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# Savings, Child Support, Pensions and Endogenous (and Heterogeneous) Fertility

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# Savings, Child Support, Pensions and Endogenous (and Heterogeneous) Fertility

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Abstract

van Groezen, Leers and Meijdam (2003) (for short, GLM) analyzed combination of public pension and child support in an OLG model. We impose credit constraint on workers, and extend GLM's analysis from the case where workers do not understand the cost also to the case where they do. GLM's infinite stream of generations is simplified into three generations but heterogeneity of rearing costs and of enjoying children is introduced. Two major results: (i) excluding negative savings, fertility decreases with pension contributions and increases with taxes; (ii) the introduction of fertility-dependent pensions may strengthen heterogeneity in fertility.

Keywords: Child support, Endogenous fertility, Overlapping generations, Pensions

JEL classification: D10, H55, J13, J14, J18, J26

# Megtakarítások, gyermektámogatások, nyugdíjak és endogén (valamint heterogén) termékenység

Simonovits András

Összefoglaló

van Groezen–Leers–Meijdam (2003) (rövidítve, GLM) az együttélő nemzedékek modelljében elemezte a társadalombiztosítási nyugdíj és a gyermektámogatás kombinációját. Ebben a dolgozatban hitelkorláttal szembesítjük a dolgozókat, és kiterjesztjük GLM elemzését – arról az esetről, amikor a dolgozók nem értik a költségeket – arra az esetre is, amikor értik. GLM végtelen számú nemzedékfolyamát három nemzedékre egyszerűsítjük, de figyelembe vesszük a gyermeknevelési költségek és a gyermekkedvelés heterogenitását. Két fontos eredményünk a következő. (i) Kizárva a negatív megtakarításokat, a termékenység a nyugdíjjárulék csökkenő és a személyi jövedelemadó növekvő függvénye. (ii) A termékenységfüggő nyugdíjak bevezetése növelheti a termékenységi különbségeket.

Tárgyszavak: gyermektámogatás, endogén termékenység, együttélő nemzedékek, nyugdíjak

JEL kód: D10, H55, J13, J14, J18, J26

# Savings, Child Support, Pensions and Endogenous (and Heterogeneous) Fertility

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October 26, 2013

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**Abstract***

van Groezen, Leers and Meijdam (2003) (for short, GLM) analyzed combination of public pension and child support in an OLG model. We impose credit constraint on workers, and extend GLM's analysis from the case where workers do not understand the cost also to the case where they do. GLM's infinite stream of generations is simplified into three generations but heterogeneity of rearing costs and of enjoying children is introduced. Two major results: (i) excluding negative savings, fertility decreases with pension contributions and increases with taxes; (ii) the introduction of fertility-dependent pensions may strengthen heterogeneity in fertility.

Keywords: Child support, Endogenous fertility, Overlapping generations, Pensions, Pension reform

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* I express my indebtedness for Róbert Iván Gál for his permanent encouragement to pursue this topic, Volker Meier and János Vincze for their thorough critique. The author has received generous financial support from OTKA K 67853. I must admit honestly that I have strong reservations on the applicability of the theoretical results.

## 1. Introduction

The worldwide population aging is a great challenge to the pension systems of the 21st century (cf. e.g. Barr and Diamond, 2008). One cause of aging is the drop in the total fertility rate. The other cause is the steep rise in life expectancy even after eliminating infant mortality. The indexation of retirement age to the life expectancy is an obvious tool to fight this second problem. But the solution of the first problem needs another tool, namely to enhance fertility. There are numerous empirical studies (e.g. Cigno (1992)claiming that the more generous the pension system (equivalently, the higher the replacement rate, i.e. the ratio of the pension to the net wage), the lower the endogenous fertility. This gives rise to the policy proposal to introduce child allowances (more generally, support) and especially fertility-dependent old-age pensions.

A lot of theoretical papers have analyzed this problem. Some of them (starting with Leibenstein (1957) and Becker (1960)) simply put the number of children into the utility function without studying the issue of pensions (cf. Eckstein and Wolpin, 1985). In contrast, van Groozen, Leers and Meijdam (2003) (for short, GLM) set up an overlapping generations model, where the government pays child allowance proportionally to fertility from labor taxes; charges pension contributions (called PAYG-taxes by GLM) in young age and pays pension benefits in old age. Assuming away productivity growth and identifying the lengths of old age and of young age, the family's pension benefit is equal to the product of the contribution and the number of children. Identifying the lengths of raising children and of young age, the tax rate is equal to the product of the per-child family allowance and of the fertility. (Note also that in such models usually a single adult is assumed rearing half of his/her children.)

Furthermore, retirees decumulate their private savings to add to their benefits. Considering a small open economy, the interest rate is given and it is frequently assumed that private savings are more efficient than public pensions. Choosing their fertility and private saving, families maximize their discounted lifetime utility functions, consisting of logarithmic utilities derived from young- and old-age consumption and having children.

The above-mentioned equilibrium model was only a building block of GLM (Section 3), and their main emphasis was how to generate long-term social optimum with its help. Using a Millian rather than a Benthamite social welfare function, they determined the first-best optimum and tried to achieve it by introducing an appropriate child support and old-age pension system. (A very ambitious study by Rangel (2003) applied the tools of game theory to model the interaction of forward and backward directed intergenerational transfers.)

Among the followers, we mention Fenge and Meier (2009) who confined their attention to the equilibrium part. So doing they proved that the family allowances and fertility-related pensions are equivalent for earnings-related pensions but the former is inferior to the latter for flat pensions. (Note the very specific meaning of the words fertility- and earnings-relatedness in case of homogeneous populations, where every worker has the same fertility and the same earning!)

The present paper also focuses on the equilibrium part and discusses four models of the GLM family.

In model 1, "individual agents [in fact, families, A.S.] act atomistically and do not take [the] budget constraint into account when they choose how many children to raise" (GLM, p. 240). Making the GLM model more realistic, we introduce a credit constraint, excluding negative savings. The credit constraint is called slack, when the saving intention is positive; tight, when it is negative; but in the second case, the actual saving is zero. In our model 1, the optimal fertility is an increasing function of the tax rate and—apart from high tax rates and low contribution rates—a decreasing function of the contribution rate. Introducing paternalistic social welfare function á la Feldstein (1985), where discounting is eliminated, the socially optimal contribution and tax rates can also be determined explicitly. Due to the special features of the model, the two optimal transfer rates are equal!

In model 2, deciding on fertility, the social planner takes into account the impacts neglected so far: the explicit connection between the number of children and the two types of benefits. This gives rise to higher fertility for low contribution rates and lower fertility for high contribution rates than in the basic model but the qualitative picture remains the same. The possible cause of the latter, paradoxical behavior is as follows: the families with externalities do not realize that after all, they pay the taxes which cover the child support.

To give more sense to fertility-dependent child support and pensions, in models 3 and 4 we introduce *heterogeneous* actors. There are two types, the first has lower rearing cost and higher utility of having children than the second. We shall assume that the pension benefit is a convex linear combination of two schemes: a common and an individualized one, the latter being proportional to the type-specific fertility. We shall see that the equilibrium fertilities satisfy our expectations: type 1 has more children than type 2 has. (In a separate paper, Simonovits (2013) concentrates on the fertility block and demonstrates that with growing heterogeneity in rearing costs, the higher cost type pays more and more net transfer to the other type.) To get rid of the problems of slack credit constraint for one type and a tight one for the other, we concentrate on the simplest and probably most relevant case of tight–tight combination.

Though model 3 neglects the incentive effects of fertility-specific pensions, raising the transfer rates may diminish the less fertile type's optimal fertility rate. Model 4 is a combination of models 2 and 3. In the simplest and most interesting case of total separation of the two types' pensions, model 4 can be reduced to model 3 and the social welfare is further increased but the tensions arising between low- and high fertility types may prevent the introduction of such a system.

Studying a similar model with general rather than logarithmic utility function, Cremer, Gahvari, and Pestieau (2008), for short, CGP) also took into account that the government has only no information on cost differences. Applying the methods of mechanism design, they also analyzed the problem of moral hazard (when parents have not full control over their fertility) and adverse selection (when low-cost type pretend to be high-cost type). They found that "in the absence of moral hazard problems the case for a positive link between pension benefits and fertility is not as strong it may at first appear and as it has been advocated in some recent work" (CGP, p. 962.).

To have a feeling for the order of magnitudes, we shall display several numerical calculations following our theoretical results. For example, in the basic model, the maximal fertility is achieved when the contribution rate is 0.1 and the tax rate is 0.4, while the maximal social welfare is achieved when both rates are approximately equal to 0.3. Or in the model with heterogeneous rearing costs and relative child utilities and fertility-related pensions, with the rise of the common transfer rates from 0.15 to 0.25

(to the social optimum), the lower fertility drops from 2.04 to 1.93!

We call attention to several limitations of the models: (a) the role of social norms are totally neglected, (b) wage differences are looked over, (c) proportional taxes and benefits are assumed and (d) the impact of survivor's pensions are neglected. Concerning (a), it is obvious that deciding on the number of children, the would-be parents are much more influenced by their social environment than when they decide on the type of car they buy. (For the issue of social norms in the welfare state, see Lindbeck, Nyberg and Weibull (1999).) Concerning (b), uniform family allowances redistribute from the higher-paid to the lower-paid, while tax allowances do the opposite. Concerning (c), in some countries, the family benefit and especially the tax allowance are strongly nonlinear. (For example, since 2011, in Hungary, the birth of the third child more than triples the maximal per-child tax allowance!) Concerning (d), the existence of survivor's pensions is a significant pro-family tool and there are proposals to diminish or eliminate it.

The structure of the remaining part of the paper is as follows. Sections 2 and 3 discuss homogeneous fertility with and without externalities on public transfers, respectively. Section 4 studies heterogeneous endogenous fertility. Section 5 concludes. An Appendix contains the more technical proofs.

## 2. Homogeneous fertility with externality (second-best)

In this Section we recapitulate the equilibrium part of GLM based on the usual two-period overlapping generations model with homogeneous fertility and externality, modify and develop it further.

### Framework

We keep their notations except for replacing their young and old-age consumption $c^y$ and $c^o$ by $c$ and $e$, respectively and abbreviating their interest factor $1+r$ by $R$. It is heroically assumed that the tax and contribution rates do not influence the labor supply which is equal to unity. Furthermore, in our small open economy, the world interest rate determines the domestic interest rate.

Total wage: $w = 1$, pension contribution (rate): $\tau$, tax (rate): $\theta$, saving: $s$. The cost of rearing a child: $p$, per child benefit: $\varphi$, fertility, i.e. the number of children in a family: $n$, the pay-as-you-go benefit: $\eta$; all positive real numbers. Another heroic assumption: the number of children can be any positive real!

Young-age ($c$) and old-age consumptions ($e$) are respectively

$$c + (p - \varphi)n = 1 - \tau - \theta - s \quad \text{and} \quad e = Rs + \eta. \tag{1}$$

The lifetime utility function is logarithmic:

$$U(c, n, e) = \log c + \gamma \log n + \beta \log e, \tag{2}$$

where $\gamma$ is the coefficient of the relative utility of having children and $\beta$ is the discount factor, $\gamma > 0$ and $0 \le \beta \le 1$. Introducing the net of contribution wage $\hat{\tau} = 1 - \tau$ and inserting (1) into (2) yield the reduced utility function:

$$u(s, n) = \log(\hat{\tau} - \theta - s - (p - \varphi)n) + \gamma \log n + \beta \log(Rs + \eta). \tag{2*}$$

Denoting the individual optimum by $c(\tau,\theta), n(\tau,\theta), e(\tau,\theta)$ and following the idea of Feldstein (1985), we define a *paternalistic* social welfare function by replacing the discount factor $\beta$ with 1 in (2). Then our social welfare function is

$$V(\tau,\theta) = \log c(\tau,\theta) + \gamma \log n(\tau,\theta) + \log e(\tau,\theta).$$

As we have indicated in the Introduction, in our basic model, the workers neglect the impact of their decisions on the balances described in (4) below. Filling the gap left in GLM, from now on we exclude negative savings, and distinguish two cases: either slack or tight credit constraint.

To learn if the credit constraint is slack or tight, we must determine the *separatrix curve* $\theta(\tau)$ which separates the two domains in the $(\tau,\theta)$-plane.

**Lemma 1.** *a) The separatrix curve $\theta(\tau)$ is given by*

$$\theta(\tau) = \frac{[\beta p - \gamma R^{-1}\tau]\hat{\tau}}{\beta p + R^{-1}\tau}, \quad \text{where} \quad 0 \le \tau < \gamma^{-1}\beta p R.$$

*b) The separatrix curve starts from $\theta(0) = 1$ and ends at $\theta(\tau_M) = 0$, where $\tau_M = \gamma^{-1}\beta p R$; and $\theta(\tau)$ is declining in $[0,\tau_M]$.*
*c) If $0 \le \tau < \tau_M$ and $0 < \theta < \theta(\tau)$, then $s(\tau,\theta) > 0$: slack.*
*d) If $0 \le \tau \le \tau_M$ and $\theta(\tau) \le \theta \le 1-\tau$, then $s(\tau,\theta) = 0$: tight.*
*e) If $\tau_M < \tau \le 1-\theta$, then $s(\tau,\theta) = 0$: tight.*

**Slack credit constraint (S)**

We continue the analysis with the slack credit constraint. Copying GLM, we take the partial derivatives of (2) with respect to $s$ and $n$ yields the first-order necessary conditions for optimum:

$$0 = u'_s(s,n) = \frac{-1}{\hat{\tau} - \theta - s - (p-\varphi)n} + \frac{\beta R}{Rs + \eta} \tag{3$a$}$$

and

$$0 = u'_n(s,n) = \frac{-(p-\varphi)}{\hat{\tau} - \theta - s - (p-\varphi)n} + \frac{\gamma}{n}. \tag{3$b$}$$

For the time being, we do not exclude negative savings. Then the conditional optima are

$$c(\tau,\theta) = \frac{\hat{\tau} - \theta + R^{-1}\eta}{1+\beta+\gamma} \quad \text{and} \quad n(\tau,\theta) = \frac{\gamma c(\tau,\theta)}{p-\varphi}. \tag{4}$$

Here pension $\eta$ and per-child allowance $\varphi$ depend on fertility $n$, therefore we introduce the transfer equations mentioned in the Introduction:

$$\theta = \varphi n \quad \text{and} \quad \eta = \tau n. \tag{5}$$

In words: a) the family's pension benefit is equal to the product of the contribution and the number of children; b) the tax rate is equal to the product of the per-child family allowance and of the fertility.

It is time to explain the role of these transfers. There is a general presumption—which we follow—that these two transfer systems should be balanced separately but in some countries this is not the case. For example, countries like Denmark, operate flat public pensions and finance the system from personal income taxes rather than from pension contributions. On the other hand, there are countries like Hungary, which use pension contributions for correcting distortions in their personal income tax systems.

The introduction of per-child support $\varphi$ reduces the *private* cost of rearing a child from $p$ to $p-\varphi$. If (5a) holds, there is no income effect. The introduction of the contribution rate $\tau$ diminishes the young-age consumption by the same amount and increases the old-age consumption according to $\eta = \tau n$ [(5b)]. In a stationary economy with $n = 1$, the two changes cancel each other; for falling/growing population, the reduction of young-age consumption is greater/lower than the increase of the old-age consumption. Using the concept of dynamic efficiency: $R > n$, the comparison above changes. As is known, in a dynamically efficient economy, the introduction of a pay-as-you-go pension is suboptimal etc.

Inserting the transfer equations (5a)–(5b) into (3a)–(3b) yields the final optima (see GLM):

**Theorem 1.** *a) When the credit constraint is slack (Lemma 1c), then the individually optimal young-age consumption and fertility are given respectively:*

$$c_S^* = \frac{\hat{\tau} - \theta[1 - \tau/(pR)]}{1 + \beta + \gamma - \gamma\tau/(pR)} \quad \text{and} \quad n_S^* = \frac{\gamma\hat{\tau} + (1+\beta)\theta}{(1+\beta+\gamma)p - \gamma R^{-1}\tau}. \tag{6}$$

*b) For a given contribution rate $\tau$, the optimal fertility is an increasing function of the tax rate $\theta$. For a given tax rate $\theta$, the optimal fertility is a decreasing/increasing function of the contribution rate if the tax rate is lower/higher than the critical tax rate*

$$\theta^{\text{o}} = \frac{(1+\beta+\gamma)pR - \gamma}{1+\beta}.$$

*c) The family support is always lower than the expenditure on children:* $0 \le \varphi < p$.

**Remark.** If $\theta^{\text{o}} < 0$, then the first case (decreasing fertility: normal) is empty; if $\theta^{\text{o}} > \theta(\tau)$, then the second case (increasing fertility: paradox) is empty. Denoting by $\tau^{\text{o}}$ the root of $\theta^{\text{o}} = \theta(\tau)$, to be called critical contribution rate, normalcy prevails in interval $\tau^{\text{o}} < \tau < 1$.

### Tight credit constraint (T)

Now we turn to the tight credit constraint: $s = 0$. We must replace condition $u_s'(0,n) = 0$ [(3a)] by $u_s'(0,n) < 0$ and then (3b) becomes $u_n'(0,n) = 0$. By easy calculation we obtain

$$(p-\varphi)n = \gamma c = \gamma[\hat{\tau} - \theta - (p-\varphi)n]. \tag{7}$$

Inserting the budget condition $\theta = \varphi n$, and the notation $\bar{\gamma} = 1 + \gamma$, we end-up with

**Theorem 2.** *a) When the credit constraint is tight (Lemma 1d+e), the optimal fertility and the corresponding young-age consumption are respectively*

$$n_T^* = \frac{\gamma\hat{\tau} + \theta}{\bar{\gamma}p} \quad \text{and} \quad c_T^* = \frac{\hat{\tau} - \theta}{\bar{\gamma}}. \tag{8}$$

*b) The optimal fertility is a decreasing function of the contribution rate and an increasing function of the tax rate.*

*c) For $0 < \gamma < 1$, the equal increase of the transfer rates increases the optimal fertility: $\Delta\tau = \Delta\theta > 0$ implies $\Delta n_T^* > 0$.*

Note that only having determined the optimal fertility in the two regions, can we prove Lemma 1.

Next we determine the socially optimal pair of contribution and tax rates in the tight region.

**Theorem 3.** *In the region of tight credit constraint, the socially optimal contribution and tax rates are equal respectively to*

$$\tau^* = \theta^* = \frac{1}{3 + \gamma}. \tag{9}$$

**Remarks.** 1. It is easy to show that the optimal pair of transfer rates in (9) generate tight credit constraint.

2. Tables 1 and 2 suggest that probably there is no local maxima of the social welfare function in the region of slack credit constraint.

3. In a prototype model without pensions, Simonovits (2013) also introduces a paternalistic coefficient $\gamma^*$, to derive extra children.

### Single instruments

It may be helpful further analyzing the two cases when either a) there is no child-support or b) there is no old-age pension.

*a) No child-support: $\theta = 0 = \varphi$*

By Lemma 1, the optimal saving $s > 0$ if $0 \le \tau < \gamma^{-1}\beta pR$, zero otherwise. Then the optimal adult consumption and fertility are respectively equal to

$$c_S^* = \frac{\hat{\tau}}{1 + \beta + \gamma - \gamma\tau/(pR)} \quad \text{and} \quad n_S^* = \frac{\gamma\hat{\tau}}{(1 + \beta + \gamma)p - \gamma R^{-1}\tau} \tag{6$a$}$$

and

$$n_T^* = \frac{\gamma\hat{\tau}}{\bar{\gamma}p} \quad \text{and} \quad c_T^* = \frac{\hat{\tau}}{\bar{\gamma}}. \tag{8$a$}$$

b) *No pensions: $\tau = 0$ and $\eta = 0$*

By Lemma 1, $\theta(0) = 1$, i.e. for any $0 \le \theta \le 1$, the optimal saving is positive and the optimal adult consumption and fertility are respectively equal to

$$c_S^* = \frac{1 - \theta}{1 + \beta + \gamma} \quad \text{and} \quad n_S^* = \frac{\gamma + (1 + \beta)\theta}{(1 + \beta + \gamma)p}. \tag{6$b$}$$

Note that the extreme pair $(\tau, \theta) = (0, 1)$ provide the highest fertility: $n_S^* = 1/p$.

**Numerical illustrations**

To have a feeling for the order of magnitudes, we shall display several illustrative numerical calculations. Note that our contribution and tax rates are higher than they should be, because we do not scale down the length of the periods spent with raising children (length of 20 years) and drawing pensions (length of 20 years) with respect to the working period (length of 40 years).

**Example 1.** Assume a 30-year long period between accumulation and decumulation, and choose $\beta = 0.4$ (annual discount rate of 3%), $p = 0.2$, $\gamma = 0.5$. To give room for an unfunded pension system, we assume $\beta R < 1$. We choose a low enough interest factor, namely $R = 1.5$ (annual interest rate being 1.4%). Then changing the contribution and the tax rates, their impact can be studied also numerically.

First we present the separatrix (discussed in Lemma 1). It declines from 1 to 0 while the contribution rate $\tau$ rises from 0 to 0.24. The critical tax rate $\theta^{\mathrm{o}} = 0.05$ and the corresponding contribution rate $\tau^{\mathrm{o}} = 0.2$. Second, from Theorem 3, $\tau^* = \theta^* = 0.286$, close 0.3, appearing in the Tables 1 and 2 below.

As is well-known, the numerical value of the social welfare function has no economic content. To obtain meaningful numbers, we shall occasionally compare the welfare provided by a $(\tau, \theta)$-system with the no-transfer system's (0, 0) as follows. Let us define the *relative efficiency* of the former with respect to the latter by the positive number $\varepsilon(\tau, \theta)$ if multiplying the unit wage by $\varepsilon$ in the no-transfer system, the welfare would reach that value provided by the transfer system with unitary wages. In formula:

$$V[\varepsilon, 0, 0] = V[1, \tau, \theta].$$

Due to the simple utility function (2), the optimal fertility is independent of the wage and the optimal consumption pair are homogeneous linear functions of the wage. Therefore

$$2 \log \varepsilon(\tau, \theta) + \log c(0, 0)) + \gamma \log n(0, 0) + \log e(0, 0) = \log c(\tau, \theta) + \gamma \log n(\tau, \theta) + \log e(\tau, \theta).$$

Hence $\log \varepsilon(\tau, \theta)$ or $\varepsilon(\tau, \theta)$ can simply be determined:

$$2 \log \varepsilon(\tau, \theta) + V[1, 0, 0] = V[1, \tau, \theta] \quad \text{i.e.} \quad \varepsilon(\tau, \theta) = \exp[0.5(V[1, \tau, \theta] - V[1, 0, 0])].$$

**Example 1, continued.** To stay in the real world, we limit the contribution and the tax rate by 0.4–0.4, respectively. Starting with the optimal fertility, the first line of Table 1 (no pension) shows a marked rise in fertility in parallel with the tax rate from 1.3 (at $\theta = 0$) to 2.79 (at $\theta = 0.4$). The first column (not considering the column of contribution rates) of Table 1 (no child allowance) displays a marked sink in fertility while the contribution rate rises: from 1.32 (at $\tau = 0$) to 1.0 (at $\tau = 0.4$). Similar tendencies can be observed in the other lines and columns, respectively, though for low enough tax rates (with positive savings) the fertility is paradoxically an increasing function of the contribution rate: for example for $\theta = 0.1$, the fertility rises from 1.684 to 1.702 while the contribution rate $\tau$ rises from 0 to 0.1. The maximal fertility (2.83) is achieved around $(\tau, \theta) = (0.1, 0.4)$.

**Table 1.** *Transfer rates and fertility*

| Contribution rate ($\tau$) | Tax rate ($\theta$) 0.0 | 0.1 | 0.2 | 0.3 | 0.4 |
|---|---|---|---|---|---|
| 0.0 | 1.316 | 1.684 | 2.053 | 2.421 | 2.789 |
| 0.1 | 1.298 | 1.702 | 2.106 | 2.500 | *2.833* |
| 0.2 | 1.277 | 1.667 | 2.000 | 2.333 | 2.667 |
| 0.3 | 1.167 | 1.500 | 1.833 | 2.167 | 2.500 |
| 0.4 | 1.000 | 1.333 | 1.667 | 2.000 | 2.333 |

**Remark.** $p = 0.2$ and $\gamma = 0.5$

Next we display the dependence of the relative efficiency of the transfer system on the transfer rates. The first line of Table 2 (no pension system) displays the drop in the relative welfare as the tax rate rises: from 1 (at $\theta = 0$) to 0.72 (at $\theta = 0.4$). The first column of Table 2 (no child allowance) displays the drop in the relative welfare as the contribution rate rises: from 1 (at $\tau = 0$) to 0.92 (at $\tau = 0.4$). No other column and only the next row, however, show similar monotonicity. The fourth line and the fourth column contains the entry of the approximately maximal efficiency, namely 1.16 at $(\tau, \theta) = (0.3, 0.3)$. In this sense, the social optimum is achieved for the careful harmonization of the contribution and the tax rates.

**Table 2.** *Transfer rates and relative efficiency*

| Contribution rate ($\tau$) | Tax rate ($\theta$) 0.0 | 0.1 | 0.2 | 0.3 | 0.4 |
|---|---|---|---|---|---|
| 0.0 | 1.000 | 0.957 | 0.894 | 0.815 | 0.724 |
| 0.1 | 0.983 | 0.974 | 0.945 | 0.911 | 0.913 |
| 0.2 | 0.963 | 1.026 | 1.089 | 1.116 | 1.104 |
| 0.3 | 0.962 | 1.075 | 1.141 | *1.157* | 1.115 |
| 0.4 | 0.916 | 1.038 | 1.097 | 1.089 | 0.999 |

To get a more detailed picture, we repeat the calculations along the diagonal with $\tau = \theta$ but display not only the fertility and the relative efficiency but also the young- and the old-age consumption and the subjective utility. First of all, note that the optimal saving is only positive for low enough contribution and tax rates, namely below $\tau = \theta = 0.15$. The subjective utility function is decreasing for low and high rates, its maximum is achieved in the interval $[0.15, 0.2]$. The relative efficiency behaves similarly but its maximum is reached in the interval $[0.25, 0.3]$.

**Table 3.** *The impact of the equal transfer rates with externality*

| Transfer rates $\tau = \theta$ | Fertility $n^*$ | Saving $s^*$ | Young-age consumption $c^*$ | Old-age consumption $e^*$ | LT utility function $U^*$ | Relative efficiency $\varepsilon$ |
|---|---|---|---|---|---|---|
| 0.00 | 1.316 | 0.211 | 0.526 | 0.316 | –0.966 | 1.000 |
| 0.05 | 1.500 | 0.150 | 0.500 | 0.300 | –0.972 | 0.982 |
| 0.10 | 1.702 | 0.079 | 0.481 | 0.288 | –0.964 | 0.974 |
| 0.15 | 1.917 | 0 | 0.467 | 0.288 | *–0.935* | 0.987 |
| 0.20 | 2.000 | 0 | 0.400 | 0.400 | *–0.936* | 1.089 |
| 0.25 | 2.083 | 0 | 0.333 | 0.521 | –0.993 | 1.146 |
| 0.30 | 2.167 | 0 | 0.267 | 0.650 | –1.107 | *1.157* |
| 0.35 | 2.250 | 0 | 0.200 | 0.788 | –1.300 | 1.113 |
| 0.40 | 2.333 | 0 | 0.133 | 0.933 | –1.619 | 0.999 |

LT=Lifetime

Finally, we shall consider the sensitivity of fertility to the relative utility of children ($\gamma$) and the cost of raising children ($p$), while holding the transfer rates fixed at 0.3. Table 4 displays the optimal fertility as a function of these parameter values. If the child utility parameter drops by 16% (from 0.5 to 0.42) and the cost of raising a child rises by 20% (from 0.2 to 0.24), then the fertility sinks by 20% (from 2.167 to 1.743). Similarly, if the child utility parameter rises by 16% (from 0.5 to 0.58) and the cost of raising a child decreases by 20% (from 0.2 to 0.16), then the fertility increases by 29% (from 2.167 to 2.793).

**Table 4.** *Dependence of fertility on utility and cost*

| Utility of children $\gamma$ | Cost of child $p$ | Fertility $n$ |
|---|---|---|
| 0.420 | 0.240 | 1.743 |
| 0.460 | 0.220 | 1.936 |
| 0.500 | 0.200 | 2.167 |
| 0.540 | 0.180 | 2.446 |
| 0.580 | 0.160 | 2.793 |

**Remark.** $\tau = \theta = 0.3$.

## 3. Homogeneous fertility without externalities (first best)

In Section 2, we assumed that there are externalities (second best). Now we turn to the diametrically opposite case, when the representative agent or the government fully takes into account the macrorelations (first best). Then the child allowance disappears and the investment side of the fertility is fully taken into account. We display the modified equations (with tilde) and the corresponding optimality conditions.

Consumption functions:

$$c = \hat{\tau} - s - pn \quad \text{and} \quad e = Rs + \tau n. \tag{$\tilde{1}$}$$

Derived utility function

$$u(s,n) = \log(\hat{\tau} - s - pn) + \gamma \log n + \beta \log(Rs + \tau n). \tag{$\tilde{2}^*$}$$

Taking the partial derivatives of $u$ with respect to $s$ and $n$ yields the first-order necessary conditions for optimum:

$$0 = u'_s(s,n) = \frac{-1}{\hat{\tau} - s - pn} + \frac{\beta R}{Rs + \tau n} \tag{$\tilde{3}a$}$$

$$0 = u'_n(s,n) = \frac{-p}{\hat{\tau} - s - pn} + \frac{\gamma}{n} + \frac{\beta \tau}{Rs + \tau n}. \tag{$\tilde{3}b$}$$

We separate the two cases again, and obtain the optimal fertilities. Introducing the new separating contribution rate (which is independent of the tax rate $\theta$)

$$\tilde{\tau}_M = \frac{p\beta R}{\beta + \gamma},$$

we have

**Theorem 4.** *If there is no externality, then the optimal fertility is given for slack credit constraint,*

$$\tilde{n}^*_S = \frac{\beta R \hat{\tau}}{\gamma^{-1}(1+\beta)[p\beta R - (\beta+\gamma)\tau] + p\beta R + \tau}, \qquad \tau \le \tilde{\tau}_M \tag{$\tilde{6}$}$$

*and for tight credit constraint,*

$$\tilde{n}^*_T = \frac{(\beta+\gamma)\hat{\tau}}{(1+\beta+\gamma)p}; \qquad \tau > \tilde{\tau}_M. \tag{$\tilde{8}$}$$

Continuing the numerical explorations, as a function of the contribution rate, Table 5 reports again increasing fertility for the slack case and decreasing one for the tight case but at a higher/lower level than before. Indeed, comparing (8) and ($\tilde{8}$), this paradoxical behavior can be proven. (The comparison of (6) and ($\tilde{6}$) would be much more tiring.)

**Lemma 2.** *If for a given transfer rates pair, the systems with and without externality have tight credit constraint and if $\beta\hat{\tau} < (1+\beta+\gamma)\theta$ holds, then the optimal fertility is lower with externality than without externality:*

$$n^*_T < \tilde{n}^*_T.$$

The proof is left to the Appendix, but the intuitive justification is as follows: in our model family, the lack of externality means that the tax rate has no influence on the fertility. With our parameter values, for $\tau = 0.2 = \theta$, $\tilde{\tau}_M = 0.133$ and then the condition of Lemma 2 holds:

$$0.4 \times 0.8 = 0.32 < 0.38 = 1.9 \times 0.2.$$

We make the following observations: individual fertility is maximized around the common transfer rates 0.15, while the socially optimal pairs are still around 0.3.

**Table 5.** *The impact of the equal transfer rates, no externality*

| Transfer rates $\tau = \theta$ | Fertility $n^*$ | Saving $s^*$ | Young-age consumption $c^*$ | Old-age consumption $e^*$ | LT utility function $U^*$ | Relative efficiency $\varepsilon$ |
|---|---|---|---|---|---|---|
| 0.00 | 1.316 | 0.211 | 0.526 | 0.316 | –0.966 | 1.000 |
| 0.05 | 1.500 | 0.150 | 0.500 | 0.300 | –0.972 | 0.999 |
| 0.10 | 1.776 | 0.071 | 0.474 | 0.284 | –0.963 | 1.015 |
| 0.15 | 2.013 | 0 | 0.447 | 0.302 | *–0.933* | 1.158 |
| 0.20 | 1.895 | 0 | 0.421 | 0.379 | *–0.934* | 1.239 |
| 0.25 | 1.776 | 0 | 0.395 | 0.444 | –0.967 | 1.278 |
| 0.30 | 1.658 | 0 | 0.368 | 0.497 | –1.025 | *1.285* |
| 0.35 | 1.539 | 0 | 0.342 | 0.539 | –1.104 | 1.265 |
| 0.40 | 1.421 | 0 | 0.316 | 0.568 | –1.203 | 1.223 |

LT=lifetime

## 4. Heterogeneous fertility with externalities

Having discussed the simplest cases of homogeneous fertility with and without externality, we turn to the more realistic and more interesting case of heterogeneous fertility.

### Framework

We confine attention to the simpler case when there are externalities. To derive heterogeneity in the optimally chosen number of children, we shall assume the existence of two types indexed as $i = 1, 2$. They differ in the following parameter values: the cost of a child $p_i$ and the relative utility of a child $\gamma_i$, with the following plausible orderings (cf. CGP):

$$p_1 > p_2 > 0 \quad \text{and} \quad 0 < \gamma_1 < \gamma_2.$$

In words: type 1 is less efficient than type 2 is. The corresponding population shares are denoted by $f_1, f_2 > 0$, their sum being equal to 1: $f_1 + f_2 = 1$. Let the average number of children be

$$n = f_1 n_1 + f_2 n_2. \tag{10a}$$

It is a weakness of our static model that the initial distribution of the two types changes during a generation: $(f_1, f_2)$ becomes $(f_1 n_1/n, f_2 n_2/n)$.

Note that here we assume that type $i$'s transfers are dependent on the type-specific fertility $n_i$, but for the time being, the types do not take this into account. We define the new transfer rules. The aggregate child allowance rule simply remains $\theta = \varphi n$.

We assume that the type-specific pension formula is an increasing inhomogeneous linear function of the type-specific fertility. Let $\delta \in [0, 1]$ be the weight of the common and $\hat{\delta} = 1 - \delta$ be the weight of the individual pension:

$$\eta_i = \delta \tau n + \hat{\delta} \tau n_i, \qquad i = 1, 2. \tag{10$b$}$$

If $\delta = 0$, then we have two separate pension systems for the low and the high types. If $\delta = 1$, then we have a unified pension system, studied above. In general, we shall assume $0 < \delta < 1$.

Young-age and old-age consumptions are respectively

$$c_i = \hat{\tau} - \theta - (p_i - \varphi)n_i - s_i \quad \text{and} \quad e_i = Rs_i + \eta_i. \tag{11}$$

The lifetime utility functions are as follows:

$$U_i(c_i, n_i, e_i) = \log c_i + \gamma_i \log n_i + \beta \log e_i. \tag{12}$$

Substituting (11) into (12) yields the type-specific reduced utility function:

$$u_i(s_i, n_i) = \log(\hat{\tau} - \theta - (p_i - \varphi)n_i - s_i) + \gamma_i \log n_i + \beta \log(Rs_i + \eta_i). \tag{13}$$

We solve the individual optimization problems for slack and tight credit constraints separately.

In the case of slack credit constraint, taking the partial derivatives with respect to $s_i$ and $n_i$ (cf. (6)), yields

$$c_i^* = \frac{\hat{\tau} - \theta + R^{-1}\eta_i}{1 + \beta + \gamma_i} \quad \text{and} \quad n_i^* = \frac{\gamma_i c_i^*}{p_i - \varphi}. \tag{14}$$

In the case of tight credit constraint, we return to (7):

$$(p_i - \varphi)n_i = \gamma_i c_i = \gamma_i[\hat{\tau} - \theta - (p_i - \varphi)n_i]. \tag{15}$$

With rearrangement, and using notation $\bar{\gamma}_i = 1 + \gamma_i$,

$$\bar{\gamma}_i(p_i - \varphi)n_i = \gamma_i(\hat{\tau} - \theta), \qquad i = 1, 2. \tag{16}$$

Note that it is possible that one type has a slack credit constraint and the other has a tight one. There are four combinations:

$$s_1, s_2 > 0, \quad s_1 > 0 = s_2, \quad s_2 > 0 = s_1, \quad s_1 = 0 = s_2.$$

We omit the discussion of the separation of the cases and for a while, suppress the notation of the interaction of the two types' optima. We have the following trivial observation for the two pure combinations of types (cf. (14) and (16)).

**Lemma 3.** *If both types have either slack: $s_1, s_2 > 0$ or tight credit constraint: $s_1 = s_2 = 0$, then the less efficient type has lower fertility than the more efficient has:*

$$p_1 > p_2 > 0 \quad \text{and} \quad 0 < \gamma_1 < \gamma_2 \quad \text{implies} \quad n_1^* < n_2^*.$$

Due to Lemma 3, we may simply call the less/more efficient type less/more fertile.

**Fertility depednent pensions without incentives**

Due to its relevance and simplicity, we shall confine our attention to the fourth combination, when both types' savings are zero:

$$s_i = 0 \quad \text{and} \quad u'_{i,n_i}(0, n_i) = \frac{-(p_i - \varphi)}{\hat{\tau} - \theta - (p_i - \varphi)n_i} + \frac{\gamma_i}{n_i} = 0, \qquad i = 1, 2. \tag{17}$$

To determine the optimum, we must determine the equilibrium value of $\varphi$.

**Theorem 5.** *Consider a fertility-specific transfer system, where the workers do not take into account this feature of the pension system. In the tight–tight optimum, the per-child allowance $\varphi$ is the smaller positive root of the following quadratic equation:*

$$A\varphi^2 + B\varphi + C = 0, \tag{18}$$

*where the coefficients in (18) are respectively*

$$A = \omega\bar{\gamma}_1\bar{\gamma}_2 + f_1\gamma_1\bar{\gamma}_2 + f_2\gamma_2\bar{\gamma}_1, \tag{19a}$$

$$B = \omega\bar{\gamma}_1\bar{\gamma}_2(p_1 + p_2) + f_1\gamma_1\bar{\gamma}_2 p_2 + f_2\gamma_2\bar{\gamma}_1 p_1, \tag{19b}$$

$$C = \bar{\gamma}_1\bar{\gamma}_2 p_1 p_2, \tag{19c}$$

*where*

$$\omega = \frac{\theta}{\tau - \theta} \quad \text{and} \quad \bar{\gamma}_i = 1 + \gamma_i, \qquad i = 1, 2. \tag{20}$$

Finally, we define the paternalistic and utilitarian social welfare function:

$$V = \sum_{i=1}^{2} f_i u_i^*[s_i^*, n_i^*], \quad \text{where} \quad u_i^*[s_i^*, n_i^*] = \log c_i^* + \gamma \log n_i^* + \log e_i^*. \tag{21}$$

**Example 2.** We assume that the shares of the two types are equal: $f_1 = f_2 = 0.5$; and differentiate the costs and utilities of child between the two types: $p_1 = 0.22$, $p_2 = 0.18$ and $\gamma_1 = 0.46$, $\gamma_2 = 0.54$ (cf. Table 4). For the time being, the pension benefits are uniform: $\delta = 1$.

Table 6 shows the difference with the homogeneous cases. In the heterogeneous case, the equal increase of the transfer rates (above 0.2–0.2) diminishes type 1's fertility, though still increases the average fertility. The basic reason for this paradoxical behavior is probably connected to the introduction of net transfers from type 1 to type 2. Furthermore, the rise in the transfer rates above 0.2 diminishes both types' subjective lifetime utilities but the socially optimal contribution rate remains close 0.3. Since we do not deal with slack constraints, the value of the social welfare function at no transfer is unknown. Therefore we must take the maximum as our starting point for measuring the relative efficiency of the system.

**Table 6.** *The impact of equal transfer rates: heterogeneous types, uniform pensions*

| Transfer rates $\tau = \theta$ | Fertility-1 $n_1^*$ | Fertility-2 $n_2^*$ | Utility-1 function $U_1^*$ | Utility-2 function $U_2^*$ | Relative efficiency $\varepsilon$ |
|---|---|---|---|---|---|
| 0.15 | 1.539 | 2.375 | *–1.027* | –0.812 | 0.850 |
| 0.20 | 1.545 | 2.554 | –1.046 | *–0.793* | 0.940 |
| 0.25 | 1.523 | 2.765 | –1.127 | –0.825 | 0.990 |
| 0.30 | 1.460 | 3.028 | –1.279 | –0.908 | *1.000* |
| 0.35 | 1.328 | 3.375 | –1.530 | –1.057 | 0.961 |

**Remark.** $p_1 = 0.22$, $p_2 = 0.18$ and $\gamma_1 = 0.46$, $\gamma_2 = 0.54$

Next we turn to the differentiated pensions: $0 \leq \delta < 1$. For the time being, we exclude incentive effects. To obtain a marked difference, we consider the maximal differentiation with $\delta = 0$. The outcomes displayed in Table 7 are basically the same: the less efficient type receives even less utility, the more efficient receives even higher utility and in balance, the maximal social welfare is slightly less than with uniform pensions and reached at transfer rates close to 0.25 rather than to 0.3.

**Table 7.** *The impact of transfer rates: heterogeneous types, differentiated pensions*

| Transfer rates $\tau = \theta$ | Fertility-1 $n_1^*$ | Fertility-2 $n_2^*$ | Utility-1 function $U_1^*$ | Utility-2 function $U_2^*$ | Relative efficiency $\varepsilon$ |
|---|---|---|---|---|---|
| 0.15 | 1.539 | 2.375 | *–1.123* | –0.734 | 0.867 |
| 0.20 | 1.545 | 2.554 | –1.159 | *–0.705* | 0.955 |
| 0.25 | 1.523 | 2.765 | –1.264 | –0.723 | *1.000* |
| 0.30 | 1.460 | 3.028 | –1.451 | –0.788 | 0.999 |
| 0.35 | 1.328 | 3.375 | –1.758 | –0.912 | 0.941 |

What happens if the types do take into account the fertility dependence of the child allowance and the pension?

**Fertility-dependent pensions with incentives**

Introducing the game-theoretic notation $-i$ as the 'other' player's index, similarly to (3b) and ($\tilde{3b}$), (17) is modified into

$$u'_{i,n_i}(0, n_i, n_{-i}) = \frac{-(p_i - \varphi)}{\hat{\tau} - \theta - (p_i - \varphi)n_i} + \frac{\gamma_i}{n_i} + \frac{\hat{\delta}\beta}{\delta n + \hat{\delta} n_i} = 0, \qquad i = 1, 2. \tag{17*}$$

If $0 < \delta < 1$, then we have a system of two nonlinear equations, which is probably only solvable with numerical methods. Moreover, we do not want to consider a Cournot-game between less and more fertile types.

However, for total separation of pensions: $\delta = 0$, i.e. $\hat{\delta} = 1$, (17*) breaks up into two independent equations:

$$\frac{-(p_i - \varphi)}{\hat{\tau} - \theta - (p_i - \varphi)n_i} + \frac{\gamma_i}{n_i} + \frac{\beta}{n_i} = 0, \qquad i = 1, 2. \tag{$\tilde{17}$}$$

Comparing (17) and ($\tilde{17}$), the only difference between them is that $\gamma_i$ in (17) is replaced by $\gamma_i + \beta$ in ($\tilde{17}$). This observation proves

**Theorem 6.** *a) For total separation pensions ($\delta = 0$), introducing the incentive effects is equivalent to add the discount factor $\beta$ to the relative child utility coefficient $\gamma_i$:*

$$\tilde{\gamma} = \gamma_i + \beta.$$

*b) The activation of incentives obviously raises the total fertility but decreases the lower type's fertility.*

We repeat the numerical calculations with the new $\tilde{\gamma}_i$s. Table 8 shows that the socially optimal contribution and tax rates get lower, around 0.25 and the two types' utilities and social welfare are further increased.

**Table 8.** *The impact of transfer rates:*
*heterogeneous types, totally differentiated pensions with incentives*

| Transfer rates $\tau = \theta$ | Fertility-1 $n_1^*$ | Fertility-2 $n_2^*$ | Utility-1 function $U_1^*$ | Utility 2 function $U_2^*$ | Relative efficiency $\varepsilon$ |
|---|---|---|---|---|---|
| 0.15 | 2.039 | 2.857 | *–0.838* | –0.372 | 0.911 |
| 0.20 | 1.997 | 2.938 | –0.904 | *–0.373* | 0.978 |
| 0.25 | 1.934 | 3.045 | –1.037 | –0.418 | *1.000* |
| 0.30 | 1.837 | 3.193 | –1.253 | –0.505 | 0.979 |
| 0.35 | 1.680 | 3.414 | –1.591 | –0.641 | 0.908 |

The picture is quite difficult to understand. To help better understanding, in Table 9 we look for the *critical heterogeneity* between $\gamma$s and $p$s which keeps the low fertility–transfer rate function $n_1(\tau, \tau)$ almost constant. By trial and error we choose the following coefficients:

$$\gamma_1 = 0.496, \quad \gamma_2 = 0.504, \quad \text{and} \quad p_1 = 0.202, \quad p_2 = 0.198.$$

Note how small the differences between the two types are. The per child allowance $\varphi$ and the net transfer $t_2 = \varphi n_2^* - \theta$ *received* are also displayed in Table 9. While the per child allowance is quite impressive, due to the minimal differences in the parameter values, the net transfer received remains definitely modest.

**Table 9.** *Critical heterogeneity, totally differentiated pensions with incentives*

| Transfer rates $\tau = \theta$ | Fertility-1 $n_1^*$ | Fertility-2 $n_2^*$ | Per child allowance $\varphi$ | Net transfer received by 2 $t_2$ |
|---|---|---|---|---|
| 0.15 | 2.368 | 2.449 | 0.062 | 0.003 |
| 0.20 | 2.375 | 2.468 | 0.083 | 0.004 |
| 0.25 | 2.379 | 2.490 | 0.103 | 0.006 |
| 0.30 | 2.379 | 2.517 | 0.123 | 0.008 |
| 0.35 | 2.371 | 2.552 | 0.142 | 0.013 |

$\gamma_1 = 0.496$, $\gamma_2 = 0.504$, $p_1 = 0.202$, $p_1 = 0.198$.

As a summary of this Section, we make the following remarks. The policy results are quite impressive but there are well-known limits to its application: a) the less fertile part of the population would not vote for such a policy; b) even if such a policy were accepted, it would fatally weaken the incentives of the less fertile part to supply labor and report their incomes; c) in practice, the less fertile support the more fertile within the larger family (for examples, less fertile aunts support more fertile siblings' nephews, while less fertile uncles support more fertile siblings' nieces; d) widespread divorce and the existence of widow's pension also complicate the situation.

## 5. Conclusions

We have revisited GLM's siamese twins of public pension and child support, at least in its equilibrium framework. Public pension is important because the young are short-sighted and the large families had broken up. Child support is important because the introduction of public pension may undermine fertility (externality). We have emphasized the importance of modeling credit constraints and have distinguished the slack and the tight credit constraints in developing GLM's model. Also we studied the role of endogenous and heterogeneous fertility with and without externality.

An important but largely neglected dimension of the fertility issue is heterogeneity: there are families with low fertility and others with high fertility produced by high-cost, low-utility and low-cost, high-utility types, respectively. Here the redistribution between the two types in the child allowances cannot be forgotten, regardless of the existence externalities. In various countries (including Germany and Hungary) there are also plans to make the pension benefits to be partly proportional to fertility. We are able to model such a design and at least on paper, it increases the average fertility. Nevertheless, there are strong reasons against its application. To name just one: such a policy may polarize the society into *fertiles* and *infertiles*.

## Appendix

### Proof of Theorem 1b

Take the derivative of $n^*_S(\tau,\theta)$ with respect to $\tau$ and drop the constants:

$$\frac{\partial n^*_S(\tau,\theta)}{\partial \tau} \approx \gamma[(1+\beta+\gamma)p - \gamma R^{-1}\tau] + [\gamma\hat{\tau} + \theta(1+\beta)]\gamma R^{-1}.$$

A simple calculation yields that

$$\frac{\partial n^*_S(\tau,\theta)}{\partial \tau} < 0 \quad \text{if and only if} \quad \theta < \theta^{\mathrm{o}}.$$

### Proof of Lemma 1

Inserting the formulas for $n^*_S$ and $n^*_T$ into the separatrix, $n^*_S = n^*_T$ results in

$$\frac{\gamma\hat{\tau} + (1+\beta)\theta}{(1+\beta+\gamma)p - \gamma R^{-1}\tau} = \frac{\gamma\hat{\tau} + \theta}{\bar{\gamma}p}.$$

With rearrangement, $[\gamma\hat{\tau} + (1+\beta)\theta]\bar{\gamma}p = [\gamma\hat{\tau} + \theta][(1+\beta+\gamma)p - \gamma R^{-1}\tau]$. After simplification, $\theta(\tau)$ is obtained. For $\theta > \theta(\tau)$, $s(\tau,\theta) = 0$; for $\theta < \theta(\tau)$, $s(\tau,\theta) > 0$.

The separatrix is a fraction: its numerator is the product of two decreasing positive functions and its denominator is an increasing positive function, therefore the fraction is also declining.

### Proof of Theorem 3

For simplicity, we confine our attention to the tight credit condition. Our starting point is as follows:

$$V\{\tau,\theta\} = \log\frac{1-\tau-\theta}{\bar{\gamma}} + \gamma\log\frac{\gamma\hat{\tau}+\theta}{\bar{\gamma}p} + \log\frac{\tau(\gamma\hat{\tau}+\theta)}{\bar{\gamma}p} \to \max.$$

Using the identity $\log(x/y) = \log x - \log y$, the constant denominators can be dropped. We have then an equivalent problem:

$$W\{\tau,\theta\} = \log(1-\tau-\theta) + \bar{\gamma}\log(\gamma(1-\tau)+\theta) + \log\tau \to \max.$$

Taking the partial derivatives of $W$ with respect to $\tau$ and $\theta$ and equate the derivatives to zero yield the first-order necessary conditions:

$$0 = \frac{\partial W}{\partial \tau} = \frac{-1}{1-\tau-\theta} - \frac{\bar{\gamma}\gamma}{\gamma(1-\tau)+\theta} + \frac{1}{\tau} \qquad (A1)$$

and

$$0 = \frac{\partial W}{\partial \theta} = \frac{-1}{1-\tau-\theta} + \frac{\bar{\gamma}}{\gamma(1-\tau)+\theta}. \qquad (A2)$$

From (A2),

$$\theta = \frac{1-\tau}{2+\gamma} = \kappa(1-\tau). \qquad (A3)$$

Inserting back (A3) into (A1),

$$\frac{-1}{(1-\kappa)(1-\tau)} - \frac{\bar{\gamma}\gamma}{(\gamma+\kappa)(1-\tau)} + \frac{1}{\tau} = 0.$$

Hence Theorem 3 is obtained.

**Proof of Theorem 4**

*Slack credit constraint*

Combining $(\tilde{3}a)$ and $(\tilde{3}b)$ leads to

$$\frac{p\beta R}{Rs+\tau n} = \frac{p}{\hat{\tau}-s-pn} = \frac{\gamma}{n} + \frac{\beta\tau}{Rs+\tau n}.$$

With rearrangement,

$$s = \gamma^{-1}R^{-1}[p\beta R - (\beta+\gamma)\tau].$$

Rewrite $(\tilde{3}a)$ as $(Rs+\tau n) = \beta R(\hat{\tau}-s-pn)$ or

$$R(1+\beta)s = \beta R\hat{\tau} - (p\beta R + \tau)n$$

Inserting $s$ into our last equation leads to $(\tilde{6}a)$.

*Tight credit constraint*

With $s = 0$, $(\tilde{3}b)$ reduces to

$$0 = u'_n(0,n) = \frac{-p}{\hat{\tau}-pn} + \frac{\gamma}{n} + \frac{\beta}{n}. \qquad (3b^o)$$

Solving for $\tilde{n}_T$ yields the result. To obtain the separatrix, one needs to substitute $\tilde{n}_T$ into the modified $(\tilde{3}a)$:

$$u'_s(0,n) = \frac{-1}{\hat{\tau}-pn} + \frac{\beta R}{\tau n} \le 0. \qquad (3a^<)$$

With rearrangement, $\beta R\tau \le \tilde{n}^*_T(\tau + p\beta R)$. Inserting $(\tilde{8})$ into our equation, simple calculation yields the lower bound on $\tau$.

### Proof of Lemma 2

When does $n_T^* < \tilde{n}_T^*$ hold? Using $(\tilde{6})$ and $(\tilde{8})$ results in

$$\frac{(\beta+\gamma)\hat{\tau}}{(1+\beta+\gamma)p} < \frac{\gamma\hat{\tau}+\theta}{\bar{\gamma}p}.$$

Simple algebra yields the condition stated in Lemma 2.

### Proof of Theorem 5

Using $\bar{\gamma}_i = 1 + \gamma_i$ in (16) and inserting the new formula

$$n_i = \frac{\gamma_i(\hat{\tau}-\theta)}{\bar{\gamma}_i(p_i-\varphi)}$$

into $\theta = \varphi(f_1 n_1 + f_2 n_2)$ yields an implicit equation for $\varphi$:

$$\theta = \varphi\left(f_1\frac{\gamma_1(\hat{\tau}-\theta)}{\bar{\gamma}_1(p_1-\varphi)} + f_2\frac{\gamma_2(\hat{\tau}-\theta)}{\bar{\gamma}_2(p_2-\varphi)}\right).$$

With rearrangement,

$$\frac{\theta}{\hat{\tau}-\theta} = \varphi\left(\frac{f_1\gamma_1}{\bar{\gamma}_1(p_1-\varphi)} + \frac{f_2\gamma_2}{\bar{\gamma}_2(p_2-\varphi)}\right).$$

Having eliminated the denominators yields the quadratic equation (18)–(20) for $\varphi$.

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