Partitioning 3-colored complete graphs into three monochromatic cycles *

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Abstract

We show in this paper that in every 3-coloring of the edges of K^n all but $o(n)$ of its vertices can be partitioned into three monochromatic cycles. From this, using our earlier results, actually it follows that we can partition *all* the vertices into at most 17 monochromatic cycles, improving the best known bounds. If the colors of the three monochromatic cycles must be different then one can cover $(\frac{3}{4} - o(1))n$ vertices and this is close to best possible.

1 Introduction

It was conjectured in [8] that in every r-coloring of a complete graph, the vertex set can be covered by r vertex disjoint monochromatic cycles (where vertices, edges and the empty set are accepted as cycles).

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Conjecture 1 (Erdős, Gyárfás, Pyber, [8]). *In every r-coloring of the edges of K_n its vertex set can be partitioned into r monochromatic cycles.*

For general r, the $O(r^2 \log r)$ bound of Erdős, Gyárfás, and Pyber [8] has been improved to $O(r \log r)$ by Gyárfás, Ruszinkó, Sárközy and Szemerédi [11]. The case $r = 2$ was conjectured earlier by Lehel and was settled by Łuczak, Rödl and Szemerédi [16] for large n using the Regularity Lemma. Later Allen [1] gave a proof without the Regularity Lemma and recently Bessy and Thomassé [3] found an elementary argument that works for every n.

The main result of this paper confirms Conjecture 1 in an asymptotic sense for $r = 3$.

Theorem 1. *In every 3-coloring of the edges of K_n all but $o(n)$ of its vertices can be partitioned into three monochromatic cycles.*

The history of Conjecture 1 suggests that the cycle partition problem is difficult even in the $r = 2$ case. On the other hand, if we relax the problem and allow two monochromatic cycles to intersect in at most one vertex (almost partition), then it becomes easy. Indeed, Gyárfás [9] gave a simple proof that two cycles of distinct colors that intersect in at most one vertex cover the vertex set. A similar result does not seem to be easy for $r \geq 3$ colors.

Combining Theorem 1 with some of our earlier results from [11] we can actually prove that we can partition *all* the vertices into at most 17 monochromatic cycles, improving the best known bounds for $r = 3$.

Theorem 2. *In every 3-coloring of the edges of K_n the vertices can be partitioned into at most 17 monochromatic cycles.*

Note that in the same way for a general r if one could prove the corresponding asymptotic result as in Theorem 1 (even with a weaker linear bound on the number of cycles needed; unfortunately we are not there yet), then we would have a linear bound overall. This makes the asymptotic result interesting.

In the proof of Theorem 1 our main tools will be the Regularity Lemma [17] and the following lemma. A *connected matching* in a graph G is a matching M such that all edges of M are in the same component of G.

Lemma 1. *If n is even then in every 3-coloring of the edges of K_n the vertex set can be partitioned into three monochromatic connected matchings.*

In our (now rather standard) approach Lemma 1 is needed for the 'reduced graph', where only the regular pairs of clusters of the Regularity Lemma are represented. Thus we will need a the following density version of Lemma 1.

Lemma 2. *For every $\eta > 0$ there exist n_0 and $\varepsilon > 0$ such that for $n \geq n_0$ the following holds. In every 3-edge coloring of a graph G with n vertices and more than $(1-\varepsilon)\binom{n}{2}$ edges there exist 3 monochromatic connected matchings which partition at least $(1-\eta)n$ vertices of G.*

Certain 3-colorings often occur among extremal colorings for Ramsey numbers of triples of paths, triples of even cycles and their analysis is important in the corresponding results, see e.g. [2, 12]. These colorings also play a crucial role in this paper and we call them 4-*partite colorings*, defined as follows.

The vertex set of K_n is partitioned into four non-empty parts $A_1 \cup A_2 \cup A_3 \cup A_4$, $|A_1| \leq |A_2| \leq |A_3| \leq |A_4|$ such that all edges in the complete bipartite graphs $B(A_1, A_2)$ and $B(A_3, A_4)$ are colored 1, in $B(A_1, A_3)$ and $B(A_2, A_4)$ are colored 2, and $B(A_1, A_4)$ and in $B(A_2, A_3)$ are colored 3. Inside each part the edges are colored arbitrarily.

One can easily observe that in a 4-partite coloring that has equal partite classes and within all the four partite classes all edges are colored with color 1, at most 75 percent of the vertices can be covered by three vertex disjoint cycles having *different colors.* Thus Theorem 1 fails if we insist that the monochromatic cycles must have different colors. On the other hand, Theorem 3 shows that this example is essentially best possible.

Theorem 3. *In every* 3*-coloring of the edges of* K_n*, at least* $(\frac{3}{4} - o(1))n$ *vertices can be covered by vertex disjoint monochromatic cycles having distinct colors.*

Theorem 3 relies on the following variant of Lemma 1.

Lemma 3. *In every* 3*-coloring of the edges of* K_n *vertex disjoint monochromatic connected matchings of distinct colors cover at least* $\frac{3n}{4} - 1$ *vertices.*

In fact, here again we will need the density version of Lemma 3.

Lemma 4. *For every* $\eta > 0$ *there exist* n_0 *and* $\varepsilon > 0$ *such that for* $n \geq n_0$ *the following holds. In every* 3*-edge coloring of a graph* G *with* n *vertices and more than* $(1 - \varepsilon)\binom{n}{2}$ *edges vertex disjoint monochromatic connected matchings of distinct colors cover at least* $(1 - \eta)\frac{3n}{4}$ *vertices of* G*.*

The organization of the paper is as follows. In the next section we present the proofs of Lemmas 1 and 3. Lemma 1 is the key result of the paper because the derivation of Lemma 2 and Theorem 1 from it (as well as the derivation of Theorem 3 and Lemma 4 from Lemma 3) can now be considered as a rather standard application of the Regularity Lemma, as done in [2], [10], [12] and [15]. Therefore in Sections 3 and 4 we just describe these steps briefly. In Section 5 we sketch the proof of Theorem 2.

2 Proofs of Lemmas 1 and 3

Proof of Lemma 1. Take an arbitrary coloring of the edges of K_n with colors, say, 1, 2, and 3. Let G_1, G_2, G_3 be the subgraphs spanned by the edges of colors 1, 2, 3, respectively. First assume that one of the G_i-s, say, G_1 is a connected. Then take a maximum matching M_1 in G_1. All the edges in $V(K_n) \setminus V(M_1)$ are colored 2 or 3, thus these vertices are connected in, say, color 2. Take a maximum matching M_2 in color 2. Again, since M_2 is maximal, all edges in $V(K_n) \setminus (V(M_1) \cup V(M_2))$ are colored 3. A

maximum matching M_3 here will be connected in color 3 and will contain all vertices of $V(K_n) \setminus (V(M_1) \cup V(M_2))$.

Hence from now on we assume that none of G_i-s is connected. Let H_1 be a largest monochromatic component attained in, say, color 1, and select a maximum matching $M_1 \subset H_1$. Gyárfás [7] (see also [5]) showed that every r-edge-coloring of K_n contains a monochromatic component on at least $n/(r-1)$ vertices, i.e., $|V(H_1)| \geq \frac{n}{2}$. Let $Y = V(H_1) \setminus V(M_1)$ and $X = [n] \setminus V(H_1)$. Clearly, all edges in the bipartite graph $B(V(H_1), X)$ have color 2 or 3.

Case 1: $|X| \leq |Y|$. Since M_1 is maximum in H_1, edges having both endpoints in Y are colored 2 or 3. Therefore, Y is connected in, say, color 2. Let M_2 a maximum matching in color 2 in the bipartite graph $B(X, Y)$, $Y_1 = Y \setminus V(M_2)$, $X_1 = X \setminus V(M_2)$. If $X_1 \neq \emptyset$ then $B(X_1, Y_1)$ is complete bipartite in color 3. So take a matching M_3 in color 3 of size $|X_1|$ in $B(X_1, Y_1)$. Since $|X_1| \leq |Y_1|$, we covered all vertices in X. If $|X_1| = |Y_1|$ then we are ready. If $|X_1| < |Y_1|$, regardless of $X_1 = \emptyset$ or $X_1 \neq \emptyset$ take a maximum matching in color 2 in $Y_1 \setminus V(M_3)$ and add its edges to M_2. If we did not cover all the vertices in Y_1 then the vertices yet uncovered span a complete graph in color 3. Cover them with a perfect matching and add these edges to M_3. Let $M = M_1 \cup M_2 \cup M_3$. Clearly, we got a partition into matchings and M_1, M_2, M_3 are connected in 1, 2, 3, respectively. Indeed, M_1 is connected because it is entirely in H_1, M_2 is connected because at least one of the endpoints of each of its edges is in Y which is connected in color 2. M_3 is connected because if $X_1 \neq \emptyset$ then $B(X_1, Y_1)$ is complete bipartite in color 3 and the rest of its edges have both endpoints in Y_1. If $X_1 = \emptyset$ then the edges of M_3 span a complete graph in color 3.

Case 2: $|X| > |Y|$. In this case we reduce the problem to the 4-*partite* case.

If either $V(H_1)$ or X is connected in G_2 or G_3 then we can use an argument similar to the one we used in case $|X| \leq |Y|$ to get the desired partition. Indeed, assume that, say, X is connected in G_2. Since $|V(H_1)| \geq n/2 \geq |X|$, take arbitrary $(|X| - |Y|)/2$ edges from M_1 (note that $|X| - |Y|$ is even, since n is even) and let Z be the union of their $|X| - |Y|$ endpoints and Y, $|Z| = |X|$. Let M_2 be a maximum matching in $B(Z, X)$ in color 2. Since we assumed that X is connected in G_2, the matching M_2 is connected. The yet uncovered vertices in $B(Z, X)$ form a balanced complete bipartite graph in color 3, cover them with a matching in color 3. Those edges in M_1 which do not have endpoints in Z, M_2 and M_3 give the desired partition. The same argument works if H_1 is connected in G_2 or G_3.

Let A_1 be the intersection of a component of G_2 with $V(H_1)$. We may assume that $\emptyset \neq A_1 \neq V(H_1)$, else $V(H_1)$ would be connected in G_3, G_2, respectively. Set $A_2 = V(H_1) \setminus A_1$. If that color component does not extend to X then all edges between A_1 and X are colored 3 which would imply that X is connected in G_3. So let $\emptyset \neq A_3 \neq X$ be the subset of the vertices of X which are in the same color component with A_1 in G_2, $A_4 = X \setminus A_3$. Clearly all edges in $B(A_1, A_4)$ and $B(A_2, A_3)$ are colored 3, else the color component in G_2 containing vertices of $A_1 \cup A_3$ would contain a vertex from $A_2 \cup A_4$ contradicting to the definition of A_i-s. If a single edge in $B(A_1, A_3)$ or $B(A_2, A_4)$ is colored 3 then $B(V(H_1), X)$ is connected in color 3. Therefore, we may assume that all edges in

$B(A_1, A_4)$ and $B(A_2, A_3)$ are colored 2. Finally, if a single edge in $B(A_1, A_2)$ or $B(A_3, A_4)$ is colored 2 or 3 then $B(V(H_1), X)$ is connected in color 2 or 3, respectively. Therefore, we may assume that all edges in $B(A_1, A_2)$ and $B(A_3, A_4)$ are colored 1. Thus we have a 4-partite coloring and the proof will be finished by Lemma 5 below. $\square$

We notice that the proof above gives immediately the following (so far we did not have to repeat a color).

Corollary 1. *Let n be even and assume that we have a 3-edge coloring of the edges of K_n that is* not *4-partite. Then $V(K_n)$ can be partitioned into (at most three) monochromatic connected matchings of distinct colors.*

Lemma 5. *Let n be even and assume that we have a 4-partite 3-edge coloring of the edges of K_n. Then $V(K_n)$ can be partitioned into three monochromatic connected matchings.*

Proof of Lemma 5. In the proof we consider how the orders $|A_i|$ and the orders of monochromatic matchings inside each A_i relate to each other. We reduce the number of cases to be checked to just a few. To check these we use only basic graph theory and a theorem of Cockayne and Lorimer on the Ramsey numbers of matchings.

For transparency we assume first that all $|A_i|$'s are even. A matching is called *crossing* if its edges all go between different A_i's and *inner* if its edges are all within A_i's. A crossing matching C is *proper* with respect to an inner matching M if the vertex set of C intersects any edge of M in two or zero vertices.

Let $a_i(j)$ denote the size of a maximum matching in A_i in color j. Here and through the whole proof we consider the size of a matching to be the number of vertices it covers, i.e. twice the number of edges. A matching covering all vertices of X is called perfect in X. The indices will always show the parts in or among which the matching edges are considered, the number in parenthesis is the color. For example, an inner matching $M_3(2)$ is in A_3 and its edges are colored with color 2, a crossing matching $M_{2,4}(3)$ is between A_2, A_4 in color 3.

There are two basic types for the connected components of the required partition into three connected matchings, one is when the components have three different colors, called the *star-like partition*, for example where the three matchings are in the components $A_1 \cup A_4, A_2 \cup A_4, A_3 \cup A_4$ (of color $3, 2, 1$, respectively). The other type is the *path-like partition* that repeats a color, as in the components $A_1 \cup A_3, A_3 \cup A_2, A_2 \cup A_4$ (of colors $2, 3, 2$, respectively.) The three components are referred as the *target components* in both (star-like and path-like) cases.

Claim 1. *If*

$$|A_4| \geq |A_1| + |A_2| + |A_3| - (a_1(3) + a_2(2) + a_3(1)) \tag{1}$$

then there is a star-like partition of K_n.

Proof. Let $M_1(3), M_2(2), M_3(1)$ be inner matchings of size $a_1(3), a_2(2), a_3(1)$, respectively, and let M be an arbitrary perfect matching of A_4. Condition (1) ensures that we can select a crossing matching C that is proper with respect to M and matches

$$(A_1 \setminus V(M_1(3))) \cup (A_2 \setminus V(M_2(2))) \cup (A_3 \setminus V(M_3(1)))$$

to A_4. Since the matchings not covered by C, i.e. $M_1(3), M_2(2), M_3(1)$ and the uncovered part of M, are in the same target components, the claim follows. $\square$

So we may assume

$$|A_4| < |A_1| - a_1(3) + |A_2| - a_2(2) + |A_3| - a_3(1). \tag{2}$$

Next notice that the inequalities

$$\begin{aligned} |A_2| - a_3(1) &< |A_4| - a_4(2) && (3)\\ |A_3| - a_4(2) &< |A_2| - a_2(3) && (4)\\ |A_4| - a_2(3) &< |A_3| - a_3(1) && (5) \end{aligned}$$

cannot hold at the same time. Indeed, else their sum gives $0 < 0$, a contradiction. So at least one of these inequalities is violated and we may assume that one of the following cases must hold:

$$\begin{aligned} |A_2| - a_3(1) &\geq |A_4| - a_4(2) && (6)\\ |A_3| - a_4(2) &\geq |A_2| - a_2(3) && (7)\\ |A_4| - a_2(3) &\geq |A_3| - a_3(1) && (8) \end{aligned}$$

Case 1: (6) holds. Here we will find a path-like partition in the components $A_1 \cup A_3, A_3 \cup A_2, A_2 \cup A_4$ (of colors $2, 3, 2$, respectively).

Match vertices of A_1 arbitrarily in color 2 to $|A_1|$ vertices of A_3. Denote this matching by $M_{1,3}(2)$. The rest of the vertices in A_3 can be partitioned into three monochromatic matchings, $M_3(1), M_3(2), M_3(3)$. Match the endpoints of the edges in $M_3(1)$ arbitrarily to $|M_3(1)|$ vertices in A_2, obtaining $M_{3,2}(3)$. This is feasible, since by (6)

$$|A_2| \geq |A_4| - a_4(2) + a_3(1) \geq |M_3(1)|.$$

Now take an inner matching $M_4(2)$ of size $a_4(2)$. The yet uncovered $|A_2| - |M_3(1)|$ vertices in A_2 will be matched to vertices in A_4 so that this matching $M_{2,4}(2)$ covers $A_4 \setminus V(M_4(2))$, and it is proper with respect to $M_4(2)$. This is feasible, because by (6)

$$|A_4| - a_4(2) \leq |A_2| - a_3(1) \leq |A_2| - |M_3(1)| = |A_2| - \frac{|M_{3,2}(3)|}{2}.$$

Since the part of $V(K_n)$ uncovered by the crossing matching $M_{1,3}(2) \cup M_{3,2}(3) \cup M_{2,4}(2)$ is covered by $M_3(2) \cup M_3(3) \cup M_4(2)$ which belong to the target components, we have the required partition.

Case 2: (7) holds. Here we define a path-like partition in the components $A_1 \cup A_4, A_4 \cup A_3, A_3 \cup A_2$ (of colors $3, 1, 3$, respectively).

Let $M_{1,4}(3)$ be an arbitrary crossing matching that maps A_1 to A_4 and partition the uncovered vertices of A_4 into three monochromatic matchings $M_4(1), M_4(2), M_4(3)$.

Subcase 2.1: $|M_4(2)| \leq |A_3| - |A_2|$. Let $M_{2,3}(3)$ be an arbitrary crossing matching that maps A_2 to A_3. Let $M_{4,3}(1)$ be a crossing matching from the uncovered part of A_3

into $A_4 \setminus V(M_{1,4}(3))$ such that it covers $M_4(2)$ and it is proper with respect to $M_4(1) \cup M_4(2) \cup M_4(3)$. This is feasible since

$$M_4(2) \leq |A_3| - |A_2| \leq |A_4| - |A_1|$$

and the vertex set uncovered by the union of the three crossing matchings is covered by matchings in the same target components (by $M_4(1) \cup M_4(3)$).

Subcase 2.2: $|M_4(2)| > |A_3| - |A_2|$. Now we match $V(M_4(2))$ arbitrarily into $U \subseteq A_3$ by a crossing matching $M_{4,3}(1)$. This is possible since by (7)

$$|A_3| \geq |A_2| - a_2(3) + a_4(2) \geq |A_2| - a_2(3) + |M_4(2)| \geq |M_4(2)|.$$

Then take a matching $M_2(3)$ of size $a_2(3)$ in A_2. There exists a crossing matching $M_{3,2}(3)$ from $A_3 \setminus U$ to A_2 such that it covers $A_2 \setminus V(M_2(3))$ and it is proper with respect to $M_2(3)$ because by (7)

$$\begin{aligned} |A_2| - |V(M_2(3)| &= |A_2| - a_2(3) \leq |A_3| - a_4(2) \leq |A_3| - |M_4(2)| \\ &= |A_3| - |U| < |A_2|, \end{aligned}$$

where the last inequality follows from the subcase condition. The vertex set uncovered by the union of the three crossing matchings is covered by $M_4(1) \cup M_4(3)$ so covered by matchings in the target components.

Case 3: (8) holds. A_4 is partitioned into matchings $M_4(1), M_4(2), M_4(3)$. Here we define four subcases.

Subcase 3.1: $|A_2|+|A_3|-|A_1| \geq |A_4|-(|M_4(1)|+|M_4(2)|)$. Here we use the components $A_1 \cup A_2, A_2 \cup A_4, A_4 \cup A_3$ (of colors $1, 2, 1$, respectively).

First we take $M_{1,2}(1)$ as an arbitrary crossing matching that matches all vertices of A_1 to A_2. The uncovered part of A_2 is partitioned into matchings $M_2(1), M_2(2), M_2(3)$. Take a matching $M_3(1)$ of size $a_3(1)$ in A_3.

We want to define a crossing matching M^* from $A_3 \cup (A_2 \setminus V(M_{1,2}(1))$ to A_4 such that $M^* = M_{2,4}(2) \cup M_{3,4}(1)$ and has the following two properties. On one hand, we want $M_{2,4}(2)$ to cover $M_2(3)$ and $M_{3,4}(1)$ to cover $A_3 \setminus V(M_3(1))$. This is possible since by (8)

$$|M_2(3)| + |A_3| - a_3(1) \leq |M_2(3)| + |A_4| - a_2(3) \leq |A_4|. \tag{9}$$

On the other hand, we want M^* to cover $M_4(3)$ and this is guaranteed by the condition of the present subcase. Indeed

$$|A_2| - |A_1| + |A_3| \geq |M_4(3)| = |A_4| - (|M_4(1)| + |M_4(2)|). \tag{10}$$

Therefore M^* can be defined with the required properties as a proper matching with respect to $M_2(1) \cup M_2(2) \cup M_4(1) \cup M_4(2)$. Notice that the definition of M^* ensures that the vertices uncovered by $M_{1,2}(1) \cup M^*$ are in the target components. This finishes Subcase 3.1.

Subcase 3.2: $|A_1| + (|A_3| - |A_2|) \geq |M_4(2)|$. Here we use the components $A_1 \cup A_4, A_4 \cup A_3, A_3 \cup A_2$ (of colors $3, 1, 3$, respectively) again.

Partition A_4 into matchings $M_4(1), M_4(2), M_4(3)$. First match all vertices of A_2 to A_3 to obtain $M_{2,3}(3)$.

Then $M_{1,4}(3)$ and $M_{3,4}(1)$ are defined so that their union is a crossing matching and proper with respect to $M_4(1) \cup M_4(3)$ and $M_{1,4}(3)$ matches the set A_1 to A_4 and $M_{3,4}(1)$ matches $A_3 \setminus V(M_{2,3}(3))$ to A_4. Since $|A_1| + |A_3| \leq |A_2| + |A_4|$, i.e. $|A_1| + (|A_3| - |A_2|) \leq |A_4|$, there is enough room in A_4 for $M_{1,4}(3)$ and $M_{3,4}(1)$. Moreover, by the subcase condition, we can also ensure that $M_{1,4}(3) \cup M_{3,4}(1)$ covers $M_4(2)$. Therefore the vertices uncovered by $M_{2,3}(3) \cup M_{1,4}(3) \cup M_{3,4}(1)$ are covered by $M_4(1) \cup M_4(3)$, so they are in the target components. This finishes Subcase 3.2.

We may assume that the conditions of the previous two subcases are violated. Adding their negations we get $2|A_3| < |A_4| - |M_4(1)|$, so we have

$$\begin{aligned} |A_2| + |A_3| &\leq 2|A_3| < |A_4| - |M_4(1)| && (11)\\ &< |A_1| - a_1(3) + |A_2| - a_2(2) + |A_3| - a_3(1) - |M_4(1)|, && (12) \end{aligned}$$

where the last inequality follows from (2). Therefore,

$$|A_1| > |M_4(1)|. \tag{13}$$

Subcase 3.3: $a_3(3) \geq |A_3| - |A_2|$ (or $a_3(2) \geq |A_3| - |A_1|$). This condition ensures a crossing matching $M_{2,3}(3)$ that matches the set A_2 to A_3 so that the uncovered part of A_3 has a perfect matching $M_3(3)$. On the other hand, condition (13) ensures that the set A_1 can be matched to A_4 properly by $M_{1,4}(3)$ with respect to $M_4(2) \cup M_4(3)$ so that it covers $V(M_4(1))$. Now matchings $M_{2,3}(3) \cup M_3(3)$, $M_{1,4}(3)$ and the uncovered edges of $M_4(2)$ are three matchings and the edges uncovered by these are in $M_4(3)$ i.e. in a target component. The condition $a_3(2) \geq |A_3| - |A_1|$ is completely similar, just using crossing matchings from A_1 to A_3, A_2 to A_4 respectively. This finishes Subcase 3.3.

Subcase 3.4: We may assume that the inequalities of Subcase 3.3 are violated as well and thus we have the

$$\begin{aligned} a_3(3) &< |A_3| - |A_2| = x && (14)\\ a_3(2) &< |A_3| - |A_1| = y && (15) \end{aligned}$$

upper bounds in two colors for the maximum monochromatic matching in the 3-colored complete graph spanned by A_3. Now we will use the following Theorem of Cockayne and Lorimer [4] to get a lower bound z for $a_3(1)$, in terms of $|A_3|, x, y$.

Theorem 4. *[Cockayne and Lorimer, [4]] Assume that $n_1, n_2, n_3 \geq 1$ are integers such that $n_1 = max(n_1, n_2, n_3)$. Then for $n \geq n_1 + 1 + \sum_{i=1}^{3}(n_i - 1)$ every 3-colored K_n contains a matching of color i with n_i edges for some $i \in \{1, 2, 3\}$.*

Using the notation that the size of a matching is twice the number of its edges (as we did in the proof), an easy computation from Theorem 4 gives that $z = |A_3| - \frac{x+y}{2} + 2$ if $z \geq x, y$ (i.e. z is the maximum among x, y, z). Therefore in this case

$$a_3(1) \geq z > |A_3| - \frac{x+y}{2}. \tag{16}$$

Substituting x, y to (16) we get

$$a_3(1) > |A_3| - \frac{2|A_3| - |A_1| - |A_2|}{2} = \frac{|A_1| + |A_2|}{2}, \tag{17}$$

Now choose a matching $M_3(1)$ of size $a_3(1)$ in A_3. Using (11), $|A_1| \leq |A_2|$ and (17)

$$\begin{aligned} |A_4| &> |A_2| + |A_3| \geq \frac{|A_1| + |A_2|}{2} + |A_3| = |A_1| + |A_2| + \left(|A_3| - \frac{|A_1| + |A_2|}{2}\right) \\ &\geq |A_1| + |A_2| + |A_3 \setminus V(M_3(1))| = |A_1| + |A_2| + |A_3| - a_3(1), \end{aligned}$$

thus Claim 1 finishes the proof.

If z is not maximum then from $y \geq x$ the maximum is y and from Theorem 4, $z = 2|A_3| - (x + 2y) + 4$. Thus here

$$a_3(1) \geq z > 2|A_3| - (x + 2y). \tag{18}$$

Substituting x, y to (18)

$$a_3(1) > 2|A_3| - (2(|A_3| - |A_1|) + |A_3| - |A_2|) = 2|A_1| + |A_2| - |A_3|. \tag{19}$$

Now choose a matching $M_3(1)$ of size $a_3(1)$ in A_3. Using (11) we get

$$|A_4| > 2|A_3| > 2|A_3| - |A_1| = |A_1| + |A_2| + (|A_3| - (2|A_1| + |A_2| - |A_3|)).$$

If $2|A_1| + |A_2| - |A_3|$ is negative then $|A_4| > |A_1| + |A_2| + |A_3|$, otherwise by (19), $|A_4| > |A_1| + |A_2| + (|A_3| - a_3(1))$. In both cases Claim 1 finishes the proof.

The reader who followed the proof probably agrees that the cases when two or four of the $|A_i|$'s are odd can be treated easily from the following general remark. The inequalities used in the proofs are either sharp and then determine the parity of both sides or there is a slack of at least one and that can be used to adjust the proof. $\square$

Proof of Lemma 3.

Since the proof is very straightforward, we do not address parity problems. By Corollary 1 we may assume that we have a 4-partite coloring (using the same notation as in the previous proof). Notice that equations

$$2|A_1| + a_2(1) + a_3(2) + a_4(3) < \frac{3n}{4} \tag{20}$$

$$a_1(1) + 2|A_2| + a_3(3) + a_4(2) < \frac{3n}{4} \tag{21}$$

$$a_1(2) + a_2(3) + 2|A_3| + a_4(1) < \frac{3n}{4} \tag{22}$$

$$a_1(3) + a_2(2) + a_3(1) + 2|A_4| < \frac{3n}{4}, \tag{23}$$

do not hold at the same time. Else summing them we get

$$\sum_{\substack{1 \leq i \leq 4 \\ 1 \leq j \leq 3}} a_i(j) < n,$$

a contradiction, because the union of perfect matchings within the A_i-s cover all n vertices.

We may assume that some, say the first, of the four (symmetric) inequalities fails, i.e.,

$$2|A_1| + a_2(1) + a_3(2) + a_4(3) \geq \frac{3n}{4}.$$

Select matchings $M_2(1), M_3(2), M_4(3)$ of size $a_2(1), a_3(2), a_4(3)$ in A_2, A_3, A_4, respectively.

If $|A_1| \geq |A_2| - a_2(1) + |A_3| - a_3(2) + |A_4| - a_4(3)$, then similarly to the case of Claim 1 we have a star-like partition, i.e., we cover perfectly all the vertices and all colors are different. Otherwise let M be a matching from A_1 to $B = (A_2 \cup A_3 \cup A_4) \setminus (V(M_2(1)) \cup V(M_3(2)) \cup V(M_4(3)))$. Clearly, $M \cup M_2(1) \cup M_3(2) \cup M_4(3)$ is a union of three connected monochromatic matchings in colors $1, 2, 3$ and is of size $2|A_1| + a_2(1) + a_2(3) + a_4(3) \geq \frac{3n}{4}$. $\square$

3 Moving from complete graphs to almost complete ones

In this section we prove Lemmas 2 and 4 from Lemmas 1 and 3 by outlining the technical steps needed to get the 'density version' of a 'complete graph theorem'. Since applications of the Regularity Lemma require working on the 'reduced graph' (or cluster graph), the authors and others worked out techniques to get variants of results from the complete graph K_n to $(1-\epsilon)$-dense graphs (that have at least $(1-\epsilon)\binom{n}{2}$ edges). Here we apply the method in [12] that replaces the $(1-\epsilon)$-dense graph by a more convenient subgraph H described in the next lemma. Here $\delta(G)$ denotes the minimum, $\Delta(G)$ the maximum degree of a graph G and $d(v)$ is the degree of a vertex v.

Lemma 6 (Lemma 9 in [12])**.** *Assume that G_n is $(1-\varepsilon)$-dense. Then G_n has a subgraph H with at least $(1-\sqrt{\varepsilon})n$ vertices such that: A. $\Delta(\overline{H}) < \sqrt{\varepsilon}n$; B. $\delta(H) \geq (1-2\sqrt{\varepsilon})n$; C. H is $(1-2\sqrt{\varepsilon})$-dense.*

To transform the proof of Lemma 1 to the proof of Lemma 2 we do the following. We start with a 3-edge colored $(1-\epsilon)$-dense graph G_n and we find there a subgraph H described in Lemma 6. Then one can basically follow the steps of the proof of Lemma 1 just using H instead of K_n. For example, the first paragraph of the proof of Lemma 1 can be rewritten as follows.

Suppose first that G_1, the graph with edges of color 1, has a connected component of size at least $(1-2\sqrt{\epsilon})n$. Then take a maximum matching M_1 from this component. The edges of $V(H) \setminus V(M_1)$ are colored with two colors $2, 3$, one of the colors, say color 2 almost spans its vertex set. In fact this density version of the well-known remark that a 2-colored complete graph is connected in one of the colors can be easily proved. Alternatively we can refer to an easy lemma (Lemma 11) from [12] implying that $V(H) \setminus V(M_1)$ has a connected component in color 2 covering all but at most $4\sqrt{\epsilon}n$ vertices. Take a maximum matching M_2 in color 2, now all edges of $V(H) \setminus (V(M_1) \cup V(M_2))$ are in color 3, therefore a maximum matching M_3 will be connected and covers all but at most $\sqrt{\epsilon}n$ vertices. Thus

we conclude that $M_1 \cup M_2 \cup M_3$ are three connected matchings covering all but at most $7\sqrt{\epsilon}n$ vertices of H.

Translating the second paragraph of the proof of Lemma 1, let H_1 be a largest monochromatic component of H. The statement $|V(H_1)| \geq \frac{n}{2}$ (referred to [7] for proof) should be changed to the analogous statement $|V(H_1)| \geq (\frac{1}{2} - 2\sqrt{\epsilon})n$. This can be proved easily from the following easy Lemma ([12]).

Lemma 7. *Assume H has n vertices, $\Delta(\overline{H}) < \sqrt{\varepsilon}n$ and $F = [A, B]$ is a bipartite subgraph of H with $2\sqrt{\varepsilon}n < |A| \leq |B|$. Then F is a connected subgraph of H and contains a matching of size at least $|A| - \sqrt{\varepsilon}n$.*

Let C_1 be a largest monochromatic component in H, say in color 1. At any vertex $v \in V(C_1)$ let C_2 be the largest monochromatic component containing v in a different color, say in color 2. Suppose indirectly that $|C_1| < (\frac{1}{2} - 2\sqrt{\epsilon})n$. Then

$$|C_2| > \frac{1 - \sqrt{\epsilon} - (1/2 - 2\sqrt{\epsilon})}{2} n$$

thus $|C_1| \geq |C_2| \geq 4\sqrt{\epsilon}n$ for small enough ϵ. Set $X = V(H) \setminus (C_1 \cup C_2)$. If $|X| < 2\sqrt{\epsilon}n$ then $|C_1|$ or $|C_2|$ is large enough, at least $\frac{1-3\sqrt{\epsilon}}{2}n$. If for $Y = C_1 \cap C_2$, $|Y| < 2\sqrt{\epsilon}n$ then $Z_1 = C_1 \setminus C_2$ and $Z_2 = C_2 \setminus C_1$ both have at least $2\sqrt{\epsilon}n$ vertices and from Lemma 7 $[Z_1, Z_2]$ is connected in color 3, but would be larger then C_1 - contradiction. Using Lemma 7 again, $V(H)$ (apart from $\sqrt{\epsilon}n$ vertices) covered by the connected color 3 bipartite graphs $[Z_1, Z_2], [X, Y]$ thus one of them is large enough, completing the proof.

In Case 1 of the proof of Lemma 1 the subcase $X_1 \neq \emptyset$ is replaced by $|X_1| > 2\sqrt{\epsilon}n$ and that ensures a connected matching M_3 of $B(X_1, Y_1)$ in color 3 covering all but at most $\sqrt{\epsilon}n$ vertices of X. To finish the covering of the vertices of Y_1 we have to observe only that the uncovered vertices span an almost complete graph in color 3 (instead of a complete one) and this clearly allows to extend M_3 and $M = M_1 \cup M_2 \cup M_3$ covers all but at most $2\sqrt{\epsilon}n$ vertices of H.

In Case 2 of the proof of Lemma 1 the condition 'if either $V(H_1)$ or X is connected in G_2 or G_3' is naturally changed to 'if either $V(H_1)$ or X has a connected component in G_2 or G_3 covering all but at most $2\sqrt{\epsilon}n$ vertices' and this way the case is reduced to Case 1.

This leads to reduction to 4-partite colorings with the natural change in the definition: the condition $A_i \neq \emptyset$ should be replaced by $|A_i| \geq 2\sqrt{\epsilon}n$ and the monochromatic bipartite graphs $B(A_i, A_j)$ are not complete but at most $\sqrt{\epsilon}n$ edges are missing from any of their vertices. To finish the proof of Lemma 2 we can use the following analogue of Lemma 5.

Lemma 8. *Assume that we have a 4-partite 3-edge coloring of the edges of H. Then there are three pairwise disjoint monochromatic connected matchings covering all but at most $8\sqrt{\epsilon}n$ vertices of H.*

The proof of Lemma 8 is really straightforward, repeating the steps of the proof of Lemma 5 with the obvious modification dictated by the fact that the monochromatic bipartite graphs $[A_i, A_j]$ are not complete.

Then the proof of Claim 1 and the proofs of Cases 1,2,3 can be repeated exactly as stated, the only difference is that the star-like and path-like partitions constructed may leave a negligible number of vertices uncovered. By Lemma 7 every crossing matching from A_i to A_j may leave at most $\sqrt{\epsilon}n$ vertices of A_i uncovered. Also, it is obvious that inner matchings selected within A_i may leave at most $\sqrt{\epsilon}n$ vertices of A_i uncovered. Since one uses at most seven crossing and inner matchings in the proof of Claim 1 and in all subsequent cases (and subcases), at most $7\sqrt{\epsilon}n$ vertices of H remain uncovered. We added an extra $\sqrt{\epsilon}n$ for just coping with the parity problem, i.e. crossing matchings may leave one vertex uncovered.

In the final step of the proof (Subcase 3.4) we have to apply inside A_3 the following density version of Theorem 4 (with $s+1=\sqrt{\epsilon}n$).

Theorem 5 (Theorem 3 in [12])**.** *Assume that n_1, n_2, n_3 are nonnegative integers such that we have $n_1 = max(n_1, n_2, n_3)$, s is a nonnegative integer and G is a graph on n vertices such that for each $v \in V(G)$, $d(v) \geq n-1-s$. If*

$$n \geq s + n_1 + 1 + \sum_{i=1}^{3}(n_i - 1)$$

then, in any 3-coloring of the edges of G there is a matching of color i with n_i edges for some $i \in \{1,2,3\}$.

One can derive Lemma 4 by similar modifications from the proof of Lemma 3. □

4 Moving from connected matchings to cycles

In this section we sketch how to derive Theorem 1 from Lemma 2. This technique is fairly standard by now, it has been applied and much discussed in [2], [10], [12] and [15] among others. Therefore here we just give a brief overview, the missing details can be found in these papers.

We apply the edge-colored version of the Regularity Lemma to a 3-colored K_n with a small enough ε, we define the reduced graph G^R and we introduce a majority coloring in G^R. Using Lemma 2 we find three monochromatic connected matchings which partition most of the vertices of G^R. Then we turn these connected matchings into monochromatic cycles in K_n with a procedure suggested first by Łuczak in [15]. In fact, we can just use the following abridged version of a lemma from [6] (Lemma 4.2, where it was used for k-colorings and for Berge-cycles of hypergraphs).

Lemma 9. *Assume that for some positive constant c we find a monochromatic connected matching M saturating at least $c|V(G^R)|$ vertices of G^R. Then in the original 3-edge colored K_n we find a monochromatic cycle of length at least $c(1-3\varepsilon)n$.*

Here ε is the same with which we use the Regularity Lemma. For the convenience of the reader we just sketch the proof of Lemma 9 from [6]. Using the fact that M is

connected we can connect the matching edges by monochromatic paths, following a cyclic ordering of the edges in M. Then go back to the original graph and replace these paths by short monochromatic vertex disjoint connecting paths between the cluster pairs that associated to the edges of M. These connecting paths will be parts of the final cycle. To define the rest of the cycle, remove the internal vertices of these connecting paths and remove also a small number of exceptional vertices from each cluster pair associated to M to guarantee super-regularity and to make sure that the cluster pairs are balanced. Then in each cluster pair find a monochromatic Hamiltonian path to close the connecting pairs to a cycle (the existence of this Hamiltonian path is well-known; see e.g. Lemma 2 in [11] or actually this is a very special case of the Blow-up Lemma [14]).

5 Proof of Theorem 2

In this section we sketch the proof of Theorem 2. The proof is using Theorem 1 and results from [11], hence we omit some of the details.

Again just as in Section 4 we apply the edge-colored version of the Regularity Lemma to a 3-colored K_n with a small enough ε, we define the reduced graph G^R and we introduce a majority coloring in G^R. We will need the concept of a *half dense* matching from [11]: a matching M in a graph G is called k-half dense if one can label its edges as $x_1y_1, \ldots, x_{|M|}y_{|M|}$ so that each vertex of $X = \{x_1, \ldots, x_{|M|}\}$ (called the strong end points) is adjacent in G to at least k vertices of $Y = \{y_1, \ldots, y_{|M|}\}$.

Following the proof technique in [11] with $r = 3$, we find the at most 17 monochromatic cycles in the following steps.

- Step 1: We find a sufficiently large monochromatic (say red), half-dense connected matching M in G^R (more precisely an $(l/48)$-half dense matching where l is the number of vertices in G^R).

- Step 2: Apply Theorem 1 with a small enough δ to cover by three monochromatic vertex disjoint cycles most of the vertices (at least a $(1-\delta)$-portion) of the original graph K_n outside the cluster-pairs associated to M. Denote the set of leftover vertices (not covered by the three cycles and the cluster pairs) by B. We may assume that the number of remaining vertices in B is even by removing one more vertex (a degenerate cycle) if necessary.

- Step 3: We split B into two equal parts B_1 and B_2 (this is why we needed $|B|$ to be even) and these are matched with vertices from either side of M, thereby ensuring that the bipartite graph in the final step is balanced. Applying twice a lemma about cycle covers of r-colored unbalanced complete bipartite graphs (Lemma 8 from [11], here we use it with $r = 3$) we cover B_1 and B_2 and some vertices of the clusters associated with vertices of M by $2 \cdot 6 = 12$ cycles.

- Step 4: Finally after some adjustments through alternating paths with respect to M, we find a red cycle spanning the uncovered vertices of K_n.

Steps 1, 3 and 4 are identical to the corresponding steps in [11] with $r = 3$. The only difference in the two proofs is that here in Step 2 we are using Theorem 1 instead of the greedy technique applied in [11].

For completeness we restate here the main lemmas and ideas needed in the above steps. In Step 1 the key lemma is the following.

Lemma 10 (Lemma 4 in [11])**.** *Every graph of average degree at least $8k$ has a connected k-half dense matching.*

We take the color class G_1^R that has the most edges in G^R, then the average degree in G_1^R is at least $\frac{1}{2}\frac{l}{3} = \frac{l}{6}$. Thus applying Lemma 10 we can indeed find a connected $l/48$-half dense matching M in G^R. Say M has size

$$|M| = l_1 \geq \frac{l}{48},$$

and the matching $M = \{e_1, e_2, \ldots, e_{l_1}\}$ is between the two sets of end points U_1 and U_2, where U_1 contains the strong end points, i.e. the points in U_1 have at least $l/48$ neighbors in U_2. Furthermore, define $f(e_i) = (V_1^i, V_2^i)$ for $1 \leq i \leq l_1$ where V_1^i is the cluster assigned to the strong end point of e_i, and V_2^i is the cluster assigned to the other end point. Hence we have our large, red, half-dense, connected matching M as desired in Step 1. We find short connecting paths between the consecutive matching edges and we make the matched cluster pairs balanced super-regular, as usual (see Section 4). However, for technical reasons we postpone the closing of this cycle within each pair until the end of Step 4, since in Step 3 we will use some of the vertices in $f(M)$, and we will have to make some adjustments first in Step 4.

In Step 3 we will use the following two lemmas from [11].

Lemma 11 (Lemma 5 in [11])**.** *Let $\vec{G} = \vec{G}(V, E)$ be a directed graph with $|V| = n$ sufficiently large and minimum out-degree $d_+(x) \geq cn$ for some constant $0 < c \leq 10^{-3}$. Then there are subsets $X, Y \subseteq V$ such that*

- $|X|, |Y| \geq cn/2$;
- *From every $x \in X$ there are at least $c^6 n$ internally vertex disjoint paths of length at most c^{-3} to every $y \in Y$ (denoted by $x \hookrightarrow y$).*

Lemma 12 (Lemma 8 in [11])**.** *There exists a constant n_0 such that the following is true. Assume that the edges of the complete bipartite graph $K(A, B)$ are colored with r colors. If $|A| \geq n_0$, $|B| \leq |A|/(8r)^{8(r+1)}$, then B can be covered by at most $2r$ vertex disjoint monochromatic cycles.*

We have the connected, red matching M of size l_1 between U_1 and U_2. Define the auxiliary directed graph $\vec{G}$ on the vertex set U_1 as follows. We have the directed edge from V_1^i to V_1^j, $1 \leq i, j \leq l_1$ if and only if $(V_1^i, V_2^j) \in G_1^R$. The fact that M is $l/48$-half dense implies that in $\vec{G}$ for the minimum outdegree we have

$$\min_{x \in U_1} d_+(x) \geq \frac{l}{48} \geq \frac{|U_1|}{48}.$$

Thus applying Lemma 11 for $\vec{G}$ with $c = 10^{-3}$, there are subsets $X_1, Y_1 \subset U_1$ such that

- $|X_1|, |Y_1| \geq c|U_1|/2$;
- From every $x \in X_1$ there are at least $c^6|U_1|$ internally vertex disjoint paths of length at most c^{-3} to every $y \in Y_1$ ($x \hookrightarrow y$).

Let X_2, Y_2 denote the set of the other endpoints of the edges of M incident to X_1, Y_1, respectively. Note that a path in $\vec{G}$ corresponds to an alternating path with respect to M in G_1^R.

In each cluster $V_1^i \in Y_1$ let us consider an arbitrary subset of $c^8|V_1^i|$ vertices. Let us denote by A_1 the union of all of these subsets. Similarly we denote by A_2 the union of arbitrary subsets of $V_2^j \in X_2$ of size $c^8|V_2^j|$. Then we have

$$|A_1|, |A_2| \geq c^8|f(Y_1)| \geq c^8\frac{c}{2}|f(U_1)| \geq c^8\frac{c}{2} \cdot \frac{n}{48} \geq c^{10}n.$$

Let us divide the vertices in B (B was defined in Step 2) into two equal sets B_1 and B_2. We apply Lemma 12 with $r = 3$ in $K(A_1, B_1)$ and in $K(A_2, B_2)$. The conditions of the lemma are satisfied if δ is small ($\delta \ll c^{10}$). Let us remove the at most 6 vertex disjoint monochromatic cycles covering B_1 in $K(A_1, B_1)$ and the at most 6 cycles covering B_2 in $K(A_2, B_2)$. By doing this we may create discrepancies in the number of vertices in the cluster-pairs associated to matching edges.

Thus in Step 4 we have to eliminate these discrepancies with the use of the many alternating paths. By removing the vertex disjoint monochromatic cycles covering B_1 in $K(A_1, B_1)$ we have created a 'surplus' of $|B_1|$ vertices in the clusters of Y_2 compared to the remaining number of vertices in the corresponding clusters of Y_1. Similarly by removing the cycles covering B_2 in $K(A_2, B_2)$ we have created a 'deficit' of $|B_2|(= |B_1|)$ vertices in the clusters of X_2 compared to the number of vertices in the corresponding clusters of X_1. The natural idea is to 'move' the surplus from Y_2 through an alternating path to cover the deficit in X_2. The description of this procedure can be found in [11] (Section 2.4 in [11]). Once the pairs are balanced we may close the red cycle within each pair by finding a Hamiltonian path as above (Lemma 2 in [11]). $\square$

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