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TARTU ÜLIKOOLI TOIMETISED

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Matemaatika- ja mehaanikaalaseid töid

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THE SIXTIETH JUBILEE OF PROFESSOR ÜLO LUMISTE

Professor of Tartu University Ülo Lumiste, chairman of the Estonian Mathematical Society, celebrated his sixtieth jubilee on June 30, 1989.

Professor Ülo Lumiste is a well-known specialist in the field of differential geometry. His talent and results of fruitful scientific work are well reflected in his publications. He finished Vändra Secondary School with a gold medal in 1947 and graduated from the Faculty of Mathematics of Tartu University with the cum laude diploma in 1952. His dissertation for Candidate's degree was completed under the supervision of professor Anatoli Vassilyev at Moscow State University. The academic degree of Candidate of Physical and Mathematical Sciences was confered him in 1958. In 1960 he was elected Associated Professor at Tartu University. In the following years he was actively engaged in scientific research work. He defended his doctoral dissertation in 1968. From 1970 he is Professor of Tartu University and the Head of the Department of Algebra and Geometry. During the years 1974-1980 he was the Dean of the Faculty of Mathematics at Tartu University. The list of his publications contains 168 titles, half of them have been devoted to the problems of differential geometry and the rest consists of articles of history of mathematics, of foundations of geometry and of popular-scientific articles.

Professor Ülo Lumiste has actively taken part in numerous All-Union and international conferences. He is a member of the Bureau of the All-Union German Laptev Geometrical Seminar at VINITI of the Academy of Sciences of the U.S.S.R. He is a permanent reviewer of the journal "**РЖ Математика**", a member of council which confers doctoral degrees at Tbilisi University and at the Institute of Physics of the Academy of Sciences of the Estonian S.S.R. Under the supervision of professor Ülo Lumiste R.Mullari, M.Rahula, L.Tuulmets, K.Riives, A.Fleischer, A.Parring, R.Kolde, I.Maasikas, E.Abel, V.Mirzoyan, V.Abramov, M.Väljas and T.Virovere have finished and defended their dissertations in the years 1964-1989.

A survey of the scientific research of professor Ülo Lumiste together with the complete bibliography of his scientific works until 1979 has been printed in the journal "Tartu Ülik. Toimetised", No 610, 1980, p.3-14. Now we supplement this survey of bibliographical data with the list of publications of the last ten years.

In the first place it is necessary to mention the publications of professor Ülo Lumiste in the field of connection theory and its applications to quantum theory of gauge fields in their geometrical interpretations [106,111, 113,124,130,135,141]. First the main results of his doctoral thesis in the most complete form are given, i.e. the horisontability conditions of a distribution on the associated fibre bundle, transversal to the fibres and the corresponding splitting of curvature forms with respect to the order of isotropy.

Now we shall consider the above mentioned applications. It is known that Lagrangian consists of the invariant part, of the gauge fixing part and of the so-called Faddeev-Popov part. An important roll is played by the symmetrics of this Lagrangian. Here the relation between the BRS-symmetry and the structural equations of German Laptev for the corresponding bundle has been discovered by professor Ülo Lumiste. By means of forms and antiforms the new interpretation for Yang-Mills and Faddeev-Popov fields are given as well as for the field c together with antifield $\bar{c}$. The gauge transformations are interpreted as bundle automorphisms compatible with group action. The calculation of the continual Feynmann integral in the orbit factor-space is made clear by the noneffective group action. Together with Viktor Abramov [141] the supermanifold over the connection space and the corresponding supersymmetry have been introduced. The group action becomes effective. As a result a new direction of investigation has been established in applications of differential geometry in theoretical physics. The fact that the contribution of series of articles about connection theory in the 5-volumes Mathematical Ensyclopedia [115,128,137] was confined to professor Ülo Lumiste proves that he is a leading specialist in the theory of connection. Here we can also mention his plenary session report [135] at the VII All-Union Conference on Geometry and its Applications.

A new cycle of investigations by professor Ülo Lumiste began in 1984 when he again started to deal with the problems which are devoted to the differential geometry of submanifolds in spaces of constant curvature. It was the field of his dissertation for Candidate Degree. His attention was given to the submanifolds with the third parallel fundamental form $\alpha_3 = \nabla\alpha_2$ (i.e. with the property $\overline{\nabla}\alpha_3 = 0$), where α_2 denotes the second fundamental form and $\overline{\nabla}$ the van der Waerden-Bortolotti connection [125,138,143,145, 146,151,158,161]. The complete classification of lines (m=1), surfaces (m=2) and hypersurfaces (m=n-1) with this property in the Euclidean space E^n as well as of all submanifolds M^m with $\overline{\nabla}\alpha_3 = 0$ and with flat normal connection $\nabla^{\perp}$ in E^n have been given. The condition for integrability of the system $\overline{\nabla}\alpha_3 = 0$ is $\overline{R}\cdot\alpha_2 = 0$, where $\overline{R}$ is the curvature operator of the connection $\overline{\nabla}$. All submanifolds with this property professor Ülo Lumiste calls semi-symmetric [150, 153-155,166,167]. The classification of semi-symmetric surfaces and hypersurfaces in E^n was given by J.Deprez (who called them semi-parallel). In the above-mentioned articles professor Ülo Lumiste gaves the classification of all normally flat (i.e. with flat connection $\overline{\nabla}^{\perp}$, in particular with codimension 2) semi-symmetric submanifolds, showed that a semi-symmetric M^m with maximal first normal space in E^n, $n = (m+3)\frac{m}{2}$, is a maximal symmetric submanifold or a Veronese manifold. Recently the complete classification of 3-dimensional semi-symmetric submanifolds M^3 in E^n is given (see the present volume). These M^3 are discussed from the point of view of the general results that any semi-symmetric submanifold M^m in E^n is a second order envelope of the symmetric m-dimensional submanifolds in E^n gained by professor Ülo Lumiste (will be published in Proc. Acad. Sci. Estonia, Phys. Math.).

The other recently results obtained by professor Ülo Lumiste in the field of differential geometry of submanifolds [126,129,140,142,149,162] have been devoted to the study of the generalizations of geometrical Bäcklund transformation, the establishment of new properties of quasi-umbilic submanifolds (e.g. a contra-example for a false statement of B.-Y.Chen and T.H.Teng has been constructed) and others.

Moreover, professor Ülo Lumiste is the author of the first textbook of differential geometry and topology in the Estonian language [147,148]. He is the author of popular-scientific articles [116,127,144] and of numerous publications about the history of Mathematics at Tartu University and about the scientists whose life and activities have been connected with Tartu University.

M.Rahula
M.Abel

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Publications of Professor Ülo Lumiste
(the years 1980-1989*)

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** translated into English by Amer. Math. Soc.

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THREE-DIMENSIONAL SEMI-SYMMETRIC SUBMANIFOLDS WITH AXIAL, PLANAR OR SPATIAL POINTS IN EUCLIDEAN SPACES

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1. Introduction. A Riemannian manifold M^m satisfying $R(X,Y)\cdot R = 0$ (i.e. the integrability condition of the system $\nabla R = 0$, which characterizes a symmetric Riemannian manifold), is called a semi-symmetric Riemannian manifold [16, 12]. Analogically, a submanifold M^m in a Euclidean space E^n satisfying $\bar{R}(X,Y)\cdot h = 0$ (i.e. the integrability condition of the system $\bar{\nabla} h = 0$, which characterizes a symmetric submanifold M^m in E^n ; see [5]) is called a semi-symmetric submanifold M^m in E^n. Intrinsically it is a semi-symmetric Riemannian manifold. Here h is the second fundamental form of M^m in E^n, $\bar{\nabla}$ is the van der Waerden-Bortolotti connection [2, 10] and $\bar{R}$ its curvature operator.

The semi-symmetric surfaces and hypersurfaces in Euclidean spaces E^n are classified, respectively, in [3] and [4] (where the concept of a semi-symmetric submanifold is introduced under the denomination "semi-parallel submanifold").

T H E O R E M A. ([3]). *A semi-symmetric surface M^2 in E^n is either* (1) *an open part of a sphere S^2 in a 3-plane or* (2) *a surface with flat $\bar{\nabla}$ or* (3) *a 2nd order envelope of a family of Veronese surfaces V^2.*

Here the case (3) is formulated in a new way using the general fact that a semi-symmetric submanifold is the 2nd order envelope of a family of symmetric submanifolds (see [8]).

T H E O R E M B. ([4],[6]). *A semi-symmetric hypersur-*

face M^{n-1} *in* $E^n (n \geq 3)$ *is an open part of* (1) *a hyperplane* E^{n-1} *or* (2) *a* S^{n-1} *or* (3) *a round cone* C^{n-1} (*the point-vertex of which is to be excepted*; $n \geq 4$) *or* (4) *a product* $S^m \times E^{n-m-1}$ $(2 \leq m \leq n-2)$ *or* (5) *a product* $C^m \times E^{n-m-1}$ $(3 \leq m \leq n-2)$ *or* (6) *is a* M^{n-1} *with rank* 1 (*i.e. intrinsically flat* M^{n-1} *with* $(n-2)$*-dimensional plane generators*).

The semi-symmetric submanifolds M^{n-2} in E^n are found out in [6] where the fact is used that such M^{n-2} has always flat normal connection $\nabla^\perp$. More generally, all semi-symmetric submanifolds M^m with flat $\nabla^\perp$ in E^n are determined in [10, 15].

In this paper we classify and characterize geometrically all such semi-symmetric M^3 in E^n for which the linear span of $h(X,Y)$ has in every point $x \in M^3$ the dimension 1,2 or 3, i.e. which consist of axial, planar or spatial points respectively (see [11]). In the first two cases this classification follows from the results of papers cited above. We deal with these cases in Sections 3 and 4. The same can be said about the third case by the complementary condition that the normal connection $\nabla^\perp$ is flat (see Section 5).

The essentially new investigations are made in Sections 6 and 7 for the third case (i.e. of spatial points) by nonflat $\nabla^\perp$. It will be shown (see Theorem 4 below) that only four possibilities occur:

a semi-symmetric M^3 with spatial points and nonflat $\nabla^\perp$ in E^n is either (1) a part of the product $V^2 \times E^1$ in E^6 or (2) an 2nd order envelope in E^n $(n > 6)$ of a family of such products or (3) a M^3 in E^6 with 1-family of generating concentric 2-spheres, orthogonal trajectories of which are the certain plane curves, namely either (3_1) the concentric congruent circles, which have the same centre as the family 2-spheres or (3_2) the congruent logarithmic spirals with the common pole in the centre of family 2-spheres.

Here M^3 of cases (1) and (3_1) are symmetric. The cases (2) and (3_2) give the first examples of the semi-symmetric nonsymmetric submanifolds with nonflat normal connection.

In Section 2 we give the necessary apparatus. In Section 6 some examples are constructed which generalize the cases indicated just now and present some new semi-symmetric M^m with nonflat $\nabla^\perp$ in E^n. The main classification theorem is proved in Section 7.

Note that the maximal dimension of the span $h(X,Y)$ for a M^3 in E^n ($n \geq 9$) is 6. The semi-symmetric M^3 with this maximal value can be described using a more general result obtained in [7] for the particular case $m = 3$. It follows that a semi-symmetric M^3 with dim span $h(X,Y) = 6$ is either a Veronese submanifold V^3 in E^9 (and for $n = 9$ only this case is possible) or a 2nd order envelope of a family of Veronese submanifolds V^3, if $n > 9$. We announce that there are no semi-symmetric M^3 with dim span $h(X,Y) = 5$. It remains to investigate semi-symmetric M^3 with dim span $h(X,Y) = 4$. This is done in [9], which finishes also the classification of all semi-symmetric submanifolds M^3 in Euclidean spaces.

2. Apparatus. We use the Cartan moving frame method for the orthonormal frames $\{x, e_1, \dots, e_n\}$ in E^n:

$$dx = e_I \omega^I,\ de_I = e_J \omega^J_I,\ \omega^J_I + \omega^I_J = 0;\ I,J,\dots=1,\dots,n;$$
$$d\omega^I = \omega^J \wedge \omega^I_J,\ d\omega^J_I = \omega^K_I \wedge \omega^J_K .$$

For the frames adapted to M^m in E^n we have

$$\omega^\alpha = 0\ (\alpha=m+1,\dots,n) \Rightarrow \omega^\alpha_i = h^\alpha_{ij}\omega^j\ (i,j,\dots=1,\dots,m) \Rightarrow$$
$$\Rightarrow \bar\nabla h^\alpha_{ij} = (\bar\nabla_k h^\alpha_{ij})\omega^k \Rightarrow \bar\nabla(\bar\nabla_k h^\alpha_{ij})\wedge\omega^k = \bar\Omega h^\alpha_{ij}. \qquad (2.1)$$

Here

$$\bar\nabla h^\alpha_{ij} := dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta , \qquad (2.2)$$

$$\bar\Omega h^\alpha_{ij} := - h^\alpha_{kj}\Omega^k_i - h^\alpha_{ik}\Omega^k_j + h^\beta_{ij}\Omega^\alpha_\beta , \qquad (2.3)$$

$$\Omega^k_i := d\omega^k_i - \omega^j_i\wedge\omega^k_j = -\Sigma R^k_{i,pq}\,\omega^p\wedge\omega^q , \qquad (2.4)$$

$$\Omega^\alpha_\beta := d\omega^\alpha_\beta - \omega^\gamma_\beta\wedge\omega^\alpha_\gamma = -\Sigma R^\alpha_{\beta.pq}\,\omega^p\wedge\omega^q , \qquad (2.5)$$

$$R^k_{i.pq} = \sum_\alpha h^\alpha_{i[p} h^\alpha_{q]k} ,\ R^\alpha_{\beta,pq} = \sum_i h^\alpha_{i[p} h^\beta_{q]i} . \qquad (2.6)$$

The last quantities are the components of the curvature operator $\bar R$ of the van der Waerden-Bortolotti connection $\bar\nabla = \nabla\oplus\nabla^\perp$ and (2.4), (2.5) are its curvature 2-forms, the first ones of the Levi-Civita connection ∇ and the second ones of the normal connection $\nabla^\perp$.

Note that in (2.1) every $\Rightarrow$ is verified by exterior differentiation and using the Cartan lemma (in the first two cases). Thus h^α_{ij} - the components of the second fundamental form h are symmetric in i,j and $\bar\nabla_k h^\alpha_{ij}$ - the components of the third fundamental form $\bar\nabla h$ are symmetric in i,j,k (the Peterson-Codazzi equations).

In the case of symmetric M^m we have $\bar\nabla h = 0$ due to [5].

From (2.1) it follows that this leads to

$$\bar{\Omega} h^{\alpha}_{ij} = 0 \ , \qquad (2.7)$$

which is due to (2.3)-(2.5) a component-wise form of $\bar{R}\cdot h = 0$ and consequently it is a characteristic condition of the *semi-symmetric* submanifold.

The subspace span $h(X,Y) = \mathrm{span}(h^{\alpha}_{ij} e_{\alpha})$ in the normal space $T^{\perp}M^m$, where $X,Y \in T_x M^m$, is called the *first normal space* and denoted by $N^{(1)}_x M^m$. The point $x \in M^m$ is said to be *axial*, *planar* or *spatial* if dim $N^{(1)}_x M^m$ is 1, 2 or 3, respectively.

The vector $H = H^{\alpha} e_{\alpha}$ in $N^{(1)}_x M^m$, where $H^{\alpha} = \frac{1}{m} \sum_i h^{\alpha}_{ii} e_{\alpha}$, is called the mean curvature vector. Taking in (2.7) $i=j$ and summing by $i=1,\dots,m$, we get

$$H^{\beta}\ \Omega^{\alpha}_{\beta} = 0 \ . \qquad (2.8)$$

If $H^{\beta} = 0$, then (2.7) lead to $h = 0$ as follows from a result of [1] (Theorem 2, case (iii)). So $H \neq 0$ for a semi-symmetric M^m which is not a part of a plane E^m in E^n.

3. <u>Semi-symmetric M^m with axial points.</u> Here we can use the next

T h e o r e m C. (Segre C., see [10]). *A submanifold M^m with axial points in E^n is either a hypersurface in a $E^{m+1} \subseteq E^n$ or a M^m with rank* 1 *in E^n.*

In the last case let e_{m+1} be in $N^{(1)}_x M^m$ for every $x \in M^m$. Then $h^{\rho}_{ij} = 0$ ($\rho = m+2,\dots,n$). Due to rank $\|h^{m+1}_{ij}\| = 1$ we can take $e_1,\dots,e_m$ so that $h^{m+1}_{11} \neq 0$, $h^{m+1}_{22} = \dots = h^{m+1}_{mm} = 0$, $h^{m+1}_{ij} = 0$ ($i \neq j$). It follows from (2.6) that $\Omega^j_i = \Omega^{\alpha}_{\beta} = 0$, so M^m with rank 1 has flat $\bar{\nabla}$ and thus is semi-symmetric.

We have got the following light extension of Theorem B:

T H E O R E M 1. *A semi-symmetric submanifold M^m with axial points in E^n is either one of the hypersurfaces* (1)-(5) *of Theorem B in $E^{m+1} \subseteq E^n$ or* (6) *a submanifold M^m of rank* 1 *in E^n.*

C o r o l l a r y . A semi-symmetric M^3 with axial points in E^n is a part of (i) a sphere S^3 in a $E^4 \subset E^n$ or (ii) a round cone C^3 (with excepted point-vertex) in a $E^4 \subset E^n$ or (iii) a cylinder $S^2 \times E^1$ in a $E^4 \subset E^n$ or (iv) a M^3 with rank 1 in E^n.

4. <u>Semi-symmetric M^3 with planar points.</u> Taking e_4 and e_5 in $N^{(1)}_x M^3$ we have $h^{\rho}_{ij} = 0$ ($\rho = 6,\dots,n$). Thus due to (2.4)-(2.6) among Ω^{α}_{β} only Ω^5_4 can be non-zero. The condition (2.8) reduces to

$$H^4\Omega^5_4 = 0\ ,\quad H^5\Omega^5_4 = 0\ ,\quad \text{where } H \neq 0\ .$$

So $\Omega^5_4 = 0$ and therefore $\nabla^\perp$ is flat.

We can use the results on normally flat ($\equiv\nabla^\perp$ is flat) semi-symmetric submanifolds stated in [10] (see also [6]). Taking e_1, e_2, e_3 so that $h^\alpha_{ij} = k^\alpha_i \delta_{ij}$ we have

$$(k_i - k_j)\ \langle k_i, k_j\rangle = 0$$

due to (2.7), where $k_i = k^\alpha_i e_\alpha = k^4_i e_4 + k^5_i e_5$ are three principal curvature vectors, the span of which is the 2-dimensional $N^{(1)}_x M^3$. Here all possibilities can be obtained (by permutation of 1, 2, 3) from the next two:

1) $0 \neq k_1 \perp k_2 \neq 0\ ,\quad k_3 = 0\ ,$

2) $0 \neq k_1 \perp k_2 = k_3 \neq 0\ ,$

These are the cases $M^3_{(2,0,1)}$ and $M^3_{(1,1,0)}$ of [10,15] and therefore we get (see also [6,14])

T H E O R E M 2. *A semi-symmetric M^3 with planar points in E^n is either* (1) *a M^3 with rank* 2 *and flat* $\bar\nabla$ *or* (2) *an orthogonal type canal submanifold.*

In the case (1) M^3 is generated by straight lines through the points of a Cartan M^2 with flat $\bar\nabla$ in E^n so that the field of their directions is a normal vector field on M^2, parallel with respect to the flat $\nabla^\perp$ of M^2.

In the case (2) M^3 is the envelope of a 1-parameter family of spheres S^3 in E^n with the property that the first curvature vector of the orthogonal trajectory of the family of characteristic spheres S^2 is always perpendicular to the 4-plane of the family sphere.

5. Normally flat semi-symmetric M^3 with spatial points. Here we have the case $M_{(3,0,0)}$ of [10,15] and thus get

T H E O R E M 3. *A semi-symmetric M^3 with spatial points and flat $\nabla^\perp$ in E^n is a Cartan M^3 with flat $\bar\nabla$.*

Due to [8] this M^3 is the 2nd order envelope of a family of $S^1(r_1) \times S^1(r_2) \times S^1(r_3)$ (= rectangular Clifford 3-tori). Some more information on Cartan M^m with flat $\bar\nabla$ is given in [14].

6. Some examples of normally nonflat semi-symmetric submanifolds. Here we give some constructions of such submanifolds M^m in E^n. Further we show that in case of normally nonflat semi-symmetric M^3 with spatial points these are the only possible ones.

E x a m p l e 1. Let us have $M^3 = \tilde V^2 \times E^1$ where $\tilde V^2$ is

3

a 2nd order envelope of a family of Veronese surfaces V^2 in 5-planes of E^n. Here $\tilde{V}^2$ is semi-symmetric due to Theorem A, case (3); thus $\tilde{V}^2 \times E^1$ is semi-symmetric.

In [3] it is shown that for a $\tilde{V}^2$ the frame can be adapted, so that

$$\|h^3_{ab}\| = \begin{bmatrix} \nu\sqrt{3} & 0 \\ 0 & \nu\sqrt{3} \end{bmatrix}, \quad \|h^4_{ab}\| = \begin{bmatrix} \nu & 0 \\ 0 & -\nu \end{bmatrix}, \quad \|h^5_{ab}\| = \begin{bmatrix} 0 & \nu \\ \nu & 0 \end{bmatrix}, \quad h^\rho_{ab}=0,$$

$a,b = 1, 2$; $\rho = 6,\dots,n$; $\nu \neq 0$. It follows that for $\tilde{V}^2 \times E^1$ we can obtain $\omega^3_a = 0$,

$$\|h^4_{ij}\|=\begin{bmatrix} \nu\sqrt{3} & 0 & 0 \\ 0 & \nu\sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \|h^5_{ij}\|=\begin{bmatrix} \nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \|h^6_{ij}\|=\begin{bmatrix} 0 & \nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6.1}$$

and $h^\rho_{ij} = 0$, where $i,j = 1,2,3$; $\rho = 7,\dots,n$. Now from (2.4)-(2.6) it follows that $\Omega^6_5 = -2\nu\omega^1\wedge\omega^2$, so that $\nabla^\perp$ is nonflat. All points of $\tilde{V}^2 \times E^1$ are spatial.

In [8] it is shown that $\tilde{V}^2$ in E^5 is a single V^2, i.e. in case n = 5 we have ν = const. It follows that a $V^2 \times E^1$ in E^6 is characterized by (6.1) with ν = const.

E x a m p l e 2. Let M^3 in E^n $(n > 6)$ be the 2nd order envelope of a family of products $V^2 \times E^1$ in 6-planes. As $V^2 \times E^1$ in E^6 is symmetric then such M^3 in E^n is semi-symmetric due to [8]. All its points are spatial.

The 2nd order enveloping means that M^3 in every point $x \in M^3$ has the same tangent 3-plane and second fundamental form as the family submanifold $V^2 \times E^1$ (see [8]). So M^3 is characterized by (6.1) and $h^\rho_{ij} = 0$ (without restrictions ν = = const and $\omega^3_a = 0$), consequently, it is an integral submanifold of the system $\omega^4 = \omega^5 = \omega^6 = \omega^\rho = 0$ by

$$\left.\begin{array}{llll} \omega^4_1 = \nu\sqrt{3}\omega^1, & \omega^4_2 = \nu\sqrt{3}\omega^2, & \omega^4_3 = 0, & \\ \omega^5_1 = \nu\omega^1, & \omega^5_2 = -\nu\omega^2, & \omega^5_3 = 0, & \\ \omega^6_1 = \nu\omega^2, & \omega^6_2 = \nu\omega^1, & \omega^6_3 = 0, & \\ \omega^\rho_1 = 0, & \omega^\rho_2 = 0, & \omega^\rho_3 = 0, & \nu \neq 0. \end{array}\right\} \tag{6.2}$$

The equations of the last column give after exterior differentiation:

$$\omega^3_1\wedge\omega^1 + \omega^3_2\wedge\omega^2 = 0,$$
$$\omega^3_1\wedge\omega^1 - \omega^3_2\wedge\omega^2 = 0,$$

$$\omega_1^3\wedge\omega^2 + \omega_2^3\wedge\omega^1 = 0$$

and thus due to the Cartan lemma

$$\omega_1^3 = \tau\omega^1, \quad \omega_2^3 = \tau\omega^2. \qquad (6.3)$$

In the same manner these two equations lead to

$$d\tau = \tau^2\omega^3.$$

If here $\tau = 0$, we return to Example 1. It follows that $d(x + \frac{1}{\tau}e_3) = 0$, thus this 2nd order envelope M^3 is a cone which has the vertex in the point with the radius vector $x + \frac{1}{\tau}e_3$; every straight generator is determined by $\omega^1 = \omega^2 = 0$ and goes in the direction of e_3. Similarly, as in Example 1, we have $\Omega_\sigma^5 \neq 0$, so $\nabla^\perp$ is nonflat.

Announce here that the 2-order envelope of a family of $V^2 \times E^1$ in E^6 is a single $V^2 \times E^1$, i.e. in the case $n = 6$ this family degenerates to a single submanifold.

E x a m p l e 3. Let M^m in E^n be generated simultaneously by a m_1-parametric family of congruent concentric m_2-spheres and by another m_2-parametric family of congruent concentric m_1-spheres which have the same centre as the first family and intersect its m_2-spheres orthogonally ($m = m_1 + m_2$). It follows that such M^m, if it exists, is a minimal submanifold of a hypersphere S^{n-1}.

Let e_{m+1} have the direction of the radius and R be the length of the radius. Then $d(x + Re_{m+1}) = 0$ and thus

$$\omega_i^{m+1} = \frac{1}{R}\omega^i, \quad \omega_{m+1}^\rho = 0 \ (\rho = m+2,\dots,n),$$

$$de_i = \omega_i^j e_j + \frac{1}{R}\omega^i e_{m+1} + h_{ij}\omega^j, \quad de_{m+1} = -\frac{1}{R}\omega^i e_i,$$

where $h_{ij} = h_{ij}^\rho e_\rho$. Taking $e_1,\dots,e_{m_1}$ tangent to the family m_1-sphere we get that from $\omega^{a_2} = 0$ it must follow

$$\omega_{a_1}^{b_2} = 0, \quad h_{a_1 b_1} = 0$$

$(a_1, b_1, \dots = 1,\dots,m_1;\ a_2, b_2,\dots = m_1+1,\dots,m)$, thus

$$\omega_{a_1}^{b_2} = \Gamma_{a_1 c_2}^{b_2}\omega^{c_2}.$$

Here the roles of subindices $_1$ and $_2$ can be exchanged. This leads to

$$\omega_{a_1}^{b_2} = 0, \quad h_{a_1 b_1}^\rho = h_{a_2 b_2}^\rho = 0,$$

which together with $h^{m+1}_{ij} = \frac{1}{R}\delta_{ij}$ give

$$\bar{\nabla}h^{\alpha}_{a_1b_1} = \bar{\nabla}h^{\alpha}_{a_2b_2} = 0.$$

It follows from (2.1) that

$$\bar{\nabla}_k h^{\alpha}_{a_1b_1} = \bar{\nabla}_k h^{\alpha}_{a_2b_2} = 0$$

and due to the symmetry of $\bar{\nabla}_k h^{\alpha}_{ij}$ we have $\bar{\nabla}_k h^{\alpha}_{ij} = 0$. So the considered M^m is a symmetric submanifold in E^n and thus semi-symmetric.

From $\omega^{b_2}_{a_1} = 0$ we have after exterior differentiation that

$$-\frac{1}{R^2} + \langle h_{a_1b_2}, h_{a_1b_2}\rangle = 0, \quad \langle h_{a_1b_2}, h_{c_1d_2}\rangle = 0$$

if $a_1 \neq c_1$ or $b_2 \neq d_2$. It is convinient to denote $\frac{1}{R} = k$, the first m_1m_2 vectors among e_ρ by $e_{(a_1,b_2)}$, the other by e_ξ and take $e_{(a_1,b_2)} \uparrow\uparrow h_{a_1b_2}$. Then M^m is determined by

$$\omega^{m+1} = \omega^{(a_1,b_2)} = \omega^{\xi} = 0, \quad \omega^{m+1}_{a_1} = k\omega^{a_1}, \quad \omega^{m+1}_{a_2} = k\omega^{a_2},$$

$$\omega^{(a_1,b_2)}_{a_1} = k\omega^{b_2}, \; \omega^{(a_1,b_2)}_{b_2} = k\omega^{a_1},$$

$$\omega^{(c_1,d_2)}_{a_1} = 0 \; (a_1 \neq c_1), \quad \omega^{(c_1,d_2)}_{b_2} = 0 \; (b_2 \neq d_2),$$

$$\omega^{\xi}_{a_1} = \omega^{\xi}_{a_2} = 0, \quad \omega^{b_2}_{a_1} = 0, \quad \omega^{(a_1,b_2)}_{m+1} = 0, \quad \omega^{\xi}_{m+1} = 0.$$

By differential prolongation it follows that

$$\omega^{b_1}_{a_1} = \omega^{(b_1,c_2)}_{(a_1,c_2)}, \; \omega^{b_2}_{a_2} = \omega^{(c_1,b_2)}_{(c_1,a_2)}, \; \omega^{(c_1,d_2)}_{(a_1,b_2)} = 0 \quad (a_1 \neq c_1, b_2 \neq d_2),$$

$$\omega^{\xi}_{(a_1,b_2)} = 0.$$

If we add these new equations to the previous system, then the exterior differentiation gives the covariant equations which are satisfied due to the extended system. Thus the system is completely integrable, the considered M^m exists and, as we see, lies in the $(m + m_1m_2 + 1)$-plane, spanned on x, e_i, e_{m+1} and $h_{a_1b_2}$, and therefore in a $S^{m+m_1m_2}$. This M^m

is normally nonflat because $\Omega^{(a_1,c_2)}_{(a_1,b_2)} = -k^2\omega^{b_2}\wedge\omega^{c_2}$, $\Omega^{(c_1,a_2)}_{(b_1,a_2)} = -k^2\omega^{b_1}\wedge\omega^{c_1}$.

In the particular case, if m = 3, denoting $e_{(1,2)} = e_5$, $e_{(1,3)} = e_6$, we have a normally nonflat symmetric M^3 in E_6 which is determined by the completely integrable system which consists of

$$\left.\begin{array}{lll} \omega^4 = \omega^5 = \omega^6 = 0, & & \\ \omega_1^4 = k\omega^1, & \omega_2^4 = k\omega^2, & \omega_3^4 = k\omega^3, \\ \omega_1^5 = k\omega^2, & \omega_2^5 = k\omega^1, & \omega_3^5 = 0, \\ \omega_1^6 = k\omega^3, & \omega_2^6 = 0\ , & \omega_3^6 = k\omega^1 \end{array}\right\} \qquad (6.4)$$

and

$$\omega_1^2 = \omega_1^3 = \omega_4^5 = \omega_4^6 = 0, \quad \omega_2^3 = \omega_5^6.$$

This M^3 bears an orthogonal net of generating great 2-spheres and great circles of a $S^5 \subset E^6$. All its points are spatial.

Note that in the particular case, if m = 2, our M^2 is the Clifford surface in S^3. In E^4 it is a product of two circles which are bisectors of generating great circles of the two families. But the M^3 considered above is not a product.

There exist submanifolds M^3 with orthogonal 3-net of generating great circles in S^4, S^6 or S^7 (see [13]). They all are extrinsically homogeneous, but not symmetric or semi-symmetric.

E x a m p l e 4. Let M^m in E^n be generated by an 1-parametric family of concentric (m-1)-spheres and let the family of its orthogonal trajectories consist of geodesic lines of M^m which are congruent logarithmic spirals with the common pole in the centre of family spheres.

We take $e_2, \dots, e_m$ tangent to family sphere and e_{m+1} on the plane of orthogonal logarithmic spiral. Then $\omega^1 = 0$ determines a foliation of spheres and therefore

$$\omega_a^1 = k_a\omega^1 + k\omega^a, \quad \omega_a^{m+1} = h_{1a}^{m+1}\omega^1 + l\omega^a, \quad \omega_a^{\rho} = h_{1a}^{\rho}\omega^1 + l^{\rho}\omega^a$$

(a = 2,...,m). It follows that modω^a we have

$$de_1 = -\omega^1\sum_a k_a e_a + \omega_1^{m+1}e_{m+1} + \omega_1^{\rho}e_{\rho}.$$

As every orthogonal trajectory is a plane geodesic then

$k_a = 0$ and $\omega^\rho_1 = h^\rho_{1a}\omega^a$, i.e. $h^\rho_{11} = 0$. Now $d\omega^1 = 0$ and so $\omega^1 = ds$. It follows that $\omega^{m+1}_1 = \varkappa ds + h^{m+1}_{1a}\omega^a$, where $\varkappa = h^{m+1}_{11}$ is the curvature of the orthogonal trajectory. All these trajectories are supposed to be congruent logarithmic spirals.

It is well known that the logarithmic spiral with polar equation $\rho = a^\varphi$ has the curvature $\varkappa = s^{-1}\tan\mu = (s\ln a)^{-1}$, where s is the arc length from the pole and μ is the constant angle between the tangent vector and radius vector. In this notation $\rho = s\cos\mu$ and therefore the pole is determined by

$$c = x + s\cos\mu(e_{m+1}\sin\mu - e_1\cos\mu).$$

This vector must be constant. As

$$dc = [1 + \cos\mu(k\cos\mu - l\sin\mu)]\omega^a e_a +$$
$$+ s\cos\mu[\omega^1\Sigma h^{m+1}_{am} e_a + \Sigma(\omega^\rho_{m+1}\sin\mu - h^\rho_{1a}\omega^a\cos\mu)e_\rho]$$

we have

$$k\cos\mu - l\sin\mu = -(s\cos\mu)^{-1}, \; h^{m+1}_{1a} = 0, \; \omega^\rho_{m+1} = \omega^\rho_1\cot\mu.$$

The centre of every family sphere which is determined by

$$x + \frac{ke_1 + le_{m+1} + l^\rho e_\rho}{k^2 + l^2 + \Sigma(l^\rho)^2}$$

must coincide with this fixed pole. Thus

$$l^\rho = 0, \; \frac{l}{k^2+l^2} = s\cos\mu\sin\mu, \; \frac{k}{k^2+l^2} = -s\cos^2\mu$$

and therefore

$$k = -s^{-1}, \; l = s^{-1}\tan\mu.$$

We have got that if the considered M^m in E^n exists, then for its adapted frame bundle

$$\omega^{m+1} = 0, \; \omega^\rho = 0,$$
$$\omega^1_a = -s^{-1}\omega^a, \; \omega^{m+1}_1 = s^{-1}\tan\mu\cdot ds, \; \omega^{m+1}_a = s^{-1}\tan\mu\cdot\omega^a.$$
$$\omega^\rho_a = h^\rho_{1a}ds, \; \omega^\rho_1 = h^\rho_{1a}\omega^a = \tan\mu\cdot\omega^\rho_{m+1}.$$

Differential prolongation leads to

$$\Sigma_\rho h^\rho_{1a}h^\rho_{1b} = \delta_{ab}(s^{-1}\tan\mu)^2,$$

$$dh^\rho_{1a} - h^\rho_{1b}\omega^b_a + h^\sigma_{1a}\omega^\rho_\sigma = -s^{-1}h^\rho_{1a}ds.$$

Taking now $e_{m+a} \upuparrows h^\rho_{1a}e_\rho$ we have

$$h^{m+b}_{1a} = \delta^b_a s^{-1}\tan\mu, \; h^\xi_{1a} = 0 \; (\xi = 2m + 2,\dots,n)$$

and therefore

$$\left.\begin{array}{l}\omega^{m+1} = \omega^{m+a} = \omega^{\xi} = 0,\\ \omega_c^1 = -s^{-1}\omega^a,\ \omega_1^{m+1} = s^{-1}\tan\mu\cdot ds,\ \omega_a^{m+1} = s^{-1}\tan\mu\cdot\omega^a,\\ \omega_a^{m+b} = \delta_a^b s^{-1}\tan\mu\cdot ds,\ \omega_1^{m+a} = s^{-1}\tan\mu\omega^a = \tan\mu\omega_{m+1}^{m+a},\\ \omega_1^{\xi} = \omega_a^{\xi} = \omega_{m+1}^{\xi} = 0,\ \omega_a^b - \omega_{m+a}^{m+b} = 0.\end{array}\right\} \quad (6.5)$$

Here exterior differentiation gives identities except the last row which leads to $\omega_{m+a+1}^{\xi} = 0$. Adding these equations we get the completely integrable system, thus the considered M^m exists and, as we see, lies in 2m-plane, spanned on x, e_i, e_{m+1} and e_{m+a+1}.

Now

$$\Omega_a^b = \Omega_{m+a+1}^{m+b+1} = -s^{-2}\tan^2\mu\omega^a\wedge\omega^b,\ \Omega_a^m = \Omega_{m+1}^{m+a+1} = 0$$

and (2.7) is satisfied as it is easy to see. Thus the considered M^m is semi-symmetric with nonflat $\nabla^{\perp}$. Note that this M^m is not symmetric.

In the particular case, if m = 3, our M^3 lies in E^6 and all its points are spatial.

7. <u>Main theorem.</u> We show that the list of examples in the previous section is exhausting in some sense.

T H E O R E M 4. *A 3-dimensional semi-symmetric submanifold M^3 with spatial points in E^n is either*

(1) *a Cartan submanifold with flat $\bar{\bar{\nabla}}$ or*

(2) *in the case of nonflat $\nabla^{\perp}$ one of the submanifolds of examples* 1,2,3 or 4.

P r o o f. Let M^3 be semi-symmetric with nonflat $\nabla^{\perp}$ and dim span$(h_{ij}^{\alpha}e_{\alpha}) = 3$, i.e. we have not the case (1) (see Theorem 3). Taking $e_4 \parallel H$ and e_5, e_6 also in $N_x^{(1)}M^m$ we get $H^5 = H^6 = H^{\rho} = 0$, $h_{ij}^{\rho} = 0$ $(\rho = 7,\dots,n)$ and thus due to (2.5), (2.6) and (2.8) $\Omega_{\alpha}^{\rho} = 0$, $\Omega_4^5 = \Omega_4^6 = 0$; it follows $\Omega_5^6 \neq 0$. Here $\Omega_4^5 = 0$ gives the commutability of $\|h_{ij}^4\|$ and $\|h_{ij}^5\|$ and thus e_1, e_2, e_3 can be taken so that $h_{ij}^4 = h_{ij}^5 = 0$ if $i\neq j$. We take e_5 and e_6 so that h_{11} contains in their span, then $h_{11}^6 = 0$ and now

$$\Omega_5^6 = (h_{22}^5-h_{11}^5)h_{12}^6\omega^1\wedge\omega^2+(h_{33}^5-h_{22}^5)h_{23}^6\omega^2\wedge\omega^3+(h_{11}^5-h_{33}^5)h_{23}^6\omega^3\wedge\omega^1.$$

It follows that at least one of h_{12}^6, h_{13}^6, h_{23}^6 is nonzero. Without a loss of generality we can suppose $h_{12}^6 \neq 0$. Now (2.7) or, more explicitely,

$$h^{\alpha}_{kj}\Omega^{k}_{i} + h^{\alpha}_{ik}\Omega^{k}_{j} - h^{\beta}_{ij}\Omega^{\alpha}_{\beta} = 0 \qquad (7.1)$$

gives by $\alpha=5$, $i=j=2$ that $h^{6}_{22}\Omega^{5}_{6} = 0$ and thus due to $H^{6} = \frac{1}{3}(h^{6}_{22} + h^{6}_{33}) = 0$ we have $h^{6}_{22} = h^{6}_{33} = 0$. By $\alpha=5$, $i\neq j$ we get

$$(h^{5}_{jj} - h^{5}_{ii})\Omega^{j}_{i} - h^{6}_{ij}\Omega^{5}_{6} = 0. \qquad (7.2)$$

Taking here $i=1$, $j=2$ we have

$$(h^{5}_{22} - h^{5}_{11})\Omega^{2}_{1} = h^{6}_{12}\Omega^{5}_{6} \neq 0. \qquad (7.3)$$

Thus $\Omega^{2}_{1} \neq 0$, $h^{5}_{11} \neq h^{5}_{22}$ and now (7.1) by $\alpha=4$, $i=1$, $j=2$, i.e. $(h^{4}_{22} - h^{4}_{11})\Omega^{2}_{1} = 0$, gives $h^{4}_{11} = h^{4}_{22}$.

Let $\alpha=6$, $i=1,j=3$ and then $\alpha=6$, $i=2$, $j=3$. From (7.1) it follows that

$$h^{6}_{23}\Omega^{2}_{1} - h^{6}_{12}\Omega^{3}_{2} = 0, \quad h^{6}_{13}\Omega^{2}_{1} + h^{6}_{12}\Omega^{3}_{1} = 0. \qquad (7.4)$$

As a result we get that Ω^{5}_{6}, Ω^{3}_{1} and Ω^{3}_{2} are proportional to Ω^{2}_{1}. After substituting it in (7.2) by $i=1,j=3$ and $i=2,j=3$ we have

$$h^{6}_{13}(2h^{5}_{11} - h^{5}_{22} - h^{5}_{33}) = 0,$$

$$h^{6}_{23}(2h^{5}_{22} - h^{5}_{11} - h^{5}_{33}) = 0$$

which together with $H^{5} = \frac{1}{3}(h^{5}_{11} + h^{5}_{22} + h^{5}_{33}) = 0$ give

$$h^{5}_{11}h^{6}_{13} = 0, \quad h^{5}_{22}h^{6}_{23} = 0.$$

Here in the indices the roles of 1 and 2 can be exchanged, so that we have only two essential cases (recall that $h^{5}_{11} \neq h^{5}_{22}$):

<u>A</u>. $h^{6}_{13} = h^{6}_{23} = 0$, <u>B</u>. $h^{5}_{22} = 0$, $h^{6}_{13} = 0$, $h^{6}_{23} \neq 0$,

which we treat next separately.

<u>A</u>. Here (7.1) by $\alpha=6$, $i=j=1$ and $\alpha=6$, $i=j=2$ gives

$$2h^{6}_{22}\Omega^{2}_{1} - h^{5}_{11}\Omega^{6}_{5} = 0, \quad -2h^{6}_{12}\Omega^{2}_{1} - h^{5}_{22}\Omega^{6}_{5} = 0, \qquad (7.5)$$

thus $h^{5}_{11} + h^{5}_{22} = 0$ and also $h^{5}_{33}=0$. By $\alpha=4$, $i=1$, $j=2$ we get

$$(h^{4}_{22} - h^{4}_{11})\Omega^{2}_{1} = 0.$$

Denoting $h^{4}_{11}=\mu$, $h^{5}_{11}=\nu$ we have $h^{4}_{22}=\mu$, $h^{5}_{22}=-\nu$ and now (7.3) and (7.5) give $(h^{6}_{12})^{2}=\nu^{2}$. Exchanging e_{6} by $-e_{6}$, if needed, we have $h^{6}_{12} = \nu \neq 0$. In (7.3) $\Omega^{2}_{1} = (2\nu^{2}-\mu^{2})\omega^{1}\Lambda\omega^{2}$, $\Omega^{6}_{5} =$

$= -2\nu^2\omega^1\wedge\omega^2$, therefore $\mu^2=3\nu^2\neq 0$. As (7.4) reduces to $0 = \Omega_1^3 = -\mu h_{33}^4\omega^1\wedge\omega^3$, $0 = \Omega_2^3 = -\mu h_{33}^4\omega^2\wedge\omega^3$, we have $h_{33}^4=0$. Exchanging e_4 by $-e_4$, if needed, we get finally $\mu=\nu\sqrt{3}$.

We are obtained (6.1), which lead to the system (6.2) and consequently to (6.3), too. If $\tau=0$ we get the M^3 of Example 1, as it is said in Section 6. If $\tau\neq 0$ we have the M^3 of Example 2.

B. In this case $h_{11}^5\neq 0$. From (7.4) we have $\Omega_1^3=0$, $\Omega_2^3\neq 0$ and now (7.1) by $\alpha=4$, $i=2$, $j=3$ gives $h_{33}^4=h_{22}^4$. By $\alpha=6$, $i=j=1$ and $\alpha=5$, $i=1$, $j=2$ we get

$$2h_{12}^6\Omega_1^2 - h_{11}^5\Omega_5^6 = 0,$$
$$h_{11}^5\Omega_1^2 - h_{12}^6\Omega_5^6 = 0,$$

thus $2(h_{12}^6)^2 = (h_{11}^5)^2$. Taking $\alpha=6$, $i=j=2$ we have $h_{12}^6\Omega_1^2 - h_{23}^6\Omega_2^3 = 0$, and this together with (7.4) gives $(h_{12}^6)^2 = (h_{23}^6)^2$. Denoting $h_{11}^4=h_{22}^4=h_{33}^4=k$, $h_{11}^5=l$, we have $h_{33}^5=-l$, and replacing e_6 by $-e_6$ and e_2 by $-e_2$, if needed, we get $h_{12}^6 = h_{23}^6 = \frac{1}{\sqrt{2}}\,l$. Now

$$0 = \Omega_1^3 = (l^2-k^2)\omega^1\wedge\omega^3$$

and taking $-e_4$ instead of e_4, if needed, we have $l=k$. So

$$\omega_1^4 = k\omega^1,\ \omega_2^4 = k\omega^2,\ \omega_3^4 = k\omega^3,$$

$$\omega_1^5 = k\omega^1,\ \omega_2^5 = -k\omega^2,\ \omega_3^5 = 0,$$

$$\omega_1^6 = \frac{1}{\sqrt{2}}k\omega^3,\ \omega_2^6 = \frac{1}{\sqrt{2}}k\omega^3,\ \omega_3^6 = \frac{1}{\sqrt{2}}k(\omega^1+\omega^2).$$

It remains to rotate e_1 and e_2 in their 2-plane on $\frac{\pi}{4}$ and to replace e_5 by $-e_5$ in order to get the differential system (6.4).

Thus the considered M^3 in Case B is determined by the system consisting of the equations $\omega^4=\omega^5=\omega^6=\omega^\rho=0$, (6.4) and $\omega_1^\rho=\omega_2^\rho=\omega_3^\rho=0$. The differential prolongation of this system gives us

$$d\ln k = \varkappa\omega^1,\ \omega_1^2 = -\varkappa\omega^2,\ \omega_1^3 = -\varkappa\omega^3,\ \omega_2^3 - \omega_5^6 = 0,$$
$$\omega_4^5 = -\varkappa\omega^2,\ \omega_4^6 = -\varkappa\omega^3,\ \omega_4^\rho = \omega_5^\rho = \omega_6^\rho = 0$$

and from here we get in the same manner

$$d\varkappa = \varkappa^2\omega^1.$$

The prolonged system is totally integrable, thus considered

M^3 exists and, as we see, lies in a 6-plane E^6 in E^n.

If $\varkappa=0$, we have the M^3 of Example 3, where $R = \frac{1}{k} =$ $=$ const is the radius of the sphere S^5 containing this M^3.

Let $\varkappa \neq 0$. As $d\omega^1 = 0$ we have $\omega^1 = ds$. Now $d\ln k = \varkappa ds$, $d\varkappa = \varkappa^2 ds$ lead tc

$$\varkappa = -\frac{1}{s}, \quad k = \frac{c}{s}$$

(if to take a suitable zero point of s). Replacing this in the equations of the prolonged system obtained above we get the particular case of the system (6.5) by m=3 where $\tan\mu=c$. Thus the considered M^3 in this case coincides with the M^3 of Example 4. The theorem is proved.

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ТРЕХМЕРНЫЕ ПОЛУСИММЕТРИЧЕСКИЕ ПОДМНОГООБРАЗИЯ С АКСИАЛЬНЫМИ, ПЛАНАРНЫМИ И СПАЦИАЛЬНЫМИ ТОЧКАМИ В ЕВКЛИДОВЫХ ПРОСТРАНСТВАХ

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Р е з ю м е

Подмногообразие M^m в евклидовом пространстве E^n называется полусимметрическим, если $\bar{R}(X,Y)\cdot h=0$, где $\bar{R}$ есть оператор кривизны связности $\bar{\nabla}=\nabla\oplus\nabla^{\perp}$ ван дер Вардена-Бортолот-

ти и h есть вторая фундаментальная форма. Если размерность n_1 первого нормального пространства $\operatorname{span} h(X,Y)$ в точке $x \in M^m$ равна 1,2 или 3, то x называется, соответственно, аксиальной, планарной или спациальной точкой.

Классифицируются все полусимметрические M^3 в E^n с $n_1 \le 3$. При $n_1=1$ или $n_1=2$ полученные результаты следуют из теорем в [4,6] и из теорем Сегре и они сформированы в теоремах 1 и 2. В случае $n_1=1$ полусимметрическое M^3 есть либо (i) сфера S^3 в $E^4 \subset E^n$, либо (ii) конус вращения C^3 (с исключенной точечной вершиной) в $E^4 \subset E^n$, либо (iii) цилиндр $S^2 \times E^1$ в $E^4 \subset E^n$, либо (iv) M^3 ранга 1 в E^n. В случае $n_1=2$ полусимметрическое M^3 есть либо (1) M^3 ранга 2 с плоской $\bar\nabla$, либо (2) каналово M^3 ортогонального типа. Заметим, что все такие M^3 имеют плоскую нормальную связность $\nabla^\perp$.

При $n_1=3$ доказывается, что полусимметрическое M^3 с $n_1=3$ в E^n есть либо (1) картаново подмногообразие с плоской $\bar\nabla$, либо (2) одно из M^3 следующего списка (все они имеют неплоскую $\nabla^\perp$):

(2_1) Произведение $\tilde V^2 \times E^1$, где $\tilde V^2$ является огибающей 2-го порядка семейства поверхностей Веронезе; $n \ge 6$.

(2_2) Огибающая 2-го порядка семейства произведений $V^2 \times E^1$, $n \ge 6$.

(2_3) Симметрическое M^3 в $S^5 \subset E^6$ с ортогональной сетью больших 2-сфер и больших окружностей; такое M^3 является минимальныи в S^5.

(2_4) M^3 с ортогональной сетью концентрических 2-сфер и конгруентных логарифмических спирал с полюсом в центре этих сфер в E^6.

Если в случаях (2_1) и (2_2) имеет место $n=6$, то семейство состоит из одного $V^2 \times E^1$, которое тем самым и является огибающей.

Для случаев (2_3) и (2_4) получаются обобщения: соответственно симметрическое $M^{m_1+m_2}$ в $S^{m_1+m_2+m_1m_2}$ с ортогональной сетью больших m_1- и m_2-сфер; M^m в E^{2m}, которое получается из (2_4) заменой семейства 2-сфер со семейством $(m-1)$-сфер.

CLASSIFICATION OF THREE-DIMENSIONAL SEMI-SYMMETRIC SUBMANIFOLDS IN EUCLIDEAN SPACES

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1. A submanifold M^m in a Euclidean space E^n is said to be semi-symmetric [7,8] (or semi-parallel [2,3]) if $\bar{R}(X,Y)\cdot h=0$, where $\bar{R}$ is the curvature operator of the van der Waerden-Bortolotti connection $\bar{\nabla}$ [1] and $h: TM^m \times TM^m \to T^{\perp}M^m$ is the second fundamental form. Intrinsically such a M^m is a semi-symmetric Riemannian manifold, i.e. $R(X,Y)\cdot R=0$ (see [14]). Due to the identity $\bar{\nabla}_{[X}\bar{\nabla}_{Y]}h=\bar{R}(X,Y)h$ a M^m in E^n is semi-symmetric iff its fourth fundamental form $\bar{\nabla}\bar{\nabla}h$ is symmetric, i.e. $\bar{\nabla}_{[X}\bar{\nabla}_{Y]}h=0$. Recall that the third fundamental form $\bar{\nabla}h$ is always symmetric due to the Codazzi equation: $\bar{\nabla}_X h(Y,Z)=\bar{\nabla}_Y h(X,Z)$.

Semi-symmetric surfaces and hypersurfaces in Euclidean spaces are classified respectively, in [2] and [3,7,8]. All semi-symmetric submanifolds M^m with the flat normal connection $\nabla^{\perp}$ in Euclidean spaces are listed in [11,16] (in particular, semi-symmetric M^m with codimension 2 in [8]).

Next we continue the classification of three-dimensional semi-symmetric submanifolds M^3 in Euclidean spaces , started in [12], and make it complete. The new results in the present paper concern the semi-symmetric M^3 with dim span $h(X,Y)\geq 4$ and thus with nonflat $\nabla^{\perp}$.

2. First we give a list of semi-symmetric M^3 in E^n, ordered by the dim span $h(X,Y)=n_1$; note that $0\leq n_1\leq 6$. Some of the submanifolds of this list can have singularities. In all the cases only the regular open parts of the listed submanifolds (which can coincide with the whole complete

submanifolds if they are regular everywhere) are to be considered. For the complicated cases the hints are given to find complementary information.

$n_1=0$: (1) 3-plane E^3 in E^n.

$n_1=1$: (2) Sphere $S^3 \subset E^3$ in E^n.

(3) Product $S^2 \times E^1 \subset E^4$ in E^n.

(4) Round cone $C^3 \subset E^4$ in E^n with point-vertex, i.e. the cone of revolution in E^4 around some axis.

(5) M^3 with rank 1 in E^n, $n \geq 4$, i.e. M^3 with an one-parameter family of 2-plane generators, along which the tangent 3-plane is constant.

$n_1=2$: (6) M^3 with rank 2 and flat $\bar{\nabla}$ in E^n, $n \geq 5$ (see [15]).

(7) Orthogonal type canal M^3 in E^n, $n \geq 5$ (see [8,11,16].

$n_1=3$: (8) Cartan type M^3 with flat $\bar{\nabla}$ in E^n, $n \geq 6$ (see [15]).

(9) Product $V^2 \times E^1 \subset E^6$ in E^n, where V^2 is a Veronese surface in $S^4 \subset E^5$ (see [1] and Section 6 below).

(10) 2nd order envelope of a family of $V^2 \times E^1$ in E^n, $n>6$ (see [12]).

(11) Irreducible symmetric W^3, which has an orthogonal net of great 2-spheres and great circles, in $S^5 \subset E^6 \subset E^n$ (see [12]); this W^3 is minimal in S^5.

(12) M^3 with an orthogonal net of concentric 2-spheres and congruent logarithmic spirals with the pole in the centre of these 2-spheres in $E^6 \subset E^n$ (see [12]).

$n_1=4$: (13) Product $V^2 \times M^1 \subset E^7$ in E^n, where M^1 is a plane line.

(14) 2nd order envelope of a family of $V^2 \times S^1$ in E^n, $n>7$.

$n_1=6$: (15) Veronese submanifold $V^3 \subset S^8 \subset E^9$ in E^n (see [6] and Section 5 below).

(16) 2nd order envelope of a family of congruent Veronese submanifolds V^3 in E^n, $n>9$.

Here we mean that a submanifold M^m in E^n is the 2nd order envelope of a family of submanifolds $\tilde{M}^m$ if for every point $x \in M^m \cap \tilde{M}^m$ and for every line λ in M^m through x a line $\tilde{\lambda}$ in $\tilde{M}^m$ through x is to be found which has the common tangent direction and the common curvature vector with λ at x. The equivalent condition is: $T_x\tilde{M}^m = T_xM^m$ and second fundamental forms $\tilde{h}$ a h for $\tilde{M}^m$ and M^m (as maps) coincide at x, i.e. $\tilde{h}_x = h_x$.

We have shown in [10] that a submanifold M^m in E^n (or in the space $N^n(c)$ of constant curvature) is semi-symmetric iff it is a 2nd order envelope of symmetric submanifolds, i.e. of submanifolds with $\bar{\nabla}h=0$ (see [5]).

All submanifolds of the previous list are either symmetric (as (1),(2),(3),(9),(11),(15)) or such envelopes: (4) of $S^2 \times E^1$, (5) of $S^1 \times E^2$, (6) of $S^1 \times S^1 \times E^1$, (7) of $S^2 \times S^1$, (8) of $S^1 \times S^1 \times S^1$, (12) of W^3, (13) of $V^2 \times S^1$. The cases (10),(14) and (16) are directly indicated by such envelopes. The geometrical structure of the latter needs complementary investigations.

Now we formulate the main result of this paper.

T H E O R E M. *The previous list contains all semi-symmetric submanifolds* M^3 *in Euclidean spaces* E^n, *i.e. every semi-symmetric* M^3 *in* E^n *is one of* (1)-(16).

3. For the cases $0 \le n_1 \le 3$ the Theorem is proved in [12]. The purpose of the present paper is to prove the Theorem for remaining cases $4 \le n_1 \le 6$ and thereby to complete the classification of three-dimensional semi-symmetric submanifolds M^3 in Euclidean spaces.

To do it we write the semi-symmetricity condition $\bar{R}(X,Y)\cdot h=0$ in a special form by means of the vectors h(X,Y) only.

Let $\{e_1,\dots,e_m;e_{m+1},\dots,e_n\}$ be an element of the frame bundle on M^m of the adapted orthogonal frames, i.e. $e_i \in T_xM^m$, $e_\alpha \in T_x^\perp M^m$; $i=1,\dots,m$; $\alpha=m+1,\dots,n$. Then in the formulae

$$dx = e_I\omega^I,\ de_I=e_J\omega_I^J\ ;\quad \omega_I^J + \omega_J^I = 0, \tag{3.1}$$

$$d\omega^I = \omega^J\wedge\omega_J^I\ ,\ d\omega_I^J = \omega_I^K\wedge\omega_K^J\ ;\ I,J,K = 1,\dots,n \tag{3.2}$$

we have

$$\omega^\alpha = 0,\quad \omega_i^\alpha = h_{ij}^\alpha\omega^j,\quad h_{ij}^\alpha = h_{ji}^\alpha \tag{3.3}$$

and $h(X,Y)=h_{ij}^\alpha X^iY^je_\alpha$ for $X = X^ie_i$, $Y = Y^je_j$.

A submanifold M^m is semi-symmetric iff

$$h_{kj}^\alpha\Omega_i^k + h_{ik}^\alpha\Omega_j^k - h_{ij}^\beta\Omega_\beta^\alpha = 0, \tag{3.4}$$

$$\Omega_i^j = -\sum_\alpha h^\alpha_{i[k} h^\alpha_{l]j} \omega^k \wedge \omega^l, \qquad (3.5)$$

$$\Omega_\alpha^\beta = -\sum_i h^\alpha_{i[k} h^\beta_{l]i} \omega^k \wedge \omega^l, \qquad (3.6)$$

are the curvature 2-forms of the connections ∇ and $\nabla^\perp$, respectively. If we denote $h_{ij} = h^\alpha_{ij} e_\alpha$ and $H_{ij,kl} = \langle h_{ij}, h_{kl}\rangle = \sum_\alpha h^\alpha_{ij} h^\alpha_{kl}$, then (3.4) takes the form

$$\sum_k \{h_{kj} H_{i[p,q]k} + h_{ik} H_{j[p,q]k} - H_{ij,k[p} h_{q]k}\} = 0 \qquad (3.7)$$

which can also be given as follows:

$$\mathrm{sym}_{(X,Y)} \mathrm{alt}_{[Z,W]} \{\mathrm{trace}\ h(X,\cdot)H(Y,Z;W,\cdot) - \frac{1}{2}\mathrm{trace}\ h(W,\cdot)H(X,Y;Z,\cdot)\} = 0,$$

where $H(X,Y;Z,W) = \langle h(X,Y), h(Z,W)\rangle$.

Note that (3.7) is a system of cubic homogeneous equations on the components h^α_{ij} of h and thus is a pointwise algebraic condition.

4. Let us start with the case $n_1=6$. Then all six normal vectors $h_{11}, h_{22}, h_{33}, h_{12}, h_{23}, h_{31}$ are linearly independent and (3.7) leads to

$$H_{12,12} = H_{23,23} = H_{31,31} = \varkappa^2, \qquad (4.1)$$

$$H_{11,22} = H_{22,33} = H_{33,11} = 2\varkappa^2, \qquad (4.2)$$

$$H_{11,11} = H_{22,22} = H_{33,33} = 4\varkappa^2, \qquad (4.3)$$

$$H_{aa,ab} = H_{aa,bc} = H_{ab,ac} = 0 \qquad (4.4)$$

for every three distinct a,b,c from {1,2,3}. Shortly

$$H_{ij,kl} = \varkappa^2(2\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

(see [9]). Now due to (3.5)

$$\Omega_i^j = -H_{i[k,l]j}\omega^k \wedge \omega^l = -\varkappa^2 \omega^i \wedge \omega^j$$

and the Schur theorem gives $\varkappa$=const. The procedure made in [9] yields $\langle h_{ij}, \nabla_p h_{kl}\rangle = 0$.

5. If $n_1=6$, $n=3+6=9$ then $\nabla_p h_{kl}$ are the vectors of 6-dimensional span h(X,Y) and are orthogonal to 6 linearly independent vectors $h_{11},\dots,h_{31}$. It follows that $\nabla_p h_{kl} = 0$, thus h is parallel and M^3 is a symmetric submanifold of constant curvature in E^9.

In [9] it is shown that such M^3 is a Veronese submanifold V^3 in a S^8 (see also [6]), i.e. an orbit of a 6-parametric Lie subgroup of rotations of E^9 around the centre of this S^8.

Taking into account the later use we find the equations (3.3) for V^3 in E^9 with respect to a suitable adapted frame field and prove these results anew. Due to (4.2) and (4.3) h_{11}, h_{22}, h_{33} are side vectors of a regular tetrahedron and (4.1), (4.4) show that h_{12}, h_{23}, h_{31} are mutually orthogonal vectors with the same lenght, orthogonal to the 3-plane of this tetrahedron. Let us take $e_4 \| h_{11}+h_{22}$, $e_5 \| h_{11}-h_{22}$, $e_6 \| h_{12}$ $e_7 \| e_{31}$, $e_8 \| e_{23}$ and e_9 in this 3-plane.
Then (3.3) are

$$\left.\begin{array}{llll} \omega^4=\ldots=\omega^9=0, & \omega^4_1=\varkappa\sqrt{3}\,\omega^1, & \omega^4_2=\varkappa\sqrt{3}\omega^2, & \omega^4_3=\frac{2}{3}\varkappa\sqrt{3}\,\omega^3, \\ & \omega^5_1=\varkappa\omega^1, & \omega^5_2=-\varkappa\omega^2, & \omega^5_3=0, \quad \varkappa=\text{const} \\ & \omega^6_1=\varkappa\omega^2, & \omega^6_2=\varkappa\omega^1, & \omega^6_3=0, \\ & \omega^7_1=\varkappa\omega^3, & \omega^7_2=0, & \omega^7_3=\varkappa\omega^1, \\ & \omega^8_1=0, & \omega^8_2=\varkappa\omega^3, & \omega^8_3=\varkappa\omega^2, \\ & \omega^9_1=0, & \omega^9_2=0, & \omega^9_3=\frac{2}{3}\varkappa\sqrt{6}\,\omega^3. \end{array}\right\} \quad (5.1)$$

After differential prolongation, i.e. the use of (3.2) and Cartan lemma, we get

$$\left.\begin{array}{l} 0=\omega^5_4=\omega^6_4=\omega^9_4=\omega^9_5=\omega^9_6, \\ \omega^2_1=\frac{1}{2}\,\omega^6_5=\omega^8_7, \\ \omega^3_1=\sqrt{3}\omega^7_4=\omega^8_6=\omega^7_5=\frac{1}{4}\sqrt{6}\omega^9_7, \\ \omega^3_2=\sqrt{3}\omega^8_4=-\omega^8_5=\omega^7_6=\frac{1}{4}\sqrt{6}\omega^9_8. \end{array}\right\} \quad (5.2)$$

The next differential prolongation gives the identities and thus the system (5.1),(5.2), which determines V^3 in E^9, is completely integrable.

It is seen directly that V^3 is an orbit of the 6-parametric Lie group of motions in E^9 with Maurer-Cartan forms $\omega^1, \omega^2, \omega^3, \omega^2_1, \omega^3_1, \omega^3_2$. This group has the fixed point with radius vector $c = x + \frac{1}{8\varkappa^2}(h_{11}+h_{22}+h_{33}) = x + \frac{\sqrt{3}}{3}\varkappa(e_4 + \frac{1}{4}\sqrt{2}\, e_9)$ and thus is isomorfic to the $O(4)$. Here $\frac{1}{3}(h_{11}+h_{22}+h_{33})$ is the mean curvature vector and hence V^3 is a minimal submanifold in S^8

5

We add that V^3 gives a 2nd standard minimal imbedding of S^3 (see [4]). About Veronese submanifolds V^m see [6] and references in [9]; R.Mullari [17] has shown that S^m and V^m are the only "maximal symmetric" submanifolds (i.e. those imbeddings of the constant curvature spaces, all motions of which are induced by the motions of the ambient space) in their first osculating Euclidean spaces.

So we have got the case (15) of the list in Section 2.

6. The surface determined in V^3 by the completely integrable system $\omega^3=\omega^3_1=\omega^3_2=0$ is a Veronese surface V_2 in the 5-plane E^5 spanned on x,e_1,e_2,e_4,e_5,e_6. In this E^5 it is determined by the equations $\omega^4=\omega^5=\omega^6=0$ and the equations framed by the dotted line in (5.1),(5.2); they form a completely integrable system, too. As E^5 is orthogonal to the inner normal e_3 of V^2 in V^3, this V^2 is a geodesic surface of V^3.

Such V^2 will play an important role below in Section 14. It is easy to show that V^2 lies in a hypersphere S^4 of E^5 and is its minimal surface (the 2nd standard minimal imbedding of S^2) admitting a 3-parametric Lie group of rotations around the centre of S^4 in E^5.

7. Let $n_1=6$, $n>9$. Then we have to add to (5.1) the equations $\omega^\rho_i=0$; $i=1,2,3$; $\rho=10,\dots,n$, and thus to (5.2) the equations

$$\omega^\rho_\alpha=\lambda^\rho_{\alpha 1}\omega^1 \quad ; \quad \alpha=4,\dots,9$$

with some relations between $\lambda^\rho_{\alpha 1}$: $\lambda^\rho_{63}=\lambda^\rho_{72}=\lambda^\rho_{81}$ etc. The system obtained in such a way is not completely integrable in general. The problem of compatibility of this system is rather complicated and depends on $\dim \operatorname{span}\lambda(\xi,X)=n_2\le 10$, where $\lambda(\xi,X)=\lambda^\rho_{\alpha 1}\xi^\alpha X^1 e_\rho$ for $\xi=\xi^\alpha e_\alpha$, $X=X^1e_1$. We can only say that the semi-symmetric M^3 in E^n, $n\ge 10$, in this case is a 2nd order envelope of a family of congruent Veronese V^3 and thus M^3 is a Veronese type constant isotropic submanifold (see [13]).

So we have got the case (16) of the list in Section 2.

8. Next we show there is no semi-symmetric submanifold M^3 with $n_1=5$ in E^n, $n\ge 8$.

Let $n_1=5$. We have a linear dependence between six vectors $h_{11}, h_{22}, h_{33}, h_{12}, h_{23}, h_{31}$ and thus six functions ξ^{11}, $\ldots, \xi^{31}$ exist so that $h_{ij}\xi^{ij}=0$, $\Sigma(\xi^{ij})^2 \neq 0$. Here h_{ij} determine a vector valued symmetric tensor field, hence ξ^{ij} give a symmetric tensor field determined up to a multiplier. We can take e_1, e_2, e_3 in every point $x \in M^3$ so that this dependence is $h_{11}\xi^{11} + h_{22}\xi^{22} + h_{33}\xi^{33} = 0$ which by reordering, if necessary, leads to

$$h_{33} = \mu_1 h_{11} + \mu_2 h_{22}. \tag{8.1}$$

The five vectors $h_{11}, h_{22}, h_{12}, h_{23}, h_{31}$ are linearly independent.

We have to substitute (8.1) in all semi-symmetricity conditions (3.7) and take the coefficients by these five vectors; these coefficients must be equal to zero. Next we refer to the condition (3.7) in the form $[i,j;p,q]$ and if we take the coefficient by h_{rs}, then we write $[i,j;p,q|r,s]$. So we have

$$[a,a;a,b|a,b] \; : \; 3H_{aa,bb} - 2H_{ab,ab} - H_{aa,aa} = 0,$$

where a and b are 1 or 2, $a \neq b$, which in particular give

$$H_{11,11} = H_{22,22} = \sigma^2 > 0. \tag{8.2}$$

Further

$$[a,a;a,3|a,3] \; : \; 3H_{aa,33} - 2H_{a3,a3} - H_{aa,aa} = 0. \tag{8.3}$$

Substituting (8.1) and (8.2) we obtain

$$(3\mu_a - 1)\sigma^2 + 3\mu_b H_{aa,bb} - 2H_{a3,a3} = 0, \tag{8.4}$$

Now

$$[a,a;b,3|b,3] \; : \; \mu_a\sigma^2 + (\mu_b - 1)H_{aa,bb} = 0 \tag{8.5}$$

gives a homogeneous linear system with nontrivial solution $(\sigma^2, H_{11,22})$ and thus

$$(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 1) = 0.$$

Here $\mu_1 + \mu_2 = 1$ is impossible: (8.5) gives $H_{11,22} = \sigma^2$, but this contradicts to (8.2), because h_{11} and h_{22} are linearly independent.

However, $\mu_1 = \mu_2$ leads to a contradiction, too. Due to (8.5) and (8.2) we have then $\mu_1 = \mu_2 = \mu \neq 1$, thus $H_{11,11} = \mu\sigma^2(1-\mu)^{-1}$. Now

$$[3,3;3,a|a,3] \; : \; 3H_{aa,33} - 2H_{a3,a3} - H_{33,33} = 0$$

together with (8.3) gives $H_{33,33} = \sigma^2$. Substituting (8.1) we get $2\mu^2+\mu-1=0$. The root $\mu=\frac{1}{2}$ leads to a particular case of $\mu_1+\mu_2=1$ and thus is impossible. For the root $\mu=-1$ we have $H_{aa,bb}=-\frac{1}{2}\sigma^2$, but this is impossible, too: (8.4) gives $-4\sigma^2+\frac{3}{2}\sigma^2-2H_{a3,a3}=0$ or $-\frac{5}{2}\sigma^2=H_{a3,a3}>0$.

9. It remains to consider the most complicated case $n_1=4$. In this case there is an 1-parametric family of linear dependencies between six vectors $h_{11},\dots,h_{31}$ with two independent basic relations $h_{ij}\xi^{ij}=0$ and $h_{ij}\eta^{ij}=0$. In this family $\rho(h_{ij}\xi^{ij})+\sigma(h_{ij}\eta^{ij})=0$ the singular case corresponds to a root of the cubic equation $\det|\rho\xi^{ij}+\sigma\eta^{ij}|=0$ with respect to $\rho:\sigma$ or $\sigma:\rho$. There is at least one real root and thus the basic relation can be presented in the form $h_{ij}\xi_1^{(i}\xi_2^{j)}=0$, where $\xi_1=\xi_1^i e_i$ and $\xi_2=\xi_2^j e_j$ determine some directions in T_xM^3.

A. Let these directions be distinct. After the normalization of ξ_1 and ξ_2 we can take e_2 orthogonal to them and $e_1\|\xi_1+\xi_2$, $e_3\|\xi_1-\xi_2$. Then the distinguished basic relation is $h_{11}(\xi_1^1)^2-h_{33}(\xi_1^3)^2=0$. Here $\xi_1^1\xi_1^3\neq 0$; the roles of e_1 and e_3 can be exchanged taking $-\xi_2$ instead of ξ_2. Hence the following subcases occur.

(A_1) $(\eta^{12})^2+(\eta^{23})^2\neq 0$. Here we can suppose $\eta^{23}\neq 0$ and the basic relations are

$$h_{33}=\mu h_{11},\ h_{33}=\nu_1 h_{11}+\nu_2 h_{22}+\nu_3 h_{12}+\nu_4 h_{13}.$$

(A_2) $\eta^{12}=\eta^{23}=0$, $\eta^{13}\neq 0$. Then

$$h_{33}=\mu h_{11},\ h_{13}=\nu_1 h_{11}+\nu_2 h_{22}.$$

(A_3) $\eta^{12}=\eta^{23}=\eta^{13}=0$. Here either

(A_3') $\eta^{22}\neq 0$ and $h_{33}=\mu h_{11}$, $h_{22}=\nu h_{11}$ or

(A_3'') $\eta^{22}=0$, then $\eta^{11}\neq 0$ and $h_{33}=h_{11}=0$.

B. Let ξ_1 and ξ_2 have the same direction, in which we take e_3. Then the distingueshed basic relation is $h_{33}=0$ and the roles of e_1 and e_2 can be exchanged taking $-e_3$ instead of e_3. Here the following subcases occur.

(B_1) $(\eta^{13})^2+(\eta^{23})^2\neq 0$. We can suppose $\eta^{23}\neq 0$ and analytically we get the limit case of (A_1), when $\mu=0$.

(B_2) $\eta^{13}=\eta^{23}=0$, $\eta^{12}\neq 0$. Then

$$h_{33} = 0, \ h_{12} = \lambda_1 h_{11} + \lambda_2 h_{22}.$$

(B_3) $\eta^{13} = \eta^{23} = \eta^{12} = 0$. Here we get either the limit case of (A_3'), when $\mu = 0$, or the case (A_3'').

So we have to control three cases: (A_2),(A_3''),(B_2) and two cases (A_1),(A_3') with their limit cases when $\mu = 0$.

10. Let us start with (B_2):

$$h_{33} = 0, \quad h_{12} = \lambda_1 h_{11} + \lambda_2 h_{22}$$

and $h_{11}, h_{22}, h_{13}, h_{23}$ are linearly independent.

We have $[1,1;1,3|1,3]$: $2H_{13,13} + H_{11,11} = 0$, which is a contradiction. Hence (B_2) is impossible for a semi-symmetric M^3 in E^n. The same argumentation gives, that (A_3'') is impossible, too.

Next we consider the case (A_3'):

$$h_{33} = \mu h_{11}, \ h_{22} = \nu h_{11} \tag{10.1}$$

and $h_{11}, h_{12}, h_{23}, h_{31}$ are linearly independent.

Here $[1,1;1,2]$ and $[2,2;1,2]$ give by h_{12}:

$$3H_{11,22} - 2H_{12,12} - H_{11,11} = 0, \tag{10.2}$$

$$3H_{11,22} - 2H_{12,12} - H_{22,22} = 0, \tag{10.3}$$

thus $H_{22,22} = H_{11,11}$ and due to (10.1) $\nu^2 = 1$. The case $\nu = -1$ is impossible because (10.2) gives then due to (10.1) a contradiction $2H_{11,11} + H_{12,12} = 0$. Hence $\nu = 1$ and $H_{11,22} = H_{12,12}$. Now $[1,3;1,2|2,3]$: $H_{11,22} - H_{12,12} - H_{13,13} = 0$, gives the contradiction $H_{13,13} = 0$ and so the case (A_3') is impossible for the semi-symmetric submanifold M^3 in E^n.

11. Next we show that (A_2) is impossible for a semi-symmetric submanifold M^3, too. Let

$$h_{33} = \mu h_{11}, \ h_{13} = \nu_1 h_{11} + \nu_2 h_{22}, \ \mu \neq 0$$

and $h_{11}, h_{22}, h_{12}, h_{23}$ are linearly independent.

Here as before we get (10.2) and (10.3) and thus $H_{11,11} = H_{22,22}$. On the other hand, $[3,3;1,2|1,2]$ gives $\mu(H_{11,22} - H_{11,11}) = 0$ and thus $H_{11,11} = H_{22,22} = H_{11,22}$. This is impossible because h_{11}, h_{22} are linearly independent.

12. It turns out that the semi-symmetricity conditions (3.7) can be satisfied only in the case (A_1):

$$h_{33} = \mu h_{11}, h_{23} = \nu_1 h_{11} + \nu_2 h_{22} + \nu_3 h_{12} + \nu_4 h_{13} \tag{12.1}$$

with linearly independent $h_{11}, h_{22}, h_{12}, h_{13}$. To show it we consider

$[1,1;1,2|1,1]$: $H_{11,12} - \nu_1 H_{11,13} = 0,$

$[1,1;1,2|2,2]$: $-H_{11,12} - \nu_2 H_{11,13} = 0,$

$[2,2;1,2|1,1]$: $H_{12,22} + \nu_1(2H_{12,23} - 3H_{22,13}) = 0,$

$[2,2;1,2|2,2]$: $-H_{12,22} + \nu_2(2H_{12,23} - 3H_{22,13}) = 0,$

$[1,2;1,2|1,1]$: $2H_{12,12} - H_{11,22} + \nu_1(H_{11,23} - H_{12,13}) = 0,$

$[1,2;1,2|2,2]$: $-2H_{12,12} + H_{11,22} + \nu_2(H_{11,23} - H_{12,13}) = 0,$

$[3,3;1,2|1,1]$: $H_{33,12} + \nu_1(2H_{22,13} - 2H_{12,23} - H_{13,33}) = 0,$

$[3,3;1,2|2,2]$: $-H_{33,12} + \nu_2(2H_{22,13} - 2H_{12,23} - H_{13,33}) = 0.$

Suppose $\nu_1 + \nu_2 \neq 0$. Then

$$H_{11,12} = H_{11,13} = H_{12,22} = 2H_{12,23} - 3H_{22,13} =$$
$$= 2H_{12,12} - H_{11,22} = H_{11,23} - H_{12,13} = H_{22,13} - H_{12,23} = 0.$$

It follows that $H_{22,13} = H_{12,23} = 0$, thus $\nu_3 H_{12,12} + \nu_4 H_{12,13} = 0$.

Now $[1,3;1,2]$ gives by h_{11} and h_{22}:

$$\nu_1(H_{11,22} - H_{12,12} - H_{13,13}) + H_{12,13} = 0,$$
$$\nu_2(H_{11,22} - H_{12,12} - H_{13,13}) - H_{12,13} = 0,$$

hence $H_{12,13} = H_{11,22} - H_{12,12} - H_{13,13} = 0$, thus $H_{11,23} = 0$, $H_{12,12} = H_{13,13} = a^2 \neq 0$, $\nu_3 = 0$, $H_{11,22} = 2a^2$. Taking

$[1,1;1,2|1,2]$: $3H_{11,22} - 2H_{12,12} - H_{11,11} = 0,$

$[2,2;1,2|1,2]$: $3H_{11,22} - 2H_{12,12} - H_{22,22} = 0$

we have $H_{11,11} = H_{22,22} = 4a^2$.

On the other hand $[1,3;1,2|1,3]$: $\nu_4(H_{11,22} - H_{12,12}) = 0$ gives $\nu_4 \cdot a^2 = 0$, thus $\nu_4 = 0$ and now $[2,2;1,2|1,3]$ yields $H_{22,23} = 0$. But $H_{11,23} = 0$ and $H_{22,23} = 0$ give together a contradiction:

$$4a^2\nu_1 + 2a^2\nu_2 = 0, \quad 2a^2\nu_1 + 4a^2\nu_2 = 0, \quad \nu_1 + \nu_2 \neq 0.$$

It follows that $\nu_1 = -\nu_2 = \nu$ and

$$H_{11,12} = \nu H_{11,13}, \qquad H_{22,12} = \nu(3H_{22,13} - 2H_{12,23}), \tag{12.2}$$

$$\left.\begin{aligned} 2H_{12,12} - H_{11,22} &= \nu(H_{12,13} - H_{11,23}), \\ \mu H_{11,12} &= \nu(2H_{12,23} - 2H_{22,13} + \mu H_{11,13}). \end{aligned}\right\} \tag{12.3}$$

Newt we take the following relations (3.7):

$[1,1;1,3|11]$: $(1-\mu)H_{11,13} - \nu H_{11,12} = 0,$

$[1,1;1,3|22]$: $\nu H_{11,12} = 0,$ (12.4)

$[2,2;1,3|11]$: $(1-\mu)H_{22,13} + \nu(2\mu H_{11,12} - 2H_{22,13} - H_{22,12}) = 0,$

$[2,2;1,3|22]$: $\nu(2\mu H_{11,12} - 2H_{22,13} - H_{22,12}) = 0,$

$[1,2;1,3|11]$: $2H_{12,13} - H_{11,23} - \mu H_{12,13} +$
$+ \nu(\mu H_{11,11} \; H_{12,12} - H_{13,13}) = 0,$

$[1,2;1,3|22]$: $H_{11,23} - H_{12,13} -$
$- \nu(\mu H_{11,11} - H_{12,12} - H_{13,13}) = 0,$ (12.5)

$[1,3;1,2|11]$: $\nu(H_{11,22} - H_{12,12} - H_{13,13}) + H_{12,13} +$
$+ (1-\mu)(H_{12,13} - H_{11,23}) = 0,$

$[1,3;1,2|22]$: $\nu(H_{11,22} - H_{12,12} - H_{13,13}) + H_{12,13} = 0,$ (12.6)

$[2,3;1,2|11]$: $H_{12,23} + \mu(H_{12,23} - H_{13,22}) - \nu H_{13,23} = 0,$

$[2,3;1,2|22]$: $H_{13,22} - 2H_{12,23} + \nu H_{13,23} = 0,$ (12.7)

$[2,3;1,3|11]$: $H_{13,23} + \mu(\mu H_{11,12} - 2H_{13,23}) - \nu H_{12,23} = 0,$

$[2,3;1,3|22]$: $H_{13,23} - \mu H_{11,12} + \nu H_{12,23} = 0.$ (12.8)

Suppose $\mu \neq 1$. Then from the first three pairs of relations we have $H_{11,13} = H_{22,13} = H_{12,13} = 0$ and (12.2) gives $H_{11,12} = 0$. Due to the next three pairs of relations $H_{11,23} = H_{12,23} = H_{13,23} = 0$ and from (12.2) and (12.3) $H_{22,12} = 2H_{12,12} - H_{11,22} = 0$. Now $H_{12,23} = H_{13,23} = 0$ and (12.1) give $\nu_3 H_{12,12} = 0$, $\nu_4 H_{13,13} = 0$ and thus $\nu_3 = \nu_4 = 0$. Adding

$[2,2;2,3|11]$: $-\mu H_{22,23} + \nu(3\mu H_{11,22} - 2H_{23,23} - H_{22,22}) = 0,$

$[2,2;2,3|22]$: $H_{22,23} - \nu(3\mu H_{11,22} - 2H_{23,23} - H_{22,22}) = 0$

we get $(1-\mu)H_{22,23} = 0$ and then $H_{22,23} = 0$. This together with $H_{11,23} = 0$ give

$$\nu(H_{11,11} - H_{11,22}) = 0,$$

$$\nu(H_{11,22} - H_{22,22}) = 0.$$

Here $\nu \neq 0$ yields a contradiction: $H_{11,11} = H_{11,22} = H_{22,22}$ and h_{11}, h_{22} could not be linearly independent. But $\nu = 0$ leads to a contradiction, too. Indeed, then $h_{23} = 0$ and $[2,3;1,2|13]$: $H_{12,12} - H_{11,22} = 0$ contradicts to $2H_{12,12} - H_{11,22} = 0$ obtained above.

So we have $\underline{\mu = 1}$ and $h_{33} = h_{11}$, $h_{23} = \nu(h_{11} - h_{22}) + \nu_3 h_{12} +$

$+ \nu_4 h_{13}$.

Suppose $\nu \neq 0$. Then $H_{11,12} = H_{11,13} = 0$ due to (11.6) and (11.2). Now [3,3;1,3] gives by h_{22} and h_{12} that $H_{13,23} = H_{11,23} = 0$ and [1,1;1,3] by h_{12} and h_{13} that $H_{12,13} = H_{11,11} - H_{13,13} = 0$. Substituting it in (12.5) we get a contradiction $\nu H_{12,12} = 0$.

Thus $\underline{\nu = 0}$ and $h_{33} = h_{11}$, $h_{23} = \nu_3 h_{12} + \nu_4 h_{13}$. From (12.2)-(12.8) we obtain $H_{11,12} = H_{22,12} = H_{11,22} - 2H_{12,12} = H_{12,13} = H_{13,22} - 2H_{12,23} = H_{11,23} = H_{13,23} = 0$. Thus $\nu_4 H_{13,13} = 0$, so $\nu_4 = 0$ and $h_{23} = \nu_3 h_{12}$. Taking now [1,2;2,3|1,3]: $H_{11,22} - H_{12,12} - \nu_3^2 H_{12,12} = 0$ we have $(1-\nu_3^2)H_{12,12} = 0$ and hence $\nu_3 = \pm 1$. Here the case $\nu_3 = 1$, when $h_{33} = h_{11}$, $h_{23} = h_{12}$, can be reduced to the case $\nu_3 = -1$ taking $-e_1$ instead of e_1.

For his part the case $\nu_3 = -1$: $h_{33} = h_{11}$, $h_{23} = -h_{12}$ turns after frame transformation $e'_3 = \frac{1}{\sqrt{2}}(e_1 + e_3)$, $e'_2 = \frac{1}{\sqrt{2}}(-e_1 + e_3)$, $e'_1 = e_2$ to

$$h'_{23} = h'_{13} = 0.$$

The straightforward control shows now that all semi-symmetricity conditions (3.7) by $h_{13} = h_{23} = 0$ and linearly independent $h_{11}, h_{22}, h_{12}, h_{33}$ reduce to $H_{11,12} = H_{11,33} = H_{22,12} = H_{22,33} = H_{12,33} = 0$, $H_{11,11} = H_{22,22} = 2H_{11,22} = 4H_{12,12}$.

13. It remains to analyse this only possible case thoroughly. If we denote $H_{12,12} = \varkappa^2$, $H_{33,33} = \lambda^2$ and take $e_4 \upuparrows h_{11} - h_{22}$, $e_5 \upuparrows h_{11} + h_{22}$, $e_6 \upuparrows h_{12}$, $e_7 \upuparrows h_{33}$, then we have the Pfaff system (3.3) in the form

$$\left.\begin{array}{l} \omega^4 = \omega^5 = \omega^6 = \omega^7 = \omega^\rho = 0;\ \rho = 8, 9, \ldots, n, \\ \omega_1^4 = \varkappa\sqrt{3}\omega^1,\ \omega_1^5 = \varkappa\omega^1,\ \omega_1^6 = \varkappa\omega^2,\ \omega_1^7 = 0,\ \omega_1^\rho = 0,\ \varkappa > 0, \\ \omega_2^4 = \varkappa\sqrt{3}\omega^2,\ \omega_2^5 = -\varkappa\omega^2,\ \omega_2^6 = \varkappa\omega^1,\ \omega_2^7 = 0,\ \omega_2^\rho = 0, \\ \omega_3^4 = 0\ ,\ \omega_3^5 = 0\ ,\ \omega_3^6 = 0\ ,\ \omega_3^7 = \lambda\omega^3,\ \omega_3^\rho = 0,\ \lambda > 0. \end{array}\right\} \quad (13.1)$$

This system is to be prolonged, i.e. to use the exterior differentiation and after that the Cartan lemma. We start with the equations of the last row. They give

$$\varkappa\sqrt{3}\omega_3^1 \wedge \omega^1 + \varkappa\sqrt{3}\omega_3^2 \wedge \omega^2 + \lambda\omega_4^7 \wedge \omega^3 = 0,$$

$$\varkappa\omega_3^1 \wedge \omega^1 - \varkappa\omega_3^2 \wedge \omega^2 + \lambda\omega_5^7 \wedge \omega^3 = 0,$$

$$\varkappa\omega_3^1 \wedge \omega^2 + \varkappa\omega_3^2 \wedge \omega^1 + \lambda\omega_6^7 \wedge \omega^3 = 0,$$

$$\lambda\omega_3^1\wedge\omega^1 + \lambda\omega_3^2\wedge\omega^2 + d\lambda\wedge\omega^3 = 0,$$

$$\lambda\omega_7^\rho\wedge\omega^3 = 0$$

and so

$$\varkappa\sqrt{3}\omega_3^1 = \varphi\omega^1 + \psi\omega^3,$$

$$\varkappa\sqrt{3}\omega_3^2 = \varphi\omega^2 + \chi\omega^3,$$

$$\lambda\omega_4^7 = \psi\omega^1 + \chi\omega^2 + \xi\omega^3,$$

$$\lambda\sqrt{3}\omega_5^7 = -\psi\omega^1 + \chi\omega^2 + \eta\omega^3,$$

$$\lambda\sqrt{3}\omega_6^7 = -\chi\omega^1 - \psi\omega^2 + \zeta\omega^3,$$

$$\varkappa\sqrt{3}d\ln\lambda = \psi\omega^1 + \chi\omega^2 + \sigma\omega^3,$$

$$\omega_7^\rho = \Lambda^\rho\omega^3.$$

The other equations (13.1) of the fourth column give

$$\lambda\omega_1^3\wedge\omega^3 + \varkappa\sqrt{3}\omega^1\wedge\omega_4^7 + \varkappa\omega^1\wedge\omega_5^7 + \varkappa\omega^2\wedge\omega_6^7 = 0,$$

$$\lambda\omega_2^3\wedge\omega^3 + \varkappa\sqrt{3}\omega^2\wedge\omega_4^7 - \varkappa\omega^2\wedge\omega_5^7 + \varkappa\omega^1\wedge\omega_6^7 = 0,$$

thus $\psi = \chi = \zeta = \eta = 0$, $\xi = \dfrac{\varphi\lambda^2}{3\varkappa^2}$. Denoting $-\dfrac{\varphi}{3\sqrt{3}} = \Phi$, $\dfrac{\sigma}{\varkappa\sqrt{3}} = \Psi$ we have

$$\omega_1^3 = \Phi\omega^1,\ \omega_2^3 = \Phi\omega^2,\ \omega_4^7 = -\frac{\Phi\lambda}{\varkappa\sqrt{3}}\omega^3,\ \omega_5^7 = 0,\ \omega_6^7 = 0,\ d\lambda = \Psi\omega^3. \quad (13.2)$$

In a similar way the other equations (13.1) lead to

$$-\frac{1}{2}d\ln\varkappa = A\omega^1 + B\omega^2 - \frac{1}{2}\Phi\omega^3,\ \frac{1}{6}(2\omega_1^2 - \omega_6^6) = -B\omega^1 + A\omega^2, \quad (13.3)$$

$$\frac{1}{\sqrt{3}}\omega_4^5 = A\omega^1 - B\omega^2, \qquad \frac{1}{\sqrt{3}}\omega_4^6 = B\omega^1 + A\omega^2, \quad (13.4)$$

$$\sqrt{3}\omega_4^\rho + \omega_5^\rho = P^\rho\omega^1 + Q^\rho\omega^2,\ \omega_6^\rho = Q^\rho\omega^1 + R^\rho\omega^2,$$

$$\sqrt{3}\omega_4^\rho - \omega_5^\rho = S^\rho\omega^1 + Q^\rho\omega^2.$$

14. Let $n=7$. Then the equations, in which ρ is the upper index, get lost. Now the fourth and the fifth equations (13.2) yield

$$A\Phi = B\Phi = 0. \quad (14.1)$$

Let here $\Phi=0$; then $\omega_1^3 = \omega_2^3 = 0$ and this together with (13.1) shows that M^3 is a product submanifold $M^2 \times M^1$. Here (13.3) and (13.4) give by differential prolongation

$$dA = B\omega_1^2 + \frac{1}{5}(14B^2 - 11A^2)\omega^1 - 5AB\omega^2,$$

$$dB = -A\omega_1^2 - 5AB\omega^1 + \frac{1}{5}(14A^2 - 11B^2)\omega^2$$

and now the exterior differentiation yields

$$A[\varkappa^2 + \frac{42}{25}(A^2 + B^2)] = B[\varkappa^2 + \frac{42}{25}(A^2 + B^2)] = 0;$$

thus A=B=0 and $\varkappa$ = const. Hence M^2 is the Veronese surface V^2 in a 5-plane (cf.[1]; see also Section 6). The other component M^1 is an integral line of the system $\omega^2 = \omega^3 = 0$ on M^3 and thus it is a plane line with curvature λ. So in this case semi-symmetric M^3 in E^7 is a product $V^2 \times M^1$ of a Veronese surface V^2 in E^5 and of a line M^1 in E^2, i.e. we have the case (13) in the list of Section 2.

Let A=B=0 in (14.1).Then $2\omega_1^2 = \omega_5^6$ and after the exterior differentiation we get Φ=0 and thus return to the previous case.

Let n > 7. Then (13.1) show that M^3 is a 2nd order envelope of the symmetric submanifolds $V^2 \times S^1$ and we get the case (14) of the list in Section 2.

This completes the proof of the theorem.

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КЛАССИФИКАЦИЯ ТРЕХМЕРНЫХ ПОЛУСИММЕТРИЧЕСКИХ ПОДМНОГООБРАЗИЙ В ЕВКЛИДОВЫХ ПРОСТРАНСТВАХ

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Р е з ю м е

Подмногообразие M^m в евклидовом пространстве E^n называется полусимметрическим, если $\bar{R}(X,Y)\cdot h=0$, где $\bar{R}$ есть оператор кривизны связности $\bar{\nabla}=\nabla\oplus\nabla^{\perp}$ ван дер Вардена-Бортолотти и h есть вторая фундаментальная форма. Полусимметрические поверхности ($m=2$) и гиперповерхности ($m=n-1$) в E^n классифицировал Деприи [2,3] (называя их полупараллельными; см. также [7,8]); при $m=n-2$ классификация получена в [8].

Классификация полусимметрических подмногообразий M^3 в евклидовых пространствах E^n, начатая в [12], доводится теперь до конца. Остались исследовать случаи, когда $n_1\geq 4$, где $n_1=\dim \operatorname{span} h(X,Y)$ - размерность первого нормального пространства рассматриваемого M^3; всегда $n_1\leq 6$. Метод исследования для этих случаев отличается от примененного в [12] и опирается на систему однородных кубических уравнений на компоненты второй фундаментальной формы h. Сперва устанавливается, используя результат из [9], что при $n_1=6$ полусимметрическое M^3 является либо (I) подмногообразием Веронезе в $S^8\subset E^9$, либо (II) огибающей 2-го порядка семейства конгруентных подмногообразий Веронезе; $n>9$. Затем доказывается, что полусимметрических подмногообразий M^3 с $n_1=5$ не существует.

Изучение оставшегося случая, когда $n_1=4$, оказалось технически наиболее сложным. В итоге получено, что полусимметрическое M^3 с $n_1=4$ в E^n является либо (III) произведением $V^2\times M^1$ в E^7, где V^2 - поверхность Веронезе в $S^5\subset E^6$ и M^1 - плоская линия, либо (IV) огибающей 2-го порядка семейства симметрических произведений $V^2\times S^1$ в E^n, $n>7$.

Эти результаты вместе с полученными в [12] суммированы в списке, приведенном в разделе 2 настоящей статьи. Этот список содержит 16 классов: один класс (3-плоскости) при $n_1=0$, четыре класса (2)-(5) при $n_1=1$, два класса (6) и (7) при $n_1=2$, пять классов (8)-(12) при $n_1=3$ (о всех их подробнее см. [12]), два класса (13) и (14) при $n_1=4$ - указанные выше классы (III) и (IV), два класса (15) и (16) при $n_1=6$ - указанные выше классы (I) и (II). Из них (1),(2),(3),(9),(11),(15) состоят из симметрических подмногообразий, остальные состоят и огибающих 2-го порядка семейств симметрических подмногообразий, указанных в разделе 2 для каждого класса.

THREE-DIMENSIONAL SUBMANIFOLDS WITH PARALLEL THIRD FUNDAMENTAL FORM IN EUCLIDEAN SPACES

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1. Introduction. It was proved by K.Nomizu for Riemannian manifolds that $\nabla^k R = 0$ implies $\nabla R = 0$ (for the global version that requires completeness, see [13]; cf. [6], Remark 7). In other words, a Riemannian manifold M^m with $\nabla^k R = 0$, $\nabla^{k-1} R \neq 0$, $k>1$, does not exist.

If M^m is immersed into a Euclidean space E^n as a submanifold, then R is determined, due to the Gauss equation, by the second fundamental form h (see (3.6) below) and the Levi-Civita connection ∇ is complemented by the normal connection $\nabla^\perp$ to the van der Waerden-Bortolotti connection $\bar\nabla = \nabla \oplus \nabla^\perp$. Now the condition $\bar\nabla^p h = 0$ yields $\nabla^k R = 0$ by $k = 2p-1$ and thus $\nabla R = 0$.

But can we have $\bar\nabla^p h = 0$, $\bar\nabla^{p-1} h \neq 0$, $p>1$ for a submanifold M^m in E^n ? Does such a submanifold exist?

If $\bar\nabla^p h := \bar\nabla(\bar\nabla^{p-1} h) = 0$, then M^m in E^n is said to be a submanifold with parallel (p+1)-st fundamental form $\bar\nabla^{p-1} h$ (or α_{p+1}). The well-known identity $\bar\nabla_{[X}\bar\nabla_{Y]} h = \bar R(X,Y)\cdot h$ for arbitrary $X,Y \in T_x M^m$, where $\bar R$ is the curvature operator of $\bar\nabla$, shows that a submanifold M^m with parallel third fundamental form (i.e. p=2) in E^n satisfies $\bar R(X,Y)\cdot h = 0$. The last condition is an algebraic one and characterizes the so called semi-symmetric [7,8,9] (or semi-parallel [1,2]) submanifolds.

The first results on submanifolds M^m with $\bar\nabla^p h = 0$ are obtained by V.Mirzoyan [16]. The first examples of surfaces M^2 with $\bar\nabla^2 h = 0$, $\bar\nabla h \neq 0$ in E^n, which give the affirmative answer to the question above in a particular case, are given in [15]. The simplest such surfaces are the products $C^1(a) \times \times E^1$ in E^3, $C^1(a) \times S^1(r)$ in E^4 and $C^1(a) \times C^1(a')$ in E^4, where $C^1(a)$ is a Cornu spiral (i.e. a plane line whose curvature is proportional to the arc length: $k = as$), $S^1(r)$ is a circle and E^1 is a straight line. A surface with $\bar{\nabla}^2 h = 0$, $\bar{\nabla} h \neq 0$ in E^4, which is not a product, is indicated in [15] by its Pfaff system. The geometrical construction of this surface is explained in [14] (see also [10]). It is a so called B-scroll $B^2(a,r)$ of a line in $S^3(r)$ with spherical curvature $k_S = as$ and with spherical torsion $\varkappa_S = \pm \frac{1}{r}$; here B-scroll means the surface described by binormal great circle of a line in $S^3(r)$. For getting product surfaces with $\bar{\nabla}^2 h = 0$, $\bar{\nabla} h \neq 0$ in E^n, a new line is given in [14] which is omitted in [15]: the spherical Cornu spiral $C^1_S(a,r)$ in $S^2(r)$ whose spherical curvature is $k_S = as$.

The general result of [14] is the full list of lines and surfaces with $\bar{\nabla}^2 h = 0$ in a Euclidean space E^n.

T H E O R E M 1 ([14]). *A line M^1 with $\bar{\nabla}^2 h = 0$ in E^n is either E^1, $S^1(r)$ in E^2, $C^1(a)$ in E^2 or $C^1_S(a,r)$ in E^3 or a part of one of them. A surface M^2 with $\bar{\nabla}^2 h = 0$ in E^n is a part of a complete surface which is either a product of two lines listed above (included $E^2 = E^1 \times E^1$) or $S^2(r)$ in E^3, $B^2(a,r)$ in E^4 or Veronese surface $V^2(r)$ in $S^4(r)$ in E^5.*

Here E^1 and E^2 have $h=0$, $S^1(r)$, $S^2(r)$ and $V^2(r)$ have $\bar{\nabla} h = 0$; $V^2(r)$ is minimal in $S^4(r)$ and represents the second standard immersion of $S^2(r)$ in $S^4(r)$.

All submanifolds M^m with parallel $\bar{\nabla} h$ and with flat normal connection $\nabla^{\perp}$ in E^n are classified in [10]. It turns out that the only irreducible M^m with flat $\nabla^{\perp}$ and with $\bar{\nabla}^2 h = 0$, $\bar{\nabla} h \neq 0$ are $C^1(a)$, $C^1_S(a,r)$ and $B^2(a,r)$.

T H E O R E M 2 ([10]). *A submanifold M^m with flat normal connection $\nabla^{\perp}$, and with parallel third fundamental form $\bar{\nabla} h$ in E^n is a part of a complete submanifold which is either a plane or a sphere (included straight line and circle) or one of $C^1(a)$, $C^1_S(a,r)$ and $B^2(a,r)$, or a product of several of them.*

In this paper we classify all three-dimensional submanifolds M^3 with $\bar{\nabla}^2 h = 0$ in E^n. It is remarkable that all M^3 with $\bar{\nabla}^2 h = 0$, $\bar{\nabla} h \neq 0$ in E^n are reducible (i.e. are products).

T H E O R E M 3. *A submanifold M^3 with $\bar\nabla^2 h = 0$ in E^n is a part of a complete submanifold which is either (1) a product of lines and surfaces with $\bar\nabla^2 h = 0$ listed in Theorem 1 or (2) a Veronese submanifold $V^3(r)$ in $S^8(r)$ or (3) a $W^3(r)$ in $S^5(r)$ with an orthogonal net of great 2-spheres and great circles of $S^5(r)$. Here $V^3(r)$ and $W^3(r)$ have $\bar\nabla h = 0$ (i. e. are symmetric by D.Ferus [5]) and are minimal in spheres $S^8(r)$ and $S^5(r)$, respectively.*

The proof of this theorem will be given below in the following way. The scheme of the proof is outlined in Section 2 and the apparatus is developed in Section 3. The necessary details are proved in Sections 4, 5 and 6.

Theorems 1, 2 and 3 lead us to a conjecture and to some problems which are formulated in Section 7.

2. Scheme of the proof. As we have remarked, every submanifold M^m with $\bar\nabla^2 h = 0$ in E^n is semi-symmetric. All semi-symmetric submanifolds M^3 in E^n are classified in [11,12]. They are listed in [11] by growing dimension $n_1 = \dim \operatorname{span} \{h(X,Y) | X,Y \in T_x M^3\}$ of the first normal space of the submanifold M^3.

If $0 \le n_1 \le 2$, then the semi-symmetric M^3 has flat $\nabla^\perp$ (see [8]) and we can use Theorem 2.

If $n_1=3$, we have five classes of semi-symmetric M^3 in E^n (below they are numbered as in [11]):

(8) Cartan type M^3 with flat ∇ in E^n, $n \ge 6$,

(9) Product $V^2(r) \times E^1$ in $E^6 \subseteq E^n$,

(10) 2nd order envelope of a family of $V^2(r) \times E^1$ in E^n, $n>6$,

(11) Minimal W^3 of a sphere $S^5(r)$ with an orthogonal net of great 2-spheres and great circles,

(12) M^3 with an orthogonal net of concentric 2-spheres and congruent logarithmic spirals with the pole at the centre of these 2-spheres in E^6.

In the case (8) we can use Theorem 2, too. The cases (9) and (11) are included in the formulation of Theorem 3.

Thus, only cases (10) and (12) need here a special investigation. This is done in Section 4, where we show that M^3 of the case (10) by $\bar\nabla^2 h = 0$ reduces to (9) and M^3 of the case (12) does not have $\bar\nabla^2 h = 0$.

If $n_1=4$, we have two classes of semi-symmetric M^3 in E^n:

(13) Product $V^2(r) \times M^1$ in E^7, where M^1 is a plane line,

(14) 2nd order envelope of a family of $V^2(r) \times S(r')$ in E^n, $n>7$.

In the case (13) the product $V^2(r) \times M^1$ has $\bar{\nabla}^2 h = 0$ if and only if the plane line M^1 satisfies $\bar{\nabla}^2 h = 0$, i.e. if M^1 is either $S^1(r')$ or $C^1(a)$. These products are included in Theorem 3.

The case (14) needs a special investigation. This is done in Section 5, where we show that M^3 of the case (14) with $\bar{\nabla}^2 h = 0$ is $V^2(r) \times C^1_E(a,r')$.

Finally, if $n_1=6$, then we have two classes of semi-symmetric M^3 in E^n:

(15) Veronese submanifold $V^3(r)$ in $S^8(r)$,

(16) 2nd order envelope of a family of congruent Veronese submanifolds $V^3(r)$, r = const. in E^n, $n>9$.

The case (15) is included in Theorem 3. In Section 6 we show that by $\bar{\nabla}^2 h = 0$ the case (16) reduces to (15). This finishes the proof of Theorem 3.

3. <u>Apparatus.</u> We use the Cartan moving frame method for the orthogonal frames $\{x; e_1, \dots, e_n\}$ in E_n:

$$dx = e_I \omega^I, \quad de_I = e_J \omega^J_I, \quad \omega^J_I + \omega^I_J = 0; \quad I, J, \dots = 1, \dots, n;$$

$$d\omega^I = \omega^J \wedge \omega^I_J, \quad d\omega^J_I = \omega^K_I \wedge \omega^J_K .$$

For the frames adapted to M^m in E^n (i.e. for elements of the bundle $\mathcal{O}(M^m, E^n)$; see [6], V.II, ch.VII) we have

$$\omega^\alpha = 0 \ (\alpha = m+1, \dots, n) \Rightarrow \omega^\alpha_i = h^\alpha_{ij} \omega^j \ (i, j = 1, \dots, m) \Rightarrow$$

$$\Rightarrow \bar{\nabla} h^\alpha_{ij} = (\bar{\nabla}_k h^\alpha_{ij}) \omega^k \Rightarrow \bar{\nabla}(\bar{\nabla}_k h^\alpha_{ij}) \wedge \omega^k = \bar{\Omega} h^\alpha_{ij} \qquad (3.1)$$

Here every implication $\Rightarrow$ is verified by exterior differentiation and using the Cartan lemma (in the first two cases). The second fundamental form $h\colon TM \times TM \to T^\perp M$ is given by $h(X,Y) = h^\alpha_{ij} X^i Y^j e_\alpha$ for $X = X^i e_i$, $Y = Y^j e_j$ and

$$\bar{\nabla} h^\alpha_{ij} := dh^\alpha_{ij} - h^\alpha_{kj} \omega^k_i - h^\alpha_{ik} \omega^k_j + h^\beta_{ij} \omega^\alpha_\beta \qquad (3.2)$$

are components of the covariant differential of h with respect to the connection $\bar{\nabla}$, thus $\bar{\nabla}_k h^\alpha_{ij} := h_{ijk}$ are the components of the third fundamental form $\bar{\nabla} h$; they are symmetric in indices i, j, k as follows from Cartan lemma (this symmetricity is the content of Peterson-Codazzi equations). So the third identity (3.1) is

$$dh^{\alpha}_{ij} - h^{\alpha}_{kj}\omega^{k}_{i} - h^{\alpha}_{ik}\omega^{k}_{j} + h^{\beta}_{ij}\omega^{\alpha}_{\beta} = h^{\alpha}_{ijk}\omega^{k}. \quad (3.3)$$

In the last identity (3.1)

$$\bar{\Omega}h^{\alpha}_{ij} := - h^{\alpha}_{kj}\Omega^{k}_{i} - h^{\alpha}_{ik}\Omega^{k}_{j} + h^{\beta}_{ij}\Omega^{\alpha}_{\beta},$$

where

$$\Omega^{k}_{i} := d\omega^{k}_{i} - \omega^{j}_{i}\wedge\omega^{k}_{j} = - \sum R^{k}_{i,pq}\omega^{p}\wedge\omega^{q}, \quad (3.4)$$

$$\Omega^{\alpha}_{\beta} := d\omega^{\alpha}_{\beta} - \omega^{\gamma}_{\beta}\wedge\omega^{\alpha}_{\gamma} = - \sum R^{\alpha}_{\beta,pq}\omega^{p}\wedge\omega^{q}, \quad (3.5)$$

$$R^{k}_{i,pq} = \sum h^{\alpha}_{i[p}h^{\alpha}_{q]k}, \qquad R^{\alpha}_{\beta,pq} = \sum h^{\alpha}_{i[p}h^{\beta}_{q]i} \quad (3.6)$$

So this last identity takes the form

$$\bar{\nabla}h^{\alpha}_{ijk}\wedge\omega^{k} = - h^{\alpha}_{kj}\Omega^{k}_{i} - h^{\alpha}_{ik}\Omega^{k}_{j} + h^{\beta}_{ij}\Omega^{\alpha}_{\beta}, \quad (3.7)$$

where in the left side $\bar{\nabla}h^{\alpha}_{ijk}$ is similar to (3.2).

If $\bar{\nabla}^2 h = 0$ then $\bar{\bar{\nabla}}h^{\alpha}_{ijk} = 0$ and

$$dh^{\alpha}_{ijk} = h^{\alpha}_{ljk}\omega^{l}_{i} + h^{\alpha}_{ilk}\omega^{l}_{j} + h^{\alpha}_{ijl}\omega^{l}_{k} - h^{\beta}_{ijk}\omega^{\alpha}_{\beta}. \quad (3.8)$$

The right side of (3.7) gives now

$$h^{\alpha}_{kj}\Omega^{k}_{i} + h^{\alpha}_{ik}\Omega^{k}_{j} - h^{\beta}_{ij}\Omega^{\alpha}_{\beta} = 0. \quad (3.9)$$

From (3.8) by exterior differentiation we get

$$h^{\alpha}_{ljk}\Omega^{l}_{i} + h^{\alpha}_{ilk}\Omega^{l}_{j} + h^{\alpha}_{ijl}\omega^{l}_{k} - h^{\beta}_{ijk}\Omega^{\alpha}_{\beta}. \quad (3.10)$$

Note that (3.9) characterizes the class of semi-symmetric (or semi-parallel) submanifolds M^m in E^n.

4. <u>Submanifolds M^3 with $\bar{\nabla}^2 h = 0$ by $n_1 = 3$.</u> As is explained in Section 2 we have to investigate the cases (10) and (12) of [11].

In the case (10) it is shown in [12] that the frame $\{x; e_1, \dots, e_n\}$ can be adapted to such M^3 so that

$$\|h^4_{ij}\| = \begin{pmatrix} \nu\sqrt{3} & 0 & 0 \\ 0 & \nu\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \|h^5_{ij}\| = \begin{pmatrix} \nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \|h^6_{ij}\| = \begin{pmatrix} 0 & \nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.1)$$

where $\nu \neq 0$, and

$$\|h^{\rho}_{ij}\| = 0; \ \rho, \sigma = 7, \dots, n. \quad (4.2)$$

Substituting this into (3.3) we get after some calculation that

$$
\left.\begin{aligned}
d\ln\nu &= \nu_1\omega^1 + \nu_2\omega^2 + \nu_3\omega^3,\\
2\omega_1^2 - \omega_5^6 &= \frac{5}{2}(\nu_2\omega^1 - \nu_1\omega^2),\\
\omega_1^3 &= \nu_3\omega^1,\\
\omega_2^3 &= \nu_3\omega^2,\\
\omega_4^5 &= \frac{\sqrt{3}}{2}(-\nu_1\omega^1 + \nu_2\omega^2),\\
\omega_4^6 &= -\frac{\sqrt{3}}{2}(\nu_2\omega^1 + \nu_1\omega^2),
\end{aligned}\right\} \qquad (4.3)
$$

$$
\left.\begin{aligned}
\omega_4^\rho &= p_1^\rho\omega^1 + p_2^\rho\omega^2,\\
\omega_5^\rho &= q_1^\rho\omega^1 + q_2^\rho\omega^2,\\
\omega_6^\rho &= (p_1^\rho\sqrt{3} + q_2^\rho)\omega^1 + (p_1^\rho\sqrt{3} - q_1^\rho)\omega^2
\end{aligned}\right\} \qquad (4.4)
$$

and h_{ijk}^α can be expressed by the next matrix:

α	4	5	6	ρ
1 1 1	$\frac{3}{2}\sqrt{3}\nu\nu_1$	$-\frac{1}{2}\nu\nu_1$	$-\nu\nu_2$	$\nu(p_1^\rho\sqrt{3} + q_1^\rho)$
1 1 2	$\frac{1}{2}\sqrt{3}\nu\nu_2$	$\frac{5}{2}\nu\nu_2$	$\nu\nu_1$	$\nu(p_2^\rho\sqrt{3} + q_2^\rho)$
1 1 3	$\sqrt{3}\nu\nu_3$	$\nu\nu_3$	0	0
1 2 2	$\frac{1}{2}\sqrt{3}\nu\nu_1$	$\frac{1}{2}\nu\nu_1$	$\nu\nu_2$	$\nu(p_1^\rho\sqrt{3} - q_1^\rho)$
1 2 3	0	0	$\nu\nu_3$	0
1 3 3	0	0	0	0
2 2 2	$\frac{3}{2}\sqrt{3}\nu\nu_2$	$-\frac{5}{2}\nu\nu_2$	$-\nu\nu_1$	$\nu(p_2^\rho\sqrt{3} - q_2^\rho)$
2 2 3	$\sqrt{3}\nu\nu_3$	$-\nu\nu_3$	0	0
2 3 3	0	0	0	0
3 3 3	0	0	0	0

(4.5)

If to substitute this into (3.8), we get by $i=1$ or $i=2$ and $j=k=3$ that $\nu_3=0$ and now $d\tilde{e}_3=0$. It follows that the considered M^3 is the product $\tilde{V}^2 \times \tilde{E}^1$, where $\tilde{V}^2$ is the 2nd order envelope of Veronese surfaces $V^2(r)$ which must also satisfy $\bar{\nabla}^2 h = 0$.

In [14] it is shown that the only $\tilde{V}^2$ with $\bar{\nabla}^2 h = 0$ is the $V^2(r)$ itself. For the sake of completeness we give here a short proof of this assertion. From (3.4), (3.5) and (3.6) it follows that

$$\left.\begin{array}{ll} \Omega^2_1 = -\nu^2\omega^1\Lambda\omega^2, & \Omega^6_5 = -2\nu^2\omega^1\Lambda\omega^2 \\ \Omega^5_4 = \Omega^6_4 = \Omega^\rho_4 = \Omega^\rho_5 = \Omega^\rho_6 = \Omega^\rho_\sigma = 0 & \end{array}\right\} \qquad (4.6)$$

and now (3.10) implies by $\alpha=4$ that $\nu_1 = \nu_2 = 0$ and by $\alpha=\rho$ that $p^\rho_1 = q^\rho_1 = p^\rho_2 = q^\rho_2 = 0$. Thus $h^\alpha_{ijk} = 0$, $\nu = \text{const} \neq 0$ and $\tilde{V}^2$ is $V^2(r)$ in $S^4(r)$, $r = \frac{1}{\nu\sqrt{3}}$ (see [14], [9]).

So in the case (10) the submanifold M^3 with $\bar{\nabla}^2 h = 0$ reduces to $V^2(r) \times E^1$, i.e. to the case (9).

In the case (12) it is shown in [12] that the frame $\{x; e_1, \ldots, e_6\}$ can be adapted to such M^3 in E^6 so that

$$\|h^4_{ij}\| = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}, \quad \|h^5_{ij}\| = \begin{pmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \|h^6_{ij}\| = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}, \qquad (4.5)$$

where $k \neq 0$. Substituting this into (3.3) we get after some calculation that

$$d\ln k = \varkappa\omega^1, \quad \omega^2_1 = -\varkappa\omega^2, \quad \omega^3_1 = -\varkappa\omega^3, \quad \omega^3_2 - \omega^6_5 = 0,$$

$$\omega^5_4 = -\varkappa\omega^2, \quad \omega^6_4 = -\varkappa\omega^3,$$

where in the case (12) we have $\varkappa \neq 0$, and h^α_{ijk} can be expressed by the next matrix:

ijk \ α	4	5	6
1 1 1	kϰ	0	0
1 1 2	0	kϰ	0
1 1 3	0	0	kϰ
1 2 2	kϰ	0	0
1 2 3	0	0	0
1 3 3	kϰ	0	0
2 2 2	0	-3kϰ	0
2 2 3	0	0	-kϰ
2 3 3	0	-kϰ	0
3 3 3	0	0	-3kϰ

If to substitute this into (3.8), we get by $i=j=k=1$ and $\alpha=5$ that $k\varkappa=0$ but this contradicts to $k\neq0$ and $\varkappa\neq0$. Thus the semi-symmetric submanifold M^3 of the case (12) does not have $\bar{\nabla}^2 h = 0$.

5. Submanifold M^3 with $\nabla^2 h = 0$ in the case (14).

For this case in [11] it is shown that the frame $\{x; e_1, \ldots, e_n\}$ can be adapted to M^3 so that we have (4.1), but (4.2) is to be replaced by

$$\|h^7_{ij}\| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \lambda \neq 0, \quad \|h^\xi_{ij}\| = 0; \ \xi = 8, \ldots, n$$

As further we get (4.3), but (4.4) is to be exchanged by

$$\omega^7_4 = -\frac{\lambda}{\nu\sqrt{3}} \nu_3 \omega^3, \qquad \omega^7_5 = \omega^7_6 = 0,$$

$$\omega^\xi_4 = p^\xi_1 \omega^1 + p^\xi_2 \omega^2, \qquad \omega^\xi_5 = q^\xi_1 \omega^1 + q^\xi_2 \omega^2,$$

$$\omega^\xi_6 = (p^\xi_2 \sqrt{3} + q^\xi_2)\omega^1 + (p^\xi_1 \sqrt{3} - q^\xi_1)\omega^2.$$

In addition to them

$$\omega^\xi_7 = \mu^\xi \omega^3, \ d\lambda = \lambda_3 \omega^3.$$

The last row in the matrix (4.5) is to be replaced by the next rows:

ijk \ α	7	ξ
1 1 1	0	$\nu(p^\xi_1\sqrt{3} + q^\xi_1)$
1 1 2	0	$\nu(p^\xi_2\sqrt{3} + q^\xi_2)$
1 1 3	$-\lambda\nu_3$	0
1 2 2	0	$\nu(p^\xi_1\sqrt{3} - q^\xi_1)$
1 2 3	0	0
1 3 3	0	0
2 2 2	0	$\nu(p^\xi_2\sqrt{3} - q^\xi_2)$
2 2 3	$-\lambda\nu_3$	0
2 3 3	0	0
3 3 3	λ_3	$\lambda\mu^\xi$

Now (3.8) gives by $i=1$ or $i=2$ and $j=k=3$ that $\nu_3=0$. Further (3.10) yields by $\alpha=4$ that $\nu_1 = \nu_2 = 0$ and by $\alpha=\xi$ that $p^\xi_1 = p^\xi_2 = q^\xi_1 = q^\xi_2 = 0$. So

$$de_3 = \lambda e_7 \omega^3, \qquad de_7 = (-\lambda e_3 + \mu^\xi e_\xi)\omega^3$$

and M^3 is in this case a product $V^2(r) \times M^1$, where M^1 is a line with $\bar\nabla^2 h = 0$. If $\mu^\xi=0$ then M^1 is a plane line, thus $S^1(r')$ or $C^1(a)$, and we return to the case (13), considered in Section 2. In the case (14) we have $\mu^\xi e_\xi \neq 0$ and $M^1 = C^1_s(a,r')$.

So in the case (14) the submanifold M^3 with $\bar\nabla^2 h = 0$ is $V^2(r) \times C^1_s(a,r')$ and is included in Theorem 3 under (1).

6. <u>Submanifold M^3 with $\bar\nabla^2 h = 0$ of the case (16).</u>

As is shown in [11] the frame $\{x;e_1,\dots,e_n\}$ can be adapted here to M^3 so that

$$\|h^4_{ij}\|=\begin{pmatrix}\varkappa\sqrt{3} & 0 & 0\\ 0 & \varkappa\sqrt{3} & 0\\ 0 & 0 & \frac{2}{3}\varkappa\sqrt{3}\end{pmatrix},\quad \|h^5_{ij}\|=\begin{pmatrix}\varkappa & 0 & 0\\ 0 & -\varkappa & 0\\ 0 & 0 & 0\end{pmatrix},\quad \|h^6_{ij}\|=\begin{pmatrix}0 & \varkappa & 0\\ \varkappa & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\|h^7_{ij}\|=\begin{pmatrix}0 & 0 & \varkappa\\ 0 & 0 & 0\\ \varkappa & 0 & 0\end{pmatrix},\quad \|h^8_{ij}\|=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \varkappa\\ 0 & \varkappa & 0\end{pmatrix},\quad \|h^9_{ij}\|=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \frac{2}{3}\varkappa\sqrt{6}\end{pmatrix},\quad \varkappa=\text{const}\neq 0$$

$$\|h^\tau_{ij}\| = 0;\qquad \tau = 10,\dots,n.$$

Substituting this into (3.8) we get $h^a_{ijk} = 0$; $a=4,\dots,9$. From (3.4) and (3.5) it follows that

$$\Omega^j_i = -\varkappa^2\omega^i A\omega^j,\qquad \Omega^\tau_a = \Omega^\tau_\sigma = 0$$

and now (3.10) gives that $h^\tau_{ijk} = 0$. Thus $\bar\nabla h = 0$ and our M^3 reduces to Veronese submanifold $V^3(r)$ in $S^8(r)$, $r = \frac{\sqrt{6}}{4\varkappa}$ (see [9], [11]). This finishes the proof of Theorem 3.

7. <u>Concluding remarks.</u> Theorems 1,2 and 3 formulated in Section 1 lead us to the

<u>Conjecture.</u> A submanifold M^m with $\bar\nabla^2 h = 0$ in E^n is a part of a complete submanifold which is either a symmetric submanifold (by D.Ferus [5], i.e. has $\bar\nabla h = 0$) or a line $C^1(a)$, $C^1_s(a,r)$, or a surface $B^2(a,r)$, or a product of several of them.

Theorems cited above show that this conjecture is true if $m \leq 3$ or if the normal connection $\nabla^\perp$ of M^m is flat.

The problem is: does it exist an irreducible submanifold M^m, $m>2$, with $\bar\nabla^2 h = 0$ and $\bar\nabla h \neq 0$ in E^n. We have shown (Theorems 2 and 3), that by flat $\nabla^\perp$ or by m=3 it does not.

There are some results about submanifolds M^m with $\bar\nabla^p h = 0$, $\bar\nabla^{p-1} h \neq 0$, in E^n. The first example is given by V.Mirzoyan (unpublished): the plane line M^1 whose curvature is proportional to the (p-1)st power of the arc length; he showed also that a M^m with $\bar{\nabla}^p h = 0$ in E^n is intrinsically a locally symmetric Riemannian manifold (i.e. $\nabla R = 0$) and if $\bar{\nabla}^{p-1} h \neq 0$, p>1, then noncompact and lies in its (p+1)st order osculating space.

All hypersurfaces M^{n-1} with $\bar{\nabla}^p h = 0$ in E^n are classified by F.Dillen [3]: such a hypersurface is a part of a complete M^{n-1} in E^n, which is either E^{n-1}, $S^{n-1}(r)$, $E^m \times S^{n-m-1}(r)$ or $E^{n-2} \times M^1$, where M^1 is a plane line whose curvature is a polynomial function of degree at most p-1 of the arc length. In [4] the first example of a surface M^2 with $\bar{\nabla}^p h = 0$, $\bar{\nabla}^{p-1} h \neq 0$, p>2, in E^n is given. It is the B-scroll of the line in $S^3(r)$ whose spherical curvature is a polynomial function of degree p-1 of the arc length and the spherical torsion is $\pm \frac{1}{r}$. These B-scrolls are the only surfaces with $\bar{\nabla}^p h = 0$, $\bar{\nabla}^{p-1} h \neq 0$ in $S^3(r)$.

Recently F.Dillen obtained the next generalization of our Theorem 2 (a private communication): a normally flat submanifold M^m with $\bar{\nabla}^p h = 0$ in E^n is a part of (1) E^m, (2) $S^m(r)$, (3) a complete flat normally flat submanifold with $\bar{\nabla}^k h = 0$ and with only one-fold principal curvature vectors, which do not vanish on an open dense subset, or (4) a product of submanifolds (1),(2) and (3), the sum of dimensions of which in m.

We do not know yet, which are the submanifolds with flat $\nabla^{\perp}$ and $\bar{\nabla}^p h = 0$ of the case (3) in general, i.e. if $\bar{\nabla}^{p-1} h \neq 0$, p>2. Which kind of submanifolds M^m in E^n we have, if to drop here the condition of flatness of $\nabla^{\perp}$? Is there any M^m with $\bar{\nabla}^p h = 0$ in E^n, which is not semi-symmetric (here is necessary p>2 and nonflat $\nabla^{\perp}$)?

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ТРЕХМЕРНЫЕ ПОДМНОГООБРАЗИЯ С ПАРАЛЛЕЛЬНОЙ ТРЕТЬЕЙ ФУНДАМЕНТАЛЬНОЙ ФОРМОЙ В ЕВКЛИДОВЫХ ПРОСТРАНСТВАХ

Ю.Лумисте

Р е з ю м е

Пусть E^n является евклидовым пространством, M^m подмногообразием в нем, h второй фундаментальной формой, ∇ связностью Леви-Чивита, $\nabla^{\perp}$ нормальной связностью и пусть $\bar\nabla = \nabla\oplus\nabla^{\perp}$. Третьей фундаментальной формой называется $\bar\nabla h$; говорят, что она параллельна, если $\bar\nabla^2 h: = \bar\nabla\bar\nabla h = 0$. Отсюда следует полусимметричность ([1],[7],[11]).

Все поверхности ($m = 2$) с $\bar\nabla^2 h = 0$ классифицированы в [14]. Теперь классифицируются все трехмерные подмногообразия с $\bar\nabla^2 h = 0$.

Т е о р е м а. Подмногообразие M^3 с $\bar\nabla^2 h = 0$ в E^n является частью полного подмногообразия, которое есть либо (1) произведение линий или поверхностей с $\bar\nabla^2 h = 0$, найденных в [14], либо (2) подмногообразие Веронезе $V^3(r)$ в $S^8(r)$, либо (3) подмногообразие $W^3(r)$ в $S^5(r)$ с ортогональной сетью больших 2-сфер и больших окружностей этой $S^5(r)$. Здесь $V^3(r)$ и $W^3(r)$ обладают свойством $\bar\nabla h = 0$ (т.е. они симметрические по Д.Ферусу [5]) и являются минимальными подмногообразиями в сферах $S^8(r)$ и $S^5(r)$, соответственно.

Высказана гипотеза, что любое M^m с $\bar\nabla^2 h = 0$ и E^n либо типа (1), либо симметрическое по Ферусу. Сформулированы некоторые открытые еще проблемы.

INVESTIGATION OF SUBMANIFOLDS IN THE METRIZED PROJECTIVE SPACES BY REDUCE

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1. Introduction. In this paper the m-dimensional submanifolds V_m in the Euclidean (non-Euclidean) N-dimensional spaces ${}^lE_N$(corresp. ${}^lS_N$) of index l are considered. In the case $l = 0$ we get the well-known spaces E_N and S_N, but to get a more general treatment we consider the metrized projective spaces ${}^lP_N$ [4]. The spaces ${}^lE_N$ and ${}^lS_N$ may be obtained as the special cases of ${}^lP_N$.

The submanifold V_m is given by its Pfaff system. The existence of the submanifold with some prescribed properties is established by means of the well-known exterior differential method [6]. Often the obtained equations are rather complicated. For this reason we use here the system of analytic computations REDUCE [1-3] on EC-1060 and on PC XT.

We remark that REDUCE was applied in [5] to investigate the demi-focal connection between 2-dimensional submanifolds in E_4.

2. The description of the metrized projective space. Let ${}^lP_N$ be the N-dimensional metrized projective space [4] with index l ($0 \le l \le N$). Let us have the next indeces:

$$I,J,K,\ldots = 1,2,\ldots,N; \qquad I_0,J_0,K_0,\ldots = 0,1,2,\ldots,N.$$

Let $\{e_0, e_1, \ldots, e_N\}$ be a frame in ${}^lP_N$, where in Euclidean case e_0 is the initial point and e_I the vectors of the frame. In non-Euclidean case $\{e_{I_0}\}$ is some set of geometrical points in the space.

Let $g_{I_0 J_0} = \langle e_{I_0}, e_{J_0} \rangle = \langle e_{J_0}, e_{I_0} \rangle$. Then for a suitable frame $\{e_{I_0}\}$ we get

$$g_{00} = \varepsilon \cdot \varepsilon_0 \ , \quad |\varepsilon_0| = 1 \ , \quad |g_{I_0 J}| = \delta^J_{I_0} \ , \tag{1}$$

where $\varepsilon = 0$ in Euclidean and $\varepsilon = 1$ in non-Euclidean case. Such frame is in the first case called the orthonormal and in the second one the autopolar and normed frame.

For the space ${}^1P_N$ and the frame $\{e_{I_0}\}$ the following derivation formulae hold:

$$de_0 = \omega^I_0 e_I , \qquad de_I = \varepsilon \cdot \omega^0_I e_0 + \omega^K_I e_K . \tag{2}$$

Let us denote $\varepsilon_{I_0 J_0} = g_{I_0 I_0} \cdot g_{J_0 J_0}$, then we get due to (1):

$$\omega^0_{I_0} = -\varepsilon \cdot \varepsilon_{0 I_0} \omega^{I_0}_0 , \tag{3}$$

$$\omega^J_I = -\varepsilon_{IJ} \omega^I_J , \tag{4}$$

which yield

$$\omega^{I_0}_{I_0} = 0. \tag{5}$$

By the exterior differentiation we get from (2) the following structure equations:

$$d\omega^J_0 = \omega^K_0 \wedge \omega^L_K , \tag{6}$$

$$d\omega^J_I = \varepsilon \cdot \omega^0_I \wedge \omega^J_0 + \omega^K_I \wedge \omega^J_K . \tag{7}$$

At the beginning of corresponding REDUCE-program SPACE the initial values of N, EPS = ε and the metric as an $1 \cdot (N+1)$ matrix GG (GG(1,1) = g_{00}, GG(1,2) = g_{11}, ..., GG(1,N+1) = g_{NN}) are loaded. We could not use the identifier G, because it is reserved in REDUCE for gamma matrices in high energy physics calculations [1].

Then the formulae (3)-(7) are from the program SPACE loaded. After that it is useful to clear the identifiers EPS and GG, because they are not used in the following (the evaluation of the formulae will be faster in REDUCE if we have as few as possible evaluated identifiers).

Some problems arise by programming of the exterior differentiation which is an anticommutative operation. In general the expressions in REDUCE are the associative-commutative polynomials of identifiers, operators and functions. Unfortunately in this version of REDUCE the command NONCOM, which declares some operations to be non-commutative under multiplication, is not implemented.

The second possibility is to declare a new infix operator, e.g. the symbol V (there is no key $\wedge$ on the keyboard):

```
INFIX V;  PRECEDENCE  V,-;
```

The declaration PRECEDENCE says that V should be inserted into the infix operator precedence list after the subtraction and for the multiplication operator [1]. But in our version this ability is, however, not usable.

We use the third possibility: the operator W on the left side of the operator $\wedge$ is renamed as operator V and instead of $\wedge$ the usual operation of multiplication * is used. For example, if we denote W(I) = ω_0^I, the formula (4) and anticommutativity are given as

```
FOR ALL J LET V(J)*W(J) = 0;
FOR ALL J,K SUCH THAT J>K LET V(J)*W(K) = -V(K)*W(J);
```

The expressions, involving exterior differentiation, are printed with the help of an procedure PRN, which displays the operator $\wedge$ as &, W1 instead of V and W2 instead of W. For example, the formula V(1)*W(2) is displayed in the natural way as W1&W2.

3. Submanifolds in the space ^1P_N. Let V_m (m<N) be some m-dimensional submanifold in ^1P_N. We use the following indices:

$$i,j,k,\ldots = 1,2,\ldots,m; \quad i_0,j_0,k_0,\ldots = 0,1,2,\ldots,m;$$
$$p,q,s,\ldots = m+1,\ldots,N.$$

We may choose the frame in such a way, so that the vectors e_{i_0} belong to the tangent surface of V_m. Then we may assume that the forms $\omega_0^1,\omega_0^2,\ldots,\omega_0^m$, are linearly independent and the system of Pfaff equations, which determines V_m is

$$\text{(I)}\qquad \begin{cases} \omega_0^p = 0, \\ \omega_i^p = \Gamma^p_{ij}\omega_0^j, \quad \Gamma^p_{ij} = \Gamma^p_{ji}, \end{cases}$$

If the submanifold V_m is of rank r (r < m), then V_m has n=(m-r)-dimensional generatrices and is a variety of planes of r parameters.

Assume that $e_1,e_2,\ldots,e_n$ are on the n-dimensional generatrix and $\omega_0^{n+1},\ldots,\omega_0^{n+r}$ is the maximal linearly independent subsystem of the system $\omega_0^1,\omega_0^2,\ldots,\omega_0^m$. If the indeces are

$$a,b,c,\ldots = 1,2,\ldots,n \quad\text{and}\quad t,u,v,\ldots = n+1,\ldots,m=n+r,$$

then the following system of Pfaff equations defines the submanifold V_a of rank r<m:

$$
(II) \quad \begin{cases} \omega^p_0 = \omega^p_a = 0, \\ \omega^p_u = \Gamma^p_{ut}\omega^t_0, \quad \Gamma^p_{ut} = \Gamma^p_{tu}, \\ \omega^u_a = \gamma^p_{at}\omega^t_0, \quad \gamma^t_{au}\Gamma^p_{ts} = \gamma^t_{as}\Gamma^p_{tu}. \end{cases}
$$

Remark. The system (II) follows from the system (I) if we take

$$\Gamma^p_{ab} = \Gamma^p_{at} = 0 \quad \text{and} \quad \omega^u_a = \gamma^u_{at}\omega^t_0.$$

The system (I) or (II) is in the REDUCE-program SUBMAN given. The correspondent equations would form the base system of the cycle of extension. Respectively to the realization of exterior differentiation in p.2., the submanifold is declared with the analogical formulae as all the space. Some of the equalities are also duplicated in the "left form" of the exterior differentiation, i.e. W(I,J) and W(I) are replaced with V(I,J) and V(I). Instead of the coefficients Γ^p_{ut} and γ^u_{at} only one operator B(J), which satisfies the given conditions, is used.

4. The existence of the submanifold. To determine the existance of the submanifold (I) or (II), we need to find two natural numbers Q and $\tilde{N}$ [6]. The exterior differentiation of the corresponded Pfaff system gives the following system of covariancies

$$(I') \qquad \nabla \Gamma^p_{ij} \wedge \omega^j_0 = 0,$$

where $\nabla \Gamma^p_{ij} = d\Gamma^p_{ij} - \Gamma^p_{kj}\omega^k_i - \Gamma^p_{ik}\omega^k_j + \Gamma^q_{ij}\omega^p_q$,

or

$$
(II') \quad \begin{cases} \nabla \Gamma^p_{ut} \wedge \omega^t_0 = 0, \\ \nabla \gamma^u_{at} \wedge \omega^t_0 = 0, \end{cases}
$$

where

$$\nabla \Gamma^p_{ut} = d\Gamma^p_{ut} - \Gamma^p_{st}\omega^s_u - \Gamma^p_{us}\omega^s_t + \Gamma^q_{ut}\omega^p_q,$$

$$\nabla \gamma^u_{at} = d\gamma^u_{at} - \gamma^u_{b_0 t}\omega^{b_0}_a - \gamma^u_{as}\omega^s_t + \gamma^v_{at}\omega^u_v$$

and

$$d\gamma^t_{au}\Gamma^p_{ts} + \gamma^t_{au}dP^p_{ts} = d\gamma^t_{as}\Gamma^p_{tu} + \gamma^t_{as}d\Gamma^p_{tu}.$$

From the system (I') or (II') the program computes the Cartan number Q [6] with the help of ranks of some linear forms. We remark that our procedure RNK gives $\bar{r}$ = rank A (for the matrix A) and also the row and column indeces for one minor $M \neq 0$ of degree $\bar{r}$.

Using the Cartan Lemma [6] for the systems (I') or (II'), we get

$$(I'') \qquad \nabla \Gamma^p_{ij} = \Gamma^p_{ijk}\omega^k_0, \quad \Gamma^p_{ijk} = \Gamma^p_{ikj}$$

or

$$(II'') \qquad \begin{cases} \nabla \Gamma^p_{ut} = \Gamma^p_{utq}\omega^q_0 - \Gamma^p_{uq}\gamma^q_{at}\omega^a_0, & \Gamma^p_{utq} = \Gamma^p_{uqt}, \\ \nabla \gamma^u_{at} = \gamma^u_{atq}\omega^q_0 - \gamma^u_{aq}\gamma^q_{bt}\omega^b_0, & \gamma^u_{atq} = \gamma^u_{aqt}, \end{cases}$$

where Γ^p_{ijk}, Γ^p_{utq} and γ^u_{atq} are the new variables (in the program they are denoted by B(I) for the next free values of I). Then the integer $\tilde{N}$ is the number of all these new variables.

If $Q = \tilde{N}$, then the submanifold V_m, given with Pfaff system (I) or (II), exists [6]. If $Q < \tilde{N}$, then the initial system has no solution, and if $Q > \tilde{N}$, then this process must be repeated (extended).

The corresponded REDUCE-program SOLVER has several subprograms, in which the new variables are used in some other way, because they may be chosen in a linear dependence way as simple as possible. New variables are at the beginning denoted by $X^2, X^3, \ldots$ to apply the REDUCE-functions connected with polynomials. To solve a nontrivial equation, we at first search for the free variables W(J,K). If they are absent, we search for the free variables DB(J) and then for the X^J. If they all are absent then, if the equation contains B(J)-s, we express one of them, having the lowest difficulty, via the others. This connection is named infinite. All the infinite connections are gathered together and at the end of solution process the system is resolved using these connections.

The user may to switch on a mode, in which for every equation the program asks on the screen, which variable is to be expressed (e.g. W(2,3) or DB(1) from the equation B(1)*W(2,3) - B(2)*DB(1) = 0).

In the case if the equation does not contain even the variables B(J), the program asks if it is a contradiction or whether this equation is to be ignored or which variable is to be expressed.

5. <u>An example of Euclidean space.</u> Let us consider the 2-dimensional submanifold V_2' in the 4-dimensional Euclidean space ${}^0R_4$ ($\varepsilon = 0$, $N = 4$, $g_{II} = 1$, $m = 2$, $n = 2$, $r = 0$) which is defined throw the next system:

$$(I_1)\qquad \begin{cases} \omega_3 = \omega_4 = 0, \\ \omega_1^3 = \Gamma^3_{11}\omega_1 + \Gamma^3_{12}\omega_2, \\ \omega_2^3 = \Gamma^3_{21}\omega_1 + \Gamma^3_{22}\omega_2, \\ \omega_1^4 = \Gamma^4_{11}\omega_1 + \Gamma^4_{12}\omega_2, \\ \omega_2^4 = \Gamma^4_{21}\omega_1 + \Gamma^4_{22}\omega_2, \end{cases}$$

where $\Gamma^3_{21} = \Gamma^3_{12}$ and $\Gamma^4_{21} = \Gamma^4_{12}$.

Assume that V_2' has the constant curvature and the frame is canonized. Then we have the following supplementary conditions:

$$(I_2)\qquad \begin{cases} \omega_3^4 = \Gamma^4_{31}\omega_1 + \Gamma^4_{32}\omega_2, \\ \omega_1^2 = \qquad\quad \Gamma^2_{12}\omega_2, \\ \Gamma^3_{22} = \dfrac{(\Gamma^4_{12})^2 + (\Gamma^3_{12})^2 - C^2}{\Gamma^3_{11}}, \\ \Gamma^4_{11} = 0. \end{cases}$$

These systems are by the program PRN displayed as follows:

(1.1) W(1,3) = B(1)*W1 + B(2)*W2

(1.2) W(2,3) = B(2)*W1 + $(B(4)^2 + B(2)^2 - C^2)$/B(1)*W2

(1.3) W(1,4) = B(4)*W2

(1.4) W(2,4) = B(4)*W1 + B(5)*W2

(1.5) W(3,4) = B(6)*W1 + B(7)*W2

(1.6) W(1,2) = B(8)*W2

After the exterior differentiation and simplification the program SOLVER establishes that the Cartan number Q = 8. Then the corresponding system is solved and the expressions for DB(1),...,DB(8) are found. For example,

DB(8) = $(C^2 - B(8)^2)$*W1 - B(13)*W2,

where B(13) is one of the 8 new variables B(9),...,B(16). Thus, $\tilde{N} = 8 = Q$, the system $(I_1) + (I_2)$ is compatible and the corresponding submanifold V_2' exists.

6. <u>An example of non-Euclidean space.</u> Let us consider a 3-dimensional submanifold V''_3 in the 4-dimensional non-Euclidean space $^2S_4$ ($\varepsilon = 1$, $N = 4$, $g_{00} = g_{22} = -1$, $g_{11} = g_{33} = g_{44} = 1$, $m = 3$, $n = 1$, $r = 2$):

$$(II_2)\quad \begin{cases} \omega_0^4 = \omega_1^4 = 0, \\ \omega_1^2 = \gamma_{12}^2\omega_0^2, \\ \omega_1^3 = \gamma_{13}^3\omega_0^3, \\ \omega_2^4 = \Gamma_{22}^4\omega_0^2, \\ \omega_3^4 = \Gamma_{33}^4\omega_0^3. \end{cases}$$

The program PRN rewrites this system in the form:

```
(1.1)  W(1,2) = B(1)*W2
(1.2)  W(1,3) = B(2)*W3
(1.3)  W(2,4) = B(3)*W2
(1.4)  W(3,4) = B(4)*W3
(1.5)  W(1,4) = 0
```

By (4) the program computes

W(2,1) = W(1,2), W(3,1) = -W(1,3), W(4,2) = W(2,4), W(4,3) = -W(3,4), W(3,2) = W(2,3).

Then, as in the preceding example, the program finds $\tilde{N} = 2 = Q$ and the submanifold V''_3 exists.

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ИССЛЕДОВАНИЕ ПОДМНОГООБРАЗИЙ В МЕТРИЗОВАННОМ ПРОЕКТИВНОМ ПРОСТРАНСТВЕ С ПОМОЩЬЮ REDUCE

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Р е з ю м е

Рассматриваются m-мерные подмногообразия V_m в метризованном проективном N-мерном пространстве ^1P_N индекса 1, частными случаями которого являются евклидово пространство ^1E_N и неевклидово пространство ^1S_N.

Во многих работах подмногообразие V_m задается с помощью системы Пфаффа, совместность которой определяется продолжением этой системы (внешним дифференцированием и применением леммы Картана). Решение соответствующей системы уравнений требует часто довольно громоздких преобразований формул.

В данной работе описывается система программ на языке аналитических вычислений REDUCE, позволяющая проводить подобные вычисления на ЭВМ ЕС-1060 и на PC XT. Приведены примеры использования пакета в евклидовом и неевклидовом пространствах.

ON THE GELFAND-MAZUR THEOREM FOR EXPONENTIALLY GALBED ALGEBRAS

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1. Introduction. Let A be a topological algebra (that is a linear topological space over $\mathbb{C}$ wich is an associative algebra over $\mathbb{C}$ with separately continuous multiplication). The element $a \in A$ is called to be *bounded in A* if there exists a $\lambda \in \mathbb{C}\setminus\{0\}$ such that the set $\{(a/\lambda)^n) : n \in \mathbb{N}\}$ is bounded in A. It is easy to see that $a \in A$ is bounded in A if and only if there exists a $\lambda \in \mathbb{C}\setminus\{0\}$ such that $((a/\lambda)^n)$ converges in A to zero element Θ of A.

In the papers [5,6] Turpin considered a new class of topological algebras which he called *exponentially galbed algebras* (algèbres de exponentiellement galbés). These are such topological algebras A for which for each neighbourhood of zero $\mathcal{U}$ there exists a neighbourhood of zero $\mathcal{V}$ such that

$$\left\{\sum_{k=0}^{n} \frac{a_k}{2^k} : a_0,\dots,a_n \in \mathcal{V}\right\} \subset \mathcal{U} \tag{1}$$

for each $n \in \mathbb{N}$. It is easy to show that all locally pseudoconvex algebras (consequently, all locally convex algebras and all locally bounded algebras) are exponentially galbed.

Topological algebra A is called a *Gelfand-Mazur algebra* if the quotient A/M is topologically isomorphic to $\mathbb{C}$ for each such closed two-sided regular ideal M of A which is maximal in A as a left ideal and as a right ideal. The main classes of locally pseudoconvex Gelfand-Mazur algebras have been given in [1]. Examples of topological algebras which are not Gelfand-Mazur algebras have been given in [2], p.214-217; [3], p.75; [7], p.141-148; [8]; [11], p.83-86 and [12], p.127.

The following generalization of the Gelfand-Mazur Theorem has been given by Turpin in [5].

T H E O R E M 1. *Let A be a an exponentially galbed Hausdorff division algebra for which all elements are bounded. Then A is topologically isomorphic to* $\mathbb{C}$.

He only mentioned in [5] that this result is valid by the Liouville's Theorem for A-valued holomorphic function, where A is an exponentially galbed space.

In the present paper we shall give a new proof for Theorem 1 in the case when the multiplication in A is jointly continuous. In this case there exists a locally bounded topology on every maximal commutative subalgebra of A. Therefore it is possible to prove Theorem 1 in this case by using the Gelfand-Mazur Theorem for locally bounded division algebras. Moreover, we show that every exponentially galbed algebra A, all elements of which are bounded in A, is a Gelfand-Mazur algebra.

2. <u>Proof of Theorem 1.</u> Let A be an algebra with unit e and $sp_A(a)$ be the set of all $\lambda \in \mathbb{C}$ for which $a - \lambda e$ is not invertible in A. First we prove the following results.

L E M M A 1. *Let A be a Hausdorff division algebra. If on every maximal commutative subalgebra of A with unit there exists a topology in respect of which it is a Gelfand-Mazur algebra, then A is topologically isomorphic to* $\mathbb{C}$.

P r o o f. Let $a \in A$ and A(a) be the commutative subalgebra of A generated by a and e. Then there exists a maximal commutative subalgebra $\mathfrak{A}$ of A such that $A(a) \subseteq \mathfrak{A}$. As A is a division algebra, then $\mathfrak{A}$ is also a division algebra. If there exists a topology on $\mathfrak{A}$ in respect of which $\mathfrak{A}$ is a Gelfand-Mazur algebra then there exists a topological isomorphism ν of $\mathfrak{A}$ onto $\mathbb{C}$. Since $\mathfrak{A}$ is maximal commutative subalgebra in A then $sp_A(a) = sp_{\mathfrak{A}}(a)$ for each $a \in A$ (cf. [9], p.46). Therefore $\nu(a) \in sp_A(a)$ for each $a \in A$. Consequently (cf., for example, [1], p.381), A is topologically isomorphic to $\mathbb{C}$.

L E M M A 2. *Let A be a commutative exponentially galbed Hausdorff division algebra with jointly continuous* *multiplication. If every element of A is bounded, then there exists a such topology on A that A is a locally bounded algebra in this topology.*

P r o o f. As every element of A is bounded then for each $a \in A$ there exists a $\lambda \in \mathbb{C}\setminus\{0\}$ such that $((a/\lambda)^n)$ converges to Θ in A. For each $a \in A$ we put

$$\beta(a) = \inf\{|\lambda| : \lambda \in \mathbb{C}\setminus\{0\} \text{ and } \lim_{n\to\infty} (\tfrac{a}{\lambda})^n = \Theta \text{ in } A\}.$$

We shall show that β is a quasi-norm on A. It is clear that $0 \le \beta(a) < \infty$ for each $a \in A$ and $\beta(\mu a) = |\mu|\beta(a)$ for each $\mu \in \mathbb{C}$ and $a \in A$. To prove that $\beta(a_1 + a_2) \le 2(\beta(a_1) + \beta(a_2))$ for each $a_1, a_2 \in A$, we fix a_1 and a_2 in A and a neighbourhood of zero U of A. Since A is exponentially galbed, then there exists a circled neighbourhood of zero V such that (1) is valid. By our assumption the multiplication in A is jointly continuous. Therefore there exists a neighbourhood $\mathscr{W}$ such that $\mathscr{W}^2 \subset V$.

Let now λ_1 and λ_2 be such numbers that the sequences $((a_1/\lambda_1)^n)$ and $((a_2/\lambda_2)^n$ converge to Θ. Then the sets $\{(a_1/\lambda_1)^n : n \in \mathbb{N}\}$ and $\{(a_2/\lambda_2)^n : n \in \mathbb{N}\}$ are bounded in A. Hence there exist $\rho_1, \rho_2 \in \mathbb{C}\setminus\{0\}$ such that $(a_1/\lambda_1)^n \in \rho_1\mathscr{W}$ and $(a_2/\lambda_2)^n \in \rho_2\mathscr{W}$ for each $n \in \mathbb{N}$. Moreover, there exist $n_1, n_2 \in \mathbb{N}$ such that $(a_1/\lambda_1)^n \in \rho_2^{-1}\mathscr{W}$ if $n>n_1$ and $(a_2/\lambda_2)^n \in \rho_1^{-1}\mathscr{W}$ if $n>n_2$. Let now $N = n_1 + n_2$ and let $n>N$. Then (by reason of commutativity of A) we have

$$z_n = \left(\frac{a_1 + a_2}{2(|\lambda_1| + |\lambda_2|)}\right)^n = \sum_{k=0}^{n} \frac{b_k}{2^k},$$

where

$$b_k = \frac{\binom{n}{k} \lambda_1^k \lambda_2^{n-k} 2^k}{(|\lambda_1| + |\lambda_2|)^n 2^n} \left(\frac{a_1}{\lambda_1}\right)^k \left(\frac{a_2}{\lambda_2}\right)^{n-k}$$

for each $k \in \mathbb{N}_n = \{0,1,\ldots,n\}$. As

$$\left(\frac{a_1}{\lambda_1}\right)^k \left(\frac{a_2}{\lambda_2}\right)^{n-k} \in (\rho_2^{-1}\mathscr{W})(\rho_2\mathscr{W}) = \mathscr{W}^2 \subset V$$

if $k>n_1$,

$$\left(\frac{a_1}{\lambda_1}\right)^k \left(\frac{a_2}{\lambda_2}\right)^{n-k} \in (\rho_1\mathscr{W})(\rho_1^{-1}\mathscr{W}) = \mathscr{W}^2 \subset V$$

if $k<n-n_2$ and

$$\frac{\binom{n}{k}\ |\lambda_1|^k|\lambda_2|^{n-k}\ 2^k}{(|\lambda_1| + |\lambda_2|)^n\ \ 2^n} \le 1$$

for each $k \in \mathbb{N}_n$ then $b_k \in V$ for each $k \in \mathbb{N}_n$. Hence $(z_n) \in U$ for each $n > \mathbb{N}$. It means that (z_n) converges to Θ in A. Hence

$$\beta(a_1 + a_2) \le 2(|\lambda_1| + |\lambda_2|)$$

for each such λ_1 and λ_2 for which $((a_1/\lambda_1)^n)$ and $((a_2/\lambda_2)^n)$ converge to Θ in A. Now it is easy to show that $\beta(a_1+a_2) \le \le 2(\beta(a_1) + \beta(a_2))$. Moreover, $\beta(a_1 . a_2) \le \beta(a_1)\beta(a_2)$ for each $a_1, a_2 \in A$ because the multiplication in A is jointly continuous. As A is a division algebra then $\beta(a) > 0$ for each $a \in \in A\setminus\{\Theta\}$ as $1 \le \beta(e) \le \beta(a)\beta(a^{-1})$. Consequently, β is a quasi-norm on A. Endowing A with the topology, defined by means of neighbourhood of zero $\{a \in A: \beta(a) \le \alpha\}$ where $\alpha>0$, we see that A is a locally bounded algebra in this topology (see [4], p.159).

Now by Lemmas 1 and 2 we prove Theorem 1 in the case when the multiplication in A is jointly continuous. For it let $\mathfrak{A}$ be a maximal commutative subalgebra of A. Then $\mathfrak{A}$ is a commutative exponentially galbed Hausdorff division algebra with jointly continuous multiplication, each element of which is bounded. Therefore by Lemma 2 there exists a topology τ on $\mathfrak{A}$ in respect of which $\mathfrak{A}$ is a locally bounded Hausdorff division algebra. Thus $\mathfrak{A}$ is a Gelfand-Mazur algebra in topology τ (cf.[10], p.345, or [11], p.18). Consequally, A is topologically isomorphic to $\mathbb{C}$ by Lemma 1 and in the considerable case Theorem 1 is proved.

3. New class of Gelfand-Mazur algebras. Let A be an exponentially galbed algebra and M be a closed two-sided regular ideal of A which is maximal as a left ideal and as a right ideal of A. Then the quotient algebra A/M is an exponentially galbed Hausdorff division algebra. Moreover, if all elements of A are bounded in A, then all elements of A/M are bounded in A/M. Therefore by Theorem 1 we have

T H E O R E M 2. *Every exponentially galbed algebra, all elements of which are bounded in A, is a Gelfand-Mazur algebra.*

In particular, for locally pseudoconvex algebras, Theorem 2 has been proved in [1].

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О ТЕОРЕМЕ ГЕЛЬФАНДА-МАЗУРА ДЛЯ ЭКСПОНЕНЦИАЛЬНО ИЗОГНУТЫХ АЛГЕБР

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Р е з ю м е

Топологическая $\mathbb{C}$-алгебра A (с раздельно непрерывным умножением) называется *экспоненциально изогнутой алгеброй*, если для каждой окрестности нуля $\mathcal{U}$ алгебры A найдется такая окрестность нуля $\mathcal{V}$ алгебры A, что справедливо (1) для каждого $n \in \mathbb{N}$, и *алгеброй Гельфанда-Мазура*, если факторалгебра A/M алгебры A является топологически изоморфной полю $\mathbb{C}$ для каждого такого замкнутого двустороннего регулярного идеала M алгебры A, который максимален как левый идеал и как правый идеал этой алгебры.

В статье [5] утверждено, что каждая отделимая экспоненциально изогнутая алгебра A с делением является топологически изоморфной полю $\mathbb{C}$, если каждый ее элемент a ограничен в смысле Аллана (т.е. найдется такое число $\lambda \in \mathbb{C}\backslash\{0\}$, что множество $\{(a/\lambda)^n\colon n \in \mathbb{N}\}$ ограниченно в A). В данной заметке дается простое доказательство для этого утверждения в случае когда в рассматриваемой алгебре умножение элементов непрерывно. Кроме того, показывается, что каждая экспоненциально изогнутая алгебра, все элементы которой ограничены в смысле Аллана, является алгеброй Гельфанда-Мазура.

STRONGLY SPECTRALLY BOUNDED ALGEBRAS

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The concept of "strong spectral boundedness" for locally m-convex algebras is extended in the present paper to the case of locally pseudoconvex (not necessarily of locally m-pseudoconvex) algebras. It is shown that every sequentially complete pseudobarrelled strongly spectrally k-bounded Hausdorff algebra is (within a topological isomorphism) a k-Banach algebra (Theorem 1) and every commutative sequentially complete pseudobarrelled strongly spectrally k-bounded *-algebra with the weakened C^*-property and with jointly continuous multiplication is (within a topological isomorphism) a C^*-algebra if $0 < k \leq 1$ (Theorem 2a). Moreover, an analogue of Gelfand-Naimark Theorem (Theorem 2c)) and an analogue of *-isomorphism theorem of C^*-algebras (Theorem 3) are given.

Preliminaries

1. Let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$. A topological algebra is a (non-zero) topological linear space together with an associative separately continuous multiplication making it an algebra over $\mathbb{K}$. If the respective topological linear space is, in particular, locally pseudoconvex (locally convex) we shall speak of a *locally pseudoconvex* (resp. *locally convex*) algebra. On the other hand, if the multiplication of a given topological algebra is jointly continuous, we speak of a *topological* (resp. *locally pseudoconvex and locally convex*) *algebra with continuous multiplication.*

The topology of a locally pseudoconvex algebra A is usually given by a family of k_α-homogeneous seminorms $P_A = \{p_\alpha : \alpha \in \mathfrak{A}\}$ where $0 < k_\alpha \le 1$ for each $\alpha \in \mathfrak{A}$. In particular, when $k_\alpha = k$ ($k_\alpha = 1$) for each $\alpha \in \mathfrak{A}$, algebra A is said to be a *locally k-pseudoconvex* (resp. *a locally convex*) algebra. At this, a locally pseudoconvex algebra is said to be *locally m-pseudoconvex* (*locally k-(m-pseudoconvex)*) if any k_α- homogeneous (resp. any k-homogeneous) seminorm in P_A is submultiplicative. Moreover, a locally pseudoconvex algebra A is said to be *A-pseudoconvex* or *absorbingly pseudoconvex* if for each $p_\alpha \in P_A$ and $a \in A$ there exist $M > 0$ and $N > 0$ such that $p_\alpha(ab) \le M\, p_\alpha(b)$ and $p_\alpha(ba) \le N\, p_\alpha(b)$ for all $b \in A$. In particular, when constants M and N depend only on a, but not on p_α, then algebra A is called a *uniformly A-pseudoconvex.* In addition to this, a locally pseudoconvex (m-pseudoconvex, k-pseudoconvex and k-(m-pseudoconvex)) $*$-algebra A (i.e. an algebra A with continuous involution $a \to a^*$) is said to be a *locally pseudoconvex* (resp. *m-pseudoconvex, k-pseudoconvex* and *k-(m-pseudoconvex)*) *C^*-algebra* if $p_\alpha(a)^2 = p_\alpha(a^*a)$ for each $p_\alpha \in P_A$ and $a \in A$. At this, if the family P_A satisfies a weakened C^*-condition (i.e. there exists a constant C (not depending on p_α) such that $p_\alpha(a)^2 \le C\, p_\alpha(a^*a)$ for each $a \in A$) then we speak of a *locally pseudoconvex* (resp. *m-pseudoconvex, k-pseudoconvex* and *k-(m-pseudoconvex)*) *$*$-algebra with a weakened C^*-property. Locally m-convex algebras, A-convex algebras, uniformly A-convex algebras, locally m-convex C^*-algebras* and etc., are defined similarly.

Further on, we call a locally pseudoconvex (locally k-pseudoconvex) algebra A *strongly spectrally bounded* (resp. *strongly spectrally k-bounded*) if $\sup_\alpha p_\alpha(a) < \infty$ for each $a \in A$.

In the case, when the topology of an algebra A is given by a submultiplicative k-homogeneous ($0 < k \le 1$) norm $\|\cdot\|$ and A is complete with respect to this norm, we speak of a k-*Banach algebra* A. A k-Banach $*$-algebra A we call a *k-(C^*-algebra)* if $\|a\|^2 = \|a^*a\|$ for each $a \in A$ and call a *k-Banach $*$-algebra with weakened C^*-property* if $\|a\|^2 \le C\, \|a^*a\|$ for each $a \in A$. In a particular case, when $k = 1$, we speak about Banach algebras, C^*-algebras and Banach $*$-algebras with weakened C^*-property.

2. A topological algebra A is said to ba a *Gelfand-Mazur algebra* if the quotient algebra A/M is topologically isomorphic with $\mathbb{K}$ for each such closed regular two-sided ideal M which is maximal as a left ideal and as a right ideal in A. In the case, when the set $\mathbb{Q}$invA of all quasi-invertible elements of algebra A is open in A, A is called a *$\mathbb{Q}$-algebra*. By a *Fréchet algebra* we mean a topological al-algebra for which the respective topological linear space is a Fréchet space (i.e., metrizable and complete). Moreover, a topological algebra A is called a *barrelled algebra* if every barrel of A (i.e. a closed circled convex and absorbing set) is a neighbourhood of zero in A. Similarly, a topological algebra A is called a *pseudobarrelled algebra* if every pseudobarrel of A (i.e. a closed circled pseudoconvex and absorbing set) is a neighbourhood of zero in A. It is clear that every pseudobarrelled algebra is a barrelled algebra. Moreover, every Fréchet algebra is a pseudobarrelled algebra. To prove it, let T be a pseudobarrel in a Fréchet algebra A. Then $A = \bigcup\{kT: k \in \mathbb{N}\}$ (T is an absorbing set). Therefore according to Baire's theorem there exist $a_0 \in A$ and an open set V in A such that $a_0 \in V \subset T$. Let now $\lambda > 0$ be such a real number that $T + T \subseteq \lambda T$ and f be a mapping of A onto A such that

$$f(a) = \frac{a-a_0}{2\lambda}$$

for each $a \in A$. Then f is a homeomorphism. Now it is clear that f(V) is an open neighbourhood of zero in A. As $f(V) \subset T$ then T is a neighbourhood of zero in A.

Sequentially complete pseudobarrelled strongly spectrally bounded algebras

It is known that any complete barrelled locally m-convex strongly spectrally bounded C^*-algebra is topologically isomorphic with a C^*-algebra ([3,4,6]). Moreover, any complete barrelled uniformly A-convex Hausdorff algebra with unit is topologically isomorphic with a Banach algebra with unit [13,14]. As all uniformly A-convex algebras with unit are strongly spectrally bounded algebras, it is interesting to know for which strongly spectrally bounded algebras there exists a k-homogeneous norm λ on A for some $k \in (0,1]$ such that (A,λ) (i.e. the algebra A endowed with the topology defined by norm λ) is a k-Banach algebra. The following theorem gives an answer to this question:

T H E O R E M 1. *Let $k \in (0,1]$ and A be a sequentially complete pseudobarrelled (barrelled in the case, when $k = 1$) strongly spectrally k-bounded Hausdorff algebra. Then there exists a k-homogeneous submultiplicative norm λ on A such that*

a) *(A,λ) is a k-Banach algebra*

and

b) *(A,λ) is topologically isomorphic with A.*

P r o o f. Let $\{p_\alpha : \alpha \in \mathfrak{A}\}$ be the family of k-homogeneous seminorms on A which defines the topology of A and let

$$\mu(a) = \sup_\alpha p_\alpha(a)$$

for each $a \in A$. Then μ is a k-homogeneous norm on A. As the multiplication of elements in A is separately continuous, there exist for each $a \in A$ constants $M > 0$ and $N > 0$ such that $\mu(ab) \le M\,\mu(b)$ and $\mu(ba) \le N\,\mu(b)$ for each $b \in A$. Therefore, (A,μ) is a k-normed algebra.

Next, we shall show the completeness of the algebra (A,μ). Let (a_n) be a Cauchy sequence in (A,μ). Then (a_n) is also a Cauchy sequence in A (since $p_\alpha(a) \le \mu(a)$ for each $a \in A$ and seminorm p_α). By virtue of sequentially completeness of algebra A, the sequence (a_n) converges in A to an element a_0. Let now O be a neighbourhood of zero in (A,μ). Then there exists $\delta > 0$ such that

$$O_0 = \{a \in A: \mu(a) \le \delta\} \subset O.$$

As the seminorms p_α are continuous on A and

$$O_0 = \bigcap_{\alpha\in\mathfrak{A}} \{a \in A: p_\alpha(a) \le \delta\},$$

then O_0 is a closed subset of A. It is clear that O_0 is pseudoconvex (in the case, when $k = 1$, O_0 is convex) circled and absorbing. Consequently, O_0 (as a pseudobarrel in A) is a neighbourhood of zero in A. Therefore there exists $N \in \mathbb{N}$ such that $\mu(a_n - a_0) \le \delta$ for each $n > N$. It means that the sequence (a_n) converges in (A,μ) to a_0. So (A,μ) is a Fréchet algebra. By this reason, the multiplication in (A,μ) is jointly continuous, i.e. there exists $M > 0$ such that

$\mu(ab) \le M\,\mu(a)\mu(b)$ for each $a,b \in A$. Putting now $\lambda(a) = M\,\mu(a)$ for each $a \in A$, it is clear that λ is a k-homogeneous submultiplicative norm on A. Therefore (A,λ) is a k-Banach algebra

Let now ε_A be the identity mapping on A. As all pseudobarrelled algebras are barrelled, then A is a barrelled algebra. Therefore, ε_A is an open mapping of (A,λ) onto A by the Open Mapping Theorem ([8], Theorem 7). Consequently, ε_A is a topological isomorphism of (A,λ) onto A.

The next result follows directly from Theorem 1.

C O R O L L A R Y 1. *Let $k \in (0,1]$ and A be a sequentially complete pseudobarrelled (barrelled in the case, when $k = 1$) strongly spectrally k-bounded Hausdorff *-algebra with the weakened C^*-property. Then there exists such a k-homogeneous submultiplicative norm λ on A that (A,λ) is a k-Banach *-algebra with weakened C^*-property within a topological algebraic isomorphism.*

Later on we shall use the following notions. For each element a of a given topological algebra A we put

$$sp_A(a) = \{\lambda \in \mathbb{C}\setminus\{0\} : \tfrac{a}{\lambda} \notin \mathrm{QinvA}\} \cup \{0\}$$

and

$$\rho_A(a) = \sup_{\lambda \in sp_A(a)} |\lambda|.$$

C O R O L L A R Y 2. *Let $k \in (0,1]$ and A be a complete pseudobarrelled (barrelled in the case, when $k = 1$) locally k-(m-pseudoconvex) Hausdorff C^*-algebra. If A is a Q-algebra then there exists such a k-homogeneous submultiplicative norm λ on A that (A,λ) is a k-(C^*-algebra) within a topological algebraic isomorphism.*

P r o o f. By Theorem 1 it suffices to show that A is a strongly spectrally k-bounded algebra. For it let $\{p_\alpha : \alpha \in \mathfrak{A}\}$ be the family of k-homogeneous seminorms on A which defines the topology of A. For each $\alpha \in \mathfrak{A}$ let $N_\alpha = \ker p_\alpha$, $A_\alpha = A/N_\alpha$, $\pi_\alpha(a) = a + N_\alpha$ and $q_\alpha(\pi_\alpha(a)) = p_\alpha(a)$ for each $a \in A$. Then q_α is a k-homogeneous submultiplicative norm on A_α. Let $\tilde{A}_\alpha$ be the completion of A_α, τ_α be the topological isomorphism of A_α into $\tilde{A}_\alpha$ (defined by the completion of A_α) and $\tilde{q}_\alpha$ be the extension of q_α to $\tilde{A}_\alpha$. Now $\tilde{q}_\alpha$ is a k-homogeneous submultiplicative norm on $\tilde{A}_\alpha$ which defines its topology and

$$\tilde{q}_\alpha[\tau_\alpha o \pi_\alpha(a)] = p_\alpha(a)$$

for each $a \in A$ and $\alpha \in \mathfrak{U}$.

By Theorem 5 from [19] the mapping Φ, defined by $\Phi(a) = (\tau_\alpha o \pi_\alpha(a))_{\alpha\in\mathfrak{U}}$ for each $a \in A$, is a topological isomorphism of A onto the projective limit of k-Banach algebras $\tilde{A}_\alpha$ and

$$sp_{\Phi(A)}(\Phi(a)) = \bigcup_{\alpha\in\mathfrak{U}} sp_{\tilde{A}_\alpha}(\mu_\alpha(\Phi(a)))$$

for each $a \in A$ (cf.[19], Theorem 1), where μ_α is the projection of $\Phi(A)$ into $\tilde{A}_\alpha$. Consequently,

$$\rho_{\Phi(A)}(\Phi(a)) = \sup_{\alpha\in\mathfrak{U}} \rho_{\tilde{A}_\alpha}(\mu_\alpha(\Phi(a)))$$

for each $a \in A$. It is easy to see that $\tilde{A}_\alpha$ is in the present case a k-(C^*-algebra). Therefore (in the same way as in the case of C^*-algebras) it is true that

$$\rho_{\tilde{A}_\alpha}[\mu_\alpha(\Phi(a))] = \tilde{q}_\alpha[\mu_\alpha(\Phi(a))] = p_\alpha(a)$$

for each $a \in A$. As A is a Q-algebra, then $\Phi(A)$ is also a Q-algebra. Consequently (cf.[11], p.60),

$$\sup_{\alpha\in\mathfrak{U}} p_\alpha(a) = \rho_{\Phi(A)}(\Phi(a)) < \infty$$

for each $a \in A$.

In a particular case, when $k = 1$, Corollary 2 is known (cf.[10], Theorems 1 and 3, [4], Corollary 2.1. and [5], Theorem 4.3.).

C O R O L L A R Y 3. *Let $k \in (0,1]$ and A be a sequentially complete pseudobarrelled (barrelled in the case, when $k = 1$) strongly spectrally k-bounded Hausdorff algebra with left (or right) approximate identity (e_λ), the topology of which has been defined by the family $\{p_\alpha : \alpha \in \mathfrak{U}\}$ of k-homogeneous seminorms. If*

$$\sup_\lambda \sup_{\alpha\in\mathfrak{U}} p_\alpha(e_\lambda) < \infty,$$

then

a) *for every $a \in A$ and neighbourhood of zero $\mathcal{U}$ in A there exist elements $b, c \in A$ such that $a = bc$ (resp. $a = cb$) and $c-a \in \mathcal{U}$*

and

b) *every maximal two-sided ideal in A is a prime ideal.*

P r o o f. By Theorem 1 there exists a k-homogeneous submultiplicative norm λ on A such that (A,λ) is a k-Banach algebra. As (A,λ) satisfies all hypotheses of Theorem 1 and Corollary 1 from [18], then the statements a) and b) are true by these results.

In a particular case, when k = 1, an analogical result with Corollary 3a) has been proved in [2], p.187.

Gelfand-Naimark theorem for commutative strongly spectrally bounded algebras

Let A be a topological $\mathbb{C}$-algebra, $\hom_0 A$ be the set of all continuous complex homomorphisms of A and hom A be the subset of all non-zero homomorphisms in $\hom_0 A$, endowed with the relative weak topology induced on it by the weak topological dual of A. We assume that A is a topological algebra for which the set hom A is not empty. If now $a \in A$, then $a^\wedge$ will denote the $\mathbb{C}$-valued function on hom A defined by $a^\wedge(\varphi) = \varphi(a)$ for each $\varphi \in \hom A$. It is easy to see that $a^\wedge$ is a continuous function on hom A for each $a \in A$. Let $\mathcal{G}_A$ denote the mapping of A into the algebra C(hom A) (of all continuous $\mathbb{C}$-valued functions on hom A) defined by $\mathcal{G}_A(a) = a^\wedge$ for each $a \in A$. Moreover, let

$$\operatorname{rad} A = \bigcap_{\varphi \in \hom A} \ker\varphi$$

and for every $\varphi_0 \in \hom A$, $\varepsilon > 0$ and $a_1, \dots, a_n \in A$ let

$$\Phi_A(\varphi_0; a_1, \dots, a_n, \varepsilon) = \bigcap_{k=1}^{n} \{\varphi \in \hom A: |(\varphi - \varphi_0)(a_k)| < \varepsilon\}.$$

Next, we prove some lemmas.

L E M M A 1. *Let $k \in (0,1]$ and A be a commutative k-Banach *-algebra with the weakened C^*-property. Then there exists $\gamma > 0$ such that*

$$\gamma\|a\| \le \lim_{n \to \infty} \|a^n\|^{1/n} \le \|a\| \qquad (1)$$

for each $a \in A$.

P r o o f. Similarly, as in the case of Banach algebras, it is easy to prove that the limit

$$\nu(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

exists for each $a \in A$. As A is a k-Banach *-algebra with the weakened C^*-property, then there exists $\beta \geq 1$ such that

$$\|a\|^2 \leq \beta\|a^*a\| \tag{2}$$

for each $a \in A$. Therefore from

$$\|a^n\|^2 \leq \beta\|(a^n)^*a^n\| = \beta\|(a^*a)^n\| \leq \beta\|(a^*)^n\| \; \|a^n\|$$

follows

$$\|a^n\|^{2/n} \leq \beta^{1/n}\|(a^*a)^n\|^{1/n} \leq \beta^{1/n}\|(a^*)^n\|^{1/n}\|a^n\|^{1/n}$$

for each $a \in A$ and $n \in \mathbb{N}$. Going now in this inequality to the limit as $n \to \infty$, we have

$$\nu(a)^2 \leq \nu(a^*a) \leq \nu(a^*)\nu(a). \tag{3}$$

To prove that $\nu(a^*) = \nu(a)$ for each $a \in A$, we first assume that a is a self-adjoint element in A (that is, $a^* = a$). Then

$$\|a\| \leq \beta^{1/2}\|a^2\|^{1/2} \leq \ldots \leq \beta^{1/2+1/4+\ldots+2^{-n}}\|a^{2^n}\|^{2^{-n}}$$

for each $n \in \mathbb{N}$ by the inequality (2). Going again to the limit as $n \to \infty$, we have that

$$\|a\| \leq \beta\nu(a). \tag{4}$$

Taking this into account, we have

$$\|a\|^2 \leq \beta\|a^*a\| \leq \beta^2\nu(a^*a) \leq \beta^2\nu(a^*)\nu(a) \tag{5}$$

for each $a \in A$ by inequalities (2) and (3).

Since A is a commutative Gelfand-Mazur Q-algebra (cf. [20], Theorems 3.3 and 4.2, [16], Lemma 3.6 and [17], p.10), then in the case when A has a unit, the set hom A is not empty and

$$\rho_A(a) = \sup_{\varphi \in \text{hom } A} |\varphi(a)|$$

for each $a \in A$ (cf. [1], Theorem 1c)). Moreover,

$$\nu(a) = \sup_{\varphi \in \text{hom } A} |\varphi(a)|^k$$

for each $a \in A$ (cf. [16], Theorem 4.9, or [17], Theorem 4.8) Therefore

$$\nu(a) = \rho_A(a)^k \tag{6}$$

for each $a \in A$. As

$$sp_A(a^*) = \{\bar{\lambda}: \lambda \in sp_A(a)\}$$

for each $a \in A$, then

$$\nu(a^*) = \rho_A(a^*)^k = \rho_A(a)^k = \nu(a) \tag{7}$$

for each $a \in A$ by (6). Consequently, the inequality (5) is in the form (4) for each $a \in A$. When A has not a unit we shall consider $A \times \mathbb{K}$ instead of the algebra A. Similarly as in the case of C^*-algebras (cf.[22], p.242) it is easy to show that $A \times \mathbb{K}$ is a commutative k-Banach *-algebra with unit and with the weakened C^*-property relative to the norm defined by

$$\|(a,\lambda)\| = \sup_{b\in A,\|b\|\le 1} \|ab + \lambda b\| \tag{8}$$

for each $(a,\lambda) \in A \times \mathbb{K}$. As

$$\|(a,0)\| \le \|a\| \le \beta^2\|(a,0)\| \tag{9}$$

for each $a \in A$, then

$$\|a\| \le \beta^2\|(a,0)\| \le \beta^3\nu((a,0)) = \beta^3\lim_{n\to\infty} \|(a^n,0)\|^{1/n} \le \beta^5\nu(a)$$

by (4) for each $a \in A$. On the other hand, $\nu(a) \le \|a\|$ for each $a \in A$. Therefore the inequality (1) is valid for each $a \in A$.

L E M M A 2. *Let $k \in (0,1]$ and A be a commutative k-Banach *-algebra with the weakened C^*-property. Then there exists a submultiplicative norm λ on A such that (A,λ) is a C^*-algebra within a topological algebraic isomorphism.*

P r o o f. First we assume that A has a unit. Let $\lambda(a) = \nu(a)^{1/k}$ for each $a \in A$. Then λ is a submultiplicative norm on A by the inequality (4) (cf. [17], p.17). As $\nu(a)^2 = \nu(a^*a)$ for each $a \in A$ by (3) and (7) and there exists $C > 0$ such that

$$C\|a\|^{1/k} \le \lambda(a) \le \|a\|^{1/k} \tag{10}$$

for each $a \in A$ by Lemma 1, then (A,λ) is a commutative C^*-algebra with unit. When A has not an unit then $A \times \mathbb{K}$ (as a commutative k-Banach algebra with unit and with the weakened C^*-property relative to the norm (8)) is a C^*-algebra with unit relative to the norm λ' defined by $\lambda'((a,\mu)) = \nu((a,\mu))^{1/k}$ for each $(a,\mu) \in A \times \mathbb{K}$. Let now $\lambda(a) = \lambda'((a,0))$ for each $a \in A$. Then there exists $K > 0$ such that

$$K\,\|(a,0)\|^{1/k} \le \lambda(a) \le \|(a,0)\|^{1/k} \tag{11}$$

for each a ∈ A by Lemma 1. As (10) is valid for each a ∈ A by (9) and (11) then (A,λ) is a commutative C^*-algebra without unit. It is easy to see that

$$\{a \in A: \lambda(a) < \delta\} \subseteq \{a \in A: \|a\| < \varepsilon\}$$

if $0 < \delta < C\varepsilon^{1/k}$ and

$$\{a \in A: \|a\| < \delta\} \subseteq \{a \in A: \lambda(a) < \varepsilon\}$$

if $0 < \delta < \varepsilon^k$. Therefore, the identity mapping ε_A on A is a topological isomorphism of (A,λ) onto A.

L E M M A 3. *Let A be a topological algebra for which hom A is non-empty and equicontinuous and S be a everywhere dense subset of A. Then the sets* $\Phi_A(\varphi_0;a,\varepsilon)$ *where* $a \in S$ *and* $\varepsilon > 0$, *form in hom A a neighbourhood subbase of* $\varphi_0 \in$ *hom A.*

P r o o f. Let $\varphi_0 \in$ hom A and $O(\varphi_0)$ be a neighbourhood of it in hom A. Then there exist $n \in \mathbb{N}$, $\varepsilon > 0$ and fixed elements $a_1,\ldots,a_n \in A$ such that $\mathcal{U} = \Phi_A(\varphi_0;a_1,\ldots,a_n,\varepsilon) \subseteq O(\varphi_0)$. As hom A is equicontinuous, then there exists a neighbourhood O of zero in A such that $|\varphi(a)| < \varepsilon/4$ for each $a \in O$ and $\varphi \in$ hom A. Now for each $k = 1,2,\ldots,n$ O defines a $b_k \in S$ such that $b_k - a_k \in O$ and the set

$$\mathcal{V} = \Phi_A(\varphi_0;b_1,\ldots,b_n,\varepsilon/4)$$

is a neighbourhood of φ_0 in hom A. As

$$|(\varphi - \varphi_0)(b_k)| \le |\varphi(b_k - a_k)| + |(\varphi - \varphi_0)(a_k)| + \\ + |\varphi_0(b_k - a_k)| < \frac{3\varepsilon}{4} < \varepsilon$$

for each $\varphi \in \mathcal{V}$ then $\mathcal{V} \subseteq \mathcal{U} \subseteq O(\varphi_0)$.

Let now A and B be topological algebras, the spaces hom A and hom B of which are non-empty, T be a homomorphism of A into B and T^* be the mapping hom B into hom A which is defined by relation $T^*(\psi) = \psi \circ T$ for each $\psi \in$ hom B.

L E M M A 4. *Let A and B be topological algebras the spaces hom A and hom B of which are non-empty and T be a continuous homomorphism of A onto an everywhere dense subalgebra of B. Then*

1) T^* *is a homeomorphism of hom B into hom A if*

a) *hom B is equicontinuous*

or

b) T is open and hom A is locally equicontinuous;

2) $T^*(\text{hom } B)$ is closed in hom A if the multiplication in B is jointly continuous

and

3) $T^*(\text{hom } B) = \text{hom } A$ if

a) T is an open one-to-one mapping and the multiplication in B is jointly continuous

or

b) T is an one-to-one mapping, A is a regular algebra (that is for each closed subset S of hom A and $\varphi_0 \in \text{hom } A \setminus S$ there exists $a \in A$ such that $\varphi_0(a) = 1$ and $\varphi(a) = 0$ for each $\varphi \in S$) with $\text{rad } A = \{\theta_A\}$ for which for each $\varphi_0 \in \text{hom } A$ and its neighbourhood $\mathcal{U}$ there exist an open set $\mathcal{V}$ and an element $a \in A$ such that $\varphi_0 \in cl_{\text{hom } A} \mathcal{V} \subset \mathcal{U}$, $\varphi(a) = 1$ for each $\varphi \in cl_{\text{hom } A} \mathcal{V}$ and $\varphi(a) = 0$ for each $\varphi \in \in \text{hom } A \setminus \mathcal{U}$, *and B is a such topological algebra with jointly continuous multiplication, for which $\text{rad } B \subset \mathbb{Q}\text{inv}B$.*

P r o o f. 1) It is clear that T^* is one-to-one. Let $X = T^*(\text{hom } B)$, $\psi_0 \in \text{hom } B$, $\varphi_0 = T^*(\psi_0)$ and let $O(\varphi_0)$ be an open neighbourhood of φ_0 in X. Then there exist $n \in \mathbb{N}$, $\varepsilon > 0$ and $a_1, \dots, a_n \in A$ such that $\mathcal{V} = X \cap \mathbb{O}_A(\varphi_0; a_1, \dots, a_n, \varepsilon) \subseteq \subseteq O(\varphi_0)$. Now

$$\mathcal{U} = (T^*)^{-1}(\mathcal{V}) = \mathbb{O}_B(\psi_0; T(a_1), \dots, T(a_n), \varepsilon)$$

is a neighbourhood of ψ_0 in hom B and $T^*(\mathcal{U}) = \mathcal{V} \subseteq O(\varphi_0)$. Hence T^* is continuous.

To prove that T is open, we first assume that hom B is equicontinuous. Let $\mathcal{U}$ be an open subset of hom B and let $\varphi_0 \in T^*(\mathcal{U})$. Then $\varphi_0 = T^*(\psi_0)$ for a $\psi_0 \in \text{hom } B$ and there exists a neighbourhood $O(\psi_0)$ of ψ_0 in hom B such that $O(\psi_0) \subseteq \mathcal{U}$. As $T(A)$ is everywhere dense in B, then there exist $m \in \mathbb{N}$, $\varepsilon > 0$ and $a_1, \dots, a_m \in A$ such that

$$W = \mathbb{O}_B(\psi_0; T(a_1), \dots, T(a_m), \varepsilon) \subseteq O(\psi_0)$$

by Lemma 3. Now

$$T^*(W) = X \cap \mathbb{O}_A(\varphi_0; a_1, \dots, a_m, \varepsilon)$$

is a neighbourhood of φ_0 in X and $T^*(W) \subseteq T^*(O(\varphi_0)) \subseteq T^*(\mathcal{U})$. Hence T^* is an open mapping.

Let now T be an open mapping and hom A be locally equicontinuous. Let again $\mathcal{U}$ be an open set in hom B and $\varphi_0 \in$

$\in T^*(\mathcal{U})$. Then $\varphi_0 = T^*(\psi_0)$ for a $\psi_0 \in$ hom B and there exists a neighbourhood $O(\psi_0)$ of ψ_0 in hom B such that $O(\psi_0) \subseteq \mathcal{U}$. Now

$$\Phi = \Phi_B(\psi_0; b_1, \ldots, b_n, \varepsilon) \subseteq O(\psi_0)$$

for some $n \in \mathbb{N}$, $\varepsilon > 0$ and $b_1, \ldots, b_n \in B$. As hom A is locally equicontinuous, then there exists in hom A an equicontinuous neighbourhood $O(\varphi_0)$ of φ_0. Therefore for each $\delta > 0$ there exists an open neighbourhood of zero O_A in A such that $|\varphi(a)| < \delta/2$ if $a \in O_A$ and $\varphi \in O(\varphi_0)$. In view of the fact that T is open, $T(O_A)$ is an open neighbourhood of zero in $T(A)$. Hence $T(O_A) = T(A) \cap O_B$ for an open neighbourhood of zero O_B in B. Let now $b \in O_B$ and $O(b)$ be a neighbourhood of it in B. Then $O_B \cap O(b)$ is also a neighbourhood of b in B. As T(A) is everywhere dense in B, then the intersection

$$O(b) \cap T(O_A) = (O(b) \cap O_B) \cap T(A)$$

is not empty. Hence $b \in B_0 = cl_B T(O_A)$. Thus, from $O_B \subseteq B_0$ follows that B_0 is a neighbourhood of zero in B.

Further on, let $Y = (T^*)^{-1}(O(\varphi_0))$ and $\psi \in Y$. Then $\psi(B_0) \subseteq cl_{\mathbb{K}}[T^*(\psi)(O_A)]$ and $T^*(\psi) \in O(\varphi_0)$. Therefore $|\psi(b)| \leq \delta/2 < \delta$ if $b \in B_0$ and $\psi \in Y$. That means, Y is equicontinuous at zero. In view of this, Y is an equicontinuous subset of hom B. Hence, for each k there exists a neighbourhood $O(b_k)$ of $b_k \in B$ such that $|\psi(b-b_k)| < \varepsilon/3$ for each $b \in O(b_k)$ and $\psi \in Y$. Again, as T(A) is everywhere dense in B then for each k there exists an element $a_k \in A$ such that $T(a_k) \in O(b_k) \cap T(A)$, by virtue of which

$$|\psi(T(a_k)-b_k)| < \varepsilon/3$$

for each k and each $\psi \in Y$.

Now $O_1(\varphi_0) = X \cap O(\varphi_0) \cap \Phi_A(\varphi_0; a_1, \ldots, a_n, \varepsilon)$ is a neighbourhood of φ_0 in X. To show that $O_1(\varphi_0) \subseteq T^*(\mathcal{U})$, let $\varphi \in O_1(\varphi_0)$. Then $\varphi = T^*(\psi)$ for a $\psi \in$ hom B. As

$$|(\psi - \psi_0)(b_k)| \leq |\psi(b_k - T(a_k))| + |(\varphi - \varphi_0)(a_k)| + \\ + |\psi_0(T(a_k) - b_k)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

then $\varphi \in T^*(\Phi)$. In view of this we have that $O_1(\varphi_0) \subseteq T^*(\Phi) \subseteq T^*(\mathcal{U})$, i.e. that $T^*(\mathcal{U})$ is an open set in X. Thus T^* is open. Consequently (in the both cases), T^* is a homeomorphism of hom B into hom A.

2) Let $\varphi_0 \in cl_{\text{hom } A} X$, $\varepsilon > 0$ and $a \in A$. Then

$$\mathcal{V} = \mathcal{O}_A(\varphi_0;\ a,\varepsilon)$$

is a neighbourhood of φ_0 in hom A. Therefore the intersection $\mathcal{V} \cap X$ is not empty. In view of this we have

$$|(T^*(\psi) - \varphi_0)(a)| < \varepsilon$$

for each fixed $\psi \in (T^*)^{-1}(\mathcal{V} \cap X)$. As

$$|\varphi_0(a)| \le |(\varphi_0 - T^*(\psi))(a)| + |\psi(T(a))| < \varepsilon + |\psi(T(a))|$$

then $|\varphi_0(a)| < \varepsilon$ if $a \in \ker T$. From this follows that $\ker T \subseteq \ker \varphi_0$. Therefore, there exists a $\psi \in$ hom T(A) such that $\varphi_0 = \psi o T$. By our assumption T(A) is everywhere dense in B and the multiplication in B is jointly continuous. Hence ψ has an extension $\psi_0 \in$ hom B by Proposition 1 from [21] and $T^*(\psi_0) = \varphi_0$. Thus $\varphi_0 \in X$.

3) First we suppose that T is an open one-to-one mapping and $\varphi \in$ hom A. Then $\varphi o T^{-1} \in$ hom T(A). As T(A) is everywhere dense in B and the multiplication in B is jointly continuous then $\varphi o T^{-1}$ has an extension $\psi \in$ hom B. Therefore $T^*(\psi) = \varphi$. Thus X = hom A.

Last we suppose that the condition b) is fulfilled. Then X is a closed subset of hom A by the statement 2). If $X \ne$ hom A, then there exist $\varphi_0 \in (\text{hom } A)\setminus X$ and an open neighbourhood $O(\varphi_0)$ of it, such that $O(\varphi_0) \subseteq (\text{hom } A)\setminus X$. Now (by our assumption) there exist an open set $\mathcal{V}$ and an element $a_1 \in A$ such that $\varphi_0 \in \overline{\mathcal{V}} = cl_{\text{hom } A}\mathcal{V} \subset O(\varphi_0)$, $\varphi(a_1) = 1$ if $\varphi \in \overline{\mathcal{V}}$ and $\varphi(a_1) = 0$ if $\varphi \in (\text{hom } A) \setminus O(\varphi_0)$. Moreover (in view of regularity of algebra A), there exists an element $a_2 \in A$ such that $\varphi_0(a_2) = 1$ and $\varphi(a_2) = 0$ if $\varphi \in$ $\in (\text{hom } A)\setminus\mathcal{V}$. Therefore $\varphi(a_2 a_1 - a_2) = 0$ for each $\varphi \in$ hom A. Thus $a_2 a_1 - a_2 \in$ rad A. Therefore $a_2 = a_2 o a_1$. As $\varphi(a_1) = 0$ for each $\varphi \in X$, then $\psi[T(a_1)] = T^*(\psi)(a_1) = 0$ for each $\psi \in$ hom B. So $T(a_1) \in$ rad B $\subset \mathcal{Q}$invB. In view of this $T(a_1)ob = boT(a_1) = \theta_B$ for a $b \in B$. By virtue of it

$$T(a_2) = T(a_2)o\theta_B = T(a_2)o(T(a_1)ob) =$$

$$= (T(a_2)oT(a_1))ob = T(a_2 o a_1)ob =$$

$$= T(a_1)ob = \theta_B.$$

So $a_2 = \theta_A$, which is not possible. Hence X = hom A.

As an application of the previous results, we now

1*

prove an analogue of Gelfand-Naimark Theorem. For it (for a given locally compact space X) by $C_0(X)$ we denote the Banach algebra of all $\mathbb{C}$-valued continuous functions on X vanishing at infinity.

T H E O R E M 2. *Let* $k \in (0,1]$ *and A be a commutative sequentially complete pseudobarrelled (barrelled in the case, when* $k=1$*) strongly spectrally k-bounded Hausdorff *-algebra with the weakened* C^**-property and with jointly continuous multiplication. Then*

a) *A is (within topological algebraic isomorphism) a commutative* C^**-algebra,*

b) *hom A is locally equicontinuous (and equicontinuous if A has an unit)*

and

c) $\mathcal{G}_A$ *is a topological isomorphism of A onto* C_0*(hom A).*

P r o o f. As it has been shown above (cf. Corollary 1 and Lemma 2), there exists a submultiplicative norm λ on A such that (A,λ) is a commutative C^*-algebra and the identity mapping ε_A is a topological isomorphism of (A,λ) onto A. Therefore ε_A^* is a homeomorphism of hom A onto hom (A,λ) by Lemma 4 1b) and 3a) as hom (A,λ) is non-empty and locally equicontinuous (and equicontinuous if A has an unit). In view of this hom A is also locally equicontinuous (and equicontinuous if A has an unit). Moreover, the mapping F, defined by the relation $F(f) = f \circ \varepsilon_A^*$ for each $f \in C_0(\text{hom }(A,\lambda))$, is a topological isomorphism of $C_0(\text{hom }(A,\lambda))$ onto $C_0(\text{hom } A)$ and $\mathcal{G}_{(A,\lambda)}$ is an isometrical isomorphism of (A,λ) onto $C_0(\text{hom }(A,\lambda))$ by Gelfand-Naimark Theorem. Hence $\Phi = F \circ \mathcal{G}_{(A,\lambda)} \circ \varepsilon_A$ is a topological isomorphism of A onto $C_0(\text{hom } A)$. As $\Phi = \mathcal{G}_A$ then Theorem 2 has been proved.

In a particular case, when A is a commutative uniformly A-convex Fréchet C^*-algebra with unit, Theorem 2 is known (cf. [12], p.8, and [13], p.80).

C O R O L L A R Y 4. *Let* $k \in (0,1]$, *A be a commutative sequentially complete pseudobarrelled (barrelled in the case, when* $k = 1$*) strongly spectrally k-bounded Hausdorff *-algebra with weakened* C^**-property, B be a commutative Gelfand-Mazur *-algebra with jointly continuous multiplica-*

tion, which is a non-radical Q-algebra, and T be a continuous isomorphism of A onto a everywhere dense subalgebra of B. Then T^* is a homeomorphism of hom B onto hom A.

P r o o f. As hom A is locally equicontinuous and $\mathcal{G}_A$ is a topological isomorphism of A onto C_0(hom A) by Theorem 2, then hom A is a locally compact space and rad A = $\{\theta_A\}$ (because C_0(hom A) is semi-simple). It is known (cf.[9], p.167) that C_0(hom A) is a regular algebra. Therefore A is also a regular algebra.

Let $\varphi_0 \in$ hom A and $\mathcal{U}$ be a neighbourhood of φ_0. Then there exist open relatively compact sets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that

$$\varphi_0 \in \mathcal{V}_1 \subset \overline{\mathcal{V}}_1 \subset \mathcal{V}_2 \subset \overline{\mathcal{V}}_2 \subset \mathcal{U},$$

where $\overline{\mathcal{V}}_k = cl_{homA}\mathcal{V}_k$ and k = 1,2. In view of this (cf.[23], p.231) there exists a function f $\in$ C(hom A,[0,1]) such that $f(\varphi) = 1$ for each $\varphi \in \overline{\mathcal{V}}_1$ and $f(\varphi) = 0$ for each $\varphi \in$ $\in$ (hom A)$\setminus\mathcal{V}_2$. As (hom A)$\setminus\overline{\mathcal{V}}_2 \subset$ (hom A)$\setminus\mathcal{V}_2$ then f $\in C_0$(hom A). Hence (by Theorem 2c)) there exists an element a $\in$ A such that $\mathcal{G}_A(a) = f$. Thus $\varphi(a) = 1$ for each $\varphi \in \overline{\mathcal{V}}_1$ and $\varphi(a) = 1$ for each $\varphi \in$ (hom A)$\setminus\mathcal{V}_2$. Moreover, hom B is not empty and rad B coincides withn the Jacobson radical of B (cf.[1], Theorem 2). Therefore rad B $\subset$ Qinv B (cf.[15], p.156). So T^*(hom B) = hom A by Lemma 4 3b). In a particular case, when B has an unit, then hom B is equicontinuous (cf. [11], p. 75). In view of this, the statement is true by Lemma 4 1a) and 3b).

Continuous *-isomorphisms

It is well known (cf., for example, [22], p.263) that every *-isomorphism T of a C^*-algebra A into an another C^*-algebra B (that is isomorphism T for which $T(a^*) = T(a)^*$ for each a $\in$ A) is an isometry. We shall now prove an analogue of this result for strongly spectrally bounded *-algebras. Before doing it we need some results which are known for the case of Banach algebras.

L E M M A 5. *Let A and B be topological algebras for which the sets hom A and hom B are non-empty and let*

$$\sigma_A(a) = \{\varphi(a): \varphi \in hom_0 A\} \qquad (12)$$

for each a $\in$ A and

$$\mathrm{sp}_B(b) = \{\psi(b): \psi \in \mathrm{hom}_0 B\}$$

for each $b \in B$. Moreover, let T be a homomorphism of A into B such that $T^*(\mathrm{hom}\ B) = \mathrm{hom}\ A$. Then

$$\mathrm{sp}_A(a) = \mathrm{sp}_B(T(a)) \qquad (13)$$

for each $a \in A$.

P r o o f. Let $a \in A$ and $\lambda \in \mathrm{sp}_A(a)$. Then $\lambda = \varphi(a)$ for a $\varphi \in \mathrm{hom}_u A$. If φ is the zero homomorphism on A and ψ is the zero homomorphism on B then $\psi|T(A) = \varphi$. Therefore $\lambda = \psi(T(a)) \in \mathrm{sp}_B(T(a))$. Let now $\varphi \in \mathrm{hom}\ A$. Then there exists $\psi \in \mathrm{hom}\ B$ such that $\varphi = T^*(\psi)$. Hence $\lambda \in \mathrm{sp}_B(T(a))$. Thus, $\mathrm{sp}_A(a) \subseteq \mathrm{sp}_B(T(a))$. The converse is proved similarly.

C O R O L L A R Y 5. *Let A and B be (not necessarily simultaneously) either commutative Gelfand-Mazur Q-algebras over $\mathbb{C}$ with unit or commutative complete locally k-(m-pseudoconvex) Hausdorff $\mathbb{C}$-algebras with unit and let T be a homomorphism of A into an everywhere dense subalgebra of B such that $T^*(\mathrm{hom}\ B) = \mathrm{hom}\ A$. Then the equality (13) is valid for each $a \in A$.*

P r o o f. It is known (cf.[1], Theorem 1, and [19], Theorem 6) that the equality (12) is valid for each $a \in A$ in the present case. Hence Corollary 5 is correct by Lemma 5.

L E M M A 6. *Let $0 \le k_1 \le k_2 \le 1$, A be a k_1-Banach $*$-algebra with unit and with weakened C^*-property, B be a k_2-Banach $*$-algebra with unit and T be a continuous $*$-isomorphism of A into B. Then T is a topological $*$-isomorphism.*

In particular, when A and B are k-(C^*-algebras) for a $k \in (0,1]$, then T is an isometrical $*$-isomorphism.

P r o o f. Let $a \in A$ and A_0 be the commutative $*$-subalgebra of A generated by a^*a. Then there exists a maximal commutative $*$-subalgebra A_1 which contains A_0. As A_1 is closed then A_1 is a commutative k_1-Banach $*$-algebra with weakened C^*-property. Hence hom A_1 is non-empty (cf.[16], p.13)

Moreover let $B_1 = \mathrm{cl}_B T(A_1)$. As the involution in B is continuous then B_1 is a commutative k_2-Banach $*$-algebra. Hence B_1 is a commutative Gelfand-Mazur Q-algebra with continuous multiplication (cf.[16], p.17-18). Therefore

$T^*(\hom B_1) = \hom A_1$ by Corollary 4. In view of this

$$sp_{A_1}(a^*a) = sp_{B_1}(T(a)^*T(a))$$

for each $a \in A$ by Corollary 5. Thus

$$\rho_{A_1}(a^*a) = \rho_{B_1}(T(a)^*T(a)) \qquad (14)$$

for each $a \in A$. Now by Lemma 1 and equations (4), (6) and (14) we have

$$\|a\|_A^2 \le \beta\|a^*a\|_A = \beta\|a^*a\|_{A_1} \le \beta^2\nu_{A_1}(a^*a) =$$

$$= \beta^2\rho_{A_1}(a^*a)^{k_1} = \beta^2\rho_{B_1}(T(a)^*T(a))^{k_1} \le$$

$$\le \beta^2\rho_{B_1}(T(a)^*T(a))^{k_2} \le \beta^2\|T(a)^*T(a)\|_{B_1} =$$

$$= \beta^2\|T(a)\|_B^2$$

for each $a \in A$ and some $\beta \ge 1$. Thus $\|a\|_A \le \beta\|T(a)\|_B$ for each $a \in A$. Hence T^{-1} is a continuous mapping. That means that T is a topological $*$-isomorphism.

The case, when A and B are k-(C^*-algebras) for a $k \in (0,1]$, is proved similarly (then $\nu(a) = \|a\|_A$ for each $a \in A$ and $\nu(b) = \|b\|_B$ for each $b \in B$ by (5) and (7)).

Next we prove the main result.

T H E O R E M 3. *Let $0 < k_1 \le k_2 \le 1$, A be a strongly spectrally k_1-bounded Fréchet $*$-algebra with unit and with weakened C^*- property, B be a strongly spectrally k_2-bounded Fréchet $*$-algebra with unit and T be a continuous $*$-isomorphism of A into B. Then*

a) *T is a topological $*$-isomorphism*

and

b) *T(A) is a closed $*$-subalgebra of B.*

P r o o f. As every Fréchet algebra is a pseudobarrelled and a barrelled algebra then there exist a submultiplicative k_1-homogeneous norm λ_1 on A and a submultiplicative k_2-homogeneous norm λ_2 on B such that (A,λ_1) is a k_1-Banach $*$-algebra with weakened C^*-property by Corollary 1 and (B,λ_2) is a k_2-Banach $*$-algebra by Theorem 1. At this ε_A

(the identity mapping on A) is a topological *-isomorphism of (A,λ_1) onto A and ε_* is a topological *-isomorphism of (B,λ_2) onto B.

Therefore $T_0 = \varepsilon_{T(A)} o T^{-1} o \varepsilon_A$ is a continuous *-isomorphism of (A,λ_1) into (B,λ_2). Hence T_0^{-1} is a continuous mapping by Lemma 6. As $T^{-1} = \varepsilon_A o T_0^{-1} o \varepsilon_{T(A)}$ then T^{-1} is a continuous mapping. Consequently T is a topological *-isomorphism. In view of this, T(A) is a closed *-subalgebra of of B (cf.[7], p.138).

C O R O L L A R Y 6. *Let $0 < k_1 \le k_2 \le 1$, A be a strongly spectrally k_1-bounded Fréchet *-algebra with unit and with weakened C^*-property, B be a strongly spectrally k_2-bounded Fréchet *-algebra with unit and T be a continuous *-homomorphism of A into B. Then T is an open mapping and T(A) is a closed *-subalgebra in B.*

P r o o f. Let $\{p_\alpha : \alpha \in \mathfrak{U}\}$ be the system of k_1-homogeneous seminorms on A which defines its topology. Moreover let $A_1 = A/\ker T$ and π be the natural homomorphism of A onto A_1 Then $\bar{T}$ defined by $\bar{T} o \pi = T$ is a continuous *-isomorphism of A_1 into B. As usual we endowe A_1 with the topology, defined by the system $\{\tilde{p}_\alpha : \alpha \in \mathfrak{U}\}$ of k_1-homogeneous seminorms on A_1, where

$$\tilde{p}_\alpha(\pi(a)) = \inf_{a' \in \ker T} p_\alpha(a + a')$$

for each $\pi(a) \in A_1$. Then A_1 is a strongly spectrally k_1-bounded Fréchet *-algebra with weakened C^*-property. Therefore $\bar{T}$ is a topological *-isomorphism of A_1 into B by Theorem 3. Consequently, T is an open mapping and T(A) is a closed *-subalgebra of B.

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СТРОГО СПЕКТРАЛЬНО ОГРАНИЧЕННЫЕ АЛГЕБРЫ

М.Абель

Р е з ю м е

Показывается, что

а) на каждой секвенциально полной псевдобочечной строго спектрально k-ограниченной ($0 < k \le 1$) отделимой алгебре A существует такая k-однородная норма λ, что (A,λ) (т.е. алгебра A, наделенная топологией, определенной нормой λ) является k-банаховой алгеброй, топологически изоморфной алгебре A (Теорема 1)

и

б) на каждой коммутативной секвенциально полной псевдобочечной строго спектрально k-ограниченной ($0 < k \le 1$) отделимой $*$-алгебре A с непрерывным умножением и с ослабленным C^*-свойством существует такая норма λ, что (A,λ) является коммутативной C^*-алгеброй, топологически изоморфной алгебре A (Теорема 2).

Кроме того, доказываются аналоги теоремы Гельфанда-Наймарка (Теорема 2с)), теоремы об $*$-изоморфизмах C^*-алгебр (Теорема 3) и теоремы о факторизации элементов (Следствие 3) для строго спектрально k-ограниченных алгебр с $0 < k \le 1$.

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ON THE GEOMETRIC APPROACH TO TOPOLOGICAL QUANTUM FIELD THEORY

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1. Introduction. In a lecture at the Hermann Weyl Symposium ([2]), Michael Atiyah raised the problems how to give physical interpretations to Donaldson, Floer and Jones theories. These theories deal with the topological invariants of some manifold. Witten in his papers ([6],[7],[8]) constructed corresponding quantum field theories in which all the observables are topological invariants. The topological invariance of all of the observables means that they do not depend on the choice of the metric on the manifold. In physical language this is equivalent to the concept of the general covariance of the quantum field theory. In this paper we consider the topological quantum field theory (TQF) in which the quantum field representation of Donaldson invariants was given. It is known, that the Donaldson theory deals with the geometry of the moduli space, which is a submanifold in the space of all Yang-Mills fields or, in the geometrical language, in the affine space of all irreducible smooth connections on some principal fiber bundle over four-dimensional manifold. On the other hand, in the TQF theory of Witten, in addition to classical Yang-Mills fields, the set of the fermionic fields appear. The basic property of such fields $\psi_\alpha(x)$ is anti-commutativity, which can be written in the form

$$\{ \psi_\alpha(x), \psi_\beta(y) \} = 0, \qquad (1.1)$$

where α and β are some discrete indeces, x and y are some points of manifold and $\{\ ,\ \}$ is anti-commutator. It is essential to expect that if we define the fermionic fields in a strong mathematical way, then some physical objects of TQF theory will obtain the geometrical interpretation in the frame of the geometry of the space of all connections.

It follows from (1.1) that fermionic fields are the generators of some infinite dimensional Grassmann algebra, where point x is a continuous index. Such an algebra was defined by Berezin in [10]. In Section 2 we give the definition of this algebra. However, for our purposes it is necessary to generalize the Berezin construction. One of the possible generalizations was proposed by author in [1],[9]. The construction, which will be given in this paper, differ from the previous one. In Section 4, using the notation of de Ram flow ([11]), we show how to construct it. In Section 3 all necessary notations are collected. In section 5 the constructed algebra is applied for geometrical description of TQF theory of Witten.

2. The definition of infinite dimensional Grassmann algebra, which is used in this paper, was described by F. Berezin in his book [10]. The definition of the infinite dimensional Grasssmann algebra with scalar product consists of two parts. In the first part the structure of usual Grassmann algebra is described. In the second part the additional structure - scalar product, which as a rule is not considered in the case of the finite dimensional Grassmann algebra, is described and later this structure allows to introduce the fermionic fields of type (1.1) as generators of Grassmann algebra.

<u>Definition</u>. We shall call an algebra $\mathcal{G}$ Grassmann algebra with scalar product if it has the structure of the direct sum of the topological spaces $\mathcal{G}_n$ (see [10], p. 55), $\mathcal{G} = \bigoplus_{n=0} \mathcal{G}_n$ and the multiplication (denote by $\wedge$) satisfies the following additional requirements:

1_1) $dim\, \mathcal{G}_0 = 1$ and if f_0 is a basic element of $\mathcal{G}_0$, then for arbitrary element $g \in \mathcal{G}$, we have the following property

$$\alpha f_0 \wedge g = \alpha g\ , \qquad \alpha \in \mathbb{R}\ (\text{ or } \mathbb{C}\ ), \qquad (2.1)$$

2_1) $f \wedge g \in \mathcal{G}_{p+q}$ for $\forall f \in \mathcal{G}_p$ and $\forall g \in \mathcal{G}_q$,

3_1) $f \wedge g = (-1)^{pq}\, g \wedge f,\quad f \in \mathcal{G}_p,\ g \in \mathcal{G}_q.$

4_1) if elements $f_1, \dots, f_N$ are linearly independent in $\mathfrak{F}_1$ then elements $f_{i_1} \wedge \dots \wedge f_{i_n}$ $(1 \le i_1 < \dots < i_n \le N)$ linearly independent in $\mathfrak{F}_n$,

5_1) the space of the finite linear combinations such a type as $\sum a_{ij} f_i \wedge g_j$, where $f_i \in \mathfrak{F}_p$ and $g_j \in \mathfrak{F}_q$, everywhere dense in $\mathfrak{F}_{p+q}$.

Scalar product has the following structure:

1_2) for every $\mathfrak{F}_p$ there exists the topological subspace $\tilde{\mathfrak{F}}_p \subset \mathfrak{F}_p$ with own topology,

2_2) every element of $\mathfrak{F}_p$ defines continuous linear functional on $\tilde{\mathfrak{F}}_p$,

3_2) the topology of the space $\mathfrak{F}_p$ is the topology of the dual space to $\tilde{\mathfrak{F}}_p$,

4_2) there is a scalar product defined on the space $\tilde{\mathfrak{F}}_p$ and the supplement of $\tilde{\mathfrak{F}}_p$ with respect to this scalar product $\mathscr{H}_p \subset \mathfrak{F}_p$,

5_2) if $\{ f_1, \dots, f_N \}$ is an orthogonal basis in $\mathscr{H}_1$, then $\{ f_{i_1} \wedge \dots \wedge f_{i_n} \}$, where $1 \le i_1 < \dots < i_n \le N$, is an orthogonal basis in $\mathscr{H}_n$.

We shall call the algebra described above *GB*-algebra (Grassmann - Berezin).

3. In this section some notations, which will be necessary in what follows, have been collected.

Let $\mathcal{X}$ be the compact, oriented, smooth Riemmannian n-dimensional manifold, G - compact, real, semi-simple, r-dimensional Lie group and let $\mathbb{G}$ be the Lie algebra of G. Then $P(\mathcal{X},\mathrm{G})$ denotes the principal fiber bundle over $\mathcal{X}$ with structure group G and E denotes the associated vector bundle $P(\mathcal{X},\mathrm{G}) \times_{\mathrm{G}} \mathbb{G}$ with fiber $\mathbb{G}$, where G acts on $\mathbb{G}$ by adjoint representation. Since $\mathbb{G}$ is semi-simple Lie group, there exists the Killing metric $h : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ on the Lie algebra $\mathbb{G}$. The metric h allows to define the Eucledian structure on the vector bundle E. Let us denote by $\Omega^p_0(\mathcal{X}, E)$ the space of smooth differential p-forms on $\mathcal{X}$ with values in vector bundle E. Such forms are called equivariant forms ([3]). In this paper we assume that the Lie algebra $\mathbb{G}$ has been realized as the group of real, skew-symmetric matrices in the adjoint representation. Let us denote by Tr the corresponding Killing metric h on $\mathbb{G}$. The standard scalar product is defined on the space $\Omega^p_0(\mathcal{X},E)$ for all $0\le p\le n$ by the following formula ([12]):

$$(\sigma,\tau) = \int_{\mathcal{X}} Tr(\sigma \wedge *\tau) \ , \quad \sigma,\tau \in \Omega^p_0(\mathcal{X},E). \tag{3.1}$$

where $*$ - is a star operator on the Riemmannian manifold $\mathcal{X}$. Let us denote by $H^p(\mathcal{X},E)$ the Hilbert space which is the completion of the space $\Omega^p_0(\mathcal{X},E)$ with respect to scalar product (3.1).

Let ω be a connection form on the principal fiber bundle $P(M,G)$. We denote the affine space of all irreducible connections on the $P(M,G)$ by $\mathfrak{A}(P)$ and the group of gauge transformations by $GA(P)$ ([3]). The group $GA(P)$ acts on the space $\mathfrak{A}(P)$ and we denote the quotient space of gauge equivalent classes by $\mathcal{B}(P) := \mathfrak{A}(P)/_{GA(P)}$. The curvature 2-form Θ^ω, which corresponds to the connection 1-form ω, will be considered as the element of the space $\Omega^2_0(\mathcal{X},E)$. It is known that for SD-connection (self-dual) ω the curvature form satisfies the condition $\Theta^\omega = *\Theta^\omega$ and for ASD-connection (anti-self-dual) ω we have the condition $\Theta^\omega=-*\Theta^\omega$. Let us denote by $\mathfrak{A}^-(P)$ the subspace of irreducible ASD-connections on $P(M,G)$ and by $\mathfrak{M} = \mathfrak{A}^-(P)/_{GA(P)}$ the space of moduli of ASD-connections. We also need in what follows two operators on the associated fiber bundle E: the covariant differential $D^\omega:\Omega^p_0(\mathcal{X},E)\to \Omega^{p+1}_o(\mathcal{X},E)$ and the adjoint operator $\vartheta^\omega:\Omega^{p+1}_0(\mathcal{X},E)\to\Omega^p_o(\mathcal{X},E)$ where

$$\vartheta^\omega = -(-1)^{n(p+1)}*D^\omega* \tag{3.2}$$

Let us remind ([11]) that the flow on manifold $\mathcal{X}$ is a linear continuous functional T on the space $\Omega_0(\mathcal{X}) = \bigoplus_{p=0} \Omega^p_0(\mathcal{X})$ of all smooth differential forms on $\mathcal{X}$. It is said that T is the homogeneous p-dimensional flow if $T(\vartheta)$ does not equal to zero if and only if ϑ is the homogeneous, smooth, differential p-form. We have supposed that M is the oriented manifold therefore we don't need the notation of odd and even flows. Every flow will be written as the sum of some homogeneous flows, therefore the space $\mathcal{D}'(\mathcal{X})$ of all flows on $\mathcal{X}$ decompose into the sum $\mathcal{D}'(\mathcal{X}) = \bigoplus_p \mathcal{D}'^p(\mathcal{X})$ of the space of the homogeneous flows. For the further construction we need the trivial generalization of the notation of the flow to the case of the differential forms with values on the sections of the vector bundle. Let $E\to\mathcal{X}$ be a vector bundle over $\mathcal{X}$. Then we shall call a linear continuous functional T on the space of all smooth differential forms with coefficients on the bundle E flow with values on the vector bundle E. We denote the space of the homogeneous flows with values on bundle E by $\mathcal{D}'^{p}(\mathcal{X},E)$. Let us denote the vector space of all such flows by $\mathcal{D}'(\mathcal{X},E)$. Finally, we denote by $E\otimes\ldots\otimes E$ (j-time) the vector bundle over $\mathcal{X}\times\ldots\times\mathcal{X}$ (j-time) with fiber $E_{x_1}\otimes\ldots\otimes E_{x_j}$ over the point $(x_1,\ldots,x_j)\in\mathcal{X}\times\ldots\times\mathcal{X}$.

4. In this section we will describe the construction of GB-algebra $\mathcal{G}(E)$ on the vector bundle $E=P\times_{G}\mathfrak{G}$. Some elements of the algebra $\mathcal{G}(E)$ are used for the computation of Donaldson polynomials in the TQF theory of Witten.

The space of smooth differential forms on $\mathcal{X}$ with coefficients on E split into the sum $\Omega_0(\mathcal{X},E)=\underset{p=0}{\oplus}\Omega_0^p(\mathcal{X},E)$. Since $\mathcal{X}$ is the Riemannian manifold, the $*$-operator is defined and therefore we consider only the sum $\tilde{\mathcal{E}}_1(\mathcal{X},E)=\overset{[n/2]}{\underset{p=0}{\oplus}}\Omega_0^p(\mathcal{X},E)$. The first step of our construction is to the effect that we let $\tilde{\mathcal{E}}_1:=\tilde{\mathcal{E}}_1(\mathcal{X},E)$, where $\tilde{\mathcal{E}}_1$ the space from the definition of the GB-algebra, Axiom 1_2. Then, from Axiom 4_2 it follows that $\mathcal{H}_1=\mathcal{H}_1(\mathcal{X},E)=\overset{[n/2]}{\underset{p=0}{\oplus}}H^p(\mathcal{X},E)$. Analogously, from Axioms 2_2 and 3_2 it follows that the space $\mathcal{E}_1$ coincides with the space of the flows those dimension not more than $[n/2]$, i.e. $\mathcal{E}_1=\overset{[n/2]}{\underset{p=0}{\oplus}}\mathcal{D}'^{p}(\mathcal{X},E)\subset\mathcal{D}'(\mathcal{X},E)$. Thus, the space $\mathcal{E}_1=\mathcal{D}'_1(\mathcal{X},E)$ is the space of linear continuous functionals on the space $\tilde{\mathcal{E}}_1(\mathcal{X},E)$.

Let us define the production $\wedge$ (do not confuse with wedge product $\wedge$ of differential forms) of two elements $T_1,T_2\in\mathcal{D}'_1(\mathcal{X},E)$ in the following way

$$T_1\wedge T_2(\vartheta,\sigma)=\frac{1}{2}\,[T_1(\vartheta)T_2(\sigma)-T_1(\sigma)T_2(\vartheta)] \tag{4.1}$$

where $\vartheta,\sigma\in\tilde{\mathcal{E}}_1(\mathcal{X},E)$. It is not difficult to show that the production (4.1) satisfies the Axioms 1_1 and 3_1. The space $\mathcal{E}_2$ is not still defined and therefore it is not possible to check Axioms 2_2, 4_1, 5_1. In order to describe space $\mathfrak{F}_2$ as the space of flows over the some space of differential forms, we need the generators of *GB*-algebra $\mathfrak{F}(E)$. Let us consider the space $\mathcal{D}'_1(\mathcal{X},E)$. Since the space $\mathcal{D}'_1(\mathcal{X},E)$ is the sum of the spaces of homogeneous flows $\mathfrak{F}_1 = \mathcal{D}'_1(\mathcal{X},E) = \bigoplus_{p=0}^{[n/2]} \mathcal{D}'^p(\mathcal{X},E)$, then every element $T\in\mathfrak{F}_1$ will be written as

$$T = \sum_p T^{(p)}, \qquad T^{(p)} \in \mathcal{D}'^p(\mathcal{X},E). \tag{4.2}$$

In turn, the homogeneous, p-dimensional flow $T^{(p)}$ will be written as finite sum of the integrals

$$T^{(p)}(\sigma) = \sum_\mu \int_{c_\mu} Tr\ (\sigma \wedge \theta_\mu), \tag{4.3}$$

where $\sigma\in \Omega_0^p(\mathcal{X},E)$, $dim\ c_\mu = deg\ \sigma + deg\ \theta_\mu$, c_μ-some chain on $\mathcal{X}$ and θ_μ- differential form, whose coefficients, in a general case are generalized functions (distributions). In a special case, when $T^{(p)} \in H^p(\mathcal{X},E) \subset \mathcal{D}'_1(\mathcal{X},E)$, we have

$$T^{(p)}(\sigma) = \int_{\mathcal{X}} Tr(\ \sigma \wedge *\theta\ ), \qquad \theta\in H^p(\mathcal{X},E), \tag{4.4}$$

and, in this sense, functional $T^{(p)}$ will be identified with differential p-form θ. Now we can introduce the generators of *GB*-algebra or the objects which are called in quantum field theory fermionic fields. Suppose that $\{\mathcal{U}_A, \xi_A\}_{A\in\mathfrak{B}}$, where $\xi_A : \mathcal{U}_A \to \pi^{-1}(\mathcal{U}_A)$ some smooth sections and $\mathfrak{B}$ some set of indices, is the trivialization of the vector bundle and also the covering $\{\mathcal{U}_A\}_{A\in\mathfrak{B}}$ of $\mathcal{X}$ is formed by coordinate neighborhoods. Then, the several differential form $\sigma\in \Omega_0^p(\mathcal{X},E)$ will be written locally on $\mathcal{U}_A$ as

$$\sigma = \sigma_{i_1\ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{4.5}$$

Let us define the functionals $\Phi_{i_1\ldots i_p}(x)$ on the space of differential forms $\Omega_0^p(\mathcal{X},E)$ by the following way

$$\Phi_{i_1\ldots i_p}(x)(\sigma) = \sigma_{i_1\ldots i_p}(x). \tag{4.6}$$

Since G is the group of skew-symmetric, real matrices, we see that $\Phi_{i_1 \ldots i_p}(x)$ is the matrix of functionals whose elements belong to $\mathcal{D}'^{p}(\mathcal{X},E)$. The functionals (4.6) depend on the choice of the trivialization on E. Supposing that functionals (4.6) transform as the coefficients of the form (4.5), we can define the global object

$$\Phi(x) = \Phi_{i_1 \ldots i_p}(x)\ dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{4.7}$$

Is it possible to define the wedge product for two forms (4.7)? The answer is positive, but this wedge product is different from the one for ordinary differential forms. The difference, arising here, is the same as the difference between ordinary product of the real-valued functions and the product of distributions ([14]), since the coefficients of the form (4.7) are, strictly speaking, distributions. Now we can multiply the matrix element $[\Phi_{i_1 \ldots i_p}(x)]^{\alpha}_{\beta}$ of the differential form $\Phi(x)$ and the matrix element $[\Psi_{i_1 \ldots i_p}(y)]^{\varepsilon}_{\delta}$ of the differential form $\Psi(x)$ according to (4.1), since they are functionals. Evidently the anticommutative property (1.1) holds. This property allows us to identify the coefficients of the forms such (4.7) with fermionic fields. Now we will define the product of two forms. Since the forms of such a type as (4.7) depend on the point $x \in \mathcal{X}$ their wedge product is a double form ([11]) on $\mathcal{X} \times \mathcal{X}$ with values on $E \otimes E$

$$\Phi(x) \wedge \Psi(y) = \sum dx^{i_1} \wedge \ldots \wedge dx^{i_p} (\Phi_{i_1 \ldots i_p}(x) \wedge \Psi_{j_1 \ldots j_p}(x))\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

Let us write the homogeneous p-dimensional flow (4.3) using (4.7) in the form

$$T^{(p)} = \sum_{\mu} \int_{c_\mu} Tr\ (\Phi(x) \wedge \vartheta_{\mu}). \tag{4.8}$$

The expressions, similar to (4.9), were used by Witten in his quantum topological field theory to describe the Donaldson's polynomials. Now the product of two arbitrary flows has the form

$$T \wedge T' = \sum_{\mu,\nu} \int_{c_\mu \times c_\nu} Tr[\Phi(x) \wedge \Psi(y) \wedge (\vartheta_{\mu}(x) \otimes \vartheta_{\nu}(y))], \tag{4.9}$$

where Tr is the natural Killing's metric on $\mathbf{G} \otimes \mathbf{G}$ and $\otimes$ denotes the tensor product of two ordinary forms whose coefficients are distributions ([14]). The expression (4.9) allows us to define the triple of spaces $\tilde{\mathfrak{F}}_2 \subset \mathcal{H}_2 \subset \mathfrak{F}_2$ from the definition 2. Namely, the space $\tilde{\mathfrak{F}}_2$ consists of the elements as (4.9) whose coefficients $\vartheta(x,y)$ are smooth, double differential forms on the vector bundle $E \otimes E \to \mathcal{X} \times \mathcal{X}$. We define the space $\mathcal{H}_2$ in the following way $\mathcal{H}_2 = \sum_{p,q} H^{p,q}(\mathcal{X} \times \mathcal{X}, E \otimes E)$, where $H^{p,q}(\mathcal{X} \times \mathcal{X}, E \otimes E)$ is the completion of $\Omega_0^{p,q}(\mathcal{X} \times \mathcal{X}, E \otimes E)$ with respect to natural scalar product. Then the space $\mathfrak{F}_2$ is the space of the elements of such a type as (4.11) whose coefficients $\theta_\mu(x,y)$ are the double flows on $\mathcal{X} \times \mathcal{X}$ with values on $E \otimes E$. The consequent spaces $\mathfrak{F}_p$ are constructed analogously.

It should be noted in conclusion that, as well known from the de Ram theory of the flows, every double flow of the space $\mathcal{D}'(\mathcal{X} \times \mathcal{X}, E \otimes E)$ defines the linear continuous operator. The expressions (4.11), which are considered in quantum field theory, are usually sych a kind that this operator is differential. So, in this special case, we can write (4.11) in the form

$$T = \int_{\mathcal{X}} Tr\ (\Psi D \Psi), \tag{4.12}$$

where $T \in \mathfrak{F}_2$, and D -some differential operator.

5. The GB-algebra $\mathfrak{F}(E)$, which we constructed in the previous section, gives us the appropriate geometrical description of the topological quantum field theory of Witten. It is not difficult to show that the action L of the Witten's theory is the element of the space $\mathfrak{F}_2$. The integrals of the forms W_0, W_1, W_2, W_3(see [6]) over some cycles of such a type as (4.3) (with the help of which the Donaldson's polynomials will be written) are the elements of the spaces $\mathcal{D}^{0\prime}(\mathcal{X},E)$, $\mathcal{D}^{1\prime}(\mathcal{X},E)$, $\mathcal{D}^{2\prime}(\mathcal{X},E)$, $\mathcal{D}^{3\prime}(\mathcal{X},E)$. In this section we investigate the operator Q^μ of the supersymmetry of the Witten theory and suppose $dim\ \mathcal{X}=4$.

The structure of the GB-algebra $\mathfrak{F}(E)$ depends on the dimension of the base manifold $\mathcal{X}$. Therefore, for $dim\ \mathcal{X}=4$ we have the following sum of the spaces

$$\tilde{\mathfrak{F}}_1(\mathcal{X},E) = \Omega_0^0(\mathcal{X},E) \oplus \Omega_0^1(\mathcal{X},E) \oplus \Omega_0^2(\mathcal{X},E), \tag{5.1}$$

where $\Omega^0_0(\mathcal{X},E)$ is the space of the smooth sections of E, which we denote by $C(P,\mathbb{G})$. From (5.1) it follows that we have three types of generators of GB-algebra $\mathcal{G}(E)$. They are $\eta(x)$ for $\mathcal{D}^{0\prime}(\mathcal{X},E)$, $\psi(x)$ for $\mathcal{D}^{1\prime}(\mathcal{X},E)$ and $\chi(x)$ for $\mathcal{D}^{2\prime}(\mathcal{X},E)$. These notations are taken from [6], where $\{\eta(x),\psi(x),\chi(x)\}$ denotes the set of the fermionic fields of Witten theory. However, in topological quantum field theory the fermionic 2-form $\chi(x)$ is required to be selfdual, i.e. $\chi(x)=*\chi(x)$. In our approach it means, that we must take in (5.1) instead of $\Omega^2_0(\mathcal{X},E)$ the space of SD-forms $\Omega^2_0(\mathcal{X},E)_+$, i.e.

$$\mathcal{G}_1(\mathcal{X},E)= C(P,\mathbb{G}) \oplus \Omega^1_0(\mathcal{X},E) \oplus \Omega^2_0(\mathcal{X},E)_+ . \qquad (5.2)$$

Let us define the SD-flow $T^{(2)}\in \mathcal{D}^{2\prime}(\mathcal{X},E)$ by the following condition

$$(*T^{(2)})(\sigma)= T^{(2)}(*\sigma)= T^{(2)}(\sigma). \qquad (5.3)$$

It is easy to show that condition (5.3) is equivalent to condition $\chi(x)=*\chi(x)$ from the Witten theory.

The action L of TQF theory is a functional on the set of fermionic fields $\{\eta(x),\psi(x),\chi(x)\}$, which we consider as generators of the infinite dimensional GB-algebra $\mathcal{G}(E)$ and on the set of bosonic fields $\{A,\phi,\lambda\}$, where A is a local connection 1-form on the base $\mathcal{X}$ (i.e. if $\zeta_\mu: U_\mu \to \pi^{-1}(U_\mu)$ is a local section then $A = \zeta^*_\mu(\omega)$, where ω is a connection 1-form on the bundle $P(M,\mathbb{G})$) and ϕ,λ are some sections of the vector bundle E. Action L is invariant under the supersymmetry which depends on the connection ω. The operator of this supersymmety will be denoted by Q^ω (see [6]). This operator is a differentiation on the space of functionals which depend on the fields. In this paper, we will consider only the "fermionic" part of Q^ω which has the form

$$\begin{aligned} Q^\omega\eta(x)&=\tfrac{1}{2}\,[\ \phi(x),\lambda(x)\ ],\\ Q^\omega\psi(x)&= D^\omega\phi(x),\\ Q^\omega\chi(x)&= (\Theta^\omega + *\Theta^\omega). \end{aligned} \qquad (5.4)$$

Expressions (5.4) define the operator Q^ω on the GB-algebra $\mathcal{G}(E)$ if we assume, in addition that for two arbitrary, homogeneous elements f and h of GB-algebra $\mathcal{G}(E)$ the differentiation rule

$$Q^\omega(f\wedge h)= Q^\omega f \wedge h + (-1)^{p(f)} f \wedge Q^\omega h, \qquad (5.5)$$

where $p(f)$ is a parity of the element f, is satisfied.

T H E O R E M. *If we consider the action of the operator* $\mathcal{Q}^{\omega}$ *only on the space* $\mathfrak{F}_1(\mathcal{X},E)\subset \mathcal{G}(E)$ *then* $\mathcal{Q}^{\omega}$ *is a linear functional depending on the connection* ω. *The kernel of this functional on the subspace of elements such as (4.4) coincide with the kernel of the elliptic operator* $\mathfrak{X} : \Omega^1_0(\mathcal{X},E)\to\Omega^0_0(\mathcal{X},E)\oplus \Omega^2_0(\mathcal{X},E)_+$, *where*

$$\mathfrak{X} = \partial^{\omega} \oplus (1+*)D^{\omega}. \tag{5.6}$$

P r o o f. Indeed, let us consider $\mathfrak{A}$ as the affine space $\mathfrak{A} = \omega + \Omega^1(\mathcal{X},E)$. Then, for every $\omega'\in \mathfrak{A}$ there exists a unique 1-form $\tau \in \Omega^1(\mathcal{X},E)$ such as $\omega' = \omega + \tau$. The "linear part" of $\mathcal{Q}^{\omega}$ with respect to ω for $\eta(x)$ is trivial, because $\mathcal{Q}^{\omega}$ does not depend on the ω in this case. For generator $\psi(x)$ we have

$$\mathcal{Q}^{\omega}\int_{\mathcal{X}} Tr(\psi \wedge *\sigma) = \int_{\mathcal{X}} Tr(D^{\omega}\phi \wedge *\sigma) = \int_{\mathcal{X}} Tr(\phi \wedge * \partial^{\omega}\sigma). \tag{5.7}$$

So, if σ is an element of the tangent space of the space $C(P)$ of gauge equivalent classes of connection, i.e. $\sigma\in T_{\omega}C(P) = \{ \sigma\in \Omega^1(\mathcal{X},E): \partial^{\omega}\tau = 0 \}$, then expression (5.7) is equal to zero. For generator $\chi(x)$ we have

$$\mathcal{Q}^{\omega + \tau(t)}\int_{\mathcal{X}} Tr(\chi \wedge *\theta) = \int_{\mathcal{X}} Tr\{[\Theta^{\omega} + *\Theta^{\omega} + \tag{5.8}$$
$$+ (1+*)(D^{\omega}\tau + \tfrac{1}{2}[\tau,\tau])] \wedge *\theta\},$$

where $\tau(t)$ is a smooth curve on the space $\Omega^1(\mathcal{X},E)$ such as $\tau(0)=0$, $\partial^{\omega}\tau'(0)=0$. If we take ω belonging to $\mathfrak{M}$ (i.e. that ω is a ASD-connection) and differentiate with respect to t at $t=0$, we get the second term of (5.6).■

It should be noticed in conclusion that elements of *GB*-algebra $\mathcal{G}(E)$ such as (4.11), where c_{μ} some cycle on $\mathcal{X}$, appear in the computation of Donaldson polynomials. Such elements of *GB*-algebra $\mathcal{G}(E)$ will be considered in the separate paper.

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ГЕОМЕТРИЧЕСКИЙ ПОДХОД К КВАНТОВОЙ ТОПОЛОГИЧЕСКОЙ ТЕОРИИ ПОЛЯ.

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Резюме

В работе [6] Виттеном была построена квантовая топологическая теория поля (в дальнейшем КТТП) наблюдаемые которой являются топологическими инвариантами. Для построения КТТП Виттеном были использованы хорошо известные инварианты Дональдсона 4-мерных многообразий.

Пусть $P(M,G)$ – главное расслоение, базой которого является 4-мерное риманово многообразие X, а структурной группой полупростая компактная группа Ли G, реализованная кососимметрическими матрицами в присоединенном представлении. Обозначим через $E = P\times_G \mathcal{G}$ присоединенное векторное расслоение со слоем $\mathcal{G}$ (алгебра Ли группы G) на котором группа G действует присоединенным образом. В КТТП дано представление полиномов Дональдсона в виде континуальных интегралов от функционалов, содержащих интегралы от фермионных форм W_i, $i=0, 1, 2, 3$ (см. [6]), где $\deg W_i = i$. В данной работе дается основная идея построения бесконечномерной алгебры Грассмана $G(E)$ на расслоении E, позволяющей дать строгое геометрическое описание фермионных форм и показать, что W_i есть элементы этой алгебры. В построении использована конструкция работы [10] и понятие потока де Рама [11]. При этом оказывается, что лагранжиан L теории является элементом пространства F_ω алгебры $G(F)$, зависящим от связности ω, то есть имеет смысл рассматривать алгебру $G_\omega(F)$, присоединенную к некоторой точке ω пространства неприводимых гладких связностей $\mathfrak{A}$ расслоения P. Сечения получающегося расслоения дают бесконечномерные дифференциальные формы на $\mathfrak{A}$. Оператор суперсимметрии Q также зависит от связности ω. Построенная геометрическая картина позволяет показать, что в точках ω многообразия классов калибровочно эквивалентных связностей (по действию группы калибровочных преобразований), которые удовлетворяют условию антисамодуальности (антиинстантоны) ядро оператора суперсимметрии Q совпадает с ядром оператора $K = d^{\omega}\delta(1+*)d^{\omega}$ (здесь d^{ω} – ковариантный дифференциал, δ^{ω} ковариантный кодифференциал и $*$-оператор Ходжа.

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A CLASSIFICATION OF COMPLETE UNIFORM SPACES

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The paper presents a new classification of uniform spaces. According to this classification with each complete uniform space a cardinal called the completeness index of the space is associated. The properties of completeness indices are studied.

Let $(X,\mathcal{U})$ be a uniform space, X be the corresponding Tychonoff space, and $\mathcal{U}$ be the uniformity defined by means of covers. Let $H \subset \mathcal{U}$ be a system of covers. A filter $\mathcal{F}$ in X is called H-Cauchy filter if $\alpha \cap \mathcal{F} \neq \emptyset$ for each $\alpha \in H$.

D E F I N I T I O N 1. *A uniform space $(X,\mathcal{U})$ is called H-complete where $H \subseteq \mathcal{U}$ if each H-Cauchy filter $\mathcal{F}$ in X has at least one accumulation point, i.e.* $\cap \{[F]: F \in \mathcal{F}\} \neq \emptyset$.

In case H is a base of the uniformity $\mathcal{U}$, then each H-Cauchy filter $\mathcal{F}$ is also a Cauchy filter in $(X,\mathcal{U})$ and each accumulation point of a Cauchy filter $\mathcal{F}$ is its limit (see e.g. [6], p.223), i.e. H-completeness of a uniform space reduces to its usual completeness.

Notice that if $H \subset H_1 \subset \mathcal{U}$ and a uniform space $(X,\mathcal{U})$ is H-complete, then it is H_1-complete and hence complete, too. The converse does not generally hold.

D E F I N I T I O N 2. *Let $(X,\mathcal{U})$ be a complete uniform space. Its completeness index $ic(X,\mathcal{U})$ is defined by the equality $ic(X,\mathcal{U}) = \min \{m$: there exists $H \subset \mathcal{U}$ s.t. $|H| = m$ and $(X,\mathcal{U})$ is H-complete$\}$.*

For each complete uniform space $(X,\mathcal{U})$ either $ic(X,\mathcal{U}) = 1$, or $\aleph_0 \le ic(X,\mathcal{U}) \le w(X,\mathcal{U})$, where $w(X,\mathcal{U})$ denotes the uniform weight of $(X,\mathcal{U})$, i.e. the minimal cardinality of bases of the uniformity $\mathcal{U}$.

P R O P O S I T I O N 1*. *For a uniform space $(X,\mathcal{U})$ the following conditions are equivalent:*

1) $ic(X,\mathcal{U}) = 1$;

2) *the space $(X,\mathcal{U})$ is uniformly locally compact (i.e. there exists a uniform cover $\alpha \in \mathcal{U}$, consisting of compact subsets of the space X).*

P r o o f. 1) $\Rightarrow$ 2). Let H consist of a single uniform cover $\alpha \in \mathcal{U}$ and a uniform space $(X,\mathcal{U})$ be H-complete. We shall show that there exists a uniform cover $\beta \in \mathcal{U}$, consisting of compact sets. Let $\beta = \{[A]: A \in \alpha\}$. Consider an arbitrary filter $\mathcal{F}_A$ in [A]. Then $\mathcal{F} = \{F \subset X$: there exists $P \in \mathcal{F}_A$ s.t. $F \supset P\}$ is an H-Cauchy filter in X. Therefore $\cap\{[P]: P \in \mathcal{F}_A\} = \cap\{[F]: F \in \mathcal{F}\} \ne \emptyset$. Hence $\mathcal{F}_A$ has an accumulation point in [A], i.e. [A] is compact.

2) $\Rightarrow$ 1). Assume that the elements of a uniform cover $\alpha \in \mathcal{U}$ are compact. Then each $\{\alpha\}$-Cauchy filter has at least one accumulation point and hence $ic(X,\mathcal{U}) = 1$.

D E F I N I T I O N 3 ([4]). *A uniformly continuous mapping f of a uniform space $(X,\mathcal{U})$ onto a uniform space $(Y,\mathcal{V})$ is called twice uniformly continuous if for each $\alpha \in \mathcal{U}$ there exists $\beta \in \mathcal{V}$ such that the cover $f^{-1}\beta$ is a refinement of the cover $\alpha^{<}$, where $\alpha^{<} = \{\cup\alpha_0 : \alpha_0$ runs all finite subfamilies of $\alpha\}$.*

B.A.Pasynkov has proved in [7] that an image of a complete uniform space under a twice uniformly continuous mapping is complete. We strengthen this result as follows:

P R O P O S I T I O N 2. *Let $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a twice uniformly continuous mapping. If the uniform space $(X,\mathcal{U})$ is H-complete, then there exists $B \subset \mathcal{V}$ such that the uniform space $(Y,\mathcal{V})$ is B-complete and $|B| \le |H|$.*

P r o o f. Assume that the uniform space $(X,\mathcal{U})$ is H-complete and for each $\alpha \in H$ find $\beta_\alpha \in \mathcal{V}$ such that $f^{-1}\beta_\alpha$ refines $\alpha^{<}$. Let $B = \{\beta_\alpha: \alpha \in H\}$. It is obvious that

* The same fact in different terms was stated without proofs also in [3].

$|B| \le |H|$. To show that the uniform space $(Y,\mathcal{V})$ is B-complete consider a B-Cauchy filter $\mathcal{F}$ in Y and let ξ be an ultrafilter in X containing the centered system $f^{-1}\mathcal{F}$. Since f is twice uniformly continuous, it follows that $\alpha^{<} \cap \xi \ne \emptyset$ for any $\alpha \in H$, and therefore $\alpha \cap \xi \ne \emptyset$ for any $\alpha \in H$ by maximality of ξ. Now H-completeness of $(X,\mathcal{U})$ implies that $\cap\{[P]: P \in \xi\} \ne \emptyset$. By continuity of f it follows now that $\cap\{[F]: F \in \mathcal{F}\} \ne \emptyset$. Hence the uniform space $(Y,\mathcal{V})$ is B-complete.

C O R O L L A R Y. *If a mapping $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ is twice uniformly continuous and a uniform space $(X,\mathcal{U})$ is complete, then the uniform space $(Y,\mathcal{V})$ is complete, too, and besides $ic(Y,\mathcal{V}) \le ic(X,\mathcal{U})$.*

P R O P O S I T I O N 3. *Let $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a uniformly continuous perfect surjection. If $(Y,\mathcal{V})$ is B-complete, then there exists $H \subset \mathcal{U}$ such that $(X,\mathcal{U})$ is H-complete and $|H| \le |B|$.*

P r o o f. Let the uniform space $(Y,\mathcal{V})$ be B-complete and let $H = \{f^{-1}\beta: \beta \in B\}$. Then $H \subset \mathcal{U}$ and $|H| \le |B|$. To show that the uniform space $(X,\mathcal{U})$ is H-complete consider an H-Cauchy filter $\mathcal{F}$ in X. Then $f\mathcal{F} = \{fF: F \in \mathcal{F}\}$ is a B-Cauchy filter in Y and hence, by B-completeness of the space $(Y,\mathcal{V})$, the filter $f\mathcal{F}$ has an accumulation point $y \in Y$. Since f is perfect it follows that the filter $\mathcal{F}$ has an accumulation point $x \in f^{-1}y$ (see e.g. [6]) and therefore $(X,\mathcal{U})$ is H-complete.

C O R O L L A R Y. *Let $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a uniformly continuous perfect surjection. If $(Y,\mathcal{V})$ is complete, then $(X,\mathcal{U})$ is complete, too, and besides $ic(X,\mathcal{U}) \le ic(Y,\mathcal{V})$.*

From Propositions 2 and 3 it follows

P R O P O S I T I O N 4. *Let $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a twice uniformly continuous perfect mapping. If one of the spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is complete, then the second space is complete, too, and besides $ic(X,\mathcal{U}) = ic(Y,\mathcal{V})$.*

L E M M A 1. *Let $(X,\mathcal{U})$ be a complete uniform space and $ic(X,\mathcal{U}) \le m$ $(m \ge \aleph_0)$. Then there exists a system $B \subset \mathcal{U}$ such that B is a base of a pseudouniformity on X, $|B| \le m$, and the space $(X,\mathcal{U})$ is B-complete.*

P r o o f. Let $H \subset \mathcal{U}$ be a system of covers such that $|H| \le m$, and the space $(X,\mathcal{U})$ is H-complete. For each cover

$\alpha \in H$ take a normal sequence $\varphi_\alpha = \{\alpha_n\}$ of covers $\alpha_n \in \mathcal{U}$ such that $\alpha_1 = \alpha$ and let $H' = \cup\{\varphi_\alpha: \alpha \in H\}$. Let B denote the family of all finite inner intersections of covers from H'. Then $|B| \le m$ and B is a base of a pseudouniformity on X.

T H E O R E M 1. *For a uniform space $(X,\mathcal{U})$ the following conditions are equivalent:*

(1) *the uniform space $(X,\mathcal{U})$ is complete and $ic(X,\mathcal{U}) \le m$;*

(2) *there exists a uniform space $(Y,\mathcal{V})$ and a uniformly continuous perfect surjection $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ such that $w(Y,\mathcal{V}) \le m$.*

P r o o f. (1) $\Rightarrow$ (2). Let $(X,\mathcal{U})$ be complete and $ic(X,\mathcal{U}) \le m$. Then by Lemma 1 there exists a system $B \subset \mathcal{U}$ such that B is a base of a pseudouniformity on X, $|B| \le m$ and $(X,\mathcal{U})$ is B-complete. For each $x \in X$ let $[x] = \cap\{\alpha(x): \alpha \in B\}$. It is easy to notice that either $[x] = [y]$ or $[x] \cap \cap [y] = \emptyset$ for any $x,y \in X$. Let $Y = \{[x]: x \in X\}$ and define a mapping $f: X \to Y$ by $fx = [x]$. For each $\alpha \in B$ let $[\alpha] = = \{Y\backslash f(X\backslash A): A \in \alpha\}$. One can directly show that $\mathcal{V}_0 = \{[\alpha]: \alpha \in B\}$ is a base of some uniformity $\mathcal{V}$ on Y. Notice that if α is a star refinement of β and $\alpha,\beta \in B$, then α is a refinement of the cover $f^{-1}[\beta] = \{f^{-1}(Y\backslash f(X\backslash B)): B \in \beta\}$ and hence $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous. To show that f is perfect take $x \in X$ and an arbitrary filter $\mathcal{F}_x$ in $[x] = = f^{-1}f(x)$. Let $\mathcal{F} = \{F \subset X: F \supset F_x$ for some $F_x \in \mathcal{F}_x\}$. Then $\alpha \cap \mathcal{F} \ne \emptyset$ for any $\alpha \in B$ and hence by B-completeness of the space $(X,\mathcal{U})$ it follows that $\cap\{[F]: F \in \mathcal{F}\} \ne \emptyset$. Since $[x]$ is closed in X we conclude that $\cap\{[F]: F \in \mathcal{F}\} = \cap\{[F_x]: F_x \in \in \mathcal{F}_x\} \ne \emptyset$, i.e. $\mathcal{F}_x$ has an accumulation point in $[x]$ and hence $[x]$ is compact. To show that f is closed it is sufficient to check that for each point $x \in X$ and every open set $O \supset [x]$ there exists $\alpha \in B$ such that $\alpha([x]) \subset O$. Assume that there is no such $\alpha \in B$, i.e. that $P_\alpha = \alpha([x]) \cap (X\backslash O) \ne \emptyset$ for every $\alpha \in B$. Since B is a base of uniformity it follows that the system $\xi = \{P_\alpha: \alpha \in B\}$ is centered.

Take an ultrafilter $\mathcal{F}_\xi$ in X containing the centered system ξ. Then $\alpha \cap \mathcal{F}_\xi \ne \emptyset$ for any $\alpha \in \mathcal{U}$. Really, for each $\alpha \in B$ there exists $\beta \in B$ such that $\beta(x) = \beta([x]) \subset A$, where $A \in \alpha$. However $\beta([x]) \in \mathcal{F}_\xi$ because $P_\beta \in \mathcal{F}_\xi$ and hence $A \in \alpha \cap \cap \mathcal{F}_\xi$. Therefore $\cap\{[F]: F \in \mathcal{F}_\xi\} \subset \cap\{[\alpha(x)]: \alpha \in B\} = \cap\{\alpha(x): \alpha \in B\} \subset \cap\{P_\alpha: \alpha \in B\} = \emptyset$. On the other hand B-completeness of the space $(X,\mathcal{U})$ implies that $\cap\{[F]: F \in \mathcal{F}_\xi\} \ne \emptyset$ The obtained contradiction means that there exists $\alpha \in B$ such that $\alpha([x]) \subset O$ and hence the mapping f is closed.

Since $w(Y,\mathcal{V}) \le m$ by construction of $(Y,\mathcal{V})$, to complete the proof we have to show only that the space $(Y,\mathcal{V})$ is complete. Let $\mathcal{F}_Y$ be an arbitrary Cauchy filter in $(Y,\mathcal{V})$, and let $\mathcal{F}_X = \{F \subset X: F \supset f^{-1}M \text{ for some } M \in \mathcal{F}_Y\}$. By construction of the uniform space $(Y,\mathcal{V})$ it is clear that $\alpha \cap \mathcal{F}_X \neq \emptyset$ for each $\alpha \in B$. By B-completeness of the uniform space $(X,\mathcal{U})$ we conclude that the filter $\mathcal{F}_X$ has an accumulation point in X. Since $f\mathcal{F}_X = \mathcal{F}_Y$ and f is continuous, it follows that the filter $\mathcal{F}_Y$ has an accumulation point in Y. However each accumulation point of a Cauchy filter is its limit and therefore the space $(Y,\mathcal{V})$ is complete.

Implication (2) ⇒ (1) follows from Proposition 3.

C O R O L L A R Y. *A uniform space $(X,\mathcal{U})$ is complete and $ic(X,\mathcal{U}) \le \aleph_0$ iff there exists a perfect uniformly continuous mapping from $(X,\mathcal{U})$ onto a complete metric space.*

A.V.Arhangelsky [5] and Z.Frolik [8] proved independently that a Tychonoff space X is Čech complete iff there exists a seqtence $H = \{\alpha_n\}$ of open covers of X such that each H-Cauchy filter has at least one accumulation point. Therefore it is natural to call a uniform space $(X,\mathcal{U})$ uniformly Čech complete in case it is complete and $ic(X,\mathcal{U}) \le \aleph_0$ (cf. also [3]).

Let $(X,\mathcal{U})$ be a uniform space, C(X) be the set of its compact subsets and $C(\mathcal{U})$ be the Hausdorff uniformity on C(X) (see e.g. [6]). It is known that the space $(C(X),C(\mathcal{U}))$ is complete iff the space $(X,\mathcal{U})$ is complete [2]. The next theorem supplements this result:

T H E O R E M 2. *A uniform space $(X,\mathcal{U})$ is complete and $ic(X,\mathcal{U}) \le m$ iff the hyperspace $(C(X),C(\mathcal{U}))$ is complete and $ic(C(X),C(\mathcal{U})) \le m$.*

P r o o f. Let $(X,\mathcal{U})$ be a uniform complete space and $ic(X,\mathcal{U}) \le m$. Then by Theorem 1 there exists a complete uniform space $(Y,\mathcal{V})$ such that $w(Y,\mathcal{V}) \le m$ and a uniformly continuous perfect surjection $f: (X,\mathcal{U}) \to (Y,\mathcal{V})$. Then the natural mapping $C(f): (C(X),C(\mathcal{U})) \to (C(Y),C(\mathcal{V}))$ is uniformly continuous, too (see [9]). Since the mapping f is perfect and sutjective, and uniformities $C(\mathcal{U})$ and $C(\mathcal{V})$ induce Vietoris topology on C(X) and C(Y) respectively, by a result of M.M.Čoban the mapping C(f) is a perfect surjection (see [9], p.356). According to the result stated above this implies that the space $(C(Y),C(\mathcal{V}))$ is complete. From the definition of Hausdorff uniformity it is clear that $w(C(Y), C(\mathcal{V})) = w(Y,\mathcal{V})$. The corollary of Proposition 3 allows to conclude now, that the uniform space $(C(X),C(\mathcal{U}))$ is complete and $ic(C(X),C(\mathcal{U})) \le m$. The converse follows from the fact that $(X,\mathcal{U})$ is a closed uniform subspace of the hyperspace $(C(X),C(\mathcal{U}))$ and the completeness index does not increase when taking closed subspaces.

C O R O L L A R Y. *A uniform space $(X,\mathcal{U})$ is uniformly Čech complete iff the hyperspace $(C(X),C(\mathcal{U}))$ is uniformly Čech complete.*

T H E O R E M 3. *For a complete uniform space the following conditions are equivalent:*

(1) *a uniform space $(X,\mathcal{U})$ is uniformly Čech complete;*

(2) *a uniform space $(X,\mathcal{U})$ is the limit of an inverse system $\{(X_a,\mathcal{U}_a),\pi^b_a\colon a \in M\}$, the members of which are complete metrizable uniform spaces and all projections are uniformly continuous perfect surjections.*

P r o o f. (1) ⇒ (2). Let a space $(X,\mathcal{U})$ be uniformly Čech complete and let a system $H = \{\gamma_1,\gamma_2,\dots,\gamma_m,\dots\} \subset \mathcal{U}$ be such that $(X,\mathcal{U})$ is H-complete. By Lemma 1 we can assume that H is a normal sequence of uniform covers. If $a = \{\alpha_n\}$ and $b = \{\beta_n\}$ are normal sequences in $\mathcal{U}$, then following K.Morita [1] we shall write $b > a$ if for each $\alpha_i \in a$ there exists $\beta_j \in b$ which is a (star) refinement of α_i. Let M denote the set of all normal sequences a in $\mathcal{U}$ such that $a > H$. Since $a > H$ it follows that the space $(X,\mathcal{U})$ is a-complete. Let $X_a = \{[x]_a\colon x \in X\}$, where $[x]_a = \cap\alpha_n(x)$, $a = \{\alpha_n\}$. Since a is normal it follows that for all $x,y \in X$ either $[x]_a = [y]_a$, or $[x]_a \cap [y]_a = \emptyset$. Define mappings $\pi_a\colon X \to X_a$ by $\pi_a(x) = [x]_a$ and let $[\alpha_n] = \{X_a\setminus\pi_a(X\setminus A)\colon A \in \alpha_n\}$. Then $B_a = \{[\alpha_n]\}$ is a base of some uniformity $\mathcal{U}_a$ on X_a (see the proof of Theorem 1 in [1]). As in the proof of Theorem 1 one can show that the mapping $\pi_a\colon (X,\mathcal{U}) \to (X_a,\mathcal{U}_a)$ is uniformly continuous and perfect and the space $(X_a,\mathcal{U}_a)$ is complete. Since the uniform space $(X_a,\mathcal{U}_a)$ has a countable base it is metrizable (see e.g. [9]). If $b > a$, then the partition $\{[x]_b\colon x \in X\}$ refines the partition $\{[x]_a\colon x \in X\}$ and hence the mapping $\pi^b_a\colon (X_b,\mathcal{U}_b) \to (X_a,\mathcal{U}_a)$ can be naturally defined; besides it is easy to notice that π^b_a is uniformly continuous, and if $b > a$, then $\pi_a = \pi^b_a \circ \pi_b$. Since π_a is perfect, the mapping π^b_a is perfect, too (see e.g. [9]). Therfore we have an inverse system $\{(X_a,\mathcal{U}_a), \pi^b_a, a \in M\}$ with the desired properties.

Since the uniform space $(X,\mathcal{U})$ is complete, a theorem of K.Morita [1] implies that the limit of this system is exactly the uniform space $(X,\mathcal{U})$.

(2) $\Rightarrow$ (1). Let a uniform space $(X,\mathcal{U})$ be the limit of an inverse system $\{(X_a,\mathcal{U}_a),\pi^b_a, a \in M\}$ where every $(X_a,\mathcal{U}_a)$ is a complete uniform metrizable space, and each π^b_a is a uniformly continuous perfect surjection. Then each $\pi_a : (X,\mathcal{U}) \to (X_a,\mathcal{U}_a)$ is a uniformly continuous perfect surjection, too (see e.g. [4] p.148). The Corollary of Proposition 3 implies that the uniform space $(X,\mathcal{U})$ is uniformly Čech complete.

R E M A R K. If in Theorem 3 the property "uniformly Čech complete" is substituted by the property "uniformly locally compact" the statement of the theorem remains true.

P R O P O S I T I O N 5. *Let* $(X,\mathcal{U}) = \Pi\{(X_a,\mathcal{U}_a): a \in M\}$ *be the product of* H_a*-complete uniform spaces* $(X_a,\mathcal{U}_a)$, $H_a \subset \mathcal{U}_a$, $a \in M$. *Then the uniform space* $(X,\mathcal{U})$ *is H-complete, where* $H = \{\pi^{-1}_a\alpha_a : \alpha_a \in H_a,\ a \in M\}$ *and* $\pi_a : X \to X_a$ *is the projection.*

P r o o f. Let $\mathcal{F}$ be an H-Cauchy filter in X and ξ be an ultrafilter in X containing $\mathcal{F}$. Then $\xi_a = \pi_a\xi$ is an ultrafilter in X_a and $\xi_a \cap \alpha_a \neq \emptyset$ for each $\alpha_a \in H_a$, $a \in M$. Since the space $(X_a,\mathcal{U}_a)$ is H_a-complete the ultrafilter ξ_a has an accumulation point $x_a \in X_a$, $a \in M$, and each accumulation point of an ultrafilter is its limit. Hence the ultrafilter ξ_a converges to a point $x_a \in X_a$ $(a \in M)$. Therefore the ultrafilter converges to a point $x = \{x_a : a \in M\}$ (see e.g. [6], Ch.I, §7) and hence $\cap\{[F]: F \in \mathcal{F}\} \neq \emptyset$.

C O R O L L A R Y. *Let* $(X,\mathcal{U}) = \Pi\{(X_a,\mathcal{U}_a): a \in M\}$ *be the product of complete uniform spaces* $(X_a,\mathcal{U}_a)$, $a \in M$. *Then* $ic(X,\mathcal{U}) \leq \max\{|M|, \sup\{ic(X_a,\mathcal{U}_a): a \in M\}\}$.

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КЛАССИФИКАЦИЯ ПОЛНЫХ РАВНОМЕРНЫХ ПРОСТРАНСТВ

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Р е з ю м е

В работе дана классификация полных равномерных пространств по отношению кардинальных чисел, т.е. каждому полному равномерному пространству естественным образом сопоставлено кардинальное число, названное его индексом полноты и изучены их свойства. В частности, доказана равносильность следующих условий: 1) равномерное пространство $(X,\mathcal{U})$ полно и его индекс полноты счетен; 2) равномерное пространство $(X,\mathcal{U})$ посредством равномерно непрерывного совершенного отображения можно отобразить на полное метрическое пространство; 3) равномерное пространство $(X,\mathcal{U})$ является пределом обратного спектра, составленного из полных метрических пространств с равномерно непрерывными проекциями.

TRIPLE SYSTEMS AND SUBMANIFOLDS IN A HOMOGENEOUS SPACE

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1. Introduction. Algebraically various triple systems often arise in the study of nonassociative algebras from the associator function and other multilinear objects. In particular Lie triple systems arise in the study of Jordan algebras and their generalization arises in Malcev algebras [4].Geometrically Lie triple systems are used at firstly to study of totally geodesic submanifolds of a Riemannian symmetric space [1]. In this article we shall show how a generalization of Lie triple systems arises from the study of curvature and geodesics of a torsion free connection. Using the notion of Lie triple algebra (L.t.a.) of the reductive homogeneous space M there is shown that the finding of reductive subspaces of M is equivalent to the finding of subalgebras in L.t.a. of M. Moreover, if M = G/H is a reductive space with canonical connection of the 2-nd kind [3], then its submanifold N through the origin o = H is autoparallel if and only if its tangent space To(N) is a subalgebra in L.t.a. of M.

2. Basics. Here we review some basic facts about covariant differentation, torsion and curvature as given in [1],[2]. Let M be a C^∞ manifold and let D be a covariant differentation operator (i.e. connection) defined of M.Thus for each pair X,Y of C^∞ vector fields defined on a suitable domain $A \subset M$ [2] we have a C^∞ vector field $D_X Y = D(X,Y)$ with

domain A such that if Z is a C^∞ vector field on A and f is a C^∞ real valued function on A, then D satisfies

$$D(X,Y + Z) = D(X,Y) + D(X,Z),$$

$$D(X + Y,Z) = D(X,Z) + D(Y,Z),$$

$$D(fX,Y) = fD(X,Y)$$

$$D(X,fY) = (Xf)Y + fD(X,Y).$$

The torsion tensor, Tor, assigns to each pair of C^∞ vector fields X and Y with domain A is a C^∞ vector field

$$Tor(X,Y) = D(X,Y) - D(Y,X) - [X,Y]$$

with the same domain A. For such field we have

$$Tor(X + Y,Z) = Tor(X,Z) + Tor(Y,Z),$$

$$Tor(fX,Y) = f\,Tor(X,Y),$$

$$Tor(X,Y) = - Tor(Y,X).$$

The curvature tensor, R, is defined for C^∞ vector fields X,Y and Z with domain A by

$$R(X,Y)Z = D(X,D(Y,Z)) - D(Y,D(X,Z)) - D([X,Y],Z) =$$

$$= [D_X,D_Y]Z - D_{[X,Y]}Z.$$

If f is a C^∞ real valued function which is defined on A, then R satisfies

$$R(fX,Y)Z = fR(X,Y)Z = R(X,Y)(fZ),$$

$$R(X,Y)Z = - R(Y,X)Z$$

and R(X,Y)Z is additive in each of its variables.

An abstract Lie triple system (L.t.s.) has been defined in [6] as a vector space V over a field P with an operation [X,Y,Z] defined on $V \times V \times V$ into V satisfying

(2.1) [X,Y,Z] is trilinear over F,

(2.2) $[X,Y,Z] = - [Y,X,Z]$

(2.3) $[X,Y,Z] + [Y,Z,X] + [Z,X,Y] = 0.$

In particular if we set $[X,Y,Z] = R(X,Y)Z$ for C^∞ vector fields X,Y,Z on M, we see that [X,Y,Z] satisfies (2.1) and (2.2) over the algebra of C^∞ functions on M.

One of the properties of semi-Riemannian connection on M is that $Tor(X,Y) = 0$ for vector fields X and Y. In this case we have the 1-st and 2-nd Bianchi identities which are respectively

$$[X,Y,Z] + [Y,Z,X] + [Z,X,Y] = 0,$$

$$(D_X R)(Y,Z) + (D_Y R)(Z,X) + (D_Z R)(X,Y) = 0,$$

where $D_{\mathcal{U}}R$ denotes covariant differentation of R relative to $\mathcal{U}$. The 1-st Bianchi identity is (2.3) above. Thus if $\mathcal{X}(M)$ denotes the set of all C^{∞}-vector fields on M and Tor(X,Y) = = 0 for all $X,Y \in \mathcal{X}(M)$, then $\mathcal{X}(M)$ is an abstract L.t.s. over F(M) relative to the operation [X,Y,Z] = R(X,Y)Z.

Let M be a C^{∞} semi-Riemannian manifold with nondegenetate metric $\langle , \rangle$; that is for each $m \in M$, $\langle , \rangle$ induces a nondegenerate bilinear form $\langle , \rangle_m$ on each tangent space M_m. Now [2] there exists a unique connection D on the semi-Riemannian manifold M such that for $X,Y,Z \in \mathcal{X}(M)$

(2.4) $0 = \mathrm{Tor}(X,Y) = D(X,Y) - D(Y,X) - [X,Y]$,

(2.5) $Z \langle X,Y\rangle = \langle D(Z,X),Y\rangle + \langle X,D(Z,Y)\rangle$,

i.e. D preserves the metric tensor under parallel translation. In this paper we shall assume that the connection D on a semi-Riemannian manifold satisfy (2.4) and (2.5). Thus since Tor(X,Y) = 0 we see that $\mathcal{X}(M)$ is an abstract L.t.s. with other relations using the metric tensor [2].

A nonsingular submanifold M' of a C^{∞} semi-Riemannian manifold M is a submanifold of M such that the metric tensor of M restricted to the tangent space M'_m for all $m \in M'$ is nondegenerate on M'_m. If σ is a C^{∞} curve in the C^{∞} semi-Riemannian manifold M with tangent vector field T, then it is geodesic if D(T,T) = 0 on σ. A nonsingular submanifold M' of M is totally geodesic at a point $m \in M'$ if for every $X \in M'_m$ the geodesic $\sigma(t)$ of M passing through M and with tangent vector X lies in M' for small value of the parameter t. If M' is totally geodesic in every point of M', then it is called a totally geodesic submanifold of M. From [2] follows the well-known result.

T H E O R E M 1 ([2]). Let M' be a nonsingular C^{∞} submanifold of a semi-Riemannian C^{∞} manifold M. Then the following properties are equivalent:

(T1) M' is totally geodesic;

(T2) geodesics in M' are geodesics in M;

(T3) parallel translation in M' and M are the same.

Anyone of the above imply

(T4) if X,Y,Z are in $\mathcal{X}(M')$, then [X,Y,Z] = R(X,Y)Z is in $\mathcal{X}(M')$.

3. Submanifolds of a reductive homogeneous spaces. Let G be a connected Lie group and H a closed Lie subgroup so that the homogeneous space is reductive [3]; that is, if g (resp. h) is the Lie algebra of G (resp. H), there exists a subspace m of g such that g = m + h (subspace direct sum) where $(\mathrm{ad}H)m \subset m$ (i.e. $[m,h] \subset m$ when H is connected). For $X,Y \in m$ an anticommutative multiplication XY is defined in m by $[X,Y] = XY + h(X,Y)$ where XY (resp. h(X,Y)) is the component of [X,Y] in m (resp. h) relative to a fixed decomposition g = h + m [4]. The Lie algebra identities of g yield the following identities for m and h. For $X,Y,Z \in m$ and $h \in h$ we have

(3.1) $XY = -YX$

(3.2) $h(X,Y) = - h(Y,X)$,

(3.3) $[Z,h(X,Y)] + [X,h(Y,Z)] + [Y,h(Z,X)] =$
$= (XY)Z + (YZ)X + (ZX)Y$,

(3.4) $h(XY,Z) + h(YZ,X) + h(ZX,Y) = 0$,

(3.5) $[h,h(X,Y)] = h([h,X],Y) + h(X,[h,Y])$,

(3.6) $[h,XY] = [h,X]Y + X[h,Y]$.

The above algebra will be denoted by (m,XY) and it and other nonassociative algebras are related to the differential geometry of G/M by using the following result of Nomizu ([3]).

THEOREM 2. Let G/H be a reductive homogeneous space with fixed decomposition g = h + m. Then there exists a one-to-one correspondence between the set of all G-invariant connections on G/H and the set of all bilinear functions α: $m \times m \to m$ such that the mappings

$$\mathrm{ad}_m h: m \to m: X \to [h,X]$$

are derivations of the resulting algebra.

If the reductive space M = G/H is a semi-Riemannian manifold, then the form $\langle X,Y\rangle_0$ on m is ad(H)-invariant and we have

$$\langle [X,h],Y\rangle_0 + \langle X,[Y,h]\rangle_0 = 0$$

for $X,Y \in m$ and $h \in h$. In [3] it's also shown that if a connection D on M = G/H satisfies (2.4) and (2.5) for the above metric tensor and if the 1-parameter subgroup $x(t)$ of G generated by an element X in m projects by π: $G \to G/H$: $x(t) \to x^*(t)$ so that $x^*(t)$ is a geodesic in G/H, then the connection function for D is given by $\alpha(X,Y) = 1/2XY$ for $X,Y \in m$ (canonical connection of the first kind). In this case it is also shown that $\langle XY,Z\rangle_o = \langle X,YZ\rangle_o$, i.e. m is an anticommutative algebra with a nondegenerate invariant form. Also according to [3] the curvature formula for M = G/H evaluated at o in M is given by

$$(3.7)\quad R(X,Y)Z = 1/4X(YZ) - 1/4Y(XZ) - 1/2(XY)Z - [h(X,Y),Z]$$

for $X,Y,Z \in m$. Note that (3.7) shows a subalgebra n of (with the same multiplication) is an abstract L.t.s. relative to $[X,Y,Z] = R(X,Y)Z$ iff n is h(n,n)-invariant.

Following theorem generalizes the result on totally geodesic submanifolds and Lie triple systems in symmetric space ([1]).

T H E O R E M 3. Let M = G/H be a reductive semi-Riemannian homogeneous space with fixed decomposition g = = h + m and with canonical connection of the first kind. If N is a totally geodesic nonsingular submanifold of M containing o = H and if $N_o = n \subset m$ denotes the tangent space of o in N, then n is an abstract L.t.s. with [X,Y,Z] given by (3.7). Conversely, n is a nonsingular subalgebra of m which is at the same time also an abstract L.t.s. given by (3.7), then there exists a totally geodesic nonsingular submanifold N of M = G/H with $o \in N$ and $N_o = n$.

P r o o f. The first part follows directly from the statement (T4) in Theorem 1. For the converse we note that from the remarks following (3.7) we see that the subalgebra n is h(n,n)-invariant. This yields that the subspace $\bar{g} = n +$ $+ h(n,n)$ is a Lie subalgebra in g since

$$[\bar{g},n] = [n,n] + [h(n,n),n] \subset$$
$$\subset nn + h(n,n) + n \subset n + h(n,n) = \bar{g}$$
$$[\bar{g},h(n,n)] = [n,h(n,n)] + [h(n,n),h(n,n)] \subset$$
$$\subset n + h(n,n) = \bar{g}.$$

The last inclusion follows from (3.5) and from the assumption that n is h(n,n)-invariant. Now we may proceed analogously to the case of a symmetric space [1]. Let $\bar{G}$ be the connected subgroup of G with Lie algebra $\bar{g}$ and let $N = \bar{G}\cdot o$. Then N is a submanifold of M containing o and is diffeomorphic to $\bar{G}/\bar{K}$ where $\bar{K}$ is the closed subgroup of $\bar{G}$ leaving o fixed. $N_o = n$ so that N is a nonsingular manifold.

Next by the assumption concerning the connection we have that the geodesics in M through o are of the form $\exp tX\cdot o$ for $t \in \mathbb{R}$ and $X \in m$. This geodesic is tangent to N at o iff $X \in N_o = n$; thus from the definition N is totally geodesic at o. Next note that the connection D is $\overline{G}$-invariant (because it is G-invariant) and $\overline{G}$ is a group of isometries of M and N relative to the corresponding semi-Riemannian metrics, and $\overline{G}$ acts transitively on N. Using these we see that a geodesic at any point of N is taken by some isometry induced by $\overline{G}$ into a geodesic through o in N and consequently N is geodesic at each of its points; i.e. totally geodesic.

4. Reductive subspaces and Lie triple algebras. Let now $M = G/H$ be a reductive homogeneous space with decomposition $g = h + m$ and let G' be a subgroup of G. If for the Lie algebra g' of G' holds $g' = g' \cap h + g' \cap m = h' + m'$, then the corresponding homogeneous space $M' = G'/H'$ (where $H' = G' \cap H'$) is called *a homogeneous subspace of M*. By virtue of $[h', m'] \subset g' \cap m = m'$ is the subspace M also a reductive homogeneous space and is called further *a reductive subspace of M* ([8]).

An anticomutative algebra A over a field F with multiplication $(X,Y) \in A \times A \to X*Y \in A$ is called *a Lie triple algebra* (L.t.a.) ([7]) if there is given a trilinear operation $(X,Y,Z) \in A \times A \times A \to [X,Y,Z] \in A$ for which

$$[X,X,Y] = 0,$$
$$\sigma\{[X,Y,Z] + (X*Y)*Z\} = 0,$$
$$\sigma\{[X*Y,Z,\mathcal{U}]\} = 0,$$
$$[X,Y,\mathcal{U}*V] = [X,Y,\mathcal{U}]*V + \mathcal{U}*[X,Y,V],$$
$$[\mathcal{U},V,[X,Y,Z]] = [[\mathcal{U},V,X],Y,Z] + [X,[\mathcal{U},V,Y],Z] + [X,Y,[\mathcal{U},V,Z]].$$

The symbol σ denotes here the cyclic sum for $X,Y,Z \in A$.

From the definition follows that in the case $[X,Y,Z] = 0$ the anticommutative algebra A reduces to the Lie algebra. In the case $X*Y = 0$ we receive a L.t.s.

A subspace of L.t.a. A is called *a subalgebra in A* if it´s closed with respect to the operations in A.

T H E O R E M 4 [4]. Let $M = G/H$ be a reductive homogeneous space with decomposition $g = h + m$. If we define on m the bilinear and trilinear operations by

$$X*Y = XY, \quad [X,Y,Z] = -[h(X,Y),Z]$$

for $X,Y,Z \in m$, then m becomes a L.t.a.

The L.t.a. m with the operations introduced in the Theorem 4. is called further *a L.t.a. of the reductive homogeneous space M.*

T H E O R E M 5. Let $M = G/H$ be a reductive homogeneous space with fixed decomposition $g = h + m$. If $M' = G'/H'$ is a reductive subspace of M, then its tangent space $T_o(M') = m'$ in the origin $o = H$ is a subalgebra in the L.t.a. m. Conversely, if m' is a subalgebra in the L.t.a. m of M, then there exists a reductive subspace $M' \subset M$ such that $o \in M'$ and $T_o(M') = m'$.

P r o o f. Let $M' = G'/H'$ be a reductive subspace with decomposition $g' = h' + m' = g' \cap h + g' \cap m$. Then $m'm' \subset g' \cap m = m'$. In addition from $h(m',m') \subset g' \cap h = h'$ follows $[h(m',m'),m'] \subset m'$ and therefore m' is a subalgebra in m. Let now m' be a subalgebra in m and we shall show that $g' = h(m',m') + m'$ is a Lie subalgebra in g. Really $[m',m'] = h(m',m') + m'm' \subset h(m',m') + m' = g'$ and $[h(m',m'), m'] \subset m'$ since m' is a subalgebra. Further

$$[h(m',m'),h(m',m')] \subseteq [h(m',m'),[m',m']] - [h(m',m'),m'm'],$$

$$[h(m',m'),m'm'] \subset [h(m',m'),m'] \subset m',$$

$$[h(m',m',[m',m']] \subseteq [m',[h(m',m'),m']] + [m',[m',h(m',m')] \subseteq$$

$$\subseteq [m',m'] \subset g'$$

which proves that $[g',g'] \subset g'$. We may now proceed analogously the case of symmetric space ([1]). Let $G' \subset G$ be the connected Lie subgroup with Lie algebra g' and let $M' = G' \cdot o$ (i.e. G'-orbit of o). Then M' is a submanifold of M containing o and is diffeomorphic to G'/H' where $H = G' \cap H$ is the closed subgroup of G' leaving o fixed ($g^*o \to g^*H$, $g^* \in G'$). Thus M' is reductive and $T_o(M') = m'$.

5. <u>Canonical connections and autoparallel submanifolds.</u> Among all of the G-invariant connections on the reductive homogeneous space G/H the most important are the canonical connections of the first and second kind preferred by Nomizu ([3]). For the canonical connection of the second kind the curvature and torsion tensors in the origin $o = H$ are given by

(5.1) $R(X,Y)Z = -[h(X,Y),Z], \quad T(X,Y) = -XY.$

Comparing these formulas with the result of Theorem 4 we have following

P r o p o s i t i o n. Let $M = G/H$ be a reductive homogeneous space with fixed decomposition $g = h + m$ and canonical connection of the second kind. The subspace $n \subset m$ is a subalgebra in L.t.a. m iff for $X,Y,Z \in n$ we have $R(X,Y)Z \in n$ and $T(X,Y) \in n$.

T H E O R E M 6 ([5]). A reductive subspace M' of the reductive homogeneous space M is a connected complete autoparallel submanifold in M.

T H E O R E M 7. Let $M = G/H$ be a reductive homogeneous space with fixed decomposition $g = h + m$ and canonical connection of the 2-nd kind. Let n be a linear subspace of n such that for all $X,Y,Z \in n$ we have $R(X,Y)Z \in n$ and $T(X,Y) \in n$ where R and T denote the curvature and torsion tensors of the canonical connection. Then there exists a unique connected complete autoparallel submanifold N of M such that $T_o(N) = n$.

P r o o f. From the theorem conditions and the formulas (5.1) we have

$$nn \subset n, \quad [h(n,n),n] \subset n.$$

Analogously to the proof of Theorem 5 we can show that the subspace $g' = n + h(n,n)$ is a subalgebra in g. Let $G' \subset G$ be a connected Lie subgroup of G with Lie algebra g' and $H' = G' \cap H$. Then $N = G'/H'$ is a reductive subspace of M and (by Theorem 6) a connected complete autoparallel submanifold with $T_o(N) = n$. The uniqueness of N can be seen as follows. Let N' be another connected complete autoparallel submanifold of M such that $T_o(N') = n$ and let $x \in N'$. There exists a broken geodesic in N' joining o to x. Since N' is autoparallel, the broken geodesic is a broken geodesic in M. Since N is also autoparallel and complete and $T_o(N) = T_o(N')$ the broken geodesic lies in N. Therefore we have $N' \subset N$ and similarly $N \subset N'$.

T H E O R E M 8. A connected complete autoparallel submanifold N of a reductive homogeneous space $M = G/H$ through the origin $o = H$ is a reductive subspace in M.

P r o o f. Since N is autoparallel, $T_o(N)$ satisfies the assumption for n in Theorem 7. Now the result follows immediately from the proof of Theorem 7.

C O R O L L A R Y 1. Let M = G/H be a reductive homogeneous space with the canonical connection of the 2-nd kind. Then each connected autoparallel submanifold $N \subset M$ containing the origin o = H is of the form N = G'/H' where G' is a subgroup in G with the Lie subalgebra g' and H' = = G' $\cap$ H.

From the Proposition and Theorems 7 and 8 follows also the important

C O R O L L A R Y 2. Let M = G/M be a reductive homogeneous space with decomposition g = h + m and the canonical connection of the 2-nd kind. If N is a connected autoparallel submanifold in M through the origin o = H, then its tangent space $T_o(N)$ = n is a subalgebra in L.t.a. m. Conversely if n is a subalgebra in L.t.a. m, then there exists a connected autoparallel submanifold $N \subset M$ containing o = H such that $T_o(N)$ = n.

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ТРОЙНЫЕ СИСТЕМЫ И ПОДМНОГООБРАЗИЯ В ОДНОРОДНОМ ПРОСТРАНСТВЕ

А.Фляйшер

Р е з ю м е

Пусть $M = G/H$ - редуктивное однородное пространство с размножением $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. В случае полуриманова пространства подпространство $\mathfrak{m}$ превращается в тройную систему Ли относительно операции $[X,Y,Z] = R(X,Y)Z$. Для таких пространств обобщается известный результат [1] о вполне геодезических подмногообразиях и тройных системах Ли в случае симметрических пространств. Для произвольного редуктивного пространства оснащение $\mathfrak{m}$ можно превратить в тройную алгебру Ли [7]. Вводится понятие редуктивного подпространства и доказывается, что нахождение таких пространств эквивалентно нахождению подалгебр в тройной алгебре Ли, естественно ассоциирующейся с подпространством $\mathfrak{m}$ [3]. Кроме того, если M является редуктивным пространством с канонической связностью 2-го рода [3], то его подмногообразие N, проходящее через $o = H$, будет автопараллельным тогда и только тогда, когда его касательное пространство $T_o(N)$ будет подалгеброй в тройной алгебре Ли пространства M.

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ON ALMOST COMMUTATIVE ALGEBRAS

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It is well-known that the following conditions are equivalent on any complex unital Banach algebra A ([2], [10], [15], [23], [28], [33], [36, §13], [40]):

1) A is commutative modulo its Jacobson radical RadA;

2) the spectral radius r_A *is submultiplicative on A;*

3) the Harte joint spectrum σ^A *has the projection property;*

4) $\sigma^A((a_1,a_2,\dots,a_n)) = \{(\Lambda(a_1),\dots,\Lambda(a_n)) : \Lambda \in HomA\}$, *where HomA is the Gelfand space of A* $(a_1,a_2,\dots,a_n \in A)$;

5) $r_A(a) = \sup\{|\Lambda(a)| : \Lambda \in HomA\}$ $(a \in A)$;

6) for every closed subalgebra $B \subset A$ *with* $e_A \in B$ *and for every extremal spectral state* $f \in ext(\Omega(B))$ *there exists an extension of f belonging to* $ext(\Omega(A))$.

Furthermore, as it is shown in [5], *if A is a spectrally convex complex unital Banach algebra, then A/RadA is isomorphic to* $\mathbb{C}$.

The purpose of the present paper is to prove analogous results without assuming that the algebra under consideration is banachable. In section 1 we give fundamental definitions and some preliminary lemmas, section 2 is devoted to the class of spectrally bounded algebras and in section 3 we consider the case of Fréchet algebras.

1. Preliminaries. Throughout this note, all algebras are assumed to be associative, unital and over the field $\mathbb{C}$.

Let A be an algebra. For any linear subspace B of A we denote by B^* the algebraic dual of B equipped with the weak *-topology. We also denote by HomA the subset of all non-zero multiplicative functionals of A^* and by $c(A)$ the set of all n-tuples $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of elements of $a_i \in A$ with arbitrary finite length n. If HomA is non-empty and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is an n-tuple in $c(A)$, then the Gelfand transform $\hat{\mathbf{a}}$ of $\mathbf{a}$ is a function on the space HomA with values in $\mathbb{C}^n$:

$$\hat{\mathbf{a}}(\Lambda) = \Lambda(\mathbf{a}) = (\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n)) \quad (\Lambda \in \mathrm{HomA}).$$

For each subset S of A, $L(S)$ is the linear span of S and $[S]$ is the subalgebra of A generated by the set $S \cup \{e_A\}$, where e_A stands for the identity of A. If $\mathbf{a} = (a_1, \ldots, a_n)$ and

$$S = \bigcup_{i=1}^{n} \{a_i\},$$

we write $L(\mathbf{a})$ (resp. $[\mathbf{a}]$) in place of $L(S)$ (resp. $[S]$). Moreover, if $1 \le k \le n$ $(k, n \in \mathbb{N})$, then x_k denotes the map on the set A^n of all n-tuples $(a_1, a_2, \ldots, a_n)$ of A, which sends $(a_1, a_2, \ldots, a_n)$ to a_k and $P(A^n)$ denotes the subalgebra of $F(A^n, A)$ generated by the functions x_k, where $F(A^n, A)$ is the algebra of all mappings from A^n to A, equipped with pointwise operations (cf. [12, p.99]).

By a *joint spectrum* sp on A we mean a rule which assigns to each element $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in c(A)$ a subset $sp(\mathbf{a})$ of $\mathbb{C}^n$ (possibly empty) in such a way that

(1) $sp((a_1, a_2, \ldots, a_n, e_A)) \supseteq$
$\supseteq \{(\alpha_1, \alpha_2, \ldots, \alpha_n, 1) \in \mathbb{C}^{n+1} : (\alpha_1, \alpha_2, \ldots, \alpha_n) \in sp(\mathbf{a})\}$;

(2) for every $m, k \in \mathbb{N} \cup \{0\}$, with $1 + m + k \le n$, the relation $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in sp(\mathbf{a})$ implies $(\alpha_{1+m}, \ldots, \alpha_{1+m+k}) \in$ $\in sp((a_{1+m}, \ldots, a_{1+m+k}))$;

(3) if $p(\mathbf{a}) = \theta_A$ $(p \in P(A^n))$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in sp(\mathbf{a})$, then $p((\alpha_1 e_A, \alpha_2 e_A, \ldots, \alpha_n e_A)) = \theta_A$.

If sp is a joint spectrum on an algebra A and elements $a_1, a_2, \ldots, a_n$ belong to A, we shall write in the sequel $sp(a_1, a_2, \ldots, a_n)$ in place of $sp((a_1, a_2, \ldots, a_n))$.

We say that a joint spectrum sp on A has the *projec-*

tion property if for every $(a_1,a_2,\dots,a_n) \in c(A)$, one has the relation

$$\pi^n_k(sp(a_1,a_2,\dots,a_n)) = sp(a_1,a_2,\dots,a_k),$$

where $1 \le k \le n$ and π^n_k is the projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$ given by

$$\pi^n_k((\lambda_1,\lambda_2,\dots,\lambda_n)) = (\lambda_1,\lambda_2,\dots,\lambda_k) \quad ((\lambda_1,\lambda_2,\dots,\lambda_n) \in \mathbb{C}^n).$$

Moreover, we say that a joint spectrum sp on A is *bounded* if sp(a) is bounded for all a in A.

R E M A R K 1. The axiomatic approach to joint spectra is due to W. Żelazko [32] (see also [25]) who considered different spectral systems on the set of all commuting families of elements of a given Banach algebra. A definition of joint spectrum, similar to that of given above, in case of Banach algebras is accepted in [18], [26].

The *left* (resp. *right*) *joint spectrum* $\sigma^A_l(\underline{a})$ (resp. $\sigma^A_r(\underline{a})$) of an n-tuple $\underline{a} = (a_1,a_2,\dots,a_n) \in c(A)$ with respect to A is defined to be the set of all those $(\alpha_1,\alpha_2,\dots,\alpha_n) \in \mathbb{C}^n$ for which the n-tuple $(a_1 - \alpha_1 e_A,\dots,a_n - \alpha_n e_A)$ generates a proper left (resp. right) ideal in A. The *Harte joint spectrum* $\sigma^A(\underline{a})$ of $\underline{a} \in c(A)$ with respect to A is the set $\sigma^A_l(\underline{a}) \cup \sigma^A_r(\underline{a})$. It is an immediate consequence of [12, p.99], that for σ^A, σ^A_l, σ^A_r the properties (1) - (3) are satisfied.

We call A to be *spectrally bounded* if $\sigma^A(a) \subset \mathbb{C}$ is non-empty and bounded for each $a \in A$, and if $\sigma^A(a)$ is non-empty and bounded, then $r_A(a) = \sup\{|\lambda| : \lambda \in \sigma^A(a)\}$ is the *spectral radius* of an element $a \in A$.

If B is a linear subspace of A, we let $\Omega(B,A)$ denote the set of all *spectral states* of B with respect to A, that is

$$\Omega(B,A) = \{f \in B^* : f(b) \in \mathrm{conv}(\sigma^A(b)) \text{ for all } b \in B\}$$

(here $\mathrm{conv}(\sigma^A(b))$ is the convex hull of $\sigma^A(b)$ in $\mathbb{C}$). $\Omega(B,A)$ is a convex subset of B^* and the set of all its extreme points is denoted by $\mathrm{ext}(\Omega(B,A))$.

Finally, InvA denotes the group of all *invertible elements* in A, RadA is the *Jacobson radical* of A and algebra A is said to be *almost commutative* if the algebra A/RadA is commutative.

A *topological algebra* is an algebra, which is also a topological vector space in such a way that the ring multiplication is separately continuous. A topological algebra A is called a *Q-algebra* if the set InvA is open. Every complex Q-algebra is always spectrally bounded [16, p.60]. In particular, a complex normed algebra A is a Q-algebra if and only if $\sup\{r_A(a): \|a\| \leq 1\} < \infty$ [17]. But there exist spectrally bounded topological algebras which are not Q-algebras [11, p.49-52]. Moreover, as it is well-known, every linear multiplicative functional on a Q-algebra is automatically continuous [16, p.72].

A *Gelfand-Mazur algebra* A is a complex topological algebra A such that for every proper closed maximal (maximal as a left and as a right) ideal $M \subset A$ the algebra A/M is topologically isomorphic to $\mathbb{C}$. For different classes of Gelfand-Mazur algebras see, for example, [16], [34].

For the sake of clarity we now collect some lemmas we shall need later.

L E M M A 1 [35, p. 68]. *Let sp be a joint spectrum on an algebra A, let* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in c(A)$ *and suppose that* $sp(\mathbf{a})$ *is non-empty. Then for every* $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in sp(\mathbf{a})$ *there is a homomorphism* $\Lambda(\underline{\alpha}) \in Hom[\mathbf{a}]$ *with* $\Lambda(\underline{\alpha})(a_k) = \alpha_k$ $(k = 1, \ldots, n)$ *and the mapping* $\underline{\alpha} \longrightarrow \Lambda(\underline{\alpha})$ *is a homeomorphism of* $sp(\mathbf{a})$ *into* $Hom[\mathbf{a}]$.

L E M M A 2 [35, p.74]. *Let A be an algebra, let* Δ *be a compact subset of HomA and let S be a non-empty subset of A. The following conditions on a map* $\lambda : S \longrightarrow \mathbb{C}$ *are equivalent:*

(a) λ *admits an extension to a functional on A belonging to* Δ;

(b) for each $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in c(S)$, *one has* $\hat{\mathbf{s}}(\lambda) = (\lambda(s_1), \ldots, \lambda(s_n)) \in \hat{\mathbf{s}}(\Delta)$.

L E M M A 3 [35, p.72]. *Suppose sp is a bounded joint spectrum on an algebra A. If sp has the projection property and* $\hat{\mathbf{a}}(HomA) \subset sp(\mathbf{a})$ *for all* $\mathbf{a} \in c(A)$, *then HomA is compact and* $sp(\mathbf{a}) = \hat{\mathbf{a}}(HomA)$ *for all* $\mathbf{a}$ *in* $c(A)$.

L E M M A 4 [24, Lemmas 1 and 2]. *Let A be a spectrally bounded algebra, let B be a subalgebra of A, and suppose* $\varphi \in B^*$ *is such that* $\varphi(b) \in \sigma^A(b)$ *for every b in B.*

Then $\varphi \in HomB$.

2. Almost commutative spectrally bounded algebras. Our main object in this section is to prove:

T H E O R E M 1. *Let A be a complex spectrally bounded algebra. Consider the following assertions:*

1) the algebra A is almost commutative;

2) $r_A(a) = \sup\{|\Lambda(a)| : \Lambda \in HomA\} \quad (a \in A)$;

3) $InvA = \{a \in A : \Lambda(a) \neq 0$ *for any* Λ *in* A^* *satisfying*

$$\Lambda(e_A) = 1 \text{ and } 0 \notin \Lambda(InvA)\};$$

4) if $a \in A$ *and* $\lambda \in L(\{a\})^*$ *is such that* $\lambda(a) \in \sigma^A(a)$, *then* λ *can be extended to an element of HomA;*

5) if $S \subset A$, $\lambda \in L(S)^*$ *and if* $\hat{\mathbf{s}}(\lambda) \in \sigma^A(\mathbf{s})$ *for all* $\mathbf{s} \in c(S)$, *then* λ *can be extended to an element of HomA;*

6) if $\mathbf{a} \in c(A)$, *then every* $\lambda \in L(\mathbf{a})^*$ *satisfying* $\lambda(\mathbf{a}) \in \sigma^A(\mathbf{a})$ *can be extended to an element of HomA;*

7) $\sigma^A(\mathbf{a}) = \hat{\mathbf{a}}(HomA) \quad (\mathbf{a} \in c(A))$;

8) σ^A *has the projection property;*

9) there exists a bounded joint spectrum sp on A possessing the projection property such that $\sigma^A(\mathbf{a}) \subset sp(\mathbf{a})$ *for all* $\mathbf{a}$ *in* $c(A)$;

10) the spectral radius r_A *is submultiplicative on A, that is* $r_A(ab) \leq r_A(a)r_A(b)$ *for all* $a,b \in A$;

11) for every subalgebra $B \subset A$ *with* $e_A \in B$, *the set* $\Omega(B,A)$ *is compact and each* $f \in ext(\Omega(B,A))$ *admits an extension to an element of HomA;*

12) for each $a \in A$, $\Omega([\{a\}],A)$ *is compact and each* $f \in ext(\Omega([\{a\}],A))$ *admits an extension to an element of HomA;*

Then the following implications hold:

$$1) \Leftarrow 2) \Leftrightarrow 3) \Leftrightarrow 4) \Leftrightarrow 5) \Leftrightarrow 6) \Leftrightarrow 7),$$
$$7) \Leftrightarrow 8) \Leftrightarrow 9) \Leftrightarrow 10) \Rightarrow 11) \Rightarrow 12) \Rightarrow 1).$$

In particular, if A is also a Gelfand-Mazur Q-algebra, then 1) $\Rightarrow$ *7) as well, so that in this case all the preceding assertions 1) - 12) are equivalent.*

P r o o f. 2) $\Rightarrow$ 1). If $a,b,c \in A$, then $r_A(c[a,b]) = 0$, where $[a,b] = ab - ba$. It readily follows that the algebra A/RadA is commutative.

2) $\Rightarrow$ 10). Trivial.

3) $\Rightarrow$ 2). We first observe that if $\Lambda \in M = \{\Lambda \in A^* : \Lambda(e_A) = 1,\ 0 \notin \Lambda(\mathrm{Inv}A)\}$ and $a \in A$, then $\Lambda(a) \in \sigma^A(a)$. Thus, according to Lemma 4, $M = \mathrm{Hom}A$. To establish 2), it remains to notice that for each element a in A, one has $\sigma^A(a) = \{\Lambda(a) : \Lambda \in M\}$.

4) $\Rightarrow$ 3). Clearly

$$\mathrm{Inv}A \subset Q = \{a \in A : \Lambda(a) \neq 0 \text{ for every } \Lambda \in \mathrm{Hom}A\}.$$

On the other hand, if $a \notin \mathrm{Inv}A$, then $0 \in \sigma^A(a)$, and so there is $\Lambda \in \mathrm{Hom}A$ with $\Lambda(a) = 0$. Hence $Q = \mathrm{Inv}A$, as required.

5) $\Rightarrow$ 4). Trivial.

6) $\Rightarrow$ 5). Note that by Lemmas 1 and 3 HomA is compact. Then apply Lemma 2.

7) $\Rightarrow$ 6). Trivial.

8) $\Rightarrow$ 7). Clear by Lemma 3.

9) $\Rightarrow$ 8). Use again Lemma 3.

10) $\Rightarrow$ 9). It is sufficient to prove that $\sigma^A(\underline{a}) = \hat{\underline{a}}(\mathrm{Hom}A)$ for all $\underline{a}$ in $c(A)$.

First of all, we observe that $\mathrm{Rad}A = \ker r_A = \{a \in A : r_A(a) = 0\}$. Furthermore, it is easy to see that if r_A is submultiplicative on A, then it is also subadditive (cf., for example [4, p.121]). Hence, we can define a norm $\|\ \|_C$ on $C = A/\mathrm{Rad}A$ by $\|\pi(a)\|_C = r_A(a)$, $a \in A$ (here π is the natural homomorphism of A onto C). Now, the theorem of Hirschfeld-Żelazko [14], [4, p.113] yields that the algebra C is commutative, since $\|\pi(a)\|_C = r_A(a) = r_C(\pi(a))$ for every a in A. Thus, by [16, p.103], C is a commutative Gelfand-Mazur Q-algebra with identity, and so every maximal ideal in C is, in fact , a kernel of some $\Lambda \in \mathrm{Hom}C$. Consequently, $\sigma^C(\underline{c}) = \hat{\underline{c}}(\mathrm{Hom}C)$ ($\underline{c} \in c(C)$), which gives $\sigma^A(\underline{a}) = \hat{\underline{a}}(\mathrm{Hom}A)$ for all $\underline{a}$ in $c(A)$.

10) $\Rightarrow$ 11). Suppose B is a subalgebra of A such that $e_A \in B$. By virtue of the remarks above, $C = A/\mathrm{Rad}A$ is commutative, so that $\pi(B) \subset C$ is commutative also. Let D be the completion of C and let $\overline{\pi(B)}$ be the closure of $\pi(B)$ in D (we consider C as a subalgebra of D).Next we shall prove that if λ is an extremal point of $\Omega(B,A)$, then there is $\overline{\Lambda} \in \mathrm{ext}(\Omega(\overline{\pi(B)},D))$ with $\lambda(b) = \overline{\Lambda}(\pi(b))$ for all b in B. So suppose λ is such a functional. For each b in B define

$\Lambda(\pi(b)) = \lambda(b)$. Clearly $|\Lambda(\pi(b))| \leq \|\pi(b)\|_D$ $(b \in B)$, so Λ admits an extension $\bar{\Lambda} \in (\overline{\pi(B)})^*$. Using now [6, Lemma 13.2] and the fact that $\|d\|_D = r_D(d)$ $(d \in D)$, we get $\bar{\Lambda} \in \Omega(\overline{\pi(B)},D)$. Further, if $\varphi_1,\varphi_2 \in \Omega(\overline{\pi(B)},D)$ and $t \in (0,1)$ satisfy $t\varphi_1 + (1-t)\varphi_2 = \bar{\Lambda}$, then $t(\varphi_1 \circ \pi) + (1-t)(\varphi_2 \circ \pi) = \lambda$ and $\varphi_i \circ \pi \in \Omega(B,A)$ $(i = 1,2)$ since $\sigma^A(b) = \sigma^C(\pi(b)) = \sigma^D(\pi(b))$ for all b in B (the last equality is due to [3, p.176]). Thus, $\varphi_i \circ \pi = \lambda$ $(i = 1,2)$ because $\lambda \in \mathrm{ext}(\Omega(B,A))$. Consequently, $\bar{\Lambda}$ is in $\mathrm{ext}(\Omega(\overline{\pi(B)},D))$ and since $\mathrm{ext}(\Omega(\overline{\pi(B)},D))$ belongs to the Shilov boundary of $\overline{\pi(B)}$ (as it is shown in [15, p.112], [21, p.181]), $\bar{\Lambda}$ has an extension to an element of $\Phi \in \mathrm{Hom}D$. A direct calculation shows now that $\Phi \circ \pi$ is an extension of λ belonging to HomA.

For showing that $\Omega(B,A)$ is compact let

$$\pi^* : \Omega(\pi(B),C) \longrightarrow \Omega(B,A)$$

be the map defined by $\pi^*(\Lambda) = \Lambda \circ \pi$ $(\Lambda \in \Omega(\pi(B),C))$. Then π^* is a continuous surjection and we conclude that $\Omega(B,A)$ is compact since $\Omega(\pi(B),C)$ is.

11) $\Rightarrow$ 12). Trivial.

12) $\Rightarrow$ 1). Let $a,b,c \in A$ and set $d = c[a,b]$. If $\alpha \in \sigma^A(d)$, then from Lemma 1 it follows that $\alpha = \lambda(d)$ for suitable $\lambda \in \mathrm{Hom}[\{d\}]$. Now, the spectral mapping theorem yields that $\lambda \in \Omega([\{d\}],A)$, and so, by the Krein-Milman theorem, for every $\varepsilon > 0$ there are $n \in \mathbb{N}$, $f_i \in \mathrm{ext}(\Omega([d],A))$ and $\alpha_i \in \mathbb{C}$ $(i = 1,2,\dots,n)$ such that

$$\left| \sum_{i=1}^{n} \alpha_i f_i(d) - \lambda(d) \right| < \varepsilon.$$

But the hypothesis implies that $f_i(d) = 0$ for all $i = 1, 2,\dots,n$. Consequently, $\alpha = \lambda(d) = 0$. Thus, $-1 \notin \sigma^A(c[a,b])$ for every c in A and therefore, $[a,b] \in \mathrm{Rad}A$. Equivalently, the algebra $A/\mathrm{Rad}A$ is commutative.

We conclude by showing that if A is also a Gelfand-Mazur Q-algebra, then 1) implies 7). To this end we recall that RadA is contained in every maximal one-sided ideal of A. Thus, $\sigma_l^A(a) = \sigma_r^A(a) = \sigma^A(a)$ $(a \in c(A))$ because, by assertion, A/RadA is commutative. As every maximal ideal in A is closed, it follows readily that $\sigma^A(a) = \hat{a}(\mathrm{Hom}A)$ for any a in $c(A)$. This finishes the proof.

R E M A R K 2. Note that if $\pi_1^4(\sigma^A(a,b,c,d)) = \sigma^A(a)$ for every $a,b,c,d \in A$, then the algebra $A/\mathrm{Rad}A$ is commutative. Indeed, let $a,b,c \in A$ and suppose $\alpha \in \sigma^A(c[a,b])$. Then $(\alpha, \beta,\gamma,\delta) \in \sigma^A(c[a,b],a,b,c)$ for some $\beta,\gamma,\delta \in \mathbb{C}$, and so Lemma 1 gives us a homomorphism $\Lambda \in \mathrm{Hom}[\mathbf{d}]$ ($\mathbf{d} = (c[a,b],a,b,c)$) such that $\Lambda(c[a,b]) = \alpha$. Thus, $\alpha = 0$ and we conclude that $[a,b] \in \mathrm{Rad}A$ for every $a,b \in A$. So, if A is a Gelfand-Mazur Q-algebra then the assertion 8) in Theorem 1 may be replaced by

8′) $\pi_1^4(\sigma^A(a,b,c,d)) = \sigma^A(a)$ $(a,b,c,d \in A)$.

Let A be a topological algebra and let $\mathrm{hom}A = \{ \Lambda \in \mathrm{Hom}A : \Lambda$ is continuous on $A \}$. Following W. Želazko [30], we say that a commutative algebra A *has the ES-property* if for every subalgebra B of A, containing the identity of A, every homomorphism belonging to homB can be extended to an element of homA. In [30] it is proved that a commutative complex Banach algebra A has the ES-property if and only if for each $a \in A$ the set $\sigma^A(a)$ is totally disconnected[1]. Moreover, in [19] it is shown , among other things, that for a given unital complex Banach algebra A the algebra A/RadA is commutative and has the ES-property if and only if the joint spectrum Sp defined on A by $Sp(\mathbf{a}) = \hat{\mathbf{a}}(\mathrm{hom}[\mathbf{a}])$ $(\mathbf{a} \in c(A))$ admits the projection property.

In this respect for Q-algebras we have:

T H E O R E M 2. *Let A be a Q-algebra , let* $Sp(\mathbf{a}) = \hat{\mathbf{a}}(\mathrm{hom}[\mathbf{a}])$ $(\mathbf{a} \in c(A))$ *and suppose that*[2] $r_A(a) = \sup \{|\Lambda(a)| : \Lambda \in \mathrm{hom}[\{a\}]\}$ *for every a in A. Then the following conditions are equivalent:*

1) the joint spectrum Sp has the projection property;

2) HomA is non-empty and $Sp(\mathbf{a}) = \mathbf{a}(\mathrm{Hom}A)$ *for all* $\mathbf{a}$ *in* $c(A)$;

3) for every subalgebra B of A containing the identity of A every homomorphism belonging to homB has an extension to

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[1] In this respect see also [20], [31], [37].

[2] Clearly $r_A(a) = \sup \{|\Lambda(a)| : \Lambda \in \mathrm{hom}[\{a\}]\}$ $(a \in A)$ in any Banach algebra A. However, such a condition is also satisfied in some other classes of topological algebras [16], [29], [38].

an element of homA;

4) r_A is submultiplicative and A/RadA has the ES-property.

P r o o f. 1) ⇒ 2). Use Lemma 3.

2) ⇒ 3) Use Lemma 2.

3) ⇒ 4). Since $r_A(a) = \sup\{|\Lambda(a)| : \Lambda \in \mathrm{Hom}A\}$, by Theorem 1, r_A is submultiplicative and the algebra C = A/RadA is commutative. We have only to prove that C has the ES-property. To this end suppose that B is a subalgebra of C containing e_C and suppose that $\lambda \in \mathrm{hom}B$. Then $\lambda\circ\pi \in \mathrm{hom}(\pi^{-1}(B))$, where π is the natural homomorphism of A onto C. Thus, $\lambda\circ\pi$ has an extension $\Lambda \in \mathrm{hom}A$. But then there is $\Lambda_1 \in \mathrm{hom}C$ such that $\Lambda_1(\pi(a)) = \Lambda(a) = \lambda(\pi(a))$ for every $a \in \pi^{-1}(B)$ [16, p.339].

4) ⇒ 1). If $a \in c(A)$ and $\lambda \in \mathrm{hom}[a]$ then $|\lambda(b)| \le r_A(b)$ for all $b \in [a]$. Hence there is $\Lambda \in \mathrm{Hom}(\pi([a]))$, defined by $\Lambda(\pi(b)) = \lambda(b)$ $(b \in [a])$. Now, as A/RadA is a Q-algebra [22, p.290], Λ is continuous on $\pi[a]$ [16, p. 59]. Consequently, there is $\overline{\Lambda} \in \mathrm{hom}C$ such that $\overline{\Lambda}(\pi(b)) = \lambda(b)$ for all $b \in [a]$. Finally, $\overline{\lambda} = \overline{\Lambda}\circ\pi \in \mathrm{hom}A$ and its restriction to [a] equals λ.

R E M A R K 3. As it easily follows from [20, p. 106], condition 4) of Theorem 2 on any Q-algebra A, satisfying $r_A(a) = \sup\{|\Lambda(a)| : \Lambda \in \mathrm{Hom}[\{a\}]\}$ $(a \in A)$, is equivalent to

4′) r_A is submultiplicative and $\sigma^A(a)$ is totally disconnected for any a in A.

We conclude this section by considering the class of spectrally convex algebras. In [5] S.Bhatt proved that a complex unital Banach algebra each element of which has a convex spectrum is isomorphic to $\mathbb{C}$ modulo the Jacobson radical. This leads us to the following:

T H E O R E M 3. *The following conditions on any complex algebra A are equivalent:*

1) A/RadA is isomorphic to $\mathbb{C}$;

2) A is spectrally bounded with $\sigma^A(a)$ non-empty and *convex for any a in A.*

P r o o f. 1) ⇒ 2). Trivial.

2) ⇒ 1). Fix $a \in A$. Similarily as above there is a sub-

set $\Delta \subset \mathrm{Hom}[\{a\}]$ such that $\hat{a}(\Delta) = \sigma^A(a)$. Now suppose that Λ_1, $\Lambda_2 \in \Delta$, $t \in (0,1)$ and that $\Lambda_3 = t\Lambda_1 + (1 - t)\Lambda_2$. Then $\Lambda_3 \in [\{a\}]^*$ and, by the spectral mapping theorem, $\Lambda_i(b) \in \sigma^A(b)$ $(i = 1,2)$ for every b in $[\{a\}]$. It follows that $\Lambda_3(b) \in \sigma^A(b)$ since $\sigma^A(b)$ is convex $(b \in [\{a\}])$. Thus, according to Lemma 4, $\Lambda_3 \in \mathrm{Hom}[\{a\}]$, so that $\Lambda_1 = \Lambda_2 = \Lambda_3$ (see, for example, [27, p.1029]). Hence $\sigma^A(a)$ consists of a single point.

Next we shall show that the spectral radius r_A is subadditive on A. More precisely, we shall show that

$$\sigma^A(a + b) = \{\alpha + \beta : \alpha \in \sigma^A(a),\ \beta \in \sigma^A(b)\} \quad (a,b \in A).$$

In fact, if $a,b \in A$ and $\alpha \in \sigma^A(a)$, then by a simple calculation we get $\sigma^A((a - \alpha e_A)b) = \{0\}$. If, in addition, $c \in A$ and $\beta \in \mathbb{C}$ satisfy $\beta \notin \sigma^A(c)$, then $\beta e_A - ((a - \alpha e_A) + c) = (e_A - (a - \alpha e_A)(\beta e_A - c)^{-1})(\beta e_A - c) \in \mathrm{Inv}A$. It readily follows that $\sigma^A(a + c) = \{\alpha + \gamma : \alpha \in \sigma^A(a),\ \gamma \in \sigma^A(c)\}$. Hence, r_A is subadditive on A. Moreover,

$$r_A(ab) = r_A((a - \alpha e_A)b + \alpha b) = |\alpha| r_A(b) = r_A(a) r_A(b)$$

for all $a,b \in A$ and $\alpha \in \sigma^A(a)$. So, according to Theorem 1, $\sigma^A(a) = \hat{a}(\mathrm{Hom}A)$ for all a in A. Consequently, since $\sigma^A(a)$ is a singelton, HomA consists of a single point as well. Now $A/\mathrm{Rad}A = A/\ker\Lambda_0 \cong \mathbb{C}$, where $\{\Lambda_0\} = \mathrm{Hom}A$. The proof is completed.

3. Almost commutative Fréchet algebras. In the previous section we restricted our attention to the class of spectrally bounded algebras. However, in the general case the algebra under consideration might be spectrally unbounded as this can be true even for normed algebras [9, p. 24] or for Fréchet algebras [8]. So, we are lead to seek conditions which make it possible to prove analogous results to Theorems 1 - 4 for algebras not necessarily spectrally bounded. In what follows we shall do this within the frame of Fréchet algebras.

A *locally multiplicatively convex* algebra is a topological algebra A whose topology is defined by a saturated family $\{p_i : i \in I\}$ of submultiplicative seminorms (i.e. $p_i(ab) \le p_i(a)p_i(b)$ for all $a,b \in A$ and $i \in I$). If moreover, the underlying locally convex space is complete and metrizable, then A is said to be a *Fréchet algebra.*

According to the well-known theorem due to R. Arens [1], if A is a commutative complex Fréchet algebra, then $\sigma^A(\underline{a}) = \hat{\underline{a}}(\mathrm{hom}A)$ for any $\underline{a}$ in c(A). So, for complex Fréchet algebras we get the following theorem:

T H E O R E M 4. *Let B be a complex Fréchet algebra. The following conditions are equivalent:*

1) $\sigma^B(\underline{b}) = \hat{\underline{b}}(\mathrm{hom}B)$ $(\underline{b} \in c(B))$;

2) the joint spectrum σ^B *has the projection property;*

3) B is almost commutative;

4) $\mathrm{Rad}B = \{b \in B : \Lambda(b) = 0$ *for every* Λ *in* $\mathrm{hom}B\}$.

The proof is standard and therefore will be omitted.

Now, let A be a complex Fréchet algebra and let S be a subset in A. We denote by H(S) the minimal closed full[3] subalgebra of A containing the set $S \cup \{e_A\}$. A subset $S \subset A$ is said to be *m-commutative* $(m \in \mathbb{N})$ if

$$[a_0,[a_1,\ldots,[a_{m-1},a_m]\ldots]] = 0$$

for all $a_0,a_1,\ldots,a_m \in A$ [39].

The following theorem is analogous to those proved in [13, p.141], [39, p.77], [41, p.158].

T H E O R E M 5. *Let A be a Frechet algebra, let S be a m-commutative* $(m \in \mathbb{N})$ *subset of A, and suppose B is a closed subalgebra of A such that* $[S] \subset B \subset H(S)$. *Then the equivalent conditions 1) - 4) of Theorem 4 are valid.*

P r o o f. In view of Theorem 4 it is sufficient to prove only 3).

Let $\{p_i : i \in I\}$ be a family of submultiplicative seminorms defining the topology of B. Set $B_i = B/\ker p_i$ $(i \in I)$ and define on B_i an algebra norm $\|\ \|_i$ by $\|\pi_i(b)\|_i = p_i(b)$ $(b \in B,\ i \in I)$, where π_i is the natural homomorphism of B onto B_i. The completion of B_i is denoted by C_i and in the sequel we consider B_i as a subalgebra of the Banach algebra C_i $(i \in I)$. Moreover, for each i in I let D_i be the algebra $C_i/\mathrm{Rad}C_i$, ν_i be the natural homomorphism C_i onto D_i and let $\mu_i = \nu_i \circ \pi_i$. Now, suppose $s_0,s_1,\ldots,s_{m-2} \in S$ and put y =

[3] Recall that a subalgebra B of A is *full* if B contains the unity of A and if, whenever $b \in B$ has an inverse b^{-1} in A, b^{-1} is in B.

$= [s_0, \ldots, [s_{m-3}, s_{m-2}] \ldots]$. By hypothesis $[s,[s,y]] = 0$ for every s in S, so that $[\mu_i(s),[\mu_i(s),\mu_i(y)]] = 0$ for all $s \in S$ and $i \in I$. The Kleinecke-Shirokov theorem (see, for instance, [7, p.91]) gives now that

$$r_{D_i}(\mu_i[s,y]) = 0$$

for every s in S and i in I. Consequently , $\mu_i([s,y]) = 0$ since, as it easily seen to be verified , $\mu_i([s,y])$ is contained in the centre of D_i . Thus, for each i in I , the set $\{\mu_i(s) : s \in S\}$ is (m - 1)-commutative. Now , analogously as above, we obtain that the set $\{\mu_i(s) : s \in S\}$ is (m - 2)-commutative and so on. We conclude that the algebra D_i $(i \in I)$ is commutative, and so $\pi_i([s_1,s_2]) \in RadC_i$ for all $s_1,s_2 \in S$ and $i \in I$. But this, in turn, implies that $[s_1,s_2] \in RadB$ for all $s_1,s_2 \in S$ [16, p. 93]. Equivalently, B is almost commutative because $B \subset H(S)$ and RadB is a closed and two-sided ideal in B.

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О ПОЧТИ КОММУТАТИВНЫХ АЛГЕБРАХ

А.Кокк

Р е з ю м е

Пусть A - комплексная алгебра с единицей e_A, HomA - множество всех нетривиальных линейных мультипликативных функционалов на A и InvA - множество обратимых элементов алгебры A. Пусть, далее, [S] - подалгебра алгебры A, порождаемая подмножеством $S \subset A$ и элементом e_A, A^n - прямое произведение n экземпляров A,

$$c(A) = \bigcup_{n=1}^{\infty} A^n,$$

$[\mathbf{a}] = [\bigcup_{k=1}^{n} \{a_n\}]$ для каждого $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ и $\hat{\mathbf{a}}(\Lambda) = (\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n))$ для всех $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ и $\Lambda \in \mathrm{Hom}[\mathbf{a}]$. Для каждого $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ через $\sigma^A(\mathbf{a})$ обозначим *совместный спектр Харта* семейства $\mathbf{a}$, т.е.

$$\sigma^A(\mathbf{a}) = \sigma_l^A(\mathbf{a}) \cup \sigma_r^A(\mathbf{a}),$$

где

$$\sigma_l^A(\mathbf{a}) = \{ (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{K}^n : \sum_{k=1}^{n} b_k(a_k - \alpha_k e_A) \neq e_A \text{ для всех } b_1, b_2, \ldots, b_n \in A \},$$

$$\sigma_r^A(a) = \{ (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{K}^n : \sum_{k=1}^{n} (a_k - \alpha_k e_A) b_k \neq e_A \text{ для всех } b_1, b_2, \dots, b_n \in A \},$$

Говорят, что совместный спектр σ^A на алгебре A *обладает свойством проекции*, если для всех $n, k \in \mathbb{N}$ $(k \le n)$ и $a = (a_1, a_2, \dots, a_n) \in A^n$ справедливо

$$\sigma^A((a_1, a_2, \dots, a_k)) = \pi_n^k(\sigma^A(a)),$$

где π_n^k - проекция $\mathbb{C}^n$ на $\mathbb{C}^k$ определяемое равенством

$$\pi_n^k((\alpha_1, \dots, \alpha_k, \dots, \alpha_n)) = (\alpha_1, \dots, \alpha_k)$$

для всех $(\alpha_1, \dots, \alpha_k, \dots, \alpha_n) \in \mathbb{C}^n$.

Если множество $\sigma^A(a)$ $(a \in A)$ не пусто и ограничено, то через $r_A(a)$ обозначим спектральный радиус элемента $a \in A$ т.е.

$$r_A(a) = \sup \{ |\alpha| : \alpha \in \sigma^A(a) \}.$$

Алгебра A называется *спектрально ограниченной*, если множество $\sigma^A(a)$ ограничено для всех $a \in A$.

Т е о р е м а А. *Пусть A комплексная спектрально ограниченная алгебра с единицей. Тогда следующие условия эквивалентны:*

а) $r_A(a) = \sup \{ |\Lambda(a)| : \Lambda \in HomA \}$ $(a \in A)$;

б) $InvA = \{ a \in A : \Lambda(a) \neq 0$ *для каждого линейного функционала* Λ *на A, удовлетворяющего условиям* $\Lambda(e_A) = 1$ *и* $0 \notin \Lambda(InvA) \}$;

в) если $S \subset A$ *и* λ *- линейный функционал на* $[S]$ *такой, что* $\mathbf{a}(\lambda) \in \sigma^A(\mathbf{a})$ *для всех* $\mathbf{a} \in c(S)$, *то* λ *можно продолжить до гомоморфизма* $\bar{\lambda} \in HomA$;

г) если $\mathbf{a} \in c(A)$ *и* λ *- линейный функционал на* $[\mathbf{a}]$ *такой, что* $\mathbf{a}(\lambda) \in \sigma^A(\mathbf{a})$, *то* λ *можно продолжить до гомоморфизма* $\bar{\lambda} \in HomA$;

д) $\sigma^A(\mathbf{a}) = \mathbf{a}(HomA)$ $(\mathbf{a} \in c(A))$;

е) σ^A *обладает свойством проекции;*

з) $r_A(ab) \le r_A(a) r_A(b)$ *для всех* $a, b \in A$;

Т е о р е м а Б. *Пусть A комплексная алгебра с единицей. Тогда следующие условия эквивалентны:*

а) факторалгебра A/RadA (по радикалу Джекобсона RadA алгебры A) изоморфна полю $\mathbb{C}$;

б) алгебра A является спектрально ограниченной и спектр $\sigma^A(a) \subset \mathbb{C}$ *непуст и выпукл для всех* $a \in A$.

REMARK from the Editorial Board

Having received Mr. U.Umirbayev's letter, the Managing Editor of the Vol. 878 (Monoids, Rings and Algebras) apologizes that there were misprints in Mr. Umirbayev's article: "Об аппроксимации свободных алгебр Ли относительно вхождения" ("On the approximation of free Lie algebras with respect to entry") // Tartu Ülik. Toimetised. Acta et comm. Univ. Tartuensis .- 1990 .-№ 878 . P .- 147-152). In several places in §1 (PP. 148-149) f_i instead of $\bar{f}_i$ are printed. The correct version of the §1 is given below:

§1. Включение в правый идеал конечной коразмерности

Пусть A свободная ассоциативная алгебра с единицей 1, свободно порожденная элементами $x_1, x_2, \ldots, x_n$.

На множестве базисных слов алгебры A введем линейный порядок $\leq$ и частичный порядок $\ll$. Пусть v и w произвольные слова от $x_1, x_2, \ldots, x_n$. Считаем, что $v < w$, если $d(v) < d(w)$, где d - функция длины. На слова равной длины порядок $\leq$ распространим лексикографически, исходя из неравенств: $x_1 < x_2 < \ldots < x_n$. Положим $v \ll w$, если найдется слово t такое, что $vt = w$.

Через $\bar{f}$ будем обозначать старший член элемента f алгебры A относительно $\leq$. Далее считаем, что коэффициенты старших членов рассматриваемых элементов равны единице.

Пусть $I = (f_1, f_2, \ldots, f_k)_r$ - правый идеал алгебры A, порожденный элементами $f_1, f_2, \ldots, f_k$. Можно считать, что элементы $f_1, f_2, \ldots, f_k$ удовлетворяют условию:

$\bar{f}_i, \bar{f}_j$ при $i \neq j$ несравнимы по $\ll$. (1)

Действительно, если $\bar{f}_i \ll \bar{f}_j$ $(i \neq j)$, то найдется слово t такое, что $\bar{f}_i t = \bar{f}_j$. Тогда элемент f_j можно заменить элементом $f_j - f_i t$. Так как $\overline{f_j - f_i t} < \bar{f}_j$, то, несколько раз повторяя указанный процесс, мы добъемся выполнения условия (1).

Далее считаем, что для порождающих правого идеала I выполнено условие (1). Тогда справедлива следующая

Лемма 1. Пусть f произвольный элемент алгебры A. Если f принадлежит I, то найдется f_i такой, что $\bar{f}_i \ll \bar{f}$.

Лемма доказывается стандартными рассуждениями (см. например [4]).

Лемма 2. Если f не принадлежит правому идеалу I, то найдется к.п. правый идеал J алгебры A конечной коразмерности, такое, что $I \subseteq J$ и $f \notin J$.

Доказательство. Считаем, что порождающие правого идеала I удовлетворяют условию (1).

Далее, можно считать, что элемент f по отношению к идеалу I удовлетворяет условию:

для всех f_i не выполняется $\bar{f}_i \ll \bar{f}$, $1 \le i \le k$. (2)

Действительно, если существует f_i такое, что $\bar{f}_i \ll \bar{f}$, то найдется слово t, удовлетворяющее равенству $\bar{f}_i t = \bar{f}$. Тогда, заменяя элемент f элементом $f-f_i t$, мы уменьшаем старший член f, так как $\overline{f-f_i t} < \bar{f}$. Несколько раз повторяя этот процесс, мы добъемся выполнения условия (2).

Пусть $S = \max\{d(\bar{f}_1), d(\bar{f}_2), \ldots d(\bar{f}_k), d(\bar{f})\}$. Если $f_{k+1}, f_{k+2}, \ldots, f_l$ - множество всех слов длины s+1 , которые несравнимы с элементами $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_k$ по $\ll$, то положим

$$J = (f_1, f_2, \ldots, f_k, f_{k+1}, f_{k+2}, \ldots, f_l)_r.$$

Заметим, что множество порождающих правого идеала J также удовлетворяет условию (1). Действительно, элементы $\bar{f}_i, \bar{f}_j$ $(i \ne j)$ при $i, j \le k$ несравнимы по $\ll$, так как порождающие правого идеала I удовлетворяют условию (1). Если $i \le k < j$, то $\bar{f}_i, \bar{f}_j$ несравнимы по $\ll$ в силу выбора f_j. Наконец, если $k < i, j$, то f_i, f_j слова длины s+1. Тогда $f_i \ll f_j$ невозможно при $i \ne j$.

Теперь покажем, что элемент f по отношению к идеалу J удовлетворяет условию (2). Неравенство $\bar{f}_i \ll \bar{f}$ невозможно при $i \le k$, так как f по отношению к идеалу I удовлетворяет условию (2). Если $i > k$, то неравенство $\bar{f}_i \ll \bar{f}$ противоречит тому, что $d(\bar{f}_i) = s+1$, $d(\bar{f}) \le s$.

Сопоставляя вышесказанное с леммой 1, получаем, что $f \notin J$.

Заметим, что порождающие правого идеала J были выбраны так, чтобы для любого слова v длины s+1 существовало f_i $(1 \le i \le l)$ такое, что $\bar{f}_i \ll v$. Отсюда следует, что алгебра A по модулю J линейно порождается словами длины $\le s$.

Лемма доказана.

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