Constructing union-free pairs of k-element subsets

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Abstract

It is proved that one can choose $\lfloor \frac{1}{2}\binom{n}{k} \rfloor$ disjoint pairs of k-element subsets of an n-element set in such a way that the unions of the pairs are all different, supposing that $n > n(k)$.

1 Introduction

The following notations will be used: $[n] = \{1, 2, \dots, n\}$ for an n-element set, and $\binom{[n]}{k}$ for the family of its all k-element subsets.

The main aim of the present paper is to prove the following theorem.

Theorem 1 *If $1 \le k, n$ are integers and n is large enough, that is $n \ge n(k)$, then one can find*

$$\left\lfloor \frac{1}{2}\binom{n}{k} \right\rfloor$$

unordered pairs $\{A_i, B_i\}$ so that all these sets are distinct elements of $\binom{[n]}{k}$, $A_i \cap B_i = \emptyset$ holds for every pair and

$$A_i \cup B_i \ne A_j \cup B_j \text{ holds for all } i \ne j. \tag{1}$$

*The work of the author was supported by the Hungarian National Foundation for Scientific Research grant number NK78439,

Our method however proves the following stronger, perhaps less natural statement.

Theorem 2 *If $n \geq n(k)$, then the members of $\binom{[n]}{k}$ can be listed in the following way:*

$$A_1, A_2, \ldots, A_N, A_{N+1} = A_1$$

where $N = \binom{n}{k}$, $A_i \cap A_{i+1} = \emptyset$ $(1 \leq i \leq N)$, and

$$A_i \cup A_{i+1} \neq A_j \cup A_{j+1} \text{ holds for all } i \neq j. \tag{2}$$

Theorem 2 trivially implies Theorem 1: take the pairs $\{A_{2\ell-1}, A_{2\ell}\}$ $(\ell = 1, 2, \ldots)$.

The proof of Theorem 2 will be based on a Hamiltonian type theorem that is a generalization of the following theorem of Dirac [2].

Theorem 3 *If G is a simple graph on $N \geq 3$ vertices and every degree in G is at least $\frac{N}{2}$, then G has a Hamiltonian cycle.*

We will use a similar theorem for the graph

$$G_1 = \left(\binom{[n]}{k}, E\right)$$

where two vertices are adjacent (their pair is in E) if the k-element sets are disjoint. The simple application of Dirac's theorem for this graph G_1 would lead to a cyclic listing of all k-element sets in such a way that the neighboring ones are disjoint. However we have to forbid simultaneous occurrences of certain pairs along the cycle. This will be formulated in a more general context.

The forbidden pairs of edges form a new graph with vertex set E. Let $G_2 = (E, F)$ be the new graph where F is the set of forbidden pairs of non-adjacent edges (that is edges without common vertices). We need to find a Hamiltonian cycle in G_1 which contains no two edges whose pair is in F.

Theorem 4 *Let $G_1 = (V, E)$ and $G_2 = (E, F)$ be graphs where F contains pairs of edges non-adjacent in G_1. Let further the minimum degree in G_1 be δ and the maximum degree in G_2 be Δ. Suppose that*

$$\textit{a path of three edges in } G_1 \textit{ cannot contain an element of } F \tag{i}$$

and that the inequality

$$\delta \geq N\left(1 - \frac{1}{2(\Delta+1)}\right) \qquad (ii)$$

holds. Then there is a Hamiltonian cycle in G_1 containing no pair of edges $\in F$.

Historical comments and a list of similar results will be given in Section 4.

2 Proof of Theorem 4

We will use an indirect way. Suppose that the desired Hamiltonian cycle does not exist. Delete elements of F one by one. When G_2 becomes empty then a Hamiltonian cycle is to be found without any restriction. Therefore (ii) and Dirac's theorem implies its existence. On the way from G_2 to the empty graph there is a place of jump: there is no Hamiltonian cycle if G_2 is replaced by $G_2^* = (E, F^* \cup \{f\})$ but there is one if it is replaced by $G_2^{**} = (E, F^*)$. The Hamiltonian cycle in the latter case must contain two edges which are the elements of f. Deleting one of theses edges from the Hamiltonian cycle a Hamiltonian path is obtained that contains no pair of edges from F^*.

Let the vertices in this Hamiltonian path be ordered in the following way: $v_1, v_2, \ldots, v_N$. Let P denote the set of edges along this path: $P = \{\{v_i, v_{i+1}\} : 1 \leq i < N\}$. Then no element of F^* is a subset of P. Define $A^1 \subset P$ as the set of edges along the path having a pair in F^* which is incident to v_1:

$$A^1 = \{\{v_i, v_{i+1}\} : \ 1 < i < N, \{\{v_1, v_\ell\}, \{v_i, v_{i+1}\}\} \in F^* \text{ holds for some } \ell\}.$$

Let L denote the set of endpoints of edges in G_1 with the starting point v_1:

$$L = \{\ell : \ \{v_1, v_\ell\} \in E\}.$$

Suppose that $\{v_i, v_{i+1}\} \in A^1$ and either $v_i \in L$ or $v_{i+1} \in L$ holds. Then $\{\{v_1, v_\ell\}, \{v_i, v_{i+1}\}\} \in F^*$ holds for some $\ell \neq i, i+1$. Hence $\{v_1, v_\ell\}, \{v_i, v_{i+1}\}$ and either $\{v_1, v_i\}$ or $\{v_1, v_{i+1}\}$ form a path of length 3 in E. Two of them form a pair in F^* by the definition of A^1. This contradicts (i) therefore $\{v_i, v_{i+1}\}$ cannot be in A^1. The conclusion is that the edges $\{v_i, v_{i+1}\}(1 < i)$ in the path cannot be in A^1 if one of their endpoints is in L. The number of such edges along the path is at least $|L| \geq \delta$. Hence we have $|A^1| \leq N-2-\delta$. Here $A^1 \subset E$ is a set of vertices of G_2. The degrees in G_2 are $\leq \Delta$ the total number of vertices in G_2 adjacent to at least one element of A^1 is $\leq \Delta(N-2-\delta)$. There are at least δ edges in E of the form $\{v_1, v_\ell\}$, we have at least $\delta - \Delta(N-2-\delta)$ of them which do not form an edge in G_2 with an edge in P. Here $\delta - \Delta(N-2-\delta) \geq \frac{N}{2}$ holds by (ii). If B^1 denotes the set of edges $\{v_1, v_\ell\}$ in G_1 which are not vertices in G_2 adjacent to vertices $\in P$ then we obtain the conclusion of this paragraph

$$|B^1| \geq \frac{N}{2}. \tag{3}$$

This can be repeated with the other end of the path. Let B^N denote the set of edges $\{v_N, v_\ell\}$ in G_1 which are not vertices in G_2 adjacent to vertices $\in P$. We have

$$|B^N| \geq \frac{N}{2}. \tag{4}$$

The edge $\{v_i, v_{i+1}\}$ is called *start-pinned* if $\{v_N, v_i\} \in B^N$. On the other hand, it is *end-pinned* if $\{v_1, v_{i+1}\} \in B^1$. By (3) and (4) there is an edge $\{v_r, v_{r+1}\}$ which is both start- and end-pinned. Hence the edges $\{v_1, v_{r+1}\}$ and $\{v_N, v_r\}$ can be added to P without violating the conditions, that is $P \cup \{\{v_1, v_{r+1}\}, \{v_N, v_r\}\}$ contains no element of F^*. (The relation $\{\{v_1, v_{r+1}\}, \{v_N, v_r\}\} \notin F^*$ is a direct consequence of (i).) The sequence of vertices $v_1, v_{r+1}, v_{r+2}, \ldots, v_N, v_r, v_{r-1}, \ldots, v_2, v_1$ determines a Hamiltonian cycle satisfying the conditions. This is a contradiction, the statement is proved. □

3 Proof of Theorem 2

Use Theorem 4 for the following graphs G_1 and G_2.

$$G_1 = \left(\binom{[n]}{k}, E\right)$$

where $\{A, B\} \in E$ if and only if $A \cap B = \emptyset$, F consists of the pairs $\{\{A, B\}, \{C, D\}\}$ ($\{A, B\} \neq \{C, D\}$) satisfying

$$A, B, C, D \in \binom{[n]}{k}, A \cap B = C \cap D = \emptyset, A \cup B = C \cup D. \tag{5}$$

Check the conditions of Theorem 4 for these graphs. (i) is obvious: if (5) holds then A and C cannot be disjoint.

The number of vertices of G_1 is $N = \binom{n}{k}$. Both graphs are regular. $\delta = \binom{n-k}{k}$. Observe that G_2 is a vertex disjoint union of clicks of size $\frac{1}{2}\binom{2k}{k}$. Therefore $\Delta = \frac{1}{2}\binom{2k}{k} - 1$. (ii) has the following form:

$$\binom{n-k}{k} \geq \binom{n}{k}\left(1 - \frac{1}{\binom{2k}{k}}\right). \tag{6}$$

For fixed k and large n the two binomial coefficients are asymptotically equal. The coefficient on the right hand side is less than 1. Therefore (6) holds for large n. □

More detailed analysis (see below) of (6) gives that $n(k) = k^2\binom{2k}{k} + k$ is a possible threshold. The following inequality is an equivalent form of (6).

$$\frac{(n-k)(n-k-1)\dots(n-2k+1)}{n(n-1)\dots(n-k+1)} \geq \left(1 - \frac{1}{\binom{2k}{k}}\right). \tag{7}$$

Apply the inequality

$$\frac{n-k-i}{n-i} \geq \frac{n-2k}{n-k} \ (0 \leq i < k)$$

on the left hand side of (7) then use the Bernoulli-inequality ($n \geq 2k$ can be supposed):

$$\frac{(n-k)(n-k-1)\dots(n-2k+1)}{n(n-1)\dots(n-k+1)} \geq$$

$$\left(\frac{n-2k}{n-k}\right)^k \geq \left(1 - \frac{k}{n-k}\right)^k \geq 1 - \frac{k^2}{n-k}.$$

Here the right hand side gives the right hand side of (7) for $n(k) = k^2\binom{2k}{k} + k$.

4 Historical comments, similar results

The method of using a Hamiltonian type theorem for constructing a family of disjoint pairs satisfying certain properties goes back to [1]. The following theorem was proved there.

Theorem 5 *If* $1 \leq k, n$ *are integers and* n *is large enough, that is* $n \geq n(k)$*, then one can find* $\left\lfloor \frac{1}{2}\binom{n}{k} \right\rfloor$ *unordered pairs* $\{A_i, B_i\}$ *of disjoint* k*-element subsets* ($A_i \cap B_i = \emptyset, |A_i| = |B_i| = k$) *of* $[n]$ *such that*

$$\min\{|A_i \cap A_j|, |B_i \cap B_j|\} \leq \frac{k}{2}$$

which implies

$$\min\{|A_i \cap B_j|, |B_i \cap A_j|\} \leq \frac{k}{2}$$

by the symmetry.

The proof of this theorem is based on the following Hamiltonian type statement.

Theorem 6 [1] *Let* $G_0 = (V, E_0)$ *and* $G_1 = (V, E_1)$ *be simple graphs on the same vertex set* $|V| = N$*, such that* $E_0 \cap E_1 = \emptyset$*. Let* r *be the minimum degree of* G_0 *and let* s *be the maximum degree of* G_1*. Suppose, that*

$$2\,r - 8\,s^2 - s - 1 > N$$

holds, then there is a Hamiltonian cycle in G_0 *such that if* (a, b) *and* (c, d) *are two vertex-disjoint edges of the cycle, then they do not form an alternating cycle with two edges of* G_1*.*

It was discovered in [3] that Theorem 5 can be made stronger in the following way.

Theorem 7 *If* $1 \leq k, n$ *are integers and* n *is large enough, that is* $n \geq n(k)$*, then one can find* $\left\lfloor \frac{1}{2}\binom{n}{k} \right\rfloor$ *unordered pairs* $\{A_i, B_i\}$ *of disjoint* k*-element subsets* ($A_i \cap B_i = \emptyset, |A_i| = |B_i| = k$) *of* $[n]$ *such that*

$$|A_i \cap A_j| + |B_i \cap B_j| \leq k$$

which implies

$$|A_i \cap B_j| + |B_i \cap A_j| \leq k$$

by the symmetry.

Its proof was based on a theorem similar to Theorem 6, but it is much more complicated.

Observe that Theorems 5 and 7 require the disjoint pairs $\{A_i, B_i\}$ to be "close" to each other. On the other hand our present Theorem 1 does not allow them to be too "close".

Both Theorems 4 and 6 can be interpreted in the following way. Given a graph and a 4-graph (4-uniform hypergraph) on the same vertex set. Give conditions ensuring the existence of a Hamiltonian cycle in the graph avoiding the given 4-edges as the union of two edges of the cycle. These interpretations lead to some Hamiltonian problems and theorems involving hypergraphs. The first result of this type was [5]. The survey paper [4] contains similar questions and answers. There are some important newer results in the following papers: [7], [8], [9], [6].

The author is indebted to the anonymous referee for finding several typos and grammatical inaccuracies.

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