1980

MATHEMATICAL NOTES

G.H. Meisters
University of Nebraska-Lincoln

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

PRINCIPAL VERTICES, EXPOSED POINTS, AND EARS

G. H. MEISTERS

In [1] Guggenheimer states that *a Jordan polygon has two principal vertices that are exposed points of its convex hull*, and he refers to Meisters's paper [3]. Such a statement cannot be found in Meisters's paper, and in fact it is false. The polygon illustrated in the figure below provides a counterexample.

By a *Jordan polygon* $P = V_1 \ldots V_N$ is meant a simple closed polygonal plane curve with N *sides* $V_1V_2, V_2V_3, \ldots, V_{N-1}V_N, V_NV_1$ joining the N *vertices* $V_1, \ldots, V_N$. In [3] any consecutive vertices V_{i-1}, V_i, and V_{i+1} of a Jordan polygon P are said to form an *ear* (regarded as the region enclosed by the triangle $V_{i-1}V_iV_{i+1}$) *at the vertex* V_i if the open chord joining V_{i-1} and V_{i+1} lies entirely inside the polygon P. Two such ears are called *nonoverlapping* if the interiors of their triangular regions are disjoint. The following Two-Ears Theorem was proved in [3].

TWO-EARS THEOREM. *Except for triangles, every Jordan polygon has at least two nonoverlapping ears.*

Guggenheimer's false statement was perhaps an attempt to express this Two-Ears Theorem in terms of the concept of "principal vertex." A vertex V_i of the polygon $P = V_1 \ldots V_N$ is called *principal* if no vertex of P is in the interior of the triangle $V_{i-1}V_iV_{i+1}$ or on the open chord $(V_{i-1}V_{i+1})$. See [1]. But it is doubtful that the concept of "ear vertex" (i.e., a vertex at which there is an ear) can be expressed in terms of the concept of "principal vertex" without in some way referring to the *interior* of the polygon, because the definition of the former depends on the Jordan Curve Theorem for polygons while that of the latter depends only on the Jordan Curve Theorem for triangles. For example, a principal vertex is an ear vertex if and only if the *interior angle* at this vertex is less than a straight angle. Every ear vertex is a principal vertex, but there need not be an ear at every principal vertex. In fact, every Jordan polygon has at least three principal vertices but need have no more than two ears.

An "exposed point" of a set X is defined by Klee in [2] and can also be found in many books on convex sets. (An *exposed point* of a set X in a topological linear space is a point p in X such

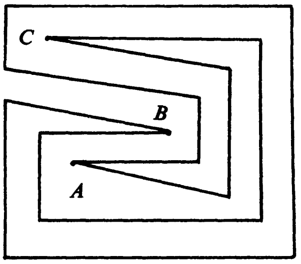

FIG. 1. This Jordan polygon has three principal vertices (A, B, and C) and has ears at two of these vertices (A and B), but it has no principal vertex on the boundary of its convex hull.

The American Mathematical Monthly, Vol. 87, No. 4 (Apr., 1980), pp. 284-285

that X is supported at p by a closed hyperplane which intersects X only at p.) The important thing here is that an exposed point is a special kind of boundary point. But the Jordan polygon in Figure 1 above has no principal vertex on the boundary of its convex hull, so that Guggenheimer's statement (italicized in the first sentence of this article) is false.

References

1. H. Guggenheimer, The Jordan and Schoenflies theorems in axiomatic geometry, this MONTHLY, 85 (1978) 753–756.
2. V. L. Klee, Extremal structure of convex sets, II, Math. Z., 69 (1958) 90–104.
3. G. H. Meisters, Polygons have ears, this MONTHLY, 82 (1975) 648–651.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEBRASKA, LINCOLN, NE 68588.