Note
Erdős–Ko–Rado from intersecting shadows

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Abstract

A set system is called t-intersecting if every two members meet each other in at least t elements. Katona determined the minimum ratio of the shadow and the size of such families and showed that the Erdős–Ko–Rado theorem immediately follows from this result. The aim of this note is to reproduce the proof to obtain a slight improvement in the Kneser graph. We also give a brief overview of corresponding results.

1 Introduction

Throughout the paper we will investigate subsets of an n-element underlying set $[n] = \{1, 2, \ldots, n\}$. $\binom{[n]}{k}$ will denote the collection of all k-element subsets of $[n]$. A family $\mathcal{F}$ is said to be k-uniform if $\mathcal{F} \subseteq \binom{[n]}{k}$.

$\mathcal{F} \subseteq \binom{[n]}{k}$ is called *intersecting* if it does not contain disjoint sets. In general, $\mathcal{F}$ is *t-intersecting* if $|F_1 \cap F_2| \geq t$ for all $F_1, F_2 \in \mathcal{F}$.

The *Kneser graph*, $\mathrm{Kn}(n, k)$, is the graph whose vertices are the k-element subsets of $[n]$, i.e. $V(\mathrm{Kn}(n, k)) = \binom{[n]}{k}$ and two vertices are connected iff the two corresponding sets are disjoint. A *coclique* in a graph is a set of vertices, such that no two vertices in the set are adjacent. An intersecting family is a coclique in the corresponding Kneser graph. The maximum size of a coclique in a graph G is denoted by $\alpha(G)$.

The following theorem is one of the famous results in extremal combinatorics:

Theorem 1. (Erdős, Ko, Rado [3]) *If $k \leq n/2$, then*

$$\alpha(\mathrm{Kn}(n,k)) = \binom{n-1}{k-1}.$$

Obviously, the family consisting of the k-subsets that contain 1 has size $\binom{n-1}{k-1}$, so only the $\leq$ part is interesting.

Let $\mathcal{F} \subseteq \binom{X}{k}$ be a family of k-element sets; for $l \leq k$, the *l-shadow* of $\mathcal{F}$ is defined as $\Delta_l \mathcal{F} = \{G : |G| = l, \exists F \in \mathcal{F} \text{ such that } G \subset F\}$. It is clear that $\mathcal{F} = \binom{[2k-t]}{k}$ is t-intersecting and $\Delta_l \mathcal{F} = \binom{[2k-t]}{l}$. The next theorem shows that this is the extremal case in some sense.

Theorem 2. (Katona [5]) *Assume that $\mathcal{F}$ is a k-uniform, t-intersecting family. Then for $l \geq k-t$,*

$$\frac{|\Delta_l \mathcal{F}|}{|\mathcal{F}|} \geq \frac{\binom{2k-t}{l}}{\binom{2k-t}{k}}.$$

2 A generalization of the EKR theorem

In this section we deduce a slight generalization of the EKR theorem from Theorem 2.

For a set $A \subseteq V(\mathrm{Kn}(n,k))$, the neighborhood of A is denoted by $N(A)$. Similarly, for a given k-uniform family $\mathcal{F}$, let us introduce the notation $\mathcal{N}(\mathcal{F}) = \{H \in \binom{[n]}{k} : \exists F \in \mathcal{F} \text{ such that } H \cap F = \emptyset\}$ as the "neighborhood" of $\mathcal{F}$.

Theorem 3. *If $k \leq n/2$ and C is a coclique in the Kneser graph,* $\mathrm{Kn}(n,k)$, *then*

$$\frac{|C|}{|C| + |N(C)|} \leq \frac{k}{n}.$$

Since C is a coclique, C and $N(C)$ are disjoint, so $|C| + |N(C)| \leq |V(\mathrm{Kn}(n,k))| = \binom{n}{k}$ and the EKR theorem follows.

Proof. To apply Theorem 2, let $\mathcal{F}$ be the intersecting k-uniform family that corresponds to C. Let $\mathcal{F}^c$ be the family of complements, i.e. $\mathcal{F}^c = \{[n] \setminus F : F \in \mathcal{F}\} \subseteq \binom{[n]}{n-k}$. For each pair $F_1, F_2 \in \mathcal{F}$, we have $|F_1 \cup F_2| \leq 2k-1$, thus $\mathcal{F}^c$ is t-intersecting for $t = n-2k+1$. By Theorem 2,

$$\frac{|\Delta_k \mathcal{F}^c|}{|\mathcal{F}^c|} \geq \frac{\binom{2(n-k)-(n-2k+1)}{k}}{\binom{2(n-k)-(n-2k+1)}{n-k}} = \frac{n-k}{k}.$$

$|\mathcal{F}^c| = |\mathcal{F}|$ and $\Delta_k \mathcal{F}^c \subseteq \mathcal{N}(\mathcal{F})$, because for every $H \in \Delta_k \mathcal{F}^c$, $H \subseteq [n] \setminus F$ for some $F \in \mathcal{F}$ and $H \cap F = \emptyset$. Thus,

$$\frac{|N(C)|}{|C|} = \frac{|\mathcal{N}(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{n-k}{k}$$

and we are done. □

3 Similar results

Let $A \subseteq V(\mathrm{Kn}(n,k))$. For another slight generalization, we denote by $I(A)$ the family of isolated points in A, that is, $I(A) = \{a \in A : (a,b) \notin E(\mathrm{Kn}(n,k))$ for all $b \in A\}$. In his paper, Borg [1] extended Daykin's proof [2] of the EKR theorem to obtain the following improvement:

Theorem 4. (Borg) *If $A \subseteq V(\mathrm{Kn}(n,k))$ and $k \leq n/2$, then*

$$|I(A)| + \frac{k}{n}\,|A \setminus I(A)| \leq \binom{n-1}{k-1}.$$

It is easy to see that Theorems 3 and 4 are equivalent:

First, let A be an arbitrary subgraph of $\mathrm{Kn}(n,k)$. $C := I(A)$ is a coclique, so by Theorem 3,

$$\frac{|I(A)|}{|I(A)| + |N(I(A))|} \leq \frac{k}{n}.$$

By definition, $I(A)$, $A \setminus I(A)$ and $N(I(A))$ are disjoint, hence

$$|I(A)| + |A \setminus I(A)| + |N(I(A))| \leq \binom{n}{k}.$$

These two inequalities now imply Theorem 4.

On the other hand, if C is a coclique, let $A := V(\mathrm{Kn}(n,k)) \setminus N(C)$. By definition, C and $N(C)$ are disjoint, and $C \subseteq I(A)$. Thus, by Theorem 4,

$$|C| + \frac{k}{n}\,|V(\mathrm{Kn}(n,k)) \setminus N(C) \setminus C| \leq \binom{n-1}{k-1},$$

and Theorem 3 follows.

Remember that though the two theorems are equivalent, their proofs are quite different: while Theorem 3 is proved as a consequence of the theorem on shadows of intersecting families, Borg uses the Kruskal–Katona theorem [6, 7] to verify Theorem 4.

Remark 1. In [1], Borg also showed that Theorem 4 (and so Theorem 3) yields Hilton's theorem [4] for cross-intersecting sub-families of $\binom{[n]}{k}$.

Recently, J. Wang and H. Zhang [8, 9] investigated similar problems in general circumstances. A graph $G = (V, E)$ is called *vertex-transitive* if its automorphism group, $\mathrm{Aut}(G)$, acts transitively on V, i.e. for every $u, v \in V$ there exists a $\gamma \in \mathrm{Aut}(G)$ such that $\gamma(u) = v$.

The following theorem is the analogue of Theorem 3 for arbitrary vertex-transitive graph.

Theorem 5. (Zhang) *Let $G = (V, E)$ be a vertex-transitive simple graph. If $C \subseteq V$ is a coclique, then*

$$\frac{|C|}{|C| + |N(C)|} \leq \frac{\alpha(G)}{|V|}.$$

Note that the EKR theorem and Theorem 5 together imply Theorem 3.

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