13} = H_{12,12} = H_{22,22} = 0$. Moreover, $[1,1;1,2]$ and $[1,1;1,3]$ by h_{23} lead to $H_{11,13} = H_{11,12} = 0$ and the equations

$$\begin{aligned}
&[1,1;2,3] \quad \text{by } h_{12}: \ 2H_{12,23} - 2H_{22,13} + H_{11,13} = 0,\\
& \quad \text{by } h_{13}: \ 2H_{12,33} - 2H_{13,23} - H_{11,12} = 0,\\
&[2,3;1,3] \quad \text{by } h_{12}: \ H_{13,13} - H_{11,33} + H_{23,23} = 0,\\
&[1,1;1,3] \quad \text{by } h_{12}: \ 3H_{11,23} - 2H_{12,13} = 0,\\
&[1,2;1,2] \quad \text{by } h_{23}: \ H_{11,23} - 2H_{12,13} = 0
\end{aligned}$$

lead to $H_{12,23} = H_{13,23} = H_{23,23} = H_{11,23} = H_{12,13} = 0$. So the metric of the first normal subspace $N_x M^3$ in the case (B) vanishes completely. The same result will be obtained supposing $\mu = 0$.

Let us consider now the case (C). Here on the same way as in previous case one has $H_{11,11} = H_{22,22}$. If $H_{11,11} \neq 0$, then $[3,3;1,2]$ by h_{12}: $H_{22,33} - H_{11,33} = 0$ on supposition $\mu \neq 0$ leads to $H_{11,11} = H_{11,22}$, which contradicts with linear independence of h_{11} and h_{22}.

So one has $H_{11,11} = H_{22,22} = 0$, where from

$$
\begin{aligned}
&[1,2;1,2] \quad \text{by } h_{23}: \ H_{11,23} - 2H_{12,13} = 0,\\
&[1,1;1,3] \quad \text{by } h_{12}: \ 3H_{11,23} - 2H_{12,13} = 0,\\
&[1,2;1,3] \quad \text{by } h_{23}: \ H_{11,23} - H_{12,13} - H_{12,12} = 0,
\end{aligned}
$$

i.e. $H_{11,23} = H_{12,13} = H_{12,12} = 0$. The latter together with (5.11) leads to $H_{11,22} = 0$.Now equations $[1,2;1,3]$ and $[1,1;1,3]$ by h_{12} give $H_112,23 = H_{11,12} = 0$. Substitution into $[1,2;1,2]$ by h_{12}: $H_{11,12} + H_{22,12} = 0$ gives $H_{22,12} = 0$. At last, equation $[1,3;2,3]$ by h_{12}: $H_{22,23} - H_{13,13} - H_{23,23} = 0$ together with (5.4) leads to $H_{23,23} = 0$. One has that all $H_{ij,kl}$ are zero, i.e. the metric of the first normal subspace in this case is completely vanishing.

The consideration of the case (D) in a system of relations

$$
\begin{aligned}
&[1,1;2,3] \quad \text{by } h_{23}: \ H_{11,22} - H_{11,33} = 0,\\
&[1,3;1,3] \quad \text{by } h_{13}: \ H_{11,13} - H_{13,33} = 0,\\
&[2,3;2,3] \quad \text{by } h_{23}: \ H_{22,23} - H_{22,33} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&[3,3;1,3] \quad \text{by } h_{13}: \ 2H_{13,13} - 3H_{11,33} + H_{33,33} = 0\\
&[3,3;1,3] \quad \text{by } h_{23}: \ 2H_{13,23} - 3H_{12,33} = 0,\\
&[3,3;2,3] \quad \text{by } h_{23}: \ 2H_{23,23} - 3H_{22,33} + H_{33,33} = 0
\end{aligned}
$$

due to (5.5) leads to $H_{11,22} = H_{11,13} = H_{22,23} = H_{13,13} = H_{13,23} = H_{23,23} = 0$. In the equalities

$$
\begin{aligned}
&[1,1;1,3] \quad \text{by } h_{13}: \ 3H_{11,33} - 2H_{13,13} + H_{11,11} = 0,\\
&[2,2;2,3] \quad \text{by } h_{23}: \ 3H_{22,33} - 2H_{23,23} + H_{22,22} = 0
\end{aligned}
$$

it gives that $H_{11,11} = H_{22,22} = 0$. Now from

$$
\begin{aligned}
&[1,1;1,3] \quad \text{by } h_{13}: \ 3H_{11,33} - 2H_{13,13} + H_{11,11} = 0,\\
&[2,2;2,3] \quad \text{by } h_{23}: \ 3H_{22,33} - 2H_{23,23} + H_{22,22} = 0
\end{aligned}
$$

one has $H_{11,23} = H_{12,13} = 0$. At last, $[2,2;1,2]$ by h_{23} and $[3,3;1,2]$ by h_{23} lead to $H_{22,13} = H_{22,12} = 0$. As a result, the metric of the first normal subspace is completely vanishing.

It remains to consider the case (E). The equalities (5.11) act here and due to (5.6) one has that $H_{12,12} = H_{22,22} = 0$. On the same way as it is done in the previous case one obtains

$$H_{13,13} = H_{23,23} = H_{22,12} = H_{22,13} = H_{22,23} = H_{12,13} = H_{12,23} = H_{13,23} = 0.$$

It means that the metric of the first normal subspace in this last case vanishes completely, too. $\square$

Proposition 5.6. *A normally non-flat parallel space-like M^3 in $E^8_{0,5}$ with $dim N_x M^3 = 4$ is either*

1) a product $V^2 \times M^1$ in $E^7_\tau \subset E^n_s$ $\tau \in \{0,3\}$, or

2) a submanifold in $E^7_{0,1}$ with two families of generators; one of them is $V^2 \in E^5_\tau$ and another is a parabola, or

3) a submanifold in $E^7_{0,4}$ with three families of parabola generators (some of them can degenerate into a straight line) and can be represented by the equation (2.4), or

4) a second order envelope of submanifolds above with some arbitrariness.

Proof. Let us start with case (A). If to take $e_4 \parallel h_{11} + h_{22}$, $e_5 \parallel h_{11} - h_{22}$, $e_6 \parallel h_{12}$ and $e_7 \parallel h_{33}$. Thus

$$h_{11} = \alpha e_4 + \beta e_5,\ h_{22} = \alpha e_4 - \beta e_5,\ h_{12} = \gamma e_6,\ h_{33} = \lambda e_7$$

and Pfaff system is

$$\begin{aligned}
&\omega 1^4 = \alpha\omega^1, \quad \omega_1^5 = \beta\omega^1, \quad \omega_1^6 = \gamma\omega^2, \quad \omega_1^7 = 0, \quad \omega_1^\xi = 0,\\
&\omega 2^4 = \alpha\omega^2, \quad \omega_2^5 = \beta\omega^2, \quad \omega_2^6 = \gamma\omega^1, \quad \omega_2^7 = 0, \quad \omega_2^\xi = 0,\\
&\omega 3^4 = 0, \quad \omega_3^5 = 0, \quad \omega_3^6 = 0, \quad \omega_3^7 = \lambda\omega^3, \quad \omega_3^\xi = 0,
\end{aligned}$$

In case of the regular metric from the semiparallelity condition one has $\alpha = \gamma\sqrt{3}$ and $\beta = \gamma$. The existence of a such submanifold and the description of the corresponding parallel submanifolds in [11] are done; it gives a product $V^2 \times M^1$ in $E^7_\tau \subset E^n_s$. $\tau \in \{0,3\}$. In case of the singular non-vanishing metric the generators are not necessary to be orthogonal and the second component is a parabola.

In cases (B)–(E) one has submanifolds from Proposition 2.2. $\square$

5.4 The case of three-dimensional principal normal subspace

If to take the vector e_4 so that $e_4 \parallel H$ and the vectors e_5, e_6 in $N_x M^3$ so that h_{11} contains to the span of H and e_5, then

$$H^4 \neq 0, \quad H^5 = H^6 = H^\xi = 0,\ h^\xi_{ij} = h^6_{11} = 0,\ (\xi = 7, ..., n).$$

If in the semiparallellity condition (2.8) to take $i = j$ and summarize by $i = 1, 2, 3$, one has

$$H^{\beta}\Omega^{\alpha}_{\beta} = 0, \tag{5.12}$$

which leads now to

$$\Omega^{\alpha}_{4} = 0. \tag{5.13}$$

The metric of the principal normal space generated by pseudo-Euclidean space can be either regular, or singular non-vanishing, or completely vanishing.

Proposition 5.7. *Let M^3 be a normally non-flat semiparallel space-like submanifolds with $dimN_xM^3 = 3$ and regular metric of the principal normal space in pseudo-Euclidean space E^n_s. Thus for h^{α}_{ij} one has either*

$$(A)\ h^6_{13} = h^6_{23} = 0,\ h^5_{22} \neq 0,\ \text{or}\quad (B)\ h^5_{22} = h^6_{13} = 0,\ h^6_{23} \neq 0, \tag{5.14}$$

and the frame vectors e_4, e_5, e_5 are such that their scalar square are 1, or -1 and pairwise scalar products are zero.

Proof. The mean curvature vector $H \neq 0$ can be either non-isotropic or isotropic (i.e. $H^2 = 0$). It turns out that the second case is impossible for the considered M^3. In this case the frame vectors e_4, e_5, e_6 are such that $\varepsilon_4 = \varepsilon_a = 0$, $\varepsilon_b \neq 0$, $g_{4a} = 1,\ a, b = 5, 6$ and have distinct values, all others $g_{\alpha\beta} = 0$, $(\alpha \neq \beta)$. Without loss of generality can be taken $a = 6$, $b = 5$, thus $\Omega^5_4 = -g_{46}\omega^6_i \wedge \omega^5_i = 0$ and matrices $\parallel h^6_{ij} \parallel$, $\parallel h^5_{ij} \parallel$ commute, the same way as it was done above can be madden $h^6_{ij} = h^5_{ij} = 0$ if $i \neq j$. Now 2-forms $\Omega^5_6 = -\varepsilon_5\Omega^4_5$, where

$$\Omega^4_5 = -\varepsilon_5\Big[(h^5_{22} - h^5_{11})h^4_{12}\omega^1 \wedge \omega^2 + (h^5_{22} - h^5_{11})h^4_{13}\omega^1 \wedge \omega^3 + (h^5_{33} - h^5_{22})h^4_{23}\omega^2 \wedge \omega^3\Big]$$

must be non-zero only. On the other hand the semiparallelity condition by $i = j$, $\alpha = 6$ gives $h^6_{ii}\Omega^5_6 = 0$, i.e. all $h^6_{22} = h^6_{33} = 0$ and the dimension of N_xM^3 is smaller than 3.

Therefore it remains to consider the case with $H^2 \neq 0$, only. Here the frame vectors $e_4,\ e_5,\ e_6$ are such that their scalar square are either 1 or -1 and all $g_{\alpha\beta} = 0$ $(\alpha \neq \beta)$. Since here $\varepsilon_4 \neq 0$ and together with (5.13) it leads to $\Omega^4_{\alpha} = 0$. In particular, $\Omega^4_5 = 0$, thus due to (1.6) and (1.11) the matrices $\parallel h^4_{ij} \parallel$ and $\parallel h^5_{ij} \parallel$ are commutative. Therefore the vectors e_1, e_2, e_3 can be taken so that $h^4_{ij} = h^5_{ij} = 0$ if $i \neq j$. Since $\nabla^{\perp}$ is supposed to be non-flat, at least, one from the 2-forms

$$\Omega^6_5 = \varepsilon_5\Big[(h^5_{22} - h^5_{11})h^6_{12}\omega^1 \wedge \omega^2 + (h^5_{33} - h^5_{11})h^6_{13}\omega^1 \wedge \omega^3 + (h^5_{33} - h^5_{22})h^6_{23}\omega^2 \wedge \omega^3\Big],$$

and

$$\Omega^5_6 = -\varepsilon_6\Big[(h^5_{11} - h^5_{22})h^6_{12}\omega^1 \wedge \omega^2 + (h^5_{11} - h^5_{33})h^6_{13}\omega^1 \wedge \omega^3 + (h^5_{22} - h^5_{33})h^6_{23}\omega^2 \wedge \omega^3\Big].$$

must be non-zero. Without loss of generality it can be supposed that

$$h^5_{11} \neq h^5_{22},\ h^6_{12} \neq 0. \tag{5.15}$$

The semiparallelity condition by $i = j$ and $\alpha = 5$ gives $h^6_{ii}\Omega^5_6 = 0$, which leads to $h^6_{22} = h^6_{33} = 0$. Moreover, equation by $i = 1,\ j = 2,\ \alpha = 5$ can be written as

$$(h^5_{22} - h^5_{11})\Omega^2_1 = h^6_{12}\Omega^5_6 \neq 0, \tag{5.16}$$

it is easy to see that here $\Omega^2_1 \neq 0$. Moreover, from this equation and from equations by $i = 1, 2\ j = 3,\ \alpha = 6$

$$h^6_{12}\Omega^3_2 - h^6_{23}\Omega^2_1 = 0, \qquad h^6_{12}\Omega^3_1 + h^6_{13}\Omega^2_1 = 0, \tag{5.17}$$

one can get that $\Omega^3_1,\ \Omega^3_2,\ \Omega^6_5$ are proportional to Ω^2_1. Substitution of these 2-forms into $(h^5_{jj} - h^5_{ii})\Omega^j_i - h^6_{ij}\Omega^5_6 = 0$, where $i = 1, 2,\ j = 3,\ \alpha = 5$ together with

$$H^5 = \tfrac{1}{3}(h^5_{11} + h^5_{22} + h^5_{33}) = 0$$

gives $h^5_{11}h^6_{13} = 0, \quad h^5_{22}h^6_{23} = 0$. This leads to the cases (A) and (B) in (5.14). □

The similar result was obtained in [11] for Euclidean spaces E^n.

Proposition 5.8. *There exists no normally non-flat semiparallel space-like M^3 with a three-dimensional principal normal space of the singular non-vanishing metric and the isotropic mean curvature vector H.*

Proof. On supposition $H^2 = 0$ one has $\varepsilon_4 = 0$ and the frame vector e_7 can be taken so that $\varepsilon_7 = 0,\ g_{47} = 1$. Now the vector e_4 can be canonized so that $h^4_{ij} = 0,\ i \neq j$ and for the vectors $e_5,\ e_6$ one has the following possibilities:

- (i) $\varepsilon_5,\ \varepsilon_6$ are either 1 or -1, and all others $g_{\alpha\beta} = 0$;
- (ii) $\varepsilon_a = 0,\ \varepsilon_b \neq 0,\ (a, b = 5, 6)$ and the frame vector e_8 can be taken so that $\varepsilon_8 = 0,\ g_{a8} = 1$ all others $g_{\alpha\beta} = 0$.

Let us start with the case (i). Thus among 2-forms Ω^β_α can be non-zero the three following

$$\Omega^6_5 = -\varepsilon\Omega^5_6, \quad \Omega^4_5 = -\varepsilon_5\Omega^5_7, \quad \Omega^4_6 = -\varepsilon_6\Omega^6_7,$$

only. Here $\varepsilon = 1$, if $\varepsilon_5 = \varepsilon_6$ and $\varepsilon = -1$, if $\varepsilon_5 = -\varepsilon_6$. Supposing that at least $\Omega^4_5 \neq 0$ in the semiparallelity condition (2.8) by $i = j = 1,\ \alpha = 4$ one has $h^5_{11} = 0$. Moreover, without loss of generality can be taken

$$h^4_{11} \neq h^4_{22},\ h^5_{12} \neq 0 \tag{5.18}$$

The semiparallelity condition by $i=1,\ j=2,\ \alpha=4$: $(h^4_{22}-h^4_{11})\Omega^2_1+h^5_{12}\Omega^4_5-h^6_{12}\Omega^4_6=0$, gives for the coefficient of $\omega^1\wedge\omega^2$

$$(h^4_{22}-h^4_{11})\Big[\varepsilon_5(h^5_{12})^2+\varepsilon_6(h^6_{12})^2\Big]=0, \tag{5.19}$$

i.e. the version $\varepsilon_5=\varepsilon_6$ leads to the contradiction with (5.18). It remains to consider the case where $\varepsilon_5=-\varepsilon_6$, thus from (5.19) one has

$$h^5_{12}=h^6_{12}=a, \qquad a\neq 0. \tag{5.20}$$

(case $h^5_{12}=-h^6_{12}=a$ can be obtained from the previous if to take the vector $-e_6$ instead of e_6). Now from equations by $i=1,2,\ j=3,\ \alpha=4$

$$\begin{aligned}(h^4_{33}-h^4_{11})\Omega^3_1-h^5_{13}\Omega^4_5-h^6_{13}\Omega^4_6=0,\\ (h^4_{33}-h^4_{22})\Omega^3_2-h^5_{23}\Omega^4_5-h^6_{23}\Omega^4_6=0,\end{aligned}$$

as a coefficients of $\omega^2\wedge\omega^3$ and $\omega^1\wedge\omega^3$, accordingly, one has

$$\begin{aligned}(h^4_{33}-h^4_{11})\Big[h^5_{12}h^5_{33}-h^6_{12}h^6_{33}\Big]+(h^4_{11}+h^4_{22}-2h^4_{33})\Big[h^5_{13}h^5_{23}-h^6_{13}h^6_{23}\Big]=0,\\ (h^4_{33}-h^4_{22})\Big[h^5_{12}h^5_{33}-h^6_{12}h^6_{33}\Big]+(h^4_{11}+h^4_{22}-2h^4_{33})\Big[h^5_{13}h^5_{23}-h^6_{13}h^6_{23}\Big]=0,\end{aligned}$$

where from on supposition (5.20) can be obtained $h^5_{33}=h^6_{33}$. Due to $H^5=H^6=0$ it leads to $h^5_{22}=h^6_{22}=-h^5_{33}=-h^6_{33}=\mu$.

Denoting $h^5_{13}=b, h^6_{13}=\beta,\ h^5_{23}=c,\ h^6_{23}=\gamma$ in $i=j=1,\ \alpha=5,6$ as a coefficients of $\omega^1\wedge\omega^2$ one has $a(b^2-b\beta)=0,\ a(b\beta-\beta^2)=0$, accordingly, i.e.

$$b(b-\beta)=0, \qquad \beta(b-\beta)=0.$$

Here $b\neq\beta$ is impossible because it leads to $b=\beta=0$, so one has $\beta=b$.

Now let us investigate a relation between γ and c. From the equations by $i=j=2,\alpha=5$ and $i=j=3,\alpha=6$ as a coefficients for $\omega^1\wedge\omega^3$ one has

$$(c-\gamma)(a\mu-2bc)=0, \qquad (c-\gamma)(a\mu-2b\gamma)=0,$$

accordingly. If here $\gamma\neq c$ then $b(c-\gamma)=0$ and $b=0$. Thus $\mu=0$ and from $i=j=2,\alpha=6$ as a coefficient of $\omega^1\wedge\omega^2:\quad (c-\gamma)(2a\gamma-b\mu)=0$, one has $\gamma=0$. The substitution into $i=j=2,\alpha=5,\ \omega^2\wedge\omega^3:\quad (c-\gamma)(a^2-\mu^2+c\gamma+c^2)=0$, gives $a^2+c^2=0$, which leads to contradiction with $a\neq 0$. Now $\gamma=c$ and the vectors h_{ij} are following

$$\begin{aligned}&h_{11}=h^4_{11}e_4, && h_{12}=a(e_5+e_6), && h_{13}=b(e_5+e_6),\\ &h_{22}=h^4_{22}e_4+\mu(e_5+e_6), && h_{23}=c(e_5+e_6), && h_{33}=h^4_{33}e_4-\mu(e_5+e_6).\end{aligned}$$

It means that not more then two vectors h_{ij} can be linearly independent, i.e. the dimension of the principal normal space is smaller then 3.

The next step is to consider possibility $\Omega_5^4 = 0$. It means that at least one from 2-forms

$$\Omega_5^6 = \varepsilon_5 \sum_{i,j=1}^{3} (h_{jj}^5 - h_{ii}^5) h_{ij}^6 \omega^i \wedge \omega^j, \quad \Omega_6^4 = -\varepsilon_6 \sum_{i,j=1}^{3} (h_{jj}^4 - h_{ii}^4) h_{ij}^6 \omega^i \wedge \omega^j,$$

$(i > j)$ is non-zero. Due to the semiparallelity condition by $i = j = 2, 3,\ \alpha = 4, 5$, where $h_{22}^6 \Omega_6^{4,5} = 0,\ h_{33}^6 \Omega_6^{4,5} = 0$ leads to $h_{22}^6 = h_{33}^6 = 0$. Now the equation

$$i = 1, 2\ j = 3,\ \alpha = 6 \qquad h_{12}^6 \Omega_2^3 - h_{23}^6 \Omega_1^2 = 0, \quad h_{12}^6 \Omega_1^3 + h_{13}^6 \Omega_1^2 = 0,$$

as a coefficients for $\omega^1 \wedge \omega^2$ give $h_{13}^6 (h_{12}^6)^2 = 0, \qquad h_{23}^6 (h_{12}^6)^2 = 0.$

Here the relation $h_{12}^6 = h_{13}^6 = h_{23}^6 = 0$ is impossible due to supposition that at least one of Ω_6^5, Ω_6^4 is non-zero. The possibility $h_{13}^6 = h_{23}^6 = 0,\ h_{12}^6 \neq 0$ is impossible due to the same reason. Because the equations

$$\begin{array}{ll}
i = 1, j = 3, \alpha = 6,\ \omega^2 \wedge \omega^3 & h_{12}^6 \Big[\varepsilon_5 l(k + l) + 2\varepsilon_6 (h_{23}^6)^2\Big] = 0, \\
i = 2, j = 3, \alpha = 6,\ \omega^1 \wedge \omega^3 & h_{12}^6 \Big[\varepsilon_5 k(k + l) + 2\varepsilon_6 (h_{13}^6)^2\Big] = 0, \\
i = 1, j = 2, \alpha = 5,\ \omega^1 \wedge \omega^2 & (l - k)(h_{12}^6)^2 = 0 \\
i = 1, j = 2, \alpha = 4,\ \omega^1 \wedge \omega^2 & (h_{12}^6)^2 (h_{22}^4 - h_{11}^4) = 0,
\end{array}$$

lead to $\Omega_6^5 = \Omega_6^4 = 0$.

It remains to consider the possibility $h_{12}^6 = 0$. The case with $h_{13}^6 h_{23}^6 \neq 0$, (i.e. where h_{13}^6, h_{23}^6 are non-zero simultaneously) is impossible due to the system

$$\begin{array}{ll}
i = 1, j = 3, \alpha = 5,\ \omega^2 \wedge \omega^3 & h_{13}^6 h_{23}^6 (2k + l) = 0, \\
i = 2, j = 3, \alpha = 5,\ \omega^1 \wedge \omega^3 & h_{13}^6 h_{23}^6 (2l + k) = 0, \\
i = 1, j = 2, \alpha = 6,\ \omega^2 \wedge \omega^3 & h_{13}^6 \Big[\varepsilon_5 l(k + l) + 2\varepsilon_6 (h_{23}^6)^2\Big] = 0,
\end{array} \tag{5.21}$$

where from $h_{13}^6 h_{23}^6 = 0$. Now without loss of generality can be considered $h_{23}^6 = 0$, $h_{13}^6 \neq 0$. The such supposition give in (5.21) that $h_{13}^6 l(k + l) = 0$ and the system

$$\begin{array}{ll}
i = 2, j = 2, \alpha = 6,\ \omega^1 \wedge \omega^3 & h_{13}^6 l(2k + l) = 0, \\
i = 1, j = 3, \alpha = 4,\ \omega^1 \wedge \omega^3 & (h_{33}^4 - h_{11}^4) \Big[\varepsilon_5 k(k + l) + 2\varepsilon_6 (h_{13}^6)^2\Big] = 0,
\end{array}$$

where from both 2-forms Ω_6^4 and Ω_6^5 are zero, i.e. it is obtained a contradiction with condition of normally non-flat connection.

It remains to consider the case (ii), where one from vectors e_5, e_6 is isotropic. Without loss of generality here can be supposed $\varepsilon_5 = 0$, $\varepsilon_6 \neq 0$ and the vector e_8 is such that $\varepsilon_8 = 0$, $g_{58} = 1$. For the such type of metric one has three 2-forms $\Omega^4_6 = -\varepsilon_6 \Omega^6_7$, $\Omega^5_6 = -\varepsilon_6 \Omega^6_8$, $\Omega^5_7 = -\Omega^4_8$, which can be non-zero.

Let suppose at first, that all these forms Ω^4_6, Ω^5_6, Ω^5_7 are non-zero, simultaneously. If $\Omega^2_1 = 0$, then due to

$$\Omega^2_1 = -\varepsilon_6[-(h^6_{12})^2\omega^1 \wedge \omega^2 - h^6_{12}h^6_{13}\omega^1 \wedge \omega^3 + (h^6_{12}h^6_{23} - h^6_{22}h^6_{13})\omega^2 \wedge \omega^3],$$

one has $h^6_{12} = 0,\ h^6_{22}h^6_{13} = 0$. Together with $h^6_{22} = -h^6_{33} = \mu$ (from $H^6 = 0$), it gives in semiparallity condition (2.8) by $i = 2,\ j = 3,\ \alpha = 6$

$$(h^6_{33} - h^6_{22})\Omega^3_2 - h^6_{12}\Omega^3_1 - h^6_{13}\Omega^2_1 = 0,$$

i.e. $\mu\Omega^3_2 = 0$, where $\Omega^3_2 = \varepsilon_6 h^6_{13}h^6_{23}\omega^1 \wedge \omega^3 + \varepsilon_6(\mu^2 + (h^6_{23})^2)\omega^2 \wedge \omega^3$. The latter leads to $\mu(\mu^2 + (h^6_{23})^2) = 0$, i.e.$\mu = 0$. Now the semiparallelity condition by $i = j = 1, 2,\ \alpha = 6$ transforms into $h^6_{13}\Omega^3_1 = 0$, $h^6_{23}\Omega^3_2 = 0$, i.e. $h^6_{13} = h^6_{23} = 0$. But it gives a contradiction with non-zero forms $\Omega^{4,5}_6$. As a result one has $\Omega^2_1 \neq 0$.

Denoting $h^6_{12} = a,\ h^6_{13} = b$ in $i = j = 1,\ \alpha = 6 : a\Omega^2_1 + b\Omega^3_1 = 0$ as coefficients by $\omega^1 \wedge \omega^2$ and $\omega^1 \wedge \omega^3$ one has

$$a(a^2 + b^2) = 0,\ \ b(a^2 + b^2) = 0. \tag{5.22}$$

It leads to $a = b = 0$, which is impossible due to $\Omega^2_1 \neq 0$. One obtained that at least one among 2-forms Ω^4_6, Ω^5_6, Ω^5_7 must be zero.

Let at first, $\Omega^5_7 = 0$, then $h^4_{ij} = h^5_{ij} = 0,\ i \neq j$. Since at least one of Ω^4_6, Ω^5_6 is non-zero from $i = j = 2, 3,\ \alpha = 4, 5$ one has $h^6_{22} = h^6_{33} = 0$ and

$$\begin{aligned}\Omega^{4,5}_6 = -\varepsilon_6[&(h^{4,5}_{22} - h^{4,5}_{11})h^6_{12}\omega^1 \wedge \omega^2 + (h^{4,5}_{33} - h^{4,5}_{11})h^6_{13}\omega^1 \wedge \omega^3 + \\ &(h^{4,5}_{33} - h^{4,5}_{22})h^6_{23}\omega^2 \wedge \omega^3].\end{aligned}$$

Without loss of generality, it is enough to consider $\Omega^5_6 \neq 0$, only. If suppose here $h^5_{11} \neq h^5_{22},\ h^6_{12} \neq 0$, then the same way as it was done in previous Proposition 5.7 can be obtained a contradiction with a such supposition.

Let us suppose now that $\Omega^4_6 = 0$. Then one has $h^6_{ij} = h^4_{ij} = 0,\ i \neq j$. Due to the metric one has $\Omega^2_1 = \Omega^3_1 = 0$ and $\Omega^3_2 = -\varepsilon_6 h^6_{22}h^6_{33}\omega^2 \wedge \omega^3 = \varepsilon_6\mu^2\omega^2 \wedge \omega^3$. The substitution into

$$i = 2,\ j = 3,\ \alpha = 6 : \qquad (h^6_{33} - h^6_{22})\Omega^3_2 = 0$$

leads to $\mu = 0$, i.e. $h^6_{ij} = 0$ for every i,j and $\dim N_x M^3 \leq 2$. □

Proposition 5.9. *Let M^3 be a normally non-flat semiparallel space-like submanifold with three-dimensional principal normal space of singular non-vanishing metric in pseudo-Euclidean space E^n_s. Then the frame vectors e_4, e_5, e_6 are such that $\varepsilon_4 \neq 0$, $\varepsilon_5 = \varepsilon_6 = 0$ and the Pfaff system is*

$$\begin{array}{lll} \omega_1^4 = 0, & \omega_1^5 = k\omega^1, & \omega_1^6 = \alpha\omega^2, \\ \omega_2^4 = 0, & \omega_2^5 = l\omega^2, & \omega_2^6 = \alpha\omega^1 + \mu\omega^2, \\ \omega_3^4 = \rho\omega^3, & \omega_3^5 = -(k+l)\omega^3, & \omega_3^6 = -\mu\omega^3, \end{array} \tag{5.23}$$

all others $\omega_i^{\overline{a}}$, ω_i^{ξ}, $\overline{a} = 7, 8$; $\xi = 9, ..., n$ are zero.

Proof. It is enough to consider the case where the mean curvature vector is non-isotropic. On this assumption and due to $H \parallel e_4$ one has that $\varepsilon_4 \neq 0$. Moreover, the 2-form $\Omega_4^5 = -\varepsilon_4 \omega_i^4 \wedge \omega_i^5$ is zero due to (5.13) and the matrices $\parallel h_{ij}^4 \parallel$, $\parallel h_{ij}^5 \parallel$ commute and it can be made $h_{ij}^4 = h_{ij}^5 = 0$ $(i \neq j)$. Now by the frame vectors e_5, e_6 there are two possibilities: only one of them is isotropic, or the both vectors are isotropic, i.e.

(i) ε_a is either 1 or -1, $\varepsilon_b = 0$, $(a, b = 5, 6)$; the next frame vector e_7 can be taken so that $\varepsilon_7 = 0$ and $g_{b7} = 1$, all others $g_{\alpha\beta} = 0$;

(ii) $\varepsilon_5 = \varepsilon_6 = 0$; the two next frame vectors e_7, e_8 can be taken so that $\varepsilon_7 = \varepsilon_8 = 0$ and $g_{57} = g_{68} = 1$, all others $g_{\alpha\beta} = 0$.

Let us start with the case (i), where only one from vectors e_5, e_6 is isotropic. Without loss of generality here can be taken $\varepsilon_6 \neq 0$, $\varepsilon_5 = \varepsilon_7 = 0$ and $g_{57} = 1$, all others $g_{\alpha\beta} = 0$. Now 2-form

$$\Omega_6^5 = -\varepsilon_6[(h_{22}^5 - h_{11}^5)h_{12}^6\omega^1 \wedge \omega^2 + (h_{33}^5 - h_{11}^5)h_{13}^6\omega^1 \wedge \omega^3 + (h_{33}^5 - h_{22}^5)h_{23}^6\omega^2 \wedge \omega^3]$$

can be non-zero, only. The same way as done in Proposition 5.7 (the case $H^2 \neq 0$) one can suppose (5.15) and obtain that $h_{11}^5 h_{13}^6 = 0$, $h_{22}^5 h_{23}^6 = 0$. On the other hand the equations by $i = j = 1, 2$, $\alpha = 6$ for the present case

$$h_{12}^6\Omega_1^2 + h_{13}^6\Omega_1^3 = 0, \qquad h_{12}^6\Omega_1^2 - h_{23}^6\Omega_2^3 = 0,$$

show that h_{13}^6 and h_{23}^6 are non-zero, i.e. $h_{11}^5 = h_{22}^5 = 0$, which leads to contradiction with (5.15).

It remains to consider the case (ii), where $\varepsilon_4 \neq 0$, $\varepsilon_5 = \varepsilon_6 = 0$ and the vectors e_7, e_8 are such that $\varepsilon_7 = \varepsilon_8 = 0$, $g_{57} = g_{68} = 1$. Among 2-forms Ω_α^β can be non-zero

$$\Omega_8^5 = -(h_{22}^5 - h_{11}^5)h_{12}^6\omega^1 \wedge \omega^2 - (h_{33}^5 - h_{11}^5)h_{13}^6\omega^1 \wedge \omega^3 - (h_{33}^5 - h_{22}^5)h_{23}^6\omega^2 \wedge \omega^3 = -\Omega_7^6,$$

only. Supposing here $h^5_{11} \neq h^5_{22}$ and $h^6_{12} \neq 0$, from the semiparallelity condition by $i = 1$ $j = 2$, $\alpha = 5$: $(h^5_{22} - h^5_{11})\Omega^2_1 = 0$ one has $\Omega^2_1 = 0$. The latter gives

$$h^4_{11}h^4_{22} = 0. \tag{5.24}$$

On the other hand, since $\Omega^6_4 = 0$ one has $h^6_{ij}(h^4_{jj} - h^4_{ii}) = 0$, which on supposition $h^6_{12} \neq 0$ together with (5.24) leads to $h^4_{11} = h^4_{22} = 0$ and as a corollary

$$h^4_{33}h^6_{13} = 0, \quad h^4_{33}h^6_{23} = 0.$$

But in case $h^4_{33} = 0$ the dimension of N_xM^3 is smaller than 3, which means that $h^4_{33} \neq 0$ and $h^6_{13} = h^6_{23} = 0$. Denoting

$$h^5_{11} = k, \quad h^5_{22} = l, \quad h^6_{22} = \mu, \quad h^4_{33} = \rho, \quad h^6_{12} = \alpha,$$

due to $H^5 = H^6 = 0$ one has $h^5_{33} = -(k + l)$, $h^6_{33} = -\mu$. Thus the Pfaff system (5.23) is obtained. □

Proposition 5.10. *Let M^3 be a normally non-flat space-like semiparallel submanifold in E^n_s with three-dimensional principal normal subspace. Then it is either*

1) a product $V^2 \times E^1 \in E^6_s$, $s = 0, 3$, where V^2 is a Veronese surface in $S^4_\tau \in E^5_\tau$, $\tau \in \{0, 2\}0, 2$, or

2) a second order envelope of a family of $V^2 \times E^1 \in E^n_s$, $n > 6$, or

3) a $3-$dimensional Segre submanifold M^3 with orthogonal net of great 2-spheres and great circles in a sphere $S^5_s \in E^6_s$, (s is 0 or 3), or

4) a submanifold M^3 in E^6_s generated by an $1-$parametric family of concentric $2-$spheres, the orthogonal trajectories of which are the congruent logarithmic spirals (specially circles) with the common pole in the center of family spheres, or

5) a translation submanifold of S^1_τ $\tau \in \{0, 1\}$ and parallel M^2 with two families of parabola generators, which can be represented by equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + h_{01}u + h_{02}v$, where coefficients h_{0i} are some constant vectors, or

6) a 2nd order envelope of a family of submanifolds 5) with some arbitrariness.

Proof. The consideration of submanifolds 1)–4) is done in [11] and [18]. It remains to consider submanifolds from Proposition 5.9 with the singular non-vanishing metric and the Pfaff system (5.23). Here $\Omega^j_i = 0$ and the same way as in Proposition 2.2 can be obtained $\omega^{j'}_{i'} = 0$. The exterior differentiation of $\omega^7_i = \omega^8_i = 0$ together with $\omega^7_5 = \omega^8_6 = 0$ and $\omega^8_5 = -\omega^7_6$ leads to

$$k\omega^7_6 \wedge \omega^1 = 0, \quad \alpha\omega^7_6 \wedge \omega^2 = 0, \quad \omega^7_6 \wedge (\alpha\omega^1 + \mu\omega^2) = 0,$$

$$(\rho\omega^7_4 - \mu\omega^7_6) \wedge \omega^3 = 0, \quad (\rho\omega^8_4 + (k + l)\omega^7_6) \wedge \omega^3 = 0,$$

i.e. applying the Cartan's lemma one has $\omega^7_6 = 0$, $\omega^7_4 = X\omega^3$, $\omega^8_4 = Y\omega^3$. On the other hands $\omega^7_4 = -\varepsilon_4\omega^4_5$, $\omega^8_4 = -\varepsilon_4\omega^4_6$ and together with the exterior differentiation of the first column from (5.23)

$$\begin{aligned} &k\omega^4_5 \wedge \omega^1 + \alpha\omega^4_6 \wedge \omega^2 = 0 \\ &\alpha\omega^4_6 \wedge \omega^1 + (\mu\omega^4_6 + l\omega^4_5) \wedge \omega^2 = 0 \\ &(d\rho - (k + l)\omega^4_5 - \mu\omega^4_6) \wedge \omega^3 = 0 \end{aligned}$$

it gives $\omega_5^4=\omega_6^4=\omega_4^7=\omega_4^8=0$, $d\rho=A\omega^3$. Now the exterior differentiation of two others columns from (5.23) and $\omega_i^\xi=0$ gives

$$\begin{aligned}
&(dk+k\omega_5^5)\wedge\omega^1+\alpha\omega_6^5\wedge\omega^2=0,\\
&\alpha\omega_6^5\wedge\omega^1+(dl+l\omega_5^5+\mu\omega_6^5)\wedge\omega^2=0,\\
&(d(k+l)-\rho\omega_4^5+(k+l)\omega_5^5+\mu\omega_6^5)\wedge\omega^3=0,\\
&k\omega_5^6\wedge\omega^1+(d\alpha+\alpha\omega_6^6)\wedge\omega^2=0,\\
&(d\alpha+\alpha\omega_6^6)\wedge\omega^1+(d\mu+l\omega_5^6+\mu\omega_6^6)\wedge\omega^2=0,\\
&(d\mu-\rho\omega_4^6+(k+l)\omega_5^6+\mu\omega_6^6)\wedge\omega^3=0,\\
&k\omega_5^\xi\wedge\omega^1+\alpha\omega_6^\xi\wedge\omega^2=0,\\
&\alpha\omega_6^\xi\wedge\omega^1+(l\omega_5^\xi+\mu\omega_6^\xi)\wedge\omega^2=0,\\
&(\rho\omega_4^\xi-(k+l)\omega_5^\xi-\mu\omega_6^\xi)\wedge\omega^3=0.
\end{aligned}$$

Due to the Cartan's lemma

$$\begin{aligned}
&d\rho=A\omega^3,\quad \alpha\omega_6^5=B\omega^1+C\omega^2,\quad dl+l\omega_5^5+\mu\omega_6^5=C\omega^1+D\omega^2\\
&dk+k\omega_5^5=E\omega^1+B\omega^2,\quad d(k+l)-\rho\omega_4^5+(k+l)\omega_5^5+\mu\omega_6^5=F\omega^3,\\
&k\omega_5^6=K\omega^1+L\omega^2,\quad d\alpha+\alpha\omega_6^6=L\omega^1+M\omega^2,\\
&d\mu+l\omega_5^6+\mu\omega_6^6=M\omega^1+N\omega^2,\quad d\mu-\rho\omega_4^6+(k+l)\omega_5^6+\mu_6^6=R\omega^3,\\
&k\omega_5^\xi=X^\xi\omega^1+Y^\xi\omega^2,\quad \alpha\omega_6^\xi=Y^\xi\omega^1+(\tfrac{l}{k}X^\xi+\tfrac{\mu}{\alpha}Y^\xi)\omega^2,\\
&\rho\omega_4^\xi-(k+l)\omega_5^\xi-\mu\omega_6^\xi=W^\xi\omega^3.
\end{aligned}$$

The basis of the left sides consists of $d\rho$, ω_4^5, ω_6^5, $dl+l\omega_5^5$, $dk+k\omega_5^5$, ω_4^6, ω_5^6, $d\alpha+\alpha\omega_6^6$, $d\mu+\mu\omega_6^6$, ω_4^ξ, ω_5^ξ, ω_6^ξ. The ranks s_1, s_2 of the polar systems are $s_1=7+3(n-9)$, $s_2=2$. The Cartan's criterion is satisfied, the Pfaff system (5.23) is compatible and determines considered space-like M^3 with arbitrariness of two real function of two variables.

For the corresponding parallel submanifolds the equations

$$\begin{aligned}
&d\rho=\omega_5^4=\omega_6^4=\omega_4^5=\omega_4^6=0,\quad \omega_4^\xi=\omega_5^\xi=\omega_6^\xi=0,\\
&\omega_5^5=-\tfrac{dk}{k}=-\tfrac{dl}{l},\quad \omega_6^6=-\tfrac{d\alpha}{\alpha}=-\tfrac{d\mu}{\mu},
\end{aligned}$$

can be added. In particular, it gives that $l=c_1k$, $\mu=c_2\alpha$, $c_1\neq 1$, c_2 are some constants. Now one has

$$\begin{aligned}
&dx=e_1\omega^1+e_2du^2+e_3\omega^3,\\
&de_1=ke_5\omega^1+\alpha e_6\omega^2,\\
&de_2=\alpha e_6\omega^1+(c_1ke_5+c_2\alpha e_6)\omega^2,\\
&de_3=(\rho e_4-k(1+c_1)e_5-c_2\alpha e_6)\omega^3,\\
&de_4=-\varepsilon_4\rho e_3\omega^3,\quad d(ke_5)=0,\quad d(\alpha e_6)=0.
\end{aligned}$$

The considered parallel M^3 lies in a $E^6_{0,2}$ spanned by the point x and vectors e_1, e_2, e_3, e_4, ke_5, αe_6. The lines $\omega^1 = \omega^2 = 0$ on the parallel M^3 are a S^1_τ $\tau \in \{0, 1\}$ in E^2_τ. The surfaces $\omega^3 = 0$ are parallel M^2 in $E^4_{0,2}$ with two families of parabola generators (see Proposition 3.2). □

5.5 The case of two-dimensional principal normal space

Unlike the Euclidean space the semiparallel submanifolds M^3 in E^n_s with $\dim N_x M^3 = 2$ can have a flat normal connection or non-flat normal connection.

Proposition 5.11. *Let M^3 be a semiparallel space-like submanifold in E^n_s with $\dim N_x M^3 = 2$. Then the normal connection is non–flat only in case of completely vanishing metric by the principal normal space.*

Proof. The frame vectors e_4, e_5 are in $N_x M^3$. Moreover, the mean curvature vector can be taken so that $H = H^4 e_4$, (i.e. $H^4 \neq 0$). Let show that the normally non–flat connection is possible in case of the completely vanishing metric of $N_x M^3$.

In case of the regular metric among 2-forms Ω^β_α can be non-zero $\Omega^5_4 = \varepsilon \Omega^4_5$ ($\varepsilon = -1$, if $\varepsilon_4 = \varepsilon_5$ and $\varepsilon = -1$ if $\varepsilon_4 = -\varepsilon_5$), only. But due to (5.12) a such 2-form is zero, i.e. normal connection is flat.

If the metric of the principal normal space is singular non-vanishing, then

$$\varepsilon_a \neq 0,\ \varepsilon_b = 0\ (a, b \text{ are } 4, 5 \text{ and } a \neq b),$$

the next frame vector e_6 can be taken so that $\varepsilon_6 = 0$ and $g_{b6} = 1$. Due to (1.11) among Ω^β_α only $\Omega^a_b (= -\Omega^a_6)$ can be non-zero. If $a = 4,\ b = 5$, then system (5.12) shows that a such 2-form is zero, i.e. normal connection in this case is flat only. On supposition $a = 5,\ b = 4$ the investigation is more complicated. Here only 2-form Ω^4_5 can be non-zero. After a rotation of the tangent part e_1, e_2, e_3 we can achieve $h^4_{ij} = 0,\ i \neq j$. Now the semiparallelity condition gives

$$\begin{aligned}
i = j,\ \alpha = 4: &\quad h^5_{11}\Omega^4_5 = 0,\ h^5_{22}\Omega^4_5 = 0,\ h^5_{33}\Omega^4_5 = 0, && (5.25)\\
i = 1,\ j = 2,\ \alpha = 4: &\quad (h^4_{22} - h^4_{11})\Omega^2_1 - h^5_{12}\Omega^4_5 = 0, && (5.26)\\
i = 1,\ j = 3,\ \alpha = 4: &\quad (h^4_{33} - h^4_{11})\Omega^3_1 - h^5_{13}\Omega^4_5 = 0, &&\\
i = 2,\ j = 3,\ \alpha = 4: &\quad (h^4_{33} - h^4_{22})\Omega^3_2 - h^5_{23}\Omega^4_5 = 0, && (5.27)
\end{aligned}$$

$$\begin{aligned}
i = j = 2,\ \alpha = 5: &\quad h^5_{12}\Omega^2_1 - h^5_{23}\Omega^3_2 = 0, && (5.28)\\
i = j = 1,\ \alpha = 5: &\quad h^5_{12}\Omega^2_1 + h^5_{13}\Omega^3_1 = 0, && (5.29)\\
i = j = 3,\ \alpha = 5: &\quad h^5_{13}\Omega^3_1 + h^5_{23}\Omega^3_2 = 0, &&\\
i \neq j,\ \alpha = 5: &\quad (h^5_{jj} - h^5_{ii})\Omega^j_i = 0, h^5_{23}[h^4_{11} - h^4_{33}] = 0 &&
\end{aligned}$$

The equalities (5.25) leads to $h^5_{11} = h^5_{22} = h^5_{33} = 0$. Thus one has

$$\Omega^4_5 = -\varepsilon_5(h^4_{ii} - h^4_{jj})h^5_{ij}\omega^i \wedge \omega^j \neq 0, \tag{5.30}$$

and without loss of generality can be supposed $h^4_{11} \neq h^4_{22}$ and $h^5_{12} \neq 0$. Now the substitution of

$$\Omega^2_1 = \varepsilon_5\Big[(h^5_{12})^2\omega^1 \wedge \omega^2 + h^5_{12}h^5_{13}\omega^1 \wedge \omega^3 - h^5_{12}h^5_{23}\omega^2 \wedge \omega^3\Big] \tag{5.31}$$

and (5.30) into (5.26) as coefficients by $\omega^1 \wedge \omega^3$ and $\omega^2 \wedge \omega^3$ gives

$$h^5_{13}[h^4_{22} - h^4_{33}] = 0, \quad h^5_{23}[h^4_{11} - h^4_{33}] = 0.$$

If here $h^5_{13} \neq 0$, then $h^4_{22} - h^4_{33} = 0$ and together with (5.27) it gives $h^5_{23} = 0$. The latter due to (5.28) leads to $\Omega^2_1 = 0$, from (5.31) it is easy to see that $h^5_{12} = 0$, i.e. one has a contradiction. In case $h^5_{13} = 0$ then due to (5.29) one has $\Omega^2_1 = 0$, which leads to the contradiction with $h^5_{12} \neq 0$, too.

It remains to consider the case when the metric of the principal normal space is completely vanishing. Here the frame vectors e_4, e_5 and the next two vectors e_6, e_7 can be taken so that

$$\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = 0 \text{ and } g_{46} = g_{57} = 1, \text{ all others } g_{\alpha\beta} = 0. \tag{5.32}$$

Taking, without loss of generality $h_{11} = e_4$ and $h_{12} = e_5$ for others h_{ij} one has

$$h_{22} = a_1h_{11} + a_2h_{12}, \; h_{33} = b_1h_{11} + b_2h_{12},$$

$$h_{13} = c_1h_{11} + c_2h_{12}, \; h_{23} = f_1h_{11} + f_2h_{12}$$

Thus the Pfaff system

$$\begin{array}{lll} \omega^4_1 = \omega^1 + c_1\omega^3, & \omega^4_2 = a_1\omega^2 + f_1\omega^3, & \omega^4_3 = c_1\omega^1 + f_1\omega^2 + b_1\omega^3 \\ \omega^5_1 = \omega^2 + c_2\omega^3, & \omega^5_2 = \omega^1 + a_2\omega^2 + f_2\omega^3, & \omega^5_3 = c_2\omega^1 + f_2\omega^2 + b_2\omega^3, \end{array} \tag{5.33}$$

and $\omega^{\overline{a}}_i = \omega^\xi_i = 0$, where $\overline{a} = 6, 7$, $\xi = 8, \ldots, n$, gives that among 2-forms Ω^β_α only

$$\begin{aligned} \Omega^5_6 = -\Omega^4_7 \;&=\; \big(a_1 - 1 + c_2f_1 - c_1f_2\big)\omega^1 \wedge \omega^2 + \\ &+\; \big(f_1 - c_2 + b_1c_2 - c_1b_2\big)\omega^1 \wedge \omega^3 + \\ &+\; \big(c_1 - a_1f_2 + f_1a_2 - f_1b_2 + b_1f_2\big)\omega^2 \wedge \omega^3 \end{aligned} \tag{5.34}$$

can be non-zero. □

Now can be considered a normally non-flat semiparallel space-like M^3 in a limit case with $a_q = b_q = c_q = f_q = 0$, i.e. $h_{22} = h_{33} = h_{13} = h_{23} = 0$. From (5.34) follows that $\Omega^5_6 = -\Omega^4_7$ is non-zero.

Proposition 5.12. *A normally non-flat semiparallel space-like M^3 with $dim N_x M^3 = 2$ is either*

1) a submanifold in $E^5_{0,2}$ with three families of generators (straight line and two parabolas); it can be represented by the equation

$$x = \frac{1}{2}h_{11}(u^1)^2 + h_{12}u^1u^2 + h_{01}u^1 + h_{02}u^2 + h_{03}u^3, \tag{5.35}$$

where coefficients h_{0i} are some constant vectors, or

2) a second order envelope of such submanifolds.

Proof. The Pfaff system (5.33) by exterior differentiation together with equalities $\omega_a^{\bar{b}} = 0$ leads to

$$\omega_4^{4,5,\xi} \wedge \omega^1 + \omega_5^{4,5,\xi} \wedge \omega^2 = 0, \quad \omega_5^{4,5,\xi} \wedge \omega^1 = 0, \tag{5.36}$$

Thus the basis of the secondary forms consists of ω_4^4, ω_5^4, ω_4^5, ω_5^5, ω_4^ξ, ω_5^ξ and the rank of polar matrices are $s_1 = 4+2(n-8)$, $s_2 = 0$, i.e the Cartan's number Q is $2n-12$. On the other hand applying the Cartan's lemma to (5.36) one has $4 + 2(n-8)$ independent coefficients. It means that the Cartan's criterion is satisfied and the considered submanifold exists with arbitrariness of $2n - 12$ real functions of one variable.

The corresponding parallel space-like M^3 is a submanifold from Proposition 2.2 with three families of parabola generators; it lies in $E^5_{0,2}$ and can be represented by the equation (5.35). □

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Summary in Estonian

Paralleelsed ja semiparalleelsed ruumisarnased madalamõõtmelised alammuutkonnad pseudoeukleidilises ruumis

Tähistame konstantse kõverusega ruumi vormi $N_s^n(c)$, kui tema kõverus on c. See on Riemanni ruum, kui $s = 0$ või $s = n$, ja pseudo-Riemanni ruum, kui $0 < s < n$. Riemanni ruumi ruumisarnast alammuutkonda M^m nimetatakse semiparalleelseks, kui suvaliste puutujavektorite X, Y korral kehtib $\overline{R}(X,Y)h = 0$, kus $\overline{R}$ on van der Waerdeni-Bortolotti seostuse $\overline{\nabla} = \nabla \oplus \nabla^{\perp}$ kõverusoperaator ja h on teine fundamentaalvorm.

Semiparalleelsete alammuutkondade klassis võib välja eraldada paralleelsete alammuutkondade alamklassi, kuhu kuuluvad paralleelse teise fundamentaalvormiga alammuutkonnad.

Eukleidilises ruumis, mille puhul $s = 0$ ja $c = 0$, on paralleelsed alammuutkonnad klassifitseeritud Feruse artiklis [2]. Takeuchi [3] ning Backesi ja Reickziegeli [4] artiklites on need tulemused laiendatud juhule $s = 0$, $c \neq 0$ ja Blomstromi [5] ja Naitoh′i [6] töödes pseudo-Riemanni ruumi vormi $N_s^n(c)$, $s > 0$ juhule. Mõned paralleelsete alammuutkondade klassid ruumides E_1^n ja E_2^n on kirjeldatud Magidi töös [7].

Semiparalleelsed alammuutkonnad ruumi vormis $N_s^n(c)$, kus $s = 0$, on klassifitseeritud ja kirjeldatud järgmistel juhtudel: pinnad ($m = 2$), kus $c = 0$ Deprez′i töös [8]; pinnad ($m = 2$), kus $c > 0$, Mercuri ja Asperti töödes [9] ja [10]. Kolmemõõtmelisi alammuutkondi juhul $m = n-2$ ning hüperpindu ($m = n-1$), kui $c = 0$, on põhjalikult uuritud Lumiste [11], [13], Lumiste ja Riivese [12] ning Deprez′i [14] artiklites. Alammuutkondi, kus normaalseostuse kõverus on võrdne nulliga, on käsitlenud Lumiste poolt [15], kui $c = 0$ ning Dilleni ja Nölkeri poolt [16], kui $c > 0$ töödes. Töös [17] on Lumiste poolt näidatud, et iga semiparalleelne alammuutkond on paralleelsete alammuutkondade teist järku mähismuutkond. Tema on esitanud üldise paralleelsete ja semiparalleelsete alammuutkondade teooria eukleidilises ruumis artiklites [18] ja [19]. Semiparalleelsed ajasarnased pinnad Lorentzi ruumis on kirjeldatud Lumiste artiklis [20].

Käesoleva väitekirja uurimisobjektideks on paralleelsed ja semiparalleelsed ruumisarnased madalamõõtmelised (M^1, M^2 ja M^3) alammuutkonnad pseudoeukleidilises ruumis E^n_s. Alammuutkondade geomeetriliste omaduste kirjeldamiseks kasutatakse Cartani liikuva reeperi meetodit ja Cartani välisdiferentsiaalarvutust.

Väitekirja **esimeses peatükis** tutvustatakse lugejale semiparalleelsete alammuutkondade teooria aluseid. On ära toodud põhimõisted, alates semipseudoeukleidilise alamruumi $E^k_{l,d}$ indeksi l ja defekti d definitsioonidest. Peatüki *esimeses paragrahvis* on konstrueeritud adapteeritud liikuv reeper ruumisarnasele alammuutkonnale $M^m \subset E^n_s$ ning Levi-Civita, normaal- ja van der Waerdeni-Bortolotti seostuste 2-vormid. Peale selle tuuakse sisse alammuutkonna M^m punktis $x \in M^m$ määratud normaalvektorruumi mõiste. *Teises paragrahvis* on esitatud ruumisarnase alammuutkonna M^m struktuurivormid pseudoeukleidilises ruumis E^n_s. Näidatakse, kuidas tekib isotroopne koonus C_x ja milline on selle võrrand. Näidatakse, et tasandeid, mis läbivad punkti x, saab liigitada selle järgi, kuidas nad asetsevad isotroopse koonuse suhtes, kui koonuse tipp asub punktis x. *Kolmandas paragrahvis* tegeldakse alammuutkonna M^m punktis $x \in M^m$ määratud peanormaalalamruumi N_xM^m mõistega ja uuritakse, milliste tingimuste korral peanormaalalamruum on regulaarse, singulaarse või kidunud meetrikaga.

Teises peatükis keskendutakse paralleelsete ja semiparalleelsete ruumisarnaste alammuutkondade teooria üldistele aspektidele. Eelkõige on siin toodud tulemused, mis on tõestatud m-mõõtmeliste ruumisarnaste alammuutkondade jaoks. *Esimeses paragrahvis* tõestatakse, et kui peanormaalalamruum on kidunud meetrikaga, siis paralleelne ruumisarnane alammuutkond $M^m \subset E^n_s$ on m paraboolse moodustaja parv, kusjuures mõni neist moodustajatest võib kiduda sirgjooneks. On tuletatud ka selle paralleelse ruumisarnase alammuutkonna võrrand. *Teises paragrahvis* on toodud semiparalleelseid alammuutkondi iseloomustavad tingimused ja on tõestatud, et iga ruumisarnane alammuutkond M^m pseudoeukleidilises ruumis E^n_s, mille peanormaalalamruum on kidunud meetrikaga, on semiparalleelne. Lisaks tõestatakse, et ruumi $E^{m+2n_1}_{0,n_1}$ semiparalleelse ruumisarnase alammuutkonna M^m normaalseostuse kõverus on null, kui selle peanormaalalamruum on n_1-mõõtmeline ja kidunud meetrikaga. *Peatüki lõpuosas* on toodud mõned mõisted, mis on seotud jooneparve teist järku mähispinna leidmisega ja esitatud Lumiste ja Riivese poolt saadud tulemused vastavalt Veronese ([18], [21], [22], [23], [24]) ja Segre ([25], [26], [27]) alammuutkondade teist järku mähismuutkondade kohta.

Kolmas peatükk on pühendatud semiparalleelsete ruumisarnaste joonte ja pindade klassifikatsioonile pseudoeukleidilises ruumis E^n_s ning nende geomeetrilisele kirjeldamisele. Need tulemused on publitseeritud autori poolt töös [**?**]. *Esimeses paragrahvis* on antud semiparalleelsete ruumisarnaste pindade klassifikatsiooniteoreem. On tõestatud, et ruumisarnane pind M^2 on semiparalleelne siis ja ainult siis, kui kas selle pinna iga punkt on ümaruspunkt (erijuhul on pind täielikult geodeetiline) või pinna seostuse $\overline{\nabla}$ kõverus võrdub nulliga või see on isotroopne pind, mille keskmine kõverus H rahuldab tingimust $||H||^2 = 3K$, kusjuures K on pinna Gaussi kõverus. Lisaks sellele on *teises paragrahvis* antud kahemõõtmeliste pindade M^2 detailsem klassifikatsioon juhul, kui seostus $\overline{\nabla}$ on tasane, kasutades asjaolu, et igaüks neist on paralleelsete pindade teist järku mähispind, ning on tõestatud, et iga paralleelne ruumisarnane joon M^1 pseudoeukleidilises ruumis E^n_s on kas sirgjoon või ringjoon, mille raadius võib olla nii reaalne kui imaginaarne, või parabool. Peatüki *kolmas paragrahv* annab ülevaate paralleelsete ruumisarnaste pindade teist järku mähispindade olemasolust ning suvast. Viimases paragrahvis on tõestatud, et eksisteerivad sellised maksimaalsed pinnad, mis pole täielikult geodeetilised.

Neljas peatükk on pühendatud tasase normaalseostusega semiparalleelsetele ruumisarnastele alammuutkondadele M^3. *Peatüki alguses* tuuakse sisse peasihtide ja peakõveruste mõisted ning kirjeldatakse peakõveruste vektoreid juhul, kui peanormaalalamruum on regulaarse, singulaarse või kidunud meetrikaga. On näidatud, et kui normaalseostus on tasane, siis peanormaalalamruum on ühe-, kahe- või kolmemõõtmeline. *Järgmises kolmes paragrahvis* antakse uuritavate alammuutkondade klassifikatsiooni.

Väitekirja **viimase peatüki** eesmärgiks on semiparalleelsete ruumisarnaste 3-mõõtmeliste mittetasase normaalseostusega alammuutkondade uurimine. *Esimeses paragrahvis* on antud vaadeldavate alammuutkondade M^3 geomeetriline kirjeldus juhul, kui peanormaalalamruum on 6-mõõtmeline. Erandlik olukord peatükis on seotud juhtumiga, kui $\dim N_xM^3 = 5$. Artiklis [11] on Lumiste näidanud, et eukleidilises ruumis E^n sellised semiparalleelsed alammuutkonnad M^3 puuduvad. Peatüki *teises paragrahvis* on tõestatud, et pseudoeukleidilises ruumis E^n_s $(s > 0)$ semiparalleelsed ruumisarnased alammuutkonnad M^3 leiduvad, ja on antud nende geomeetriline kirjeldus. *Järgmised kaks paragrahvi* käsitlevad tulemusi, mis on saadud juhul, kui N_xM^3 mõõde on vastavalt 4 või 3. *Viimases paragrahvis* uuritakse 2-mõõtmelise peanormaalalamruumi juhtumit. Selles olukorras tehakse kindlaks, et vaadeldavatel semiparalleelsetel alammuutkondadel M^3 on olemas kas parv kolmest moodustajast, täpsemalt sirgjoon ja kaks parabooli ruumis $E^5_{0,2}$, või niisugusteste alammuutkondade teist järku mähismuutkond. Sellised alammuutkonnad puuduvad eukleidilises ruumis E^n.

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