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THÈSE

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Contents

Introduction

The objects studied in this thesis are the germs of singular holomorphic foliations in $(\mathbb{C}^2, 0)$. It is divided into three parts. The first part is devoted to preliminaries concerning singular foliations. In the second part, we study formal normal forms of topologically quasi-homogeneous foliations. In the last part, we solve the problem of classification for a class of non-dicritical foliations by introducing a new invariant called "set of sliding". Moreover, we investigate the finite determinacy property of some classes of foliations by showing that this new invariant is finitely determined.

Position of the problem

For a germ of holomorphic singular foliation in $(\mathbb{C}^2, 0)$, there are three main analytic invariants which are

- The separatrices,
- The corresponding holonomies,
- The Camacho-Sad indices at the singularities corresponding to strict transform of separatrices after desingularization.

Are they a complete set of invariants for the analytic type of a foliation? This question was proposed by Thom at seminar of the IHES in the years 74-75 [14, 9, 17]. The affirmative answer holds for a class of non-degenerated reduced foliations [13]. However, Moussu in [14] gave a counterexample for this conjecture. He considered two foliations defined by

$$\omega_1 = d(y^2 + x^3),$$
$$\omega_2 = d(y^2 + x^3) + x(2ydx - 3xdy).$$

These two foliations share the same Camacho-Sad index, which is equal to $-\frac{1}{6}$, the same separatrix, which is $\{y^2 + x^3 = 0\}$ and the same corresponding holonomy, which is the identity map. However, these two foliations are not conjugated because the foliation defined by ω_2 does not admit a holomorphic first integral. Moussu in this paper also suggested to replace the holonomy by the vanishing holonomy representation, namely the holonomy representation of the invariant components of the exceptional divisor. The vanishing holonomy representation gives us not only the information about the holonomies corresponding to each separatrix but also the information about their relations. He proved that the analytic type of a cuspidal type singular point is totally determined by its separatrix and its vanishing holonomy representation. Note that the Camacho-Sad index of the cuspidal singular point is always equal to $-\frac{1}{6}$. This result is generalized to the quasi-homogeneous foliation by Genzmer [6]. However, by computing the dimension of the equisingular unfolding moduli space [11, 12], Mattei proved the following result:

Theorem (Mattei [11])**.** Let $\mathcal{F}$ be a non-dicritical foliation without saddle-node singularity after desingularization. Suppose that $\mathcal{F}$ is defined by a 1-form $\omega = a(x,y)dx + b(x,y)dy$ and denote by $f(x,y)$ an equation of its separatrices. Then the following properties are equivalent:

1. $\mathcal{F}$ is a quasi-homogeneous foliation.
2. f belongs to the ideal $(a,b) \subset \mathbb{C}\{x,y\}$.
3. f belongs to its jacobian ideal, $(f'_x, f'_y) \subset \mathbb{C}\{x,y\}$.
4. There exist coordinates z, w such that f is written as a quasi-homogeneous polynomial function with (k,ℓ) weight and there exist holomorphic functions g, h with $g(0) \neq 0$ such that
$$g\omega = df + h(\ell z dw - kw dz).$$
5. $\mathcal{F}$ satisfies the following equivalence: any equisingular unfolding of $\mathcal{F}$ is analytically trivial if and only if the underlying deformation of separatrices is analytically trivial.

Roughly speaking, the moduli space of equisingular unfoldings is the space of all foliations having the same transversal structure up to analytic conjugation. In particular, two foliations linked by an equisingular unfolding have their vanishing holonomy representations conjugated. Thus, the statement 5 above implies that the quasi-homogeneous case is the most general case in which the triple of invariants is complete for the analytic classification problem. There must be other invariants for the non quasi-homogeneous foliations. This conclusion is confirmed by Genzmer and Paul in [8], [7]. In their work, they construct some normal forms for topologically homogeneous and quasi-homogeneous foliations admitting first integrals. The number of free coefficients in their normal forms is equal to the dimension of Mattei's moduli space which is strictly bigger than the number of free coefficients in the normal forms of the separatrices. For the topologically homogeneous foliations in general (with or without first integrals), Ortiz-Bobadilla, Rosales-González, and Voronin in [9] provided also a formal normal form under some generic condition:

Theorem ([9])**.** Let $\omega = \omega_n + \omega_{n+1} + \dots$ which defines a topologically homogeneous foliation. Under some generic conditions of ω_n, ω is strictly formally orbitally equivalent to a unique germ $\omega_{h,b}$ of the form

$$\omega_{h,s} = \omega_n + dh + s(ydx - xdy),$$

where $h(x,y) = \sum h_{ij}x^iy^j$, $0 \leq i \leq n-1$, $0 \leq j \leq n-1$, $i + j \geq n+2$, and $s(x,y) = \sum_{i=0}^{n-2} s_i(x)y^ix^{n-i}$ is a polynomial in the variable y of degree less or equal to $n-2$ whose coefficients s_i are formal series in the variable x.

Here, strict conjugation means up to a diffeomorphism tangent to identity. These normal forms have a form similar to those introduced in the quasi-homogeneous case [11] (statement 4 above). The number of free coefficients of the polynomial h which is called the *hamiltonian part* is consistent with the dimension of Mattei's moduli space. Moreover, in [9] the authors also proved under a generic condition that two topologically homogeneous foliations are strictly conjugated if they have their vanishing holonomy representation strictly conjugated and the same hamiltonian part after normalization. The dimension of Mattei's moduli space and the works in [11, 12, 8, 7, 9] also gave us a must-have property of the missing invariant (at least under some assumptions): finite determinacy.

The aims of this thesis are:

- To generalize the theorem of [9] in the topologically quasi-homogeneous case. Through the number of free coefficients in the hamiltonian part, to confirm again the existence of an invariant beside the three mentioned above.
- To find the missing invariant and show that it has the finite determinacy property.

Thesis structure and main results

Beside Chapter 0 of preliminaries concerning singular foliations in $(\mathbb{C}^2, 0)$, this thesis is divided into two parts.

Formal normal forms of topologically quasi-homogeneous foliations

The whole Chapter 1 is devoted to investigate some formal normal forms for topologically quasi-homogeneous foliations, which is a generalization of the result in [9]. A foliation $\mathcal{F}$ is called topologically quasi-homogeneous with axis branches if it is a generalized curve whose separatrices are topologically conjugated with the zero locus of $xy\prod_{i=1}^{n}(y^k - c_ix^\ell)$. The axis branches correspond to the invariant curves which are topologically conjugated to $\{x = 0\}$ and $\{y = 0\}$.

Theorem A. Let ω be a 1-form which defines a topologically quasi-homogeneous foliation. Under some generic condition, ω is strictly formally orbitally equivalent to a unique form $\omega_{h,s}$

$$\omega_{h,s} = \omega_d + dh + s(kydx - \ell xdy)$$

where

$$\omega_d = c_0 xy \prod_{i=1}^{n}(y^k - c_i x^\ell)\left(\sum_{i=1}^{n} \lambda_i \frac{d(y^k - c_i x^\ell)}{y^k - c_i x^\ell} + (\ell\lambda_0 + \ell - u)\frac{dx}{x} + (k\lambda_\infty + v)\frac{dy}{y}\right),$$

$$h(x,y) = xy \sum_{\substack{ki+\ell j \geq k\ell n+1 \\ 0\leq i\leq \ell n-1 \\ 0\leq j\leq kn-1}} h_{ij}x^i y^j, \; s(x,y) = \sum_{j=0}^{kn-1} s_j(x) x^{\ell n+1+]\frac{1-\ell j}{k}[} y^j,$$

$s_i(x)$ are formal series in the variable x, $\left]\frac{1-\ell j}{k}\right[$ stands for strict integer part of $\frac{1-\ell j}{k}$.

The proof uses a classical method for constructing the normal forms: eliminating the terms degree by degree. First of all, we show that after a conjugacy by a diffeomorphism, a topologically quasi-homogeneous foliation with axis branches has the same process of desingularization as the polynomial $xy\prod_{i=1}^{n}(y^k - c_i x^\ell)$. By that, we can assume that ω can be extended into (k,ℓ) quasi-homogeneous terms

$$\omega = \omega_d + \omega_{d+1} + \omega_{d+2} + \cdots ,$$

where $d = k\ell n + k + \ell$ which is the quasi-homogeneous degree of $xy\prod_{i=1}^{n}(y^k - c_i x^\ell)$. After that, we prove that ω can be decomposed as

$$\omega = \omega_d + d\,(xyh) + s(kydx - \ell xdy),$$

where h, s are two holomorphic functions. Now, by some changes of the coordinates and multiplication by units, we will eliminate the undesirable terms of h degree by degree. The last step consists in choosing the diffeomorphisms and the units that does not modify h to normalize s. The number of free coefficients of h, as we will see, is consistent with the dimension of Mattei's moduli space.

Sliding invariants and classification of singular foliations

In Chapter 2, we introduce a new invariant called *set of slidings* and solve the problem of the strict classification for the class of non-dicritical foliations.

Let us first consider the case $\mathcal{F}$ is a nondegenerate reduced foliation with two separatrices S_1, S_2 and $\mathcal{L}$ is a regular foliation such that its separatrix is transverse to the two separatrices of $\mathcal{F}$. Then the tangent curve of $\mathcal{F}$ and $\mathcal{L}$, denoted by $T(\mathcal{F},\mathcal{L})$, is smooth and transverse to S_1, S_2 and $\mathcal{L}$. The *sliding* of $\mathcal{F}$ and $\mathcal{L}$ on S_1, denoted

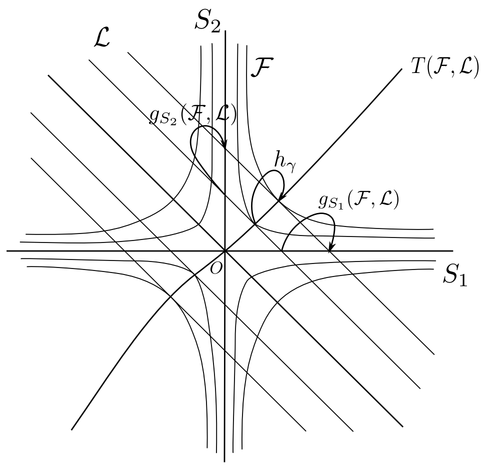

Figure 1: Sliding of $\mathcal{F}$ and $\mathcal{L}$

$g_{S_1}(\mathcal{F},\mathcal{L})$, (resp. on S_2, denoted $g_{S_2}(\mathcal{F},\mathcal{L})$) is the projection of the holonomy of $\mathcal{F}$ on $T(\mathcal{F},\mathcal{L})$ following the leaves of $\mathcal{L}$ to S_1 (resp. S_2)(figure 1).

In the case $\mathcal{F}$ is a non-dicritical foliation whose all singularities are not saddle-node after desingularization by the map σ, we will show that there exists a σ-absolutely dicritical foliation $\mathcal{L}_0$ such that after pull-back by σ, at each singularity of $\tilde{\mathcal{F}} = \sigma^*\mathcal{F}$, the separatrix of $\tilde{\mathcal{L}}_0 = \sigma^*\mathcal{L}_0$ is transverse to $\tilde{\mathcal{F}}$. Then the *slidings* of $\mathcal{F}$ and $\mathcal{L}_0$, denoted by $S(\mathcal{F},\mathcal{L}_0)$, is the set of all $g_{p,D}(\tilde{\mathcal{F}},\tilde{\mathcal{L}}_0)$ where p runs on the set of singularities of $\tilde{\mathcal{F}}$ and D runs on the set of irreducible components of the divisor $\mathcal{D} = \sigma^{-1}(0)$.

Now, denote by $\mathcal{R}(\mathcal{L}_0)$ the set of all σ-absolutely dicritical foliations $\mathcal{L}$ such that $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}_0$ have the same Dulac maps at any corner of $\mathcal{D}$, and at each singularity p of $\tilde{\mathcal{F}}$, the invariant curves of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}_0$ through p are tangent (figure 2). We define the set of slidings of $\mathcal{F}$ relative to the direction $\mathcal{L}_0$ by

$$\mathcal{S}_0(\mathcal{F}) = \cup_{\mathcal{L}\in\mathcal{R}(\mathcal{L}_0)} S(\mathcal{F},\mathcal{L}).$$

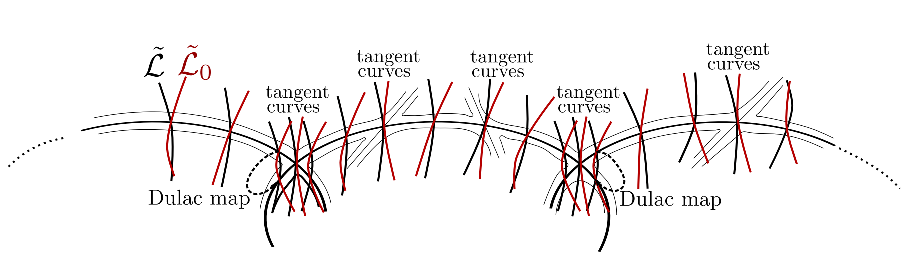

Figure 2: Element $\mathcal{L}$ of $\mathcal{R}(\mathcal{L}_0)$

Theorem B. Let $\mathcal{F}$ and $\mathcal{F}'$ be two non-dicritical foliations. Suppose that they have the same desingularization map and the same set of singularities. Moreover, at each singularity the separatrices of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are tangent and their Camacho-Sad indices are not rational (figure 3). Then the three following statements are equivalent:

(i) $\mathcal{F}$ and $\mathcal{F}'$ are strictly analytically conjugated.

(ii) Their vanishing holonomy representations are strictly analytically conjugated, $\mathrm{CS}(\tilde{\mathcal{F}}) = \mathrm{CS}(\tilde{\mathcal{F}}')$ and $\mathcal{S}_0(\mathcal{F}) = \mathcal{S}_0(\mathcal{F}')$.

(iii) Their vanishing holonomy representations are strictly analytically conjugated, $\mathrm{CS}(\tilde{\mathcal{F}}) = \mathrm{CS}(\tilde{\mathcal{F}}')$ and $\mathcal{S}_0(\mathcal{F}) \cap \mathcal{S}_0(\mathcal{F}') \neq \emptyset$.

Here $\mathrm{CS}(\tilde{\mathcal{F}}) = \mathrm{CS}(\tilde{\mathcal{F}}')$ means the Camacho-Sad indices of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are equal at each singularity.

We sketch the idea of the proof. Thanks to the equality of the slidings, we prove that there exists a strict local conjugation of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ in a neighborhood of each singularity which respects the two absolutely dicritical foliation and fixes the points of the divisor. Then, we use the non-rational property of the Camacho-Sad indices to prove that these local conjugations can be glued together to become a global strict conjugation.

Figure 3: Strict transforms $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$

The finite determinacy property of the sliding invariants is given in the following theorem:

Theorem C. Let $\mathcal{F}$ be a non-dicritical foliation without saddle-node singularities after desingularization. There exists a natural N such that if there is a non-dicritical foliation $\mathcal{F}'$ satisfying the following conditions:

(i) $\mathcal{F}$ and $\mathcal{F}'$ have the same set of singularities after desingularization and at a neighborhood of each singularity, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are locally strictly analytically conjugated,

(ii) There exist $\mathcal{L}, \mathcal{L}'$ in $\mathcal{R}(\mathcal{L}_0)$ such that $J^N(S(\mathcal{F}, \mathcal{L})) = J^N(S(\mathcal{F}', \mathcal{L}'))$, then there exists $\mathcal{L}''$ such that $\mathcal{L}''$ is strictly conjugated with $\mathcal{L}$ and $S(\mathcal{F},\mathcal{L}'') = S(\mathcal{F}',\mathcal{L}')$.

Here $J^N(S(\mathcal{F},\mathcal{L})) = J^N(S(\mathcal{F}',\mathcal{L}'))$ means $J^N(g_{D,p}(\tilde{\mathcal{F}},\tilde{\mathcal{L}})) = J^N(g_{D,p}(\tilde{\mathcal{F}}',\tilde{\mathcal{L}}'))$ for all $g_{D,p}(\tilde{\mathcal{F}},\tilde{\mathcal{L}})$ in $S(\mathcal{F},\mathcal{L})$, $g_{D,p}(\tilde{\mathcal{F}}',\tilde{\mathcal{L}}')$ in $S(\mathcal{F}',\mathcal{L}')$, where $J^N(g_{D,p}(\tilde{\mathcal{F}},\tilde{\mathcal{L}}))$ stands for the regular part of degree N in the Taylor expansion of $g_{D,p}(\tilde{\mathcal{F}},\tilde{\mathcal{L}})$. Although we need to choose a coordinate system for the Taylor expansion, writing $J^N(S(\mathcal{F},\mathcal{L})) = J^N(S(\mathcal{F}',\mathcal{L}'))$ does not depend on the coordinates.

To prove this result, we will control the terms of high order of each local sliding $g_{D,p}(\tilde{\mathcal{F}},\tilde{\mathcal{L}})$ by a local function defined in a neighborhood of each singularity. If the orders are big enough, we show that these local functions are induced by a global function from which we build conjugation of the slidings.

The two theorems in this Chapter also give the following two corollaries:

Corollary D. Let $\mathcal{F}$ be a non-dicritical foliation satisfying that after desingularization, the Camacho-Sad index of $\tilde{\mathcal{F}}$ at each singularity is not rational. Suppose that $\mathcal{F}$ is defined by a 1-form ω. Then there exists a natural N such that if $\mathcal{F}'$ is defined by a 1-form ω' satisfying that $J^N\omega = J^N\omega'$ and the vanishing holonomy representations of $\mathcal{F}$ and $\mathcal{F}'$ are strictly analytically conjugated, then $\mathcal{F}$ and $\mathcal{F}'$ are strictly analytically conjugated.

Corollary E. Let $\mathcal{L}$ be a σ-absolutely dicritical foliation defined by 1-form ω. There exists a natural N such that if $\mathcal{L}'$ is a σ-absolutely dicritical foliation defined by ω' satisfying $J^N\omega = J^N\omega'$ and the Dulac maps of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ are the same then $\mathcal{L}$ and $\mathcal{L}'$ are strictly analytically conjugated.

Chapter 0

Preliminaries of singular foliations in dimension two

0.1 Singular foliations, separatrices and holonomies

A germ of singular foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ is defined by the local integral curves of a germ of vector field

$$X = b(x, y)\frac{\partial}{\partial x} - a(x, y)\frac{\partial}{\partial y}, \tag{0.1.1}$$

where a and b are germs of holomorphic functions vanishing at the origin. The *singularity* of $\mathcal{F}$ is the zero locus of X which corresponds to common zero locus of a and b. When $\gcd(a, b)$ is not a unity, we can associate $\mathcal{F}$ a foliation defined by

$$\frac{b(x, y)}{\gcd(a, b)}\frac{\partial}{\partial x} - \frac{a(x, y)}{\gcd(a, b)}\frac{\partial}{\partial y}$$

that has an isolated singularity 0. From now on we can suppose that the singularity of $\mathcal{F}$ is isolated. It means that we require a and b have no common factor in $\mathbb{C}\{x, y\}$.

We can also consider the foliation $\mathcal{F}$ is generated by the kernel of holomorphic 1-form that can be seen as "dual" to the vector field (0.1.1)

$$\omega = a(x, y)dx + b(x, y)dy. \tag{0.1.2}$$

Two vector fields, or two 1-forms, that differ by multiplication by a unity define the same foliation.

A *separatrix* of a foliation $\mathcal{F}$ is an analytic irreducible curve S passing through the singularity p and invariant by $\mathcal{F}$. It means that vector X is tangent to S or the pull back of ω to S is identically zero. An important theorem of Camacho and Sad [3] says that any germ of foliation in dimension two has at least one separatrix. The

foliation is called *dicritical* if it has an infinite number of separatrices. Otherwise, it is called *non-dicritical.* The notion separatrix also make locally sense at p when $\mathcal{F}$ is regular: $\mathcal{F}$ has exactly one separatrix which is the smooth invariant curve passing through p.
Suppose that $\mathcal{F}$ is defined in a small neighborhood U of the isolated singularity p. Let S be a separatrix of $\mathcal{F}$ and denote $S^* = S \setminus \{p\}$. Then the curve S^*, that is isomorphic to the punctured disk $\mathbb{D}^*$, is a leaf of the regular foliation $\mathcal{F}$ defined in $U \setminus \{p\}$. Therefore we can define the *holonomy of the separatrix* S as the holonomy of $\mathcal{F}$ along an oriented loop $\gamma \in S^*$ generating $\pi_1(S^*) = \mathbb{Z}$.

The separatrices and their corresponding holonomies are two analytical invariants of foliation.

0.2 Reduced foliations

A germ of singular foliation $\mathcal{F}$ defined by a vector field X in (0.1.1) is called *reduced* if the linear part of X has at least one non-zero eigenvalue, say λ_2 and the quotient of two eigenvalues $\lambda = \frac{\lambda_1}{\lambda_2}$ is not a positive rational number.

Remark that λ is unchanged by multiplication of a unit. It is an important invariant of $\mathcal{F}$ called the Camacho-Sad index. $\mathcal{F}$ is called *nondegenerate* if both λ_1 and λ_2 are not zero, i.e. $\lambda \neq 0$. Otherwise, we say that $\mathcal{F}$ has a saddle-node singularity or $\mathcal{F}$ is a saddle-node foliation. We distinguish several cases depending on λ:

0.2.1 Poincaré domain: $\lambda \notin \mathbb{R}_{\leq 0} \cup \mathbb{Q}_{>0}$

According to a classical result of Poincaré, $\mathcal{F}$ is linearizable: there exist some coordinates (z, w) such that the foliation is defined by the linear 1-form $\lambda w dz + z dw$. There are exactly two separatrices: $\{z = 0\}$ and $\{w = 0\}$. The holonomy of each separatrix is conjugated to the diffeomorphism $h(x) = \exp(2\pi i \lambda^{\pm 1})x$ and the exponent of λ depends on which of the separatrices has been chosen. It is easy to see that, in this case, the Camacho-Sad index determines uniquely the foliation up to biholomorphism.

0.2.2 Siegel domain: $\lambda \in \mathbb{R}_{<0}$

In this case, the foliation $\mathcal{F}$ is not always linearizable. However, as in previous case, $\mathcal{F}$ has also exactly two separatrices. The holonomy is now written $h(x) = \exp(2\pi i \lambda^{\pm 1})x + \ldots$ and not always linearizable. Thus, the equality of the Camacho-Sad indices is not sufficient to imply the conjugation of holonomies. However, both index and holonomy determine uniquely the analytic class of the singularity.

Theorem 0.2.1 ([13]). *Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations having the same Camacho-Sad index $\lambda \in \mathbb{R}_{>0}$. Then they are analytically conjugated if and only if their holonomies are analytically conjugated.*

Although $\mathcal{F}$ is not always linearizable, ω still admits the *Poincaré-Dulac formal normal form*: there exists a formal transformation of coordinates that puts ω in some normal form. A 1-form is said to be in Poincaré-Dulac normal form if ω does not contain any terms $x^{\alpha_1}y^{\beta_1}dx$ or $x^{\alpha_2}y^{\beta_2}dy$, where

$$\lambda(\beta_1 - 1) - \alpha_1 = 0, \;\; \lambda\beta_2 - (\alpha_2 - 1) = 0,$$

in its Taylor expansion. In particular, if $\lambda \notin \mathbb{Q}$ then ω is always formally linearizable.

0.2.3 Saddle-node: $\lambda = 0$

According to Dulac, in suitable coordinates $\mathcal{F}$ is expressed by the vector field

$$\Big(z(1 + \nu w^k) + wF(z, w)\Big)\frac{\partial}{\partial z} + w^{k+1}\frac{\partial}{\partial w} \tag{0.2.1}$$

where $k \in \mathbb{N}_{>0}$, $\nu \in \mathbb{C}$ and F is a holomorphic function vanishing at $(0, 0)$ up to order k. The couple (k, ν) is a formal invariant of $\mathcal{F}$. More precisely, there exists a formal transformation of coordinates that puts all vector fields in (0.2.1) in *Dulac normal form*: it means that we can eliminate $F(z, w)$ in (0.2.1) by a suitable formal transformation of coordinates. The curve $\{w = 0\}$ is a separatrix, called the *strong separatrix*. Its holonomy has the form $h(x) = x + x^{k+1} + o(k + 1)$. A result of [10] affirms that this holonomy determines uniquely the germ of foliation up to biholomorphism. Beside the strong separatrix, there exists a second one, called the *weak one*, that can in general be non analytic. It is formally conjugated to a smooth curve transverse to the strong one (the curve $\{z = 0\}$ in Dulac normal form). When it is convergent, its holonomy has the form $h(x) = e^{2\pi i\nu}x + o(1)$ but gives a relatively small amount of information about the full structure of the foliation.

0.3 Blowing-up and resolution

A *Blowing-up* is a type of geometric transformation which replaces a subspace of a given space with all the directions pointing out of that subspace. In particular, the blowing-up of a point in a plane replaces the point with the projectivized tangent space at that point. Repeatedly blowing up the singular points of a curve will eventually resolve the singularities of curves. This is also true for the resolution of foliations. We now explain the desingularization process of foliations in the plane.

Definition 0.3.1. Let $\mathcal{F}$ be a germ of foliation and S be a germ of curve. The couple $(\mathcal{F}, S)$ is called *reduced* if it satisfies one of following conditions

1. $\mathcal{F}$ is singular, reduced and S is an invariant curve of $\mathcal{F}$.
2. $\mathcal{F}$ is regular and S is an invariant curve of $\mathcal{F}$.
3. $\mathcal{F}$ is regular, S is not invariant and all the leaves of $\mathcal{F}$ are transverse to S.

A *process of blowing-ups at the origin of* $\mathbb{C}^2$ is a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{M}^h & \xrightarrow{\sigma^h} \cdots \longrightarrow & \mathcal{M}^j & \xrightarrow{\sigma^j} & \mathcal{M}^{j-1} & \longrightarrow \cdots \xrightarrow{\sigma^1} & \mathcal{M}^0 = \mathbb{C}^2 \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ \mathcal{D}^h & \longrightarrow \cdots \longrightarrow & \mathcal{D}^j & \longrightarrow & \mathcal{D}^{j-1} & \longrightarrow \cdots \longrightarrow & \mathcal{D}^0 = \{0\} \\ \bigcup & & \bigcup & & \bigcup & & \bigcup \\ S^h & \longrightarrow \cdots \longrightarrow & S^j & \longrightarrow & S^{j-1} & \longrightarrow \cdots \longrightarrow & S^0 = \{0\} \end{array}$$

where $\mathcal{M}^j$ is a complex manifold of dimension 2;

$$\mathcal{D}^j := (\sigma^1 \circ \cdots \circ \sigma^j)^{-1}(0)$$

called the j^{th} *exceptional divisor* of the blowing-up; σ^{j+1} is the standard blowing-up at the center S^j. The natural h is called the *height* of the process. Let us denote

$$\sigma_j = \sigma^1 \circ \cdots \circ \sigma^j.$$

Theorem 0.3.2 (Seidenberg's desingularization of foliations [13, 16])**.** *Let $\mathcal{F}$ be a germ of foliation in* $(\mathbb{C}^2, 0)$*. There is a process of blowing-ups of height h such that*

1. *For all $j = 0, \ldots, h$, S^j is the set of non-reduced singularities of $(\sigma_j^*\mathcal{F}, \mathcal{D}^j)$.*
2. *$S^h = \emptyset$.*

Given a germ of foliation $\mathcal{F}$, let h be the smallest number satisfying the theorem above and denote by $\mathcal{M}$ the manifold $\mathcal{M}^h$. The map $\sigma = \sigma^1 \circ \cdots \sigma^h$ is called the *desingularization map* of $\mathcal{F}$, $\mathcal{D} = \sigma^{-1}(0)$ is called the *exceptional divisor*. The divisor $\mathcal{D}$ is a union of irreducible components which is homeomorphic to $\mathbb{CP}^1$. A point which is the intersection of two irreducible components of $\mathcal{D}$ is called a *corner*. Note that, after a blowing-up, a reduced singularity gives rise to two reduced singularities. Therefore, we can say the reduced singularities are the simplest singularities of foliations from the desingularization point of view. The condition (3) in Definition 0.3.1 avoids the situation when the foliation is regular at a point in the divisor and the invariant curve through this point is tangent to the divisor.

The pull-back of $\mathcal{F}$ by the desingularization map is called the *strict transform* of $\mathcal{F}$ and denoted by $\tilde{\mathcal{F}} = \sigma^*\mathcal{F}$. If all the singularities of strict transform $\tilde{\mathcal{F}}$ are not saddle-node, we say that $\mathcal{F}$ is a *generalized curve*. Note that a generalized curve is not necessary non-dicritical. Since the separatrices of a singular foliations is a union of its singular invariant curves, they admits also a desingularization process. In the case of a generalized curve, the desingularization process of the foliation and of its separatrices are equal [2]: this also explains the terminology *generalized curve*. Moreover, in the case $\mathcal{F}$ is non-dicritical generalized curve, if we denote by $\nu(\mathcal{F})$ the multiplicity of $\mathcal{F}$ and $\nu(f)$ the multiplicity of its separatrices then we have the relation [2]:

$$\nu(\mathcal{F}) + 1 = \nu(f).$$

Suppose that $\sigma : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ is the desingularization map of the foliation $\mathcal{F}$. The dual tree $\mathbb{A}^*(\mathcal{F})$ of $\mathcal{F}$ is a graph which is defined as following: set of vertices is the set of all irreducible components of the divisor $\mathcal{D}$; two vertices D and D' are connected by a edge if $D \cap D' \neq \emptyset$; the vertices are weighted by their auto-intersection number $D.D$; each vertex D which is corresponding to a component non-dicritical is attached with the arrows which corresponding to the non-corner singularities on D; each vertex which is corresponding to a dicritical component is attached with a double arrow. The dual tree is a topological invariant of foliation.

0.4 Vanishing holonomy representations

After desingularization by σ, denote by $\mathrm{Sing}(\tilde{\mathcal{F}})$ the set of all singularities of the strict transform $\sigma^*\mathcal{F} = \tilde{\mathcal{F}}$. Let D be a non-dicritical irreducible component of the exceptional divisor $\mathcal{D}$, then $D^* = D \setminus \mathrm{Sing}(\tilde{\mathcal{F}})$ is a leaf of $\tilde{\mathcal{F}}$. Let m be a regular point in D^* and Σ be a small analytic section through m transverse to $\tilde{\mathcal{F}}$. For any loop γ in D^* based on m there is a germ of a holomorphic return map

$$h_\gamma : (\Sigma, m) \to (\Sigma, m)$$

which only depends on the homotopy class of γ in the fundamental group $\pi_1(D^*, m)$. The map

$$h : \pi_1(D^*, m) \to \mathrm{Diff}(\Sigma, m)$$

is called the *vanishing holonomy representation* of $\mathcal{F}$ on D. Suppose that $\mathcal{F}'$ is a foliation that also admits σ as its desingularization map. Assume that $\mathrm{Sing}(\tilde{\mathcal{F}}') = \mathrm{Sing}(\tilde{\mathcal{F}})$ where $\mathrm{Sing}(\tilde{\mathcal{F}}')$ is the set of singularities of the strict transform $\tilde{\mathcal{F}}'$. Denote by h' the vanishing holonomy representation of $\mathcal{F}'$ on D. We say that the vanishing holonomy representation of $\mathcal{F}$ and $\mathcal{F}'$ on D are conjugated if there exists $\phi \in \mathrm{Diff}(\Sigma, m)$ such that

$$\phi \circ h_\gamma = h'_\gamma \circ \phi.$$

The vanishing holonomy representation of $\mathcal{F}$ and $\mathcal{F}'$ are called conjugated if they are conjugated on every non-dicritical irreducible components of $\mathcal{D}$.

Comparing to holonomy, vanishing holonomy representation contains more information of foliations. To see that, let us consider the non-dicritical foliation $\mathcal{F}$ whose

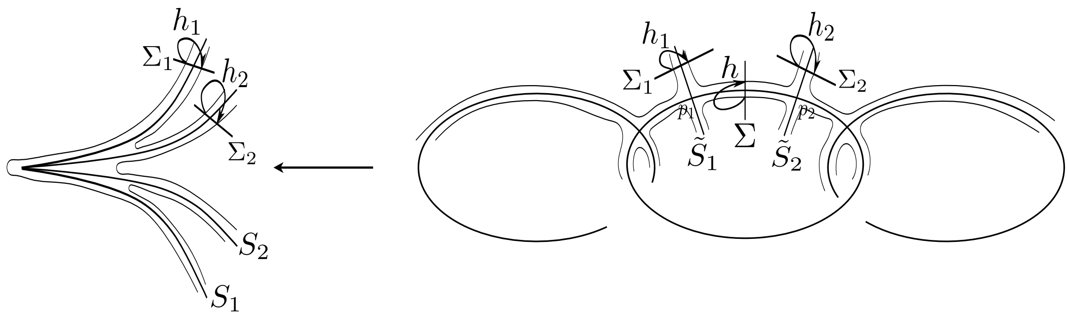

Figure 1: Holonomy vs vanishing holonomy representation

separatrices are two cuspidal curves S_1, S_2 (figure 1). Let Σ_1, Σ_2 be two transversal sections of S_1, S_2 respectively and h_1, h_2 be the corresponding holonomies. After desingularization, denote by h the vanishing holonomy representation. Consider at the singularity $p_1 = \tilde{S}_1 \cap \mathcal{D}$. If p_1 is saddle-node, the holonomy h_1 is corresponding to the weak separatrix, which contains a relative small amount of information concerning the structure of the foliation in a neighborhood of p_1. In contrast, h contains the holonomy of strong separatrix. In the case p_1 is nondegenerate, h_1 and h have the same role in the local picture of $\tilde{\mathcal{F}}$ around p_1. Nevertheless, h also contains the information about the relation of the holonomy around p_1 with the holonomies of other singularities.

0.5 Equisingular unfolding

Equisingular unfolding is a method of deforming of foliations without changing their vanishing holonomy representations.

Definition 0.5.1. An unfolding of $\mathcal{F}$ with parameters in $(\mathbb{C}^p, 0)$ is a germ of foliation $\mathcal{G}$ of $(\mathbb{C}^{2+p}, 0)$ of codimension one such that

1. The leaves of $\mathcal{G} \setminus \mathrm{Sing}(\mathcal{G})$ are transverse to the vertical foliation given by the fibers of the projection on the space of parameters $\pi : (\mathbb{C}^{2+p}, 0) \to (\mathbb{C}^p, 0)$.

2. If ν_0 stands for the embedding $\nu_0 : (\mathbb{C}^2, 0) \to (\mathbb{C}^{2+p}, 0), \nu_0(x) = (x, 0)$ then $\nu_0^*\mathcal{G} = \mathcal{F}$.

An unfolding induced in a natural way a deformation in the standard sense: indeed, one can set $\mathcal{F}_t = \nu_t^*\mathcal{G}$ where $\nu_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^{2+p}, 0)$, $\nu_t(x) = (x, t)$. The transversality conditions ensures that $\mathcal{F}_t$ is actually a foliation. The family $\mathcal{F}_t$ is an analytical deformation of $\mathcal{F}_0 = \mathcal{F}$.

Definition 0.5.2. An equisingular unfolding $\mathcal{G}$ of $\mathcal{F}$ with parameters in $(\mathbb{C}^p, 0)$ is said equisingular if there exists a manifold $\mathfrak{A}$ of dimension $2 + p$ which is a neighborhood of a compact divisor $\mathcal{D}$ such that

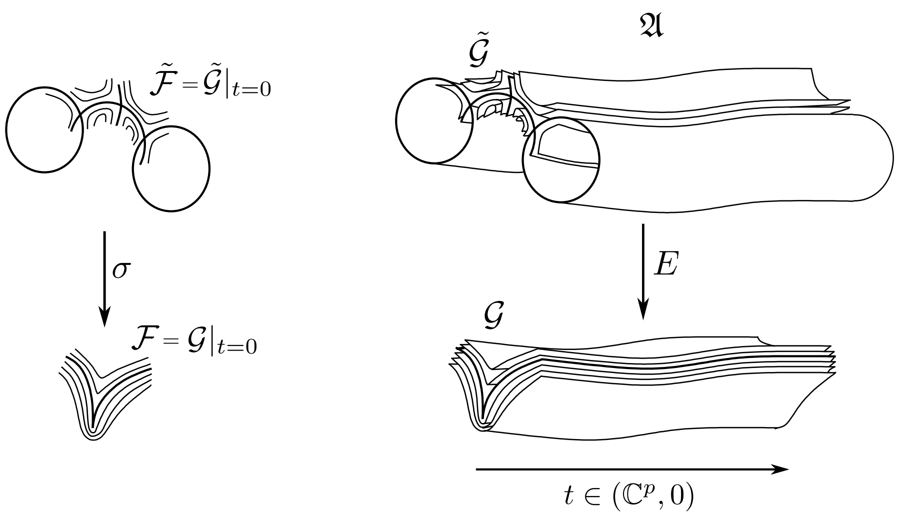

Figure 2: Equisingular unfolding

1. There is a holomorphic map $\Pi : (\mathfrak{A}, \mathcal{D}) \to (\mathbb{C}^p, 0)$ which is a surjective submersion over $(\mathbb{C}^p, 0)$ whose fibers are transverse to $\mathcal{D}$ and that is a surjective submersion on any irreducible component of $\mathcal{D}$.
2. There is a holomorphic map $E : (\mathfrak{A}, \mathcal{D}) \to (\mathbb{C}^{2+p}, 0)$ such that
   (a) $\pi \circ E = \Pi$ where $\pi : (\mathbb{C}^{2+p}, 0) \to (\mathbb{C}^p, 0)$ defined by $\pi(x, t) = t$
   (b) The leaves of $E^*\mathcal{G}$ are transverse to the fiber of Π
   (c) For any $t \in \mathbb{C}^p$, $E|_{\Pi^{-1}(t)}$ is the process of reduction of singularities of $\mathcal{F}_t$

Roughly speaking, an equisingular unfolding of $\mathcal{F}$ is a deformation $\mathcal{F}_t$ of $\mathcal{F}$ such that there is a "big" desingularization for all $\mathcal{F}_t$.

0.5.1 Cohomology interpretation of unfolding

Suppose that

$$\sigma : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$$

is the desingularization map of a foliation $\mathcal{F}$. Consider the sheaf of non-abelian group G_p as follows: the base space is the divisor $\mathcal{D}$. For each $z \in \mathcal{D}$, the fibers $G_{p,z}$ is the set of germs of automorphisms ϕ of $(M \times \mathbb{C}^p, (z, 0))$ satisfying

- $\phi(x, t) = (\bar{\phi}(x, t), t)$, $\phi(x, 0) = (x, 0)$
- In a neighborhood of $(z, 0)$ we have $\phi^*(\mathcal{F} \times \mathbb{C}^p) = \mathcal{F} \times \mathbb{C}^p$.

The last condition implies that for $x \in \mathcal{M}$ sufficient near z, the image of map $t \mapsto \bar{\phi}(x, t)$ contains in the leaf of $\mathcal{F}$ through x.

Since all equisingular unfoldings of a reduced foliation are trivial [11], there exists a natural way to construct the a cocycle in $Z^1(\mathcal{D}, G_p)$ from an equisingular unfoldings $\mathcal{G}$ of $\mathcal{F}$ as follows: let $\tilde{\mathcal{G}} = E^*\mathcal{G}$ be the desingularization of $\mathcal{G}$. There exists a covering of $\mathcal{D}$ by the set of connected open set (U_α) and the local trivializations

$$\phi_\alpha : (\tilde{\mathcal{G}}, U_\alpha) \to (\tilde{\mathcal{F}} \times \mathbb{C}^p, U_\alpha \times 0).$$

Then the maps $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ satisfy the cocycle condition. This corresponding is a bijective from the moduli space of equisingular unfolding of base space $(\mathbb{C}^p, 0)$ to the cohomology group $H^1(\mathcal{D}, G_p)$ [11].

0.5.2 Infinitesimal equisingular unfolding

Consider the sheaf $\Theta_{\mathcal{F}}$ of all the germs of vector fields on $\mathcal{M}$ tangent to the strict transform $\tilde{\mathcal{F}}$. Each equisingular unfolding $\mathcal{G}_p$ of the base $(\mathbb{C}^p, 0)$ is corresponding to $[\phi_{\alpha\beta}(x,t)] = [(\bar{\phi}_{\alpha\beta}(x,t), t)] \in H^1(\mathcal{D}, G_p)$. This induces a $\mathbb{C}$-linear map

$$\left[\frac{\partial \mathcal{G}_p}{\partial t}\bigg|_{t=0} \right] : T_0\mathbb{C}^p \to H^1(\mathcal{D}, \Theta_{\mathcal{F}})$$

$$\sum_{i=1}^{p} a_i \frac{\partial}{\partial t_i} \mapsto \left[\sum_{i=1}^{p} a_i \frac{\partial \bar{\phi}_{\alpha\beta}}{\partial t_i}\bigg|_{t=0} \right]. \tag{0.5.1}$$

Each element $H^1(\mathcal{D}, \Theta_{\mathcal{F}})$ can be interpreted as a speed of an equisingular unfolding of $\mathcal{F}$ at the moments $t = 0$. Moreover, $H^1(\mathcal{D}, \Theta_{\mathcal{F}})$ is a vector space of dimension

$$\delta(\mathcal{F}) := \sum_p \frac{(\nu_p - 1)(\nu_p - 2)}{2},$$

where p runs on the set $\bigsqcup_{i=0,\ldots,h} \mathcal{D}^i$ of all i^{th} exceptional divisor (including the origin $0 = \mathcal{D}^0$) and ν_p is the multiplicity at $p \in \mathcal{D}^i$ of $\sigma^{i*}\mathcal{F}$.

Theorem 0.5.3. *[11] Each foliation $\mathcal{F}$ admits a equisingular unfolding $\mathcal{G}^U$, of base $(\mathbb{C}^{\delta(\mathcal{F})}, 0)$, which is universal in the following sense: every equisingular unfolding $\mathcal{G}$ of some base $(\mathbb{C}^p, 0)$ is analytically conjugated to a unfolding of type $\lambda^*(\mathcal{G}^U)$, where $\lambda : \mathbb{C}^p \to \mathbb{C}^{\delta(\mathcal{F})}$. Moreover, $\mathcal{G}^U$ is unique up to a diffeomorphism and the linear map* (0.5.1) *is an isomorphism.*

$H^1(\mathcal{D}, \Theta_{\mathcal{F}})$ can be seen as the "tangent space" of the universal equisingular unfolding moduli space of $\mathcal{F}$ and it is called the *infinitesimal equisingular unfolding.* The number $\delta(\mathcal{F})$ is *the dimension of Mattei's moduli space.*

Chapter 1

Formal normal forms of topologically quasi-homogeneous foliations

In this chapter, we give a formal normal form for quasi-homogeneous foliations with axis branches under some generic conditions. One of required conditions is that all the Camacho-Sad indices after desingularization are not rational. This implies that the foliations have to admit the two invariant curves that are corresponding with $\{x = 0\}$ and $\{y = 0\}$. That is also the reason why we just consider the topologically quasi-homogeneous foliations which have the axis branches.

This chapter is divided into two sections. In the first section we recall the notation topologically quasi-homogeneous foliations and give some their properties. The results of this section work for both topologically quasi-homogeneous with and without axis branches. The formal normal form is given in the second section. In this section we restrict our attention only with the topologically quasi-homogeneous foliations with axis branches.

1.1 Topologically quasi-homogeneous foliations

1.1.1 Definitions

A germ of holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is *quasi-homogeneous* if f belongs to the jacobian ideal $J(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. If f is quasi-homogeneous, there exist coordinates (x, y) and positive coprime integers k, ℓ such that $R(f) = d \cdot f$, where $R = kx\frac{\partial}{\partial x} + \ell y\frac{\partial}{\partial y}$ is the quasi-radial vector field and d is the quasi-homogeneous degree of f [15]. In these coordinates, f can be written, up to a multiple of a constant, as

$$f = x^{n_0} y^{n_\infty} \prod_{i=1}^{n} (y^k - f_i x^\ell)^{n_i},$$

where the multiplicities satisfy $n_0 \geq 0, n_\infty \geq 0, n_i > 0$ and the coefficients f_i are non vanishing such that $f_i \neq f_j$. A germ of holomorphic function f is called *topologically quasi-homogeneous* if its zero level set is topologically conjugated to the zero level set of a quasi-homogeneous function. We also say that a germ of non-dicritical holomorphic foliations $\mathcal{F}$ is *topologically quasi-homogeneous* if after desingularization by successive blowing-ups, none of singularities of strict transform $\tilde{\mathcal{F}}$ are saddle-node and the separatrices of $\mathcal{F}$ is the zero level set of a topologically quasi-homogeneous function. The separatrices of $\mathcal{F}$ that are conjugated to these curves $\{y^k - c_i x^\ell = 0\}$ are called the *cuspidal branches*. The one (if exists) that is conjugated to $\{x^{n_0} = 0\}$ or $\{y^{n_\infty} = 0\}$ is called the *y-axis branch* or *x-axis branch* respectively. We call $\mathcal{F}$ *topologically quasi-homogeneous with axis branches* if it admits both x-axis and y-axis branches. A 1-form ω is called *topologically quasi-homogeneous* if it defines a topologically quasi-homogeneous foliation.

1.1.2 Desingularization process of quasi-homogeneous functions

In what follows, we fix a reduced quasi-homogeneous function f which is given by

$$f = x^{\varepsilon_0} y^{\varepsilon_\infty} \prod_{i=1}^{n} (y^k - f_i x^\ell). \tag{1.1.1}$$

Let us recall the algorithm of desingularization of f and its atlas of charts. On the blowing-up of $(\mathbb{C}^2, 0)$ endowed with the chart (x, y), we will use the standard charts $(x, \bar{y}), (\bar{x}, y)$ together with the transition functions $\bar{x} = \bar{y}^{-1}, y = x\bar{y}$. The center of the first chart $(x, \bar{y})$ is denoted by 0 and the center of the second one is denoted by ∞ (figure 1.1). We denote by

$$\sigma = \sigma_1 \circ \ldots \circ \sigma_p : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$$

the desingularization map of f obtained by composition of the blowing-up's σ_i, $1 \leq i \leq p$, and $\mathcal{D} = \sigma^{-1}(0)$ the exceptional divisor. Let us sketch some properties of σ. In the desingularization process, we only have to use blowing-up of 0 or ∞. Therefore, the tree of exceptional divisor is a totally ordered sequence of N components covered by $N+1$ charts and the map σ is monomial in each chart. Before the last blowing-up, all cuspidal branches share the same infinitesimal point. After the last blowing-up, they appear on the same component of $\mathcal{D}$ called the principal component. If $\varepsilon_0 \neq 0$ or $\varepsilon_\infty \neq 0$, the corresponding strict branches appear on the end components. Let us number the components of $\mathcal{D}$ and their charts in such a way that D_1 corresponds to the strict branches which appears if $\varepsilon_\infty \neq 0$. Then, we obtain $N + 1$ chart $(x_i, y_i), i = 0, \ldots, N$, such that each component $D_i, i = 1 \ldots N$ is covered by domains

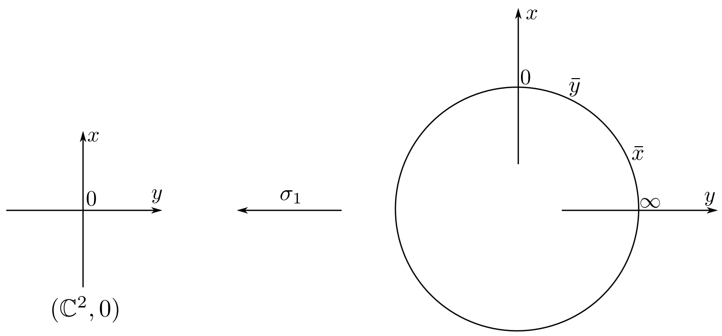

Figure 1.1: Blowing-up at origin of $(\mathbb{C}^2, 0)$

V_{i-1} and V_i of the charts (x_{i-1}, y_{i-1}) around $(D_i, 0)$ and (D_i, ∞) (figure 1.2). The change of charts is given by

$$x_i = y_{i-1}^{-1}, y_i = x_{i-1} y_{i-1}^{e_i}$$

where $-e_i$ is the self intersection number of the component D_i. We denote by c the index corresponding to the principal component. Then, the desingularization map σ is given in the chart (x_c, y_c) by ([7]):

$$(x, y) = (x_c^{k-v} y_c^k, x_c^{\ell-u} y_c^\ell),$$

where u, v two non-negative integers such that

(1.1.2) $$ku - \ell v = 1 \text{ and } u \leq \ell, v \leq k.$$

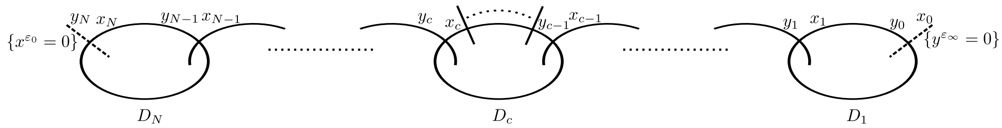

Figure 1.2: Desingularization of f

1.1.3 Infinitesimal equisingular unfolding of quasi-homogeneous foliations

Let $\mathcal{F}$ be a germ of quasi-homogeneous foliations which is defined by a 1-form ω. Let S be its separatrices which are defined by

$$f = x^{\varepsilon_0} y^{\varepsilon_\infty} \prod_{i=1}^{n} (y^k - f_i x^\ell).$$

Recall that (see [11, 12])

1. The moduli space of infinitesimal equisingular unfoldings of $\mathcal{F}$ is the first cohomology group $H^1(\mathcal{D}, \Theta_{\mathcal{F}})$, where $\Theta_{\mathcal{F}}$ is the sheaf on $\mathcal{D}$ of germs of vector fields tangent to strict transform $\tilde{\mathcal{F}}$.

2. Denote by Θ_S the sheaf on $\mathcal{D}$ of germs of vector fields tangent to $\sigma^{-1}(S) = \tilde{S} \cup \mathcal{D}$, where $\tilde{S}$ is the strict transform of the separatrices of $\mathcal{F}$. The first cohomology group $H^1(\mathcal{D}, \Theta_S)$ is the moduli space of infinitesimal equisingular unfoldings of the separatrices S.

3. Denote by $(f \circ \sigma)$ the ideal of the sheaf $\mathcal{O}_{\mathcal{D}}$ of all germs of holomorphic function on $\mathcal{D}$ generated by the global section $f \circ \sigma$. Then we have the followings exact sequence

   $$0 \to \Theta_{\mathcal{F}} \xrightarrow{i} \Theta_S \xrightarrow{\tilde{\omega}} (f \circ \sigma) \to 0 \tag{1.1.3}$$

   where i is the inclusion and $\tilde{\omega}$ is defined by $\tilde{\omega}(X) = \sigma^* \omega(X)$. The inclusion i induces an isomorphism between $H^1(\mathcal{D}, \Theta_{\mathcal{F}})$ and $H^1(\mathcal{D}, \Theta_S)$.

Let us denote by $\mathcal{F}_0$ the foliation defined by df. The inclusion i_0 induces an isomorphism between $H^1(\mathcal{D}, \Theta_{\mathcal{F}_0})$ and $H^1(\mathcal{D}, \Theta_S)$. Therefore, the two first cohomology groups $H^1(\mathcal{D}, \mathcal{F})$ and $H^1(\mathcal{D}, \mathcal{F}_0)$ are isomorphic. This isomorphism can be explicitly described as the following proposition

Proposition 1.1.1. *There is a canonical isomorphism Λ between $H^1(\mathcal{D}, \mathcal{F}_0)$ and $H^1(\mathcal{D}, \mathcal{F})$ satisfying the following commutative diagram:*

$$\begin{array}{ccc} H^1(\mathcal{D}, \Theta_{\mathcal{F}_0}) & \xrightarrow{\quad\Lambda\quad} & H^1(\mathcal{D}, \Theta_{\mathcal{F}}) \\ & {\scriptstyle i_0}\searrow \quad \swarrow{\scriptstyle i} & \\ & H^1(\mathcal{D}, \Theta_S) & \end{array}$$

Proof. Let U_0 be a open set in $\mathcal{M}$ which covers $\cup_{i=1}^{c} D_i$ minus a small disk around $D_c \cap D_{c+1}$ in D_c, and U_∞ an open set covering $\cup_{i=c}^{N} D_i$ minus a small disk around $D_{c-1} \cap D_c$ in D_c. Let $\mathcal{U}$ be the covering of $\mathcal{D}$ by these two open sets U_0 and U_∞. By [7], we have

$$H^1(\mathcal{D}, \Theta_{\mathcal{F}}) = H^1(\mathcal{U}, \Theta_{\mathcal{F}}),\ H^1(\mathcal{D}, \Theta_{\mathcal{F}_0}) = H^1(\mathcal{U}, \Theta_{\mathcal{F}_0}) \text{ and } H^1(\mathcal{D}, \Theta_S) = H^1(\mathcal{U}, \Theta_S).$$

Hence, we only need to describe the isomorphism between $H^1(\mathcal{U}, \Theta_{\mathcal{F}_0})$ and $H^1(\mathcal{U}, \Theta_{\mathcal{F}})$. Let $X_{0,\infty}$ be an element of $\mathcal{Z}^1(\mathcal{U}, \Theta_{\mathcal{F}_0})$. Then

$$g_{0,\infty} := \tilde{\omega}(X_{0,\infty}) \in \mathcal{Z}^1(\mathcal{U}, (f \circ \sigma)).$$

Since $H^1(\mathcal{U}, (f \circ \sigma)) = H^1(\mathcal{U}, \mathcal{O}_{\mathcal{D}}) = 0$, there exist $g_i \in \Gamma(U_i, (f \circ \sigma))$ for $i = 0, \infty$, such that

$$g_{0,\infty} = g_0 - g_\infty.$$

The exact sequence (1.1.3) implies the following long exact sequence

(1.1.4) $$0 \to \Gamma(U_i, \Theta_{\mathcal{F}}) \to \Gamma(U_i, \Theta_S) \xrightarrow{\tilde{\omega}} \Gamma(U_i, (f \circ \sigma)) \to H^1(U_i, \Theta_{\mathcal{F}}),\ i = 0, \infty.$$

By [7], we have
$$H^1(U_0, \Theta_{\mathcal{F}}) = H^1(U_\infty, \Theta_{\mathcal{F}}) = 0.$$
Therefore, there exist $X_i \in \Gamma(U_i, \Theta_S)$, $(i = 0, \infty)$, such that $\tilde{\omega}(X_i) = g_i$. We have $X_{0,\infty} + X_\infty - X_0$ belongs to $\mathcal{Z}^1(\mathcal{U}, \Theta_F F)$ because

$$\tilde{\omega}(X_{0,\infty} + X_\infty - X_0) = g_{0,\infty} + g_\infty - g_0 = 0.$$

Let us consider Λ defined by

$$\begin{aligned} \Lambda : H^1(\mathcal{U}, \Theta_{\mathcal{F}_0}) &\longrightarrow H^1(\mathcal{U}, \Theta_{\mathcal{F}}) \\ [X_{0,\infty}] &\longmapsto [X_{0,\infty} + X_\infty - X_0]. \end{aligned}$$

Since the equality $[X_{0,\infty}] = [X_{0,\infty} + X_\infty - X_0]$ holds in the cohomology group $H^1(\mathcal{U}, \Theta_S)$, the proof is reduced to show that Λ is well defined. Indeed, if $g'_i \in \Gamma(U_i, (f \circ \sigma))$, $X'_i \in \Gamma(U_i, \Theta_S)$, $(i = 0, \infty)$, satisfying $g_{0,\infty} = g'_0 - g'_\infty$ and $\tilde{\omega}(X'_i) = g'_i$, then we have
$$g_0 - g'_0|_{U_0 \cap U_\infty} = g_\infty - g'_\infty|_{U_0 \cap U_\infty}.$$
Therefore, there exists a global section $g \in \Gamma(\mathcal{D}, (f \circ \sigma))$ such that

$$g_{|U_0} = g_0 - g'_0 \text{ and } g_{|U_\infty} = g_\infty - g'_\infty.$$

By (1.1.3), we have the following exact sequence

$$0 \to \Gamma(\mathcal{D}, \Theta_{\mathcal{F}}) \to \Gamma(\mathcal{D}, \Theta_S) \xrightarrow{\tilde{\omega}} \Gamma(\mathcal{D}, (f \circ \sigma)) \to H^1(\mathcal{D}, \Theta_{\mathcal{F}}) \to H^1(\mathcal{D}, \Theta_S).$$

Since the spaces $H^1(\mathcal{D}, \Theta_{\mathcal{F}})$ and $H^1(\mathcal{D}, \Theta_S)$ are isomorphic, we must have

$$\mathrm{Im}(\tilde{\omega}) = \tilde{\omega}(\Gamma(\mathcal{D}, \Theta_S)) = \Gamma(\mathcal{D}, (f \circ \sigma)).$$

Therefore, there exists $X \in \Gamma(\mathcal{D}, \Theta_S)$ such that $\tilde{\omega}(X) = g$. Let Y_i be defined by

$$Y_i = X'_i - X_i + X,\ i = 0, \infty.$$

Since $\tilde{\omega}(Y_i) = g'_i - g_i + g = 0$, we obtain $Y_i \in \Gamma(U_i, \Theta_{\mathcal{F}})$. Moreover, since

$$(X_{0,\infty} + X_\infty - X_0) - (X_{0,\infty} + X'_\infty - X'_0) = Y_0 - Y_\infty,$$

we have the equality

$$[X_{0,\infty} + X_\infty - X_0] = [X_{0,\infty} + X'_\infty - X'_0] \in H^1(\mathcal{D}, \Theta_{\mathcal{F}}).$$

Hence, Λ is well defined. □

Exemple 1.1.2. Now we will describe the isomorphism Λ when $\mathcal{F}$ admits a first integral which is

$$x^{n_0}y^{n_\infty}\prod_{i=1}^{n}(y^k-f_ix^\ell)^{n_i}.$$

Recall that (see [7]):

1. The universal equisingular unfolding of $\mathcal{F}$ is given by

$$N_c^{(m)}=x^{n_0}y^{n_\infty}\prod_{i=1}^{n}(y^k-c_{i0}x^\ell+\sum_{j\in\mathbb{T}_i}c_{ij}p^{k\ell+j})^{n_i},$$

   where $(m)=(m_0,m_\infty,m_1,\ldots,m_n)$; $c=(c_{ij})_{\substack{i=1,\ldots,n\\ j\in\{0\}\cup\mathbb{T}_i}}$, $\mathbb{T}_i$ is a finite subset of $\mathbb{N}_{\geq1}$; $p^{k\ell+j}$ is the unique monomial $x^{\alpha_j}y^{\beta_j}$ such that $k\alpha_j+\ell\beta_j=k\ell+j$ and $\beta_j\leq k$; $c_{i0}\in(\mathbb{C},f_i)$, $c_{ij}\in(\mathbb{C},0)$. For the sake of simplicity, in the computation to come, we denote by $c^0=(c^0_{ij})$ where $c^0_{i0}=f_i$ and $c^0_{ij}=0$ for all $i=1,\ldots,n$ and $j\in\mathbb{T}_i$. Then

$$N_{c^0}^{(m)}=x^{n_0}y^{n_\infty}\prod_{i=1}^{n}(y^k-f_ix^\ell)^{n_i}.$$

2. Suppose that there exist holomorphic functions $\alpha^{(m)}_{0ij}(x_{c-1},y_{c-1})$, $\beta^{(m)}_{0ij}(x_{c-1},y_{c-1})$, $\alpha^{(m)}_{\infty ij}(x_c,y_c)$ and $\beta^{(m)}_{\infty ij}(x_c,y_c)$, $i=1,\ldots n, j\in\{0\}\cup\mathbb{T}_i$, such that

$$\frac{\partial\widetilde{N}_c^{(m)}}{\partial c_{ij}}\bigg|_{c=c^0}=\alpha^{(m)}_{0ij}\cdot\frac{\partial\widetilde{N}_c^{(m)}}{\partial x_{c-1}}\bigg|_{c=c^0}+\beta^{(m)}_{0ij}\cdot\frac{\partial\widetilde{N}_c^{(m)}}{\partial y_{c-1}}\bigg|_{c=c^0},\tag{1.1.5}$$

$$\frac{\partial\widetilde{N}_c^{(m)}}{\partial c_{ij}}\bigg|_{c=c^0}=\alpha^{(m)}_{\infty ij}\cdot\frac{\partial\widetilde{N}_c^{(m)}}{\partial x_{c}}\bigg|_{c=c^0}+\beta^{(m)}_{\infty ij}\cdot\frac{\partial\widetilde{N}_c^{(m)}}{\partial y_{c}}\bigg|_{c=c^0},\tag{1.1.6}$$

   where $\widetilde{N}_c^{(m)}=\sigma^*N_c^{(m)}=N_c^{(m)}\circ\sigma$. We define

$$X^{(m)}_{0ij}=\alpha^{(m)}_{0ij}\frac{\partial}{\partial_{x_{c-1}}}+\beta^{(m)}_{0ij}\frac{\partial}{\partial_{y_{c-1}}},\ X^{(m)}_{\infty ij}=\alpha^{(m)}_{\infty ij}\frac{\partial}{\partial_{x_{c}}}+\beta^{(m)}_{\infty ij}\frac{\partial}{\partial_{y_{c}}},$$

   and

$$X^{(m)}_{ij}=X^{(m)}_{0ij}-X^{(m)}_{\infty ij}.$$

   Then, $\mathcal{B}^{(m)}=\{[X^{(m)}_{ij}]\}_{\substack{i=1,\ldots,n\\ j\in\{0\}\cup\mathbb{T}_i}}$ is a basis of $H^1(\mathcal{U},\Theta_{\mathcal{F}})$, and $[X^{(m)}_{ij}]$ does not depend on choosing $\alpha^{(m)}_{0ij}$, $\alpha^{(m)}_{\infty ij}$, $\beta^{(m)}_{0ij}$, $\beta^{(m)}_{\infty ij}$ in (1.1.5) and (1.1.6) .

Denote by $(\varepsilon)=(\varepsilon_0,\varepsilon_\infty,1,\ldots,1)$, we obtain similarly a basis $\mathcal{B}^{(\varepsilon)}=\{[X^{(\varepsilon)}_{ij}]\}_{\substack{i=1,\ldots,n\\ j\in\{0\}\cup\mathbb{T}_i}}$ of $H^1(\mathcal{U},\Theta_{\mathcal{F}_0})$.

Proposition 1.1.3. *The presentation matrix of Λ in these bases is the unit matrix.*

Proof. Let us first compute $\mathcal{B}^{(m)}$. Blowing-up the deformations yields

$$\widetilde{N}_c^{(m)} = x_{c-1}^{m_0k+m_\infty\ell+|m|k\ell} y_{c-1}^{m_0v+m_\infty u+|m|\ell v} \prod_{i=1}^{n}\left(y_{c-1} - c_{i0} + \sum_{j\in\mathbb{T}_i} c_{ij}x_{c-1}^{j}y_{c-1}^{\gamma_j}\right)^{m_i},$$

where $|m| = \sum_{i=1}^{n} m_i$; $\gamma_j = v\alpha_j + u\beta_j - \ell v$ with $p^{k\ell+j} = x^{\alpha_j}y^{\beta_j}$ and (u, v) is defined as in (1.1.2). We have

$$\begin{aligned}
\frac{\partial\widetilde{N}_c^{(m)}}{\partial x_{c-1}}\bigg|_{c=c^0} &= \frac{m_0k + m_\infty\ell + |m|k\ell}{x_{c-1}} \cdot \widetilde{N}_{c^0}^{(m)},\\
\frac{\partial\widetilde{N}_c^{(m)}}{\partial y_{c-1}}\bigg|_{c=c^0} &= \left(\frac{m_0v + m_\infty u + |m|\ell v}{y_{c-1}} + \sum_{i=1}^{n}\frac{m_i}{y_{c-1} - f_i}\right) \cdot \widetilde{N}_{c^0}^{(m)},\\
\frac{\partial\widetilde{N}_c^{(m)}}{\partial c_{i0}}\bigg|_{c=c^0} &= \frac{-m_i}{y_{c-1} - f_i} \cdot \widetilde{N}_{c^0}^{(m)},\\
\frac{\partial\widetilde{N}_c^{(m)}}{\partial c_{ij}}\bigg|_{c=c^0} &= \frac{m_i x_{c-1}^{j} y_{c-1}^{\gamma_j}}{y_{c-1} - f_i} \cdot \widetilde{N}_{c^0}^{(m)} = -x_{c-1}^{j}y_{c-1}^{\beta_j} \cdot \frac{\partial\widetilde{N}_c^{(m)}}{\partial c_{i0}}\bigg|_{c=c^0},\ j \in \mathbb{T}_i.
\end{aligned}$$

Then, for $j = 0$, (1.1.5) is equivalent to

$$\frac{-m_i}{y_{c-1} - f_i} = \frac{m_0k + m_\infty\ell + |m|k\ell}{x_{c-1}}\cdot\alpha_{0i0} + \left(\frac{m_0v + m_\infty u + |m|\ell v}{y_{c-1}} + \sum_{i=1}^{n}\frac{m_i}{y_{c-1} - f_i}\right)\cdot\beta_{0i0}.$$

Denote

$$P(y_{c-1}) = y_{c-1}^{m_0v+m_\infty u+|m|\ell v}\prod_{i=1}^{n}(y_{c-1} - f_i)^{m_i}.$$

Consider the unique solution (U_0, V_0) of the following Bézout identity in $\mathbb{C}[y_{c-1}]$:

$$U_0P + V_0P' = P \wedge P' = y_{c-1}^{m_0v+m_\infty u+|m|\ell v-1}\prod_{i=1}^{n}(y_{c-1} - f_i)^{m_i-1}, \tag{1.1.7}$$
$$\deg(U_0) < n,\ \deg(V_0) < n + 1.$$

Let us denote

$$R(y_{c-1}) = \frac{P}{P \wedge P'} = y_{c-1}\prod_{i=1}^{n}(y_{c-1} - f_i).$$

We obtain a holomorphic solution of (1.1.5) by setting

$$\alpha_{0i0}^{(m)} = \frac{-m_i x_{c-1}RU_0}{(m_0v + m_\infty u + |m|\ell v)(y_{c-1} - f_i)} \quad\text{and}\quad \beta_{0i0}^{(m)} = \frac{-m_iRV_0}{y_{c-1} - f_i}.$$

Doing a similar computation in the coordinates (x_c, y_c), we have

$$\begin{aligned}
\widetilde{N}_c^{(m)} = {} & x_c^{m_0(k-v)+m_\infty(\ell-u)+|m|(k\ell-ku)} y_c^{m_0k+m_\infty\ell+|m|k\ell}\\
& \times\prod_{i=1}^{n}\left(1 - c_{i0}x_c + \sum_{j\in\mathbb{T}_i} c_{ij}x_c^{\eta_j}y_c^{j}\right)^{m_i},
\end{aligned}$$

where $\eta_j = (k-v)\alpha_j + (\ell-u)\beta_j - k\ell + ku$. As before, we denote

$$Q(x_c) = x_c^{m_0(k-v)+m_\infty(\ell-u)+|m|(k\ell-ku)} \prod_{i=1}^{n}(1-f_ix_c)^{m_i} \text{ and } S(x_c) = x_c\prod_{i=1}^{n}(1-f_ix_c).$$

Let (U_∞, V_∞) be the unique solution of the following Bézout identity in $\mathbb{C}[y_{c-1}]$:

$$U_\infty Q + V_\infty Q' = Q\wedge Q' = x_c^{m_0(k-v)+m_\infty(\ell-u)+|m|(k\ell-ku)-1}\prod_{i=1}^{n}(1-f_ix_c)^{m_i-1}, \tag{1.1.8}$$
$$\deg(U_\infty) < n,\ \deg(V_\infty) < n+1.$$

Similarly, we obtains a holomorphic solution of (1.1.6) by setting

$$\alpha^{(m)}_{\infty i0} = \frac{-m_ix_cSV_\infty}{1-f_ix_c} \text{ and } \beta^{(m)}_{\infty i0} = \frac{-m_ix_cy_cSU_\infty}{(m_0k+m_\infty\ell+|m|k\ell)(1-f_ix_c)}.$$

Therefore, $X^{(m)}_{i0}$ is given by

$$\begin{aligned} X^{(m)}_{i0} =& \frac{-m_ix_{c-1}RU_0}{(m_0v+m_\infty u+|m|\ell v)(y_{c-1}-f_i)}\frac{\partial}{\partial_{x_{c-1}}} + \frac{-m_iRV_0}{y_{c-1}-f_i}\frac{\partial}{\partial_{y_{c-1}}} \\ &+ \frac{m_ix_cSV_\infty}{1-f_ix_c}\frac{\partial}{\partial_{x_c}} + \frac{m_ix_cy_cSU_\infty}{(m_0k+m_\infty\ell+|m|k\ell)(1-f_ix_c)}\frac{\partial}{\partial_{y_c}}. \end{aligned}$$

Now, replace (m) by (ε), we obtain

$$\begin{aligned} X^{(\varepsilon)}_{i0} =& \frac{-x_{c-1}R\bar{U}_0}{(\varepsilon_0v+\varepsilon_\infty u+|\varepsilon|\ell v)(y_{c-1}-f_i)}\frac{\partial}{\partial_{x_{c-1}}} + \frac{-R\bar{V}_0}{y_{c-1}-f_i}\frac{\partial}{\partial_{y_{c-1}}} \\ &+ \frac{x_cS\bar{V}_\infty}{1-f_ix_c}\frac{\partial}{\partial_{x_c}} + \frac{x_cy_cS\bar{U}_\infty}{(\varepsilon_0k+\varepsilon_\infty\ell+|\varepsilon|k\ell)(1-f_ix_c)}\frac{\partial}{\partial_{y_c}}, \end{aligned}$$

where

$$\bar{P}(y_{c-1}) = y_{c-1}^{\varepsilon_0v+\varepsilon_\infty u+|\varepsilon|\ell v}\prod_{i=1}^{n}(y_{c-1}-f_i),$$
$$\bar{Q}(x_c) = x_c^{\varepsilon_0(k-v)+\varepsilon_\infty(\ell-u)+|\varepsilon|(k\ell-ku)}\prod_{i=1}^{n}(1-f_ix_c)^{m_i},$$

and $(\bar{U}_0, \bar{V}_0)$, (U_∞, V_∞) are the unique solutions of the following Bézout identity:

$$\bar{U}_0\bar{P} + \bar{V}_0P' = \bar{P}\wedge\bar{P}',\ \bar{U}_\infty\bar{Q} + \bar{V}_\infty\bar{Q}' = \bar{Q}\wedge\bar{Q}',$$
$$\deg(\bar{U}_0), \deg(\bar{U}_\infty) < n,\ \deg(\bar{V}_0), \deg(\bar{V}_\infty) < n+1.$$

Let us consider $X_{0i0} = X^{(m)}_{0i0} - X^{(\varepsilon)}_{0i0}$ and $X_{\infty i0} = X^{(m)}_{\infty i0} - X^{(\varepsilon)}_{\infty i0}$. We will show now that

$$X_{0i0} \in \Gamma(U_0, \Theta_S), \text{ and } X_{\infty i0} \in \Gamma(U_\infty, \Theta_S).$$

This will leads to $[X_{i0}^{(m)} - X_{i0}^{(\varepsilon)}] = 0$ in $H^1(\mathcal{U}, \Theta_S)$. Consequently, the isomorphism Λ sends $X_{i0}^{(\varepsilon)}$ to $X_{i0}^{(m)}$.

Indeed, by (1.1.7) we get

$$\begin{aligned} V_0(f_i) &= \frac{P \wedge P'}{P'}(f_i) = \frac{1}{R\left(\frac{m_0 v + m_\infty u + |m|\ell v}{y_{c-1}} + \sum_{i=1}^n \frac{m_i}{y_{c-1} - f_i}\right)}(f_i) \\ &= \frac{1}{m_i f_i \prod_{\substack{j=1,\dots,n \\ j\neq i}} (f_i - f_j)}. \end{aligned}$$

Since $m_i V_0(f_i)$ does not depend on (m), we have $(\bar{V}_0 - m_i V_0)(f_i) = 0$. Therefore, we can write X_{0i0} as

$$\begin{aligned} X_{0i0} &= \frac{x_{c-1} R}{y_{c-1} - f_i}\left(\frac{\bar{U}_0}{\varepsilon_0 v + \varepsilon_\infty u + |\varepsilon|\ell v} - \frac{m_i U_0}{m_0 v + m_\infty u + |m|\ell v}\right)\frac{\partial}{\partial_{x_{c-1}}} + \frac{R(\bar{V}_0 - m_i V_0)}{y_{c-1} - f_i}\frac{\partial}{\partial_{y_{c-1}}} \\ &= x_{c-1}\hat{U}_0 \frac{\partial}{\partial_{x_{c-1}}} + R\hat{V}_0 \frac{\partial}{\partial_{y_{c-1}}}, \end{aligned}$$

where $\hat{U}_0$ and $\hat{V}_0$ are in $\mathbb{C}[y_{c-1}]$. It is obvious that X_{0i0} is tangent to the divisor $\{x_{c-1} = 0\}$ and the separatrices $\{y_{c-1} = f_i\}$, $i = 1, \dots, n$.
Without loss of generality, we can suppose that $k > \ell$. Then, the desingularization map σ is given in the charts (x_0, y_0) and (x_N, y_N) by

$$(x, y) = (x_0, x_0 y_0) = (x_N y_N^{[\frac{k}{\ell}[+1}, y_N),$$

where $[\frac{k}{\ell}[$ stands for the usual integer part of $\frac{k}{\ell}$. Therefore, the changes of coordinates are written as follows

$$\begin{aligned} (x_0, y_0) &= (x_{c-1}^k y_{c-1}^v, x_{c-1}^{\ell-k} y_{c-1}^{u-v}), \\ (x_{c-1}, y_{c-1}) &= (x_0^{u-v} y_0^{-v}, x_0^{k-\ell} y_0^k), \\ (x_c, y_c) &= \left(x_N^\ell y_N^{\left([\frac{k}{\ell}[+1\right)\ell - k}, x_N^{u-\ell} y_N^{k-v-\left([\frac{k}{\ell}[+1\right)(\ell-u)}\right), \\ (x_N, y_N) &= (x_c^{k-v-\left(\frac{k}{\ell}+1\right)(\ell-u)} y_c^{k-\left([\frac{k}{\ell}[+1\right)\ell}, x_c^{\ell-u} y_c^\ell). \end{aligned}$$

This leads to

$$\begin{aligned} \frac{\partial}{\partial_{x_{c-1}}} &= \frac{k}{x_{c-1}} x_0 \frac{\partial}{\partial_{x_0}} + \frac{k-\ell}{x_{c-1}} y_0 \frac{\partial}{\partial_{y_0}}, \\ \frac{\partial}{\partial_{y_{c-1}}} &= \frac{v}{y_{c-1}} x_0 \frac{\partial}{\partial_{x_0}} + \frac{u-v}{y_{c-1}} y_0 \frac{\partial}{\partial_{y_0}}, \\ \frac{\partial}{\partial_{x_c}} &= \frac{k - v - \left([\frac{k}{\ell}[+1\right)(\ell - u)}{x_c} x_N \frac{\partial}{\partial_{x_N}} + \frac{\ell - u}{x_c} y_N \frac{\partial}{\partial_{y_N}}, \\ \frac{\partial}{\partial_{y_c}} &= \frac{k - \left([\frac{k}{\ell}[+1\right)\ell}{y_c} x_N \frac{\partial}{\partial_{x_N}} + \frac{\ell}{y_c} y_N \frac{\partial}{\partial_{y_N}}. \end{aligned}$$

Hence, in the coordinates (x_0, y_0), we have

$$X_{0i0} = \left(k\hat{U}_0 + v\hat{R}\hat{V}_0\right) \circ (x_0^{k-\ell} y_0^k) \cdot x_0 \frac{\partial}{\partial_{x_0}} + \left((k-\ell)\hat{U}_0 + (u-v)\hat{R}\hat{V}_0\right) \circ (x_0^{k-\ell} y_0^k) \cdot y_0 \frac{\partial}{\partial_{y_0}},$$

where $\hat{R} = \frac{R}{y_{c-1}} \in \mathbb{C}[y_{c-1}]$. Therefore, X_{0i0} extends to a holomorphic vector field in a neighborhood of $(x_0 = 0, y_0 = 0)$. By [7], X_{0i0} can be extended to a holomorphic vector field in U_0. Moreover, it is obvious that X_{0i0} is tangent to $\{x_0 = 0\}$ and $\{y_0 = 0\}$. Consequently, $X_{0i0} \in \Gamma(U_0, \Theta_S)$.

We perform a similar computation for $X_{\infty i0}$. We have

$$V_\infty \left(\frac{1}{f_i}\right) = \frac{-f_i^{n-1}}{m_i \prod_{\substack{j=1,\ldots,n \\ j\neq i}} (f_i - f_j)},$$

which implies that $(m_i V_\infty - \bar{V}_\infty)\left(\frac{1}{f_i}\right) = 0$. Hence, $X_{\infty i0}$ is written as

$$\begin{aligned} X_{\infty i0} =& \frac{x_c S(m_i V_\infty - \bar{V}_\infty)}{1 - f_i x_c} \frac{\partial}{\partial_{x_c}} + \frac{x_c y_c S}{1 - f_i x_c} \left(\frac{m_i U_\infty}{m_0 k + m_\infty \ell + |m| k\ell} - \frac{\bar{U}_\infty}{\varepsilon_0 k + \varepsilon_\infty \ell + |\varepsilon| k\ell} \right) \frac{\partial}{\partial_{y_c}} \\ =& \, x_c \left(S\hat{V}_\infty \frac{\partial}{\partial_{x_c}} + y_c \hat{U}_\infty \frac{\partial}{\partial_{y_c}} \right), \end{aligned}$$

where $\hat{U}_\infty$ and $\hat{V}_\infty$ are in $\mathbb{C}[x_c]$. We denote

$$\hat{X}_{\infty i0} = S\hat{V}_\infty \frac{\partial}{\partial_{x_c}} + y_c \hat{U}_\infty \frac{\partial}{\partial_{y_c}}.$$

It is obvious that $\hat{X}_{\infty i0}$ is tangent to the divisor $\{y_c = 0\}$ and the separatrices $\{x_c = \frac{1}{f_i}\}$, $i = 1, \ldots, n$. Moreover, in the coordinates (x_N, y_N), we have

$$\begin{aligned} \hat{X}_{\infty i0} =& \left(\left(k - v - \left([\frac{k}{\ell}[+1\right)(\ell - u)\right)\hat{S}\hat{V}_\infty + \left(k - \left([\frac{k}{\ell}[+1\right)\ell\right)\hat{U}_\infty \right) \circ \left(x_N^\ell y_N^{\left([\frac{k}{\ell}[+1\right)\ell - k} \right) \\ & \times x_N \frac{\partial}{\partial_{x_N}} + \left((\ell - u)\hat{S}\hat{V}_\infty + \ell \hat{U}_\infty\right) \circ \left(x_N^\ell y_N^{\left([\frac{k}{\ell}[+1\right)\ell - k} \right) \cdot y_N \frac{\partial}{\partial_{y_N}}, \end{aligned}$$

where $\hat{S} = \frac{S}{x_c} \in \mathbb{C}[x_c]$. This implies $\hat{X}_{\infty i0} \in \Gamma(U_\infty, \Theta_S)$. Since the function $x_c = x_N^\ell y_N^{\left([\frac{k}{\ell}[+1\right)\ell - k}$ is holomorphic on U_∞, $X_{\infty i0} = x_c \hat{X}_{\infty i0} \in \Gamma(U_\infty, \Theta_S)$.

Now, we will prove that Λ sends $X_{ij}^{(\varepsilon)}$ to $X_{ij}^{(m)}$ for all $j \in \mathbb{T}_i$. As before, it reduces to show that $X_{0ij} = X_{0ij}^{(m)} - X_{0ij}^{(\varepsilon)} \in \Gamma(U_0, \Theta_S)$ and $X_{\infty ij} = X_{\infty ij}^{(m)} - X_{\infty ij}^{(\varepsilon)} \in \Gamma(U_\infty, \Theta_S)$.

Since

$$\begin{aligned}
\frac{\partial \widetilde{N}_c^{(m)}}{\partial c_{ij}}\Big|_{c=c^0} &= -x_{c-1}^j y_{c-1}^{\gamma_j} \cdot \frac{\partial \widetilde{N}_c^{(m)}}{\partial c_{i0}}\Big|_{c=c^0}, \\
\frac{\partial \widetilde{N}_c^{(\varepsilon)}}{\partial c_{ij}}\Big|_{c=c^0} &= -x_{c-1}^j y_{c-1}^{\gamma_j} \cdot \frac{\partial \widetilde{N}_c^{(\varepsilon)}}{\partial c_{i0}}\Big|_{c=c^0}, \\
\frac{\partial \widetilde{N}_c^{(m)}}{\partial c_{ij}}\Big|_{c=c^0} &= -x_c^{\eta_j-1} y_c^j \cdot \frac{\partial \widetilde{N}_c^{(m)}}{\partial c_{i0}}\Big|_{c=c^0}, \\
\frac{\partial \widetilde{N}_c^{(\varepsilon)}}{\partial c_{ij}}\Big|_{c=c^0} &= -x_c^{\eta_j-1} y_c^j \cdot \frac{\partial \widetilde{N}_c^{(\varepsilon)}}{\partial c_{i0}}\Big|_{c=c^0},
\end{aligned}$$

X_{0ij} and X_∞ are obtained by setting

$$X_{0ij} = -x_{c-1}^j y_{c-1}^{\gamma_j} X_{0i0},\ X_{\infty ij} = -x_c^{\eta_j} y_c^j \hat{X}_{\infty i0}.$$

Therefore, we only need to show that $x_{c-1}^j y_{c-1}^{\gamma_j}$ (resp. $x_c^{\eta_j} y_c^j$) can be extended to a holomorphic function on U_0 (resp. U_∞). Let us first write $x_{c-1}^j y_{c-1}^{\gamma_j}$ in the coordinates (x_0, y_0):

$$x_{c-1}^j y_{c-1}^{\gamma_j} = x_0^{\gamma_j(k-\ell)+j(u-v)} y_0^{\gamma_j k - jv}.$$

By substituting $\alpha_j = \frac{k\ell + j - \ell\beta_j}{k}$ into the formula of γ_j, we get

$$\gamma_j = v\frac{k\ell + j - \ell\beta_j}{k} + u\beta_j - \ell v = \frac{\beta_j + vj}{k}.$$

This implies that

$$\begin{aligned}
\gamma_j k - jv &= \beta_j > 0, \\
\gamma_j(k-\ell) + j(u-v) &= \frac{\beta_j(k-\ell)+j}{k} > 0.
\end{aligned}$$

By [7], $x_{c-1}^j y_{c-1}^{\gamma_j}$ extends to a holomorphic function on the whole U_0. Similarly, we have

$$x_c^{\eta_j} y_c^j = x_N^{\eta_j\ell + j(u-\ell)} y_N^{\eta_j\left(([\frac{k}{\ell}[+1)\ell - k\right) + j\left(k - v - ([\frac{k}{\ell}[+1)(\ell-u)\right)}.$$

By substituting $\beta_j = \frac{k\ell + j - k\alpha_j}{\ell}$ into the formula of η_j, we get

$$\eta_j = \frac{(\ell-u)j + \alpha_j}{\ell}.$$

This leads to

$$\begin{aligned}
\eta_j\ell + j(u-\ell) &= \alpha_j > 0, \\
\eta_j\left(([\frac{k}{\ell}[+1)\ell - k\right) + j\left(k - v - ([\frac{k}{\ell}[+1)(\ell-u)\right) &= \alpha_j\left([\frac{k}{\ell}[+1 - \frac{k}{\ell}\right) + \frac{j}{\ell} > 0.
\end{aligned}$$

Hence, $x_c^{\eta_j} y_c^j$ extends to a holomorphic function on U_∞. □

1.1.4 Desingularization of topologically quasi-homogeneous functions

If two germs of holomorphic functions are topologically conjugated, they admit the same dual tree of desingularization. In particular, their desingularization maps have the same number of blowing-ups but they are not necessarily equal. The following lemma shows that in the case topologically quasi-homogeneous, they share the same desingularization map after a local change of coordinates.

Lemma 1.1.4. *If a reduced function f' is topologically conjugated to f, which is given as* (1.1.1)*, then there exists a local change of coordinates ϕ such that $f' \circ \phi$ has the same desingularization map as f. Moreover, ϕ can be chosen such that $x^{\varepsilon_0} y^{\varepsilon_\infty}$ divides $f' \circ \phi$.*

Proof. Without loss of generality, we can assume that $k \geq \ell$ and denote by m the integer part of $\frac{k}{\ell}$. Denote by

$$\sigma' = \sigma'_1 \circ \ldots \circ \sigma'_h : (\mathcal{M}', \mathcal{D}') \to (\mathbb{C}^2, 0)$$

the desingularization maps of f',

$$\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i \text{ and } \sigma'^i = \sigma'_1 \circ \ldots \circ \sigma'_i, \text{ for } i = 1, \ldots, h,$$

the composition of i first blowing-ups of the desingularization maps of f and f' respectively. It is easy to see that if $\sigma^{m+1} = \sigma'^{m+1}$ then $\sigma = \sigma'$.

Let us first consider the case $\varepsilon_\infty = 0$. If $\varepsilon_0 = 1$, we will show that any diffeomorphism ϕ that sends $\{x = 0\}$ to the y-axis branch L_y of f' is the desired diffeomorphism. Indeed, the center of the blowing-up σ'_i, with $2 \leq i \leq m+1$, is the intersection point of the transform $(\sigma'^{i-1})^*(L_y)$ and the divisor $(\sigma'^{i-1})^{-1}(0)$. Hence, after the local change of coordinate ϕ, the $m+1$ first blowing-ups of the desingularization map of $f' \circ \phi$ and f are the same. Consequently, $f' \circ \phi$ and f have the same desingularization map. In the case $\varepsilon_0 = 0$, after m first blowing-ups, all the strict transforms of cuspidal branches of f' share the same intersection point z with divisor $(\sigma'^m)^{-1}(0)$. We take a smooth curve $\tilde{L}$ transverse to the divisor at z and denote by L the image by σ^m of $\tilde{L}$. Then L is a germ of smooth curve of $(\mathbb{C}^2, 0)$. With the same reason as above, any diffeomorphism that sends $\{x = 0\}$ to L is the desired diffeomorphism.

Now, we consider the case $\varepsilon_\infty = 1$. Using the same argument as above, if $\varepsilon_0 = 1$ then ϕ is a diffeomorphism that sends x-axis branch L_x and y-axis branch L_y to $\{y = 0\}$ and $\{x = 0\}$ respectively. In the case $\varepsilon_0 = 0$, we define L as above then the desired diffeomorphism is the one that sends L_x and L to $\{y = 0\}$ and $\{x = 0\}$ respectively. □

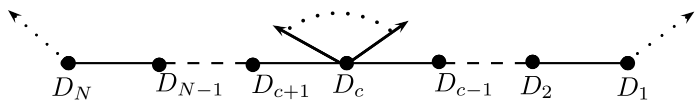

Figure 1.3: Dual tree of topological quasi-homogeneous functions

1.1.5 The criteria of topologically quasi-homogeneous foliation

Notation 1.1.5. Let us denote by $\mathcal{Q}(f)$ the set of all germs of topologically quasi-homogeneous 1-forms in $(\mathbb{C}^2, 0)$ whose separatrices are topologically conjugated to f satisfy that they admits $\sigma : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ as the desingularization map and $x^{\varepsilon_0} y^{\varepsilon_\infty}$ as invariant curves.

By the lemma 1.1.4, instead of considering the class of all topologically quasi-homogeneous 1-forms we can restrict our attention to the subset $\mathcal{Q}(f)$. We will denote by $\mathcal{O}(k, \ell, d)$ the set of germs of holomorphic functions q satisfying

$$q = q_d + q_{d+1} + q_{d+2} \ldots = q_d + h.o.t.$$

where $q_d \not\equiv 0$, q_m is (k, ℓ)-quasi-homogeneous polynomial of degree m and "h.o.t." stands for higher order term. In what follows, denote by

$$d = nk\ell + k\varepsilon_0 + \ell\varepsilon_\infty$$

the quasi-homogeneous degree of f.

Lemma 1.1.6. *If a germ of* 1*-form* $\omega = adx + bdy$ *is an element of* $\mathcal{Q}(f)$ *then*

(i) y^{ε_∞} *divides* a, x^{ε_0} *divides* b.

(ii) Let $q := kxa + \ell yb$, $p := (k - v)xa + (\ell - u)yb$, *where* u, v *satisfy* (1.1.2). *Then,* q, p *are in* $\mathcal{O}(k, \ell, d)$ *and*

$$q_d = c_0 x^{\epsilon_0} y^{\varepsilon_\infty} \prod_{i=1}^{n} (y^k - c_i x^\ell), \; c_0 \neq 0, \tag{1.1.9}$$

such that all the complex numbers c_i, $i \in \{1, \ldots n\}$ *are non-zero, different from each other. Moreover,* $\gcd\left(\frac{q_d}{x_0^{\varepsilon} y_\infty^{\varepsilon}}, \frac{p_d}{x_0^{\varepsilon} y_\infty^{\varepsilon}}\right) = 1$.

Proof. It is obvious that (i) is always satisfied. To prove (ii) by induction on the number of blowing-ups, we will replace the notation $\mathcal{Q}(f)$ by $\mathcal{Q}(k, \ell, n, \varepsilon_0, \varepsilon_\infty)$. For $k = \ell = 1$, in the blowing-up coordinates $(\bar{x}, y)$, the pullback of ω is given by

$$\begin{aligned}
\sigma^*\omega(\bar{x}, y) &= ya(\bar{x}y, y)d\bar{x} + (\bar{x}a(\bar{x}y, y) + b(\bar{x}y, y))dy \\
&= \frac{1}{\bar{x}} p(\bar{x}y, y)d\bar{x} + \frac{1}{y} q(\bar{x}y, y)dy \\
&= y^{d-1}\left(\left(\frac{y}{\bar{x}} p_d(\bar{x}, 1) + y^2(\ldots)\right) d\bar{x} + (q_d(\bar{x}, 1) + y(\ldots))\, dy\right).
\end{aligned}$$

Because after desingularization all the singularities are not saddle-node, the roots of $q_d(\bar{x}, 1)$ are distinct and none of them are a root of $q_d(\bar{x}, 1)$. Therefore, we have

$$q_d(x, y) = c_0 x^{\varepsilon_0} y^{\varepsilon_\infty} \prod_{i=1}^{n} (y - c_i x) \text{ and } \gcd\left(\frac{q_d}{x_0^{\varepsilon} y_\infty^{\varepsilon}}, \frac{p_d}{x_0^{\varepsilon} y_\infty^{\varepsilon}}\right) = 1.$$

In general, we can assume without loss of generality that $k > \ell$. Let σ_1 be the standard blowing-up at the origin. By [2], the multiplicity of $\sigma_1^*\omega$ equals to the multiplicity of $f \circ \sigma_1$ minus 1. Hence, there exists a 1-form $\omega'(\bar{x}, y)$ such that $\sigma_1^*\omega$ can be written as follows

$$\sigma_1^*\omega(\bar{x}, y) = ya(\bar{x}y, y)d\bar{x} + (\bar{x}a(\bar{x}y, y) + b(\bar{x}y, y))dy = y^{\varepsilon_0+\varepsilon_\infty+n\ell-1}\omega'(\bar{x}, y).$$

If ω is in $\mathcal{Q}(k, \ell, n, \varepsilon_0, \varepsilon_\infty)$ then ω' is in $\mathcal{Q}(k - \ell, \ell, n, \varepsilon_0, 1)$. Using the induction hypothesis for ω', we obtain

$$\begin{aligned} &(k-\ell)\bar{x}(ya(\bar{x}y, y)) + \ell y(\bar{x}a(\bar{x}y, y) + b(\bar{x}y, y)) \\ &= y^{\varepsilon_0+\varepsilon_\infty+n\ell-1}(c_0'\bar{x}^{\varepsilon_0} y \prod_{i=1}^{n}(y^{k-\ell} - c_i'\bar{x}^{\ell})) + h.o.t. \end{aligned} \tag{1.1.10}$$

where the numbers c_i', $i = 1, \ldots, n$, are non-zero, different from each other. Replace $\bar{x}y$ by x, (1.1.10) is equivalent to

$$kxa(x, y) + \ell yb(x, y) = c_0' x^{\varepsilon_0} y^{\varepsilon_\infty} \prod_{i=1}^{n}(y^k - c_i' x^l) + h.o.t.$$

Moreover, we have

$$\begin{aligned} y^{\varepsilon_0+\varepsilon_\infty+n\ell-1}p'(\bar{x}, y) &= (k - \ell - v + u)\bar{x}(ya(\bar{x}y, y)) + (\ell - u)y(\bar{x}a(\bar{x}y, y) + b(\bar{x}y, y)) \\ &= (k - v)\bar{x}ya(\bar{x}y, y) + (\ell - u)yb(\bar{x}y, y). \end{aligned}$$

Replace $\bar{x}y$ by x and use the induction hypothesis for p' and q', we obtains

$$\gcd\left(\frac{q_d}{x_0^{\varepsilon} y_\infty^{\varepsilon}}, \frac{p_d}{x_0^{\varepsilon} y_\infty^{\varepsilon}}\right) = 1.$$

□

Remark 1.1.7. The conditions (i) and (ii) in Lemma 1.1.6 are not sufficient for characterize the elements of $\mathcal{Q}(f)$. In fact, a 1-form ω satisfies these conditions if and only if the foliation $\mathcal{F}$ defined by ω satisfies:

(i) $\{x^{\varepsilon_0} y^{\varepsilon_\infty} = 0\}$ is an invariant curve of $\mathcal{F}$,

(ii) Let σ be the desingularization map of f. After pullback by σ, except the corners, the strict transform $\sigma^*(\mathcal{F})$ has n singularities on principal component D_c. Moreover at each singularity $\sigma^*(\mathcal{F})$ is defined by a 1-form whose linear part is

$$\lambda ydx + xdy, \text{ where } \lambda \neq 0.$$

There exists a dicritical 1-form satisfying these two conditions above. Therefore, for obtaining $\omega \in \mathcal{Q}(f)$ we need the non-dicritical condition: all the Camacho-Sad indices of all singularities of $\sigma^*(\mathcal{F})$ are not in $\mathbb{Q}_{>0}$.

Remark 1.1.8. An element $\omega \in \mathcal{Q}(f)$ can be written as

$$\omega = \omega_d + \omega_{d+1} + \omega_{d+2} + \dots \tag{1.1.11}$$

where $\omega_m = a_{m-k}dx + b_{m-\ell}dy$, a_{m-k}, $b_{m-\ell}$ are (k,ℓ)-quasi-homogeneous polynomials of degrees $m-k$, $m-\ell$ respectively and $\omega_d \not\equiv 0$.

1.1.6 Logarithmic representation of the initial part

Let ω be a element of $\mathcal{Q}(f)$. With the notation as in Lemma 1.1.6, the points $z_i = (\frac{1}{c_i}, 0)$ $(1 \leq i \leq n)$ in the coordinates (x_c, y_c) stand for the intersections of strict transforms of separatrices of ω with the principal component of the divisor. Denote by λ_i the Camacho-Sad indices of strict transform foliation $\tilde{\mathcal{F}} = \sigma^*\mathcal{F}$ defined by $\sigma^*\omega = \hat{a}dx + \hat{b}dy$ and the principal component $\mathcal{D}_c$. It means that

$$\lambda_i = i_{z_i}(\tilde{\mathcal{F}}, \mathcal{D}_c) = -\mathrm{Res}_{z_i}\frac{\partial}{\partial y}\left(\frac{\hat{a}}{\hat{b}}\right)(x,0).$$

We also denote by λ_0 and λ_∞ the indices of $\tilde{\mathcal{F}}$ and $\mathcal{D}_c$ at $z_0 = (x_c = 0, y_c = 0)$ and $z_\infty = (x_c = \infty, y_c = 0)$ respectively. Then, by [3] we have the relation:

$$\sum_{i=1}^{n}\lambda_i + \lambda_0 + \lambda_\infty = -1, \tag{1.1.12}$$

and the projective holonomy h_i of $\tilde{\mathcal{F}}$ at the point z_i satisfying

$$h_i'(0) = exp(2\pi i\lambda_i).$$

Actually, (λ_i) only depend on the quasi-homogeneous part ω_d. Moreover, ω_d is completely determined by (λ_i) and (c_i) as in the following lemma:

Lemma 1.1.9. *With the notation as above, we have*

$$\omega_d = -q_d\left(\sum_{i=1}^{n}\lambda_i\frac{d(y^k - c_i x^\ell)}{y^k - c_i x^\ell} + \varepsilon_0(\ell(\lambda_0+1) - u)\frac{dx}{x} + \varepsilon_\infty(k\lambda_\infty + v)\frac{dy}{y}\right),$$

where

$$\sum_{i=1}^{n}\lambda_i + \varepsilon_0(\lambda_0 + \frac{l-u}{l}) + \varepsilon_\infty(\lambda_\infty + \frac{v}{k}) + \frac{1}{k\ell} = 0.$$

Proof. When $\varepsilon_0 = 0$ (resp. $\varepsilon_\infty = 0$), the value of λ_0 (resp. λ_∞) is totally determined by the couple (k,ℓ). Therefore, we can compute the value of λ_0 (resp. λ_∞) when $\varepsilon_0 = 0$ (resp. $\varepsilon_\infty = 0$) by considering the 1-form $d(y^k - x^\ell)$. Since $(x,y) = (x_c^{k-v}y_c^k, x_c^{\ell-u}y_c^\ell) = (x_{c-1}^k y_{c-1}^v, x_{c-1}^\ell y_{c-1}^u)$, we have

$$\sigma^* d(y^k - x^\ell) = x_c^{k\ell-ku-1}y_c^{k\ell-1}\Big(y_c\Big(k\ell - ku - (k\ell-ku+1)x_c\Big)dx_c + k\ell x_c(1-x_c)dy_c\Big),$$
$$\sigma^* d(y^k - x^\ell) = x_{c-1}^{k\ell-1}y_{c-1}^{\ell v-1}\Big(k\ell y_{c-1}(y_{c-1}-1)dx_{c-1} + \ell v x_{c-1}(y_{c-1}-1)dy_{c-1}\Big).$$

It implies that

$$\lambda_0 = -\frac{v}{k} \text{ when } \varepsilon_0 = 0 \text{ and } \lambda_\infty = -\frac{\ell-u}{\ell} \text{ when } \varepsilon_\infty = 0. \tag{1.1.13}$$

Now, in the coordinates (x_c, y_c), we get

$$\sigma^*\omega_d = \frac{1}{x_c}p_d(x_c^{k-v}y_c^k, x_c^{\ell-u}y_c^\ell)dx_c + \frac{1}{y_c}q_d(x_c^{k-v}y_c^k, x_c^{\ell-u}y_c^\ell)dy_c,$$

where $p_d = (k-v)xa_{d-k} + (\ell-u)yb_{d-\ell}$, $q_d = kxa_{d-k} + \ell yb_{d-\ell}$. By Lemma 1.1.6, $\sigma^*\omega_d$ can be written as

$$\sigma^*\omega_d = x_c^{e-1}y_c^{d-1}(y_c\bar{p}(x_c)dx_c + c_0x_c\prod_{i=1}^n(1-c_ix_c)dy_c),$$

where $\bar{p}$ is a polynomial of degree n satisfying $\bar{p}(x_c^\ell) = \frac{1}{x^{\varepsilon_0}}p(x_c,1)$ and $e = nk(\ell - u) + \varepsilon_0(k-v) + \varepsilon_\infty(\ell-u)$. The definition of λ_i leads to

$$\bar{p}(\frac{1}{c_i}) = c_0\lambda_i\prod_{\substack{j=1\\ j\neq i}}^n\left(1-\frac{c_j}{c_i}\right),\ i = 1,\ldots,n,$$
$$\bar{p}(0) = -c_0\lambda_0.$$

Thanks to the formula of Lagrange polynomial, $\bar{p}$ is given by

$$\bar{p}(x_c) = c_0\left(\sum_{i=1}^n\lambda_ic_ix_c\prod_{\substack{j=1\\ j\neq i}}^n(1-c_jx_c) - \lambda_0\prod_{j=1}^n(1-c_jx_c)\right).$$

It implies that

$$\begin{aligned}p_d(x,y) &= c_0x^{\varepsilon_0}y^{\varepsilon_\infty}\left(\sum_{i=1}^n\lambda_ic_ix^\ell\prod_{\substack{j=1\\ j\neq i}}^n(y^k-c_jx^\ell) - \lambda_0\prod_{j=1}^n(y^k-c_jx^\ell)\right)\\ &= q_d\left(\sum_{i=1}^n\frac{\lambda_ic_ix^\ell}{y^k-c_ix^\ell} - \lambda_0\right).\end{aligned}$$

Consequently, we have

$$a_{d-k}(x,y)=\frac{\ell p_d-(\ell-u)q_d}{x}=\frac{q_d}{x}\left(\sum_{i=1}^{n}\lambda_i\frac{\ell c_i x^\ell}{y^k-c_ix^\ell}+(u-\ell(\lambda_0+1))\right),$$

$$b_{d-l}(x,y)=\frac{(k-v)q_d-kp_d}{y}=-\frac{q_d}{y}\left(\sum_{i=1}^{n}\lambda_i\frac{ky^k}{y^k-c_ix^\ell}+(v+k\lambda_\infty)\right).$$

Because (1.1.13), we can replace λ_0 by $\varepsilon_0\lambda_0+(\frac{u}{\ell}-1)(1-\varepsilon_0)$ and λ_∞ by $\varepsilon_\infty\lambda_\infty-\frac{v}{k}(1-\varepsilon_\infty)$. Then $u-\ell(\lambda_0+1)$ becomes $\epsilon_0(u-\ell(\lambda_0+1))$, $v+k\lambda_\infty$ becomes $\varepsilon_\infty(v+k\lambda_\infty)$ and the relation (1.1.12) becomes

$$\sum_{i=1}^{n}\lambda_i+\varepsilon_0(\lambda_0+\frac{l-u}{l})+\varepsilon_\infty(\lambda_\infty+\frac{v}{k})+\frac{1}{k\ell}=0.$$

□

1.1.7 Decomposition of topologically quasi-homogeneous foliations

Lemma 1.1.10. *Let ω be a germ of 1-form in $(\mathbb{C}^2,0)$. Then there exist unique holomorphic functions h and s such that*

$$\omega=dh+s(\ell ydx-kxdy). \tag{1.1.14}$$

Proof. Suppose that $\omega=adx+bdy$. Let $R=kx\frac{\partial}{\partial x}+\ell ykx\frac{\partial}{\partial y}$ the quasi-radial vector field. Denote by $q=\omega(R)=kxa+\ell yb$. Suppose that there exist h and s satisfying (1.1.14). We have

$$q=\omega(R)=R(h).$$

It implies that

$$h=\sum_{i=0}^{\infty}\frac{q_i}{i}. \tag{1.1.15}$$

where $q=q_0+q_1+q_2+\ldots$ is the decomposition of q into the (k,ℓ) quasi-homogeneous polynomials. This proves the uniqueness part.

Now assume that h is defined as (1.1.15). Decompose a, b, h into the (k,ℓ) quasi-homogeneous polynomials, we have

$$\begin{aligned}a_{i+\ell}-\partial_x h_{i+k+\ell}&=a_{i+\ell}-\frac{\partial_x(kxa_{i+\ell}+\ell yb_{i+k})}{i+k+\ell}\\&=\frac{(i+\ell)a_{i+\ell}-(kx\partial_x a_{i+\ell}+\ell y\partial_x b_{i+k})}{i+k+\ell}=\frac{\ell y(\partial_y a_{i+\ell}+\partial_x b_{i+k})}{i+k+\ell},\\ b_{i+k}-\partial_y h_{i+k+\ell}&=b_{i+k}-\frac{\partial_y(kxa_{i+\ell}+\ell yb_{i+k})}{i+k+\ell}\\&=\frac{(i+k)b_{i+k}-(kx\partial_x a_{i+\ell}+\ell y\partial_x b_{i+k})}{i+k+\ell}=-\frac{kx(\partial_y a_{i+\ell}+\partial_x b_{i+k})}{i+k+\ell}.\end{aligned}$$

This implies the existence of s, which is defined by

$$s_i = \frac{\partial_y a_{i+\ell} + \partial_x b_{i+k}}{i+k+\ell}.$$

□

Using Lemma 1.1.10 for the elements in $\mathcal{Q}(f)$ we obtains:

Corollary 1.1.11. *For each $\omega \in \mathcal{Q}(f)$, there exist unique holomorphic functions h and s such that*

$$\omega = \omega_d + dh + s(\ell y dx - kx dy) \tag{1.1.16}$$

where ω_R is the quasi-radial form $lydx - kxdy$. Moreover, we have

$$h(x,y) = \sum_{i=1}^{\infty} \frac{kxa_{d+i-k} + lyb_{d+i-\ell}}{d+i}, s(x,y) = \sum_{i=1}^{\infty} \frac{\partial_y a_{d+i-k} - \partial_x b_{d+i-\ell}}{d+i}. \tag{1.1.17}$$

1.2 Formal normal forms of topologically quasi-homogeneous foliations

In this section, we only consider the case $\varepsilon_0 = \varepsilon_\infty = 1$. Then $f = xy\prod_{i=1}^{n}(y^k - f_i x^\ell)$ is a homogeneous polynomial of degree $d = k\ell n + k + \ell$.

The process of normalization is follows: let ω be a topologically quasi-homogeneous 1-form. By Lemma 1.1.4, we can assume that ω is an element of $\mathcal{Q}(f)$. Decompose ω as in (1.1.16). Then the process is divided into two steps. Firstly, we apply consecutively the diffeomorphisms and the unit multiplications to simplify the hamiltonian part h degree by degree. After that, by using the diffeomorphisms and the unit functions that do not change the term h we will normalize the function s.

Let us denote by

$$\mathcal{Q}^d(f) = \{\omega_d = a_{d-k}dx + b_{d-\ell}dy : \omega_d \in \mathcal{Q}(f)\}$$

the set of initial parts of elements of $\mathcal{Q}(f)$. For each $\omega_d \in \mathcal{Q}^d(f)$, we also denote by $\mathcal{Q}(\omega_d)$ the subset of $\mathcal{Q}(f)$ containing the 1-forms admitting ω_d as their initial part:

$$\mathcal{Q}(\omega_d) = \{\omega' \in \mathcal{Q}(f) : \omega' = \omega_d + \omega'_{d+1} + \omega'_{d+2} \ldots\}.$$

Now, let $\omega(x,y) = a(x,y)dx + b(x,y)dy \in \mathcal{Q}(f)$. For each integer $m \geq 1$, we consider a local change of coordinates $\phi(x,y) = (x + \alpha(x,y), y + \beta(x,y))$ and a unity function $u(x,y) = 1 + \delta(x,y)$ where α, β, δ are (k,ℓ)-quasi-homogeneous polynomials of degrees $m+k, m+\ell, m$ respectively. The local change of coordinates ϕ and the multiplication by u take the form ω into the form

$$\tilde{\omega} = u.\phi^*\omega = \tilde{a}dx + \tilde{b}dy.$$

Lemma 1.2.1. *Denote by $\Delta q = \tilde{q} - q$, $\Delta a = \tilde{a} - a$, $\Delta b = \tilde{b} - b$ we have $\Delta q_{d+m'} = \Delta a_{d+m'-k} = \Delta b_{d+m'-\ell} = 0$ for all $m' < m$ and*

$$\Delta q_{d+m} = AU + BV \tag{1.2.1}$$

$$\Delta b_{d+m-\ell} = W + \frac{1}{d+m}\left(mb_{d-\ell}\delta - q_d\partial_y\delta\right), \tag{1.2.2}$$

where $A = ma_{d-k} + \partial_x q_d, B = mb_{d-\ell} + \partial_y q_d, U = \alpha + \frac{k}{d+m}x\delta, V = \beta + \frac{l}{d+m}y\delta$ and $W = \partial_y(a_{d-k}U + b_{d-\ell}V) - (\partial_y a_{d-k} - \partial_x b_{d-\ell})U$.

Proof. We have

$$\phi^*\omega = \left((1+\partial_x\alpha)\cdot a\circ\phi + \partial_x\beta\cdot b\circ\phi\right)dx + \left(\partial_y\alpha\cdot a\circ\phi + (1+\partial_y\beta)\cdot b\circ\phi\right)dy.$$

It implies that

$$\begin{aligned}\tilde{q} &= u\left((kx + kx\cdot\partial_x\alpha + \ell y\cdot\partial_y\alpha)a\circ\phi + (\ell y + kx\cdot\partial_x\beta + \ell y\cdot\partial_y\beta)b\circ\beta\right)\\ &= u\left((kx + (k+m)\alpha)a\circ\phi + (\ell y + (\ell+m)\beta)b\circ\phi\right).\end{aligned}$$

We also have

$$\begin{aligned}a\circ\phi - a &= \sum_{ki+\ell j\geq d-k} a_{ij}(x+\alpha)^i(y+\beta)^j - \sum_{ki+\ell j\geq d-k} a_{ij}x^iy^j\\ &= \sum_{ki+\ell j\geq d-k} a_{ij}ix^{i-1}\alpha y^j + \sum_{ki+\ell j\geq d-k} a_{ij}x^ijy^{j-1}\beta + h.o.t.\\ &= \alpha\partial_x a + \beta\partial_y a + h.o.t., \qquad (1.2.3)\\ b\circ\phi - b &= \sum_{ki+\ell j\geq d-\ell} b_{ij}(x+\alpha)^i(y+\beta)^j - \sum_{ki+\ell j\geq d-\ell} b_{ij}x^iy^j\\ &= \sum_{ki+\ell j\geq d-\ell} b_{ij}ix^{i-1}\alpha y^j + \sum_{ki+\ell j\geq d-\ell} b_{ij}x^ijy^{j-1}\beta + h.o.t.\\ &= \alpha\partial_x b + \beta\partial_y b + h.o.t.. \qquad (1.2.4)\end{aligned}$$

It follows that $\Delta q_{d+m'} = 0$ for all $0 \leq m' < m$ and

$$\begin{aligned}\Delta q_{d+m} &= kx(\alpha\partial_x a + \beta\partial_y a) + \ell y(\alpha\partial_x b + \beta\partial_y b) + (k+m)\alpha a + (\ell+m)\beta b + \delta q_d\\ &= ((k+m)a + kx\partial_x a + \ell y\partial_x b)\alpha + ((\ell+m)b + kx\partial_y a + \ell y\partial_y b)\beta + \delta q_d\\ &= \left(ma_{d-k} + \partial_x q_d\right)\alpha + \left(mb_{d-\ell} + \partial_y q_d\right)\beta + \delta q_d.\end{aligned}$$

Therefore, we obtain (1.2.1) by substituting

$$q_d = \frac{k}{d+m}(ma_{d-k} + \partial_x q_d) + \frac{\ell}{d+m}(mb_{d-\ell} + \partial_y q_d).$$

Using again (1.2.3) and (1.2.4), we obtain that $\Delta b_{d-l+m'} = 0$ for all $0 \leq m' < m$ and

$$\Delta b_{d-l+m} = \alpha\partial_x b_{d-\ell} + \beta\partial_y b_{d-\ell} + \partial_y\alpha a_{d-k} + \partial_y\beta b_{d-\ell} + \delta b_{d-\ell}. \tag{1.2.5}$$

Substituting $\alpha = U - \frac{k}{d+m}x\delta$ and $\beta = V - \frac{\ell}{d+m}y\delta$ into (1.2.5), we get

$$\Delta b_{d-l+m} = \partial_x b_{d-\ell}U + \partial_y b_{d-\ell}V + a_{d-k}\partial_y U + b_{d-\ell}\partial_y V + \frac{1}{d+m}\left(m b_{d-\ell}\delta - q_d \partial_y \delta\right).$$

This equality implies (1.2.2) by using the fact that

$$\partial_x b_{d-\ell}U + \partial_y b_{d-\ell}V + a_{d-k}\partial_y U + b_{d-\ell}\partial_y V = \partial_y(a_{d-k}U + b_{d-\ell}V) - (\partial_y a_{d-k} - \partial_x b_{d-\ell})U.$$

□

Denote by $[a[$ the usual integer part a: $[a[\leq a < [a[+1$, and $]a]$ the strict integer part of a defined by $]a] < a \leq]a] + 1$. Then the number of integer points in a closed interval $[a, b]$ is given by $[b[-]a]$.

Lemma 1.2.2. *Let e_m be the cardinality of the set $\{(i,j) \in \mathbb{N}^2 : ki + \ell j = m\}$. Then $e_m = [\frac{mu}{\ell}[-]\frac{mv}{k}]$.*

Proof. Denote by $e'_m = [\frac{mu}{\ell}[-]\frac{mv}{k}]$ the number of integer points in the closed interval $[\frac{mu}{\ell}, \frac{mv}{k}]$. For each integer c in $[\frac{mu}{\ell}, \frac{mv}{k}]$. Let us denote $i = mu - c\ell$, $j = ck - mv$, then

$$ki + \ell j = kmu - ck\ell + ck\ell - \ell mv = m(ku - \ell v) = m.$$

Consequently, $e'_m \leq e_m$. Now, if there exist two positive integers i, j such that $ki + \ell j = m$ then

$$mu = kui + \ell uj = (\ell v + 1)i + \ell uj = \ell(vi + uj) + i,$$
$$mv = kvi + \ell vj = kvi + (ku - 1)j = k(vi + uj) - j.$$

Therefore, we have

$$vi + uj = \frac{mu - i}{\ell} = \frac{mv + j}{k} \in [\frac{mu}{\ell}, \frac{mv}{k}].$$

It implies that $e_m \leq e'_m$. Thus, $e_m = [\frac{mu}{\ell}[-]\frac{mv}{k}]$. □

Lemma 1.2.3. *If $\lambda_i \notin \mathbb{Q}$ for $i = 0, 1, \ldots, n-1, n, \infty$ then $\gcd(A, B) = 1$ for all $m \in \mathbb{N}$.*

Proof. By Lemma 1.1.9,

$$A = m a_{d-k} + \partial_x q_d = q_d \left(\sum_{i=1}^{n} (m\lambda_i - 1) \frac{\ell c_i x^{\ell-1}}{y^k - c_i x^\ell} + \frac{1}{x}(u - \ell(\lambda_0 + 1) + 1) \right),$$
$$B = m b_{d-\ell} + \partial_y q_d = q_d \left(\sum_{i=1}^{n} (1 - m\lambda_i) \frac{k y^{k-1}}{y^k - c_i x^\ell} + \frac{1}{y}(1 - (v + k\lambda_\infty)) \right).$$

Suppose that $g = \gcd(A, B)$. Since $\lambda_i \notin \mathbb{Q}$, we have $\gcd(A, x) = 1$, $\gcd(B, y) = 1$ and $\gcd(A, y^k - c_i x^\ell) = 1$ for all $i = 1, \ldots, n$. Therefore $\gcd(g, q_d) = 1$. Moreover, we also have $g | q_d$ since $kxA + \ell yB = (d+m)q_d$. It implies that $\gcd(A, B) = 1$. □

The following lemma will be used to normalize the hamiltonian part:

Notation 1.2.4. We say that a 1-form $\omega_d \in \mathcal{Q}^d(f)$ satisfies the *generic condition* if $\lambda_i \notin \mathbb{Q}$ for all $i = 0, 1, \ldots, n-1, n, \infty$ and the coefficients of A and B satisfy the condition of non-vanishing determinant of the matrix M_m in (1.2.7) for $m = 1, \ldots, k\ell n - 1$.

Lemma 1.2.5. *Let $\omega_d \in \mathcal{Q}^d(f)$ satisfy the generic condition and $\omega \in \mathcal{Q}(\omega_d)$. Using the same notation as in Lemma 1.2.1, for each $m \geq 1$ there exist a diffeomorphism $\phi(x,y) = (x + x\alpha, y + y\beta)$ and a unity $u = 1 + \delta$ where α, β, δ are quasi-homogeneous polynomials of degree m such that $\tilde{q}_{d+m} = q_{d+m} + AU + BV = xy\tilde{q}'_{d+m}$ satisfies the conditions*

- $deg_x \tilde{q}'_{d+m} \leq \ell n - 1$ *and* $deg_y \tilde{q}'_{d+m} \leq kn - 1$ *if* $1 \leq m \leq k\ell n - 1$,
- $\tilde{q}_{d+m} = 0$ *if* $m \geq k\ell n$.

Proof. Denote by $QP(m)$ the set of all (k,ℓ)-quasi-homogeneous polynomials of degrees m. Then $QP(m)$ is a vector space of dimension e_m. For simplicity of notation, we denote $\bar{A} = \frac{A}{y}$, $\bar{B} = \frac{B}{x}$, $\bar{U} = \frac{U}{x}$, $\bar{V} = \frac{V}{y}$. By Lemma 1.2.1, we have

$$\frac{\tilde{q}_{d+m}}{xy} = \frac{q_{d+m}}{xy} + \bar{A}\bar{U} + \bar{B}\bar{V}.$$

Consider the linear map

$$\Psi_m : QP(m) \times QP(m) \to QP(k\ell n + m)$$

$$(\bar{U}, \bar{V}) \mapsto \bar{A}\bar{U} + \bar{B}\bar{V}.$$

Case $m \geq k\ell n$. By Lemma 1.2.3, $\bar{A}$ and $\bar{B}$ are coprime. It implies that

$$\text{Ker}\Psi_m = \{(Z\bar{B}, -Z\bar{A}), Z \in QP(m - k\ell n)\}. \qquad (1.2.6)$$

We obtain the surjectivity of Ψ_m due to the following equality of dimensions of vector spaces

$$\begin{aligned} \text{dimIm}\Psi_m &= \text{dim}QP(m) \times QPm - \text{dimKer}\Psi_m \\ &= e_m + e_m - e_{m+k\ell n} = e_{m-k\ell n} \\ &= \text{dim}QP(k\ell n + m). \end{aligned}$$

Consequently, there exists ϕ such that $\tilde{q}_{d+m} = AU + BV + q_{d+m} = 0$.

Case $1 \leq m \leq k\ell n - 1$. In this case, $\text{Ker}\Psi_m = \{0\}$. Denote by $NQP(k\ell n + m)$ the subspace of $QP(k\ell n + m)$ generalized by all the monomials $g(x,y)$ satisfying

$$deg_x g \leq \ell n - 1, \; deg_y g \leq kn - 1.$$

We also denote by $NQP^{\perp}(k\ell n+m)$ the subspace of $QP(k\ell n+m)$ generalized by all the monomials $g(x,y)$ such that

$$deg_x g \geq \ell n \text{ or } deg_y g \geq kn.$$

Denote by pr_m the standard projection

$$pr_m : QP(k\ell n+m) \rightarrow NQP^{\perp}(k\ell n+m).$$

The proof is reduced to show that in a generic condition for all $q \in QP(k\ell n+m)$ there exists $\bar{A}\bar{U}+\bar{B}\bar{V} \in \mathrm{Im}\Psi_m$ such that

$$q+\bar{A}\bar{U}+\bar{B}\bar{V} \in NQP(k\ell n+m).$$

Since

$$\dim NQP^{\perp}(k\ell n+m) = 2e_m = \dim QP(m) \times QP(m),$$

this is equivalent to prove that in a generic condition $pr_m \circ \Psi_m$ is bijective.

Because $\bar{A},\bar{B} \in QP(k\ell n)$, we can write $\bar{A}=\sum_{i=0}^{n} A_i x^{\ell(n-i)} y^{ki}$, $\bar{B}=\sum_{i=0}^{n} B_i x^{\ell i} y^{k(n-i)}$. Then the matrix representation M_m of the linear map $pr_m \circ \Psi_m$ is given by (1.2.7)

$$M_m = \left[\begin{array}{ccccc|ccccc}
A_0 & 0 & \dots & 0 & 0 & B_n & 0 & \dots & 0 & 0 \\
A_1 & A_0 & \dots & 0 & 0 & B_{n-1} & B_n & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
A_{e_m-2} & A_{e_m-3} & \dots & A_0 & 0 & B_{n-e_m+2} & B_{n-e_m+3} & \dots & B_n & 0 \\
A_{e_m-1} & A_{e_m-2} & \dots & A_1 & A_0 & B_{n-e_m+1} & B_{n-e_m+2} & \dots & B_{n-1} & B_n \\
\hline
A_n & A_{n-1} & \dots & A_{n-e_m+2} & A_{n-e_m+1} & B_0 & B_1 & \dots & B_{e_m-2} & B_{e_m-1} \\
0 & A_n & \dots & A_{n-e_m+3} & A_{n-e_m+2} & 0 & B_0 & \dots & B_{e_m-3} & B_{e_m-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & A_n & A_{n-1} & 0 & 0 & \dots & B_0 & B_1 \\
0 & 0 & \dots & 0 & A_n & 0 & 0 & \dots & 0 & B_0
\end{array}\right]$$

The determinant $\det M_m$ is a polynomial in A_i and B_j. Since $\frac{\partial^{2e_m}\det M_m}{(\partial A_0)^{e_m}(\partial B_0)^{e_m}} = (e_m!)^2 \neq 0$, such polynomial is not identically zero. Therefore, the condition $\det M_m \neq 0$ is satisfied for generic $\omega_d \in \mathcal{Q}^d(f)$. □

The following lemma will be used to normalize the radial part:

Lemma 1.2.6. *If $\omega_d \in \mathcal{Q}^d(f)$ satisfies $\lambda_i \notin \mathbb{Q}$ for all $i = 0,1,\dots,n-1,n,\infty$, then there exist $\phi(x,y) = (x+x\alpha, y+y\beta)$ and $u(x,y) = 1+\delta(x,y)$ where α,β and δ are quasi-homogeneous polynomials of degree m such that $\Delta q_{d+m} = 0$ and $\tilde{b}_{d+m-\ell} = b_{d+m-\ell} + \Delta b_{d+m-\ell}$ satisfies the following condition*

$$deg_y \tilde{b}_{d+m-\ell} \leq kn-1.$$

Proof. Suppose that $\phi(x,y) = (x+x\alpha, y+y\beta)$ and $u(x,y) = 1+\delta(x,y)$ where α,β and δ are quasi-homogeneous polynomials of degree m such that $\Delta q_{d+m} = 0$. By the proof of Lemma 1.2.5, we have $(\bar{U},\bar{V}) \in \mathrm{Ker}\Psi_m$. It implies that $(\bar{U},\bar{V}) = (Z\bar{B}, -Z\bar{A})$

where $Z = 0$ if $m \le k\ell n$ and $Z \in QP(m-k\ell n)$ if $m > k\ell n$. Therefore, W in Lemma 1.2.1 can be written as follows:

$$\begin{aligned} W &= \partial_y(a_{d-k}xZ\bar{B} - b_{d-\ell}yZ\bar{A}) - (\partial_y a_{d-k} - \partial_x b_{d-\ell})xZ\bar{B} \\ &= \partial_y(Z(a_{d-k}B - b_{d-\ell}A)) - (\partial_y a_{d-k} - \partial_x b_{d-\ell})ZB. \end{aligned}$$

We denote $C = \partial_y a_{d-k} - \partial_x b_{d-\ell}$. Then

$$\begin{aligned} a_{d-k}B - b_{d-\ell}A &= a_{d-k}(mb_{d-k} + \partial_y q_d) - b_{d-\ell}(ma_{d-k} + \partial_x q_d) \\ &= a_{d-k}\partial_y q_d - b_{d-\ell}\partial_x q_d \\ &= \frac{(q_d - \ell y b_{d-\ell})\partial_y q_d - kxb_{d-\ell}\partial_x q_d}{kx} \\ &= \frac{q_d\partial_y q_d - b_{d-\ell}(\ell y\partial_y q_d + kx\partial_x q_d)}{kx} \\ &= \frac{q_d\partial_y q_d - db_{d-\ell}q_d}{kx} \\ &= \frac{q_d(\partial_y q_d - db_{d-\ell})}{kx} \\ &= \frac{q_d(kx\partial_y a_{d-k} + \ell y\partial_y b_{d-\ell} - (d-\ell)b_{d-\ell})}{kx} \\ &= \frac{q_d(kx\partial_y a_{d-k} - kx\partial_y b_{d-\ell})}{kx} \\ &= q_d(\partial_y a_{d-k} - \partial_y b_{d-\ell}) \\ &= q_d C. \end{aligned}$$

Therefore, Δb_{d+m-l} can be written as

$$\begin{aligned} \Delta b_{d+m-l} &= \partial_y(Zq_dC) - CZB + \frac{mb_{d-\ell}\delta - q_d\partial_y\delta}{d+m} \\ &= (\partial_y q_d - B)ZC + q_d(\partial_y(ZC)) + \frac{mb_{d-\ell}\delta - q_d\partial_y\delta}{d+m} \\ &= mb_{d-\ell}\left(\frac{\delta}{d+m} - ZC\right) - q_d\partial_y\left(\frac{\delta}{d+m} - ZC\right). \end{aligned}$$

Consider the linear map

$$\Phi_m : QP(m) \to QP(k\ell n + m)$$

$$T \mapsto m\frac{b_{d-\ell}}{x}T - \frac{q_d}{x}\partial_y T.$$

We claim that Φ_m is injective. Indeed, assume that there exists $T \in \mathrm{Ker}\Phi_m$ and $T \ne 0$. Decompose $T = r\left(\frac{q_d}{x}\right)^\gamma$ where $\gamma \in \mathbb{N}$ and r does not divide $\frac{q_d}{x}$. It follows from $T \in \mathrm{Ker}\Phi_m$ that

$$r(mb_{d-\ell} - \gamma\partial_y q_d) = q_d\partial_y r. \tag{1.2.8}$$

By Lemma 1.1.9,

$$mb_{d-\ell} - \gamma\partial_y q_d = -q_d\left(\sum_{i=1}^{n}(m\lambda_i+\gamma)\frac{ky^{k-1}}{y^k - c_i x^l} + \frac{m(k\lambda_\infty + v)+\gamma)}{y}\right).$$

If $\lambda_i \notin \mathbb{Q}$ for all $i = 1, \ldots, n, \infty$ then $\frac{1}{x}(mb_{d-\ell} - \gamma\partial_y q_d)$ and $\frac{q_d}{x}$ are coprime. It implies that $r\frac{(mb_{d-\ell}-\gamma\partial_y q_d)}{x}$ does not divide $\frac{q_d}{x}$ and this is contradictory to (1.2.8).

Denote by $NQP_y(k\ell n + m)$ the subspace of $QP(k\ell n + m)$ generalized by the monomials $g(x, y)$ such that

$$deg_y g \leq kn - 1.$$

Then for all $g \in NQP_y(k\ell n + m)$ we have

$$x^{[\frac{m}{k}[+1}|g(x, y). \tag{1.2.9}$$

Denote by $NQP_y^{\perp}(k\ell n+m)$ the subspace of $QP(k\ell n+m)$ generalized by the monomials $g(x, y)$ such that

$$deg_y g \geq kn.$$

We also denote by pr_{ym} the standard projection

$$pr_{ym} : QP(k\ell n + m) \to NQP_y^{\perp}(k\ell n + m).$$

The proof is done if we can show that $pr_{ym} \circ \Phi_m$ is bijective from $QP(m)$ to $NQP_y^{\perp}(k\ell n + m)$. It reduces to show that $pr_{ym} \circ \Phi_m$ is injective due to the following equality

$$\dim NQP_y^{\perp}(k\ell n + m) = e_m = \dim QP(m).$$

We claim that this is equivalent to prove that

$$\mathrm{Im}\Phi_m \cap NPQ_y(k\ell n + m) = \{0\}. \tag{1.2.10}$$

Indeed, if (1.2.10) is true, suppose that $g \in \mathrm{Ker}(pr_{ym}\circ\Phi_m)$. Then $\Phi_m(g) \in NPQ_y(k\ell n+m)$ which implies that $\Phi_m(g) = 0$. It follows by the injectivity of Φ_m that $g = 0$.

Now, let us prove (1.2.10). Assume that there exists a non-zero element T of $QP(m)$ such that

$$\Phi_m(T) \in NPQ_y(k\ell n + m).$$

Decompose $T = x^\theta \bar{T}$ where x and $\bar{T}$ are coprime. Because $T \in QP(m)$, we have $\theta \leq [\frac{m}{k}[$. By (1.2.9), x is a divisor of $m\frac{b_{d-\ell}}{x}\bar{T} - \frac{q_d}{x}\partial_y\bar{T}$. By Lemma 1.1.9, we have

$$m\frac{b_{d-\ell}}{x}(0, y)\bar{T}(0, y) - \frac{q_d}{x}(0, y)\partial_y\bar{T}(0, y) = \tag{1.2.11}$$

$$c_0 y^{nk}\left(m(v - k(1 + \lambda_0))\bar{T}(0, y) - y\partial_y\bar{T}(0, y)\right)$$

It is a contradiction because condition $\lambda_0 \notin \mathbb{Q}$ forces that the right hand side of (1.2.11) is different from 0 for all non-zero polynomial $\bar{T}(0, y)$. □

Theorem A. For generic $\omega_d \in \mathcal{Q}^d(f)$, each germ $\omega \in \mathcal{Q}(w_d)$ is strictly formally orbitally equivalent to a unique form $\omega_{h,s}$

$$\omega_{h,s} = \omega_d + dh + s(\ell y dx - kx dy),$$

where

$$h(x,y) = xy \sum_{\substack{ki+\ell j \geq k\ell n+1 \\ 0 \leq i \leq \ell n-1 \\ 0 \leq j \leq kn-1}} h_{ij} x^i y^j,\ s(x,y) = \sum_{j=0}^{kn-1} s_j(x) x^{\ell n+1+]\frac{1-\ell j}{k}]} y^j,$$

$s_i(x)$ are formal series on x.

Proof. By Corollary 1.1.11, we can decompose

$$\omega = \omega_d + dh + s(\ell y dx - kx dy),$$

where $h_{i+k+\ell} = \frac{q_{i+k+\ell}}{i+k+\ell} = xy\frac{q'_i}{i+k+\ell}$, $q' = \frac{q}{xy}$ and s is given as in (1.1.17). Let us rewrite $s(x,y)$ as follows

$$\begin{aligned} s_i &= \frac{\partial_y a_{i+\ell} + \partial_x b_{i+k}}{i+k+\ell} \\ &= \frac{1}{i+k+\ell}\left(\frac{\partial_y(q_{i+k+\ell} - \ell y b_{i+k})}{kx} - \partial_x b_{i+k}\right) \\ &= \frac{\partial_y q_{i+k+\ell}}{(i+k+\ell)kx} - \frac{\ell b_{i+k} + \ell y \partial_y b_{i+k} + kx\partial_x b_{i+k}}{(i+k+\ell)kx} \\ &= \frac{\partial_y q_{i+k+\ell}}{(i+k+\ell)kx} - \frac{b_{i+k}}{kx}. \end{aligned}$$

By Lemma 1.2.5 and 1.2.6, we can eliminate all the monomials $x^i y^j$, $i \geq \ell n$, $j \geq kn$ in the components of q' and all the monomials $x^i y^j$, $j \geq kn$ in the components of b. This implies that we can normalize h and s such that

$$h(x,y) = xy \sum_{\substack{ki+\ell j \geq k\ell n+1 \\ 0 \leq i \leq \ell n-1 \\ 0 \leq j \leq kn-1}} h_{ij} x^i y^j,\ s(x,y) = \sum_{j=0}^{kn-1} s'_j(x) y^j,$$

where $s'_j(x)$ are formal series of x. Because $s(x,y)$ only contains the monomials of degree at least $k\ell n+1$, for each $j = 0, \dots, kn-1$ $s'_j(x)$ divides x^{i_j} where i_j is the minimal integer such that $ki_j + \ell j \geq k\ell n + 1$ and. Moreover, $ki_j + \ell j \geq k\ell n + 1$ if and only if $i_j \geq \ell n + 1 + \left]\frac{1-\ell j}{k}\right]$. Therefore, we have

$$s'_j(x) = s_j(x) x^{\ell n+1+]\frac{1-\ell j}{k}]},$$

where $s'_j(x)$ are formal series of x. The uniqueness part is straightforward by the uniqueness in Lemma 1.2.5 and 1.2.6. $\square$

Remark 1.2.7. Since every formal diffeomorphism ϕ can be decomposed as $\phi = \phi' \circ \phi_0$ where ϕ_0 is a linear transformation and ϕ' is tangent to identity, the formal normal form for the case in which we do not require the strict condition can be easily obtained. The slightly difference is in the initial part. For the strict case we have n free coefficients corresponding to the coordinates of the non-corner singularities in the principal component of the divisor, but for the general case we can normalize one of them and let the others free.

Exemple 1.2.8. For $n = 2, k = 3$, $\ell = 2$, the strict formal normal form is given by

$$\omega_d + dh + s(\ell y dx - kx dy),$$

where

$$\begin{aligned}\omega_d =& c_0 xy(y^3 - c_1x^2)(y^3 - c_2x^2)\\ &\times \left(\lambda_1 \frac{d(y^3 - c_1x^2)}{y^3 - c_1x^2} + \lambda_2 \frac{d(y^3 - c_2x^2)}{y^3 - c_2x^2} + (2\lambda_0 + 1)\frac{dx}{x} + (3\lambda_\infty + 1)\frac{dy}{y}\right),\end{aligned}$$

$$h(x, y) = xy(h_{3,2}x^3y^2 + h_{1,5}xy^5 + h_{2,5}x^2y^4 + h_{3,3}x^3y^3 + h_{2,5}x^2y^5 + h_{3,4}x^3y^4 + h_{3,5}x^3y^5),$$

$$s(x, y) = s_0(x)x^5 + s_1(x)x^4y + s_2(x)x^3y^2 + s_3(x)x^3y^3 + s_4(x)x^2y^4 + s_5(x)xy^5.$$

In the formula of ω_d we have two free coefficients c_1, c_2 corresponding to the position of two singularities of the cuspidal branches. The formal normal form (non-strict) is given by

$$\bar{\omega}_d + dh' + s'(\ell y dx - kx dy),$$

where h' and s' have the same form as h and s respectively, but we can normalize one singularity, which has the coordinates $(x_c, y_c) = (1, 0)$. Thus, $\bar{\omega}_d$ has only one free coefficient c:

$$\begin{aligned}\bar{\omega}_d =& xy(y^3 - x^2)(y^3 - cx^2)\\ &\times \left(\lambda_1 \frac{d(y^3 - x^2)}{y^3 - x^2} + \lambda_2 \frac{d(y^3 - cx^2)}{y^3 - cx^2} + (2\lambda_0 + 1)\frac{dx}{x} + (3\lambda_\infty + 1)\frac{dy}{y}\right).\end{aligned}$$

Remark 1.2.9. The number of free coefficients of h in the normal form is consistent with the dimension of Mattei's moduli space. Indeed, for $m = 1, \ldots, k\ell n - 1$ we have

$$\begin{aligned}\#\{(i, j) \in \mathbb{N}^2 | ki + \ell j = k\ell n + m, i \leq \ell n - 1, j \leq kn - 1\} &= e_{k\ell n+m} - 2e_m\\ &= n - e_m\end{aligned}$$

Therefore, the number of free coefficients of h is given by

$$\delta'(\omega) = \sum_{m=1}^{k\ell n-1} (n - e_m) = n(k\ell n - 1) - \sum_{m=1}^{k\ell n-1} e_m.$$

By [8], the dimension of Mattei's moduli space is given by

$$\begin{aligned}\delta(\omega) &= \sum_{m=0}^{k\ell n-1}\left( ]\frac{\ell-u}{\ell}(m-k\ell n)] - [\frac{k-v}{k}(m-k\ell n[\right) \\ &= \sum_{m=0}^{k\ell n-1}\left(n+] - \frac{mu}{\ell}] - [-\frac{mv}{k}[\right).\end{aligned}$$

We will show that for any real number a, $[a[+]-a] = -1$. Indeed, since $[a[\leq a < [a[+1$, $]-a] < -a \leq]-a]+1$, we have

$$[a[+]-a] < 0 < [a[+]-a]+2.$$

Therefore, $[a[+]-a] = -1$. It follows that

$$\delta(\omega) = \sum_{m=0}^{k\ell n-1}\left(n-]\frac{mu}{\ell}] + [\frac{mv}{k}[\right) = n^2k\ell - \sum_{m=0}^{k\ell n-1} e_m.$$

The difference of $\delta(\omega)$ and $\delta'(\omega)$ is given by

$$\delta(\omega) - \delta'(\omega) = n - e_0 = n - 1.$$

The existence of this difference comes from the fact that we just consider the strict conjugation. The number $n-1$ corresponds to the number of free coefficients of the position of non-corner singularities in the non-strict formal normal form.

Chapter 2

Sliding invariants and strict classification of holomorphic foliations

In this chapter, by adding a new invariant called the *set of slidings*, we solve the problem of the strict classification for the non-dicritical foliations whose Camacho-Sad indices are not rational. Here, the strict classification means up to diffeomorphisms tangent to identity.

2.1 Absolutely dicritical foliations

In this section, we will prove that for each non-dicritical foliation there exists an absolutely dicritical foliation which is transverse to its separatrices. This existence is the key point to define our new invariant.

Let us first recall the notation of absolutely dicritical foliations. Let σ be a composition of a finite number of blowing-ups at points:

$$\sigma : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0). \tag{2.1.1}$$

A germ of singular holomorphic foliation $\mathcal{L}$ is said σ-*absolutely dicritical* if the strict transform $\tilde{\mathcal{L}} = \sigma^*(\mathcal{L})$ is a regular foliation and the exceptional divisor $\mathcal{D} = \sigma^{-1}(0)$ is completely transverse to $\tilde{\mathcal{L}}$. In particular, when σ is the standard blowing-up at the origin, we called $\mathcal{L}$ a *radial foliation.* At each corner $p = D_i \cap D_j$ of $\mathcal{D}$, the diffeomorphism from (D_i, p) to (D_j, p) that follows the leaves of $\tilde{\mathcal{L}}$ is called the *Dulac map* of $\tilde{\mathcal{L}}$ at p. The existence of such foliations for any given σ is proved in [4]. In fact, in [4] the authors showed that if in each smooth component of $\mathcal{D}$ we take any two smooth curves transverse to $\mathcal{D}$ then there is always an absolutely dicritical foliation admitting them as their integral curves. We will denote by $\mathrm{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}$ if at any point $p \in \mathrm{Sing}(\tilde{\mathcal{F}})$ the separatrices of $\tilde{\mathcal{F}}$ through p are transverse to $\tilde{\mathcal{L}}$ (figure 2.1).

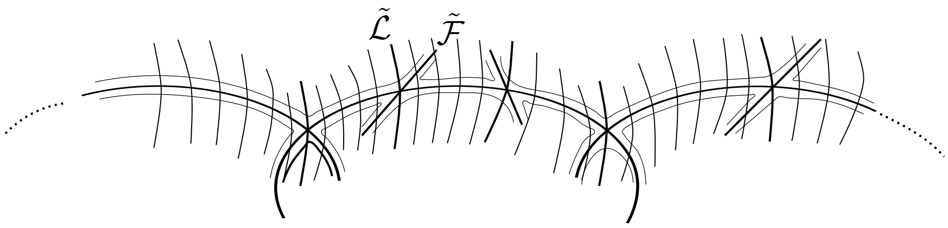

Figure 2.1: $\mathrm{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}$

Lemma 2.1.1. *Let $\mathcal{F}$ be a non-dicritical foliation and σ be its desingularization map. Then there exists a σ-absolutely dicritical foliation $\mathcal{L}$ satisfying* $\mathrm{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}$

Proof. Denote by $L_1, \ldots, L_k$ the strict transforms of the separatrices of $\mathcal{F}$. On each component D of $\mathcal{D}$ which does not contain any singularity of $\tilde{\mathcal{F}}$ except the corners, we take a smooth curve L_{k+j} transverse to D. Then we have the set of curves $\{L_1, \ldots, L_k, \ldots, L_n\}$ such that each component of $\mathcal{D}$ is transverse to at least one curve L_i. We denote $p_i = L_i \cap \mathcal{D}$. By [4], for each i there exists a σ-absolutely dicritical foliation $\mathcal{L}_i$ defined by a 1-form ω_i verifying that L_i is transverse to $\tilde{\mathcal{L}}_i$ at a neighborhood of p_i. Choose a local chart (x_i, y_i) at p_i such that $\mathcal{D} = \{x_i = 0\}, L_i = \{y_i = 0\}$ and

$$\sigma^*\omega_i(x_i, y_i) = x_i^{m_i} d(x_i + y_i) + h.o.t..$$

Write down ω_i in the local chart (x_j, y_j)

$$\sigma^*\omega_i(x_j, y_j) = x_j^{m_j} d(a_{ij}x_j + b_{ij}y_j) + h.o.t.. \tag{2.1.2}$$

Let us define $a_{ii} = b_{ii} = 1$. For each $j = 1, \ldots, n$, consider two subsets A_j, B_j of $\mathbb{C}^n$ given by

$$A_j = \{(c_1, \ldots, c_n) \in \mathbb{C}^n | \sum_{i=1}^{n} c_i a_{ij} \neq 0\},\ B_j = \{(c_1, \ldots, c_n) \in \mathbb{C}^n | \sum_{i=1}^{n} c_i b_{ij} \neq 0\}.$$

Because $a_{jj} = b_{jj} = 1 \neq 0$, A_j and B_j are dense open subsets of $\mathbb{C}^n$. Therefore, $A_1 \cap \ldots \cap A_n \cap B_1 \cap \ldots \cap B_n \neq \emptyset$. This implies that there exists a vector $(c_1, \ldots, c_n) \in \mathbb{C}^n$ such that for $j = 1, \ldots, n$,

$$a_j = \sum_{i=1}^{n} c_i a_{ij} \neq 0 \text{ and } b_j = \sum_{i=1}^{n} c_i b_{ij} \neq 0.$$

Denote by $\omega_0 = \sum c_i \omega_i$. Then, in the local chart (x_j, y_j), we have

$$\sigma^*\omega_0(x_j, y_j) = x_j^{m_j} d(a_j x_j + b_j y_j) + h.o.t., \text{ where } a_j \neq 0, b_j \neq 0.$$

Because ω_0 and ω_i, $i = 1, \ldots, n$, have the same multiplicity on each component of $\mathcal{D}$, they have the same vanishing order. Denote by $\mathcal{L}$ the foliation defined by ω_0. Since each component of $\mathcal{D}$ contains at least one point p_i and the strict transform $\tilde{\mathcal{L}}$ of $\mathcal{L}$ is transverse to $\mathcal{D}$ in each neighborhood of each p_i, $\tilde{\mathcal{L}}$ is generically transverse to $\mathcal{D}$. By [4], $\mathcal{L}$ is an absolutely dicritical foliation which satisfies $\mathrm{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}$. □

2.2 Definition of the slidings of foliations

As mentioned in the previous section, a sliding of a foliation $\mathcal{F}$ is defined through an absolutely dicritical foliation $\mathcal{L}$ satisfying $\text{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}$. If $\mathcal{F}$ is a nondegenerate reduced foliation, the desingularization map σ is the identity map. Hence, this absolutely dicritical foliation becomes a regular foliation transverse to the two separatrices of $\mathcal{F}$.

2.2.1 Nondegenerate reduced case

Let $\mathcal{F}$ be a germ of nondegenerate reduced foliation in $(\mathbb{C}^2, 0)$. By [13], there exists a coordinate system in which $\mathcal{F}$ is defined by

$$\lambda y(1 + A(x, y))dx + xdy, \ \lambda \notin \mathbb{Q}_{\leq 0}, \tag{2.2.1}$$

where $A(0, 0) = 0$. Let $\mathcal{L}$ be a germ regular foliation whose invariant curve through the origin (we call it the separatrix of $\mathcal{L}$) is transverse to the two separatrices of $\mathcal{F}$, which are denoted by S_1 and S_2. Then we have the following lemma:

Lemma 2.2.1. *The tangent curve of $\mathcal{F}$ and $\mathcal{L}$, denoted $T(\mathcal{F}, \mathcal{L})$, is a smooth curve transverse to the two separatrices of $\mathcal{F}$. Moreover, if the separatrix of $\mathcal{L}$ is tangent to $\{x + cy = 0\}$ then $T(\mathcal{F}, \mathcal{L})$ is tangent to $\{x - c\lambda y = 0\}$.*

Proof. Suppose that $\mathcal{L}$ is defined by the level sets of $x+cy+\ell(x, y)$, where $\ell \in (x, y)^2$. Then $T(\mathcal{F}, \mathcal{L})$ is zero locus of the following function:

$$\frac{1}{dx \wedge dy} d(x + cy + \ell(x, y)) \wedge (\lambda y(1 + A(x, y))dx + xdy) = x - c\lambda y + h.o.t.$$

which is transverse to two separatrices of $\mathcal{F}$. □

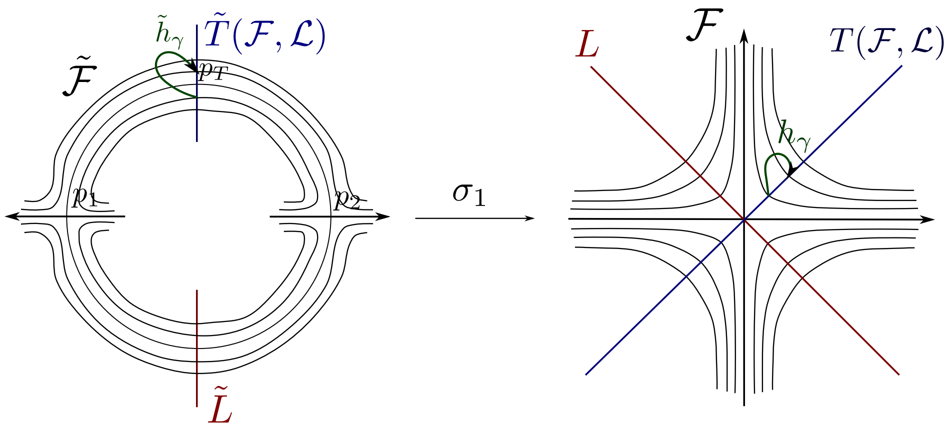

Figure 2.2: Holonomy on the tangent curve $T(\mathcal{F}, \mathcal{L})$

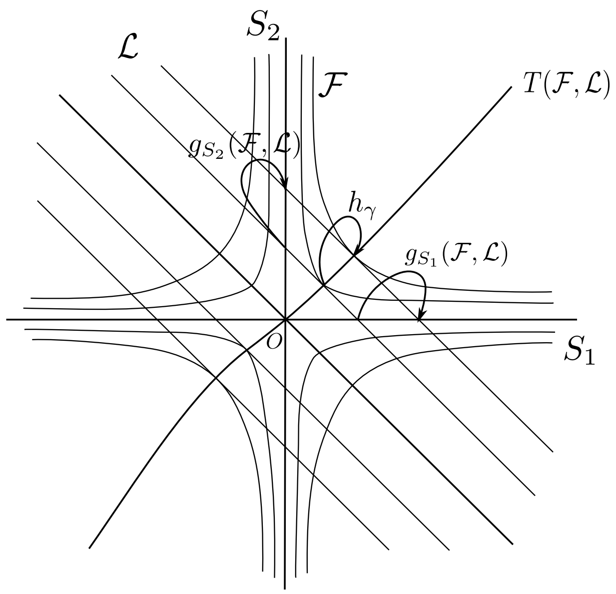

Figure 2.3: Sliding of $\mathcal{F}$ and $\mathcal{L}$

After a standard blowing-up σ_1 at the origin, the strict transform $\tilde{T}(\mathcal{F},\mathcal{L})$ of $T(\mathcal{F},\mathcal{L})$ is transverse to $\tilde{\mathcal{F}}$ and cut $D_1 = \sigma_1^{-1}(0)$ at p. We denote by $D_1^* = D_1 \setminus \mathrm{Sing}(\sigma_1^*(\mathcal{F}))$ and $\tilde{h} : \pi_1(D_1^*, p) \to \mathrm{Diff}(\tilde{T}(\mathcal{F},\mathcal{L}), p)$ the vanishing holonomy representation of $\mathcal{F}$. We choose a generator γ for $\pi_1(D_1^*, p) \cong \mathbb{Z}$. Then σ_1 induces

$$h_\gamma = \sigma_1 \circ \tilde{h}(\gamma) \circ \sigma_1^{-1} \in \mathrm{Diff}(T(\mathcal{F},\mathcal{L}), 0).$$

We call h_γ the *holonomy on the tangent curve* $T(\mathcal{F},\mathcal{L})$ (figure 2.2). Denote by π_{S_1} and π_{S_2} the projection by the leaves of $\mathcal{L}$ from $T(\mathcal{F},\mathcal{L})$ to S_1 and S_2 respectively.

Definition 2.2.2. The *sliding* of a reduced foliation $\mathcal{F}$ and a regular foliation $\mathcal{L}$ on S_1 (resp., S_2) is the diffeomorphism

$$g_{S_1}(\mathcal{F},\mathcal{L}) = \pi_{S_1*}(h_\gamma) = \pi_{S_1} \circ h_\gamma \circ \pi_{S_1}^{-1}$$
$$\Big(\text{resp., } g_{S_2}(\mathcal{F},\mathcal{L}) = \pi_{S_2*}(h_\gamma) = \pi_{S_2} \circ h_\gamma \circ \pi_{S_2}^{-1}\Big).$$

Let $d : S_1 \to S_2$ be the Dulac map of $\mathcal{L}$ (Section 2.1). Since $d = \pi_{S_2} \circ \pi_{S_1}^{-1}$, it is obvious that the sliding $g_{S_2}(\mathcal{F},\mathcal{L})$ is totally determined by $g_{S_1}(\mathcal{F},\mathcal{L})$ and the Dulac map by the following relation

$$g_{S_2}(\mathcal{F},\mathcal{L}) = d_*\left(g_{S_1}(\mathcal{F},\mathcal{L})\right) = d \circ g_{S_1}(\mathcal{F},\mathcal{L}) \circ d^{-1}. \tag{2.2.2}$$

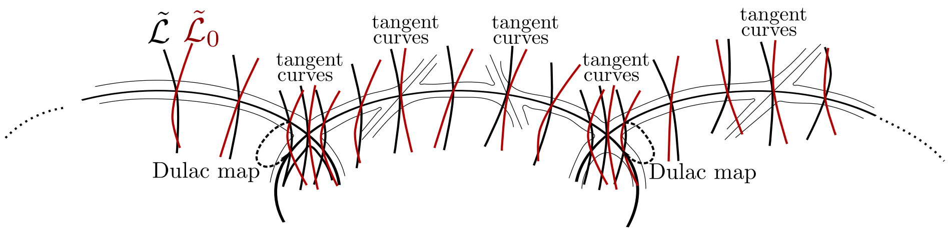

Figure 2.4: Element $\mathcal{L}$ of $\mathcal{R}(\mathcal{L}_0)$

2.2.2 General case

Now, let $\mathcal{F}$ be a non-dicritical foliation such that after desingularization by the map σ all singularities of $\sigma^*(\mathcal{F}) = \tilde{\mathcal{F}}$ are nondegenerate. By Lemma 2.1.1 there exists a σ-absolutely dicritical foliation $\mathcal{L}_0$ such that $\mathrm{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}_0$.

Notation 2.2.3. We denote by $\mathcal{R}(\mathcal{L}_0)$ the set of all σ-absolutely dicritical foliations $\mathcal{L}$ satisfying the following two properties:

- $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}_0$ have the same Dulac maps at any corner of $\mathcal{D}$.
- At each singularity p of $\tilde{\mathcal{F}}$, the invariant curves of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}_0$ through p are tangent (figure 2.4).

Let $\mathcal{L}$ be in $\mathcal{R}(\mathcal{L}_0)$ and D be an irreducible component of $\mathcal{D}$. Suppose that $p_1, \ldots, p_m$ are the singularities of $\tilde{\mathcal{F}}$ on D. Then, we denote

$$S_D(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}) = \{g_{D,p_1}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}), \ldots, g_{D,p_m}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})\},$$

where $g_{D,p_i}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ is the sliding of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{L}}$ in a neighborhood of p_i.

Definition 2.2.4. The *sliding* of $\mathcal{F}$ and $\mathcal{L}$ is

$$S(\mathcal{F}, \mathcal{L}) = \cup_{D \in \mathrm{Comp}(\mathcal{D})} S_D(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}),$$

where $\mathrm{Comp}(\mathcal{D})$ is the set of all irreducible components of $\mathcal{D}$. The *set of slidings* of $\mathcal{F}$ relative to the direction $\mathcal{L}_0$ is the set

$$\mathcal{S}_0(\mathcal{F}) = \cup_{\mathcal{L} \in \mathcal{R}(\mathcal{L}_0)} S(\mathcal{F}, \mathcal{L}).$$

Remark 2.2.5. For each singularity p of $\tilde{\mathcal{F}}$ that is a corner, i.e., $p = D_i \cap D_j$, there are two slidings $g_{D_i,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ and $g_{D_j,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})$. However, by (2.2.2), $g_{D_j,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ is completely determined by $g_{D_i,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ and the Dulac map of $\tilde{\mathcal{L}}$ at p.

Although $S(\mathcal{F}, \mathcal{L})$ is the set of local diffeomorphisms, it also contains some additional information through the global foliation $\mathcal{L}$.

2.3 Local conjugacy of the pair $(\mathcal{F}, \mathcal{L})$

Let $\mathcal{F}$, $\mathcal{F}'$ be two germs of nondegenerate reduced foliations in $(\mathbb{C}^2, 0)$. Denote by S_1, S_2 and S_1', S_2' the separatrices of $\mathcal{F}$ and $\mathcal{F}'$ respectively. Let $\mathcal{L}$ and $\mathcal{L}'$ be two germs of regular foliations such that their separatrices L and L' are transverse to the two separatrices of $\mathcal{F}$ and $\mathcal{F}'$ respectively. Suppose that Φ is a diffeomorphism conjugating $(\mathcal{F}, \mathcal{L})$ and $(\mathcal{F}', \mathcal{L}')$, then the restriction of Φ on the tangent curves commutes with the holonomies on $T(\mathcal{F}, \mathcal{L})$ and $T(\mathcal{F}', \mathcal{L}')$ of $\mathcal{F}$ and $\mathcal{F}'$. The converse is also true:

Proposition 2.3.1. *Suppose that $\mathcal{F}$ and $\mathcal{F}'$ have the same Camacho-Sad index. If $\phi : T(\mathcal{F}, \mathcal{L}) \to T(\mathcal{F}', \mathcal{L}')$ is a diffeomorphism commuting with the holonomies of $\mathcal{F}$ and $\mathcal{F}'$ then ϕ extends to a diffeomorphism Φ of $(\mathbb{C}^2, 0)$ sending $(\mathcal{F}, \mathcal{L})$ to $(\mathcal{F}', \mathcal{L}')$. Moreover, if we require that Φ sends S_1 (resp. S_2) to S_1' (resp. S_2') then this extension is unique.*

Proof. By Lemma 2.1.1, the curves S_1, S_2, L, $T(\mathcal{F}, \mathcal{L})$ (resp., S_1', S_2', L', $T(\mathcal{F}', \mathcal{L}')$) are four transverse smooth curves. It is well known that there exist two radial foliations $\mathcal{R}$ and $\mathcal{R}'$ such that S_1, S_2, L, $T(\mathcal{F}, \mathcal{L})$ and S_1', S_2', L', $T(\mathcal{F}', \mathcal{L}')$ are the invariant curves of $\mathcal{R}$ and $\mathcal{R}'$ respectively. We denote by σ_1 the standard blowing-up at the origin

$$\sigma_1 : (\tilde{\mathbb{C}}^2, \mathbb{CP}^1) \to (\mathbb{C}^2, 0).$$

After pulling back by σ_1 , denote by p_1, p_2, p_L, p_T (resp., p_1', p_2', p_L', p_T') the intersections of strict transforms $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{\mathcal{L}}$, $\tilde{T}(\mathcal{F}, \mathcal{L})$ (resp., $\tilde{S}_1'$, $\tilde{S}_2'$, $\tilde{\mathcal{L}}'$, $\tilde{T}(\mathcal{F}', \mathcal{L}')$) with $\mathbb{CP}^1$. Take ϕ_1 in $\mathrm{Aut}(\mathbb{CP}^1)$ that sends p_1, p_2, p_L to p_1', p_2', p_L' respectively. By Lemma 2.1.1, the direction of $T(\mathcal{F}, \mathcal{L})$ (resp., $T(\mathcal{F}', \mathcal{L}')$) is completely determined by the Camacho-Sad index and the direction of L (resp., L'). Therefore, $\phi_1(p_T) = p_T'$.

Now, we will show that ϕ extends to a diffeomorphism Φ_1 of $(\mathbb{C}^2, 0)$ sending $(\mathcal{F}, \mathcal{R})$ to $(\mathcal{F}', \mathcal{R}')$ by using the path lifting method after a blowing-up as in [13]. Indeed, take a point z that doesn't belong to $\tilde{S}_1 \cup \tilde{S}_2$. Denote by z_0 the intersection of the leaf of $\tilde{\mathcal{R}}$ through z with $\mathbb{CP}^1$. Let γ be a path in $\mathbb{CP}^1 \setminus \{p_1, p_2\}$ that joins z_0 to p_T. Then the lifting of γ by the foliation $\tilde{\mathcal{F}}$ joins z to a point w in $\tilde{T}(\mathcal{F}, \mathcal{L})$. The point $\phi(w)$ goes back by the lifting of the path $\phi_1(\gamma^{-1})$ to the point $\bar{z}$. By [13], the map $\tilde{\Phi}_1$ which is defined by $\tilde{\Phi}_1(z) = \bar{z}$ can extend to a diffeomorphism of $(\tilde{\mathbb{C}}^2, \mathbb{CP}^1)$. Therefore, Φ_1 is the push forward of $\tilde{\Phi}_1$ by σ_1.

Now, denote by $\mathcal{L}_0 = \Phi_{1*}^{-1}(\mathcal{L}')$. Because Φ_1^{-1} sends L' and $T(\mathcal{F}', \mathcal{L}')$ to L and $T(\mathcal{F}, \mathcal{L})$ respectively, L is also the separatrix of $\mathcal{L}_0$ and $T(\mathcal{F}, \mathcal{L}_0) = T(\mathcal{F}, \mathcal{L})$. For simplicity of notation, we denote by T the tangent curve $T(\mathcal{F}, \mathcal{L}_0)$. The proof is reduced to show that there exists a diffeomorphism fixing points in T sending $(\mathcal{F}, \mathcal{L})$ to $(\mathcal{F}, \mathcal{L}_0)$. Choose a system of coordinates (x, y) such that $\mathcal{L}_0$ is defined by $f_0 = x + y$ and $\mathcal{F}$

is defined by a 1-form

$$\omega(x,y)=\lambda y(1+A(x,y))dx+xdy,\ \lambda\notin\mathbb{Q}_{\leq 0}.$$

Then T is defined by

$$\tau(x,y)=x-\lambda y(1+A(x,y))=0.$$

We claim that there exist a natural $n\geq 2$ and a holomorphic function h such that $\mathcal{L}$ is defined by

$$f(x,y)=\left(1+\tau^n(x,y)h(x,y)\right)(x+y).$$

Indeed, assume that $\mathcal{L}$ is defined by

$$\bar{f}(x,y)=u(x,y)(x+y),$$

where u is invertible. Rewrite the equation of T as

$$x-\bar{\tau}(y)=0,$$

where $\bar{\tau}(y)=\lambda y+\ldots$. Because $\lambda\neq -1$, the maps $u(\bar{\tau}(y),y).(\bar{\tau}(y)+y)$ and $\bar{\tau}(y)+y$ are diffeomorphic. Hence there exists a diffeomorphism $g\in\mathbb{C}\{y\}$ such that

$$g\Big(u(\bar{\tau}(y),y).(\bar{\tau}(y)+y)\Big)=\bar{\tau}(y)+y.$$

This equality is equivalent to

$$\Big(g\circ\bar{f}-(x+y)\Big)_{|\tau=0}=0. \tag{2.3.1}$$

Therefore, there exist a natural $n\geq 1$ and a function h satisfying $h_{|\tau=0}\not\equiv 0$ such that

$$g\circ\bar{f}(x,y)=\left(1+\tau^n(x,y)h(x,y)\right)(x+y).$$

Because g is a diffeomorphism, $\mathcal{L}$ is also defined by $f=g\circ\bar{f}$. Let us prove $n\geq 2$. We have

$$\begin{aligned} df\wedge\omega &= \tau(x,y)(\ldots)+n(x+y)h(x,y)\tau^{n-1}d\tau\wedge\omega\\ &= \tau(x,y)(\ldots)+n(x+y)h(x,y)\tau^{n-1}(x+\lambda^2y+\ldots)dx\wedge dy. \end{aligned}$$

Because T is defined by $\tau(x,y)=0$, we have

$$n(x+y)h(x,y)\tau^{n-1}(x+\lambda^2y+\ldots)|_{\tau=0}\equiv 0$$

The fact $\lambda\neq 0,-1$ forces to $x+\lambda^2y\neq x-\lambda y$ and $x+y\neq x-\lambda y$. This implies $\tau^{n-1}|_{\tau=0}\equiv 0$. Consequently, $n\geq 2$.
Now, let

$$X=x\frac{\partial}{\partial x}-\lambda y(1+A(x,y))\frac{\partial}{\partial y}$$

tangent to $\mathcal{F}$. We next claim that there exists $\alpha \in \mathbb{C}\{x, y\}$ such that the diffeomorphism $\exp[\tau^{n-1}\alpha]X$ satisfies

$$(x + y) \circ \exp[\tau^{n-1}\alpha]X(x, y) = \sum_{i\geq 0} \frac{\tau^{i(n-1)}\alpha^i}{i!} \mathrm{ad}^i_X(x + y) = f(x, y), \tag{2.3.2}$$

where $\mathrm{ad_X}$ is the adjoint representation. Indeed, since

$$\sum_{i\geq 0} \frac{\tau^{i(n-1)}\alpha^i}{i!} \mathrm{ad}^i_X(x + y) = x + y + \tau^n\alpha + \frac{n}{2}\tau^{2n-2}\alpha^2 X(\tau) + \tau^{2n-1}(\ldots),$$

(2.3.2) becomes

$$\alpha + \frac{n}{2}\tau^{n-2}\alpha^2 X(\tau) + \tau^{n-1}(\ldots) = (x + y)h(x, y).$$

Hence, the existence of α comes from the implicit function theorem. Thus, the diffeomorphism we need is $\exp[\tau^{n-1}\alpha]X(x, y)$.

Now, we will prove the uniqueness of Φ. In fact, we only need to show that if there exists a diffeomorphism Ψ that sends $(\mathcal{F}, \mathcal{L}_0)$ to itself, preserves the two separatrices of $\mathcal{F}$ and fixes the points of T then $\Psi = \mathrm{Id}$. Since $\Psi_{|T} = \mathrm{Id}$, Ψ sends every leaf of $\mathcal{F}$ into itself. By [1], there exists $\beta \in \mathbb{C}\{x, y\}$ such that

$$\Psi = \exp[\beta]X.$$

Because $\mathcal{L}_0$ is defined by the function $x + y$ and Ψ fixes points in T, we get

$$(x + y) \circ \exp[\beta]X = x + y. \tag{2.3.3}$$

Decompose β into the homogeneous terms

$$\beta = \beta_0 + \beta_1 + \beta_2 + \ldots = \beta_0 + \bar{\beta},$$

where $\bar{\beta} = \beta_1 + \beta_2 + \ldots$. Since $\mathrm{ad}^i_X(x) = x$ and $\mathrm{ad}^i_X(y) = (-\lambda)^i y + c_i$ for all i, where $c_i \in (x, y)^2$, we have

$$\begin{aligned}(x + y) \circ \exp[\beta]X &= \sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} \mathrm{ad}^i_X(x) + \sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} \mathrm{ad}^i_X(y) \\ &= \sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} x + \sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} ((-\lambda)^i y) + h.o.t. \\ &= \exp(\beta_0)x + \exp(-\lambda\beta_0)y + h.o.t..\end{aligned}$$

This follows by (2.3.3) that

$$\exp(\beta_0) = \exp(-\lambda\beta_0) = 1.$$

Hence, we have the equalities

$$x \circ \exp[\beta_0]X = \sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} x = \exp(\beta_0)x = x, \tag{2.3.4}$$

$$y \circ \exp[\beta_0]X = \sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} \Big((-\lambda)^i y + c_i\Big) = \exp(-\lambda\beta_0)y + c = y + c, \tag{2.3.5}$$

where $c \in (x, y)^2$. We claim that

$$\exp[\beta]X = \exp[\beta_0]X \circ \exp[\bar{\beta}]X. \tag{2.3.6}$$

Indeed, for any $h \in \mathbb{C}\{x, y\}$ we have

$$\begin{aligned} h \circ \exp[\beta_0]X \circ \exp[\bar{\beta}]X &= \left(\sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} \mathrm{ad}_X^i(h)\right) \circ \exp[\bar{\beta}]X = \sum_{j=0}^{\infty} \frac{\bar{\beta}^j}{j!} \mathrm{ad}_X^j \left(\sum_{i=0}^{\infty} \frac{\beta_0^i}{i!} \mathrm{ad}_X^i(h)\right) \\ &= \sum_{k=0}^{\infty} \sum_{i+j=k} \frac{\bar{\beta}^j \beta_0^i}{j!i!} \mathrm{ad}_X^k(h) = \sum_{k=0}^{\infty} \frac{(\bar{\beta} + \beta_0)^k}{k!} \mathrm{ad}_X^k(h) = h \circ \exp[\beta]X. \end{aligned}$$

Now, let us write $\mathrm{ad}_X^i(y + c) = (-\lambda)^i y + d_i$ where $d_i \in (x, y)^2$. By (2.3.4), (2.3.5) and (2.3.6) we get

$$\begin{aligned} (x + y) \circ \exp[\beta]X &= x \circ \exp[\beta_0]X \circ \exp[\bar{\beta}]X + y \circ \exp[\beta_0]X \circ \exp[\bar{\beta}]X \\ &= x \circ \exp[\bar{\beta}]X + (y + c) \circ \exp[\bar{\beta}]X \\ &= \sum_{i=0}^{\infty} \frac{\bar{\beta}^i}{i!} \mathrm{ad}_X^i(x) + \sum_{i=0}^{\infty} \frac{\bar{\beta}^i}{i!} \mathrm{ad}_X^i(y + c) \\ &= \sum_{i=0}^{\infty} \frac{\bar{\beta}^i}{i!} x + \sum_{i=0}^{\infty} \frac{\bar{\beta}^i}{i!} ((-\lambda)^i y + d_i) \\ &= \exp(\bar{\beta})x + \exp(-\lambda\bar{\beta})y + \sum_{i=0}^{\infty} \frac{\bar{\beta}^i}{i!} d_i \\ &= x \prod_{i=1}^{\infty} \exp(\beta_i) + y \prod_{i=1}^{\infty} \exp(-\lambda\beta_i) + \sum_{i=0}^{\infty} \frac{\bar{\beta}^i}{i!} d_i. \end{aligned} \tag{2.3.7}$$

We will prove $\bar{\beta} = 0$ by induction. From (2.3.7), we have

$$(x + y) \circ \exp[\beta]X = x(1 + \beta_1) + y(1 - \lambda\beta_1) + h.o.t.$$

Hence, (2.3.3) forces $\beta_1 = 0$. Suppose that $\beta_1 = \ldots = \beta_{k-1} = 0$, we have

$$(x + y) \circ \exp[\beta]X = x(1 + \beta_k) + y(1 - \lambda\beta_k) + h.o.t..$$

Then (2.3.3) again leads to $\beta_k = 0$ and consequently $\beta = \beta_0$. This implies that

$$\Psi = \exp[\beta_0]X = (x, y + c).$$

Finally, (2.3.3) again gives $c = 0$ which implies $\Psi = \mathrm{Id}$. □

Corollary 2.3.2. *Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are two nondegenerate reduced foliations that are analytically conjugated. Let $\mathcal{L}$ and $\mathcal{L}'$ be two regular foliations that are transverse to the two separatrices of $\mathcal{F}$ and $\mathcal{F}'$ respectively. Then there exists a diffeomorphism that sends $(\mathcal{F}, \mathcal{L})$ to $(\mathcal{F}', \mathcal{L}')$.*

Proof. Let Ψ be the conjugacy of $\mathcal{F}$ and $\mathcal{F}'$. Let us denote $T' = \Psi(T(\mathcal{F}, \mathcal{L}))$. Then the restriction $\Psi_{|T(\mathcal{F},\mathcal{L})}$ commutes with the holonomies of $\mathcal{F}$ on $T(\mathcal{F}, \mathcal{L})$ and $\mathcal{F}'$ on T'. Moreover, by the holonomy transport, the holonomies of $\mathcal{F}'$ on T' and on $T(\mathcal{F}', \mathcal{L}')$ are conjugated. Hence, the holonomies of $\mathcal{F}$ on $T(\mathcal{F}, \mathcal{L})$ and $\mathcal{F}'$ on $T(\mathcal{F}', \mathcal{L}')$ are conjugated. By Proposition 2.3.1 there exists a diffeomorphism that sends $(\mathcal{F}, \mathcal{L})$ to $(\mathcal{F}', \mathcal{L}')$ □

By projecting on S_1 and S_2 the holonomies defined on $T(\mathcal{F}, \mathcal{L})$ and $T(\mathcal{F}', \mathcal{L}')$ respectively, we can obtain

Corollary 2.3.3. *If Φ is a diffeomorphism conjugating $(\mathcal{F}, \mathcal{L})$ and $(\mathcal{F}', \mathcal{L}')$, then*

$$\Phi_{|S_1} \circ g_{S_1}(\mathcal{F}, \mathcal{L}) = g_{S_1'}(\mathcal{F}', \mathcal{L}') \circ \Phi_{|S_1}.$$

Reciprocally, if $\mathcal{F}$ and $\mathcal{F}'$ have the same Camacho-Sad index and $\phi : S_1 \to S_1'$ is a diffeomorphism satisfying

$$\phi \circ g_{S_1}(\mathcal{F}, \mathcal{L}) = g_{S_1'}(\mathcal{F}', \mathcal{L}') \circ \phi$$

then ϕ uniquely extends to a diffeomorphism Φ of $(\mathbb{C}^2, 0)$ sending $(\mathcal{F}, \mathcal{L})$ to $(\mathcal{F}', \mathcal{L}')$.

Proof. Because Φ conjugates $(\mathcal{F}, \mathcal{L})$ and $(\mathcal{F}', \mathcal{L}')$, the restriction $\Phi_{|T(\mathcal{F},\mathcal{L})}$ commutes with the holonomies h_γ and h'_γ of $\mathcal{F}$ and $\mathcal{F}'$. Denote by π_{S_1} (resp., $\pi_{S_1'}$) the projection by the leaves of $\mathcal{L}$ (resp., $\mathcal{L}'$) from $T(\mathcal{F}, \mathcal{L})$ (resp., $T(\mathcal{F}', \mathcal{L}')$) to S_1 (resp., S_1'). Since Φ sends $(\mathcal{F}, \mathcal{L})$ to $(\mathcal{F}', \mathcal{L}')$, we have

$$\pi_{S_1'} \circ \Phi_{|T(\mathcal{F},\mathcal{L})} = \Phi_{|S_1} \circ \pi_{S_1}.$$

Therefore

$$\begin{aligned}
\Phi_{|S_1} \circ g_{S_1}(\mathcal{F}, \mathcal{L}) &= \Phi_{|S_1} \circ \pi_{S_1} \circ h_\gamma \circ \pi_{S_1}^{-1} = \pi_{S_1'} \circ \Phi_{|T(\mathcal{F},\mathcal{L})} \circ h_\gamma \circ \pi_{S_1}^{-1} \\
&= \pi_{S_1'} \circ h'_\gamma \circ \Phi_{|T(\mathcal{F},\mathcal{L})} \circ \pi_{S_1}^{-1} = g_{S_1'}(\mathcal{F}', \mathcal{L}') \circ \pi_{S_1'} \circ \Phi_{|T(\mathcal{F},\mathcal{L})} \circ \pi_{S_1}^{-1} \\
&= g_{S_1'}(\mathcal{F}', \mathcal{L}') \circ \Phi_{|S_1}.
\end{aligned}$$

Reciprocally, suppose $\phi : S_1 \to S_1'$ is a diffeomorphism commuting with the slidings of $\mathcal{F}$ and $\mathcal{F}'$. We denote $\psi = \pi_{S_1'}^{-1} \circ \phi \circ \pi_{S_1}$ then

$$\begin{aligned}
\psi \circ h_\gamma &= \pi_{S_1'}^{-1} \circ \phi \circ \pi_{S_1} \circ h_\gamma = \pi_{S_1'}^{-1} \circ \phi \circ g_{S_1}(\mathcal{F}, \mathcal{L}) \circ \pi_{S_1} \\
&= \pi_{S_1'}^{-1} \circ g_{S_1'}(\mathcal{F}', \mathcal{L}') \circ \phi \circ \pi_{S_1} = h'_\gamma \circ \pi_{S_1'}^{-1} \circ \phi \circ \pi_{S_1} = h'_\gamma \circ \psi.
\end{aligned}$$

By Proposition 2.3.1, ψ uniquely extends to a diffeomorphism Φ that sends $(\mathcal{F}, \mathcal{L})$ to $(\mathcal{F}', \mathcal{L}')$. □

Remark 2.3.4. This corollary implies that $\mathcal{S}_0(\mathcal{F})$ is an invariant of $\mathcal{F}$: if $\mathcal{F}$ and $\mathcal{F}'$ are conjugated by Φ then $S(\mathcal{F},\mathcal{L})$ and $S(\mathcal{F}',\Phi_*\mathcal{L})$ are conjugated:

$$S(\mathcal{F},\mathcal{L}) = \tilde{\Phi}_{|\mathcal{D}} \circ S(\mathcal{F}',\Phi_*\mathcal{L}) \circ \tilde{\Phi}^{-1}_{|\mathcal{D}},$$

where $\mathcal{D}$ is the divisor of $\mathcal{F}$ after desingularization. In many cases, the restriction $\tilde{\Phi}_{\mathcal{D}} = \mathrm{Id}$ (e.g. $\mathcal{F}$ and $\mathcal{F}'$ have the same divisor and does not admit a dead component (see the proof of Theorem B)). Then we will have $S(\mathcal{F},\mathcal{L}) = S(\mathcal{F}',\Phi_*\mathcal{L})$. Consequently, $\mathcal{S}_0(\mathcal{F}) = \mathcal{S}_0(\mathcal{F}')$.

If in Corollary 2.3.3 we have $S_1 = S_1'$ and $g_{S_1}(\mathcal{F},\mathcal{L}) = g_{S_1'}(\mathcal{F}',\mathcal{L}')$ then there exists a diffeomorphism sending $(\mathcal{F},\mathcal{L})$ to $(\mathcal{F}',\mathcal{L}')$ and fixing points in S_1. Hence, the sliding invariant gives an obstruction for the construction of local conjugacy of two foliations that fixes the points of the exceptional divisor. This is the reason why we named it "sliding".

2.4 Strict classification of foliations

In this section we will prove that the set of slidings is the missing invariant in the problem of classification for the class of non-dicritical foliations whose Camacho-Sad indices after desingularization are not rational.

Notation 2.4.1. We denote by $\mathcal{A}$ the set of all non-dicritical foliations $\mathcal{F}$ defined on $(\mathbb{C}^2, 0)$ such that the Camacho-Sad index of strict transform $\tilde{\mathcal{F}}$ at each singularity is not rational.

In this thesis an irreducible component of $\mathcal{D}$ is called a *dead component* if in this component there is a unique singularity of $\tilde{\mathcal{F}}$ that is a corner. By [3], the Chern class of an irreducible component of the divisor is equal to the sum of Camacho-Sad indices of the singularities in this component. Consequently, if $\mathcal{D}$ admits a dead component then the Camacho-sad index of the unique singularity of $\tilde{\mathcal{F}}$ on this component is an integer, which is given by this component's Chern class. Therefore, every element in $\mathcal{A}$ after desingularization admits no dead component in its exceptional divisor. Moreover, it is obvious that if $\mathcal{F}$ is in $\mathcal{A}$ then after desingularization all the singularities of $\tilde{\mathcal{F}}$ are nondegenerate.

Let $\mathcal{F},\mathcal{F}'$ be two foliations of A. We say that their *strict separatrices are tangent*, denoted $\mathrm{Sep}(\tilde{\mathcal{F}}) /\!\!/ \mathrm{Sep}(\tilde{\mathcal{F}}')$, if they have the same desingularization map and the same set of singularities. Moreover, at each singularity which is not a corner of the divisor the separatrices of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are tangent (figure 2.5). If $\mathrm{Sep}(\tilde{\mathcal{F}}) /\!\!/ \mathrm{Sep}(\tilde{\mathcal{F}}')$ and $\mathcal{L}_0$ is an absolutely dicritical foliation satisfying $\mathrm{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}_0$ then $\mathrm{Sep}(\tilde{\mathcal{F}}') \pitchfork \tilde{\mathcal{L}}_0$. Therefore, we can define sets of slidings of both $\mathcal{F}$ and $\mathcal{F}'$ relative to the direction $\mathcal{L}_0$. We also denote by $\mathrm{CS}(\tilde{\mathcal{F}})$ the set of Camacho-Sad indices of $\tilde{\mathcal{F}}$ at all singularities, and denote $\mathrm{CS}(\tilde{\mathcal{F}}) = \mathrm{CS}(\tilde{\mathcal{F}}')$ if at each singularity, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ have the same Camacho-Sad index.

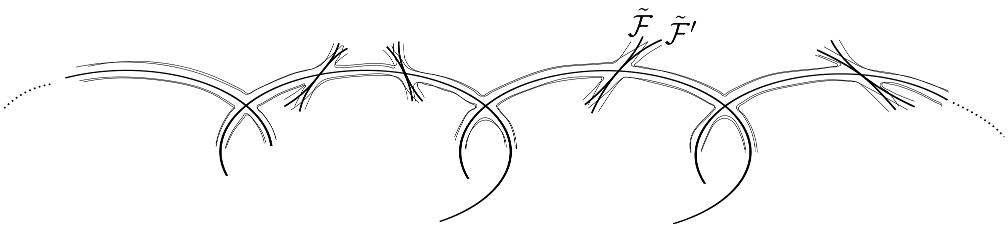

Figure 2.5: Strict transforms $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$

Theorem B. Let $\mathcal{F}$ and $\mathcal{F}'$ be two foliations of $\mathcal{A}$ such that $\mathrm{Sep}(\tilde{\mathcal{F}}) /\!/ \mathrm{Sep}(\tilde{\mathcal{F}}')$. Suppose that $\mathcal{L}_0$ is an absolutely dicritical foliation satisfying $\mathrm{Sep}(\tilde{\mathcal{F}}) \pitchfork \tilde{\mathcal{L}}_0$. Let $\mathcal{R}(\mathcal{L}_0)$ be as in Notation 2.2.3 and $\mathcal{S}_0(\mathcal{F})$, $\mathcal{S}_0(\mathcal{F}')$ the corresponding set of slidings. Then the three following statements are equivalent:

(i) $\mathcal{F}$ and $\mathcal{F}'$ are strictly analytically conjugated.

(ii) Their vanishing holonomy representations are strictly analytically conjugated, $\mathrm{CS}(\tilde{\mathcal{F}}) = \mathrm{CS}(\tilde{\mathcal{F}}')$ and $\mathcal{S}_0(\mathcal{F}) = \mathcal{S}_0(\mathcal{F}')$.

(iii) Their vanishing holonomy representations are strictly analytically conjugated, $\mathrm{CS}(\tilde{\mathcal{F}}) = \mathrm{CS}(\tilde{\mathcal{F}}')$ and $\mathcal{S}_0(\mathcal{F}) \cap \mathcal{S}_0(\mathcal{F}') \neq \emptyset$.

Here, a strict conjugacy means a conjugacy tangent to identity.

Proof. The direction ((ii)⇒(iii)) is obvious.

((i)⇒(ii)) Since the Camacho-sad index is an analytic invariant, it is obvious that $\mathrm{CS}(\tilde{\mathcal{F}}) = \mathrm{CS}(\tilde{\mathcal{F}}')$.
Let Φ be the strict conjugacy and $\tilde{\Phi} : (\mathcal{M}, \mathcal{D}) \to (\mathcal{M}, \mathcal{D})$ be its lifting by σ. Suppose that a non-corner point m of $\mathcal{D}$ is a fixed point of $\tilde{\Phi}$. Then the linear map $D\tilde{\Phi}(m)$ has two eigenvalues. One corresponds to the direction of the divisor. We will denote by $v(\tilde{\Phi})(m)$ the other eigenvalue and define $v(\tilde{\Phi})(m) = 1$ for each corner m.

Lemma 2.4.2. $\tilde{\Phi}_{|\mathcal{D}} = \mathrm{Id}$*, therefore,* $v(\tilde{\Phi})$ *is a function defined on* $\mathcal{D}$*, and moreover* $v(\tilde{\Phi}) \equiv 1$*.*

Proof. Denote by σ_1 the standard blowing-up at the origin of $(\mathbb{C}^2, 0)$

$$\sigma_1 : (\mathcal{M}_1, D_1) \to (\mathbb{C}^2, 0).$$

On D_1, we use the two standard chart $(x, \bar{y})$ and $(\bar{x}, y)$ together with the transition functions $\bar{x} = \bar{y}^{-1}$, $y = x\bar{y}$. Suppose that

$$\Phi(x, y) = (x + \alpha(x, y), y + \beta(x, y)),\ \alpha, \beta \in (x, y)^2.$$

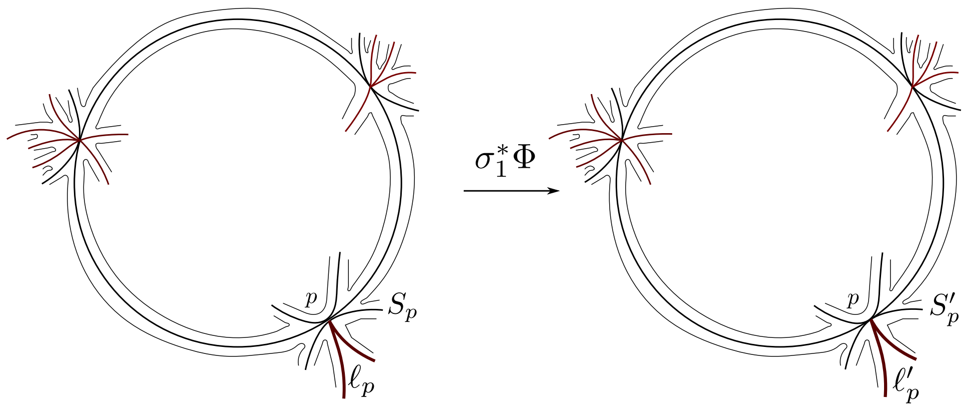

Figure 2.6: Lifting of conjugation map by σ_1

Then in the coordinate system $(x, \bar{y})$ we have

$$\Phi_1(x,\bar{y}) = \sigma_1^*\Phi(x,\bar{y}) = \left(x + \alpha(x, x\bar{y}), \frac{x\bar{y} + \beta(x, x\bar{y})}{x + \alpha(x, x\bar{y})}\right)$$
$$= (x(1+\ldots), \bar{y} + x(\beta_0 + \ldots)),$$

where $\beta_0 = \frac{\partial^2\beta}{\partial x^2}(0,0)$. Therefore $\Phi_1 : (\mathcal{M}_1, \mathcal{D}_1) \to (\mathcal{M}_1, D_1)$ fixes points in D_1 and $v(\Phi_1) \equiv 1$. Let p be a non-reduced singularity of $\sigma_1^*\mathcal{F}$ on D_1. We will show that $D\Phi_1(p) = \mathrm{Id}$ and apply the inductive hypothesis for Φ_1 in a neighborhood of p. Indeed, let σ_2 be the blowing-up at p and $D_2 = \sigma_2^{-1}(p)$. Denote by S_p and S'_p all invariant curves of $\sigma_1^*\mathcal{F}$ and $\sigma_1^*\mathcal{F}'$ through p. Because every element in $\mathcal{M}$ after desingularization admits no dead component in its exceptional divisor, D_2 is not a dead component. Therefore there is at least one irreducible component ℓ_p of S_p that are not tangent to D_1 (figure 2.6). Because $\Phi_{1*}(S_p) = S'_p$ and $\mathrm{Sep}(\tilde{\mathcal{F}}) /\!/ \mathrm{Sep}(\tilde{\mathcal{F}}')$, $D\phi_1(p)$ has an eigenvector different from the direction of D_1, which is corresponding to the direction of ℓ_p. Hence, $D\Phi_1(p)$ has two eigenvectors. Since both of their eigenvalues are 1, we have $D\Phi_1(p) = \mathrm{Id}$. □

Now let $\mathcal{L} \in \mathcal{R}_0$ and denote by $\mathcal{L}' = \Phi_*(\mathcal{L})$. Since $\tilde{\Phi}_{|\mathcal{D}} = \mathrm{Id}$, the strict transforms $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ have the same Dulac maps. Moreover, because $\mathrm{Sep}(\tilde{\mathcal{F}}) /\!/ \mathrm{Sep}(\tilde{\mathcal{F}}')$, at each singularity p of $\tilde{\mathcal{F}}$, $D\tilde{\Phi}(p)$ has two eigenvectors. As $v(\tilde{\Phi}) \equiv 1$ and $\tilde{\Phi}_{|\mathcal{D}} = \mathrm{Id}$ we have $D\tilde{\Phi}(p) = \mathrm{Id}$. Therefore the invariant curves of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ through p are tangent. This gives $\mathcal{L}' = \Phi_*(\mathcal{L}) \in \mathcal{R}(\mathcal{L}_0)$. Because $\tilde{\Phi}$ fixes points in $\mathcal{D}$, by Corollary 2.3.3 the identity map commutes with the slides of $\mathcal{F}$ and $\mathcal{F}'$. This leads to $S(\mathcal{F}, \mathcal{L}) = S(\mathcal{F}', \mathcal{L}')$. Consequently, $\mathcal{S}_0(\mathcal{F}) = \mathcal{S}_0(\mathcal{F}')$. Moreover, the vanishing holonomy representation of $\mathcal{F}$ and $\mathcal{F}'$ are conjugated by $\tilde{\Phi}$. Since $v(\tilde{\Phi}) \equiv 1$ this conjugacy is strict.

((iii)⇒(i)) Suppose that $\mathcal{L}$, $\mathcal{L}' \in \mathcal{R}_0$ satisfy $S(\mathcal{F}, \mathcal{L}) = S(\mathcal{F}', \mathcal{L}')$. By Corollary 2.3.3, at each singularity p_i, $i \in \{1, \ldots, k\}$, of $\tilde{\mathcal{F}}$ there exists a neighborhood U_i of

p_i and a local conjugacy

$$\Phi_i : (\tilde{\mathcal{F}}, \tilde{\mathcal{L}})_{|U_i} \to (\tilde{\mathcal{F}}', \tilde{\mathcal{L}}')_{|U_i}$$

such that $\Phi_{i|\mathcal{D}\cap U_i} = \mathrm{Id}$. Let U_0 be a neighborhood of $\mathcal{D} \setminus \cup_{i=1}^k U_i$ such that U_0 does not contain any singularity of $\tilde{\mathcal{F}}$. Note that U_0 is not connected and the restriction of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ on U_0 are regular. The strict conjugacy of the vanishing holonomy representations can be extended by path lifting method to the conjugacy

$$\Phi_0 : (\tilde{\mathcal{F}}, \tilde{\mathcal{L}})_{|U_0} \to (\tilde{\mathcal{F}}', \tilde{\mathcal{L}}')_{|U_0},$$

satisfying that the second eigenvalue function $v(\Phi_0)$ is identically 1. We will show that on each intersection $V_i = U_i \cap U_0$, Φ_i and Φ_0 coincide. Denote by

$$\Psi_i = \Phi_{i|V_i}^{-1} \circ \Phi_{0|V_i} : (\tilde{\mathcal{F}}, \tilde{\mathcal{L}})_{|V_i} \to (\tilde{\mathcal{F}}, \tilde{\mathcal{L}})_{|V_i}.$$

We claim that $v(\Psi_i) \equiv 1$ on $\mathcal{D} \cap V_i$. Let p, q in $V_i \cap \mathcal{D}$. Denote by l_p and l_q the invariant curves of $\tilde{\mathcal{L}}$ through p and q respectively. As the two maps $\Psi_{i|l_p}$ and $\Psi_{i|l_q}$ are conjugated by the holonomy transport, we have $v(\Psi_i)(p) = v(\Psi_i)(q)$. Consequently, $v(\Psi_i)$ is constant on V_i. Since $v(\Phi_0) \equiv 1$, it follows that $v(\Phi_i)$ is constant on $V_i \cap \mathcal{D}$. Therefore, $v(\Phi_i)$ is constant on $U_i \cap \mathcal{D}$. Moreover, at the singularity p_i, $D\Phi_i(p_i)$ has three eigenvectors corresponding to the directions of the divisor and the directions of invariant curves of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{L}}$ through p_i. Since $D\Phi_i(p_i)$ has also one eigenvalue 1 corresponding to the directions of the divisor, we have $D\Phi_i(p_i) = \mathrm{Id}$. This gives $v(\Phi_i) \equiv 1$ and consequently $v(\Psi_i) \equiv 1$.

Now at each point $p \in V_i \cap \mathcal{D}$, the map $\Psi_{i|l_p}$ commutes with the holonomy of $\tilde{\mathcal{F}}$ around p_i. Since the Camacho-Sad index λ_i of $\tilde{\mathcal{F}}$ at p_i is not rational, Lemma 2.4.3 below says that $\Psi_{i|l_p} = \mathrm{Id}$. Therefore, $\Psi_i = \mathrm{Id}$. Hence we can glue all Φ_i together and the strict conjugacy we need is the projection of this diffeomorphism on $(\mathbb{C}^2, 0)$ by σ. □

Lemma 2.4.3. *Let $h \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $h'(0) = \exp(2\pi i\lambda)$ where $\lambda \notin \mathbb{Q}$. If $\psi \in \mathrm{Diff}(\mathbb{C}, 0)$ satisfying $\psi'(0) = 1$ and $\psi \circ h = h \circ \psi$ then $\psi = \mathrm{Id}$.*

Proof. Since $\lambda \notin \mathbb{Q}$, there is a formal diffeomorphism ϕ such that $\phi \circ h \circ \phi^{-1}(z) = \exp(2\pi i\lambda)z$. Denote by $\tilde{\psi} = \phi \circ \psi \circ \phi^{-1}$, then $\tilde{\psi}'(0) = 1$ and $\tilde{\psi}(\exp(2\pi i\lambda)z) = \exp(2\pi i\lambda)\tilde{\psi}$. The proof is reduced to show that $\tilde{\psi} = \mathrm{Id}$. Suppose that $\tilde{\psi}(z) = z + \sum_{j=2}^{\infty} a_j z^j$. Then

$$\tilde{\psi}(\exp(2\pi i\lambda)z) = \exp(2\pi i\lambda)z + \sum_{j=2}^{\infty} a_j \exp(2j\pi i\lambda)z^j,$$

and

$$\exp(2\pi i\lambda)\tilde{\psi}(z) = \exp(2\pi i\lambda)z + \sum_{j=2}^{\infty} a_j \exp(2\pi i\lambda)z^j.$$

Since $\lambda \notin \mathbb{Q}$, it forces $a_j = 0$ for all $j \geq 2$. Hence $\tilde{\psi} = \mathrm{Id}$. □

2.5 Finite determinacy

Let S be a germ of curve at p in a surface X. we denote by $\Sigma(S)$ the set of all germs of singular curves having the same desingularization map and having the same singularities as S after desingularization. If S is smooth, we denote by $\mathfrak{m}^n(S)$ the set of all holomorphic functions on S whose vanishing orders at the origin are at least n.

Proposition 2.5.1. *Let S be a germ of curve in $(\mathbb{C}^2, 0)$ and $S_1, \ldots, S_k$ be its irreducible components. Suppose that $\sigma : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ is a finite composition of blowing-ups such that all the transformed curves $\sigma^* S_i = \tilde{S}_i$ are smooth. Then there exists a natural N such that if $f_i \in \mathfrak{m}^N(\tilde{S}_i)$, $i = 1, \ldots, k$, then there exists $F \in \mathbb{C}\{x, y\}$ such that $F \circ \sigma_{|\tilde{S}_i} = f_i$. Moreover, the same N can be chosen for all elements in $\Sigma(S)$.*

Proof. We first consider the statement when S is irreducible. If S is smooth then $\tilde{S}$ is diffeomorphic to S. Therefore, we can suppose that S is singular. Denote by $p = \tilde{S} \cap \mathcal{D}$. Choose a coordinate system (x_p, y_p) in a neighborhood of p such that $\tilde{S} = \{y_p = 0\}$ and $\mathcal{D} = \{x_p = 0\}$. Then σ^{-1} is defined by

$$x_p = \frac{\alpha(x, y)}{\beta(x, y)} \text{ and } y_p = \frac{\mu(x, y)}{\nu(x, y)},$$

where $\alpha, \beta, \mu, \nu \in \mathbb{C}\{x, y\}$, $\gcd(\alpha, \beta) = 1$ and $\gcd(\mu, \nu) = 1$. This implies the equality

$$\frac{\alpha \circ \sigma(x_p, y_p)}{\beta \circ \sigma(x_p, y_p)} = x_p \tag{2.5.1}$$

Therefore, there exist a natural k and a holomorphic function h such that

$$\alpha \circ \sigma(x_p, y_p) = x_p^{k+1} h(x_p, y_p), \;\; \beta \circ \sigma(x_p, y_p) = x_p^k h(x_p, y_p), \tag{2.5.2}$$

where $x_p \nmid h$. We claim that h is a unit. Indeed, suppose $h(0, 0) = 0$ and denote by $\tilde{L}$ the curve $\{h(x_p, y_p) = 0\}$. Let L be a curve defined in $(\mathbb{C}^2, 0)$ such that $\sigma^*(L) = \tilde{L}$. Let $\{\bar{h}(x, y) = 0\}$ be a reduced equation of L. By (2.5.2), $\bar{h}|\alpha$ and $\bar{h}|\beta$. It contradicts the fact that $\gcd(\alpha, \beta) = 1$. Now denote $u(x_p) = h(x_p, 0)$ which is a unit, we have

$$\alpha \circ \sigma(x_p, 0) = u(x_p) x_p^{k+1}, \; \beta \circ \sigma(x_p, 0) = u(x_p) x_p^k. \tag{2.5.3}$$

For each $m \geq (k-1)(k+1)$ there exists $j \in \{0, \ldots, k-1\}$ such that $k|(m - j(k+1))$. Thus

$$m = ik + j(k+1), \; i, j \in \mathbb{N}.$$

Therefore, (2.5.3) implies that if a holomorphic function $f(x_p)$ satisfies $x_p^{(k-1)(k+1)}|f(x_p)$ then there exists a holomorphic function $F(x, y)$ such that $F \circ \sigma(x_p, 0) = f(x_p)$. Consequently

$$\mathfrak{m}^{(k-1)(k+1)}(\tilde{S}) \subset \sigma^* \mathbb{C}\{x, y\}_{|\tilde{S}}. \tag{2.5.4}$$

In the general case, suppose that S_i is defined by $\{g_i = 0\}$. If $f_i \in \mathfrak{m}^N(\tilde{S}_i)$, $i = 1, \ldots, k$, with N big enough, by the above there exist F_i, $i = 1, \ldots, k$, such that $F_i \circ \sigma_{|\tilde{S}_i} = f_i$. Hence, it suffices to prove the existence of a holomorphic function F such that $F_{|S_i} = F_{i|S_i}$ for all $i = 1, \ldots, k$. This is reduced to show that there exists a natural M such that the following morphism Θ is surjective

$$\frac{(x,y)^M}{(g_1) \cap \ldots \cap (g_k) \cap (x,y)^M} \xrightarrow{\Theta} \frac{(x,y)^M}{(g_1) \cap (x,y)^M} \oplus \ldots \oplus \frac{(x,y)^M}{(g_k) \cap (x,y)^M}.$$

Indeed, by Hilbert's Nullstellensatz, there exists a natural M_1 such that

$$(x,y)^{M_1} \subset (g_i, g_j) \tag{2.5.5}$$

for all $1 \leq i < j \leq k$. We claim that for all $i = 1, \ldots, k$, $j = 0, \ldots, (k-1)M_1$ the elements $e_{ij} = (0, \ldots, x^j y^{(k-1)M_1 - j}, \ldots, 0)$, where $x^j y^{(k-1)M_1-j}$ is in the i^{th} position, are in ImΘ and then Θ is surjective if M is chosen as $(k-1)M_1$. Indeed, we decompose

$$x^j y^{(k-1)M_1-j} = \prod_{\substack{l=1,\ldots,k \\ l \neq i}} x^{j_l} y^{M_1 - j_l},$$

where $0 \leq j_l \leq M_1$. By (2.5.5), there exist $a_{il}, b_{il} \in \mathbb{C}\{x,y\}$ such that $a_{il} g_i + b_{il} g_l = x^{j_l} y^{M_1 - j_l}$. This implies that

$$e_{ij} = \Theta\left(\prod_{\substack{l=1,\ldots,k \\ l \neq i}} \left(x^{j_l} y^{M_1 - j_l} - a_{il} g_i \right) \right) \in \mathrm{Im}\Theta.$$

It remains to prove that the same N can be chosen for all elements of $\Sigma(S)$. In the case S is irreducible, let S' in $\Sigma(S)$ and $\{y_p = s(x_p)\}$ be an equation of $\sigma^*(S') = \tilde{S}'$ in a neighborhood of p. In the same manner we can obtain

$$\alpha \circ \sigma(x_p, s(x_p)) = v(x_p) x_p^{k+1},\ \beta \circ \sigma(x_p, s(x_p)) = v(x_p) x_p^k,$$

where $v(x_p) = h(x_p, s(x_p))$ which is a unit. Consequently, (2.5.4) still holds for S'. In the general case, it is sufficient to show that the same M_1 in (2.5.5) can be chosen for all elements of $\Sigma(S)$. Let M_{ij} be the smallest natural satisfying

$$(x,y)^{M_{ij}} \subset (g_i, g_j).$$

We claim that

$$M_{ij} \leq \mathrm{I}(g_i, g_j) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(g_i, g_j)}.$$

Indeed, there exists $x^l y^{M_{ij}-1-l} \notin (g_i, g_j)$. Let P_m, $m = 1, \ldots, M_{ij}$, be a sequence of monomials such that $P_1 = 1$, $P_{M_{ij}} = x^l y^{M_{ij}-1-l}$ and either $P_{m+1} = x \cdot P_m$ or $P_{m+1} = y \cdot P_m$. Since $P_m | P_{M_{ij}}$ we have $P_m \notin (g_i, g_j)$ for all $m = 1, \ldots, M_{ij}$. We will show that $\{P_1, \ldots, P_{M_{ij}}\}$ is independent in the vector space $\frac{\mathbb{C}\{x,y\}}{(g_i,g_j)}$ over $\mathbb{C}$. Suppose that

$$c_1 P_1 + \ldots + c_{M_{ij}} P_{M_{ij}} \in (g_i, g_j).$$

Suppose there exists $c_m \neq 0$. Let m_0 be the smallest natural such that $c_{m_0} \neq 0$. Then

$$c_{m_0} P_{m_0} + \ldots + c_{M_{ij}} P_{M_{ij}} = P_{m_0}(c_{m_0} + \ldots) \in (g_i, g_j).$$

This implies that P_{m_0} in (g_i, g_j) which is impossible.

Now, it is well known that the intersection number $\mathrm{I}(g_i, g_j)$ is a topological invariant. It means that if two curves $\{g_i \cdot g_j = 0\}$ and $\{g'_i \cdot g'_j = 0\}$ are topologically conjugated then $\mathrm{I}(g_i, g_j) = \mathrm{I}(g'_i, g'_j)$. Consequently, M_1 can be chosen as $max_{1 \leq i < j \leq k} \mathrm{I}(g_i, g_j)$ that doesn't depend on the elements of $\Sigma(S)$. □

Now, we will prove the finite determinacy property of the slidings of foliations.

Theorem C. Let $\mathcal{F}$ be a non-dicritical foliation without saddle-node singularities after desingularization. There exists a natural N such that if there is a non-dicritical foliation $\mathcal{F}'$ satisfying the following conditions:

(i) $\mathcal{F}$ and $\mathcal{F}'$ have the same set of singularities after desingularization and at a neighborhood of each singularity, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are locally strictly analytically conjugated,

(ii) There exist $\mathcal{L}, \mathcal{L}'$ in $\mathcal{R}(\mathcal{L}_0)$ such that $J^N(S(\mathcal{F}, \mathcal{L})) = J^N(S(\mathcal{F}', \mathcal{L}'))$,

then there exists $\mathcal{L}''$ such that $\mathcal{L}''$ is strictly conjugated with $\mathcal{L}$ and $S(\mathcal{F}, \mathcal{L}'') = S(\mathcal{F}', \mathcal{L}')$.

Here $J^N(S(\mathcal{F}, \mathcal{L})) = J^N(S(\mathcal{F}', \mathcal{L}'))$ means $J^N(g_{D,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})) = J^N(g_{D,p}(\tilde{\mathcal{F}}', \tilde{\mathcal{L}}'))$ for all $g_{D,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ in $S(\mathcal{F}, \mathcal{L})$, $g_{D,p}(\tilde{\mathcal{F}}', \tilde{\mathcal{L}}')$ in $S(\mathcal{F}', \mathcal{L}')$, where $J^N(g_{D,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}))$ stands for the regular part of degree N in the Taylor expansion of $g_{D,p}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}})$.

Proof. Suppose that $\tilde{T}(\mathcal{F}, \mathcal{L}) = \cup T_i$, $\tilde{T}'(\mathcal{F}', \mathcal{L}') = \cup T'_i$ where T_i and T'_i are irreducible components of $T(\mathcal{F}, \mathcal{L})$ and $T(\mathcal{F}', \mathcal{L}')$. Then the singularities of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are $p_i = T_i \cap \mathcal{D} = T'_i \cap \mathcal{D}$. Denote by $h_{i\gamma}$ the holonomy of $\tilde{\mathcal{F}}$ on T_i.

Now let p_i be a singularity $\tilde{\mathcal{F}}$. We first suppose that p_i is not a corner. denote by D the irreducible component of $\mathcal{D}$ through p_i. Because $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are strictly conjugated in a neighborhood of p_i, by Corollaries 2.3.2 and 2.3.3, there exists a diffeomorphism ψ_i in $\mathrm{Diff}(D, p_i)$ tangent to identity such that

$$\psi_i \circ g_{D,p_i}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}) = g_{D,p_i}(\tilde{\mathcal{F}}', \tilde{\mathcal{L}}') \circ \psi_i. \tag{2.5.6}$$

Let π_D (resp., π'_D) be the projection from T_i (resp., T'_i) to D that follows the leaves of $\tilde{\mathcal{L}}$ (resp., $\tilde{\mathcal{L}}'$). We denote

$$\phi_i = \pi_D^{-1} \circ \psi_i^{-1} \circ \pi_D. \tag{2.5.7}$$

Since $J^N(S(\mathcal{F},\mathcal{L})) = J^N(S(\mathcal{F}',\mathcal{L}'))$, ϕ_i is a diffeomorphism of (T_i, p_i) tangent to identity map at order at least N.

In the case p_i is a corner, let D be one of two irreducible components of $\mathcal{D}$ through p_i and define ϕ_i as above.

Lemma 2.5.2. *Suppose that there exists a diffeomorphism Φ such that the lifting $\sigma^*(\Phi) = \tilde{\Phi}$ satisfies*

- $\tilde{\Phi}_{|\mathcal{D}} = \mathrm{Id}$,
- $\tilde{\Phi}_{|T_i} = \phi_i$,
- $T(\mathcal{F}, \Phi_*\mathcal{L}) = T(\mathcal{F},\mathcal{L})$.

Then $\mathcal{L}'' = \Phi_*(\mathcal{L})$ satisfies $S(\mathcal{F},\mathcal{L}'') = S(\mathcal{F}',\mathcal{L}')$.

Proof. Let p_i be a singularity $\tilde{\mathcal{F}}$. In the case p_i is not a corner, we denote D, π_D, π'_D as above. Let π''_D be the projection following the leaves of $\tilde{\mathcal{L}}''$ from T_i to D, then

$$\pi''_D = \pi_D \circ \phi_i^{-1}.$$

Hence, we have the equality

$$\begin{aligned} g_{D,p_i}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}'') &= \pi''_D \circ h_{i\gamma} \circ \pi''^{-1}_D = \pi_D \circ \phi_i^{-1} \circ h_{i\gamma} \circ \phi_i \circ \pi_D^{-1} \\ &= \psi_i \circ \pi_D \circ h_{i\gamma} \circ \pi_D^{-1} \circ \psi_i^{-1} = \psi_i \circ g_{D,p_i}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}) \circ \psi_i^{-1} \\ &= g_{D,p_i}(\tilde{\mathcal{F}}', \tilde{\mathcal{L}}'). \end{aligned}$$

If p_i is a corner, $p_i = D \cap D'$, we still have

$$g_{D,p_i}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}'') = g_{D,p_i}(\tilde{\mathcal{F}}', \tilde{\mathcal{L}}').$$

Since $\tilde{\Phi}_{|\mathcal{D}} = \mathrm{Id}$ the Dulac maps of $\tilde{\mathcal{L}}''$ and $\tilde{\mathcal{L}}'$ in a neighborhood of p_i are the same. Hence, Remark 2.2.5 leads to

$$g_{D',p_i}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}'') = g_{D',p_i}(\tilde{\mathcal{F}}', \tilde{\mathcal{L}}').$$

□

What is left is to prove the existence of Φ in Lemma 2.5.2 for N big enough. Suppose that $\mathcal{F}$ and $\mathcal{L}$ are respectively defined by

$$\begin{aligned} \omega &= a(x,y)dx + b(x,y)dy, \\ \omega_{\mathcal{L}} &= c(x,y)dx + d(x,y)dy. \end{aligned}$$

Then the tangent curve $T = T(\mathcal{F},\mathcal{L})$ is defined by

$$q(x,y) = da - cb = 0.$$

Let $X_q = \frac{\partial q}{\partial y}\frac{\partial}{\partial x} - \frac{\partial q}{\partial x}\frac{\partial}{\partial y}$ be a vector field tangent to T and $\tilde{X}_q$ be its lifting by σ. By the implicit function theorem, if N is big enough, there exists f_i defined on T_i such that

$$\exp[f_i]\left(\tilde{X}_q|_{T_i}\right) = \phi_i.$$

Using Proposition 2.5.1, by choosing N big enough, there exists $f \in \mathbb{C}\{x, y\}$ such that

$$\exp[f \circ \sigma]\tilde{X}_q|_{T_i} = \phi_i.$$

For each $\Phi = (\Phi_1, \Phi_2) \in \mathrm{Diff}(\mathbb{C}^2, 0)$, let us denote

$$< \Phi >= \frac{\omega_{\mathcal{L}} \wedge \Phi^*\omega_{\mathcal{L}}}{dx \wedge dy} = c\left(c \circ \Phi\frac{\partial \Phi_1}{\partial y} + d \circ \Phi\frac{\partial \Phi_2}{\partial y}\right) - d\left(c \circ \Phi\frac{\partial \Phi_1}{\partial x} + d \circ \Phi\frac{\partial \Phi_2}{\partial x}\right).$$

It follows easily that $T(\mathcal{F}, \Phi_*\mathcal{L}) = T$ if and only if $q| < \Phi >$. Lemma 2.5.3 below implies that there exists a holomorphic function u such that $\exp[f - uq]X_q$ satisfies Lemma 2.5.2 for N big enough. Moreover, by Proposition 2.5.1, we can chose N that only depends on $\mathcal{F}$. □

Lemma 2.5.3. *If N is big enough, for all f in $(x, y)^N$ there exists a holomorphic function u such that $q| < \Phi_{f-uq} >$.*

Proof. To simplify notation, for each holomorphic function f, let us write Φ_f instead of $\exp[f]X_q$. We have

$$\begin{aligned}
\frac{\partial}{\partial x}x \circ \Phi_{f-uq}\Big|_{\{q=0\}} &= \frac{\partial}{\partial x}\sum_{i=0}^{\infty}\frac{(f-uq)^i}{i!}\mathrm{ad}^i_{X_q}(x)\Big|_{\{q=0\}} \\
&= \frac{\partial}{\partial x}x \circ \Phi_f\Big|_{\{q=0\}} - u.\frac{\partial q}{\partial x}.\sum_{i=1}^{\infty}\frac{f^{i-1}}{(i-1)!}\mathrm{ad}^i_{X_q}(x)\Big|_{\{q=0\}} \\
&= \frac{\partial}{\partial x}x \circ \Phi_f\Big|_{\{q=0\}} - u.\frac{\partial q}{\partial x}.\frac{\partial q}{\partial y} \circ \Phi_f\Big|_{\{q=0\}}.
\end{aligned}$$

Similarly, we obtains

$$\begin{aligned}
\frac{\partial}{\partial y}x \circ \Phi_{f-uq}\Big|_{\{q=0\}} &= \frac{\partial}{\partial y}x \circ \Phi_f\Big|_{\{q=0\}} - u \cdot \frac{\partial q}{\partial y} \cdot \frac{\partial q}{\partial y} \circ \Phi_f\Big|_{\{q=0\}}, \\
\frac{\partial}{\partial x}y \circ \Phi_{f-uq}\Big|_{\{q=0\}} &= \frac{\partial}{\partial x}y \circ \Phi_f\Big|_{\{q=0\}} + u \cdot \frac{\partial q}{\partial x} \cdot \frac{\partial q}{\partial x} \circ \Phi_f\Big|_{\{q=0\}}, \\
\frac{\partial}{\partial y}y \circ \Phi_{f-uq}\Big|_{\{q=0\}} &= \frac{\partial}{\partial y}y \circ \Phi_f\Big|_{\{q=0\}} + u \cdot \frac{\partial q}{\partial y} \cdot \frac{\partial q}{\partial x} \circ \Phi_f\Big|_{\{q=0\}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
< \Phi_{f-uq} > |_{\{q=0\}} &= < \Phi_f > |_{\{q=0\}} \\
&\quad - u \cdot \left(c\frac{\partial q}{\partial y} - d\frac{\partial q}{\partial x}\right) \cdot \left(\left(c\frac{\partial q}{\partial y} - d\frac{\partial q}{\partial x}\right) \circ \Phi_f\right)\Big|_{\{q=0\}}.
\end{aligned} \tag{2.5.8}$$

Let us denote $h = c\frac{\partial q}{\partial y} - d\frac{\partial q}{\partial x}$. Then $\{h = 0\}$ is the tangent curve of $\mathcal{L}$ and the foliation defined by the level sets of q. Since at each singularity p_i of $\tilde{\mathcal{F}}$, the irreducible component T_i of T is transverse to $\tilde{\mathcal{L}}$, the irreducible components of the strict transform of $\{h = 0\}$ at p_i are also transverse to T_i. This implies that $(q, h) = 1$ and the two curves $\{q \cdot h = 0\}$ and $\{q \cdot (h \circ \Phi_f) = 0\}$ are topologically conjugated. By Hilbert's Nullstellensatz and the proof of Proposition 2.5.1 there exists a natural M such that $(x, y)^M \subset (q, h)$ and $(x, y)^M \subset (q, h \circ \Phi_f)$. This implies that $(x, y)^{2M} \subset (q, h \cdot (h \circ \Phi_f))$. Hence, if $< \Phi_f >\in (x, y)^{2M}$, by (2.5.8) we can choose $u \in \mathbb{C}\{x, y\}$ such that $q| < \Phi_{f-uq} >$. □

Remark 2.5.4. If we replace the condition "$\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are locally analytically conjugated" in Theorem C by the condition "$\mathcal{F}$ and $\mathcal{F}'$ are in $\mathcal{A}$" then the conclusion in Theorem C becomes: "For all natural $M \geq N$ there exists $\mathcal{L}''_M$ such that $J^M(S(\mathcal{F}, \mathcal{L}''_M)) = J^M(S(\mathcal{F}', \mathcal{L}'))$". Indeed, in that case, because the Camacho-Sad indices are not rational, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are locally formally conjugated. Therefore, we can choose ψ in (2.5.6) such that

$$J^M(\psi_i \circ g_{D,p_i}(\tilde{\mathcal{F}}, \tilde{\mathcal{L}}) \circ \psi_i^{-1}) = J^M(g_{D,p_i}(\tilde{\mathcal{F}}', \tilde{\mathcal{L}}')).$$

The two Theorems B and C give two corollaries on finite determination of the class of isoholonomy non-dicritical foliations and absolutely dicritical foliations that have the same Dulac maps.

Corollary D. Let $\mathcal{F} \in \mathcal{A}$ defined by a 1-form ω then there exists a natural N such that if $\mathcal{F}'$ is defined by a 1-form ω' satisfying that $J^N\omega = J^N\omega'$ and the vanishing holonomy representations of $\mathcal{F}$ and $\mathcal{F}'$ are strictly analytically conjugated, then $\mathcal{F}$ and $\mathcal{F}'$ are strictly analytically conjugated.

Proof. Let $\mathcal{L} \in \mathcal{R}_0$ then $J^{m(N)}S(\mathcal{F}, \mathcal{L}) = J^{m(N)}S(\mathcal{F}', \mathcal{L})$ where $m(N)$ is an increasing function on N and $m(N) \to \infty$ when $N \to \infty$. By Theorem C if N is big enough there exists $\mathcal{L}'' \in \mathcal{R}_0$ such that $S(\mathcal{F}, \mathcal{L}'') = S(\mathcal{F}', \mathcal{L})$. By Theorem B, $\mathcal{F}$ and $\mathcal{F}'$ are strictly analytically conjugated. □

Remark 2.5.5. This Corollary is consistent with the result of Mattei in [11] which says that the dimension of moduli space of the equisingular unfoldings of a foliation is finite. Note that the vanishing holonomy representations of two foliations that are joined by a unfolding are conjugated but the converse is not true.

Corollary E. Let $\mathcal{L}$ be a σ-absolutely dicritical foliation defined by 1-form ω. There exists a natural N such that if $\mathcal{L}'$ is a σ-absolutely dicritical foliation defined by ω' satisfying $J^N\omega = J^N\omega'$ and the Dulac maps of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ are the same then $\mathcal{L}$ and $\mathcal{L}'$ are strictly analytically conjugated.

Proof. Suppose that $\mathcal{D} = \cup_{i=1,\ldots,k} D_i$ where D_i is an irreducible component of $\mathcal{D}$. We take a pair of irreducible functions f_i and g_i for each $i = 1, \ldots, k$, such that the curve $C_i = \{f_i = 0\}$ and $C'_i = \{g_i = 0\}$ satisfy the following properties:

1. The strict transforms $\tilde{C}_i$ and $\tilde{C}'_i$ cut D_i at two different points p_i, q_i, respectively, such that none of them is a corner.
2. $\tilde{C}_i$, $\tilde{C}'_i$ are smooth and transverse to the invariant curve of $\tilde{\mathcal{L}}$ through p_i, q_i respectively.

Because $[\mathbb{C}:\mathbb{Q}]$ is an infinite field extension, there exists $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{C}^k$ such that

$$\sum_{i=1}^{k} c_i \lambda_i \notin \mathbb{Q},\ \forall (c_1, \dots, c_k) \in \mathbb{Q}^k \setminus \{(0, \dots, 0)\}.$$

Now, let us consider the non-dicritical foliation $\mathcal{F}$ defined by the 1-form

$$\omega_0 = \prod_{i=1}^{k} (f_i g_i) \cdot \left(\sum_{i=1}^{k} \left(\lambda_i \frac{df_i}{f_i} + \frac{dg_i}{g_i} \right) \right).$$

Then $\mathcal{F}$ admits σ as its desingularization map and the singularities of the strict transform $\tilde{\mathcal{F}}$ are the corners of $\mathcal{D}$ and p_i, q_i, $i = 1, \dots, k$. We claim that at each singularity, the Camacho-Sad index of $\tilde{\mathcal{F}}$ is not rational. Indeed, denote by m_{ij} the multiplicity of $f_i \circ \sigma$ and $g_i \circ \sigma$ on D_j. At the corner $p_{ij} = D_i \cap D_j$, we take coordinates (x, y) such that $D_i = \{x = 0\}, D_j = \{y = 0\}$. In this coordinate system, we can write $\sigma^* \omega_0$ as

$$\sigma^* \omega_0 = u(x,y) x^{2\sum_{l=1}^{k} m_{li}} y^{2\sum_{l=1}^{k} m_{lj}} \sum_{l=1}^{k} \left((\lambda_l + 1) m_{li} \frac{dx}{x} + (\lambda_l + 1) m_{lj} \frac{dy}{y} + \alpha_{ij} \right),$$

where $u(x, y)$ is a unit and α_{ij} is a holomorphic form. Therefore, the Camacho-Sad index of $\tilde{\mathcal{F}}$ at p_{ij} is given by

$$\mathrm{I}(p_{ij}) = \frac{\sum_{l=1}^{k} (\lambda_l + 1) m_{lj}}{\sum_{l=1}^{k} (\lambda_l + 1) m_{li}} \notin \mathbb{Q}.$$

Similarly, the Camacho-Sad indices of $\tilde{\mathcal{F}}$ at p_i and q_i, respectively, are

$$\mathrm{I}(p_i) = \frac{\sum_{l=1}^{k} (\lambda_l + 1) m_{li}}{\sum_{l=1}^{k} \lambda_l} \notin \mathbb{Q},\ \mathrm{I}(q_i) = \frac{\sum_{l=1}^{k} (\lambda_l + 1) m_{li}}{k} \notin \mathbb{Q}.$$

Now if $J^N \omega = J^N \omega'$ then $J^{m(N)} S(\mathcal{F}, \mathcal{L}) = J^{m(N)} S(\mathcal{F}, \mathcal{L}')$ where $m(N)$ is an increasing function on N and $m(N) \to \infty$ when $N \to \infty$. Moreover if N is big enough the invariant curves of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ through the singularities of $\tilde{\mathcal{F}}$ are tangent. By using Theorem C for $\mathcal{F}' = \mathcal{F}$, there exists a foliation $\mathcal{L}''$ strictly conjugated with $\mathcal{L}$ such that the two couples $(\mathcal{F}, \mathcal{L}'')$ and $(\mathcal{F}, \mathcal{L}')$ are strictly conjugated. Consequently, $\mathcal{L}$ and $\mathcal{L}'$ are strictly conjugated. □

Future works

The results of this thesis point to several interesting directions for future works:

The immediate problem is to generalize the second part of the paper [9]. In Chapter 1, we have only found some formal normal forms for topologically quasi-homogeneous foliations. The next step is to prove that if topologically quasi-homogeneous foliations have the same hamiltonian part and their vanishing holonomy representation strictly conjugated then they are strictly conjugated. We think that this can be done by using the sliding invariants.

Although the sliding invariants are finitely determined, we have not found a method to compute this invariant even in the simplest case such as the nondegenerate reduced foliations. In [9], the authors shows that if topologically homogeneous two foliations, whose vanishing holonomy representations are strictly conjugated, have the same hamiltonian part after desingularization then they are strictly conjugated. It seems, at least in the topologically homogeneous case, that the sliding invariants depend only on the hamiltonian part. Proving and generalizing this result may help us solve the following problems:

- To prove that, in the topologically quasi-homogeneous case, the hamiltonian part and the vanishing holonomy representation can classify the foliations.
- To decompose a foliation (or 1-form which defines that foliation) into two parts, one which contains only the information of the finite determinacy invariant, other which contains the vanishing holonomy representation. Doing so, we may found a normal form for non-dicritical foliations (by normalizing each part) or a criteria of conjugacy for two foliations having the vanishing holonomy representations conjugated.

Another direction for our future work is to consider the case where the Camacho-Sad indices are rational. The equality of sliding invariant implies the existence of local conjugations and the non-rational condition allows us to glue them together. Without the non-rational condition, it seems that there is a hidden invariant of resonant type which controls the existence of global conjugation.

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