Making Sense of Mathematics: Supportive and Problematic Conceptions with special reference to Trigonometry

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Declaration

I, the author, declare that this thesis is entirely my original work and has not been submitted for a degree at any other institution.

Abstract

This thesis is concerned with how a group of student teachers make sense of trigonometry. There are three main ideas in this study. This first idea is about the theoretical framework which focusses on the growth of mathematical thinking based on human perception, operation and reason. This framework evolves from the work of Piaget, Bruner, Skemp, Dienes, Van Hiele and others. Although the study focusses on trigonometry, the theory constructed is applicable to a wide range of mathematics topics.

The second idea is about three distinct contexts of trigonometry namely triangle trigonometry, circle trigonometry and analytic trigonometry. Triangle trigonometry is based on right angled triangles with positive sides and angles bigger than $0°$ and less than $90°$. Circle trigonometry involves dynamic angles of any size and sign with trigonometric ratios involving signed numbers and the properties of trigonometric functions represented as graphs. Analytic trigonometry involves trigonometric functions expressed as power series and the use of complex numbers to relate exponential and trigonometric functions.

The third idea is about supportive and problematic conceptions in making sense of mathematics. This idea evolves from the idea of met-before as proposed in Tall (2004). In this case, the concept of 'met-before' is given a working definition as 'a trace that it leaves in the mind that affects our current thinking'. Supportive conception supports generalization in a new contexts whereas problematic conception impedes generalization. Furthermore, a supportive conception might contain problematic aspects in it and a problematic conception might contain supportive aspects in it. In general, supportive conceptions will give the learner a sense of confidence whereas problematic conceptions will give the learner of sense of anxiety. Supportive conceptions may occur in different ways. Some learners might know how to perform an algorithm without a grasp of how it can be related to different mathematical concepts and the underlying reasons for using such an algorithm.

Chapter 1
Thesis Overview

1.1 Introduction.

As a university lecturer who has taught several years of mathematics courses for prospective teachers, I became very interested in exploring how humans make sense of mathematics in ways that can essentially improve teacher education. Apart from this, I also have two years of teaching experience as a secondary school mathematics teacher before joining the university as a lecturer. As a consequence, I noticed that most secondary students have difficulties in understanding trigonometry. Additionally, I still remember my experience as a secondary school student who had struggled to make sense of school trigonometry. With this background, I started the journey of this study. I initiated my investigation by reading journals and books which were related to the theories in mathematical thinking.

The rationale of choosing trigonometry as a research topic arose because it involved the blending of two domains of mathematics involving visual and symbolic aspects in geometry and algebra. Moreover I found that there were very few research studies conducted in trigonometry, therefore a study in this area should be beneficial for mathematics education. Through extensive reading and relating my personal experience in learning school trigonometry I conjectured that there were certain common difficulties in learning school trigonometry across learners such as the transition from the triangle trigonometry to the unit circle. Then I designed the first version of a questionnaire to collect some informal data.

Having collected the data I sought to read further research papers to formulate a theoretical framework to make sense of the responses. The discussions with my supervisor, Professor David Tall enabled us to clarify issues in his work on three worlds of mathematics, which was subsequently published as Tall (2013), and to modify the theory to base the ideas on the blending of geometry and algebra and the more general ideas of blending perception, operation and reason, which has its origins in the ideas of Richard Skemp (1979) where he speaks of perception, action and reflection.

This study is concerned with how a group of student teachers make sense of trigonometry which they will eventually teach at different levels to pupils in school at different stages of development. The theoretical framework focusses on the growth of mathematical thinking based on human perception, operation and reason. This has its foundations in a wide range of empirical and theoretical research. In particular human perception is related to the notion of embodiment as suggested by Lakoff & Nunez (2000) meanwhile operation is related to the process-object encapsulation theories as proposed by Dubinsky & McDonald (2001), Sfard (1991), Gray & Tall (1994). Reason is closely related to the increasing sophistication in embodiment and symbolism such as the increasing sophistication in geometry of Van Hiele (1986) and in arithmetic and algorithm based on the rules of arithmetic, and the theories of advanced mathematical thinking in Tall (1991).

In this thesis there are three main ideas. The first focuses on how individuals make sense of mathematics in general and trigonometry in particular, in terms of *perception*, *operation* and *reason* (Chin & Tall, 2012). Although the study focusses on trigonometry, the theory constructed is part of a broader theory of cognitive and affective development of mathematical concepts (Tall, 2013).

The second idea is about three distinct contexts in making sense of trigonometry. The review of literature and the outcome of the pilot study (discussed in chapters 2 and 4) suggest that there are two distinct contexts for trigonometry in school and further developments in undergraduate study. These three distinct contexts of trigonometry are introduced as *triangle trigonometry, circle trigonometry and analytic trigonometry* in this study. Triangle trigonometry is based on right angled triangles with positive sides and angles bigger than $0°$ and less than $90°$. Circle trigonometry involves dynamic angles of any size and sign with trigonometric ratios involving signed numbers and the properties of trigonometric functions represented as graphs. Analytic trigonometry involves trigonometric functions expressed as power series and the use of complex numbers to relate exponential and trigonometric functions.

The third idea concerns *supportive and problematic conceptions* in making sense of trigonometry. This involves the effect of previous experience in new contexts. This idea was first introduced in Tall (2004) as a 'met-before'. In this case, 'met-before' can be regarded as a mental construct that an individual uses at a given time based on experiences they have met before. Met-befores can be *supportive* in a new situation and encourage generalization or they may become *problematic* and impede progress. For instance the conception of triangle in Euclidean geometry may impede the generalization of triangle in the Cartesian plane because the sides and angles, which previously had specific properties as unsigned quantities, can now be any size, positive or negative. In this thesis we will find that conceptions in general may involve a mixture of supportive and problematic conceptions. A supportive conception that leads to successful mathematical thinking may also involve subtle problematic aspects. On the other hand, problematic aspects may become so dominant as to cause increasing disaffection. In general, supportive conceptions will give the learner a sense of confidence whereas problematic conceptions may act as a challenge to some or lead to a sense of anxiety for others. This will result in a spectrum of performance in the whole population from those who take pleasure in success at one extreme or experience mathematical anxiety and likely failure at the other.

As the students develop their knowledge, supportive conceptions may occur in different ways. A student may 'know' how to perform an algorithm or how to use a particular concept, without 'grasping' the meaning of the idea in a manner which enables them to make sense of more sophisticated ideas. For instance, a pupil may 'know' that one uses degrees in triangle trigonometry and radians in circle trigonometry, without grasping why this is necessary (for example, that the derivative of $\sin kx$ is $k\cos kx$ when x is measured in radians and so radian measure is required in the calculus because the derivative of $\sin x°$ is not $\cos x°$).

More importantly, 'grasping' a concept may allow the thinker to imagine it as a single entity, so that it can be more easily manipulated in the mind. This also enables the thinker to think about the concept in different ways. For instance, in switching from triangle trigonometry to circle trigonometry, the notion of sinx changes from handling the complexities of ratio and proportion, to 'see' the sine of an angle in radians as the (signed) length of the y-coordinate of a point on a unit circle, to offer a conceptual extension of the idea of trigonometric ratios from circle geometry to functional representations of trigonometric functions as graphs.

The analysis of the changing meanings of the ideas in trigonometry will be investigated using carefully designed questionnaires with follow-up interviews with selected students that focus on the essential changes from one form of trigonometry to another including:

(i) Problematic aspects in learning successive levels of trigonometry;

(ii) The relative importance of the mathematical items in the questionnaire and perceived by the students;

(iii) The level of confidence in responding to the given mathematical items.

1.2 Structure of the thesis.

The relevant literature is reviewed in chapter 2. Since the main focus of the study is about developing a framework to see how humans make sense of trigonometry, this considers the extensive literature about theories of knowledge and understanding in mathematics. This chapter also presents the literature review of the past research in trigonometry, in particular the difficulties in learning trigonometry at all levels of education. The last section of chapter 2 considers the emotional affects association of mathematical thinking and wider issues in teacher education.

Chapter 3 discusses the constructs and the theoretical framework of this study in greater detail. It considers the nature of making sense of mathematics through human perception, operation and reason and relates this to the changing meanings in triangle, circle and analytic geometry. This is formulated in terms of supportive and problematic conceptions and the deeper ideas of not just knowing mathematical ideas but also grasping them within a coherent knowledge schema.

Chapter 4 discusses the details of the research design and methods that are used in this study. Research questions will be presented in this chapter and the rationale of adopting certain methods will be explained. The demography of the sample and the selection of sample will be reported. The issues of reliability, validity and limitations of the study will be explained in detail.

Chapter 5 reports the outcome of the pilot study with teacher-training students and a special case study of a highly gifted sixth-form pupil which together reveal a broad spectrum of data. This chapter begins with a description of how the pilot study is conducted, in particular, it describes the items used in the questionnaire and also the gathered responses. The data analysis of the pilot study is presented in the later part of this chapter. The theoretical hypotheses of the research are constructed by relating the relevant literature review to the preliminary data is discussed in this chapter.

Chapter 6 discusses the main data analysis collected from the questionnaires and from follow up interviews with five selected student teachers. The data collected from the questionnaires and the follow up interviews are reported and analyzed by using the theoretical framework explained in chapter 5.

Chapter 7 presents the analysis of concept maps constructed by the student teachers before and after the follow-up interviews. Concept map is used as a tool to tap into the knowledge structure of the student teachers. The process of construction of this concept map was recorded in order to trace the sequence of development.

Chapter 8 discusses the analysis of student teachers' perceptions on the importance of mastery of subject matter knowledge tested by the mathematics items in the questionnaire. Then the analysis of the student teachers' level of confidence in responding to the mathematics items of the questionnaire is presented. The data for these two constructs were collected through the questionnaire and the follow-up interviews.

Chapter 9 reports the learning difficulties as perceived by the student teachers. The first section is concerned with the student teachers' learning difficulties in trigonometry. The second section is about the perceived learning difficulties of students. Data is collected through the follow-up interviews.

The final chapter of this thesis, chapter 10, presents the summary and plans for future directions. This chapter also reiterates the three main ideas in this thesis in order to emphasize the importance of the proposed ideas. The issues of methodology are discussed in order to be aware with the limitations of the study. As a consequence of this study, suggestions for a new course in teacher education are proposed.

CHAPTER 2
Literature Review

2.1 Introduction.

This study is concerned with how student teachers make sense of trigonometry. The development of making sense through perception, operation and reason is partly based on the review of literature. In this case, various theories of mathematical thinking are presented in order to show how this framework evolves from previous work.

Student teachers learn trigonometry in school and university. This process of learning involves the changes of meaning between different contexts. In this study, three distinct contexts of trigonometry are proposed as triangle trigonometry, circle trigonometry and analytic trigonometry. The study performed by Michelle Challenger (2009) clearly shows that secondary school students have serious difficulties in coping with the changes of meaning in different context of school trigonometry. In this case, it would be interesting to investigate whether student teachers have difficulties in making sense of school trigonometry or not after learning university trigonometry. Some research done in trigonometry is presented in order to see what can be extended from previous work.

The transition in different contexts of trigonometry involves supportive and problematic conceptions. The notion of supportive and problematic conceptions can be regarded as a useful and powerful way of looking at the difficulties of student teachers in making sense of trigonometry. Rather than recognizing these difficulties as misconceptions, it might be more insightful to see these as pre-conceptions. In this study, the nature of these pre-conceptions is formulated as supportive or problematic conceptions in a new context so that the linkage between different contexts can be seen as a coherent whole. In the later stages of the sense-making process, some student teachers grasp the whole concept while most of the student teachers only manage to know the essential skills and strategies to cope with the changing of contexts.

The review of literature related to mathematical conceptions and mathematical understanding is presented in this chapter which is divided into four sections. The theories of mathematical thinking are reviewed in the first section relevant to the formulation of making sense through perception, operation and reason in this study. Most of the literature in this section is from the work of cognitive scientists. It will show how the theoretical framework in this study is evolved. The second section is about the literature review of research in trigonometry, in particular the difficulties in learning trigonometry. The third section presents the literature review of mathematical conceptions and mathematical understanding. The last section is a short review of subject matter knowledge and emotions associated to mathematical thinking.

2.2 Theories related to mathematical thinking.

Jean Piaget formulated a cognitive theory to explain the mechanisms and processes by which a child developed into an adult through successive stages of development. The first stage is known as the *sensori-motor* stage where a child develops the relationship between physical actions to perceived results of those actions. In the *pre-operational* stage, a child develops language and mental imagery from his/her own viewpoint. In a later stage of development which is known as the *concrete operational* stage, a child develops a stable conception of a physical object and understands that although the appearance of something changes, the thing itself does not. Lastly in the *formal operational* stage, a child develops the capacity to perform logical reasoning and is able to think in an abstract way without any dependence to physical objects. It should be noticed that Piaget's theory focusses on readiness and biological maturation of a learner in the learning process.

Jerome Bruner (1966), classified three modes of human representation namely *enactive, iconic* and *symbolic. Enactive* representation is action-based information and can be represented using gestures whereas *iconic* representation is image-based information and can be represented using pictures and diagrams. *Symbolic* representation develops last and this kind of information is stored and represented in the form of mathematical symbols and language.

Skemp (1979) referred to a development through *perception*, *action* and *reflection* at two levels, delta-one which involves perception (input) through the senses and action (output) operating on the external world, with internal reflection that leads to a higher order delta-two system which also involves input, output and reflection.

Pamela Liebeck (1984) proposed a theory of learning mathematics for children. The process of learning involves a sequence of abstraction which she called: E (Experience) – L (language) – P (Pictures) – S (Symbols). In this case, E refers to experience with physical objects, L refers to spoken language that describes that experience, P refers to pictures that represent the experience and S refers to written symbols that generalize the experience. E-L-P-S shares some similarities with Bruner's three modes of knowledge representation. E corresponds to *enactive mode*, P corresponds to *iconic mode,* L and S correspond to *symbolic mode* of Bruner's theory.

Efraim Fischbein (1987) considered the development of mathematics through three approaches namely *intuitive, algorithmic* and *formal.* An intuitive cognition is a kind of cognition that is accepted directly without the need for justification because it is self-evident for an individual. For instance, the whole is bigger than its parts is self-evident. *Algorithmic* aspect refers to solving techniques and standard strategies whereas the *formal* aspect refers to axioms, definitions theorems and proofs.

Tall (2004, 2013) developed the ideas of perception, action and reflection to categorize two distinct approaches to mathematics in school and a third formalist approach at university level. He spoke of three worlds of mathematics to categorize the three distinct types of mathematical thinking. The first is based on the world of *conceptual embodiment* building from human perceptions and actions through increasingly sophisticated practical activity and thought experiment to imagine perfect mental entities or platonic concepts within the mind. This world of embodiment gives us the meanings in the real world that become increasingly sophisticated in thought. The second is based on the world of *operational symbolism* building from physical actions, such as counting or measuring being symbolized as manipulable mental concepts of arithmetic and algebra. The world of symbolism gives us the power of computation and the ability to reason about the properties of numbers and algebra. The third is based on the world of *axiomatic formalism* which involves formal set-theoretic definitions and mathematical proof.

It should be noted that the three worlds of mathematics might blend in a sophisticated way so that new entities exist such as the number line which is a blending of the world of *conceptual embodiment* and the world of *operational symbolism*. In this case, the real number system blends a line (which arises from the world of conceptual embodiment) with numbers (which arises from the world of operational symbolism) so that new characteristics exist such as the numbers on the right hand side will be bigger than the numbers on the left hand side on a number line. In this case, humans need to use a blending of perception and operation in order to make sense of number line. Later in formal mathematics, they may describe the properties of the operations and formulate the definition of a complete ordered field.

2.2.1 Compression of operations.

The previous section shows some theories in mathematical thinking in a holistic view. This section will focus on the theories which explain how the mathematical concepts are learnt and understood from one level to another level.

Piaget (1972) introduced the notions of *empirical abstraction* and *pseudo-empirical abstraction*. *Empirical abstraction* focuses on objects and their properties whereas *pseudo-empirical abstraction* focuses on actions on objects and the properties of the actions. In a later stage, r*eflective abstraction* occurs through mental actions on mental concepts in which the mental operations themselves become new objects of thought (Piaget, 1972, p.70). Tall (2013) proposed a fourth type of abstraction as *platonic abstraction* where properties of objects are conceptualized as mental objects. Mitchelmore and White (1995) suggested the two faces of abstraction as a continuum between abstract-apart and abstract-general ideas. In short, abstract-general idea means the existence of links which enable learner both to recognize the idea in different contexts and to call up a variety of contexts in which the abstract idea is found. Abstract-apart idea indicates the mathematical idea is separated and apart from any context. They proposed this continuum is to explain students' difficulties in learning mathematics as they shift from familiar situations to more formal mathematical contexts.

Thurston (1990) spoke of similar ideas by using the term compression to describe how human develop thinkable concept.

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it

> quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990, p. 847)

Compression is needed in the learning of mathematics because it can free the cognitive space of a learner so that the learner will not be required to focus on certain details all the time. For instance, the learning of numbers is started from the operation of counting. When the learner has compressed this operation of counting into the number concept then he can use the numbers freely for different operations (such as multiplication and division etc) without the need to do the counting all the time.

The terms procedural knowledge and conceptual knowledge are widely used in mathematics education research (Hiebert & Lefevre, 1986; Hiebert & Carpenter, 1992). Procedural knowledge can be regarded as knowledge of performing step by step procedure. When the procedural knowledge is compressed into thinkable concepts then the linking of different thinkable concepts can be performed to build a powerful knowledge structure. Meaning is generated when relationships between units of knowledge are recognized or developed whereas conceptual knowledge by the definition in Hiebert (1986) must be learned meaningfully. Procedures may or may not be learned meaningfully. Hiebert (1986) proposed that procedures that are learned meaningfully are procedures that are linked with conceptual knowledge. On the other hand, rote learning produces knowledge that is without relationship and this knowledge is tied closely to specific context, therefore it can’t be generalized to other situations.

Cottrill et al (1996) proposed a theory to describe the sequence of learning mathematics as actions, processes, objects and schemas with acronym as APOS theory. Action is regarded as physical or mental transformation of objects that is initially seen as external to the individual. In this stage, the subject has least control over the action and every subsequent step is triggered by the results of the previous step. Conscious control can be established when an individual reflects on the actions then the actions are internalized as processes. In this stage, the individual is able to describe and reflect on all the steps in the transformation without necessarily performing them. Later on the individual might encapsulate the processes as thinkable objects so that transformations can be acted on them. These thinkable objects will be situated in a growing schema of ideas and the schema can be encapsulated as an object in further development.

Sfard (1991) spoke of operational mathematics which focusses on processes alternating with structural mathematics which focusses on objects and their properties. She identifies a constant three-step pattern in the transitions from operational to structural conceptions. These three steps of concept development are known as interiorization, condensation and reification. In the first stage there must be processes acted on objects then interiorization occurs when the individual is acquainted with the processes which eventually give rise to a new concept. The condensation stage signifies the ability of an individual to think of a given process in terms of input/output without necessarily considering its details. Reification is the later stage which solidifies process into object and this allows an individual to think about it in a structural way.

Gray & Tall (1994) proposed the term *procept* to talk about symbol as process and concept. According to Tall (2005), there are five levels of development for operational procedures to be compressed into a thinkable concept. The first level is known as the pre-procedure level which indicates that no operational procedure is developed for a problem. The second level is known as the procedure where a specific procedure is developed to solve a routine problem. A Multi-procedure level is achieved when an individual has developed more than one procedure to solve a routine problem and this allows the individual to select a more efficient procedure. Process is a level where an individual sees the process as an input/output operation and various procedures are seen to be equivalent. The procept level is achieved when an individual can think of a symbol as process or concept in a flexible way and this gives power to the individual to operate different procedures and to think about the related concepts. All these are related to Davis's ideas (1984) who formulated the transition between process and object as three stages namely visually moderated sequence (VMS), integrated sequence and a noun. The visually moderated sequence stage indicates each step prompts the next whereas the integrated sequence stage indicates that different sequences are seen as a whole and can be broken into sub-sequences. Finally the integrated sequence is encapsulated as a noun so that an individual can conceive it as an entity or an object.

Tall (2013) identifies three major methods for compression which are *categorization, encapsulation* and *definition*. In general, *categorization* is based on the recognizable properties of objects, *encapsulation* is based on repeating actions which are symbolized as mental objects, and *definition* uses language to formulate concepts as a basis for proof and mathematical reasoning. Over the years, Tall (2013) further developed and refined the notion of compression into *embodied compression* and *symbolic compression*. By focusing on the effect of operation on already known objects (which are called 'base objects'), *embodied compression* occurs. On the other hand, *symbolic compression* focuses on symbols and being able to perform the operation so that the symbols can be conceived as processes or concepts in a flexible way.

2.2.2 Compression of objects through categorization.

Van Hiele (1959) formulated a theory which consists of five levels to describe the learning of geometry. The first level is known as visualization which indicates that a child is able to classify geometry shapes based on holistic appearance. At this level, visual prototypes are developed in children minds. The second level is known as analysis and the objects of thought are classes of shapes therefore a child at this level will be able to analyze the properties of a shape. For instance, the child might say, "A triangle is a figure with three sides and three angles." The third level is known as abstraction which indicates the objects of thought are geometric properties. At this level, a child can relate different set of properties and see the implications of one set of properties to another. For instance, a child might say, "All squares are rectangles but not all rectangles are squares.", this is because the child understand the properties of square and rectangle and sees a square is just a specific example of rectangle.

Level four is known as deduction where the object of thought is deductive reasoning (Euclidean proof). At this level, Euclidean geometry is used to deduce other properties based on a figure with certain property and the understanding of geometric ideas is limited to the objects in Euclidean plane. For instance, the idea of congruent triangle arises through the laying one triangle over the other triangle. The final level which is level 5 is known as rigor. At this level, students understand that definitions are arbitrary and need not to refer to any real world concrete object so that the students can study non-Euclidean geometries.

Over the years, different researchers have renamed these levels for their research contexts. Hoffer (1981) renamed these 5 levels as recognition, analysis, ordering, deduction and rigor. Meanwhile Clements & Battista (1992) renamed these as recognition, descriptive/analytic, abstract/relational, Euclidean deduction and rigor. Tall (2013) renamed these as recognition, description, definition, Euclidean proof and rigor.

2.2.3 Distillation of the theories.

Section 2.2 describes general theories of mathematical thinking covering all the domains of mathematics. However, they do not put much emphasis in explaining the different stages of compression in mathematics. Meanwhile the theories in section 2.2.1 focus on the explanation of how the mathematical concepts are compressed from one level to another level, in particular the compression of operation to concept. On the other hand, the theories in section 2.2.2 focus on the explanation of different stages of compression through sensory input.

There is no perfect theory in this world. Different theories have different emphases to serve different purposes. For instance, Jean Piaget's stage theory focuses on the readiness of humans for learning based on their biological development from sensori-motor beginnings through concrete operational and formal operational stages of development. Meanwhile, Van Hiele (1959) proposed a theory to explain the learning of geometry. Bruner (1966) focuses on different kinds of representation of information from enactive through iconic and symbolic. Skemp (1979) focuses on the humans' innate ability to make sense of mathematics through perception, action and reflection and whether the student's understanding is instrumental (in terms of rote learning) or relational (Skemp, 1976). Efraim Fischbein (1987) focuses on three approaches to mathematics: intuitive, algorithmic and formal. Tall (2004) focuses on three distinct developments of mathematical thinking by proposing three worlds of mathematics, one focusing on the perception, recognition and construction of objects and their properties, one focusing on operations that are compressed into manipulable symbols, and the increasing sophistication through reasoning that leads to the highest level of axiomatic formal mathematics in university and in mathematical research.

Part of the theoretical framework in this study was formulated by blending these theories together in a coherent way in order to explain how humans make sense of mathematics. This gave rise to the three modes of making sense through *perception, operation* and *reason*, where operations are seen generally as actions that are performed with a specific purpose e.g. for construction in geometry or through symbolic operations in arithmetic and algebra. The aspects of operations in embodiment and symbolism are blended together in the study of trigonometry. In the following table, we consider the roles of perception, action (which becomes more sophisticated as operation) and reason.

Author	Proposed framework				Emphasis
Bruner (1966)	Iconic (*Perception)*	Enactive (*Action)*	Symbolic (*Reason)*		Representation of information
Skemp (1979)	Perception (*Perception*)	Action (*Action)*	Reflection (*Reason*)		Humans' ability to make sense
Liebeck (1984)	Experience (*Perception/ Action)*	Language (*Reason*)	Picture (*Perception*)	Symbol (*Operation)*	Sequence of abstraction
Fischbein (1987)	Intuitive (*Perception)*	Algorithmic (*Operation*)	Formal (*Reason*)		Approaches to mathematics
Tall (2004)	Embodiment (*Perception)*	Symbolism (*Operation*)	Formalism (*Reason*)		Three distinct types of thinking in maths
Cottrill et al (1996)	APOS (*Action Process Object Schema*)				Sequence of compression of operation into a concept
Sfard (1991)	Operational (*Operation*)		Structural (*Perception*)		Two different types of mathematics
Van Hiele (1959)	Successive levels in geometry involving perceptions, geometric operations and reason				Explain the hierarchy of learning geometry
Gray & Tall (1994)	Procept (*operations become mental objects represented as symbols with properties that are subject to reason*)				Symbol as process and concept

Table 2.1 Links between the proposed theoretical framework to other theories.

Based on Table 2.1, it can be noticed that the proposed theoretical framework in this study corresponds to other theories up to certain extent. The words in italics indicate the proposed mode which corresponds to different parts of other theories. However, while these modes may focus on on *perception, operation* or *reason,* usually they involve a blend of all three. For instance, the three worlds of Tall (2004) all involve perception, operation and reason while the main emphasis in conceptual embodiment is on perception of objects and their resulting properties found by operation and reason, operational symbolism is based on operations and their properties represented symbolically and axiomatic formal theory is based on verbal definition and reason, often suggested by perceptions and operations.

2.3 Research in trigonometry.

There are not many research studies in the learning of trigonometry. A few have suggested new teaching instructions as supplements for the conventional teaching methods of trigonometry (Searl, 1998; Barnes, 1999; Miller, 2001). These suggestions are potentially useful to improve students' understanding of trigonometry but they are not based on any mathematics learning theories. A study performed by Michelle Challenger (2009) on a group of 16-18 year old students found that students have certain confusions in learning trigonometry. Some of the students' comments are given as follow:

> "Are we talking about triangle trigonometry or circle trigonometry here?"

"I used to understand it when it was just triangles but now I don't know where to start."

"What is sine exactly? I thought I knew but now it is so confusing."

It is evident that secondary school students have difficulties in making sense of trigonometry. What about student teachers? Do they have the same difficulties in making sense of trigonometry after studying a mathematics degree in university? Weber (2005) conducted a study to examine the understanding of college students on trigonometry. He designed an experimental instruction which followed a learning trajectory based on the notion of *procept* proposed by Gray & Tall (1994).

The outcome of the study indicated that the students in the experimental group possess deeper understanding than the students of the control group who were taught by using conventional lecture-based classes. The main aims of Weber's study (2005) was to explore how students reason about trigonometric functions and how they justify the properties of trigonometric functions after being taught with two different approaches. In this case, the questions set for the pre-test, post-test and interviews focus on students' conceptual understanding. Below are some questions from the tests and interviews:

"What is the range of sin x? Why?"

"Approximate a. $\sin 340°$ b. $\cos 340°$. Explain your work."

"Why does $\sin^2 \theta + \cos^2 \theta = 1$?"

"What does the sentence, $\sin 40° = 0.635$ mean to you?"

"What can you tell me about $\sin 170°$? Will this number be positive or negative? How do you know? Can you give me an approximation for this number?"

Generally, the results of the pre-test were poor. All justifications provided by the students were either based on rote learning such as "it's just a mathematical law" or mnemonic devices. In the post-test, all the students of the experimental group had improved greatly.

Two limitations of students' understandings are reported in this study. The first limitation is related to the role of geometric figures in reasoning about the concept of functions. For instance, some students couldn't approximate $\sin\theta$ for a given θ only and asked for more information such as a labeled triangle so that they can answer the task. The second limitation is about the level of control felt by the students when operating with the sine function. According to Breidenbach et al. (1992), an operation performed by an individual can be categorized as either internal or external to him/her. If the perceived operation is internal then the individual will see it as a means to accomplish a mathematical goal and this implies he/she possesses deeper understanding than those who perform meaningless operations (perceived as external operations) which are triggered by external cues. Weber's (2005) study also shows that those students who are successful in the post-test have a tendency to reason about trigonometric functions by using the unit circle.

Blackett and Tall (1991) found that students of experimental group who were taught by using interactive computer graphics to relate the visual model (triangle figures) to numerical data has helped them to gain greater improvement in terms of performance than the control group. For example computers were used to show the changes of triangle for every 10° as the ratio of the side opposite to the hypotenuse of the corresponding triangle is tabulated. The purpose of doing this was to embody the dynamic relationship between visual and numerical data and this can hardly be done by using conventional teaching methods.

Hart (1981) shows that ratio is an extremely difficult concept for children to understand. This is related to the learning of triangle trigonometry which defined sine as the ratio of the side opposite a given angle (in a right angled triangle) to the hypotenuse. Ratio and proportion are complicated because they involve the comparison of 4 pieces of information and the relationships between them. When ratio is compressed to equivalent fractions then a single unified concept arises, which is the sine of an angle that is a number that can vary as the angle varies (Tall, 2013).

2.4 Construction of knowledge and understanding.

Constructivism is a worldview stating that knowledge is constructed through an active constructive process by a learner. A more traditional approach occurs when knowledge is transmitted from teacher to learner. Over recent years, there has been much support for the constructivist paradigm and also the recognition that there is a need to blend aspects of both approaches in a more connectionist style where the teacher as mentor guides the learner to grasping more subtle aspects of the mathematical ideas (Askew et al, 1997). Examining knowledge from the constructivist viewpoint, knowledge is a personal construction which is essentially different between individuals therefore it may be hypothesized that different individuals possess different nature of knowledge. The epistemological issue in mathematics knowledge is complex. The argument whether mathematics knowledge is discovered or invented is still ongoing (Tall, 2011). A mathematician, Sir Christopher Zeeman stated that he invented some aspects of mathematics in order to be able to formulate and study a certain problem and then other aspects are discovered as the consequence of its context (Arnot, 2005). This practical viewpoint has highlighted the unavoidable point that individuals need to develop their own mathematics understanding for themselves.

The term 'understanding' is a commonly used term in the process of teaching and learning. However, the meaning of understanding is rather subjective. Different person interpret understanding differently. It seems like there is no agreed meaning for the term understanding.

> To understand something means to assimilate it into an appropriate schema. (Skemp, 1971, p,46)

Skemp (1979), described schema as a structure of connected concepts. This mental model which is made up of a number of interconnected concepts, is a conceptual structure.

> A conceptual structure is called a schema. Among the new functions which a schema has, beyond the separate properties of its individual concepts, are the following: it integrates existing knowledge, it acts as a tool for future learning, and it makes possible understanding.
>
> (Skemp, 1987, p.124)

It should be noted that the function of a schema is essentially different from the functions of those individual concepts which make up this conceptual structure. Different individuals might understand something by assimilating it into their own personal schemas, thus this explains the subjective nature of understanding. Besides, assimilation into an inappropriate schema might also give a learner a subjective feeling of understanding.

According to Kotarbinski (1961) (in Sierpinska, 1994), the word 'to understand' is often claimed to be highly ambiguous. Kotarbinski pointed out that 'understanding' can be thought of as an actual or a potential mental experience. Some authors state that 'understanding' is synonymous to 'understanding why'. Piaget speaks of understanding a practical action (Sierpinska, 1994). In this case, to understand an action means to understand why it works or why it does not work. Skemp (1976) used the term 'relational understanding' to indicate a kind of understanding that involves knowing what to do and why. Another kind of understanding proposed by Skemp (1976) is instrumental understanding which can be described as 'rules without reasons'.

> Learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point.
>
> (Skemp, 1976, p.25)

National Council of Teachers of Mathematics (NCTM) states the focus of high school mathematics as reasoning and sense-making. In this context,

> Sense making may be considered as developing understanding of a situation, context, or concept by connecting it with the existing knowledge or previous experience. (NCTM, n. d.)

This is coherent to the notion of understanding as proposed by Skemp (1987). The focus of NCTM has highlighted the mechanisms and importance of sense-making in mathematics.

2.5 Conceptions of mathematics.

Sophisticated mathematical thinking builds on personal conceptions in particular mathematical concepts. This view is consistent with the view of constructivism which proposed that knowledge is constructed by the learner perhaps with guidance by a mentor, rather than only being transmitted. Tall & Vinner (1991) expressed the distinction between concept image and concept definition which is widely used in the research of mathematics education nowadays. In this context, the concept image is described as the total cognitive structure associated with the concept which might include mental images, processes, representations and properties. The concept definition is referred as a form of words used to specify the concept. Sfard proposed a similar idea by referring it as conception.

> The whole cluster of internal representations and associations evoked by the concept – the concept's counterpart in the internal, subjective "universe of human knowing". (Sfard, 1991, Vol 22, p3)

The subjective nature of personal mathematical conceptions leads us to realize that the divergence in the performance of mathematics across learners is partly due to this. The more able learners will make sense of new mathematics ideas based on the conceptions they hold to build on more sophisticated knowledge structure. The less able learners tend to learn new ideas more as discontinuous sets of facts. The question is how do we humans acquire mathematical conceptions? Personal experience leads us to have certain conceptions on certain things, in particular mathematics. Lima & Tall (2008) introduced the term *met-before* to indicate the effect of previous experience in a new situation that affects our current thinking. *Met-before* highlights the importance of previous experience in shaping mathematical conception. There might be *supportive* met-befores and *problematic* met-befores incorporated in conceptions. A supportive met-before is considered as a kind of experience which supports the development of coherent knowledge structure in a new situation. A problematic met-before impedes generalization in new situation. It should be noted that supportive or problematic met-befores can be incorporated into new conceptions, for instance, the met-before of 'take-away always gives less' in natural number is regarded as a supportive met-before when working in positive integers. On the other hand, this met-before will become problematic when working in the context of negative integers.

2.6 Subject matter knowledge of mathematics.

Subject Matter Knowledge is the 'amount and organization of the knowledge *per se* in the mind of the teacher' (Shulman, 1986, p.9). A systematic review of mathematics teacher's subject matter knowledge by Baturo and Nason in 1996, led the authors to redefine the term 'subject matter knowledge'. It is not just about substantive knowledge but it should include other important facets such as (i) understanding of knowledge about the nature and discourse of mathematics; (ii) knowledge about mathematics in culture and society; (iii) teachers' dispositions towards the subject. In this context, substantive knowledge is not limited to getting the correct answer but also has a sense of the mathematical meanings underlying the concepts and processes. It should be a collection of interconnected concepts and procedures. Knowledge about the nature and discourse of mathematics refers to (i) what counts as an 'answer', justification is part of the answer; (ii) how the truth or reasonableness of an answer is established; (iii) creative activities such as examining patterns, formulating and testing generalizations and constructing proofs; (iv) what can be derived logically versus what could be defined as mathematics convention. Knowledge about mathematics in culture and society refers to an understanding how mathematical ideas are used in our society.

A background review done by Rowland (2007) indicated that novice secondary school teachers use various coping strategies, such as relying heavily on a textbook when they lack content knowledge. At the same time, the teacher's style of instruction is also affected in order to avoid discussion and student questions. Ball (1990) discovered that prospective teachers' university experiences and understanding were commonly instrumental and not conceptual. A few researchers have shown the inadequacy of content knowledge of mathematics teachers (e.g. Martin & Harel, 1989; Ma, 1999; Rowland et al, 2001; Goulding & Sulgate, 2001; Rowland & Tsang, 2005).

Research done by Rowland and Tsang (2005) on SMK of Hong Kong primary school mathematics teachers has showed that the SMK of the respondents was quite shallow. 138 respondents were involved in this survey. A test was used as one of the instruments to collect data regarding the SMK of the participants. In this survey, an interesting finding indicated that teachers' perception of the SMK that is important in teaching was different from what they thought they would be able to solve. In reality, the respondents performed poorly with the test items that they considered important. Another unexpected finding from this survey was the relationship between the respondents' test score and the years of teaching experience. The Pearson correlation coefficient found was -0.256 (significant at $\alpha = 0.01$ (two tailed)). This indicates that respondents who have more years of teaching experience tend to have a lower test score.

2.7 Emotions associated to mathematical thinking.

Skemp (1979) proposed a theory which linked emotions to mathematics. According to this theory, human emotions to mathematics are related to the goal state and anti-goal state (see Figure 2.1).

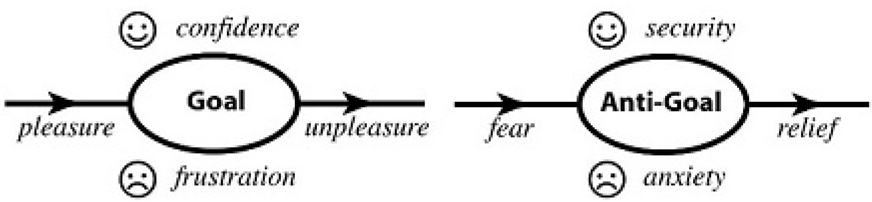


Figure 2.1: Emotions associated with goals and anti-goals (Tall, 2013).

The nature of goal state and anti-goal state can be categorized as short term or long term. For instance, a short term goal might be a wish to learn the necessary procedures to solve a routine problem without knowing why the procedures work so that the learner can pass a test. Meanwhile, a long term goal might be a wish to achieve relational understandings in mathematics. On the other hand, a short term anti-goal might be a wish to avoid failing a particular mathematics test or a longer-term desire to avoid mathematics after secondary school education.

Based on figure 2.1, there are eight categories of emotions. The first four types of emotions are pleasure, unpleasure, fear and relief which signal the changes towards or away from a state. In this case, the state can be either a goal state or an anti-goal state. Meanwhile the other four types of emotions are confidence, frustration, security and anxiety which signal the knowledge of ability or inability to change into a stage. An individual will have the pleasure of doing something if the individual is moving towards the goal state. Confidence is a positive emotion which emerges when an individual believes that he/she can achieve the goal.

For the anti-goal state, an individual will have a sense of security if the individual believes that he/she can avoid this state. On the other hand, an individual will have a sense of anxiety if the individual believes that he/she can't avoid this anti-goal state. Moving towards the anti-goal state will give us a sense of fear, while moving away gives us relief. All these are related to the notion of supportive and problematic conceptions in this study. A supportive conception gives an individual a sense of confidence in answering mathematical task because the individual believes that this conception can lead him/her to the goal state. On the other hand, a problematic conception gives an individual a sense of anxiety in answering mathematical task because the individual believes that this conception will lead him/her to the anti-goal state.

Some supportive conceptions might contain problematic aspects. In this case, an individual with this conception might have confidence in using this conception to answer a mathematical task and suppress the problematic aspects because the individual believes that it will lead him/her to the goal state. It should be noted that different individuals handle these problematic aspects differently. Some individuals might keep on using this conception without bothering or thinking about the problematic aspects. Other individuals might believe that these problematic aspects might be resolved as they learn higher level mathematics.

2.8 Emergence of research questions from the literature review.

The literature review in section 2.3 has shown that most students have confusions in making sense of trigonometry in school and college. Hence it is definitely worthwhile to see how student teachers who have studied more advanced concepts in trigonometry at university think about the forms of trigonometry that they may teach in school. For this reason, I started this study by exploring the concept image of university students concerning the concept of sine and expanded this theme to focus on various difficulties that the student may experience in dealing with trigonometry.

A broad review of theories related to mathematical thinking was presented in section 2.2. As a consequence of this, I realized that different theories have different emphases and different ways of looking at how humans make sense of mathematics. The literature review of section 2.5 focused on conceptions of mathematics that led me to rethink and characterize the nature of personal conceptions in mathematics and their long-term development. In order to explore the personal conceptions of a learner, I included a few mathematics items that cover the domains of school trigonometry and university trigonometry. By doing this, I will be able to gain relevant data on how a learner attempts to cope with the changes of contexts in trigonometry.

Past studies have shown the inadequacy of teachers' subject knowledge and their relationships to other constructs such as the perceived importance of subject knowledge. A study by Rowland & Tsang (2005) has shown that there was a decay of subject knowledge in teachers that increased over their years of teaching experience. I found this very interesting because it conflicted with my expectation. In order to explore further this theme therefore I included it as one of the research themes in this study. Emotion is associated to mathematical thinking, therefore it is definitely sensible to explore how the process of sense-making affects the emotion of a learner in order to build a bigger picture for this study.

2.9 Summary.

In this chapter, relevant reviews of literature are presented. This study concerns with how student teachers make sense of mathematics. The development of making sense of mathematics through perception, operation and reason are built from the existing theories of mathematical thinking which are presented in section 2.2. There are three distinct contexts in making sense of trigonometry which can be traced from the past research done in this field. In this case, the review of research done in trigonometry is presented in section 2.3.

The process of sense-making in mathematics involves the building of coherent links between different contexts. In this case, learners build on the supportive or problematic conceptions to make sense of a new context. Due to this point, the reviews of knowledge construction and mathematical conceptions were presented in section 2.4 and 2.5 respectively. Humans have different emotions associated to supportive or problematic conceptions. In relation to this aspect, the review of emotions associated to mathematical thinking was presented in section 2.7. This study also explores the perception of student teachers regarding the importance of subject matter knowledge tested by the research instruments, therefore it is sensible to include the review of literature in subject matter knowledge which was performed in section 2.6.

Chapter 3
Theoretical Framework

3.1 Introduction.

This chapter presents the development of theoretical framework of this study. It begins by describing the evolution of a pre-existing theory of how human makes sense of mathematics through *perception, action* and *reason*. Then the nature of trigonometry is discussed from the mathematical perspective, in particular the notion of extensional blend in mathematics. An extensional blend occurs in mathematics when a system is generalized into a broader system. Some of the aspects of the earlier system generalizes but others do not. As a consequence, learners need to cope with the changes of meaning. It is inherently difficult for most of the learners to deal with problematic conceptions that worked in the earlier context but no longer work in the new context. In this study, three distinct contexts of trigonometry are proposed to focus on the changes of meaning: triangle trigonometry, circle trigonometry and analytic trigonometry. Next, a theory of making sense of mathematics is described. Learners have problematic conceptions or supportive conceptions in making sense of mathematics in the new context. We investigate how the difficulties to cope with the changes of meaning in mathematics are due to the problematic conceptions of the learner.

3.2 Ways of making sense of mathematics.

This study evolves a framework for making sense of mathematics through *perception, operation* and *reason* and uses it in the specific case of trigonometry. It is based on a fundamental theory of how human learn to think mathematically from early childhood to adult including mathematician. Similar frameworks have developed over the years, including Piaget (1950, 1972), Bruner (1966), Skemp (1979), Liebeck (1984), Fischbein (1987), and Tall (2004, 2013). A theory of abstraction was proposed by Piaget to focus on three distinct types, namely *empirical abstraction, pseudo-empirical abstraction* and *reflective abstraction.* In this case, *empirical abstraction* focuses on how a child constructs meaning for the properties of objects. *Pseudo-empirical abstraction* focuses on the operations themselves and *reflective abstraction* focuses on how actions and operations become thematized objects of thought

Fischbein (1987) proposed the development of mathematics through three different approaches namely *intuitive, algorithmic* and *formal*. An intuitive cognition is accepted directly without the need for justification because it is self-evident for an individual. *Algorithmic* aspect refers to solving techniques and standard strategies whereas *formal* aspect refers to the use of definitions, theorems and proofs.

Bruner (1966) suggested three modes of representation of information or knowledge in human namely *enactive, iconic* and *symbolic. Enactive* representation is action-based information and can be represented using gestures whereas *iconic* representation is image-based information and can be represented using pictures and diagrams. *Symbolic* representation develops last and this kind of information is stored and represented in the form of mathematical symbols and language.

Skemp (1979) proposed three distinct types of human activity namely *perception* (input), *action* (output) and *reflection* which involve higher levels of perception and action.

The SOLO Taxonomy of Biggs and Collis (1982) is an acronym for the Structure of Observed Learning Outcomes, which is an assessment system to give credits to the types of responses in assessment. It formulates a sequence of stages following Piaget and Bruner through sensori-motor, iconic, concrete operational, formal operational and post-formal operational. Within each mode it formulates an increasing sophistication of response, categorized into *unistructural* (noticing one aspect), *multistructural* (noticing different aspects), *relational* (relating different aspects) and *extended abstract* (a coherent whole structure). A significant interpretation of SOLO taxonomy is that each successive stage is incorporated into the next, so that, for example, *enactive* and *iconic* may be incorporated into a blend that Tall (2004) calls *conceptual embodiment* involving *perceptions* and *actions* that over the long-term are reflected upon to produce mental conceptual imagery such as platonic objects.

Pamela Liebeck (1984) proposed a theory of learning mathematics for children which combines practical aspects of other theories such as those of Piaget and Bruner. The process of learning involves a sequence of abstraction which she called ELPS involving E (Experience)-L (language)-P

(Pictures)-S (Symbols). In this case, E is refers to experience with physical objects, L refers to spoken language that describes that experience, P refers to pictures that represent the experience and S refers to written symbols that formulate the experience.

The pre-existing theory of making sense of mathematics in this study is built on the theoretical framework of Tall (2004, 2013). The long-term development of mathematical thinking in human terms is based on the fundamental foundations of human *perception, operation* and *reason.* The original framework focused on three distinct type of mathematical thinking. The first is based on the world of *conceptual embodiment* building from human perceptions and actions through increasingly sophisticated practical activity and thought experiment to imagine perfect mental entities or platonic concepts within the mind. The second is based on the world of *operational symbolism* building from physical actions, such as counting or measuring being symbolized as manipulable mental concepts of arithmetic and algebra.

These two ways of operation are blended together in school mathematics. Euclidean geometry builds from embodied perception and operation that develops verbal reasoning processes through verbal definitions and Euclidean proof to support the imagination of perfect geometric figures. Physical measurement is initially an embodied operation that translates into operational symbolism. In trigonometry, visual imagination of similar figures and the properties of right-angled triangles lead to the operational symbolism of trigonometric ratios in triangle trigonometry. This blend of embodiment and symbolism extends to dynamically changing angles and trigonometric functions that can again be represented graphically and computed symbolically in the form of circle trigonometry.

At university in pure mathematics, there is a shift to more formal mathematics which includes the use of power series to compute trigonometric and exponential functions to any desired degree of accuracy and complex numbers that offer visual interpretations of relationships between exponential and trigonometric functions such as Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ and de Moivre's theorem which uses Euler's formula for the angle $\theta + \varphi$ to compute the formulae for sin(A+B), cos(A+B).

Tall (2013) pictures this development in the following figure:

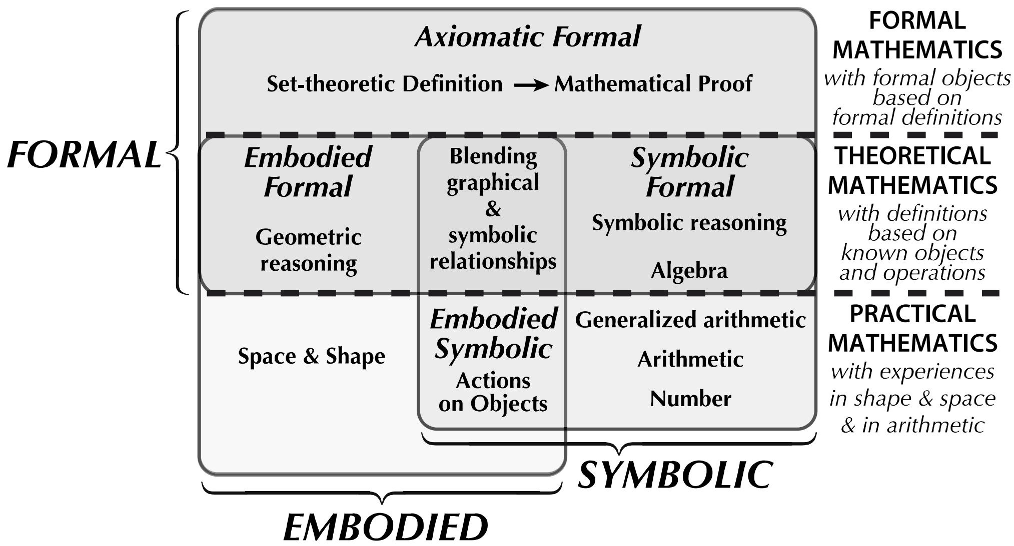


Figure 3.1: Practical, theoretical and formal mathematics (Tall, 2013).

Here practical mathematics starts with experiences in shape and space with actions on objects such as counting and sharing becoming symbolised as number, developing the operations of arithmetic and the generalized properties of arithmetic that lead to the more theoretical study of algebra.

Practical mathematics is based on recognition of figures, description of their properties and constructing them physically in geometry as measurement allows them to be related to properties of arithmetic. Theoretical mathematics introduces definitions in Euclidean geometry including properties of Euclidean triangles and trigonometric ratios, which link to arithmetical operations to solve trigonometric problems. It is possible at this level to give a Euclidean proof of the formula for sin(A+B) for $A+B < 90°$, although it is more likely that the definition will be introduced as a formula to be remembered by heart. The dynamic variation of the angle in a unit circle leads to radian measurement and the introduction of circle geometry where angles can now be any size, positive or negative, leading to the sine function and its graphical representation with its periodic repetition and visual symmetries that link visual and symbolic reasoning.

Formal mathematics is studied at university in various forms. Applied mathematicians may use a combination of visual ideas in complex numbers and symbolic calculations with power series that essentially extend the conceptual embodied world of triangle and circle geometry to higher levels of formal embodiment and formal symbolism. Pure mathematicians are more likely to study mathematical analysis that shifts to a higher level of axiomatic formalism.

Reflecting on the successive levels of thinking as the learner becomes increasingly sophisticated, the three-worlds model of development begins from the sensori-motor aspects of the child and builds upon the child's increasingly sophisticated mental connections that are enhanced through the use of language. In the world of conceptual embodiment the child perceives objects and operates upon them to discover and verbalize their properties and so thinking in this kind of operation begins with perceptions of objects, operations on those objects and reasoning about the properties concerned.

Meanwhile, in the world of operational symbolism, the child focuses on operations being performed, such as counting, sharing, measuring and the focus is initially on carrying out the operations and seeing their effects and reasoning about relationships, so that, although there is a focus on operation, this blends with perception and reason to develop the properties of arithmetic and later symbolic developments.

In axiomatic symbolism, the focus switches to formulating properties verbally, making definitions and reasoning by deducing further properties from the definitions using mathematical proof.

Therefore, in every mode of operation and development in sophistication, there is an interplay between perception, operation and reason.

Trigonometry involves the perception of geometric figures, the definition of the trigonometric properties, first as ratios of lengths, then as trigonometric functions and the interplay between geometric relationships and symbolic operations.

The purpose of this study is to investigate the conceptions that university students develop in analytic trigonometry and how this may relate to the triangle and circle trigonometry that they will teach in school. This will involve analysing how the students make sense of mathematics through *perception, operation* and *reason.* While it may seem on the surface that perception is more dominant in the embodied world, operation in the symbolic world and reason in the formal world, perception, operation and reason operate in various ways in all three. The embodied world of geometry includes operations on figures and reasoning using Euclidean proof. It is linked to the symbolic world through trigonometric definitions and relationships. The symbolic world is based on perception of real world operations and reasoning about them, using the observed rules of arithmetic to develop algebraic ideas as generalised arithmetic and to relate measurement of figures and dynamic visual graphs to corresponding operational symbolism. Most of the work in trigonometry can be performed using a blending of visual embodiment and operational symbolism, however, there are also aspects of formal definitions of complex numbers and convergence of power series that arise, linking perception, operation and reason.

In particular, over the long term, reason begins by reflecting upon perception and operation and steadily develops into more formal ways of thinking. Reason arises through making conceptual and deductive links, first based on experiment and prediction, which is consonant with Skemp's mode (i) and Liebeck's notion of experience and language in ELPS where, in the longer term, language develops into more sophisticated forms of definition and deduction. This involves a steady increase in sophistication as connections are made and higher-level concepts are labeled so that they can be grasped and mentally manipulated in more sophisticated theories.

3.3 Extensional blend in trigonometry.

Generalization in mathematics involves an extensional blend in which an old system is generalized to cover a broader domain, for instance from integers to rational numbers, from rational numbers to real numbers, from real numbers to complex numbers etc. A generalization is a blend of conceptual ideas that focuses on certain essentials that continue to apply in the extended situation. This may involve problematic aspects that no longer work in the new situation and may impede generalization. Such situations may be seen to occur in shifting from triangle trigonometry to circle trigonometry, in the transition to analytic trigonometry and also in the transition back from analytic trigonometry in university to teaching triangle and circle trigonometry in school. We now consider each of these in greater detail.

3.3.1 Triangle trigonometry.

The learning of trigonometry in school starts off with the introduction of sine as the ratio of the length of the side opposite to an angle to the length of the hypotenuse in a right-angle triangle. This involves subtle ideas of ratio and proportion that are already known to cause significant difficulties for learners (Noelting, 1980a; Noelting, 1980b; Hart, 1981). Triangle trigonometry involves seeing the lengths as magnitude and angles in between 0 and 90 degrees inside a right-angle triangle (see Figure 3.2). In this context, the right-angle triangle is operating in the Euclidean Plane. The angle involved will be considered in terms of size only without any direction of measuring (see Figure 3.2). Angle is measured using degrees.

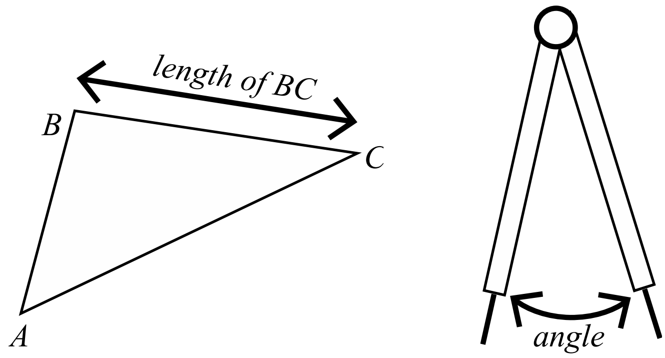


Figure 3.2: Length and angle in Euclidean geometry.

3.3.2 Circle Trigonometry

An extensional blend occurs when triangle trigonometry is extended to circle trigonometry which involves variable angles at the centre of a circle which now can be positive (in the anticlockwise direction measured from the positive x-axis) or negative (clockwise) with trigonometric ratios involving signed numbers that vary and lead to the properties of trigonometric functions. In this context, the unit circle is operating in what may be called the Modern Cartesian mode. The notion of modern Cartesian in this study is different from the notion of the Cartesian system as suggested by René Descartes. The Descartes version of Cartesian system was developed in 1637 with positive and negative values on a number line but doesn't have perpendicular axes. On the other hand, the notion of modern Cartesian in this study does have perpendicular axes and the Cartesian coordinates are signed numbers.

The modern definition evolves from Cartesian ideas rather than Euclidean ideas. Vertex A is located at the origin of circle (see Figure 3.3). The angle involves magnitude and direction of measuring (see Figure 3.3). Radians emerge as a consequence of relating the angle in a circle to the length of the arc subtended by the angle in a unit circle. The length of the opposite side and the length of the adjacent side of the right angle triangle are now signed lengths, although, by convention, the hypotenuse is always taken to be positive (see Figure 3.3).

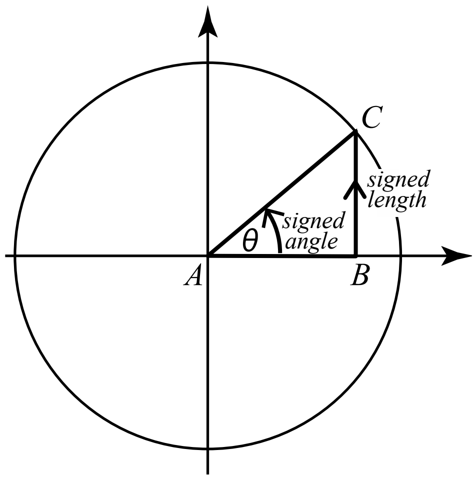


Figure 3.3: A Right angled triangle in the unit circle.

New signed lengths have been introduced into the blend: the length of sides AB and BC of the triangle which correspond to numbers in the Cartesian plane (see Figure 3.3). The hypotenuse of the right angle triangle is taken to be 1 with the sine function identified with the vertical y component and the cosine with the horizontal x-component. New mathematical functions emerge which are known as the cosine theta (whose value is the x-coordinate of C) and the sine theta (whose value is the y-coordinate of C). *Graphical trigonometry* (using the graph) is part of the *circle trigonometry* but it may be used on its own without relating back to the circle.

Differentiation and integration can be performed on these functions. The transition from circle trigonometry to analytic trigonometry needs calculus. In this case, the calculus is used to determine the relationship which leads to the possibility of expressing sine and cosine as power series. This involves the introduction of measurement of angles in radians rather than degrees. In the following section, the derivation of power series for sine function and cosine function will be demonstrated.

3.3.3 Analytic trigonometry.

Analytic trigonometry involves trigonometric functions are expressed as power series and the use complex numbers to relate exponential and trigonometric functions. Below is the demonstration of how calculus leads to the expression of sine as power series. The power series for a function is given as below:

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

When *a*=0, we have

$$f(x)=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

Let $f(x)=\sin x$ then we have

$$\sin x=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

When $x=0$, $\sin 0 = 0 = c_0$;

$$f'(x) = \frac{d \sin x}{dx} = \cos x = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

Again when $x = 0$, $\cos 0 = 1 = c_1$;

$$f''(x) = \frac{d^2 \sin x}{dx^2} = -\sin x = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$$

Again when $x = 0$, $-\sin 0 = 0 = c_2$;

$$f'''(x) = \frac{d^3 \sin x}{dx^3} = -\cos x = 6c_3 + 24c_4 x + \cdots$$

Again when $x = 0,\ -\cos 0 = -1 = 6c_3 \Rightarrow c_3 = \frac{1}{6}$;

So we see that we will have coefficients only for odd values of n. In addition, the sign of the coefficients will alternate. Thus,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \ldots$$

Then we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

If we differentiate the power series of sine x then we get cosine x as follow:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Similarly, a special function e^x can be derived as follow by using the same above method:

$$f(x) = e^x = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$f(x) = e^x$	$f(0) = 1$	$a_0 = 1$
$f'(x) = e^x$	$f'(0) = 1$	$a_1 = 1$
$f''(x) = e^x$	$f''(0) = 1$	$a_2 = \frac{1}{2}$
$f'''(x) = e^x$	$f'''(0) = 1$	$a_3 = \frac{1}{6}$
$f^{(iv)}(x) = e^x$	$f^{(iv)}(0) = 1$	$a_4 = \frac{1}{24}$

Table 3.1: Determination of a_n .

Based on the table 3.1, we get a power series for e^x as follow:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

Since $i^2 = -1$, then

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

When the unit circle is operating in the trigonometric complex plane blend, we get

$x + iy = \cos\theta + i\sin\theta$, since *r=1* (see Figure 3.4)

$e^{i\theta} = x + iy$ is known as the Euler formula (see Figure 3.5)

In general, a complex number can be expressed as

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

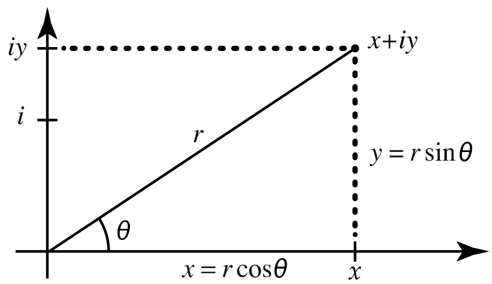


Figure 3.4: A complex number in cartesian and polar coordinates.

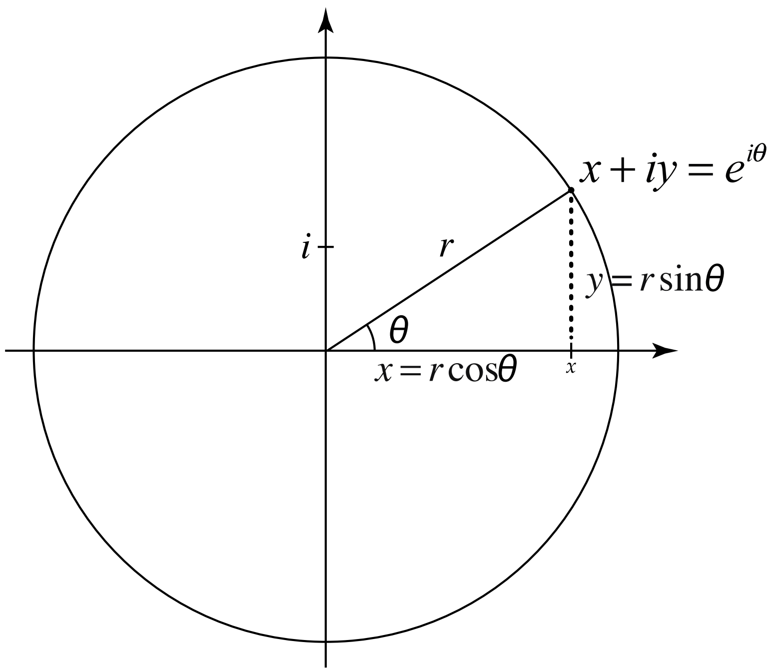


Figure 3.5: Geometric representation of Euler formula.

3.4 Supportive Conceptions and Problematic Conceptions in Changing Contexts in Trigonometry

A supportive conception supports generalization in a new context whereas a problematic conception impedes generalization. Making sense of trigonometry in three distinct contexts involves situations where a conception that is supportive in one context may become problematic in another. For instance, in triangle trigonometry, learners can imagine $\sin 70^\circ$ as a ratio of the opposite and hypotenuse in a right-angle triangle because they could visualize this triangle. However based on triangle trigonometry, it is hard for learners to make sense of $\sin 200^\circ$ because their experience in Euclidean geometry tells them that an angle in a right angled triangle must be greater than zero and less than 90°. As we saw in the pilot study in particular respondent A, she ended up drawing a weird figure (see page 80) in making sense of $\sin 270^\circ$. In this case, the Euclidean conception becomes problematic in circle geometry because it impedes the sense making of learner in the broader context.

Meanwhile a supportive conception may remain supportive in a new context. For instance, the sine graph, which is a supportive conception in circle trigonometry remains supportive in analytic trigonometry. It should be noted that there might exist problematic aspects in a supportive conception. For instance, some learners couldn't make sense of why $\sin 270^\circ = -1$ but they know that it is true solely based on the sine graph. The problematic aspect is they may not be able to relate it to the unit circle or the triangle trigonometry to see it as a coherent whole.

The examples above have shown how meanings may change from supportive to problematic or stay supportive in a new context. This is not the whole story. Other possibilities occur. For instance a problematic aspect may stay problematic but become supportive in the light of more powerful insight. A good example is the formula for $\sin(\alpha+\beta)$. In general this formula is supportive conception because it can be used in all the three distinct contexts of trigonometry. However this supportive conception has a problematic aspect in triangle trigonometry as it can only be proved for α, β positive and $\alpha+\beta$ less than $90°$. In this case, the problematic aspect is why the formula is applicable for any value of A and B because from the proof in the triangle trigonometry the angle A and B is constrained to A+B less than $90°$ (see Figure 3.6).

Meanwhile, it is not obvious how to prove it geometrically in circle trigonometry (the problematic aspect still remains problematic) in the general case and no-one ever attempts this (although it would be possible to write an app for a tablet where one could move the vertices around the circle and represent the different signs with different colours). The proof of this sum angle identity using a unit circle is problematic. It is quite easy to prove this identity in the first quadrant because we know that is AB=1 due to the radius of the unit circle and the angles α and β are inscribed in the relevant right angled triangles (see Figure 3.6). However proving this identity in second quadrant or third quadrant or fourth quadrant can be very difficult for learners to conceive and execute. The formula is supportive when used in a symbolic way in trigonometry and in developing formulae in the calculus.

Let $AB = 1$ and $\angle A = \alpha + \beta$ as in the figure.
Then in the triangle ABC, $BC = \sin(\alpha + \beta)$.
Drop the perpendicular *BD* from B to *AD*, *DE* from *D* to *BC* and *DF* from *D* to *AF*.
Then **BC = BE + DF** … (*).
In triangle ABC, $BD = AB\sin\beta = \sin\beta$,
in *BDE*, $\mathbf{BE} = BD\cos\alpha = \cos\alpha\sin\beta$.
In the same way $AD = AB\cos\beta = \cos\beta$,
And $\mathbf{DF} = AD\sin\alpha = \sin\alpha\cos\beta$
From (*), $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

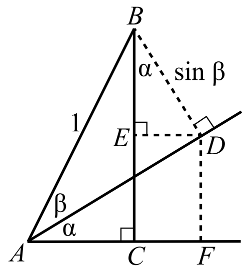


Figure 3.6: Proof of the formula for $\sin(\alpha+\beta)$ in triangle trigonometry.

In analytic trigonometry, proving this identity using the special number *e* and complex number *i* where $i^2 = -1$ is very easy (the problematic aspect becomes supportive in analytic trigonometry).

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} \ (Power\ law\ x^{m+n} = x^m x^n)$$
$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

Multiplying out the brackets gives

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + i\cos\alpha\sin\beta)$$

By comparing the real and imaginary parts then we get

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + i\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

This is a very good example to show that the journey that a learner needs to go through in order to achieve a full coherent knowledge. In learning mathematics, learners might need to suppress certain problematic aspect in order to move to another level and a full coherent knowledge structure might be achieved in later stage. The data from the special case study has shown that the respondent was attempting to sort out the confusion of complex numbers that have some aspects that make sense to him which he used and other aspects that were problematic that he hoped to sort out later. For him it was not a matter of suppression. Learners need to constantly reconstruct their knowledge structure in order to overcome problematic conceptions in new context. In general, the angle sum identity can be regarded as a supportive conception with problematic aspects (in triangle trigonometry and circle trigonometry) because it always helps learners to get the correct answer but critical learners might feel a bit uncomfortable with the proof conducted in triangle trigonometry. However this problematic aspect will be resolved once the learners have a full coherent knowledge structure after making sense of analytic trigonometry. This is a good example to show how a conception which seems not to work in an old context but can work perfectly in a new context at a later stage.

Based on the above explanation, an important question is raised. Does learning advanced mathematics in analytic trigonometry provide a learner with a more convincing and simple explanation all the time? The answer is not necessary. For instance when the respondents are asked to explain why $\sin\theta$ can never equal 2, a response based on Taylor series or complex numbers will not be easy to justify. A more direct justification is the response obtained from the unit circle. The radius of the unit circle which is always 1 can easily provide the respondents with a simple and convincing explanation of why $\sin\theta$ can never equal 2. Respondents can also give a triangle trigonometry response for why $\sin\theta$ can never equal 2. In this case, the response would be based on the ratio of opposite to hypotenuse of a right angled triangle and the respondents would justify this by saying that $\sin\theta$ can never equal 2 because the hypotenuse must be greater than the opposite of a right angled triangle. A further reflections will reveal that the response based on the triangle trigonometry will have problematic aspects if the respondents are asked to justify why $\sin\theta$ can never equal -2. In this case, the unit circle still provides the simplest and most convincing explanation for this. It should be noted that there are strengths and weaknesses in giving different responses based on different contexts of trigonometry.

3.5 Knowing and grasping.

Students often learn *how* to carry out a mathematical procedure without knowing *why*. For example they may learn particular algorithms and conventions that need to be used in a specific situation so that they are successful in solving simple problems but they do not grasp the essential ideas that will enable them to solve more sophisticated problems. As an example, students in this study often *know* that they must use degrees in triangle trigonometry and radians in circle trigonometry and do this successfully but they do not *grasp* the essential reason why this is so. In this thesis I will use the distinction between *knowing* a mathematical idea and *grasping* that idea so that it can be used in more sophisticated ways. It is analogous to being able to grasp an object in one's hand and to be able to manipulate it, to look at it from different angles, to use it in different ways. To say that one can *grasp* an abstract idea essentially means that one can think of it and speak about it as a meaningful entity in its own right. A student might *know* an idea at one level and yet not *grasp* the idea at a higher level. In analyzing the verbal data we will seek to distinguish those cases in which the student grasps the fundamental ideas rather than just knowing how to cope with them in a routine manner. On the other hand a student teacher may grasp a mathematical idea at a higher level but not know how to use it in teaching learners who are meeting the idea at an earlier level.

3.5.1 Relationships with established theories.

Hiebert (1986) distinguished two different kinds of mathematics knowledge as conceptual knowledge and procedural knowledge. Conceptual knowledge is rich in relationships and is characterized as a connected web of knowledge as a single coherent knowledge. Procedural knowledge consists of formal language or symbol representation system and rules or algorithms to solve mathematical tasks. To grasp a particular piece of mathematical knowledge requires conceptual relationships to place that knowledge in context, but it also requires a fluent ability to perform the necessary procedures where necessary.

Skemp (1976) proposed a similar notion as *instrumental understanding* and *relational understanding.* In this case, *instrumental understanding* can be summarized as knowing what to do and how to do whereas *relational understanding* is concern with knowing why it works. Again, grasping ideas requires relational understanding, with a fluent ability to perform the operations.

Skemp (1979) distinguished between conceptual links (C-links) and associative links (A-links) possessed by the learners. C-links have conceptual qualities which relate one idea to another whereas A-links are associative in nature which may be formed through rote learning and memorizing. Leron and Hazzan (2006) reported dual processing theory where the immediate response (S1 response) operates at a non-analytic or intuitive level, which is immediate, effortless and inflexible. In contrast, S2 response operates at an analytic level, which is slow, effortful and relatively flexible. Grasping ideas requires C-links which usually are S2 responses. Kahneman (2011) spoke of a similar notion by suggesting two different ways the brain forms thoughts namely system 1 which is fast, automatic, subconscious, frequent, stereotypic whereas system 2 is slow, logical, effortful, calculating, conscious and infrequent. When a C-link is used in a regular basis, it might be transformed into a A-link without consciously recognizing the underlying ideas therefore it will become a S1 response. In this case, a blending of links occurs to reduce the cognitive strains in doing mathematical tasks.

3.6 Summary.

The theoretical framework of this study is the consequence of three important ideas in this thesis. The idea of how human make sense of mathematics has motivated the work of reviewing extensive literature which leads to the hypothesized theory of making sense through *perception, operation* and *reason*. The idea of changing of meaning in mathematics has resulted in the framework of looking at trigonometry in three distinct contexts namely triangle trigonometry, circle trigonometry and analytic trigonometry. Based on the notion of extensional blend in this study, the factor which impedes the shifting between triangle trigonometry and circle trigonometry is proposed. In this context, the proposed factor is the changing of meaning between Euclidean geometry and modern Cartesian. Finally the idea of how humans cope with the changes of meaning in mathematics has resulted the notion of problematic conception and supportive conception. Human may suppress problematic conceptions and problematic aspects of supportive conceptions in order to keep on learning mathematics at a higher level. Problematic conceptions impede the sense making in new context and thus prohibit the building of coherent knowledge structure. The idea of *knowing* and *grasping* is important in the sense that it could provide a powerful explanation for the nature of knowledge possess by a respondent. The more able learners would know an idea at one level and grasp the same idea at a higher level. They can look at it from different angles and speak about it as a coherent entity in its own right. On the other hand, the less able learners would know an idea at one level but couldn't grasp the idea at a higher level. This is evident when they couldn't think of it and speak of it as a coherent entity.

Chapter 4
Research Design and Methods

4.1 Introduction.

In general, this study concerns how student teachers make sense of trigonometry. Based on the review of literature and the collected data, trigonometry can be categorized into three distinct contexts namely triangle trigonometry, circle trigonometry and analytic trigonometry. In these three contexts, student teachers use different combinations of perception, operation and reason to make sense of trigonometry. The transition in different contexts of trigonometry involves supportive and problematic conceptions. In the later stages of the sense-making process, some student teachers grasp the concept while others only manage to know the essential skills to progress to learning higher-level concepts. The focus is to explore how the student teachers cope with the changes of meanings in trigonometry after learning triangle trigonometry and circle trigonometry in school and analytic trigonometry in university.

According to the review of literature, there are a few themes that are strongly related to this study and should be taken into consideration. The central ideas relate to the nature of the knowledge structures and their difficulties in learning trigonometry together with the importance of subject matter knowledge and the level of confidence in responding to the mathematics items of the questionnaire. This chapter discusses the research design, methods of data collection and method of data analysis, together with issues related to validity and reliability.

4.2 Research questions.

In this section, the overall development will be considered to draw out important aspects for study that will be denoted *in italics*. According to Tall & Vinner (1981), the concept image is the total cognitive structure of an individual's mind that is associated to a concept. In this case, the questionnaire begins with an item to describe the concept of sine so as *to explore the evoked concept image of student teachers, which is likely to involve triangle trigonometry*. Then relevant items in circle trigonometry are set to explore the conceptions of student teachers in this context and *to see how they will make sense of circle trigonometry* in particular with reference to the changes of meaning from the triangle trigonometry to circle trigonometry. For instance, item 3 and item 4 of the questionnaire are about sense making of $\sin 200°$ and $\sin 270°$. Item 4 explores further by asking the respondents to explain why $\sin 270°$ has certain value, *which is likely to evoke a graphical explanation*. Besides, these items also aim to explore the conceptions of student teachers on the given expressions.

The student teachers are then asked to *explain the reason for using radians instead of degrees*. This item is important in the sense that radians only start to arise in circle trigonometry, in particular using the unit circle and measuring the angle in terms of the length of the arc, and this idea also links triangle trigonometry and circle trigonometry in a visually meaningful way. Since the generation of the sine curve is from the unit circle, radians can be considered as the starting point of calculus in trigonometry.

Next, some of the properties of sine are explored. For instance the questionnaire asks the respondents to state and explain why sinx is decreasing for certain values. This item is set to explore *whether the respondents have developed a coherent link between the unit circle and the sine graph*. The questionnaire also asks the respondents to explain why $\sin\theta$ can never equal 2. For this item, the respondent will have a freedom to respond in either one of the three trigonometry contexts. I also expect that some of the respondents might respond in more than one context.

Then the questionnaire asks about the ideas in calculus. For instance the questionnaire asks "What does dy/dx mean?", "What would d/dx [sinx] mean? What is d/dx [sinx]? Explain why." *All these items are set to explore the conceptions of respondents in calculus*.

After that the respondents are asked to describe $\sin 30°$, $\sin 120°$ and $\tan 90°$. *The first may invoke triangle trigonometry, the second may involve circle trigonometry and the third involves a possible singular case where the angle is no long part of a proper Euclidean triangle*.

The last item of the questionnaire asks the respondents to *explain the relationships between the concept of sine and concepts such as function, series, complex numbers and y=mx*. This item is set to see *whether the respondents have coherent links between different aspects of mathematics that arise in triangle, circle and analytic trigonometry*.

In general, the questionnaire was designed to cover the full range of development of trigonometry encountered in school including triangle trigonometry in terms of ratio and proportion, circle trigonometry in terms of radians, angles in a circle and graphical representations of trigonometric functions, with more sophisticated topics in the calculus. I was particularly interested in how students who had spent several years studying more formal analytic mathematics may respond to these questions and how this related to the development of trigonometry in school.

Follow-up interviews were conducted with selected student teachers on a voluntary basis in order to gain further insights on their written responses. For instance during the follow-up interviews, the interviewees were asked whether they can visualize a triangle with $\sin 200°$ and $\sin 270°$ or not. These questions are asked in order to explore the conceptions of interviewees and to expand the written responses for items in the questionnaire. Meanwhile it is impossible to list all the questions which were asked in the follow-up interviews because different interviewees will be asked different questions, were based on the verbal responses during the follow-up interviews. The main aim of the interviews is to gain further insights on the written responses of the interviewees.

Based on the description above, it should be evident that I am interested to research the following things:

1. What is the evoked concept image of sine?
2. How do respondents make sense of trigonometry?
3. Is there any evidence that shows student teachers working in different contexts of trigonometry in making sense of trigonometry?

4. What are the supportive and problematic conceptions involved in making sense of trigonometry and how do these conceptions affect the sense making of trigonometry?
5. What are the student teachers' conceptions on using degrees and radians in trigonometry?
6. Do the student teachers have a coherent link between the unit circle and the sine graph?
7. What are the conceptions of student teachers in calculus in trigonometry?
8. Do the student teachers grasp the knowledge of trigonometry or they just know it?
9. What is the perceived level of importance for the subject matter knowledge tested by the mathematical items?
10. What is the level of confidence in responding to the mathematical items?
11. What are the difficulties in learning trigonometry as perceived by the student teachers?

The research questions above can be regrouped into two types which are specific and general research questions. Some research questions are specific in the sense that they can be answered by using a particular item in the questionnaire. For instance, the first research question (What is the evoked concept image of sine?) can be answered by analyzing the data collected from item 1 of the questionnaire (Describe sin x in your own words.). Similarly, the fifth research question (What are the student teachers' conceptions on using degrees and radians in trigonometry?) can be answered by the data collected from item 6 (What do radians mean? Why do we need radians when we have degrees?) of the questionnaire. Likewise for the seventh research question (What are the conceptions of student teachers in calculus of trigonometry?) can be answered by the data collected from item 10 (What does dy/dx mean?) and item 11 (What would d/dx [sin x] mean? What is d/dx [sin x]? Explain why.) Research question no 9 is answered by the data collected from the part B of the questionnaire. As with research question no 10, it can be answered by analyzing the data from part C of the questionnaire. The follow-up interviews also contribute to the generation of additional data to answer these research questions.

On the other hand, research question no. 2, 3, 4, 6 and 8 as stated above are regarded as general because these questions can only be answered through analyzing data collected from a series of items in the questionnaire. Additionally, the follow-up interviews are also an important source of data to answer these research questions.

4.3 The sample.

The sample of the main study is a group of 24 student teachers attending the PGCE Secondary Mathematics Programme at a university in the UK. The university is an established university in providing a teacher training programme. Most of the PGCE student teachers possess a mathematics bachelor degree without any teaching experience. A questionnaire is distributed to all the student teachers in order to get a broad sense of the variety of responses of the student teachers. Follow-up interviews were conducted for eight student teachers based on voluntary basis from which five were selected as offering a spectrum of responses. Before conducting the follow-up interview, the student teachers were asked to construct a concept map. Then a second concept map will be constructed by the student teachers after the follow-up interview in order to see the effect of follow-up interviews on the knowledge structure of the respondents.

4.4 Methods of data collection.

According to Cohen and Manion (1980), methods mean 'the range of approaches used in educational research to gather data which are to be used as a basis for inference, interpretation for explanation and prediction' (p. 42). In short, methods refer to the techniques and procedures used for data collection. Questionnaire, follow-up interviews and concept maps are used for data collection in this study. Every research method has its strengths and weaknesses therefore it is important to discuss each of them in detail so that the appropriateness of the research methods for this study can be justified. The following sections will discuss the justification of methods of data collection in a more general sense because the details of the questionnaire and the follow-up interviews are discussed in section 4.2.

4.4.1 Questionnaire.

According to Cohen & Manion (1980), a questionnaire is a good method of data collection to gather responses in a standardised way. It is more objective than interviews and less time consuming in administering. Besides, data can be collected from a large group. However this potential might not be realised due to the possibility of low response rate; therefore in this study, the questionnaire is delivered and responded to in class time in order to minimise the possibility of low response.

There are some disadvantages in using a questionnaire. A respondent might misinterpret the meaning of an item in a questionnaire. In order to minimise the possibility of items misinterpretation in a questionnaire, a pilot study and special case study were conducted so that the item could be rephrased to reduce misinterpretation. A follow-up interview is conducted for voluntary participants to correct any misinterpretation and to gain greater insight into their written response. According to Cohen & Manion (1980), open-ended questions may generate a large amount of data but this didn't happen in this study due the specific nature of the items. The open-ended questions in the questionnaire are related to mathematical conceptions that a respondent may hold in a specific situation, therefore this is very specific and didn't generate a lot of data. A copy of the questionnaire is attached as appendix in this thesis. According to Carter & Williamson (1996), respondents might answer superficially for the items in a questionnaire. Based on Leron and Hazzan (2006, 2009), an S1 response is considered as automatic, effortless, non-conscious and inflexible response. On the other hand, an S2 response may be considered as time consuming, conscious, effortful, and relatively flexible response. Meanwhile a superficial answer may be related to the notion of S1 response in Dual Process Theory. This also further justifies the need for a follow-up interview in order to get more

S2 responses. Before the questionnaire is distributed, the respondents were told why the information is being collected and how the results of the study will be beneficial. This might invite more honest responses because the respondents will know that a negative response is just as useful as positive response. The questionnaire is anonymous so that the respondents may feel more comfortable in responding to it.

There are three sections in the questionnaire. Section A comprises 15 mathematical items. Section B is about the perceptions of respondents regarding the importance of mastery of subject matter knowledge tested by the items in section A. Section C is about the level of confidence of the respondents in responding to the mathematical items in section A.

Broadly speaking, the mathematical items were set to answer the proposed research questions. The format of the questionnaire was adapted from Rowland and Tsang (2005) whereas the mathematics items of the questionnaire were related to those from Weber (2005). The paper by Weber (2005) investigated the college students' understanding of trigonometric functions based on Gray and Tall's (1994) notion of *procept* and other process-object encapsulation theories of learning. Weber (2005) also found that the experimental instruction which was designed based on the learning trajectory of *procept* was successful in helping the learners to develop a deeper understanding of trigonometric functions as compared to the learners of the control group of the study.

This study extended Weber's research in the sense that it covers the three contexts of trigonometry namely triangle trigonometry, circle trigonometry and analytic trigonometry. Besides, this study also uncovers the difficulties of learning trigonometry from the mathematics perspective and cognitive perspective based on a newly developed theoretical framework resulting from this study. The methodology adopted in this study is partly influenced by the Piagetian view that consistent errors made by learners is related to their cognitive structures. In this study, consistent errors among respondents maybe viewed as a kind of problematic conception or problematic aspect possessed by the respondents in making sense of a new context. By using the mathematical items in the questionnaire, data on how respondents respond to the items and the conceptions that they possess are gathered which should provide insight into their cognitions. In the end, the focus is to explore how the student teachers cope with the changes of meanings in trigonometry after learning all the three distinct contexts in trigonometry namely triangle trigonometry, circle trigonometry and analytic trigonometry, to reflect on how this may impact on their future teaching.

4.4.2 Follow-up interview.

Interviewing students is a very popular way of gathering data for the research in mathematics education (Hazzan & Zazkis, 1999). This is due to the reason that an interview is a good method to discern complex thinking processes. In this study, the main aim is to explore the mathematical conceptions possessed by the student teachers and, according to Svensson (1997), personal conceptions are accessible through language therefore using interviews can serve this purpose. Piaget (1929) adopted the clinical interview method in order to understand the development of children's minds. In this case, the experimenter gave a open-ended task to the participants to complete, whilst thinking aloud. Then the experimenter asked further questions based on the responses of the participants. A clinical interview is similar to the notion of semi-structured interviews as suggested by Cohen et al (2001).

The main reason for using semi-structured follow-up interview as a research method is because of its ability to gain further insights based on the responses of the interviewees. Question followed by question can be asked in the interview session to expand particular responses. Moreover some participants might be more comfortable with verbal explanation instead of written explanation. One of the problems in using interviews as a research method is the problem of validity. According to Cohen & Manion (1980), the cause of invalidity is bias which can be defined as a systematic or persistent tendency to make errors in the same direction. Perhaps the most practical way of gaining greater validity is to minimise the bias as much as possible. In this case, the sources of bias will come from the interviewer, the interviewee and the question. For instance, a interviewer might ask questions to support his pre-conceptions; misinterpretations of the explanations of interviewee, etc. Besides, the questions asked in an interview should be crystal clear in meaning to avoid misinterpretations.

4.4.3 Concept map.

According to Novak & Govin (1984), a concept map is a reliable tool to externalise the knowledge structure of a respondent. A concept map-based assessment consists of a task that elicits a participant's connected understanding of a knowledge domain. The degree of directedness of a mapping task is set by the assessor depending on a few factors such as the objective of the research, time constraints and available resources etc. A low degree of directedness might be a task that asks the participant to construct a concept map without the researcher specifying any concept, linking labels or structure. On the other hand, a very highly directed degree of mapping task might only require the participant to fill in nodes or fill in lines for a given concept map. The amount or extent of information provided in a concept mapping task is also an important aspect in determining the degree of directedness. According to Ruiz-Primo (2004), different mapping techniques provide different information about students' connected understanding. Quantitative analysis of concept maps by using scores is reliable even when complex judgement regarding quality of proposition is involved. A researcher needs to be aware that different mapping techniques may lead to different conclusions about participants' knowledge in a subject domain (Ruiz-Primo, 2004).

In this study, a low directness mapping task is adopted. This will allow the participants to have the highest freedom and flexibility in constructing the concept maps. Student teachers are asked to construct two concept maps at different times. No concepts are given to the student teachers. The first concept map is constructed before the follow-up interview whereas the second concept map is constructed after the follow-up interview. Due to the issue of practicality, only focus questions will be given to participants prior to the concept map construction. In this case, the participants are asked to construct concept maps for the concept of trigonometry. Individual participant-generated maps offer the most open-ended approach (Bolte, 2006). It might provide the broadest picture of individual knowledge construction. However, sometimes it is difficult for a participant to generate a list that captures his/her depth of understanding. Non-structured concept maps were used in this study because this left space for the expression of individual concepts.

4.5 The quasi-judicial method of analysis.

In this study, I try to be as objective as possible in analysing the collected data by not only looking for evidence which supports my theoretical framework but also data which is contrary to the framework. Meanwhile it is possible that some data might help to extend and modify the framework. In this case, I have adopted the quasi-judicial method of analysis because this method gives equal weight to data which supports my theory and also data which denies my theory.

Bromley (1986, 1990) developed the quasi-judicial method of analysis for qualitative data in the context of psychological case-studies. This method of analysis is very different from other methods of qualitative data analysis in the sense that rather than letting the theories emerge from data, this method uses the existing data to test pre-existing theories. Besides looking for evidence which supports the theory, one of the important features of this method is the searching for sufficient evidence to eliminate as many of the proposed explanations as possible. According to Bromley (1986), psychological case-study is adequate to take account of why and how a person behaved in an interesting context provided that it "contains enough empirical evidence, marshalled by a sufficiently cogent and comprehensive argument, to convince competent investigators that they understand something that previously puzzled them" (p.37).

Bromley (1986) suggested the ten steps below in analysing a case.

1. State initial issues clearly.
2. Collect and state background context for the case.
3. Suggest prima facie explanations.
4. Examine prima facie explanations and look for additional evidence.
5. Search for sufficient evidence to eliminate as many of the proposed explanations as possible.
6. Examine the evidence and sources of evidence closely to check for consistency and accuracy.
7. Conduct a critical inquiry into the internal coherence, logic and external validity of the arguments in the favoured explanations.
8. Adopt the 'most likely' explanation.
9. Formulate, if appropriate, what implications there are for action.
10. Write a coherent report. (Adapted from Bromley, 1986, p.26.)

Bromley (1986, 1990) introduced the idea of "case law" to deal with the increasing of evidence from multiple cases. Family resemblances might exist between different cases. “By comparing and contrasting cases, a kind of case-law can be developed. Case-law provides rules, generalisations, and categories which gradually systematise the knowledge (facts and theories) gained from the intensive study of individual cases.” (Bromley, 1986, p.2.) This method also raises validity and reliability issues in the sense that how can this method generalise to other cases. Bromley argues that the generalisation based on case-study or case law (developed from several case-studies) should be seen from the validity of the case rather than its representativeness. Additionally, the interpretation of data may be seen as valid if they inform a coherent whole.

Instead of building theories from data, this method focusses on testing different theories by using data so that theories can be accepted, rejected, modified and extended.

4.6 Ethical considerations.

All the student teachers who joined the pilot study and the main study were volunteers. Ethical approval was granted by Warwick Institute of Education. A consent form with copyright assignment was distributed to every interviewee prior the follow-up interviews. The purpose of the follow-up interview was stated in the consent form so that all the interviewees will have an idea regarding this study. The copyright assignment is an important document to indicate that I have obtained the permission from the interviewees to use the recordings and written materials from the follow-up interviews. Additionally, all the interviewees are requested to agree to the use of anonymous written transcripts in publications and presentations.

4.7 Conclusion.

The research design and methods of data collection are discussed in this chapter. The rationales of adopting certain methods of data collection are presented. Questionnaires, follow-up interviews and concept maps are used in this study to collect the relevant data. The items in the questionnaire are described in order to see how the items can be used to collect the needed data for answering the research questions. Finally, quasi-judicial method of analysis for qualitative data is presented because this method fits the aim of this study which is to test a proposed theory.

Chapter 5
Preliminary Investigations

5.1 Introduction.

This chapter presents the description and the output of the pilot study conducted for this research. The pilot study was conducted early in the year 2011 with a group of student teachers who were doing Secondary Mathematics PGCE at university. This pilot study is aimed to test our pre-existing theory on how students make sense of trigonometry. The pre-existing theory suggests that human makes sense of mathematics through *perception, operation* and *reason.* In this case, *perception* is based on conceptual embodiment building from human perceptions and thought experiment. *Operation* is based on physical actions such as counting, symbolized as manipulable mental concepts in the operational symbolism of arithmetic and algebra. *Reason* is based on verbalizing relationships such as links between visual and symbolic representation and on definition and deduction. In addition, respondents' conceptions of trigonometry are explored. I hypothesised that humans have problematic conceptions and supportive conceptions in making sense of mathematics in new contexts. The changes of meanings in mathematics in new contexts are one of the factors that caused mathematics becomes a subject which is very difficult to make sense. In this context, trigonometry is a very good topic to illustrate this.

The learning of trigonometry has been categorized into three distinct contexts: triangle trigonometry, circle trigonometry and analytic trigonometry. Triangle trigonometry is based on right-angled triangles with positive sides and angles bigger than $0°$ and less than $90°$. Circle trigonometry involves dynamic angles of any size and sign with trigonometric ratios involving signed numbers and the properties of trigonometric functions represented as graphs. Analytic trigonometry involves trigonometric functions expressed as power series and the use of complex numbers to relate exponential and trigonometric functions. The initial theoretical framework was developed based on the review of literature, the discussions with my supervisor and partly based on the research questions as proposed in chapter 4. The pilot study also serves as a platform to test the feasibility of the research instrument in answering the proposed research questions of this study. In the pilot study, a set of questionnaire was distributed to the respondents.

In addition, a special case study was conducted early in the year 2012. This study involved one sixth form student. Currently this student is studying mathematics at Oxford University with scholarship. This chapter describes the pilot study and the special case study, in particular, the research instrument used, the pilot study sample details and the outcomes of the pilot study.

5.2 The pilot study.

In general, a questionnaire is a good instrument to gather data from a group of respondents at one time. The data provides insight on the various possible ways of making sense of trigonometry. Furthermore, the conceptions of respondents could be explored. This questionnaire is divided into three sections. Section A consists 8 mathematics items. Section B was aimed to explore student teachers' perception on the importance of the subject matter knowledge tested by the items in section A of the questionnaire. Section C was about the level of confidence of respondents in responding to the mathematical items. A group of 45 Secondary Mathematics PGCE student teachers participated in this pilot study.

5.3 Outcome of the pilot study.

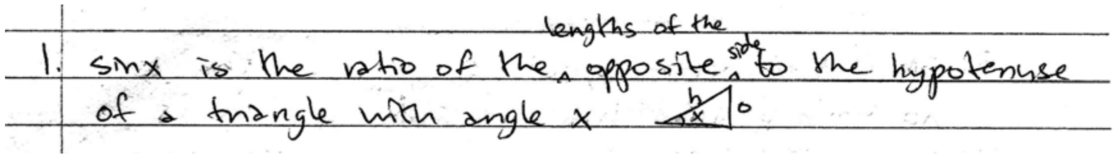
The outcome of the pilot study shows that the responses of the respondents fit nicely into the suggested pre-existing theory of making sense of mathematics. In general, the responses can be categorized into *perception, operation* and *reason.* In this study, *perception* is based on conceptual embodiment building from human perceptions and thought experiment. *Operation* is based on physical actions such as counting, symbolized as manipulable mental concepts in the operational symbolism of arithmetic and algebra. *Reason* is based on verbalizing relationships and using definition and deduction. The quasi-judicial method of analysis is employed in order to search for evidence which supports the pre-existing theory. At the same time, the emphasis will be given to evidence which doesn't fit the theory as well.

After a close examination of all the received responses, the responses of three respondents were chosen and reported in this section. The rationale for choosing these three respondents was because they showed qualitatively different responses and could cover the spectrum of responses of this group of respondents.

5.3.1 Respondent A.

Respondent A is a female PGCE student with a 2(i) bachelor degree in physics with previous employment as a medical physicist.

Describe sin x in your own words.


1. sinx is the ratio of the lengths of the opposite side to the hypotenuse of a triangle with angle x

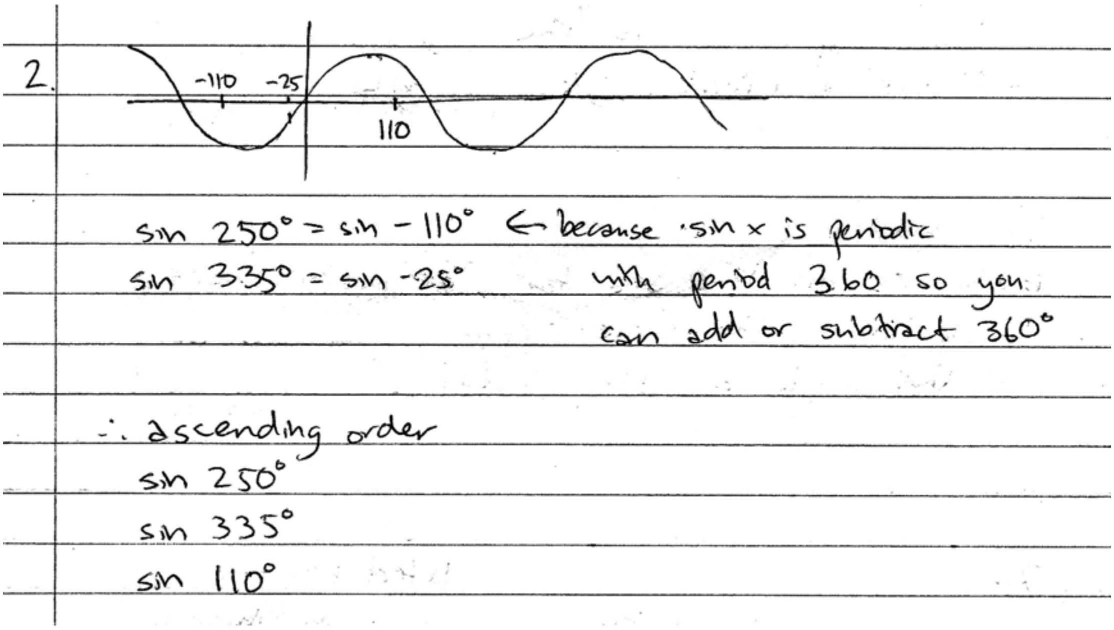
Her response was in the context of triangle trigonometry, in particular, it describes $\sin x$ as the ratio of the lengths focusing on the first stage of compression from operation to symbolic concept.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

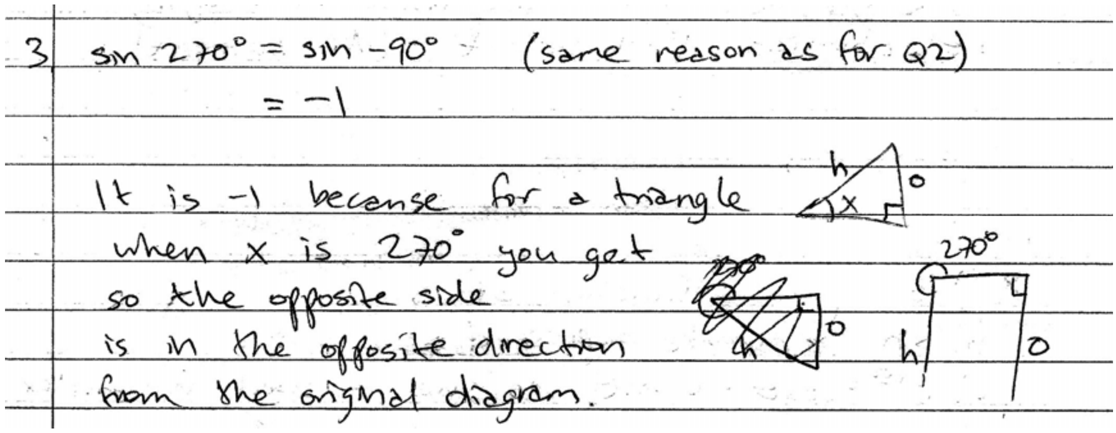
(b) $\sin 250°$

(c) $\sin 335°$



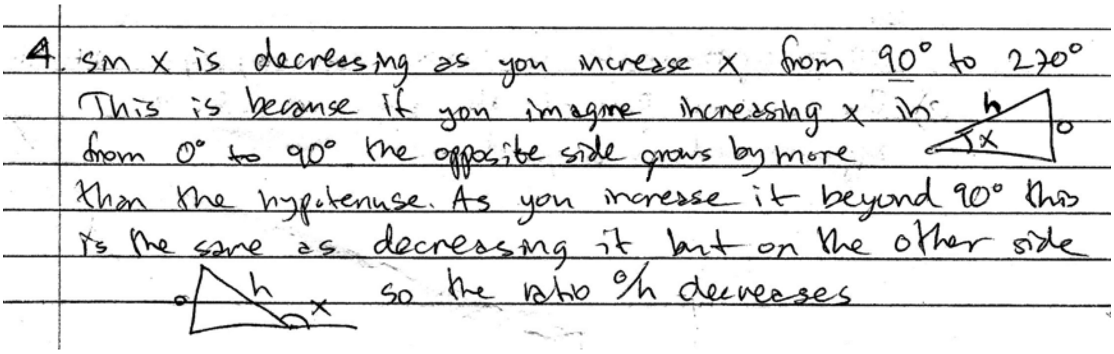
She was referring to her sine graph to get her answer. This was making sense through *perception* in the context of circle trigonometry (i.e. using the sine graph). Additionally, she must have used the symmetrical property of the sine graph to get those equivalent values. She was using the sine graph to get her answer without any indication of relating it to the unit circle. In this case, the sine graph was her supportive conception.

What is the value of $\sin 270^\circ$? Explain why $\sin 270^\circ$ has this value?



Firstly respondent A saw that $\sin 270^\circ = \sin(-90^\circ) = -1$ using the graph as in the question before. However, when attempting to draw the angle turning anti-clockwise through 270°, the first attempt has the opposite side *o* clearly drawn downward but is scribbled out because the angle is not the right size. When it is redrawn with the correct angle 270°, the radius is now vertical, with the (unsigned) hypotenuse *h* drawn over the (signed) opposite side *o,* so there is no longer a visible triangle as occurs in the context of triangle trigonometry. This clearly shows the problematic conception of respondent A in making sense of $\sin 270^\circ = -1$. She was working in the context of triangle trigonometry to make sense of $\sin 270^\circ$ which was in the context of circle trigonometry. This was essentially making sense through *perception.*

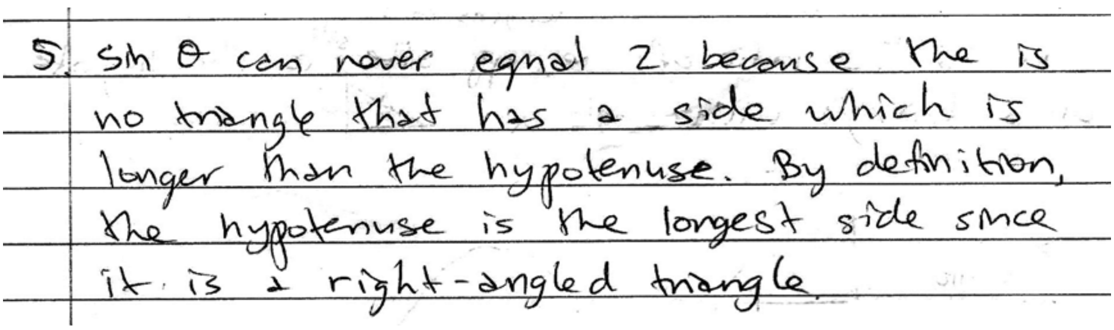
For what values is sin x decreasing? Explain why it is decreasing for these values?



4. sin x is decreasing as you increase x from 90° to 270° This is because if you imagine increasing x in from 0° to 90° the opposite side grows by more than the hypotenuse. As you increase it beyond 90° this is the same as decreasing it but on the other side so the ratio o/h decreases

She made sense through *perception* by varying the angle *x* of right angle triangle. She had problematic conception by thinking that the length of hypotenuse will change when the angle is varied. The shifting from triangle trigonometry to circle trigonometry was problematic for her. In fact, she drew a triangle for angle larger than $90°$. If she was working in circle trigonometry then she would realize that hypotenuse is always fixed at the length of 1.

Explain why $\sin\theta$ **can never equal 2.**



5. Sin θ can never equal 2 because there is no triangle that has a side which is longer than the hypotenuse. By definition, the hypotenuse is the longest side since it is a right-angled triangle

Respondent A was clearly working in the context of triangle trigonometry. She made sense of it through reasoning her *perception.* She reasoned by verbally describing the property of a right angled triangle where the hypotenuse is the longest side.

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

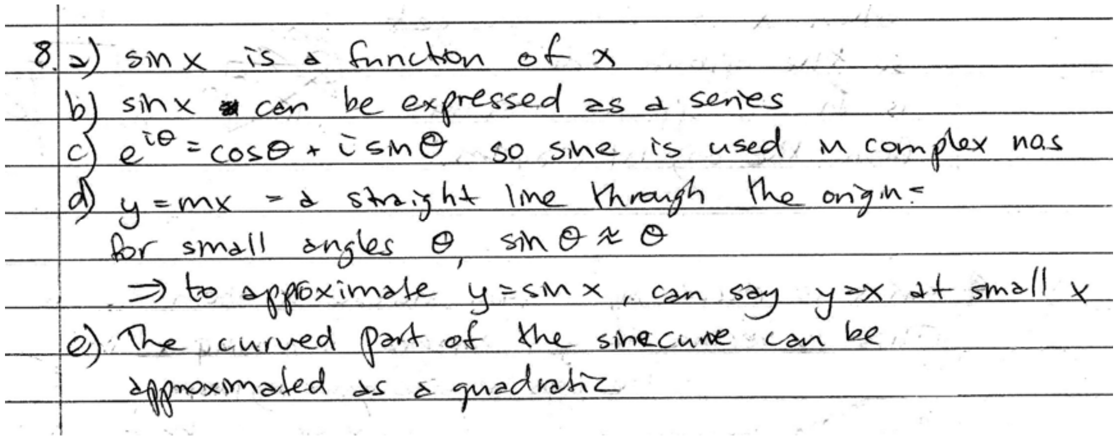
(a) Function

(b) Series

(c) Complex number

(d) y=mx

(e) $y=x^2$



8 a) sin x is a function of x
b) sin x can be expressed as a series
c) $e^{i\theta} = \cos\theta + i\sin\theta$ so sine is used in complex nos
d) y = mx = a straight line through the origin
for small angles θ, $\sin\theta \approx \theta$
⇒ to approximate y = sin x, can say y = x at small x
e) The curved part of the sinecurve can be approximated as a quadratic

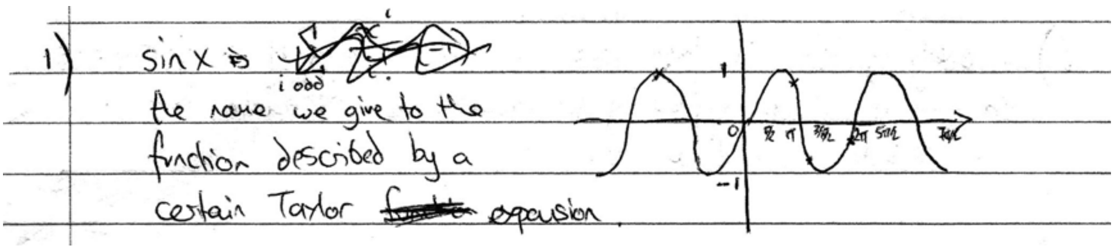
Respondent A is able to offer some links between the concept of sine and the above concepts. She knows sin x is a function which can be expressed as a power series. She states the Euler formula to show the relationship of sine with the complex numbers. She wrote y=x at small x, I speculate this might be related to her experience in learning sinx/x=1 for a limit as x approaches 0. She also related $y = x^2$ to the curved part of a sine curve and this might be due to her perception about the outlooks of both graphs. In this case, she might have observed the similarities of both graphs based on outlook of these graphs. Meanwhile I couldn't examine the nature of these links solely based the on the written answers given above.

In short, respondent A with a 2(i) degree and practical experience responds by combining triangle trigonometry and circle trigonometry with a good grasp of both yet with a problematic aspect in visualizing what happens when the angle is $270°$. She had problems in making sense of why sin x is decreasing for certain values. In this case, her right-angled triangle in the second quadrant was problematic because she didn't visualize it in a unit circle, therefore she thought that both the hypotenuse and the opposite side will grow as the angle of sine increased.

5.3.2 Respondent B.

Respondent B is male PGCE student with a first class mathematics degree.

Describe sin x in your own words.



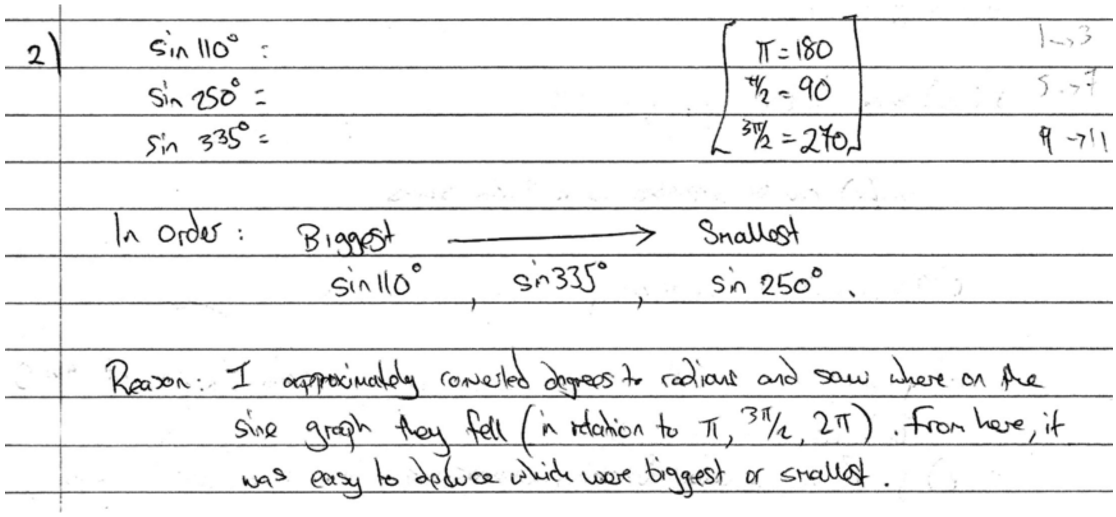
The evoked concept was a combination of circle trigonometry and analytic trigonometry. Respondent B begins with a series formula crossed out then says it is a function described by a Taylor series. The graph is said to be part of circle trigonometry but here he does not (yet) relate to it to a circle. It may be of interest here and in later examples to consider the more detailed relationship between triangle trigonometry, triangle in a circle and the use of graphs.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

(b) $\sin 250°$

(c) $\sin 335°$



2) sin 110° :
sin 250° =
sin 335° =

π = 180
π/2 = 90
3π/2 = 270

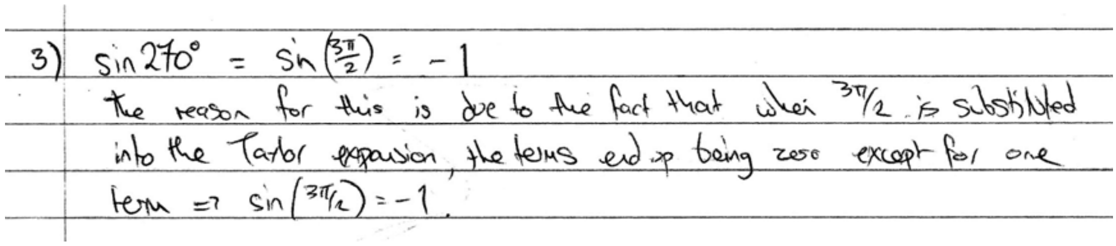
1→3
5→7
9→11

In order : Biggest ⟶ Smallest
sin 110° , sin 335° , sin 250° .

Reason: I approximately converted degrees to radians and saw where on the sine graph they fell (in relation to π, 3π/2, 2π). From here, it was easy to deduce which were biggest or smallest.

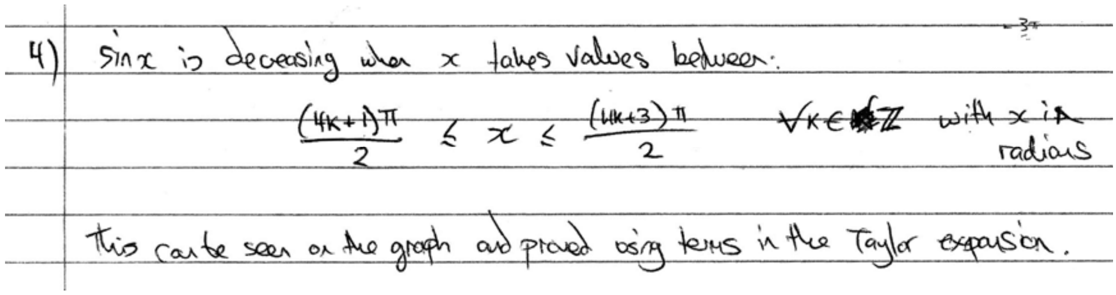
He appeared to be recalling his sine graph and saw where the points fell on the sine graph. This was essentially making sense through *perception* and *reason* in the context of circle trigonometry. His perception includes aspects of symbolism. He did the approximation by using measurement. His reason based on relating to the visual properties of the graph such as symmetry, periodicity etc.

What is the value of $\sin 270°$? Explain why $\sin 270°$ has this value?


3) $\sin 270° = \sin\left(\frac{3\pi}{2}\right) = -1$
the reason for this is due to the fact that when $3\pi/2$ is substituted into the Taylor expansion, the terms end up being zero except for one term $\Rightarrow \sin(3\pi/2) = -1$.

He sees that $\sin 270°$ is −1. His reasoning suggests that 'when $\frac{3\pi}{2}$ is substituted into the Taylor expansion, the terms end up being zero except for one term.' This is untrue if the actual number is substituted as the later terms would be clearly non-zero. So he must, in some way, have in mind a series such as that for $\cos x$ where $x = 0$ and $\cos 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = 1 - \frac{0^2}{2!} + \frac{0^4}{4!} - \cdots$. Skemp (1979) used the notion of C-links (conceptual links) and A-links (associative links) to emphasize that associative link does not work computationally. On the other hand, this also relates to the dual processing theory reported in Leron & Hazzan (2006) where the immediate response operates at an intuitive, non-analytic level. Respondent B was working in the context of analytic trigonometry with a supportive conception of Taylor expansion. There were problematic aspects in this supportive conception which he didn't realize at that moment.

For what values is sin x decreasing? Explain why it is decreasing for these values?

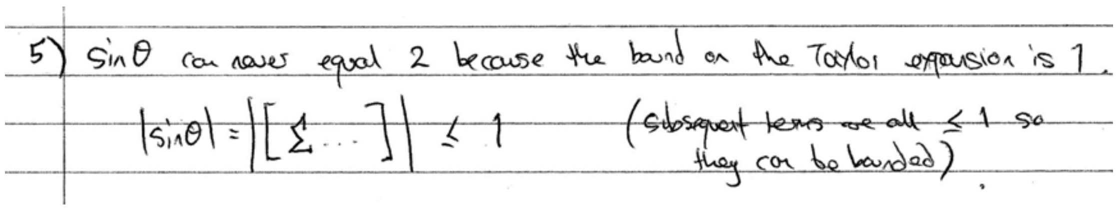

4) sin x is decreasing when x takes values between:

$$\frac{(4k+1)\pi}{2} \le x \le \frac{(4k+3)\pi}{2} \quad \forall k \in \mathbb{Z} \text{ with } x \text{ in radians}$$

this can be seen on the graph and proved using terms in the Taylor expansion.

He clearly saw this on the sine graph and he held a supportive conception that his statement can be proved by using the terms in Taylor expansion. He got his answer through *perception* by working in the context of circle trigonometry. Meanwhile, his supportive conception was in the context of analytic trigonometry. He clearly saw sin x was decreasing in the proposed region through the sine graph. He believed that he could use Taylor series to prove his answer.

Explain why $\sin\theta$ can never equal 2.


5) $\sin\theta$ can never equal 2 because the bound on the Taylor expansion is 1.

$$|\sin\theta| = \left|\left[\Sigma \ldots\right]\right| \le 1$$ (subsequent terms are all ≤ 1 so they can be bounded).

He responded with immediate associative links between the function as a Taylor series and its behaviors as a function at the context of analytic trigonometry and circle trigonometry with the conceptual links not yet reflected upon. Again he had a supportive conception on Taylor expansion and suggested that the bound on the Taylor expansion is 1. In fact it is not true that all the subsequent terms of Taylor expansion are less than or equal to 1. He just blindly believed that Taylor expansion can justify all the properties of sine without going into the details of this expansion. This was a S1 response and he was essentially working in the context of analytic trigonometry.

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

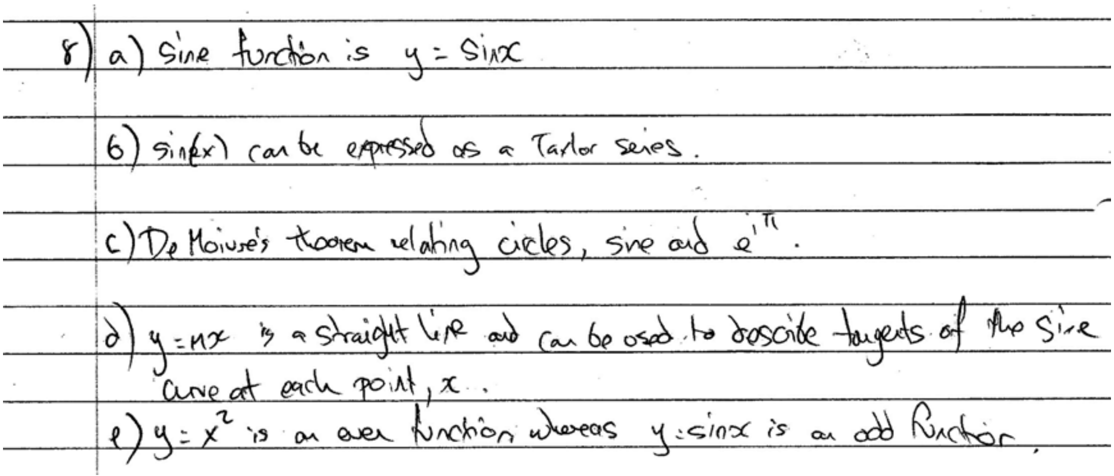
(a) Function

(b) Series

(c) Complex number

(d) y=mx

(e) $y=x^2$



8) a) Sine function is $y = \sin x$

b) $\sin(x)$ can be expressed as a Taylor series.

c) De Moivre's theorem relating circles, sine and $e^{i\pi}$.

d) $y = mx$ is a straight line and can be used to describe tangents of the sine curve at each point, x.

e) $y = x^2$ is an even function whereas $y = \sin x$ is an odd function.

He could briefly link the concept of sine to the concepts above. However, the nature of these links between concepts couldn't be identified from his written responses. Respondent B has a first class mathematics degree, he refers to analytic ideas such as power series in the context of analytic trigonometry and circle trigonometry with analytically faulty associative links between them. He never evokes triangle trigonometry. He only evokes graph and circle trigonometry and analytic trigonometry.

5.3.3 Respondent C.

Respondent C is a female PGCE student with a 2(ii) degree in mathematics.

Describe sin x in your own words.

1. Sin x is a function which takes any real number and outputs a value from −1 to 1.

It is closely related to other trig functions such as cos x & tan x. It is ~~used from~~ found in triangles, angles, etc. It has interesting links with π, is used as a tool with complex numbers, etc.

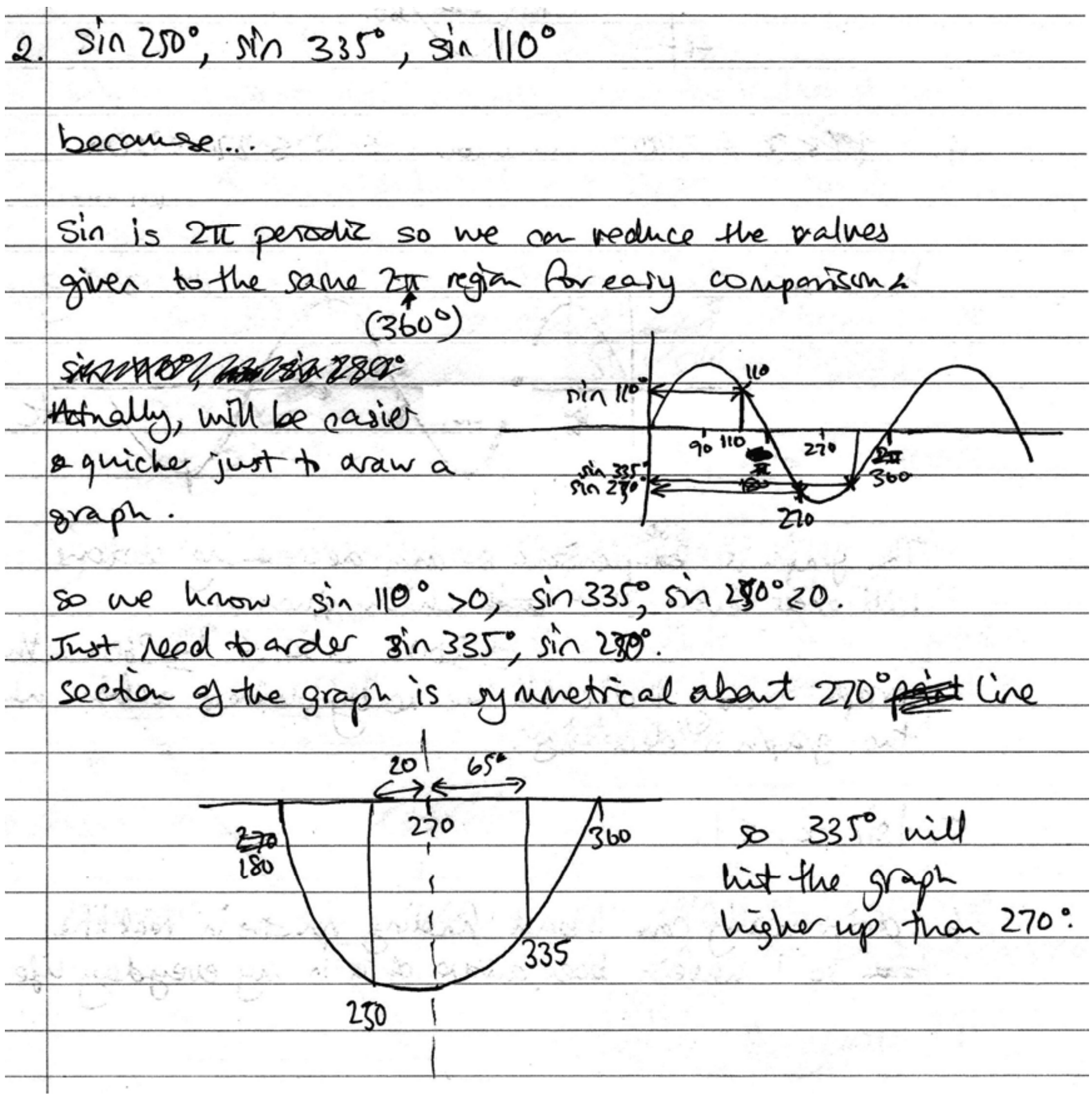
Her concept image was in the contexts of circle trigonometry and analytic trigonometry, speaking of *sin x* as a function defined for a real number, and refers to wider links involving complex numbers. However her verbal relational description only had little detail. In fact, it is a description but not a definition.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

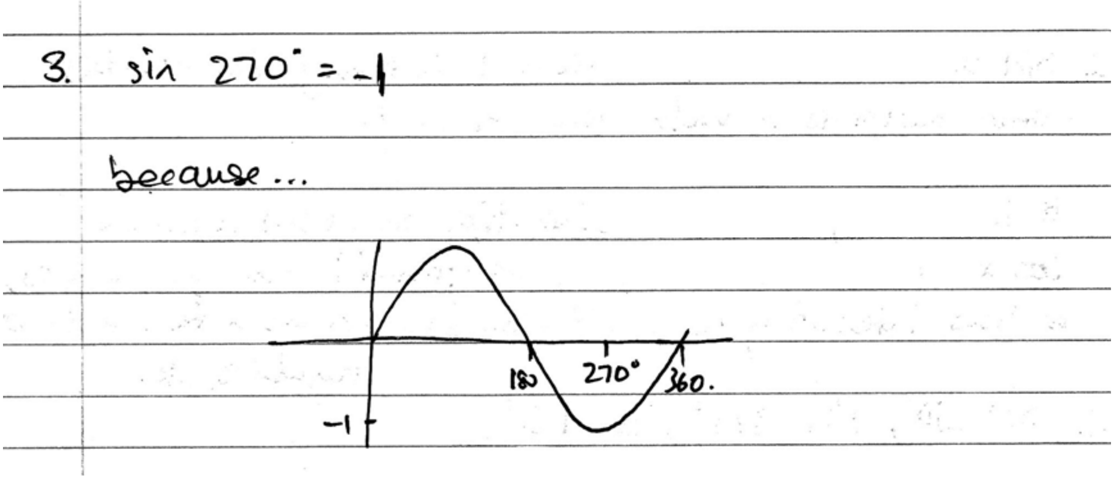
(b) $\sin 250°$

(c) $\sin 335°$



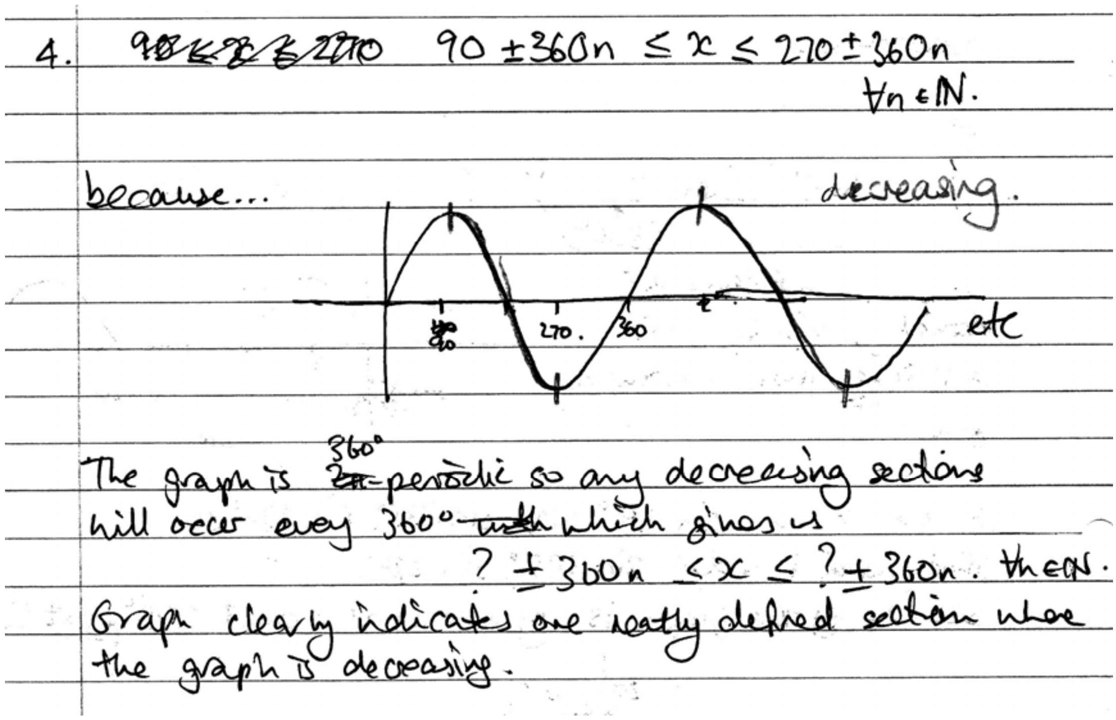
She got her answer through *perception* and she was working in the context of circle trigonometry. In addition to looking at the sine graph, she also used the properties of it such as symmetry and periodicity to approximate the locations of relevant points on the graph. Sine graph was a supportive conception for her.

What is the value of $\sin 270°$ **? Explain why** $\sin 270°$ **has this value?**



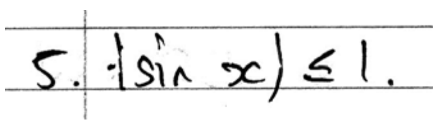
This response seems like an intuitive embodied response operating graphically in the context of circle trigonometry. She made sense of this through *perception.* Her supportive conception is the sine graph.

For what values is sin x decreasing? Explain why it is decreasing for these values?



Again, respondent C used her supportive conception which was the sine graph to get her answer. This was essentially making sense through *perception* and her familiarity with the graph in the context of circle trigonometry.

Explain why $\sin\theta$ **can never equal 2.**



Comparing this with the previous responses of respondent C, reveals a circle trigonometry response through evoking the shape of the graph. She had also responded in the context of circle trigonometry and analytic trigonometry.

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

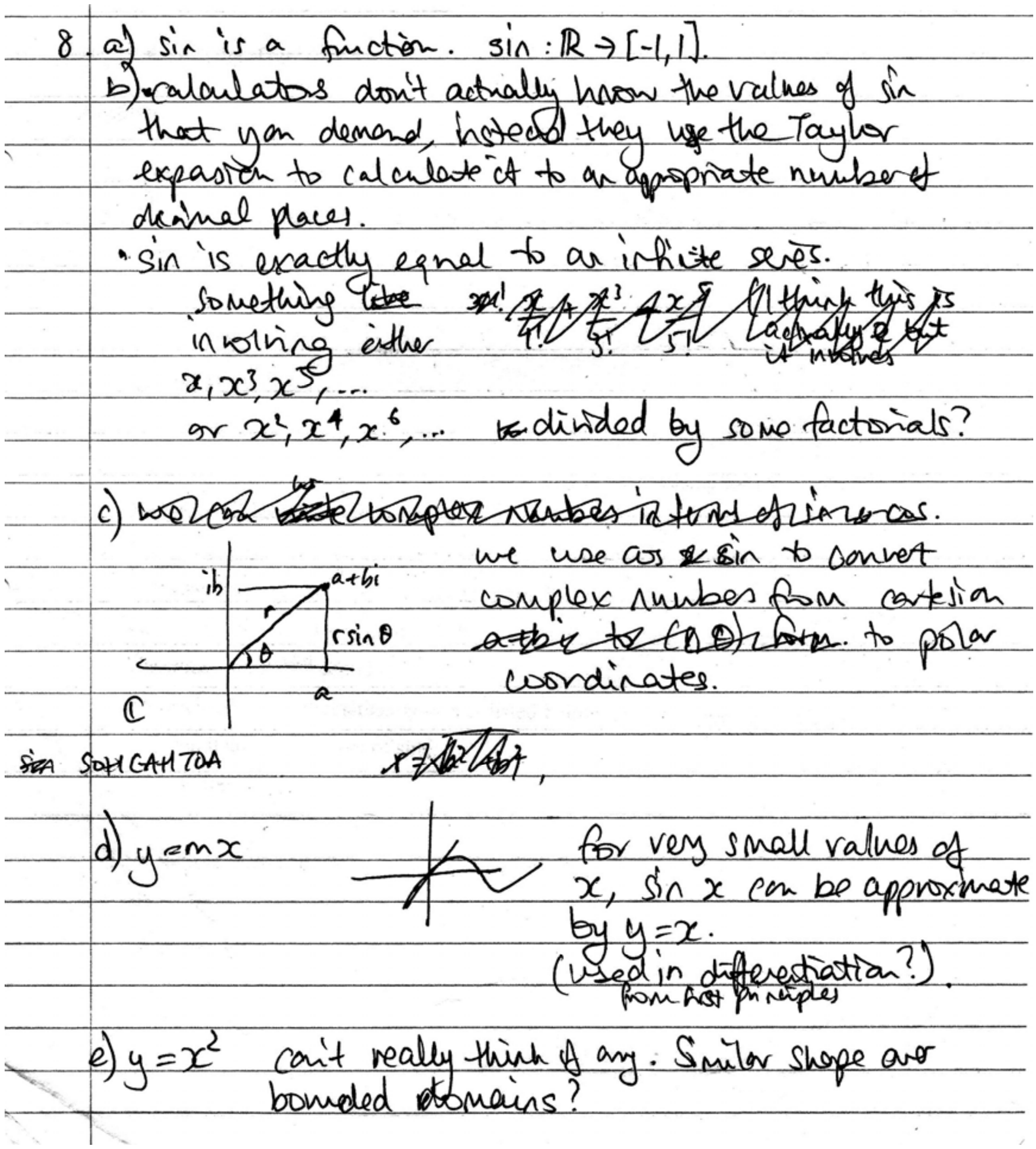
(a) Function

(b) Series

(c) Complex number

(d) y=mx

(e) y=x²



8 a) sin is a function. sin : $\mathbb{R} \to [-1,1]$.

b) calculators don't actually know the values of sin that you demand, instead they use the Taylor expansion to calculate it to an appropriate number of decimal places.

• sin is exactly equal to an infinite series. Something involving either $x, x^3, x^5, \ldots$ or $x^2, x^4, x^6, \ldots$ divided by some factorials? (I think this is actually it)

c) we use cos & sin to convert complex numbers from cartesian to polar coordinates.

SOH CAH TOA

d) y=mx

for very small values of x, sin x can be approximate by y=x. (Used in differentiation? from first principles)

e) $y=x^2$ can't really think of any. Similar shape over bounded domains?

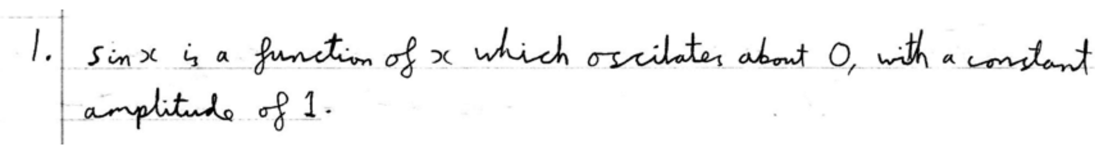
She knew that sine can be expressed as an infinite series but she wasn't sure how this expansion would look like therefore it seems like that was just an associative link to her. She used an Argand diagram to show the relationship of sine to the complex numbers. Respondent C with a 2(ii) mathematics degree operates visually at circle trigonometry with indications of possible links at circle trigonometry. She uses sine graph very often to make sense of trigonometry and it may be hypothesize that she doesn't relate the sine graph to the unit circle because there is no indication that she has related the unit circle to the sine graph based on her written response. She partially remembered links to analytic trigonometry which were the power series and complex number. She knows some links but she doesn't grasp them as a coherent whole.

5.4 The special case study.

After the pilot study, the questionnaire was further refined to have 15 main mathematical items in section A whereas section B and section C were added 7 items each. This was due to the reason that to cover a broader context of trigonometry. A follow up interview was conducted in order to gain further insights on the thinking of the respondents. Only one male sixth form student participated in this special case study. Currently he is studying mathematics at Oxford University with a scholarship. In this study, this student is given a name as SC.

5.4.1 Outcome of the special case study.

Describe sin x in your own words.


1. sin x is a function of x which oscilates about 0, with a constant amplitude of 1.

The evoked concept image was in circle trigonometry.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

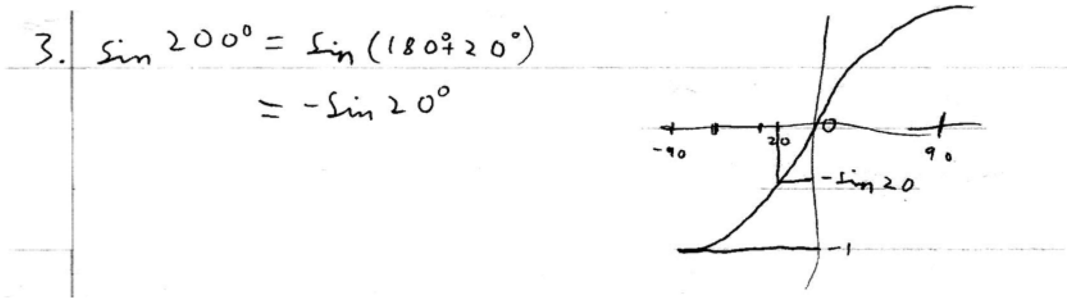
(b) $\sin 250°$

(c) $\sin 335°$

> KE: Arrange the following values of sine in ascending order and explain your answer. You are given $\sin 110°, \sin 250°, \sin 335°$. You have given me ascending order $\sin 250°, \sin 335°, \sin 110°$ [...] how do you arrive at this answer?
>
> SC: So you might think of the way… at the start it's at 0 then at 90 it's at 1 and then at 180 it's at 0 again, 270 it's -1 and then it goes back to 0… 360 and then each of those are part of that… and symmetrical… then you can work out which are higher and which are lower.
>
> KE: Ok… so you made sense through the graph basically… you imagined like the position in the graph and then you estimated roughly the values.
>
> SC: Yeah.

SC was working in graphical trigonometry and used the sine graph. He knew the properties of the sine graph such as symmetry and periodicity. By using the outlook of the graph and its properties, he got his answer for this item. This was making sense through *perception*.

How do you make sense of $\sin 200°$?



KE: […] Do you know how to derive the sine graph?

SC: You've got like a triangle… it's got hypotenuse… you've got this triangle with varying angle (pointing to his sketch, Figure 4.1)… we are looking at this vector (pointing to his sketch, Figure 5.1)… in a basic sense… it's transferring that vector…it's just an angle and magnitude into… erm… x and y value right?… erm… sine value is the y value, cos value is the x value…of theta… so as theta changes then the graph of theta on the x axis against this one being the value of y changing and the value of x changing […] I'm imagining this in my head that's 180 there and I am seeing this extending backwards and I know that is symmetrical through there the plus 20 here extends to minus 20 here in terms of where it ends… vertical axis (inaudible).

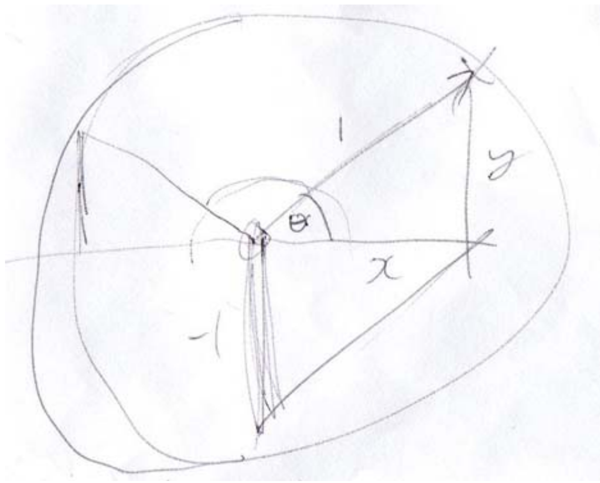
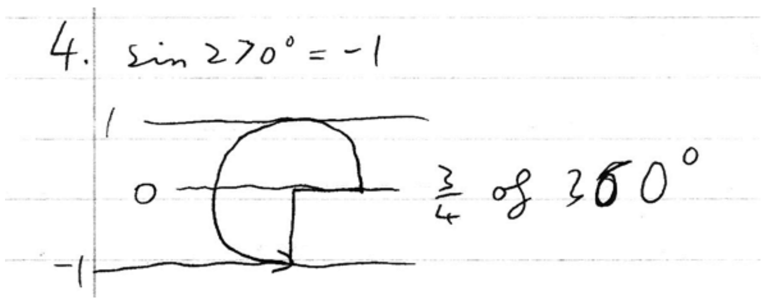


Figure 5.1: Sketch of Interviewee SC.

Based on his written response, SC was using the sine graph and its symmetrical property to get the equivalent value for $\sin 200°$ which was $-\sin 20°$. During the follow-up interview, he explained that he was working in the unit circle then to the sine graph. Finally he arrived at his answer by writing $\sin 200° = -\sin 20°$. He realized that he was dealing with vectors in the unit circle (see Figure 5.1). This indicates that he was aware of the difference between Euclidean geometry (which involves seeing the lengths as magnitude and the angle involved in term of size only) and Modern Cartesian (which involves signed lengths and signed angles). He had an awareness of doing things differently in different contexts. SC knew that sine value was the y value and cosine value was the x value. In fact the final answer was from the sine graph and he got the links between triangle trigonometry and circle trigonometry. This was making sense through perception and reason. He was able to build a coherent link between triangle trigonometry and circle trigonometry.

What is the value of $\sin 270°$ **? Explain why** $\sin 270°$ **has this value?**



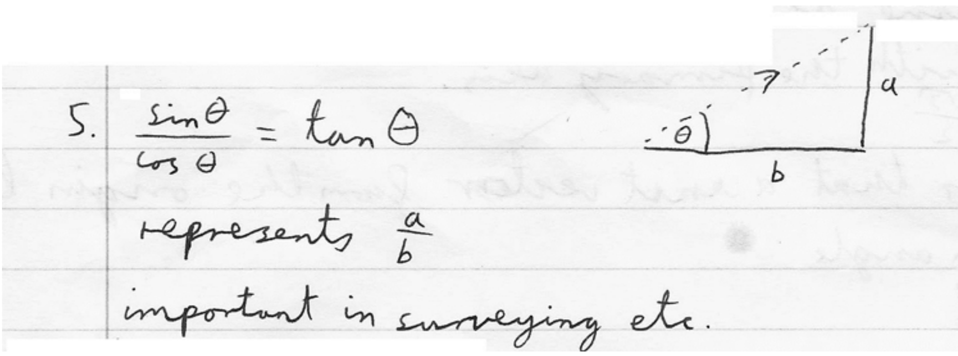
> SC: It's the same as this one here (pointing to his answer for item 4)… you've got this theta being 270 which is 3 quarters of the 360 circle… so then the y value which is sine is -1… because it is the 1 below the axis.
>
> KE: Can you visualize this triangle? […]
>
> SC: […] I guess it is more abstract. The triangle at this point, because it has gone past the 180 point and going back on itself…

KE: [...] Can you imagine what happen to the hypotenuse and the opposite?

SC: It's still this one... so erm... hypotenuse is still 1 because it's been defined as 1, theta is 270, x is 0, and so y is also 1... and so you have to define the direct... because distances are scalars not vectors so they can't have direction so you've to think about displacement in terms of the x and y coordinates and this as a distance and a direction as two separate kind of things.

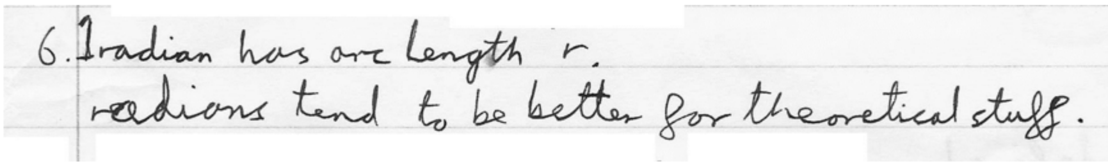
He offered a subtle description of the difference between Euclidean magnitudes and signed Cartesian coordinates (vectors). His experience of manipulating triangle in Euclidean geometry also led him to see triangle outside the first quadrant as abstract triangle. He clearly aware of doing things differently in the triangle trigonometry and the circle trigonometry. This was making sense through perception of sine and cosine as the vertical and horizontal components of the point moving around the circle in circle trigonometry. He reasons that the hypotenuse is always 1 because it is defined as 1.

What is sine over cosine? Does that mean anything?



He remembered the definition of tangent and stated the real life usage of tangent. Apparently this was making sense through *reason* in triangle trigonometry. Although interestingly he doesn't say $\frac{a/h}{b/h}$, perhaps this knowledge is already compressed and he sees it straight away.

What do radians mean? Why do we need radians when we have degrees?


6. 1 radian has arc length r.
radians tend to be better for theoretical stuff.

KE: So I am interested to know theoretical stuff in your sense, what theoretical stuff you mean?

SC: It's not right for a start… you've got everything in terms of pi…pi over 3 equals to 60 degrees. This is kind of arbitrary. This one is defined in a specific way in terms of the radius of the circle it links together better so that there is more of a certain relationship and also you've got (inaudible)… so that when you are measuring in radians… I will use the graph (pointing to the graph which he drew just now)… this gradient here is 1.

KE: You are trying to say the degree is more arbitrary compared to the radian?

SC: Yeah… it's got a lot of nice little properties which make it easier to work with.

KE: So basically do you prefer…in what context you will use radians? Usually in what context you tend to use radians?

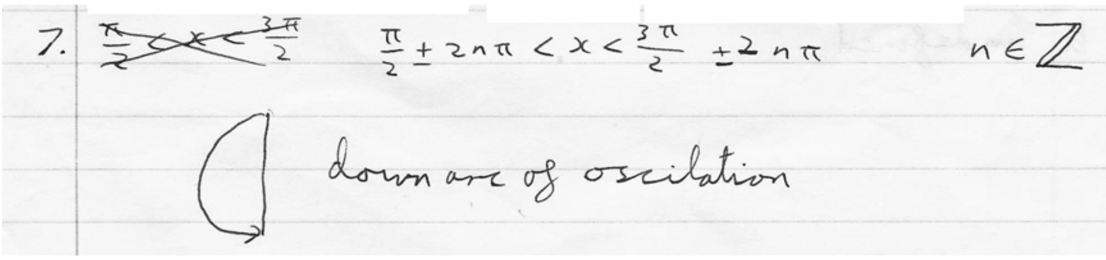
SC: Usually being theoretical things because in degrees it's easier to measure.

KE: So do you prefer to use radians or degrees?

SC: Being more a theoretical thinker rather than a practical user of angles I tend to do things more with radians.

He realized that in triangle trigonometry one is doing practical measurement and it's easier to work with degrees for instance angles like $60°$ etc. He could sense the reason of using radians in higher level mathematics but here he didn't articulate why (he does so later when using calculus).

For what values is sin x decreasing? Explain why it is decreasing for these values?



> SC: There is a realization here so there is a down arc of oscillation where the y values is decreasing and obviously you can go back to the function which relates directly to that diagram where dy by dx is negative that's where is decreasing so it's just between there and there (pointing to his answer script)… obviously the oscillation continues so it will be 360 degrees there and backwards as well […] so as it is going round and round and round where sine x which is why decreasing… sine theta so sine theta is decreasing… this bit here that's where y is going down and here is going back up again…and considering this in terms of vertical.

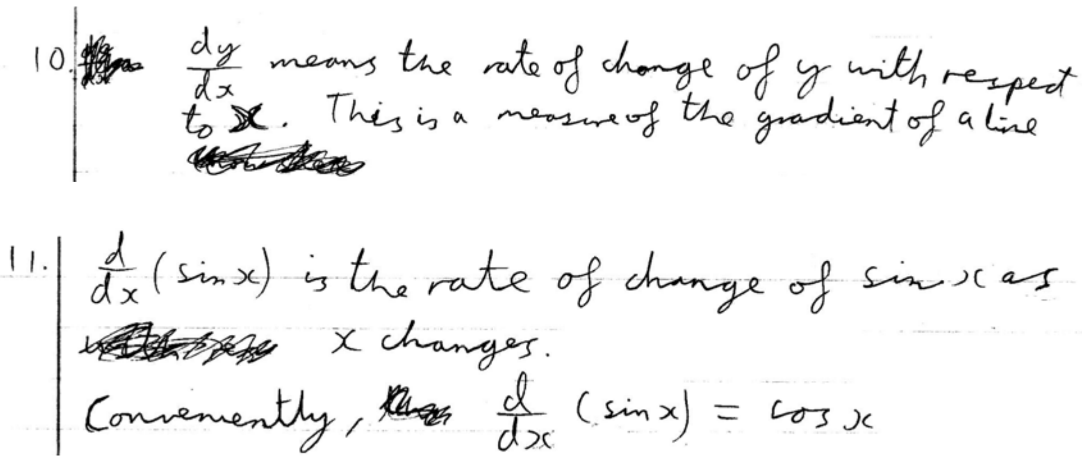
He was working in the circle trigonometry, operating the unit circle and sine graph as his base objects. This was making sense through perception and reasoning his perception which involves noticing the changes of y values (which represents the changes of magnitude of the vertical side).

Explain why $\sin\theta$ can never equal 2.

> SC: Well if we were to define it as such in this diagram (see Figure 4.1), we've got unit as hypotenuse and the point we are measuring to is always only 1 away from the origin it can't ever be more than one above the origin.

SC was working in the unit circle and clearly he thought the hypotenuse as 1 because it was defined as 1 in the unit circle. He was making sense through *perception* (noticing the hypotenuse must be 1 in the unit circle) and *reason* (verbalizing the reason of it as because of the definition).

What is d/dx [sin x]? Explain why.


10. $\frac{dy}{dx}$ means the rate of change of y with respect to x. This is a measure of the gradient of a line

11. $\frac{d}{dx}(\sin x)$ is the rate of change of sin x as x changes.
Conveniently, $\frac{d}{dx}(\sin x) = \cos x$

SC: It always oscillating and at the start, it's going up at a rate of 1 so if you mark that there as 1 and here is horizontal so it's changing at a rate of 0 so if you mark that as 0 there and here is going down at the highest rate it can go so here is the most gradient and here is the least gradient so that's going to be the lowest point and obviously… because that is symmetrical so these gradients are going be the same… so that is going to be the peaks of the waves… erm… and here again is 0 and here it's back to the original gradient so that creates another wave that's shifted and is basically the same as the sine wave except that if the sine wave is compacted then these points would be higher because the gradient would have to be greater… which is why it doesn't work when you are not measuring in radians.

KE: Yeah… ok… so you realized that if you differentiate sine x with respect to x you cannot use degrees. Can you use degrees?

SC: You could but you will need a quite complex exchange between the two because this is now… not… it's a steeper or shallower gradient it's a different gradient so it would still be a multiple of the cos x graph it's just the altitude would be

> different […] so this point is going to be quite low here so it's going to look more like this which is still a cosine curve… it's just say pi over 180 times cosine or something like that […] pi over 180 times cosine.

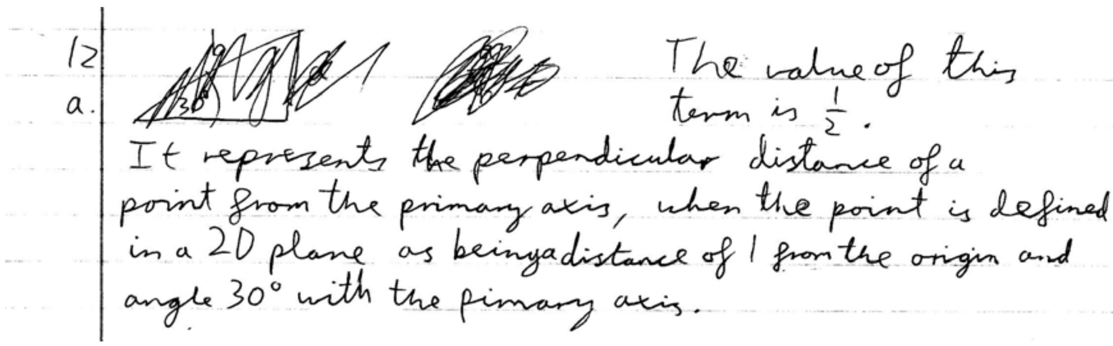
He embodied the changing of gradient in sine graph as cosine graph. This was making sense through *perception* and *reason.* He sensed the difference of results in differentiation when x was expressed in degrees and radians. SC reasoned this through the world of embodiment by imagining the difference in outlook of the sine graph which was caused by using degrees and radians. He does grasp the reason for using radians.

Describe as fully as possible what you understood by the following terms:

(a) $\sin 30°$

(b) $\sin 120°$

(c) $\tan 90°$


12
a. The value of this term is $\frac{1}{2}$.
It represents the perpendicular distance of a point from the primary axis, when the point is defined in a 2D plane as being a distance of 1 from the origin and angle 30° with the pimary axis.

SC was working in the circle trigonometry when describing $\sin 30°$. This was obvious when he saw the hypotenuse as 1. He also mentioned about origin and primary axis. These ideas only occur in the unit circle. If a respondent is working in the Euclidean geometry then the notions of origin and primary axis will not exist. This was essentially making sense through *perception.*

SC: That just an exact value [...] sine 120... it's the y value when you've got theta equals 120... this value here (pointing to his figure (see Figure 5.1))... how do I know it's that value?

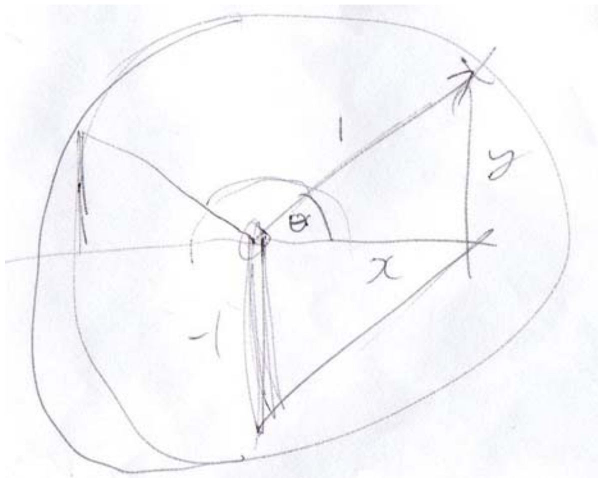


Figure 5.1 Sketch of Interviewee SC.

KE: Yes.

SC: Erm... well I've learnt it...

He was working in the circle trigonometry by mentioning unit vector and origin. He was looking at a picture of the angle in a unit circle (see Figure 4.1) and seeing the relationship between $\sin 120°$ and $\sin 60°$. His argument is a subtle compression of knowledge as a gestalt relating visual, symmetric, ratio of lengths etc.

SC: Because cosine is 0.

KE: Because you realized tangent is sine over cosine?

SC: Yeah…

SC made sense of this through *operation* and *reason.*

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

(a) Function

(b) Series

(c) Complex number

(d) y=mx

SC: Sine x is a function of x.

SC: Well you've got a series of sine function and cosine function. I can't remember exactly what they are but they are very useful in terms of finding pi to so many billions of decimal places etc.

SC: Complex numbers… erm… trigonometric functions can be used to represent complex numbers for example if you got modulus core… erm… (he was writing on a piece of paper) complex number modulus A and theta…so they are used quite often to represent complex numbers because they can be split in this way this is what it is…it is representing a modular form which is a polar coordinate form which is changing into a Cartesian form…which is the real bit plus the imaginary bit hence the sine represents the imaginary bit which is the vertical part.

SC: Because you got the whole thing about sine being oscillation this can't be related but there is a shift going on when you've got an…measurement of the angle so varying degrees or radians or whatever… whatever that is so you've got this… sin x equals… sorry it's about…about the sine so that is the sine of y which is the other units which is a linear multiple so that's erm… that's where you've got this shell of a cos curve… that is strenuous relationship.

SC knew sine is a function. He could link the concept of sine and cosine to series but those links apparently were associative links. He related sine to the complex numbers and knew that sine is used in interchanging between polar coordinate form and Cartesian form.

Do you know what is $\sin(\alpha+\beta)$? **Do you know what it is? (Question in follow up interview)**

> SC: Oh… I don't know the formula for it… sin(alpha + beta) equals sin alpha cos beta + cos alpha sin beta.
>
> KE: So can you prove this?
>
> SC: Erm… no… I can't.
>
> KE: Ok… alright… can you make sense of this? Or you just rote learning? Just remember it?
>
> SC: Erm… it was rote learning by last year but because of the sheer number… erm I just remember from the sin two alphas to two sin alpha cos alpha.
>
> KE: Oh… you relate to sin two alphas?
>
> SC: Yeap… yeap… I am thinking… that's the relationship that I am using…it's easy to remember that.

He remembered sine two alphas as two sin alpha cos alpha as an analogy to sin(alpha +beta). He remembered the simpler formula for $\sin 2\alpha$ and uses this to remind him of the more general formula for $\sin(\alpha+\beta)$. He had a complex knowledge structure that he used in flexible ways to retrieve links that are weak. It was an associative link but it was also a reverse conceptual link as he 'knew' the particular formula and used this to regenerate the general formula.

Do you feel fishy about complex number (question in the follow up interview)?

> SC: Erm… that seems logical… erm… you've got… erm… that's the basic definition of so the i is the square root of negative

> one… erm… that's what it is defined as… erm… if you multiply two in power… erm… so you got… erm… if you write j as… sorry i… erm… e as i pi over two then you square it then that just comes into there but that is i right (he was writing on a piece of paper)? Because that's just the sheer number that many degrees if you square that then that just goes into the bracket that is another… e power i pi which is just 180 degrees right? […] the reason we have complex numbers… as far as I can see… before we had this thing where you just can't divide by well you can't square root negative one…no negative number has a square root which make the system kind of incomplete it was just an experiment in terms of what happens if you square root negative numbers and it led into a lot of interesting and surprisingly useful stuff apparently. I've not come across any in a practical uses but it's interesting to me anyway.

He could sense the consequences of introducing complex number to the system but he didn't explain these explicitly. He was attempting to make sense of how the complex numbers operate by using the relationship between doubling the power and squaring the result. But he still doesn't see any practical value in using complex numbers although he found them interesting. In fact, he is a very intelligent pupil focusing on the *met-before* that a square is always positive and attempting to make sense of complex numbers.

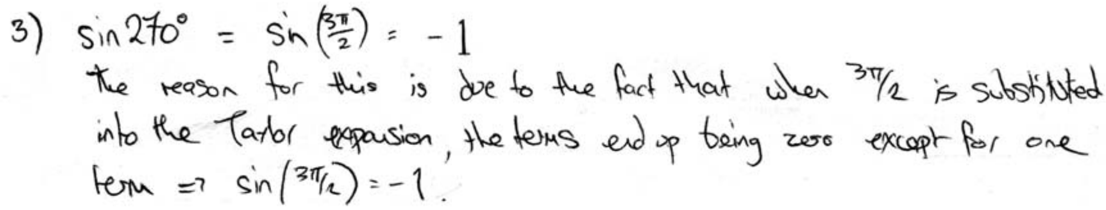
What difficulties as a student might have? (Question in the follow up interview)

> SC: Well… erm… in year six when soh cah toa and that kind of stuff first came out… I was counting a robotics course at the back of the class and I wasn't involved in the classes at the time so I missed out on the introduction of the syllabus so for several years I was behind with trigonometry and stuff like that… erm… I think the incompleteness of the first time you encounter it… it's just learn this and that's it… erm… it's not helpful I think if we approached it from this triangle and circle

> idea that's a much more useful and powerful way to express it and it can also be understood by kids and we don't give them full credit for the fact that they can properly understand it quite easily.

He is a student, fluent in triangle trigonometry and circle trigonometry who has learned the sine formula by rote yet who grasps the reasons for using radians as opposed to degrees in calculus. Yet he cannot see the point in using analytic trigonometry ideas when visual and symbolic ideas are enough for him. He reasons using visual and symbolic arguments blended well together.

SC was showed with a response as follows:


3) $\sin 270° = \sin\left(\frac{3\pi}{2}\right) = -1$
The reason for this is due to the fact that when $3\pi/2$ is substituted into the Taylor expansion, the terms end up being zero except for one term $\Rightarrow \sin(3\pi/2) = -1$.

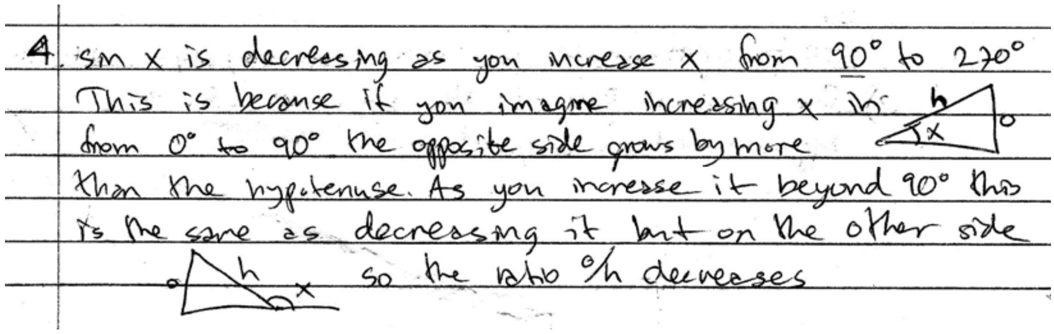
> SC: That is a whole new abstract level I mean why would you go to the Taylor series when you've got quite simple diagrams that express it all Taylor series does is expressed it in polynomial terms which is just different sort of mathematics I don't know why you would resort to that trigonometry works independently of that.

He was working in triangle trigonometry and circle trigonometry which was sufficient for his purposes. This shows the big transition from circle trigonometry to analytic trigonometry.

5.5 Interpretive dilemma in the data analysis process.

During the data analysis process, there was one instance which involved interpretive dilemmas, in particular deciding how the respondent made sense of a particular mathematics statement. Below is the mentioned instance.

For what values is sin x decreasing? Explain why it is decreasing for these values?


4. sin x is decreasing as you increase x from 90° to 270°
This is because if you imagine increasing x in
from 0° to 90° the opposite side grows by more
than the hypotenuse. As you increase it beyond 90° this
is the same as decreasing it but on the other side
so the ratio o/h decreases

One might argue that this respondent (Respondent A) was making sense through *operation* because she was increasing the angle of *x* to make sense of this mathematics statement. However if we reflect deeply then we will notice that the essence of this sense-making process is not the *operation* but is the *perception.* She got her conclusion through noticing the changes of opposite side of the right angled triangle which is related to *perception*.

5.6 Relating empirical evidence to the theoretical framework.

The data has shown that most of the respondents' evoked concept images are in the context of triangle trigonometry or circle trigonometry or a blending of both except for respondent B who has a strong link to analytic trigonometry with some aspects in circle trigonometry. This also indicates the sophisticated nature in human thinking. The evoked concept image is only part of a conceptual structure. It is sensible to say that the evoked concept image is the most direct and strongest link in humans' mind to a particular stimulus.

The initial theoretical framework is based on the theory of three world of mathematics proposed by Tall (2004). However based on the data collected from the pilot studies and special case study, it shows that when the respondents made sense of mathematics, they didn't necessary based on the formal world of mathematics. Therefore the theoretical framework of making sense of trigonometry was refined to be making sense through *perception, operation* and *reason.* In this case, *perception* means making sense through sensory input. *Operation* means making sense through physical actions.

Reason means making sense through conceptions, definitions and deductions. In this case, the reasoning is related to triangles and to dynamic relationships shape of the graph, symmetries, periodicity etc. In fact, most of the respondents reason based on the properties of visual images such as the sine graph. In general, reasoning involves verbalizing the relationships between different things such as verbalizing the relationships between certain perceptions and operations. For instance, respondent SC knows sin x is decreasing for $\frac{\pi}{2} \pm 2n\pi < x < \frac{3\pi}{2} \pm 2n\pi, n \in \mathrm{Z}$ because as he varies the angle (operation) in the unit circle and sees (perception) the length of the opposite side of the right angled triangle varies as a consequence of this action. On the other hand, respondent B who reasons some of the properties of sine by Taylor series without going into the details of it because he has a supportive conception for Taylor series (reason based on conceptions). Apparently this is a S1 response. Respondent SC also reasons based on definition when he responded that the hypotenuse is defined as 1 in the unit circle therefore $\sin\theta$ can never equal 2. This idea will be discussed in more detail in section 3.2.

The data has shown that different respondents have responded differently for certain items. For instance, respondent A was operating in triangle trigonometry when attempting to explain why $\sin\theta$ can never equal 2 by saying "the hypotenuse is the longest side since it is a right-angled triangle". Alternatively, respondent B was operating in analytic trigonometry by using the Taylor expansion without going into the details of its computation. Meanwhile, respondent C was operating in circle trigonometry probably by evoking the sine graph in order to answer this item. This shows the diversity of contexts in trigonometry.

Further reflection on the data also revealed that the conceptions that the respondent possessed were related to the notion of *met-before* as suggested by Tall (2005). *Met-before* is used to indicate the effect of previous experience in new situation that affects our current thinking. Lakoff and Nunez (2000) proposed a similar notion as *metaphor* which means speaking of new or abstract ideas in terms of familiar ideas. The notions of *met-before* and *metaphor* are related to the supportive conceptions and problematic conceptions of this study. Supportive conceptions support generalization in new context whereas problematic conceptions impede generalization in new context. As we can notice from the data, all the respondents have supportive or problematic conceptions in making sense of trigonometry. For instance, respondent A who has a strong link to triangle trigonometry, she tries to conceive $\sin 270°$ in triangle trigonometry which is clearly a problematic conception because there is no way that she can construct this triangle in triangle trigonometry.

Supportive conception might contain problematic aspects in it and problematic conception might contain supportive aspect. For instance, respondent B who has a supportive conception with problematic aspects on Taylor series, he uses this series to explain why $\sin 270° = -1$ but his explanation is not correct and he does not go into the details of the computation. Meanwhile, respondent C who has a supportive conception with problematic aspects on the sine graph, she draws the sine graph and says $\sin 270° = -1$ because she can see from the sine graph. In fact, she didn't offer an explanation of why the sine graph would look like that. Similarly respondent SC also has a supportive conception with problematic aspects. For instance he can state the formula for sin (A+B) but he couldn't prove it. In this case, it may be hypothesize that he couldn't prove this formula in the three distinct contexts of trigonometry. This idea of supportive and problematic conceptions is discussed in detail in section 3.4. Relevant literature on this idea is in section 2.5.

Apparently the three respondents in the pilot study didn't exhibit the coherent links between the three distinct contexts of trigonometry. For instance, respondent A didn't link triangle trigonometry to circle trigonometry and ended up drawing a weird figure by thinking of $\sin 270°$ in triangle trigonometry. Respondent B never evoked triangle trigonometry in answering the items and apparently he didn't build coherent links across the three contexts. He has strong links to analytic trigonometry. Meanwhile, respondent C has a strong link to the sine graph however there is no evidence showing that she has linked it to the unit circle and the triangle trigonometry. Respondent SC does exhibit coherent links between triangle trigonometry and circle trigonometry. He could link the sine graph to the unit circle and justify its properties. In short, the three respondents in the pilot study know the concepts of trigonometry but they don't grasp the relationships between them. On the other hand, respondent SC has a much better grasp of these relationships than the other three respondents.

Chapter 6
The Stories of Five Student Teachers

6.1 Introduction.

This study concerns with how student teachers make sense of trigonometry. Based on the theoretical framework as discussed in Chapter 3, humans make sense of mathematics through perception, operation and reason. There are three distinct contexts in trigonometry, namely triangle trigonometry, circle trigonometry and analytic trigonometry. The main issue would be how do the student teachers cope with the changes of meaning across different contexts as they learn more sophisticated ideas. The student teachers involved in this study were taking PGCE Secondary Mathematics at a British university when the data was collected.

This chapter presents the main data analysis of five student teachers on how they made sense of trigonometry. The responses of these five student teachers show a spectrum of responses for the collected data. Data was first collected through questionnaires, then follow-up interviews were conducted in order to gain greater insight into how the interviewees make sense of trigonometry through four important and interrelated aspects as follows:

(a) The ways that interviewees make sense of trigonometry,

(b) The contexts of trigonometry that the interviewees were operating while making sense,

(c) The supportive or problematic conceptions involved in making sense,

(d) The nature of knowledge possessed by the respondents: whether they *know* it or they *grasp* it.

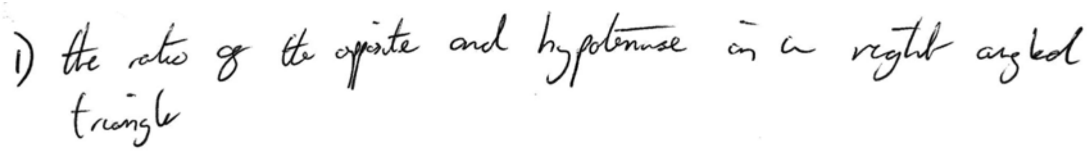
In order to guide the reader through the analysis of this chapter, the items of the questionnaire are presented first followed by relevant evidence gathered from the student teachers' responses. A summary of each student teacher is given at the end of each case and these summaries form the basis of an overall analysis. These summaries are written in a way to answer the proposed research questions in particular question 1 to 8 of chapter 4 (see page 62-63).

6.2 The story of student teacher ST1.

ST1 is a male student teacher who does not have any teaching experience.

Describe sin x in your own words.

KE: Item one sounds like this, describe sine x in your own words, so can you read your answer for item one for me?


1) the ratio of the opposite and hypotenuse in a right angled triangle

ST1: The ratio of the opposite and hypotenuse in a right angled triangle.

KE: Do you have anything else that you want to add or you feel happy about this?

ST1: I know it's the ratio because… more because I've always been told its the ratio not because I have any kind of deep understanding of why… ehem… yeah… so I can't say something that I understand fully but it's just something that I know because I have been told it.

The concept image of ST1 about sine *x* is the ratio of the opposite and hypotenuse in a right angled triangle and this indicates that his evoked concept image involves a ratio which gives a process in triangle trigonometry. He feels that he doesn't have a deep understanding on the subject.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

(b) $\sin 250°$

(c) $\sin 335°$

> KE: [...] For the second item of this questionnaire it sounds like this please arrange the following values of sine in ascending order and explain your answer. You are given three options here: sin 110 degrees, sin 250 degrees and sin 335 degrees. Can you read your answer for me for item 2?
>
> ST1: So I drew a sine graph from just the positive from 0 degrees to 360 degrees such as one period... ehem... and I knew that it's highest value is 90 degrees and lowest value is 270 and I just approximated them and I worked out which one will be bigger by the difference between those values and the crossing points and the maximum and the minimum that I already knew.

ST1 drew a sine curve and tried to approximate these values by using the sine curve. This shows that he was working in graphical trigonometry using graphs and tried to make sense of this by using his perception. The base object for his perception was the sine curve and he came out with his answer by approximating the positions of $\sin 110°, \sin 250°, \sin 335°$ on the sine curve.

How do you make sense of $\sin 200°$?

ST1: Ehem… I think I was trying to make sense of sin 200 being just taking the graph so having this first up to 90 defined by the… ehem… so defined by a ratio of the triangle and the rest of it just being a continuation of this… ehem… a continuation of the curve to make it 2 pi periodic so I tried to explain sin 200 is just this graph continued on 200.

KE: Do you want to draw? I can give you paper if you want to. You can always ask for paper from me.

ST1: I was just thinking that sin 200 would be about… ehem… so it's gonna be minus sin 20… ehem… yea (he was writing on a piece of paper (see Figure 6.1 below))… which is what I have got so for some reasons I put 200 minus 180.

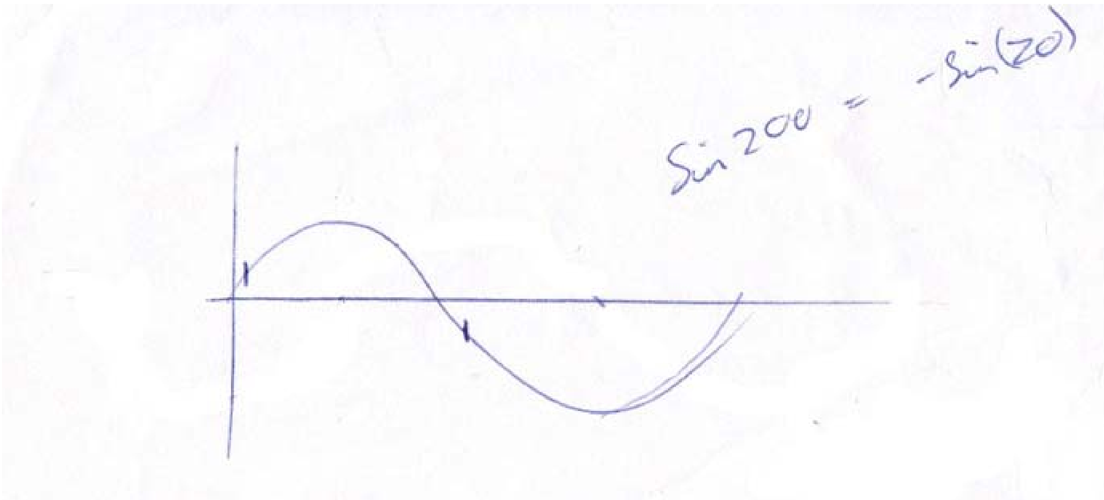


Figure 6.1: Sketch of sine graph by ST1.

KE: So basically when you are making sense of this you are trying to refer to the sine graph?

ST1: Yea.

KE: Ok. Alright. Can you visualise this triangle for sin 200 degrees?

ST1: Ehem… no.

KE: You can't visualise this triangle.

ST1: No. I can't…(pause)… no it's never really a way I have thought about it.

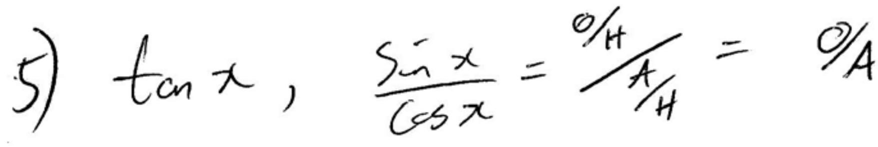
ST1 didn't refer to the unit circle when he was asked to make sense $\sin 200°$. He used the sine curve to make sense when the angle given is over 90. He linked the definition of sine as ratio of the triangle for up to $90°$ only then he switched to the sine graph. In this study, graphical trigonometry is part of circle trigonometry but it may be used without relating back to the circle. Obviously he was using his perception in graphical trigonometry. He couldn't visualize a triangle with $\sin 200°$. In this context, the right angle triangle is a problematic conception because he couldn't visualize any right angle triangle with $\sin 200°$ due to fact that angle of sine inside a right angle triangle can only be constructed when this angle is between $0°$ and $90°$. This clearly is a problematic conception due to the extensional blend from Euclidean geometry to modern Cartesian as explained in Section 3.3 of this thesis.

What is the value of $\sin 270°$? Explain why $\sin 270°$ has this value.

> KE: [...] what is the value of sin 270 degrees and explain why sin 270 degrees has this value so can you read your answer for me?
>
> ST1: Ehem... I have said sin 270 is minus 1 and I said I had no idea why and I just said is 2 pi periodic.
>
> KE: Ok. Basically you were trying to say sin 270 equals to negative 1 and then you have no idea why sin 270 equals to negative 1 and then ... why have you written here 2 pi periodic?
>
> ST1: Ehem... I was trying to justify the only real understanding why I said negative 1 purely because it's... because... actually shift it along 180 degrees, you are doing the same thing but negative... but again I can't see why it would be negative one more than because that's what the graph says.

ST1 was able to state the value of $\sin 270°$ but he couldn't explain why $\sin 270°$ has this value. He has a strong associative link to the sine graph. In most cases, he had used *graphical trigonometry* to answer the items. *Graphical trigonometry* is a supportive conception for him but there are problematic aspects in this supportive conception which do not relate to the angle in a circle. Again he didn't relate the unit circle to the generation of sine curve. For example, he couldn't explain why $\sin 270°$ equals minus 1 and he accepted this fact solely based on the sine graph. This shows that he knows the fact $\sin 270° = -1$ solely based on *graphical trigonometry* but he doesn't grasp it as an extension of circle trigonometry. The unit circle was not evoked in this mind at that moment.

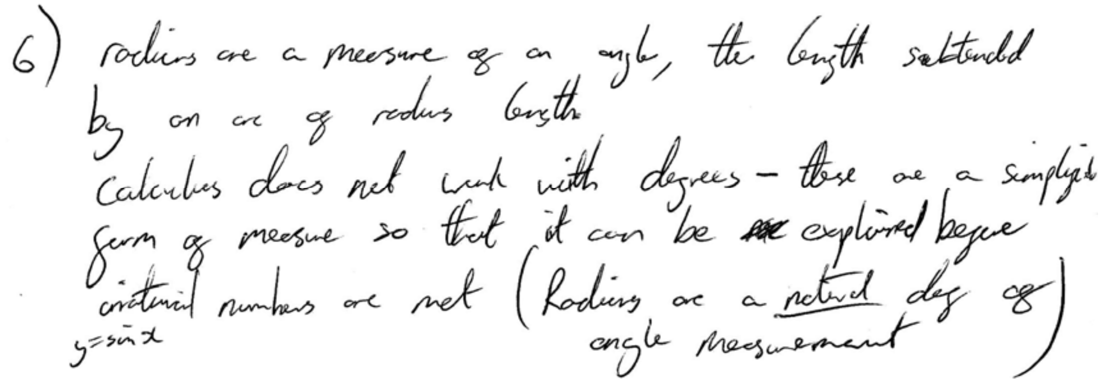
What is sine over cosine? Does that mean anything?



> KE: For item 5 of the questionnaire it sounds like this what is sine over cosine? Does that mean anything? Can you read your answer for me please?
>
> ST1: I said it was tan x: sin x divided by cos x is equal to opposite over the hypotenuse divided by the adjacent over the hypotenuse, which cancels down to give you opposite over adjacent, so that gives you the ratio between those two sides.

In Item 5, ST1 used *operation* and *reason* in triangle trigonometry to make sense of the situation. By using the relevant definitions such as $\tan x = \dfrac{\sin x}{\cos x}$, $\sin x = \dfrac{opposite}{hypotenuse}$, $\cos x = \dfrac{adjacent}{hypotenuse}$, then he operated with the mathematical symbols and finally got the answer as the ratio of opposite and adjacent. Here he is fluently switching from the visual picture of the triangle to the flexible symbolic relationship between the ratio of lengths as a process of division to the concept of trigonometric ratio as a numerical quotient. He then used symbolic operations to relate the value of the tangent to the values of sine and cosine.

What do Radians mean? Why do we need radians when we have degrees?



KE: [...] Item 6 sounds like this, what do radians mean, why do we need radians when we have degrees. Can you read your answer for me please?

ST1: Radians are the measure of an angle, the length subtended by an arc of radius length [...] I meant was if that's... ehem... so that was one radian... oh no... so if that's one that's just the length of the radius, that is the length of the radius and this is the length of the radius so that have been cut out from the circle (explaining his drawing on a piece of paper. (see Figure 6.2)) to represent one radian.

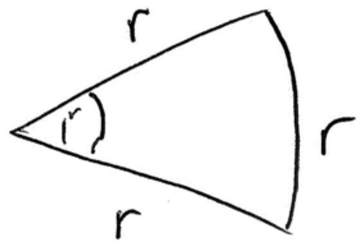


Figure 6.2: Sketch of radians by ST1.

ST1: [...] I said why we need it is the first thought came into my head was we need it for the calculus.

KE: So here you are trying to say that calculus does not work with degrees (pointing to his answer script and reading it aloud)?

ST1: Yea.

KE: Ok… and then…

ST1: I said these are a simple form of measure so that it can be explained before irrational numbers are met. The radians are not natural they're just the definition of angle measurement.

KE: So do you know why calculus doesn't work with degrees?

ST1: Ehem…

KE: Do you have any sense of why calculus doesn't work with degrees?…do you have any idea about this?

ST1: These degrees are just a number that someone's put on it because 360 has a lot of factors whereas radians, they kind of have their own natural place in mathematics, they make sense but… ehem… its not something that I thought about but I have no conclusion about it.

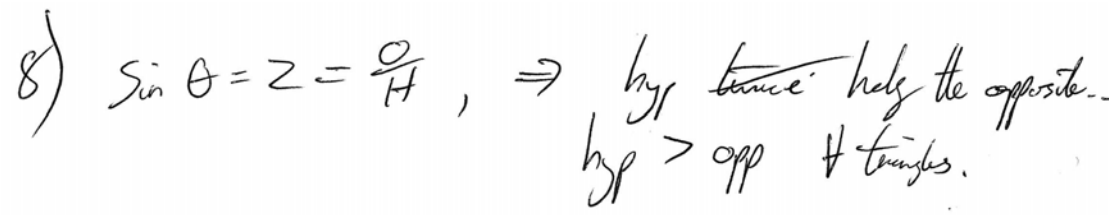
The concept image of ST1 about radians is explored in the above item. He possessed some conceptual idea about radians and recognized the problems of using degrees in calculus but he didn't *grasp* the reason. It is evident that he has a sense of conflict in his answers. Initially he said ''the radians are not natural they're just the definition of angle measurement'', then he commented again by saying ''radians they kind of have their own natural place in mathematics''. This contradiction shows that he hasn't *grasped* the idea of radians. Notice that he was describing radians in the context of general circle rather than a unit circle. This was because the radius of the sector of circle that he drew was not 1. He was describing 1 radian in a general circle sense. He is referring to ratios in triangle trigonometry and to general circles rather than signed coordinates in a unit circle in circle trigonometry. Again he didn't see radians in the context of a unit circle.

For what values is sin x decreasing? Explain why it is decreasing for these values?

> KE: […] Item 7 sounds like this for what values is sin x decreasing and then why it is decreasing for these values. Can you read your answer for me please?
>
> ST1: I have just said dy by dx is equal to positive cos x.
>
> KE: So dy by dx is equal to cos x. May I know how you came to this conclusion?
>
> ST1: Ehem… when you differentiate you get the gradient function so whenever cos… although I guess this is a bit of a circular argument…but whenever cos is positive… eeerrr… sorry but what was the question again?
>
> KE: Is this one… question 7 (Item 7: For what values is sin *x* decreasing? Explain why it is decreasing for these values?).
>
> ST1: So when cos is negative then sin is decreasing so plot a negative between… eeerrr… pi over 2 radians and 3 pi over 2 so it's in that range that sin x is decreasing.
>
> KE: Alright so you are trying to understand these things by looking at the derivative of sin x?
>
> ST1: Hmmm…

ST1 used the gradient function to get the values for sin *x* is decreasing. He knew he can get the gradient function by differentiation therefore he differentiated sin x to get cos *x*. He reasoned that when cos *x* is negative then sine is decreasing. He is blending the visual graph and the symbolic idea of derivative. Based on this, he got his answer as the range between pi over 2 radians and 3 pi over 2. This was a combination of embodied perception to see that the gradient is negative when the function is decreasing and symbolic operation to compute the derivative. He recognized that the operation of differentiation can be compressed as concept of gradient function. He reasoned that the range was between pi over 2 radians and 3 pi over 2 because cos *x* is negative in this range. He used *operation* and *reason* to make sense of this situation in the context of graphical trigonometry. However there is a high possibility that he might be using the picture (sine graph) where it is clear that the graph is decreasing at the region he stated.

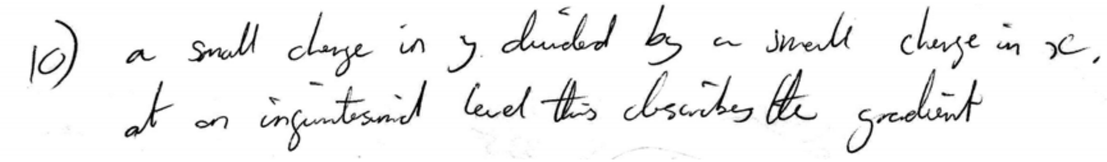
Explain why $\sin x$ can never equal 2.



> KE: [...] Item 8 sounds like this explain why sin theta can never equal 2. Can you read your answer for item 8?
>
> ST1: I put sin theta equals to 2 equals to opposite divided by hypotenuse… ammm… and if you just take those last two, so two equals to opposite over hypotenuse and multiply the hypotenuse across so you have two hypotenuse equals to the opposite, that implies that the hypotenuse is half the size of the opposite but in a triangle… in a right angled triangle… hypotenuse is always gonna be greater than… I guess greater the opposite of all the triangles.

ST1 used *operation* and *reason* to make sense of the situation. The manipulation of the mathematical symbols lead him to come out with a conclusion that hypotenuse is half the opposite. By relating this conclusion to the triangle trigonometry context, he found out that this was impossible because the size of hypotenuse should be bigger than the size of opposite for all right angle triangles. The triangle explanation was the simplest and most direct response for him in this situation. The data also shows that he was working with ratios and combining ratios as numerical entities that can be operated upon.

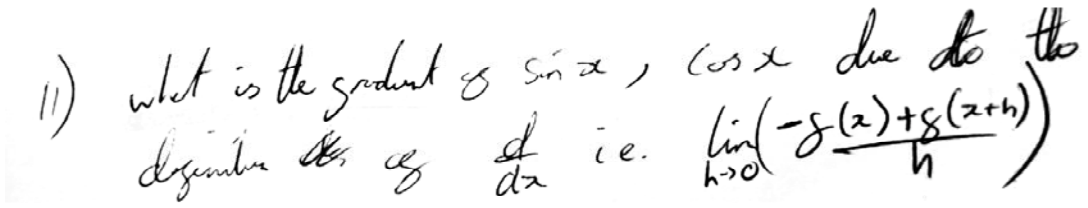
What does dy/dx mean?


10) a small change in y divided by a small change in x, at an infintesimal level this describes the gradient

> KE: I was trying to understand what did you by mean infinitesimal.
>
> ST1: [...] so at that point if you want to know the gradient of the tangent at that point then you start at the point from where we join those two up measure the gradient of that line and that should bring these points closer and closer together so the distance becomes smaller and smaller that's gonna give you an accurate representation of the gradient of the tangent at that point.
>
> KE: Ok. So how does this term (pointing to the term "infinitesimal") relate to the item here?
>
> ST1: It's so infinitesimal is just when we're essentially on that point so we are measuring the gradient of one point now... ehem... or we are measuring at the gradient of two points that are so close together that they can't be separated.

He had a problematic conception of thinking the gradient of a graph which relates to his understanding of infinitesimal (Cornu, 1983, 1991). He wasn't sure about the construction of a gradient of a graph. For instance, he said "we are measuring the gradient of one point now [...] or we are measuring at the gradient of two points ...". Obviously he was thinking in terms of dynamic process.

What does dy/dx [sin x] mean? What is d/dx [sin x]? Explain why.



KE: In your opinion, what is d by dx for you?

ST1: D by dx just by itself?

KE: Yeah… just by itself… just by this one (pointing to his answer sheet).

ST1: For me, that's just an operation when you apply it to a function and it gives you the gradient function of that original function.

KE: It seems like an operator?

ST1: Yeah… an operator.

KE: Ok. You were trying to say d by dx of sin x equals to cos x due to the definition of d by dx (pointing to his answer for Item 11)… is the whole thing the idea of this limit (pointing to his answer to get confirmation)?

ST1: Yeah… so that's the definition of d by dx when it's applied to the function f.

KE: […] In your opinion what does limit mean?

ST1: […] while limit is h tends to 0 if you take… if you take the function on either side you assess the function either side of 0 and you slowly work your way in and limit is the number that when you… so again you're getting infinitesimally close to 0… then the limit of… so we just did the limit… so the h tends to 0… (inaudible)… when these two values are infinitesimally close either side to 0 and they are equal that will be the limit of h tends to 0.

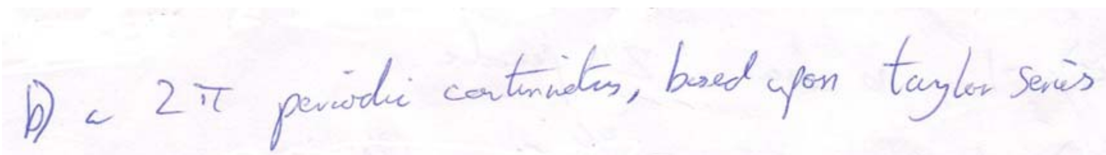
ST1 recognised the potential symbolic compression of the operation of d by dx into the gradient function. He thought d sin x by dx equals to cos x was due to the definition of d by dx i.e. $\lim_{h\to 0}\left(\frac{-f(x)+f(x+h)}{h}\right)$. This was making sense through *operation* and *reason* in the context of circle trigonometry. The concept of limit has been extensively researched by mathematics educators (Cornu, 1983; Davis & Vinner, 1986; Li & Tall, 1993; Sierpinska, 1987). Several models are proposed in order to make sense of students' understanding of limits (Tall, 1980a; Tall, 1985; Vinner, 1983; Tall & Vinner, 1981; Cornu, 1983; Tall, 1992; Gray & Tall, 1994; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas & Vidakovic, 1996).

Describe as fully as possible what you understood by the following terms:

(a) $\sin 30°$
(b) $\sin 120°$
(c) $\tan 90°$

> ST1: [...] so for the first one... I say is the ratio of two sides when your angle is 30 [...] of the right angle triangle...

ST1 was working in the context of triangle trigonometry when he was asked to describe sin 30 degrees. He made sense of this based on *reason* by describing sin 30 degrees as the ratio of two sides when the angle is 30. He was linking visual (right angled triangle with a $30°$ in it) and symbolic ($\sin 30°$) to make sense of $\sin 30°$.



> ST1: [...] for the second one I said it's a 2pi periodic continuation based upon Taylor series...so referring back to the power series definition.

KE: […] what comes into your mind when you tried to makes sense of this (pointing to item 12 (b))…I mean how do you link sin120?

ST1: […] probably the definition of the sine given by the power series …although at the moment I can't remember exactly what that is…

KE: Ok…so you have a sense that sin 120 is related to the Taylor series.

ST1: Especially in terms of getting accurate kinds of answers.

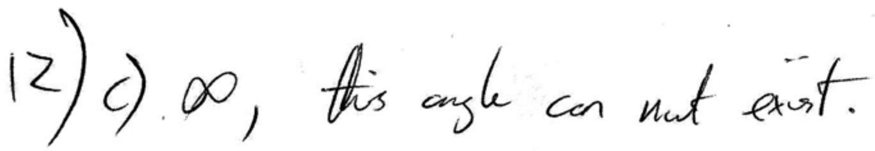
KE: Ok…so you believe that Taylor series can give you an accurate answer for?

ST1: Yeah…accurate calculations for sine specific degrees.

For sin 120 degrees, he didn't perform any computation by using the Taylor series because he couldn't remember exactly the Taylor series. He did not use circle trigonometry to compute $\sin 120°$ (which should be straightforward), instead he referred to the analytic level of Taylor series which he cannot remember. He may be attempting to make sense through *reason* (although he did not give a formal proof) which was an S1 response. In fact, it is very difficult to see the reason for the periodic nature of sine based on Taylor series, when he couldn't remember the exact Taylor series. He must have known the periodic nature of sine through the sine curve and associated this periodic nature to Taylor series because he knew that sine can be expressed as Taylor series. He believed that the Taylor series could give him an accurate calculation for the sine of a specific angle. ST1 thought that he could do the computation of Taylor series by using degrees and this is incorrect. This clearly shows that he only possessed an associative link to Taylor series and this link has problematic aspects in it. In general, the concept of Taylor series was a supportive conception for ST1 because he believed that he could get an accurate answer for any sine angle in the context of circle trigonometry. The problematic aspect was using the Taylor series for getting accurate calculations for sine of specific angles measured in degrees but not radians. In fact, the Taylor series is also problematic because he does not yet seem to know how to use it to solve specific problems. He shows no explicit ideas as to how the calculation is carried out. He didn't *grasp* the reason for using radians and he didn't realize the question of convergence as the angle gets larger and requires many more terms to get an accurate answer.

Sense making through perception can give us fundamental ideas and sense the underlying relationships whereas sense making through operation can give us the power and accuracy in computation. He was trying to work in the analytic trigonometry context by referring to the Taylor series in this situation. It seems as if he doesn't really know about Taylor series but for sure he doesn't *grasp* it.

> KE: [...] what about 12(c) for tangent 90 degrees?



> ST1: I have said it's infinite but I kind of disagree with myself... ehem... it doesn't exist because... yeap so... so that is 90 degrees there (he was pointing to his graph (see Figure 6.3 below))... so as you approach from the left hand side it goes on to infinity which is probably why I said that but if you approach from the right hand side it will go down to negative infinity so the limit does not exist because they are not equal.

Below is the graph drew by ST1:

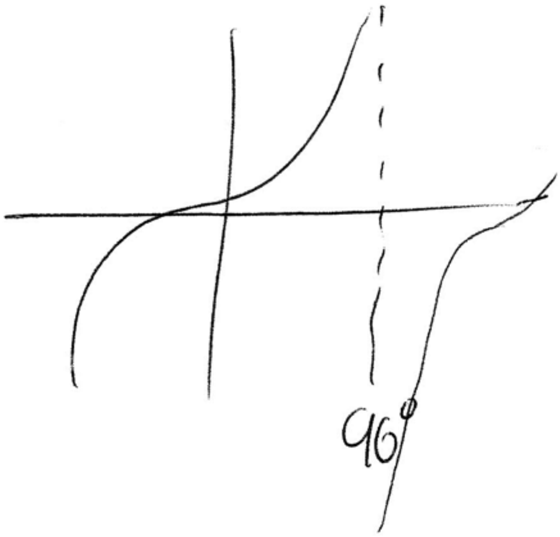


Figure 6.3: Sketch of tangent curve by ST1.

KE: So you think this one shouldn't be infinity?

ST1: No, I don't think it would be… it just doesn't exist… with that one I refer it back to the triangles as well.

KE: You mean this one 12 (c) you refer back to the triangle?

ST1: Yeah… so I kind of refer it both to the graph of tangent and how it defines how it describes the triangle.

KE: You feel like this one should be doesn't exist?

ST1: Yeah.

KE: Ok. And then how do you make sense of it? This angle cannot exist? How do you arrive at this conclusion?

ST1: Ehem… so tan x is opposite over adjacent (writing on a piece of paper)… ehem… suppose that is x… the opposite… adjacent… so you were saying this angle here is 90 degrees… but then if that was 90 degrees then you've got two parallel lines and that you can't possibly form a triangle (he was drawing a triangle on a piece of paper (see Figure 6.4)).

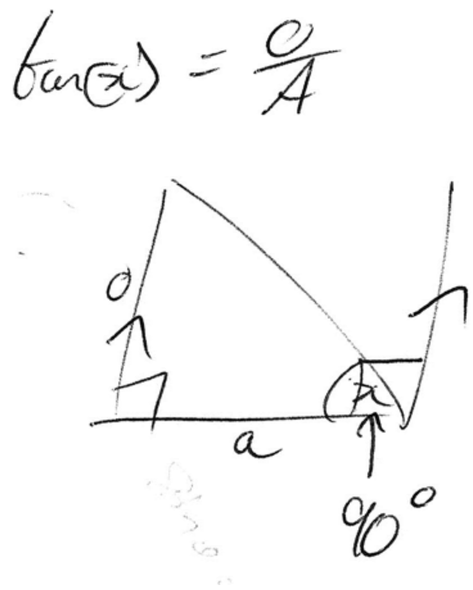


Figure 6.4: Sketch of $\tan 90°$ in triangle trigonometry by ST1.

During the follow up interview, he disagreed with his own written answer which was ∞. He drew the tangent graph and explained that the left hand limit and right hand limit when approaching to 90 degrees were not equal therefore he thought the answer should be 'doesn't exist'. This clearly shows that he had made sense through *perception* in the graphical trigonometry context. Embodied compression enables him to see the effect of operation of approaching from the left hand side and the right hand side of 90 degrees which are clearly different. He then switched to triangle trigonometry to make sense of his answer. He drew a right angle triangle and showed that if the second 90 degrees angle exist then a right angle triangle cannot be formed because there will be two parallel lines. This is a problematic aspect because he considered it is impossible to construct a right angle triangle with two right angles in it therefore he said this 90 degrees angle couldn't possibly exist. Clearly he was making sense through *perception* in Euclidean geometry. He was using both triangle trigonometry and graphical trigonometry to look at the problem in different ways to come to different conclusions.

Explain your interpretation of the following terms

(a) $\cos^{-1} 0.5$

(b) $\sin^{-1} 2.5$

KE: [...] "explain your interpretation of the following terms, so the first term is inverse cosine of 0.5".

ST1: What angle give you the sine value of 0.5, there are infinite possibilities because of the 2 pi periodic nature of sine.

He seems like was making sense through *perception* in the graphical trigonometry context by referring to the sine graph.

KE: Ok... what about 13(b) inverse sine of 2.5?

ST1: I said impossible because of the ratio of sides can never give you 2.5.

KE: So you are thinking about the triangle then you making sense of this?

ST1: I probably... probably the first thought is probably is the graph... and that bounded between minus 1 and 1 and then to explain why that happened.

ST1 interpreted $\sin^{-1} 2.5$ as impossible and he said this was because the ratio of sides can never give you 2.5. He described that he thought of the sine graph first then he proceed to explain why that happened. This clearly shows the sequence of his thinking which is from graphical trigonometry to triangle trigonometry. He did not think of the unit circle when shifting between the sine graph and triangle trigonometry. The sine graph gives him a general sense of the maximum value and minimum value of sine function and the right angle triangle helps him to explain why that happened. He was relating the visual picture to the corresponding numerical values of the function. This was making sense through *perception* and *reason* in graphical trigonometry.

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

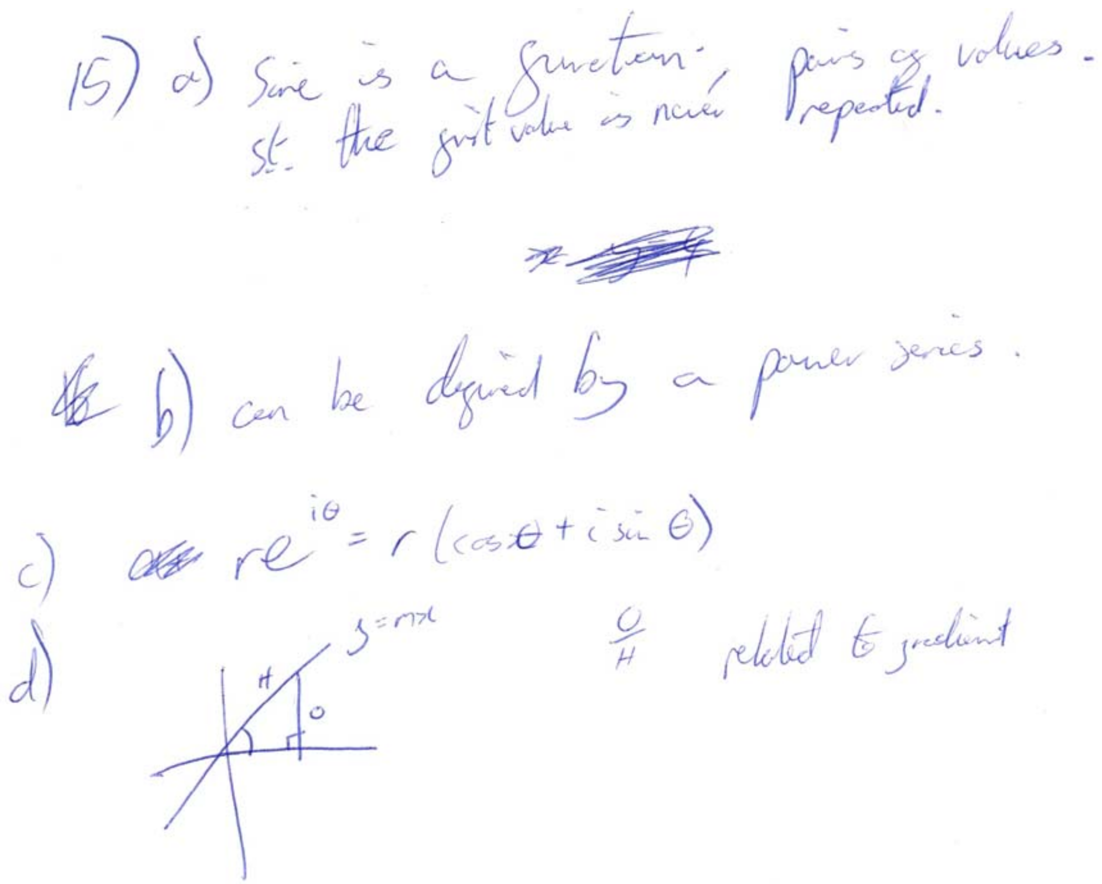
(a) Function

(b) Series

(c) Complex number

(d) y=mx

This set of concepts is highly diverse and my interest focused on how the students may relate them in their evoked knowledge structure. ST1 drew the following:



KE: [...] so what about y=mx?

ST1: I've said you can form a triangle by the y=mx graph...dropping down the perpendicular to the axis and they relate to the gradient of that line.

KE: Alright... so you are using y=mx is like a hypotenuse and then you construct a triangle on the Cartesian plane.

ST1: Yeap.

ST1 did see the complex relationships of sine with other concepts. He knew sine is a function which can be expressed as a power series but he couldn't state this power series (this was evident in the previous item of the interview). He related sine to the complex numbers by stating the De Moivre's theorem which encompasses other relationships such as the Euler formula and the exponential law. He did relate y=mx as the hypotenuse of a right angled triangle on the Cartesian plane but he didn't see this as the radius of a unit circle during the follow-up interview. His written response shows that he related y=mx to the gradient of the straight line but not that he was relating this to the dynamic gradient.

6.2.1 Summary of ST1.

ST1 is a student teacher with a second upper class degree in mathematics and he is graduated from a reputable British university. He learned analytic trigonometry recently. He knows isolated facts but he doesn't grasp the relationship between them. When he goes back to his school trigonometry, he continues to see triangle trigonometry in terms of ratio but not in terms of unit circle. Therefore his link between triangle trigonometry and circle trigonometry involves facts that he knows but not facts that he grasps. For instance, he knows he should use radians in calculus but he doesn't know why. He is still thinking of radians in term of ratio not in terms of lengths and numbers. If he has not seen it in terms of numbers then he will not be able to link triangle trigonometry and circle trigonometry.

The explanations in this paragraph and the following paragraphs are aimed to answer the proposed research questions in section 4.2 in particular research question no 1 to 8 (see page 62-63). ST1's evoked concept image of sine x included the ratio of the opposite and hypotenuse in a right angled triangle which arises in triangle trigonometry. The data shows that he has used different combinations of *perception*, *operation* and *reason* in different contexts to make sense of trigonometry. For instance when he was asked to make sense of a situation which involved angles greater than 90 degrees then he used graphical trigonometry. In this case, he used graphical trigonometry to make sense of $\sin 200°$ and $\sin 270°$ through *perception*. Meanwhile when he was asked to describe $\sin 30°$, he have made sense through *reason* in triangle trigonometry. This clearly shows that he was working in two different contexts.

For ST1, triangle trigonometry becomes a problematic conception when the angle is greater than 90 degrees. This was obvious when he couldn't make sense of the sine graph when it was bigger than 90 degrees because he didn't link it to the unit circle. This problematic conception makes him to operate in triangle trigonometry for angle equals to 90 degrees. In this case, he drew a weird figure for $\tan 90°$ (see Figure 6.4). Based on Figure 6.4, it clearly shows that he was working in the Euclidean geometry therefore he had a problematic conception in seeing tangent with 90 degrees. He didn't see the triangle trigonometry as a problematic conception when he was asked to explain certain properties of $\sin\theta$. For instance he had used the concept of ratio to make sense of why $\sin\theta$ can never equal 2 in the context of triangle trigonometry. In fact, the unit circle would be the most direct way to justify this because we can vary the angle dynamically and see the range of the $\sin\theta$ which is bounded between 1 and -1 as a function. If he was asked to make sense why $\sin\theta$ can never equal to -2 then his supportive conception (i.e triangle trigonometry) will become a problematic conception in this situation because in triangle trigonometry there are no signed lengths involved. He couldn't visualize the triangles with $\sin 200°$ and $\sin 270°$ but this is not to say that, on deep reflection and discussion he would not ever see it. But at the moment it is definitely an extensional blend in which for angles between 0 degree and 90 degrees, circle and triangle geometry are compatible, but for other values, there are problematic aspects relating to the difference between Euclidean and Modern Cartesian views.

Graphical trigonometry is a supportive conception with problematic aspects. For instance, the graphical trigonometry is a supportive conception for ST1 in the context of circle trigonometry and analytic trigonometry because the sine graph had led him to the correct answers most of the time and in turn gave him a sense of confidence. The problematic aspect is he doesn't grasp why the sine graph has the periodic nature. He believes that the Taylor series is a supportive conception in circle trigonometry. For instance he believes that he could justify the periodic nature of the graphical trigonometry by using the Taylor series. However it is problematic because he cannot use the Taylor series to complete his argument. It is sensible to say that triangle trigonometry is a supportive conception for ST1 up to 90 degrees. This is obvious when he constructed a weird figure (see Figure 6.4) to describe $\tan 90^\circ$ in the context of triangle trigonometry.

He described radians in a general circle sense as a ratio. He referred to ratios in triangle trigonometry and to general circles rather than signed coordinates in a unit circle in circle trigonometry. Meanwhile he recognized the problems of using degrees in calculus but he didn't grasp the reason why.

ST1 doesn't have a coherent link between the unit circle and the sine graph. . This was evident when he was asked to explain why $\sin 270^\circ = -1$ and he couldn't explain it solely by using the sine graph; he didn't link the sine graph to the unit circle. He also couldn't explain why the sine graph has the periodic nature when he was trying to make sense of $\sin 200^\circ$ through the sine graph. Again this shows that he didn't link the unit circle to the sine graph.

Calculus is another area that ST1 doesn't grasp. For instance he was confused with the construction of a gradient of a graph and he reproduced what he had been told. He thought of the construction of gradient as two points moving closer to each other but this conflicted with his notion of gradient of one point. ST1 knew that the derivative of sine x was cosine x by using the definition of d by dx but he didn't have an embodied sense of it. In general, ST1 knows the concepts in trigonometry but he doesn't grasp them in particular the relationships between concepts in different contexts of trigonometry.

6.3 The story of student teacher ST2.

ST2 is a female student teacher who does not have any teaching experience.

Describe sin x in your own words.

> ST2: Sin x is a function with the range brackets and then -1 and 1.
>
> KE: Is there anything you want to add or elaborate more?
>
> ST2: Hmmmm… not really.

The evoked concept image is consistent with circle trigonometry. It seems like she had compressed sine x into a set of values with range in between 1 and -1 without going into the details of sine definition and unit circle. She recognized sine is a function.

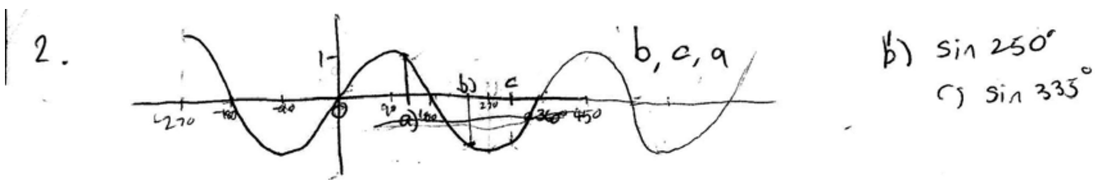
Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

(b) $\sin 250°$

(c) $\sin 335°$

Below are her written response and interview excerpt.



> ST2: I think I got… erm… what is it supposed to be in, ascending order? Erm… I've written b, c, a but on my graph it seems to be different. I think I meant to put (a), (b), (c).

KE: So what you are trying to say is…what is the largest value for these three?

ST2: Oh… no… no… no… the largest value was… (a).

KE: Ok… so you think the largest value is $\sin 110°$. What about the smallest one?

ST2: Erm…… erm…I think (b).

KE: (b) is the smallest one. Ok so you think (b) is the smallest one and then (c) is the middle one?

ST2: Yeap.

KE: Basically how did you arrive at your answer?

ST2: Ok. It looks like I drew a graph of the sine graph. Between 0 and 360 degrees… erm… so it's 1, -1 so it's 0… erm… and… I found that (a) I thought is the only positive one so that's why that one is the largest and then when looking at the negative ones…erm…which one is (b) again. Can I write on this?

KE: Ya.

ST2: (b) equal to sin 250.

KE: You can use paper if you want.

ST2: I found that it was minus 1 at 270… erm… sin 250 is 20 less than that and 335 is going to be 65 more than that but it won't go back to 0 yet. So by symmetry I thought (b) is going to be the smallest. The most negative.

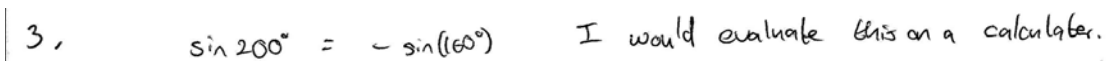
KE: Ok. So which means you used the graph to approximate the location of the point and to see which one is the biggest and the smallest. Ok so that is fine. Just now in Item 1 you write down your description for sin x and then what is the relationship between your description in Item 1 to the sine curve?

ST2: Erm… my graph is going between 1 and minus 1 on my axis so that showing the range… erm… and the fact well, that doesn't really matter much. This function is just the graph of it…(pointing to her answer script).

It is evident from the written response and interview that ST2 got the wrong answer initially because she misunderstood the instruction of the question.

ST2 used the sine curve to approximate the values of $\sin 110°, \sin 250°, \sin 335°$. This is clearly making sense through *perception* in the context of circle trigonometry. In this case, she operated the sine graph to get her answer. She gave the correct answer during the interview. Clearly ST2 is aware of the properties of sine graph such as symmetry, periodic and bounded in the range between 1 and -1 and reasons based on the visual information. It should be noted that she saw her evoked concept image of sine as the sine graph. This also confirms that her evoked concept image is in the context of circle trigonometry.

How do you make sense of $\sin 200°$?

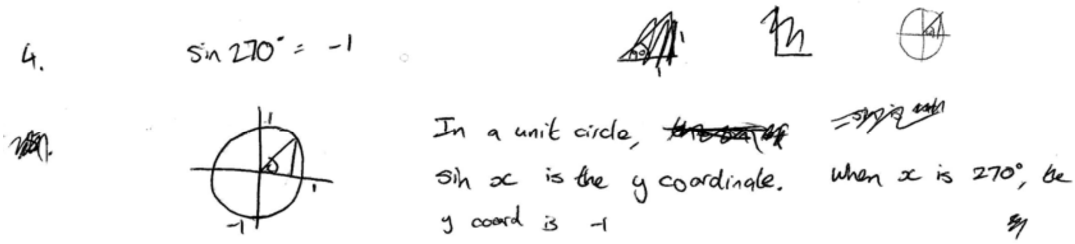

3, sin 200° = − sin(160°) I would evaluate this on a calculator.

As she wrote, she said

> ST2: Erm… to start off with I looked along on my graph to see where 200 was so sin x is 0 at 180 so 200 is just going to be a bit more negative so using the symmetry I had a look and thought it would be the same… erm… oh yes… so I said that sin 200 is 20 degrees greater than 180 so that would be the same if the negative value is 20 less than 180 and I've put evaluate this on a calculator (she was reading her own answer script) […]
>
> KE: Ok. Alright. Can you visualise this triangle with sin 200 degrees?
>
> ST2: Erm…(thinking for a while)… no.
>
> KE: So can you draw this triangle?
>
> ST2: No!

She conceptualized $\sin 200°$ as a point on the sine graph and as a numerical value given on a calculator. She used the symmetry of the sine graph to get an equivalent expression for $\sin 200°$ which was $-\sin 20°$. This is clearly making sense through *perception* in the graphical trigonometry context. She cannot draw the triangle. But at the moment it is definitely an extensional blend in which for angles between 0 degree and 90 degrees, circle and triangle geometry are compatible, but for other values, there are problematic aspects relating to the difference between Euclidean and Modern Cartesian views.

What is the value of $\sin 270°$ **? Explain why** $\sin 270°$ **has this value.**



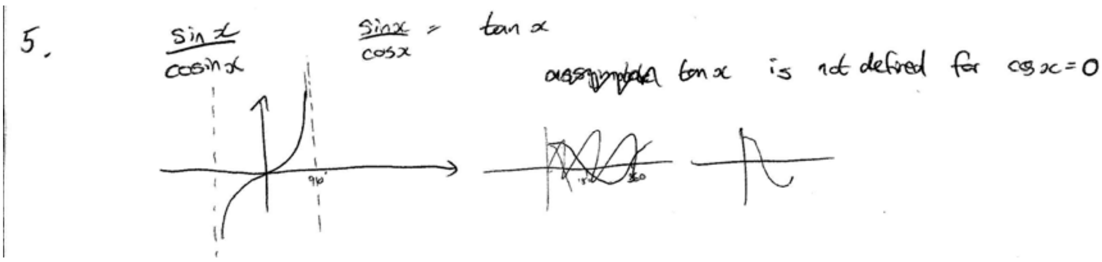
As she wrote, she said

> ST2: Sin 270 degrees equal to minus 1 and then I've drawn a unit circle on a graph. In a unit circle sin x is the y coordinate. When x equals 270, y coordinate is minus 1. Ok. You can also see it from the graph I drew (pointing to the graph that she drew for Item 2).
>
> KE: What about these few bits (pointing to her answers for item 4)? Why did you cross out these bits?
>
> ST2: I think that one might have been… I don't know what that one was a little right angle triangle with… It looks like a right angle triangle… there again it seems to be the unit circle. It's got maybe 60 degrees and I don't know what I was trying to calculate to be honest.
>
> KE: Are you trying to draw a triangle with sin 270 degrees?
>
> ST2: Erm… possibly… yeap… I am pretty sure that's what that would have been I just can't remember to be honest…yea…
>
> KE: Can you visualise this triangle with sin 270 degrees?

> ST2: Erm… looking at this angle from this line down to there so that would be 270 degrees… but I am not sure… can you draw a triangle with that?… I am not sure.

She saw sin *x* as the y coordinate in the unit circle without referring to the definition of sine *x* as opposite divided by hypotenuse. She was working in the unit circle to get the y coordinate as minus 1 when x equals to $270°$ then she related this to the sine graph which she drew for item 2. This is clearly making sense through *perception* in the circle trigonometry context. Initially she operated the unit circle to get the answer. ST2 might have tried to relate right angle triangle with $\sin 270°$. She appears to be shifting between circle trigonometry and triangle trigonometry in order to make sense of $\sin 270° = -1$. The triangle trigonometry didn't bother her so much when she saw sin x as the y coordinate in the unit circle. She is beginning to think about a triangle with 270 degrees but is still in conflict.

What is sine over cosine? Does that mean anything?



As she wrote, she said

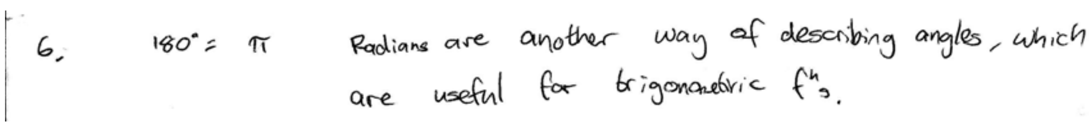
> ST2: Tan x is not defined for cos x equals 0. That is because I put sin x over cos x equals tan x and obviously you divided by 0 it's not going to be defined so that's why the tan graph looks like this.

ST2 knew that sine x divides by cosine x equals to tangent x. She wrote asymptote and then crossed it out and wrote 'tan x is not defined for cos x =

0. She then sketched a graph with the shape of the cosine function but wrote 180° and 360° where it crossed the axes, and crossed this out, replacing it by a rough sketch of cosine without any numerical information. Since tangent x is not defined for cosine x equals zero then she related this to the outlook of the tangent graph. This clearly shows that ST2 first sought to make sense through *perception* of the cosine graph then she made sense through *operation* of sine x divides by cosine x. Finally she made sense through *perception* again for the tangent graph. She was working predominantly in graphical trigonometry in this item.

What do Radians mean? Why do we need radians when we have degrees?

ST2 wrote


6. 180° = π Radians are another way of describing angles, which are useful for trigonometric f^ns.

As she wrote, she said

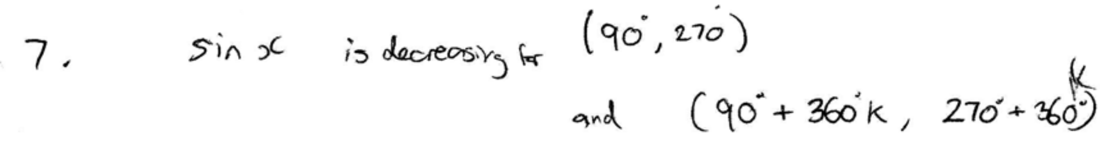
> ST2: I've put 180 degrees equals pi. Radians are another way of describing angles which are useful for trigonometric functions. [...]
>
> KE: [...] Do you know why we need radians when we have degrees?
>
> ST2: Erm... It makes writing things easier I am not sure of the exact reason why we need them, we are obviously writing pi it's a different kind of group that... like amounts of 180 degrees.
>
> KE: Do you prefer to use degrees or radians?
>
> ST2: Erm... probably in more advanced maths, I would prefer to use radians, but if I am for example looking at GCSE maths and normal triangles I would use degrees.
>
> KE: Ok. Do you know what is 1 radian?

ST2: 1 radian would be… erm… I suppose a 180 degrees divided by pi… that is pi radian.

ST2 says she is not sure of the exact reason for using radians rather than degrees. This indicates that she *knows* but does not *grasp* the reason*.* She prefers to use degrees for GCSE mathematics and normal triangles and this probably related to her experience in learning school trigonometry. Radians are only introduced in school syllabus when students learn about circle trigonometry. However there is no evidence to support this explanation. She described 1 radian as 180 degrees divided by π without explicitly relating radian to circle such as relating the radian to the circumference. It seems like she only saw radian as another kind of unit measurement (amount of 180 degrees) other than degree therefore she only described 1 radian as a mathematical operation for unit conversion from degrees. She doesn't grasp the concept of radians.

For what values is sin x decreasing? Explain why it is decreasing for these values?

ST2 wrote



As she wrote, she said

KE: Do you know why it's decreasing for these values?

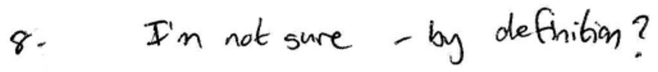
ST2: Erm………I can only describe it using this circle thing. So if I am trying to describe sin x is the y coordinate here of 0 if my angle there is going to be 0, if my angle increasing up to 90 degrees that's when the y coordinate is 1, if I carry on to

> these two quadrants as the circles going round the y coordinate is getting smaller and smaller and getting more negative and then it's increasing.

She got the correct values for sin x decreasing. It should be noted that ST2 describes why sin x is decreasing for certain values by using the unit circle. She compressed sin *x* as the y coordinate in the unit circle. Meanwhile she was relating $\sin\theta$ dynamically to the unit circle. She focused the changes on y coordinate in the unit circle when she varied the angle across the four quadrants. This is essentially embodied compression which focuses on the effect of operation. In this case the base object is the angle in the unit circle and the effect as the angle increases is the change in the y coordinate. This is clearly making sense through relating *perception*, *operation* and *reason* in circle trigonometry context.

Explain why $\sin\theta$ can never equal 2.

She wrote



As she wrote, she said

> ST2: But I am not sure, by definition?
>
> KE: Do you wish to add further?
>
> ST2: No. I don't know.
>
> KE: Why you don't know? Is there any specific reason why you don't know?
>
> ST2: I don't know. It's not something you should think about you just think it's between minus 1 and 1… and never really question why.

It is evident that ST2 did not offer a reason why sine theta can never equal to 2. She guessed that was probably because of the definition. Based on her written response for Item 1 (Item 1: Describe sin *x* in your own words), it is sensible to hypothesize that she is referring to her definition written for Item 1 which says sin *x* is a function with the range brackets and then -1 and 1. It has been 'learned' without understanding. She tried to make sense of this based on reason but it was obvious that she didn't know what the reason was. In fact, the property of sine theta can never equal 2 can easily be justified by referring to the sine definition as the ratio of the opposite to the hypotenuse of a right angle triangle or by using the unit circle but it seems like those links were not evoked at that time. Without linking the sine definition to the unit circle or sine graph, justification of this property can be difficult for learners.

What does dy/dx mean?

> ST2: Dy by dx means the differential of y with respect to x.
>
> KE: Ok. What do you understand about differential?
>
> ST2: Erm… it's to do with limits as you are approaching the curve. I am not sure how to describe it actually……………yeah.
>
> KE: Do you want to think about it? You can ask for more time if you want to.
>
> ST2: (She shakes her head).

She couldn't explain differential explicitly however she could sense there should be something to do with the concept of limit. There appeared to be some reluctance on ST2 to comment further about differential. She knows the idea of dy by dx but she doesn't grasp it.

What does dy/dx [sin x] mean? What is d/dx [sin x]? Explain why.

> ST2: Erm… d by dx sin x is the differential of sin x with respect to x. d by dx sin x is cos x and it looks like I have started to try to draw an explanation why so I've got a coordinate on a curve which I assume is the sine curve and then I've picked another point a bit further along and I was going to try and look at the limits as this got closer I think I forgotten how to do it. I didn't have time. I am not sure.
>
> KE: So basically do you have any idea why it is (pointing to d sin x by dx) cos x?
>
> ST2: No, I don't really know.
>
> KE: Do you want to think about it?
>
> ST2: Erm… I don't think I will get there.

ST2 perceived d sine x by dx to be an operation to perform by fixing a point on the curve then another point moving closer to it. She appeared to be recalling her knowledge entirely on how to operate this operator. However she have forgotten how to do it. She couldn't relate the dynamic perceptual idea to the symbolic computation of the derivative as a limit and eventually in terms of the rules of the calculus. It is evident from the excerpt that ST2 doesn't have a conceptual idea about differential. She made sense through operation in the context of circle trigonometry. She tried to operate the mathematical symbol d sin x by dx in order to make sense of it but she couldn't draw a description on this mathematic symbol out of her operation. She could sense there was a link between the idea of differential and the idea of limit but she was unable to explain explicitly. It is just an associative link. Indeed, there is an underlying emotion that she simply does not want to go there. Apparently she knows something about calculus but she doesn't grasp the relationships. ST2 was reluctant to think further when she was asked to explain why d sin *x* by dx is cos *x*. This was related to the notion of anti-goal as proposed by Skemp (1979). Moving towards the anti-goal state will give us a sense of fear. In this case, ST2 believed that she was moving to the anti-goal state (unable to explain why d sin *x* by dx is cos *x*) therefore she sensed the fear in herself.

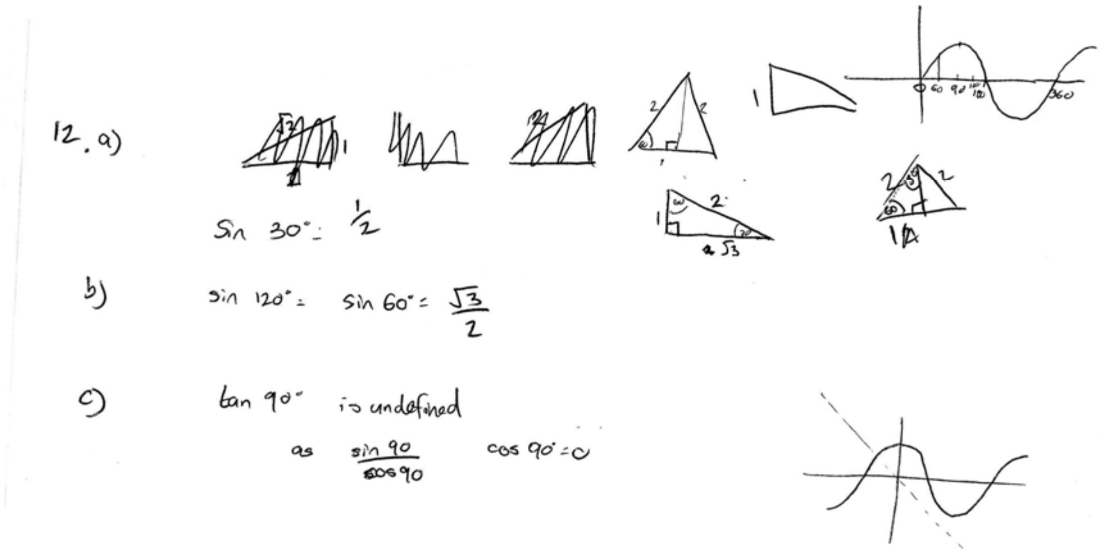
Describe as fully as possible what you understood by the following terms:

(a) $\sin 30°$

(b) $\sin 120°$

(c) $\tan 90°$

Below is her written response.



As she wrote, she said

> ST2: Ok. I drew some triangles for this, sin 30 I've got is a half.
>
> KE: Ok. Can you explain how you get this answer?

ST2: I drew a triangle, an equilateral triangle at 2 by 2 by 2 and then split into half so it went into two triangles that were, hypotenuse is 2, that length is 1 and that length was root 3. So that if it was an equilateral triangle that would still be 60 but bisecting that angle is going to be 30 and that is going to be 90 so I got a right angle triangle which is 30, 60 with these lines so I know that sine is the ratio the opposite side divided by the hypotenuse that's how I got half and then for (b) sin of 120 degrees, I've put equals sin 60 degrees I think I've put that on the graph looking back (she was turning her answer script to page 1 (item 2))… looking at the symmetry of the sine graph I saw that sin of 60 was equal to sin 120 so then I used my triangle again to do the opposite divided by the hypotenuse. […]

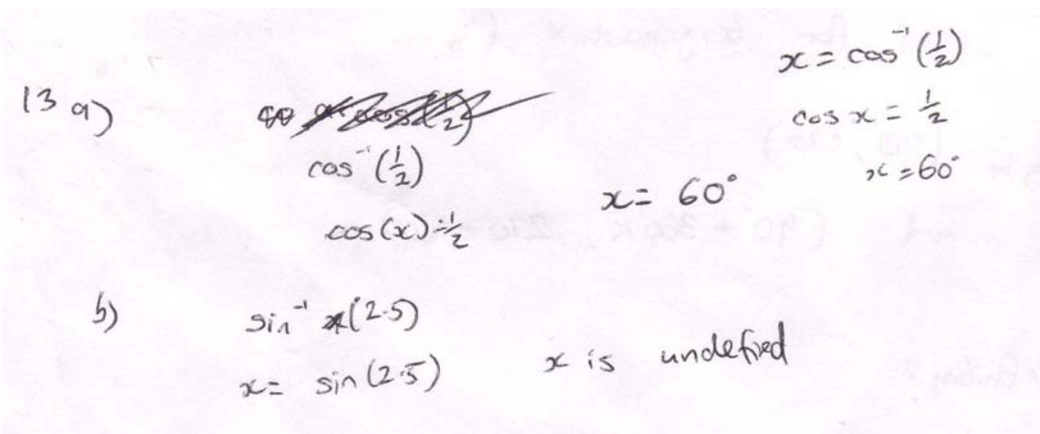
ST2: I've put tangent is undefined. Tan is sine over cos of 90 is 0 because sine divided by 0 is going to be undefined.

She worked out $\sin 30°$ using Euclidean perception of an equilateral triangle. She is able to answer $\sin 30°$ in this context. In order to get this answer, she operated an equilateral triangle through *perception* initially then she operated the sine definition through operation. She was clearly working in the context of triangle trigonometry. The problems may appear with $\sin 120°$ because it doesn't fit the Euclidean context. In the case of $\sin 120°$, ST2 referred to the sine graph to get an equivalent expression which was $\sin 60°$ based on the symmetry of the sine graph. It is interesting to note that the student is not looking at the circle, but at the graph of sin x which she may or may not, at the time, have a link between the two. In this case, she used graphical trigonometry (using the graph) to describe this incident. In this study, graphical trigonometry is considered as part of the circle trigonometry but it may be used on its own without relating back to the circle trigonometry. This was essentially making sense through *perception* in the context of graphical trigonometry and circle trigonometry. After that she went back to the right angle triangle which she drew just now to get the answer. Finally she operated the sine definition in triangle trigonometry through operation to get her answer. It should be noted that she must have realized that she couldn't draw a right angle triangle with $\sin 120°$ in the first instance therefore she looked for equivalent expression in the first quadrant. Clearly ST2 was aware of her ability to draw right angle triangle in the first quadrant but not in other quadrants. For the case of $\tan 90°$, it may not be clear what ST2 was using subconsciously to come to a conclusion. She might be using reason being based on operation, but that operation may have an internal mental representation related to a picture of a triangle or of the graph, or some other supporting evidence that is not explicitly mentioned.

Explain your interpretation of the following terms

(a) $\cos^{-1} 0.5$

(b) $\sin^{-1} 2.5$



KE: Ok. Item 13, explain your interpretation of the following terms. You are given two terms inverse cos of 0.5 and inverse sine of 2.5.

ST2: Erm… I've put inverse cos of half, I've put that equals to x, so x equals inverse cos of half and then I kind of took the cosine of both sides so then cosine x equals half and then I did try to find x so this was a half and I think I found it was 60.

KE: Ok…so what about the second term inverse sin of 2.5?

ST2: I've put… again I've tried to do x equals to… x equals to the inverse sin 2.5 so I've put that is undefined.

KE: Can you explain why it is undefined? Is there any specific reason why x is undefined?

ST2: erm………I suppose for the same reason the range of sin is minus 1 to 1 so I thought if I could figure out if that would mean anything if it was 2.5.

ST2 operated symbolically by letting x as inverse cosine of half in order to embody the idea of this mathematical symbol then she took cosine on both sides of the equation. By doing this, she found her answer as 60 degrees. It is sensible to say that she must have referred to the triangle trigonometry because she has only given an answer which is in the first quadrant. If she has referred to circle trigonometry (i.e the unit circle or the sine graph), then she should have realized or see (by looking at the sine graph) other possible answers which are outside the first quadrants. For the item inverse sine of 2.5, ST2 said it was undefined after operating the mathematical symbol in order to embody the underlying idea. Again ST2 didn't offer an explanation of why the range of sine is between 1 and -1 after an attempt to think about it.

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

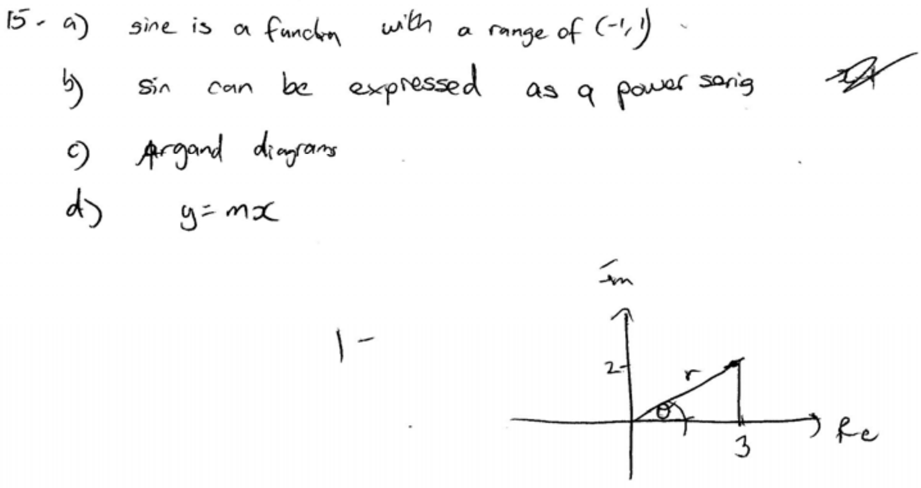
(a) Function

(b) Series

(c) Complex number

(d) y=mx

She wrote



As she wrote, she said

KE: Do you know this power series? Can you write down the power series?

ST2: Erm… I am not sure… erm… is it either the Taylor series… erm… sorry I am getting mixed up, I can't remember………………………I can't quite remember I think it's got something to do with the power of x have alternating signs and divided by factorials.

KE: Ok. It's fine. What about complex number?

ST2: Erm… I've put argand diagrams.

KE: How did you relate this?

ST2: Erm… so if you have got something like 3+2i or something you can write that on a graph with the real part (demonstrating her answer to me on her answer script).

KE: So what about the other one y equals to mx?

ST2: I didn't get anything for that one.

ST2 knew that sine is a function which can be expressed as a power series but she couldn't state this series. It was interesting to note that ST2 didn't know the relationship between power series and Taylor series and she mixed up both of them. She only showed an associative link between sine and power series without able to state this series explicitly. She seems like doesn't know about the idea and for sure she doesn't *grasp* it as a manipulable mental entity. ST2 demonstrated the relationships of sine with the complex numbers by drawing an Argand diagram but did not link it to e^{ix}. It is in the first quadrant with a triangle. It is interesting to note that she didn't link y=mx to the unit circle where at an earlier instance she did recognize sin x as the y coordinate in the unit circle.

6.3.1 Summary of ST2.

ST2 is a student teacher with a 2(ii) degree in mathematics and she is graduated from a reputable British university. She learned analytic trigonometry recently. She knows isolated facts but she doesn't grasp the relationship between them. Her link between triangle trigonometry and circle trigonometry involves facts that she knows but not facts that she grasps. For instance she didn't link the sine graph to the unit circle or the triangle trigonometry therefore she couldn't justify why $\sin\theta$ can never equal 2. She will use radians for higher level mathematics but she doesn't know why.

Her concept image of sine x was a function with range in between -1 and 1 and this was predominantly in circle trigonometry. She had compressed the definition of sine as the ratio of opposite to hypotenuse of a right angle triangle into the signed length of a unit circle then further compressed into the sine graph. She does make sense of trigonometry through *perception*, *operation* and *reason* in different contexts of trigonometry. For instance, she made sense through *perception* and *reason* in graphical trigonometry when she was asked to arrange $\sin 110°, \sin 250°, \sin 335°$ in ascending order. On the other hand, she made sense through *perception* and *operation* in triangle trigonometry when she was asked to describe $\sin 30°$.

It wasn't a problem for her to make sense of certain particular angle such as $\sin 270°$. She didn't see $\sin 270°$ as being part of a right angle triangle with an angle of $270°$. This was because she already compressed sin x as y coordinate in a unit circle therefore the problem of seeing sin x for angle x greater than $90°$ was not a problem for her. ST2 has problems in visualizing the right angled triangle with $\sin 200°$ and $\sin 270°$. In this context, the right angle triangle is a problematic conception because she couldn't visualize any right angle triangle with $\sin 200°$ due to fact that angle of sine inside a right angle triangle can only be constructed when this angle is between $0°$ and $90°$. This clearly is a problematic conception due to the extensional blend from Euclidean geometry to modern Cartesian as explained in Section 3.3 of this thesis.

ST2 saw radians as a different kind of measurement when compared to degrees. She couldn't sense radians as a natural kind of measurement. She didn't know the reason of using radians instead of degrees. It is obvious that she knows radians as a kind measurement which can be converted from degrees but she doesn't grasp the embodied sense of radians and the reason for using radians.

ST2 appears to have built a link between the unit circle and the sine graph. It appeared that the link between triangle trigonometry, unit circle and the sine graph was not available in certain instances. This was evident when she didn't link the sine definition or the unit circle to the sine graph to justify why $\sin\theta$ can never equal 2. It is evident that she has developed ways of operating with the unit circle to think about the concept sine. For instance, she knows that the value of $\sin\theta$ corresponds to the value of the y coordinate in the unit circle. Apparently her transition from triangle trigonometry to unit circle then to sine graph as a function was smooth but the reverse transition is not as smooth as the forward transition. Her knowledge in analytic trigonometry was a bit weak and she mixed up the relationship between power series and Taylor series. She couldn't state the Taylor series during the follow-up interview.

Calculus was a problematic area for ST2. She knew the differential was related to limits but she couldn't describe it. During the follow-up interview she seemed like reluctant to talk further about differential. It was obvious that the differential was not in her comfort zone. When she was asked to comment why the differential of sin x was cos x, she couldn't offer an explanation. She tried to blend the perceptual ideas with the symbolic ideas in a coherent way but she couldn't get a conclusion out from it. She didn't want to have more time to think about it because she thought she will never get the answer. Apparently calculus is another area where ST2 didn't grasp it but she did know some of the important ideas in calculus.

In general, ST2 has strong links to triangle trigonometry and circle trigonometry but there are problematic conceptions involved as mentioned above. She knows some of the important ideas in circle trigonometry and analytic trigonometry but she doesn't grasp them such as the idea of radians, differential, Taylor series etc.

6.4 The story of student teacher ST3.

ST3 is a male student teacher who has no teaching experience.

Describe sin x in your own words.

> ST3: Sin x is a trigonometric function. Given a right angle triangle with an angle x, sin x is the length of the opposite side in the triangle divided by the length of the hypotenuse.

The evoked concept image of ST3 is in the context of triangle trigonometry and circle trigonometry. He saw sin x as a trigonometric function in circle trigonometry whereas in a right angle triangle he saw it as the ratio of the opposite to the hypotenuse.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

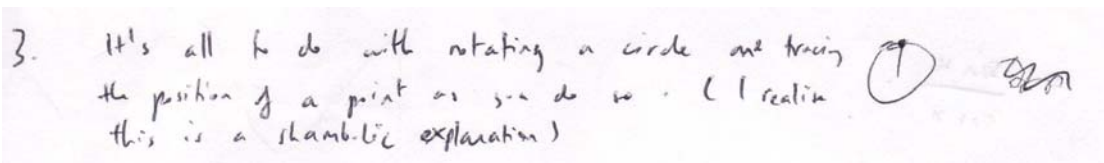
(b) $\sin 250°$

(c) $\sin 335°$

> ST3: Erm… I just sketched the sine curve… so you've got 180 in the middle… so 110's here so that's close to 1… so 250's close to 270 so it's going fairly close to minus 1, 335 is going to be a little bit bigger so that is the smallest ascending mean […]
>
> KE: What is the relationship between item 1 to the graph?
>
> ST3: Erm… well I suppose in 1, I've talked about trigonometric function and of course you are only really going to have this part of the graph when you got the triangle stuff (pointing to the first quadrant of the sine graph) that's what the graph will look like so I've just used the graph.

ST3 got the correct answer. He used the sine graph to approximate the values of $\sin 110°, \sin 250°, \sin 335°$. This was making sense through *perception* in the context of graphical trigonometry. ST3 is able to link his perceptual idea of right angled triangle to the sine graph and he is aware that he could generate the first quadrant of the sine graph by using the right angle triangle.

How do you make sense of $\sin 200°$ **?**


3. It's all to do with rotating a circle and tracing the position of a point as you do so. (I realise this is a shambolic explanation)

> ST3: Oh yea… so you get your sine curve so I just read the answer… so it's all to do with rotating a circle and tracing the

position of the point as you do so I realize this is a shambolic explanation.

KE: Ok… may be you can explain a bit of your answer…how do you make sense of this bit…do you want to use paper?

ST3: No… I will be fine… I guess what I am trying to get at is you can get this graph by taking a circle and putting a pen on a point at the top and as you move the circle round you kind of get your sine curve… that's how… that's what I think of when I think of sin 200 degrees when I get round my circle at 200 degrees… that's where I end up on the graph.

KE: […] can you visualize this triangle of 200 degrees?

ST3: No.

KE: No… ok… can you draw this triangle of sin 200 degrees?

ST3: Well a triangle got 180 degrees in it so I would have trouble doing sin 200 degrees.

It was evident from the above excerpt that ST3 was making sense of $\sin 200°$ through *perception* in the circle trigonometry context. He used the word 'shambolic', which shows that he thinks that this is not a satisfactory explanation, showing that the link between perceptual ideas and symbolic or deductive ideas is not clear. He had linked his idea of rotating a circle with a pen on a point to the generation of the sine graph. This was similar to the generation of the sine graph from the unit circle. He realized that drawing a triangle with $\sin 200°$ was impossible and he knew the problem was because a triangle had 180 degrees in it. It is impossible in to draw this triangle in Euclidean geometry but not in circle geometry hence there is a problematic extensional blend here because a property in Euclid becomes problematic in circle geometry. It was evident that he knew the need to compress the definition of sine as ratio of sides in order into another form in order to make sense of sine when the angle was greater than 90 degrees.

What is the value of $\sin 270°$? Explain why $\sin 270°$ has this value.

ST3: Sine 270 degrees is equal to minus 1.

KE: Do you have any idea why?

ST3: Well I suppose it's like… the inverse is the wrong word… but it's gonna be the inverse of 90 degrees so it's not quite inverse (his gesture shows reflection) but it's a bit outside it say the 90 degrees of the angle here… so that's your 90 degrees… and that's positive 1 the other bit is gonna be negative of it… so like with any angle whatever this is it's gonna be the negative of it…

KE: Ok… so basically same question can you visualise this triangle with sin 270 degrees?

ST3: No.

KE: You couldn't draw this triangle?

ST3: No… I couldn't do that either.

It shows the same problematic extension as mentioned in the previous item. It relates to the inability to make sense of the extended situation so it is problematic. He was making sense through *perception* in the context of circle trigonometry.

What is sine over cosine? Does that mean anything?

ST3: Sine over cos is tan.

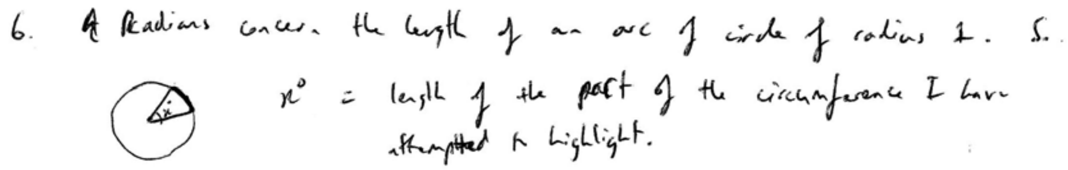
KE: Ok.

ST3: Erm… so tan of an angle is the opposite over adjacent side of a triangle.

ST3 got the correct answer and he described tangent by using the definition of tangent in the context of triangle trigonometry. This was making sense through *reason* using both *perception* and *operation* based on the definition of the trigonometric concepts. In fact, he was doing arithmetic tan=sin/cos related to picture.

What do Radians mean? Why do we need radians when we have degrees?

He wrote

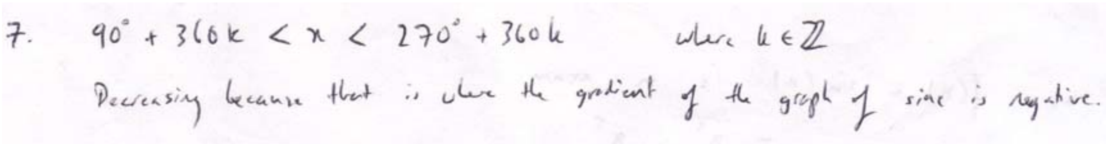


As he wrote, he said.

> KE: […] why do we need radians when we have degrees?
>
> ST3: Erm… I wouldn't say it's stuff to do with Fourier series…maybe even differentiation… erm… I can't remember… there is a good reason for this once you get further… at an advance level why you would want to use radian because degree doesn't work there is something but I can't remember what it is…
>
> KE: Alright… basically do you prefer to use degree or radians?
>
> ST3: Erm… I am not that fussed, I am quite agnostic about it but probably radians generally I would use because they are always gonna work.

ST3 doesn't grasp the reason for using radians. He could sense there was a sensible reason for the usage of radians instead of degrees but he didn't know this reason. He only knows there is something technical but he doesn't really understand why. In fact, differentiation is a very good instance for him to talk about the use of radians but he didn't think differentiation had something to do with it. Although he didn't know the real reason to use radians instead of degrees, this didn't bother him because he had a supportive conception of radians. It should be noted that he holds a supportive conception by saying that radians always work for advanced level mathematics. He knows the idea but doesn't grasp it, the link is associative rather than conceptual.

For what values is sin x decreasing? Explain why it is decreasing for these values?



> ST3: 90 degrees plus 360 k is less than x which is less than 270 degrees plus 360k… I seems to prefer degrees in this occasion…that's where k is an integer… ehem… and it's decreasing because that is where the gradient of the graph of sine is negative.
>
> KE: […] how do you interpret gradient, for you what is gradient?
>
> ST3: It's like a line… like a gradient line… it's a curve… it's like this point here (drawing a gradient line and a curve, see Figure 6.5) gradient and tangent line.

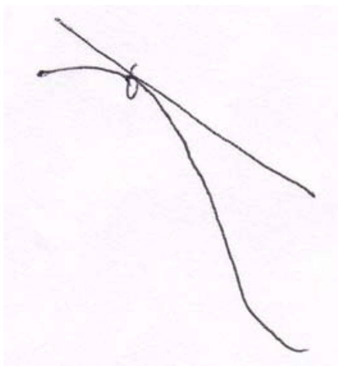


Figure 6.5: Sketch of a gradient line by ST3.

He knows the idea of gradient and the visual representation of gradient line. ST3 compressed the idea of gradient and able to conceive that the negative gradient of sine graph represents sin x is decreasing. He was clearly working in circle trigonometry because sine curve only arises in circle trigonometry. ST3 made sense of this through *perception* and *reason*.

Explain why $\sin\theta$ can never equal 2.

> ST3: Because for sine theta equal 2 would imply that the length of the opposite side was longer than the hypotenuse which is impossible.

He made sense of the above property through *perception* and *reason* in the context of triangle trigonometry. This also indicates that he holds a supportive conception that the length of the opposite side cannot be longer than the hypotenuse in a right angle triangle in any context of the trigonometry.

What does dy/dx mean?

> ST3: Dy by dx concerns the rate of change in y with respect to x.

What does dy/dx [sin x] mean? What is d/dx [sin x]? Explain why.

> ST3: […] I've said d by dx of sin x is cos x. The gradient at (x, sin x) is cos x.
>
> KE: How do you know that the gradient at (x, sin x) is cos x?
>
> ST3: I suppose going back to the infinite series we were talking about earlier you could derive it from here or anything you like… you could take the gradient lines and you can plot the gradient on a separate graph… or using Autograph… whatever you could get the gradient line moving around plotting the gradient of the slope at each point and you would get a cos graph.

ST3 knew the underlying meaning of dy/dx. He compressed the idea of dy/dx into gradient. ST3 further compressed the moving gradient line into a cosine graph and this was essentially embodied compression. The base object was the gradient line in this case. It was evident that he could link the world of embodiment to the world of symbolism. In general, he made sense of this through *perception* in the context of circle trigonometry.

Describe as fully as possible what you understood by the following terms:

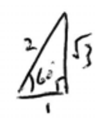
(a) $\sin 30°$

(b) $\sin 120°$

(c) $\tan 90°$

12(a) $\sin 30° = \frac{1}{2}$.

(b) $\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$



(c) $\tan 90°$ is undefined.

You can't get a ~~triangle~~ right-angled triangle where a right-angle has an opposite and adjacent side (the right angle is always opposite the hypotenuse...)

ST3: Half for the first bit, I have written square root 3 over 2 for the second bit the third one is undefined because you can't get a right angle triangle where right angle has an opposite and adjacent side because the right angle is always opposite the hypotenuse… so if you had that you would have two 90 degrees angles and you couldn't have a 0 degrees end…

KE: What about this part? Sin 30 equals to half. How did you make sense? How did you arrive at this answer?

ST3: I just imagine a right angle triangle. I just drew an equilateral triangle with side 2 and cut it into half down the middle so I got a one 2 root 3 triangle…I just used sin that opposite over hypotenuse which will give me 1 over 2.

KE: And what about this part what about sin 120 (pointing to his answer)? How do you know it is sin 60?

ST3: From the graph you just kind of reflect at 90 degrees what is you had 80 degrees that gonna to be the same as 100 degrees what is you had 70 degrees that is the same 110.

KE: Oh I see is the symmetry

ST3: Yea…the symmetry […]

KE: What about sin 120? How do you know is equal to sin 60?

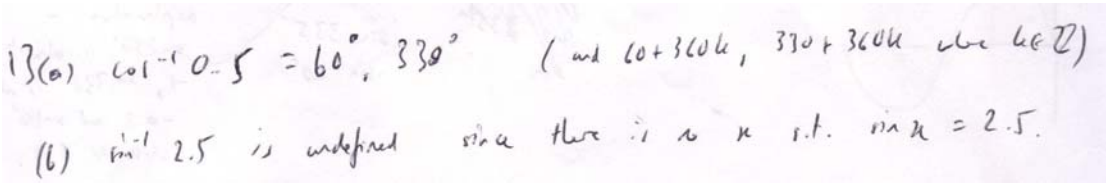
ST3: Well, from the graph obviously it's you can kind of reflect in 90 degrees so what if you have 80 degrees that's going to be the same as a 100 degrees. If you have 70 that's gonna be the same as 110 […] it's symmetry […] again I've used the same triangle…because obviously there is one 2 root 3 triangle it's going to have 60 degrees angle because it's come from the equilateral triangle will have a 30 degrees angle that were using before but now I've used the 60 degrees angle so it's opposite root 3 hypotenuse is 2 so it's root 3 over 2.

ST3 got the answer as half for $\sin 30°$ through his *perception* and *operation* by cutting an equilateral triangle into half. Then he got a right angle triangle with $30°$ in it. Finally he operated the sine definition in triangle trigonometry to get the answer as half. In the case of $\sin 120°$, ST3 did the similar thing to ST2. Initially he made sense through *perception* in circle trigonometry to get an equivalent expression then he operated the sine definition in triangle trigonometry through operation to get his answer. He clearly aware that he couldn't draw a right angled triangle with $120°$ in Euclidean geometry therefore he looked for an equivalent angle. ST3 appeared to be referring to the triangle trigonometry when he was asked to describe $\tan 90°$. He made sense of $\tan 90°$ as undefined through *perception* by indicating that he couldn't construct a triangle with two 90 degrees and a 0 degrees in it. He was referring to the Euclidean concept of a triangle in this case. He has a problematic conception in making sense of $\tan 90°$ and this problematic conception is related to the difference of views between Euclidean geometry and modern Cartesian as discussed in section 3.3 of this thesis.

Explain your interpretation of the following terms

(a) $\cos^{-1} 0.5$

(b) $\sin^{-1} 2.5$


13(a) $\cos^{-1} 0.5 = 60°, 330°$ (and $60+360k$, $330+360k$ where $k \in \mathbb{Z}$)

(b) $\sin^{-1} 2.5$ is undefined since there is no x s.t. $\sin x = 2.5$.

For the item $\cos^{-1} 0.5$, he was clearly working in the circle trigonometry because his answer involves all the possible answers in the first quadrant and the forth quadrant of the unit circle. For the item $\sin^{-1} 2.5$, there is no clear evidence showing what context he was working in. Meanwhile it is sensible to hypothesise that he must be working in either triangle trigonometry or circle trigonometry. In triangle trigonometry, the ratio of sides of a right angled triangle will not exceed 1. In circle trigonometry, in particular graphical trigonometry we can see that it is bounded between 1 and -1.

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

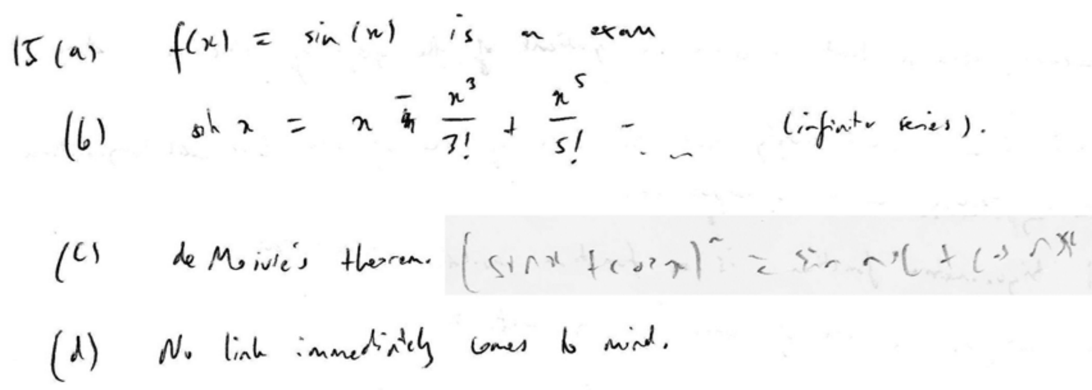
(a) Function

(b) Series

(c) Complex number

(d) y=mx

He wrote


15 (a) $f(x) = \sin(x)$ is an exam

(b) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ (infinite series).

(c) de Moivre's theorem. $(\sin x + \cos x)^n = \sin nx + \cos nx$

(d) No link immediately comes to mind.

KE: Can you explain or give the De Moivre's theorem?

ST3: Ermmmmm… that's one with the sin x + cos x to the power of n, I can't remember the exact statement in the theorem right now… I think is that (he was writing it on his answer script)… I can't remember the exact wording… it's something like that… isn't it… if you ask me earlier of the day, I probably remember…

It was interesting to note that ST3 couldn't give the correct De Moivre's theorem in the follow-up interview and he wasn't sure whether he gave the correct answer or not. He only showed an associative link between sine and complex number by relating it to the De Moivre's theorem. He knows some of the ideas in analytic trigonometry but he doesn't grasp them.

6.4.1 Summary of ST3.

ST3 is a student teacher with a first class degree in mathematics and he is graduated from a British university. He referred sine x as a trigonometric function and as the length of the opposite side in the triangle divided by the length of the hypotenuse. His concept image of sine x was clearly in the triangle trigonometry and circle trigonometry. He made sense of trigonometry through *perception*, *operation* and *reason*. For instance, he made sense of $\sin 30°$ through *perception* (looking at a right angled triangle with $\sin 30°$ in it) and *operation* (cutting a equilateral triangle into half and operates the sine definition to get the answer). This shows he uses different combinations of *perception*, *operation* and *reason* to make sense of trigonometry.

He did work in different contexts of trigonometry in order to make sense of the mathematics items. For instance, he used graphical trigonometry to approximate the values of $\sin 110°, \sin 250°, \sin 335°$. When he was asked to describe of $\sin 30°$, he was working in triangle trigonometry. From the data, he clearly aware that he could only see the triangle stuff in the first quadrant of the sine graph. When the angle is over $90°$ then he will just use graphical trigonometry. This indicates that he is aware of himself working in different contexts of trigonometry.

He has problems in visualizing the triangle with $\sin 200°$ and $\sin 270°$ which is due to the changes of meanings related to the difference between Euclidean and modern Cartesian views. When he was asked to describe $\tan 90°$, he said this was undefined because he can't get a right angled triangle with two right angles inside the triangle. In this case, triangle trigonometry becomes a problematic conception for him. On the other hand, he has a supportive conception for radians and he thinks that radians will always works in trigonometry.

ST3 knows the idea of radians but he doesn't grasp it in the sense that he is able to describe the idea of radians but he couldn't offer a reason why radians are needed for advanced level mathematics. He knows radians will always works in trigonometry. Meanwhile he is able to link the triangle trigonometry and the circle trigonometry in a coherent way.

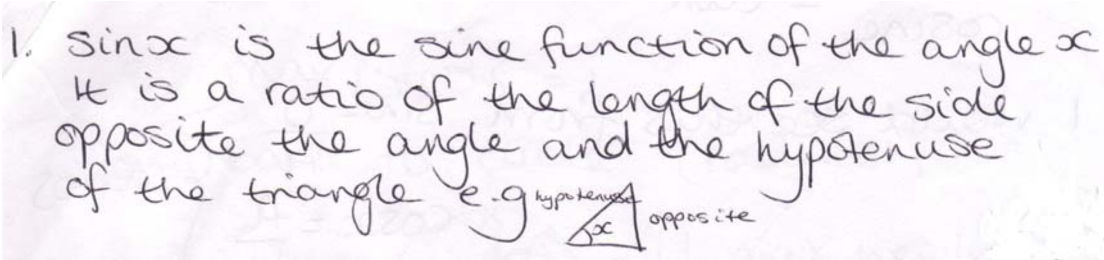
ST3 links the perceptual ideas with mathematical symbols in a flexible way which provides him the power of making sense of sophisticated ideas. For instance, he links the idea of gradient to the mathematical symbol dy/dx in particular dy/dx [sin x] so that he can sense this gradient function is cosine x. In the context of analytic trigonometry, it would appear that ST3 has some flaws in his knowledge structure. He knew there was a link between sine and complex number by saying De Moivre's theorem but he did not offer a correct De Moivre's theorem during the interview. This clearly was an associative link. He did write down the correct series for sine. This shows that he knows some of the ideas in analytic trigonometry but he doesn't grasp all of them such as the De Moivre's theorem.

In general, ST3 used different combination of *perception*, *operation* and *reason* to make sense of trigonometry. He can link triangle trigonometry and circle trigonometry in a coherent and flexible way. ST3 could grasp some of the ideas in a powerful way by linking the perceptual ideas to the mathematical symbols such as the notion of gradient with dy/dx. There are certain concepts that he doesn't grasp them such as the concept of radians and De Moivre's theorem and yet he still has a supportive conception on radians. His problematic conception is related to the transition from Euclidean geometry to modern Cartesian.

6.5 The story of student teacher ST4.

ST4 is female student teacher who has no teaching experience.

Describe sin x in your own words.


1. Sinx is the sine function of the angle x
It is a ratio of the length of the side
opposite the angle and the hypotenuse
of the triangle e.g

> ST4: Sin x is the sine function of the angle x. It is a ratio of the length of the side opposite the angle and the hypotenuse of the triangle. I've drew a little picture of the triangle (pointing to her answer for item 1).
>
> KE: Is there anything that you want to add?
>
> ST4: Obviously when you develop all the other things, there are lots of different things that come to mind so you can think of it not necessarily, think of the graph but may be that little thing (pointing to her answer script). The first thing is always the triangle for me.

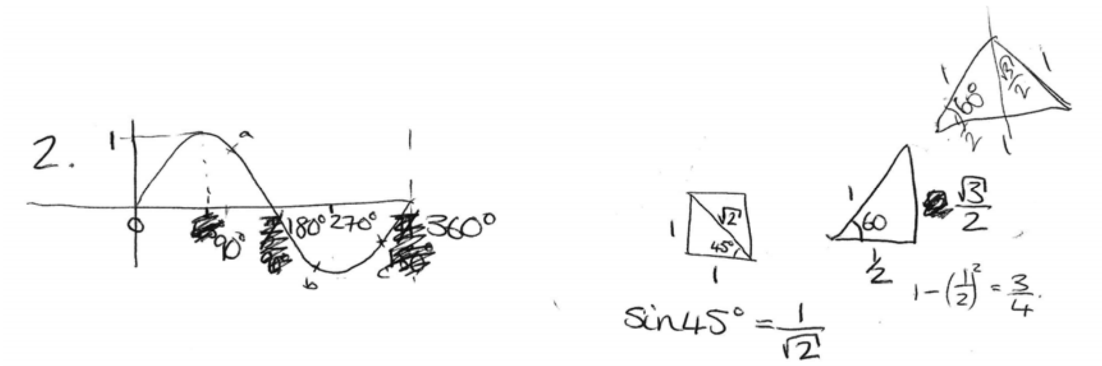
The evoked concept image of ST4 was in the context of triangle trigonometry and circle trigonometry. She could sense the complication in her cognitive structure when developing and thinking about sine.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $110°$

(b) $\sin 250°$

(c) $\sin 335°$



ST4: I put smallest is sin 250 and then sin (referring to her answer script)… but I've written the same thing twice (reading and correcting her answer script).Ok these are my points. So (a) I put there so sin110 I said was the largest because it's positive and (c) is in the middle which is the sin 335. I also remember that point there and then the lowest point is (b) (pointing to her paper) so I have done it mostly from the graph. […] I was trying to do it symmetrically… those points.

KE: Ok. Alright. What about these two figures (pointing to her answer script)?

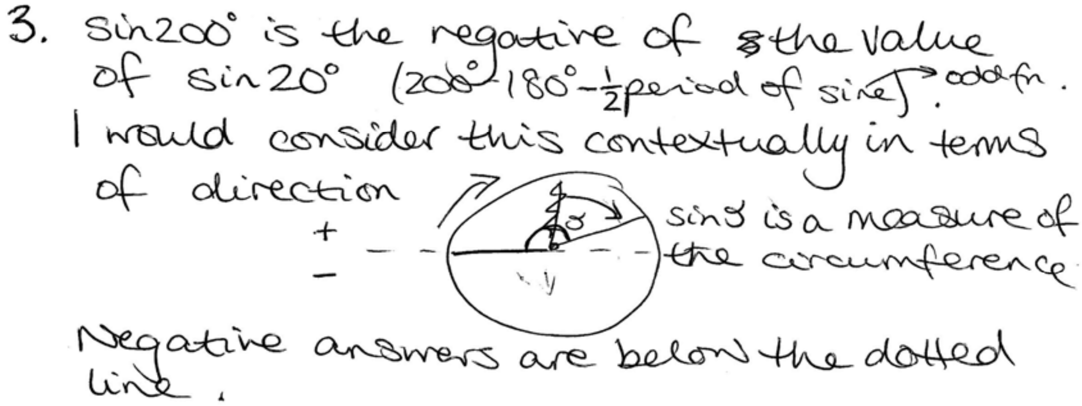
ST4: These were in order to draw the graphs (pointing to the figures besides the graph). I was just double checking to get all the points in the right place. […] I used the triangle to plot some of the key points on the graph if that makes sense. […] (c) and (b) are both negative so they must be smaller (b) is more negative because of 250 is closer to 270 degrees which is the lowest point of the graph so that's the symmetry again.

KE: […] do you see any relationship between your graph and your description in item 1?

ST4: Yeah… so that's the ratio of lengths of sides the reason I come to lengths of sides it's the fact ratio and sides to begin with because I use them to do everything so here I've used them to draw the graph but there is no maybe direct relation in my mind to the graph.

ST4 got the correct answer. She used the sine graph to get her answer and made sense of it through *perception*. She was using *graphical trigonometry* and referred back to triangle trigonometry to confirm some values. It is interesting to note that she was referring to the triangle trigonometry instead of referring to the unit circle when double checking the points on the sine graph. The advantage of using the unit circle is the ability to check all the points on the sine graph whereas the triangle trigonometry which works in the Euclidean geometry only allows learner to check the points between the angle of $0°$ and the angle less than $90°$. She is blending ideas together in a flexible way. It should be noted that she was constantly shifting between triangle trigonometry and circle trigonometry to build a sine graph that make sense for herself. This indicated that she was giving a more coherent S2 response linked to her knowledge of the visual information.

How do you make sense of $\sin 200°$?



ST4: Erm… sin 200 degrees is the negative of the value of sin 20 degrees… erm… because 200 degrees minus 180 degrees is half period of sine that’s an odd function. I would consider this contextually in terms of direction. Sine theta is a measure of

the circumference so I mean their height… I think… negative answers would be below the dotted line so what I am saying there is 200 degrees would be more than 180… the value of the height would end up going downwards so that would be negative… I think that's what I mean by that…… so because it's half of the period of sine it's not going to give you the same value… erm… that's an odd function… so that should give you the negative […] so what I was trying to get at there was similar to this that I just did that value of the height when I said circumference, I was thinking when you go round you get the sine wave… so… what I was probably meaning was the y coordinate… erm… that was just a way to try and say that 200 degrees would be around here somewhere that would then be a negative value of y […]

KE: I am trying to understand this figure. So what about this one (pointing to her figure for item 3)? You are trying to rotate from here to here, is it (pointing to her figure for item 3)?

ST4: Yeap… so I've drawn the arrow that way but I did mean that way [gesturing anti-clockwise with her hand]… I assume that I was trying to do… so if we go around that way… maybe I was going that way trying to… because it would still be negative at a time maybe I was doing from there (pointing at her answer) so starting maybe at 0 there (pointing at the positive x axis) rather than here (pointing at the negative x axis) where I would in that case […] obviously I drew it the other way round.

KE: Ok… erm… so can you visualise this triangle with sin200 degrees?

ST4: No… because it's needs to be a right angle triangle […]

KE: Can you draw this triangle?

ST4: No.

KE: Do you see any relationship between the definition of sine and the sine graph?

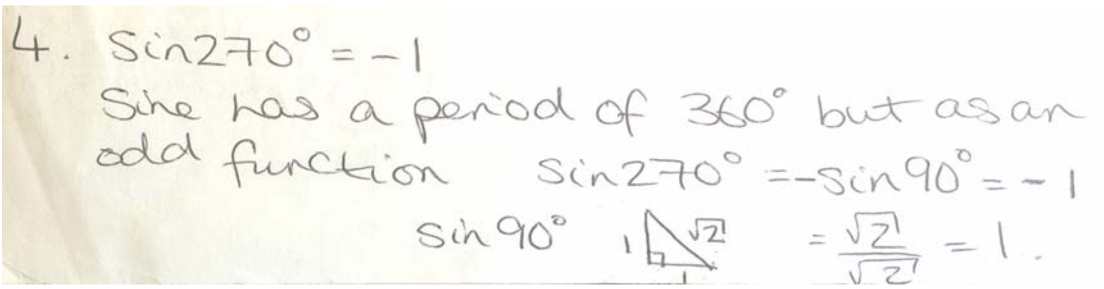
ST4: Between mine? Between what I've put there?

KE: Yup.

ST4: Not between… well for the first part I would say you've got a ratio of lengths, you can get up to almost 90 degrees I suppose from that sort of thinking… erm… but you would have to then extrapolate for the rest of it… so this when you get to larger function I'd use… erm… things like… it's an odd function and period of two pi but not really that closely linked.

She was clearly working in the unit circle through her *perception.* She operated the unit circle and observed the changes of the vertical length as sine theta varied. This was essentially embodied compression. ST4 gave a S2 response by saying that she should have rotate the radius of the unit circle the other way round. ST2 couldn't visualize this triangle and she has a problematic conception due to the reason that the sum of internal angle of a right angle triangle must be 180 degrees. This clearly shows that she was working in Euclidean geometry. It was obvious that she could sense the changing of context from triangle trigonometry to circle trigonometry when the angle get up to 90 degrees. She related the visual picture and the symbol in a meaningful way.

What is the value of $\sin 270°$**? Explain why** $\sin 270°$ **has this value.**



ST4: Sin of 270 degrees is equal to minus 1. Sine has a period of 360 degrees but is an odd function so sine of 270 degrees is equal to minus sine of 90 degrees which equals to minus 1…erm…sine 90 degrees, I've done a little triangle for opposite it would be root two because of 1 and 1 triangle… erm… the hypotenuse is actually the same value because I've chosen the right angle to do that on […]

KE: Basically, can you visualize this triangle with sin 270 degrees?

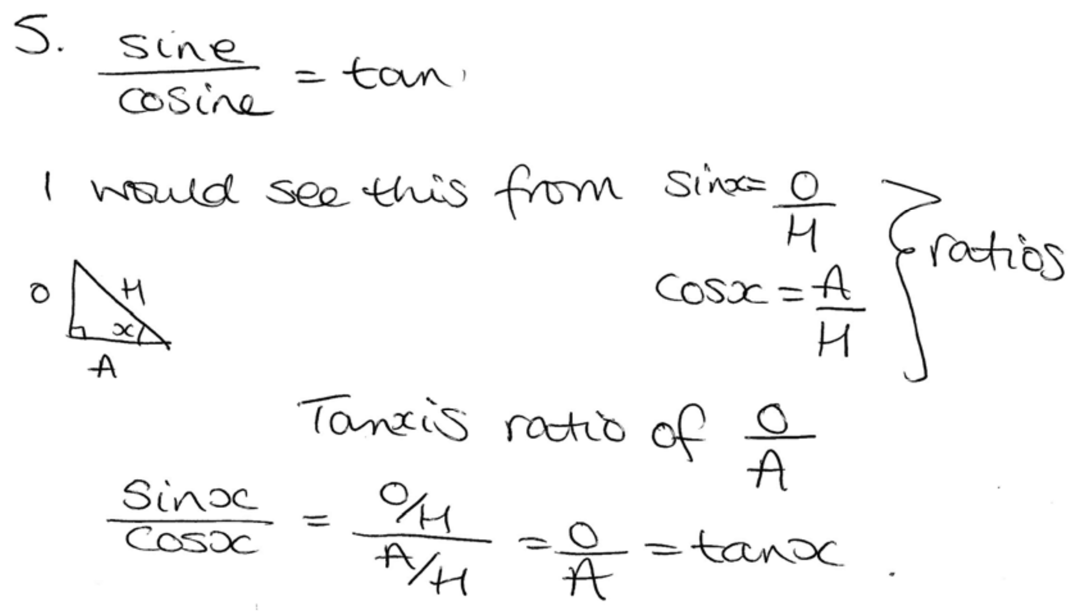
ST4: No.

KE: Can you draw the triangle?

ST4: No.

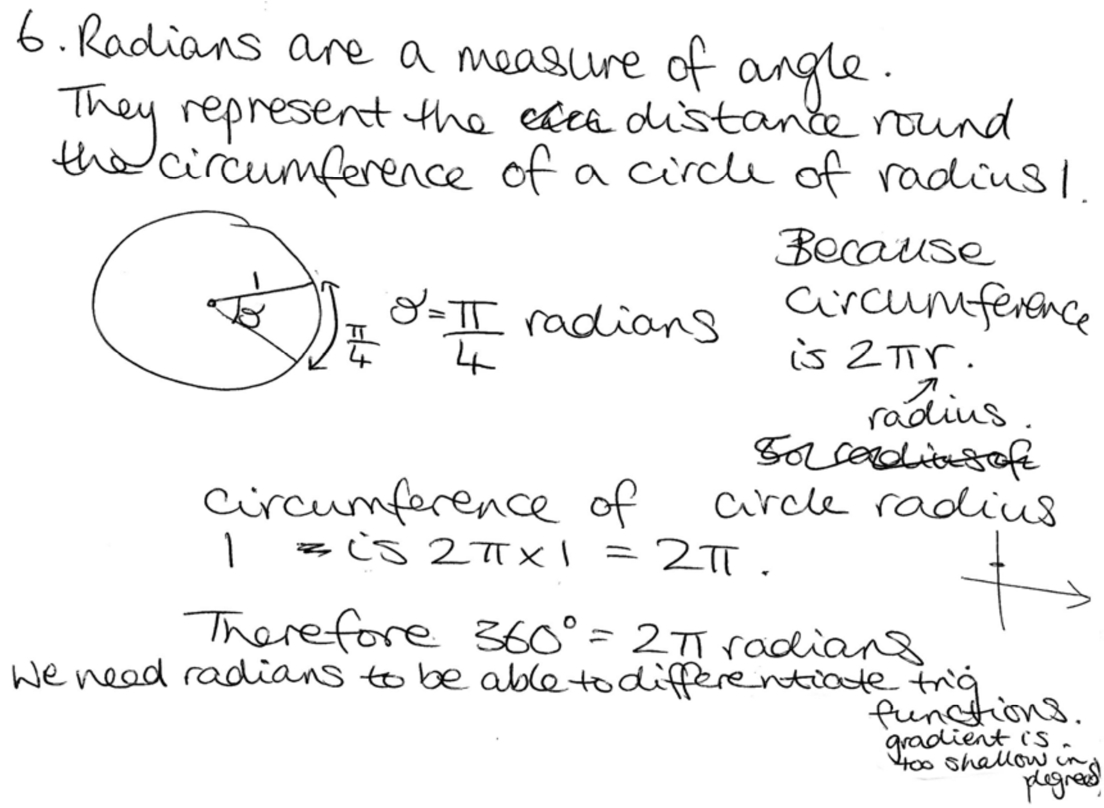
Superficially she was working in the circle trigonometry through *perception* and she did refer to the triangle trigonometry to find the value of $\sin 90°$ by operating the definition of sine. Due to the same problematic conception as mentioned above she couldn't imagine this triangle.

What is sine over cosine? Does that mean anything?



ST4 made sense of her answer through *operation* and *reason.* She operated the definition of sine and cosine then she related the answer of her operation to the tangent definition. She obviously was working in the triangle trigonometry.

What do Radians mean? Why do we need radians when we have degrees?



ST4: Radians are a measure of angle. They represent the distance round the circumference of a circle of radius 1 and because the circumference is 2 pi r so 2 pi times radius the circumference of the circle of radius 1 is 2 pi times 1 which is 2 pi therefore 360 degrees what we would normally say is all the way round is 2 pi radians. We need radians to be able to differentiate trig […] that's the circle of radius 1 so if we had an angle there and whatever it was in degrees we would know what it was in radians by the arc length (referring her picture in item 6) […]

KE: Maybe we can talk about this bit (pointing her answer for item 6). We need radians to able to differentiate trig (reading her answer for item 6). Were you trying to say that we only able to use radians to differentiate trig?

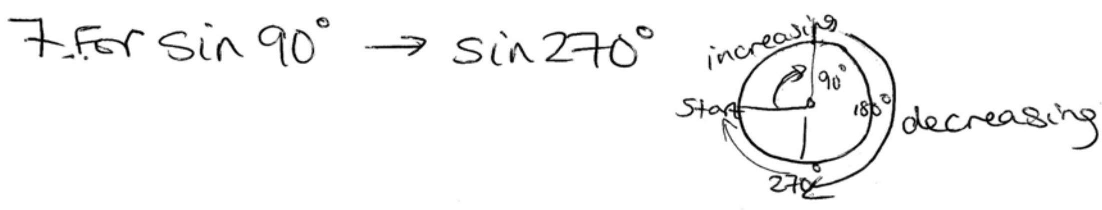
ST4: No… erm… well we usually use radians, I think because it is easier, you could do it in degrees, I don't quite know why radians confirm nicely…you could do it in degrees but you have to put in your factors of 360 they have all come out in this harder calculation…I think this is what I am going for […]

KE: Ok. So what about this bit (pointing her answer for item 6)? Gradient is too shallow in degrees (reading her answer for item 6)?

ST4: […] we need radians to differentiate… gradient is too shallow in decreasing degrees… oh no… hang on… in degrees… that's the other question… so what I mean by that is… erm… I guess that's to do with the scale factor thing in the end… erm… if you do it on autograph or something… you can do the plot sine function in degrees then the graph is actually quite shallow because you have gone 360 that way and one upwards so I think that's where is struggles for that my immediate thought was to do with that.

ST4 embodied the relationship of radians and the circumference of circle and made sense of it through perception. She sensed the reason for using radians in differentiation through perception by working in the context of circle trigonometry. In this case, she could see the sine graph will be quite 'shallow' if plotting the sine graph by using degrees instead of using radians. Apparently she grasps the idea of using radians. We might hypothesize that she was relating the dynamic gradient to the outlook of the sine graph with degrees thus she could see the reason of adding a factor if she uses degrees for differentiation.

For what values is sin x decreasing? Explain why it is decreasing for these values?

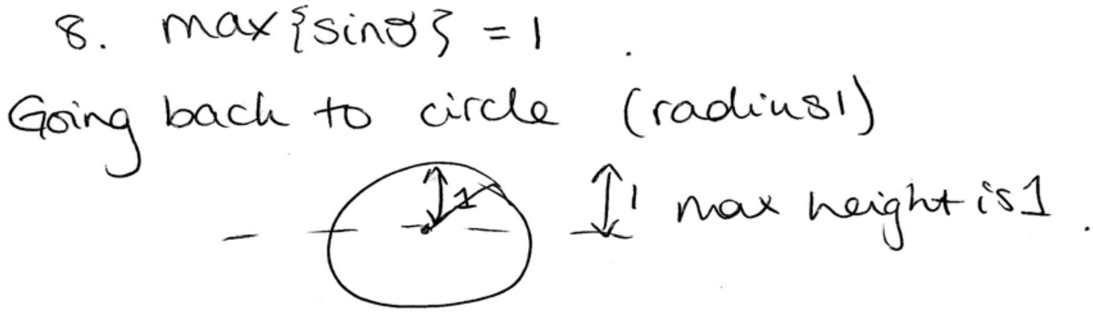


ST4: On here, I've started from this side going clockwise again… erm… so it's increasing because when we go here the

height will be going up (inaudible)… if we start from this point if we go round we are increasing the y value then on that part of the circle you get to 90 degrees so then it starts going down so once we've reached the bottom at 270 degrees then it starts going up again…

She operated the unit circle and focused on the changes of y value when she varied the angle. This was essentially making sense through perception in the context of circle trigonometry. As we can see from her answer, she only gave the answer for the first cycle only ($0°$ to $360°$) because that was pretty obvious for the unit circle. On the other hand, the properties of graphical trigonometry (the sine graph) such as periodicity and symmetry might probably easier to trigger her awareness of other answers (i.e. answers over $360°$). In this case, she may not have referred to the graphical trigonometry.

Explain why $\sin\theta$ can never equal 2.



ST4: So that's again I think… the reason I was using the circle in the first place is… yeap… so I sort of assumed that it's true that the circle gives you the sine wave… and then said the maximum value of that would be 1 […]

KE: Ok… let say if the radius of this is 2 (pointing to her unit circle in item 8), do you think is that any possibility that the maximum value will be 2?

ST4: Well not as sine… erm… well if that was 2… erm… so I think the reason I used the unit circle is because it gives those values for sine… erm… so I would probably justifying it more going back to the triangle as you do…… so…… opposite…… if you…… the biggest value you are going to get… because the hypotenuse is always longer than the opposite and the hypotenuse is always the longest… erm… so you are going to get the maximum value of sine… when the hypotenuse is or the opposite is as close to the hypotenuse as possible… so you want to shrink that triangle in… from that point which would cause that angle to get bigger… erm… and as you do that…that tends towards 1 because those two values are getting closer and closer… but that triangle doesn't actually exist…

KE: Ok…so basically are you trying to say that if let say this radius is 2 so you don't feel you can get a value over than 1? Or Is there any possibility that the maximum value of sine theta can equal to 2?

ST4: No… No… so erm… if you look at the ratio… because the hypotenuse is always bigger… erm… the maximum that can only tend towards possibly be as 1… it couldn't be 2… because the opposite it's shorter than the hypotenuse.

Initially ST4 was working in the unit circle to make sense of why sine theta can never equal 2 through *perception* and she wrote maximum of sine theta is 1. Then she referred back to triangle trigonometry to explain why she thought that sine theta can never equal 2 (assuming the radius of the unit circle has changed to 2). She has supportive conception with problematic aspect when linking the triangle trigonometry to circle trigonometry especially in the case when the hypotenuse coincides with the opposite of the right angled triangle in the unit circle. Based on the above excerpt, she clearly thought that the opposite side of the right angled triangle will never coincide with the hypotenuse but it will only get very close to the hypotenuse as she varies the angle. As a consequence, she thought that the maximum value for sine is very close to 1. This problematic conception is related to the difference in context between triangle trigonometry and circle trigonometry which arises due to the changes of meaning between Euclidean geometry and modern Cartesian. In triangle trigonometry, the opposite side of a right angled triangle will not coincide with its hypotenuse. On the other hand, the opposite side of a right angled triangle will coincide with its hypotenuse (or radius of the unit circle) at the boundaries of the four quadrants.

What does dy/dx mean?

> ST4: Dy by dx means differentiate y with respect to x and then I've put gradient next to it.

She saw dy by dx as a *procept* which acted like an operator (differentiate) and a concept (gradient). This was making sense through *operation* and *perception.*

What does dy/dx [sin x] mean? What is d/dx [sin x]? Explain why.

> ST4: Dy by dx of sin x means differentiate sin x with respect to x…gradient of sine graphs in x… not sure what that means…basically the gradient of the sine graph… d by dx of sin x is equal to cos x… not sure why both have the same maximum and minimum and involve ratios of lengths that depend on each other […] when it came to explaining I had a bit of a look at that (referring to the sine curve and cosine curve) to see if I could work out why cos x might come out of looking at multiple gradients as we round so the only way I could think of… maximum and minimum in term of gradient at 90 degrees for sin is horizontal so 0… so the 0 point would be at 90 on cos as oppose to up there so I could plot some points but I don't know exactly why […] so if you have got a triangle again… erm… so erm…(drawing)… so obviously you have got your opposite, hypotenuse and adjacent… so sine of theta is going to be equal to… and…erm… cos of theta is adjacent over hypotenuse if that was a specific angle you can use the same triangle or you would have to use the same sort of ratios so that… erm… if you fix for example O and H to get a value for sine theta then A is

determined by those because you couldn't have any other value of that so that depends on each other [...] there is a relationship there... but I am not sure exactly why some of the middle values would necessarily be so... I could do... erm...(writing) so pi by 2 to 90 I know that (drawing on a piece of paper)... that is the gradient there... which will give me cos so that ends up translating to 0 on that graph that is cos...that is sine (pointing to her drawing)... and then again at the bottom there would be a 0 and 270 and here... it gets a bit difficult... but it's the in between values that I couldn't tell you why it is curved in between again... necessarily compare to.

KE: Which means you feel you don't know why you differentiate sin and you get cos but you have a sense that they are related?

ST4: Yeap... I know that they are related but I couldn't explain why that turned out to be exactly cos.

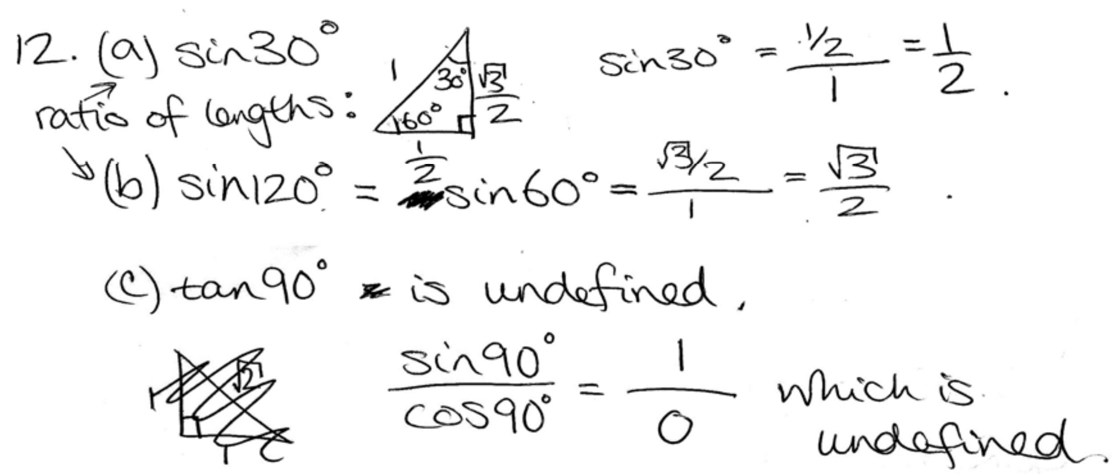
She could sense the relationship of sine and cosine by working in the context of triangle trigonometry and circle trigonometry through *perception* and aspects of symbolic computation (e.g. ratios calculated as numbers). Through the world of embodiment, she could see the relationship of sine and cosine at some specific instances but she still needs a technical way to make sure that the derivative of sine is cosine. She knows that the derivative of sin x is cos x but she doesn't grasp it. It can be noticed that the world of embodiment can give learners a sense of the possible relationship but they still need to work in the world of symbolism and formalism to get the derivative of sine. Reason can involve natural reasoning based on perception or action (in terms of manipulating objects or symbols) or axiomatic formal reasoning based on set-theoretic definitions and formal proof.

Describe as fully as possible what you understood by the following terms:

(a) $\sin 30°$

(b) $\sin 120°$

(c) $\tan 90°$



ST4: So (a) sin 30 degrees I've put ratio of lengths… I've drawn little triangle that's my 60 triangle from earlier from casting an equilateral triangle… the other angle be 30 because that is a right angle triangle so again a ratio of lengths half over 1 the opposite over hypotenuse which gives you the half so I am still using that triangle and (b) sin 120 degrees I've put that equal to sin 60 degrees because I would have looked at the graph and have gone on… so 120 is about there and that's symmetric about the middle so I would have taken…… oh ya… so it's the same distance away from the 90 to 120 as it would be to 60 because there is 30 difference between them so I am saying it's symmetry there […] (c) tan 90 degrees is undefined so I had a little bit of a look at the triangle… erm… but tan is opposite over adjacent and although it's sort of work earlier when I did it with sine… and the opposite over adjacent actually gives you a value of sine so I think maybe you can't use the right angle in a right angle triangle to prove it… haha… that's all I can conclude from that… haha… but then I went to sine of 90 degrees which I know is 1 over cos of 90 degrees which I know is 0 and you can't divide by 0.

KE: Alright. In fact, you can't really imagine this tan 90 degrees in a triangle then after that you switch to the sine 90 and cos 90 to make sense of it? You know sin 90 is 1 and cos 90 is 0 so you just…

ST4: Yeah… because I think to have… if you assume you can't use that right angle so you would have to have a right angle there and you'd have no end to them (pointing to her drawing, see Figure 6.6)… does that make sense?… it would be like infinitely.

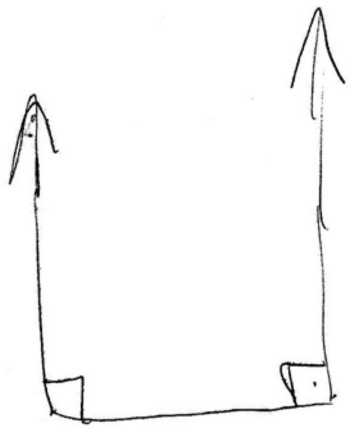


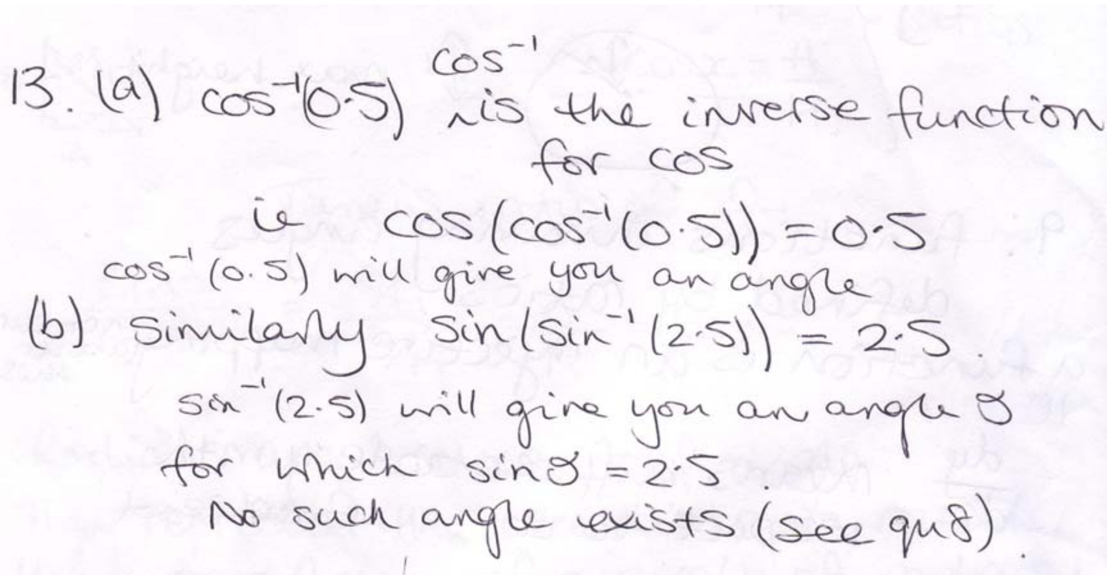
Figure 6.6: Sketch of $\tan 90°$ in triangle trigonometry by ST4.

Initially ST4 made sense of $\sin 30°$ through *perception* by looking at her right-angled triangle with a $30°$ in it then she operated the definition of sine in to get the value of $\sin 30°$. Apparently she was working in triangle trigonometry. For the case of $\sin 120°$, initially she made sense of it through perception in the circle trigonometry then she moved to triangle trigonometry to get the answer through operation of the definition of sine. She realised that she couldn't draw a right angled triangle with $\sin 120°$ in Euclidean geometry therefore she looked for an equivalent angle which was $\sin 60°$ by using the sine graph so that she could construct the right angled triangle in Euclidean geometry. Once again she had used the same right angled triangle with a $60°$ in it and operated the definition of sine to get the answer. For $\tan 90°$, she made sense of it through operation. Initially she operated the definition of tangent and realized that it was undefined because no number can be divided by 0. Then she made sense of $\tan 90°$ by trying to construct a right angle triangle in triangle trigonometry. This was problematic because she couldn't construct this right angled triangle (see Figure 6.6) due to the reason that she was working in the Euclidean geometry.

Explain your interpretation of the following terms

(a) $\cos^{-1} 0.5$

(b) $\sin^{-1} 2.5$


13. (a) $\cos^{-1}(0.5)$ $\cos^{-1}$ is the inverse function for cos

ie $\cos(\cos^{-1}(0.5)) = 0.5$

$\cos^{-1}(0.5)$ will give you an angle

(b) Similarly $\sin(\sin^{-1}(2.5)) = 2.5$.

$\sin^{-1}(2.5)$ will give you an angle θ for which $\sin\theta = 2.5$

No such angle exists (see qu8).

She knows that $\cos^{-1} 0.5$ is the inverse function of cosine then she operates this mathematical symbol by taking cos on it. From here she reasons that $\cos^{-1} 0.5$ is an angle. She is making sense through *operation* and *reason*. Similarly she sees $\sin^{-1} 2.5$ as an angle and she reasons that this angle doesn't exist because the maximum value of sine is 1. She might be working in circle trigonometry (thinking $\sin^{-1}$ and $\cos^{-1}$ as functions) at first then she related them back to the triangle trigonometry (thinking as an angle in a right angled triangle).

For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

(a) Function

(b) Series

(c) Complex number

(d) y=mx

15.

(a) function

sine is a function of the values of angles.

(b) series

sine (of a value) can be written as a Maclauren series (infinite series).

(c) complex number

$\cos\theta + i\sin\theta = 1$ → can write sine in terms of e and i.

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ De Moivre's theorem.

(d) $y = mx$ is a function

as far as I can think it has no close link to sine.

ST4: Series… sine of a value can be written as a Mclaurin series which is the infinite series.

KE: Do you know this series?

ST4: I think what I mean by that was Taylor series or one of the two… this is what I am thinking of and I am pretty sure that is Taylor series not the Mclaurin series (she have written the series in her first concept map prior the follow-up interview) […]

KE: So what about complex number?

ST4: So a complex number: I've put cos theta plus i sin theta equals 1… and you can write sine in terms of e and i, so I think it's like theta…… oh ya!… hang on… it's like e to the… something like this to pi i, there is some two's and stuff involved over there… something like this… it's like e to the i to the e minus sign (inaudible)… to the something that's why I didn't write this down!!! Because I couldn't remember it. This one said de Moivre's theorem and again I put that on there, it's just something that comes to mind when I think of complex numbers and trig, so cos theta plus i sin theta to the power of n, it's the same as cos n theta plus i sin n theta… erm… which you can use to work out…

She knows most of the ideas in analytic trigonometry. For instance she knows that it is possible to write sine in terms of *e* and *i* but she couldn't remember it. She might be trying to recall $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ during the follow-up interview which she didn't get it at that time. Obviously that is just an associative link. She did state the correct Taylor series in her first concept map prior the follow-up interview. She has also written the correct De Moivre's theorem. It seems like she remembers the concepts in analytic trigonometry as a different kind of representations to the concept of sine only. Obviously ST4 didn't grasp the idea of complex numbers.

6.5.1 Summary of ST4.

ST4 is a female student teacher with a 2(ii) mathematics degree and she is graduated from a reputable British university. Her evoked concept image of sine is in the context of triangle trigonometry. She makes sense of trigonometry through *perception*, *operation* and *reason*. She does blend the perceptual ideas with mathematics symbols in a meaningful way. For instance, she talked about the problem of using degrees in differentiation because the sine graph will be too 'shallow' and this will affect the dynamic gradient therefore a factor need to be added. Obviously she works in triangle trigonometry when the angle is less than $90°$. On the other hand, when the angle is greater than $90°$, she will work in circle trigonometry and constantly refer back to the triangle trigonometry to derive and confirm certain mathematics facts. She could move flexibly between triangle trigonometry and circle trigonometry. For instance ST4 was using graphical trigonometry to arrange the angles of sine given and she referred to the triangle trigonometry to confirm some of the values in the sine graph.

ST4's problematic conceptions arise when the angle involved is greater or equal to $90°$. This is related to the difference between Euclidean and modern Cartesian views. For instance, she drew a weird figure for $\tan 90°$ in order to justify why $\tan 90°$ was undefined and she couldn't visualize triangles with $\sin 200°$ and $\sin 270°$. Additionally, she also thought that the opposite side of the dynamic right angled triangle could only get very close to the hypotenuse but will not coincides with it. This was due to the fact that she was working in the Euclidean geometry. In this case, triangle trigonometry becomes a problematic conception in circle trigonometry when the angle involved is greater or equal to $90°$. Apparently she has developed the coherent links between unit circle and the sine graph. This is obvious when she is able to justify the properties of sine graph. For instance, she could explain why $\sin 270° = -1$ and why $\sin\theta$ can never equal 2 by linking the unit circle and triangle trigonometry.

ST4 did grasp the idea of using radians in calculus and sensed that the sine graph is very shallow if it was constructed by using degrees. She can conceive derivative as a *procept* (process and concept). She could see that at some specific angles (maximum and minimum of the graph), the derivative of sine is equal to cosine but in general she isn't sure why the derivative of sine is exactly equal to cosine especially for those angles which are in between the maximum and minimum of the graph. This might be due to the fact that she does not relate it to the formal definition of differentiation. This shows she doesn't grasp calculus in certain extent. The world of embodiment (perceptual) can give us a sense on how the gradient of a graph changes. We still need a technical way (i.e. the world of symbolism and formalism) to compute and make sure how exactly the gradient changes.

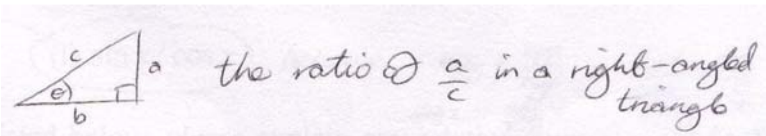
ST4 could remember some of the important facts in analytic trigonometry but she doesn't grasp them. For instance, she couldn't state the equation relating *e, i* to the concept of sine. Apparently she knows complex numbers and Taylor series as a kind of representation for sine. Meanwhile there is no clear evidence that she can link the concepts in analytic trigonometry and circle trigonometry.

In general, ST4 makes sense of trigonometry through perception, operation and reason. She possesses strong perceptual ideas in trigonometry. For instance, she could imagine the rotating unit circle to make sense the effect on y value. Furthermore, she could imagine the difference of outlook between a sine graph constructed with radians and degrees. Meanwhile, she possesses coherent links between triangle trigonometry and circle trigonometry with some problematic aspects in particular conceiving right angled triangle when the angle involved is greater or equal to $90°$. Her problems involve the shifting between Euclidean geometry and modern Cartesian. She doesn't grasp the concepts in analytic trigonometry.

6.6 The story of student teacher ST5.

ST5 is a female student teacher and she has no teaching experience.

Describe sin x in your own words.



The evoked concept image of ST5 was in the context of triangle trigonometry. She saw sin x as a ratio of sides of a right angled triangle.

Please arrange the following values of sine in ascending order and explain your answer.

(a) $\sin 110°$

(b) $\sin 250°$

(c) $\sin 335°$

ST5: Well… ya… I drew the graph and if you know where the important points are, the turning points… you know that it crosses at this point and then it crosses and then it turns at this point… and you can say… erm… so 110 is… first of all it's going to be positive, you know it's going to be there and so to work out the others… erm… you can say 335 degrees is closer to 360 degrees then it's only 25 away whereas 250 degrees is a while away from that… so you can say that this one is gonna be closer to the x axis so it's bigger then.

KE: So you approximated the location of the points on the sine graph?

ST5: Yes… because you know it's gonna be symmetrical about that bit so it's closer to the turning point. It's going to be lower down on that for that value.

KE: Alright. What is the relationship between your definition or description in item 1 to the sine curve?

ST5: Yeah… that's the thing because you don't actually do triangles of like… because whenever you draw a triangle like that… you always draw like an acute triangle… you know an acute angle here but actually you know they are big angles so in general… I don't know… I think of it… I suppose you could think of it as a really big like triangle but then you can't get a right angled triangle. So I suppose it is that but moved along so I think of this bit here like the bit between 0 and 90 degrees… erm… and then I just extrapolate it for the other values so I think of… yeah but it's not immediately I can see it's not immediately kind of accessible because these are such large values.

ST5 got the correct answer for the above item through *perception* in the context of circle trigonometry. She reasons by visual perception and links to numerical quantities. In fact she uses graphical trigonometry without relating to the unit circle. She uses the properties of the sine graph such as visual symmetry to approximate the values. It should be noted that she could sense the problematic aspect in drawing a triangle for an angle which is bigger than an acute angle therefore she switched to graphical trigonometry. She realized she could not draw a triangle when the total internal angle is bigger than $180°$ hence when she thought of a triangle with big angle, this triangle will not be a right angled triangle. This problematic conception is related to the difference between Euclidean geometry and modern Cartesian views. The above excerpt also shows the existence of two distinct contexts in trigonometry which are the triangle trigonometry and graphical trigonometry. For instance, she immediately uses the sine graph when the angle involved is greater than $90°$. She does not exhibit the link between triangle trigonometry and circle trigonometry in this particular instance.

How do you make sense of $\sin 200°$?

> ST5: I think of it as sin 180 plus 20 so it's 20 away from sin 180 degrees which is the point where it crosses.
>
> KE: Ok. So what come into your mind was the sine graph when you are asked to make sense of sin 200, was it?
>
> ST5: Hmmm…where are the turning points… so is it bigger than… its like when you are looking at angles and you've got… like I was doing bearings with my kids and so you could think of the compass points you know so you think… well this is 90 and this is 180 (referring to the picture that she had drawn) and stuff so if someone said what is 110 degrees or something then you think ok so it's bigger than 90 and it's smaller than 180 so it's got to be in this quadrant and it's closer to 90 so it's approximately about here… erm… so I can kind of think of that

like that I think you can think of well it's bigger than 180 so it must be below the axis but it's smaller than 270 so it's not yet got to the turning point and is it closer to 180 or is it closer to 270 (referring to her sine curve).

KE: Can you visualize a triangle with sine 200 degrees?

ST5: No.

KE: So can you draw a triangle with this angle?

ST5: 200 degrees??

KE: Sin 200 degrees.

ST5: You can't because it's too big and so you'd get, it's a reflex angle and it doesn't make sense… because you know infinite… you know if you were on the curve space and the lines came back to meet you again or something but no… I can't.

KE: Ok, so for example like just now…seems like you are using the properties of the sine curve to make sense of a lot big angles?

ST5: Yeah.

KE: And then just now you mentioned about symmetry because you've used the symmetric property of the graph and then to approximate those values so basically how do you make sense of this symmetry? I mean why it is symmetry?

ST5: Well… you can do this I mean rolling a cylinder as well… if you've got a point on it like say you start at the bottom but if you then mapped that (referring to her rolling cylinder that she had drawn)… erm… so this would give you a sine curve so if you do this and map the height of it then this would go… you know up and then at this point like half way you have a thing like this and your point would have reached the top so suddenly you are here… erm… and then you start going down again because like… you know… you are going up it… so that's why it is kind of even so that's why it's not squashed… you know one way or the other so you got that symmetry.

She made sense of $\sin 200°$ through *perception* in the context of circle trigonometry. In fact, she was using the properties of graphical trigonometry such as visual symmetry to make sense of $\sin 200°$. She has problematic conceptions in thinking of right angled triangle with big angles because she was thinking of right angled triangle predominantly in Euclidean geometry. She links her experience (imagining a rolling cylinder) to the construction of sine curve hence she could justify the properties of sine curve such as symmetry. She was blending her perceptual ideas and the mathematical symbol in a meaningful way.

What is the value of $\sin 270°$? Explain why $\sin 270°$ has this value.

> ST5: I said minus 1 because it's three quarters of a cycle.
>
> KE: You always imagine a rolling cylinder on a surface?
>
> ST5: Erm… I don't always imagine a cylinder, I tend to imagine the graph.
>
> KE: The graph.
>
> ST5: And I kind of know that derived from the rolling cylinder or whatever… but ya… I imagine a cycle of one single… sine cycle and then for if it's cos then it is slightly different but it's still one cycle and it's just it starts here instead it starts at 1 (referring to her sine curve).
>
> KE: Ok. Alright. Can you visualize this triangle with sin 270?
>
> ST5: No.
>
> KE: So you can't draw it as well?
>
> ST5: No. I can't really visualize anything bigger than like even… even obtuse angles I always think of acute angles of… you know triangles if it were big… if it were you know a triangle like that I'd just in my head turn it around and look at this acute angle rather than looking at that obtuse angle (she was referring to her non right angled triangle and rotating that triangle) because we are tuned by convention to have the base line on the bottom and so even… oh ya… no… the only thing I can visualize is acute angles to make these right angled triangle… make triangle in general not right angle… make it into acute angles.
>
> KE: So your explanation for this is because of 3 over 4 of circle?

ST5: Of the cycle, ya… the cycle through it… you are here and therefore you are down here (referring to her sine curve).

KE: Alright. Ok. You are constantly referring to the sine curve?

ST5: Ya.

ST5 always uses graphical trigonometry in making sense of trigonometry especially when the angle involves is greater than $90°$. It is clear that the conception of constructing right angle triangle with acute angles is problematic for ST5 to visualize any right triangle with angle greater than 90 degrees. Again, she was thinking of right-angled triangle in Euclidean geometry. She made sense of $\sin 270°$ through *perception* in the context of circle trigonometry, consisting a combination of the graph and imagining a rolling cylinder.

What do Radians mean? Why do we need radians when we have degrees?

ST5: Well… degrees is just a measure of turn and angle is a measure of turn so you can… you can measure in an arbitrary kind of amount… 360 is a bit of random number but the Babylonian you know decided that 360 was useful because you can divide it into lots of things and so for that purpose if you are just doing like I don't know… polygons (no that's not good because I was about to say 5… divided into 5 is 72.5 which is not good)… ya… if you take like angles and you just divide them into normalish things then you're talking about 30 of 60 or something which is perfectly fine but… but… because things like this… I don't know I think it is useful in talking about things in radians because then it's an obvious amounts of a whole rather than just an arbitrary number.

KE: What about the reason? Why do we need radians when we have degrees?

ST5: It's more exact like I said you can… erm… it's not like 30 degrees it isn't exact but I just… erm… I don't know I like pi. I grew up being a big geek so I memorized lots of digits of pi which of course is an approximation but… because you stop

> somewhere but I like the rational transcendental going on forever nature of pi so I like pi as a number anyway and then… ya… so it's a good measure of all the way around it […] if you have a cylinder which has a radius of 1 then then it's 2 pi all the way round so it's a good measure of general amounts of roundness, I don't know amount of turn.
>
> KE: So basically is more like… erm… you feel like you like pi. Is there anything you want to add for this one (referring to item 6)?
>
> ST5: I don't know… it's always… it's one of those questions like when people say what's a degree or are like what's an angle and you are like… it's this thing that I have always used but actually being able to describe what it is, is quite hard.

She could embody the relationship of an angle in a circle with the circumference of the circle by introducing radians therefore she felt radians are more practical than degrees. In fact, measurement is a blending of making sense through *perception* and *operation*. It blends space, shape and arithmetic into a kind measurement for our application in daily life. She knows the idea of radians and she is able to describe it. However she didn't relate radians to calculus therefore she didn't offer an explanation on why radians must be used in circle trigonometry in particular calculus. She didn't grasp the reason why radians are so important. Apparently she thought radians just another kind of measurement other than degrees.

For what values is sin x decreasing? Explain why it is decreasing for these values?

> ST5: Erm… it's decreasing from 90 degrees to 270 because it's between here and here so it goes down (referring to her sine curve).
>
> KE: So the first thing comes into your mind is the sine curve?
>
> ST5: Yeah.
>
> KE: Ok.

> ST5: I'd draw it because if you just said that to me like when is it decreasing, I'd have to draw it in order to see it…and what does it mean? It means… it's going down… if you had like this it'd mean it's on it's got to the top of the cylinder, you know the point when it's got to the top and now it's coming down but that doesn't explain what happens underneath but it's just it's, it's the down bit of the cycle.

ST5 knew sin x decreases from $90°$ to $270°$ by looking at the sine graph which is more elemental and not thinking of the derivative. This was making sense through *perception* of the graph in the context of circle trigonometry. It is evident that she only focused on the first period of the sine graph because her answer was correct only for the first period of the sine graph. She always thinks of the rolling cylinder as analogous to the dynamic visual representation of the unit circle. She blends the perceptual ideas with the symbolism in a meaningful way.

Explain why $\sin\theta$ can never equal 2.

> ST5: Erm… because sine theta is always between minus 1 and 1 by the definition of the function if you think of r equals to 1 as the unit circle, cylinder of radius 1 rolling on a surface you can't change size of the cylinder so it's always you can't suddenly have it above the point of 1 it would have to jump in the air.

She made sense of it through *perception* by using a real life analogy to the unit circle. In this case, she was working in circle trigonometry.

What does dy/dx mean?

> ST5: The derivative, ie the gradient of a curve.

What does dy/dx [sin x] mean? What is d/dx [sin x]? Explain why.

> ST5: So it's cos so d by dx of sin x is cos because it's the gradient at… like you could look at the gradient at every point but if you plot the gradient of every point on a graph it will end

up, it will be cos x and you can look at specifically things like, you know… the maximum the minimum and say you know the max… the gradient is going to be zero and the minimum is going to be zero so those are those points where it's zero on the cos graph… erm… and like at the origin where it's going like that, like gradient 1 you can plot it at 1.

KE: […] What would d sin x by dx mean?

ST5: It's just the derivative of it and the gradient of it…I mean the derivative, differentiate it… yeah.

ST5 sees the gradient of sine curve as cosine curve. She could imagine the dynamic gradient as the point varies on the sine curve and by plotting these gradients of every point then she will get a cosine graph. This is essentially making sense through *perception.* She also sees d sin x by dx as a *procept* i.e an operation (differentiate) to perform and a concept (gradient function). She compressed the gradient function of sine curve into cosine curve. In this particular instance, she did link her perceptual ideas to the mathematical symbols in a meaningful way. She was working in the circle trigonometry.

Describe as fully as possible what you understood by the following terms:

(a) $\sin 30°$

(b) $\sin 120°$

(c) $\tan 90°$

KE: […] what is your answer for sin 30 degrees?

ST5: Because these others are like the special triangle that you draw… like if this (referring to her triangle in item 12) is equilateral triangle you will have 60 on each point so if you cut it down the middle, you have 60, 30, 90 triangle which is going to have 1, 2 because that was an equilateral triangle so this whole length would be 2 so you will get 1 which gives you root 3

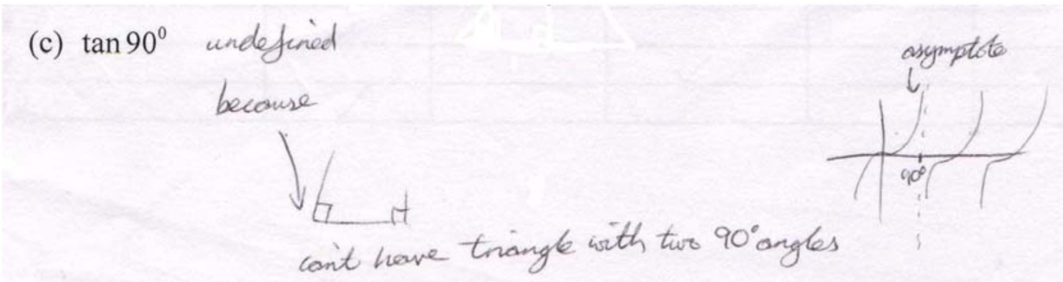
> over 2 and you can then say 30… ok so opposite over hypotenuse, that is the opposite, that is the hypotenuse so it gives you half.

Initially she made sense of this through *perception* by referring to her special triangle then she operated the sine definition to get her answer. The special triangle is a blend of perception and operation. It involves perception because you can see it is a triangle. It involves operation because she cut the equilateral triangle into half and operated the definition of sine to get the value. Additionally, the sides of the triangle are measured in term of lengths which are expressed as numbers. In this case, she was working in the context of triangle trigonometry.

> KE: […] what about sin 120?
>
> CB: Sin 120, if you look then on the graph again because it does not really make sense to say… sin 120… erm… sin 120 is like here and so you think how much is this from 180 (referring to her sine graph) and then you think well where would that be on here and so you can say well that's like 60 degrees… erm… so although 120 degrees is here because it's symmetrical you could think of it as the other end so then is 60 degrees sine 60 degrees so you can do the triangle again.

ST5 uses graphical trigonometry and relates it to the triangle trigonometry. She uses visual symmetry of sine graph rather than angle formulae for sin (A+B) or sin (180-x). Then she constructed an equivalent triangle in Euclidean geometry because she realized she couldn't construct a right angled triangle with $\sin 120°$. This is essentially making sense through *perception* and *reason*. For ST5, $\sin 120°$ does not make sense (a problematic conception) to her due to the changes of meaning between Euclidean geometry and Modern Cartesian. In fact, all the student teachers who participated the follow-up interviews used the visual symmetry of the graph which is more intuitive.

> KE: […] what about tangent 90 degrees?
>
> CB: Erm… it doesn't work.
>
> KE: Because??
>
> CB: Because on a 2D plane… on a flat 2D plane… erm… flat is the important thing you could have it on a ball but you can't have two 90 degrees angles because they never meet (referring to her picture on her answer script) and therefore like on the graph when you see it there is an asymptote because it never gets there.



ST5 was thinking in the context of triangle trigonometry when she said 'it doesn't work'. This was problematic because she couldn't construct the right angled triangle when she was working in the Euclidean geometry. Finally she ended up drawing a weird figure as above. Here her perception of tangent in a right angled triangle becomes problematic and she is unable to make sense by *perception* in triangle trigonometry. She also relates this to the tangent graph in order to get a more coherent picture.

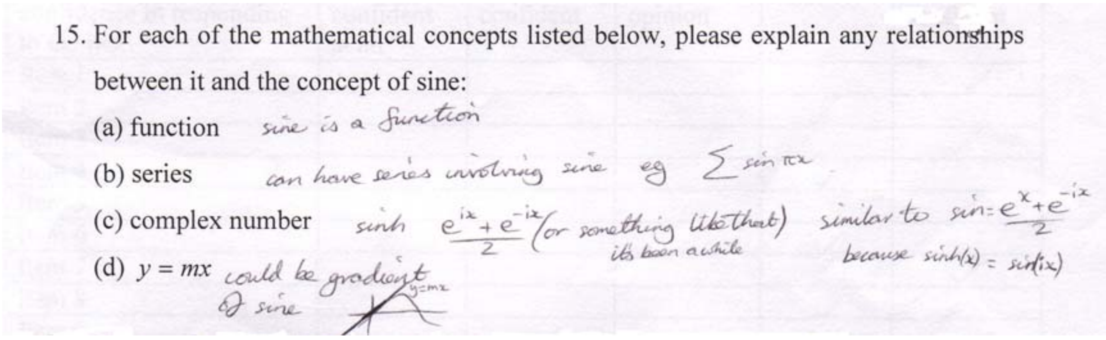
For each of the mathematical concept listed below, please explain the relationships between it and the concept of sine:

(a) Function

(b) Series

(c) Complex number

(d) y=mx


15. For each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine:

(a) function sine is a function

(b) series can have series involving sine eg $\sum \sin \pi x$

(c) complex number sinh $\frac{e^{ix}+e^{-ix}}{2}$ (or something like that) its been a while similar to sin: $\frac{e^{x}+e^{-ix}}{2}$ because sinh(x) = sin(ix)

(d) $y = mx$ could be gradient of sine

KE: Ok. Item 15, for each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine. The first one is function. Your answer is sine is a function (reading her answer script). Alright. Series, can have series involving sine for example sum of sine pi x (reading from her answer script). Other than this, do you have any other idea of series?

ST5: Of series?

KE: Yes.

ST5: Erm… you can do oh…that is pi isn't it? Erm… if you do … erm… there is a way like Maclaurin series and you can come up with … erm… a thing for pi an answer for pi… by doing a basically by Taylor expansion but that's more to do with pi than to do with sine… erm…

KE: So can you remember that series?

ST5: Erm… I could probably derive it but I can't remember it there's like a plus… like it's bigger… you know… something over two plus something over three minus something over four plus something over five minus something I don't know… it's something like that it kind of alternates maybe pi over four is this plus this over this plus this over this and minus this over this plus… minus I don't know something like that I could just about manage to derive it if I had to but I don't want to… it's too much brain power.

Based on her written response, she didn't see sine can be represented by a series. Meanwhile, she expressed that a series can involve sine. Obviously she couldn't remember Taylor series and she was confused about that.

KE: [...] what about complex number?

ST5: Complex numbers? Erm usually like pi is minus 1… yea like sinhs and things I couldn't remember the thingys but I know that theres… yea… erm… if you do like you get complex roots of things (I didn't put that one down, did I?) like if you think of like the circle (referring to her sketch, see Figure 6.7) and you are looking for roots you sometimes get like you know it's minus 1 but it's also like… erm… you know what I mean… so if this is like… erm… imaginary and this is real then whereas in if you just talking in real ones you might just get you know… minus 1 or you might not get roots but… so then it's about a round this unit circle again… finding you know this is something plus something like.

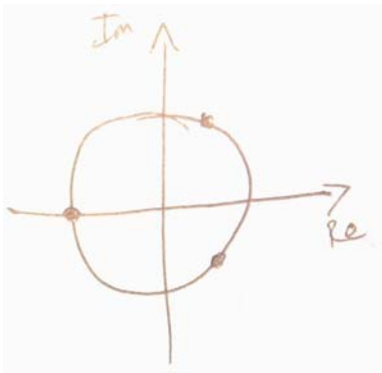


Figure 6.7: Sketch of complex numbers by ST5.

She wrote the term πx but not nx for the sub item series. In this case, πx is not a general term. If it were nx, then probably she was thinking of something which were related to Fourier series or stuff like that.

KE: Do you feel comfortable with complex number?

ST5: Yea… I think so I have known about them long enough but I mean I knew about them… I learn about them. I learnt about complex numbers when I was about 12 or 13… I don't know I only saw them at school until A level but it's just counting… I always thought of it as… the people nowadays they learn number lines so the real numbers always go side to side so you've got like 0 in the middle and you've got positive numbers and you know negative numbers but and then so imaginary numbers always going up and down instead whereas I learnt I always thought of it because we never learnt number lines erm… when I was at school that sounds like I am so old I am only 27! But yea I always thought of counting up and down erm… and then imaginary numbers as well it's like counting sideways erm… so I always thought of it as that so when I got into when I actually learnt it formally I had to re-evaluate because they always talk about it in visual terms they always thought of it as that with the real on the horizontal axis.

Based on her written answer, she mentioned sine and the hyperbolic sine. She was mixing up sin and sinh. She was mentioning things that were all confused. As we can see from her answer, she didn't state the correct formula for sine and hyperbolic sine. Obviously she mixed up these two formulae.

KE: What do you think for example like i square equals to negative 1?

ST5: Yea… I think I am happy with like I am happy with a letter as a variable or as an answer or as multiple answers like a lot of people have trouble with especially as a continuation of… like first of all people just about to deal with x plus… x plus 4 equals 1 what is x… well… ok… that is not a good example you know 10… ok so x equals 6 and they can just about to deal with one answer and then you start getting like you know x square equals 4 well what is x it could be equal 2 or it could be minus 2 and that just blows some people's minds but I don't really it might have blown my mind at the time but I don't… I am happy manipulating letters, I think pi is fine.

She was thinking about solving linear equation and x^2 which is quadratic. She was trying to recall things but she couldn't remember them properly. Apparently she was responding intuitively (S1 responses).

> KE: Ok. What about *y=mx*?
>
> ST5: Erm… y equals mx is generally going to be a straight line that goes through the origin, m is the gradient and so I mean it could be the gradient of sine at that point if m is 1 I suppose… at the origin but otherwise it's a line there is a line that goes through the origin and diverges from sine very quickly.

Apparently she was thinking of the relationships between a straight line to the sine curve.

6.6.1 Summary of ST5.

ST5 is a female student teacher with a first class mathematics degree and she is graduated from a reputable British university. Her evoked concept image is in triangle trigonometry and conceiving sine as the ratio of sides of a right angled triangle. She has a tendency of making sense through *perception* in most of the cases. For instance, she always uses graphical trigonometry to make sense of trigonometry. She is able to justify the properties of the sine graph by relating them to the rolling cylinder (a practical real life example). ST5 has strong coherent links between triangle trigonometry and circle trigonometry. She also uses *operation* to make sense of certain trigonometry statements. For instance she operated the sine definition to get the value for $\sin 30°$. Furthermore, she did use *reason* to verbalize the relationships between the visual symmetry of the sine graph to her approximation for certain values of sine which she did for item 2 in the questionnaire. She focuses on the sine graph and its properties. Meanwhile, she expresses reasoning from the shape of the graph. This shows that she has made sense of trigonometry through *perception*, *operation* and *reason*.

Apparently, she was working in the three distinct contexts of trigonometry when attempting to answer the mathematics items in the questionnaire. For instance, she used graphical trigonometry (a part of circle trigonometry) to approximate the value for $\sin 110°, \sin 250°, \sin 335°$. On the other hand, she used triangle trigonometry to make sense of $\tan 90°$ where she ended up with drawing a weird figure. She expressed that she couldn't make sense of any right angled triangle with angle bigger than acute angle. In this case, she will work in the circle trigonometry context when the angle is greater than an acute angle.

She has a problematic conception which is related to the changes of meaning between Euclidean geometry and modern Cartesian. For instance, she used triangle trigonometry to make sense of $\tan 90°$ which was clearly a problematic conception. Additionally she could not make sense of anything with angle bigger than $90°$. She can relate it back to triangle trigonometry only for angles less than $90°$. In this case, triangle trigonometry becomes a problematic conception in circle trigonometry for ST5 when the angle involved is greater or equal to $90°$. She does refer to dynamic turning of a rolling cylinder to explain how the sine curve arises. She knows that there are relationships between trigonometry and *e* and power series but cannot remember them or grasp the detail of these relationships.

ST5 can embody the notion of dy/dx by seeing it as a gradient of a point on a curve. She can see the gradients of the points on a sine curve will end up being a cosine curve after plotting these gradients. On the other hand, she couldn't relate radians to calculus therefore she didn't offer an explanation of why radians must be used in calculus. Apparently she didn't grasp this relationship.

ST5 does have coherent links between triangle trigonometry and circle trigonometry. For instance she can justify the properties of a sine graph by relating them to the unit circle or the rolling cylinder. On the other hand, there are some instances where she couldn't interchange these contexts flexibly. For instance, she couldn't imagine any triangle with angle bigger than an acute angle. Furthermore, she drew a weird figure which was resulted by conceiving $\tan 90°$ in triangle trigonometry. In general, she knows the concepts in trigonometry but she doesn't grasp them. For instance she knows that there are relationships between trigonometry and *e* and power series but cannot remember them or grasp the detail of these relationships.

6.7 Summary of the chapter.

This chapter presents the analysis of five student teachers on how they make sense of trigonometry based on the data collected from the questionnaire and the follow-up interviews. The data is analyzed by using the theoretical framework as proposed in chapter 5. A summary of student teacher is given at the end of every analysis in order to answer the stated research questions (see question 1 to 8 in section 4.2) and also to give an overview regarding how the student teacher makes sense of trigonometry.

Chapter 7
Student Teachers' Concept Maps

7.1 Introduction.

In the previous chapter, we have seen how student teachers make sense of trigonometry in the context of triangle trigonometry, circle trigonometry and analytic trigonometry, based on their supportive and problematic conceptions. In this chapter, the concept maps constructed by 5 student teachers will be presented so that we can examine their knowledge structures. Why it is important to examine the student teachers' knowledge structure? According to Skemp (1987), information needs to be assimilated into an appropriate schema in order to achieve an understanding. In this case, Skemp was referring to the relational understanding. Hence, by examining the concept maps of the student teachers then we should be able to see their conceptual structures which may provide us with a better picture of how student teachers connect different concepts of trigonometry. Furthermore this also allows the student teachers to present their understanding of trigonometry by using visual representations.

This chapter presents the concept maps constructed by the selected student teachers who had joined the follow-up interviews. Every interviewee constructed two concept maps. The first concept map was constructed before the follow-up interview began whereas the second concept map was constructed after the follow-up interview. The purpose of using concept map as a research method for data collection is to externalize the knowledge structure of the respondents. Moreover this is a good method to document the conceptual change after the follow-up interview. The observation of the process of concept map construction can give us insights on the thinking of the interviewees about the development of trigonometry in their minds.

7.2 Concept maps of ST1.

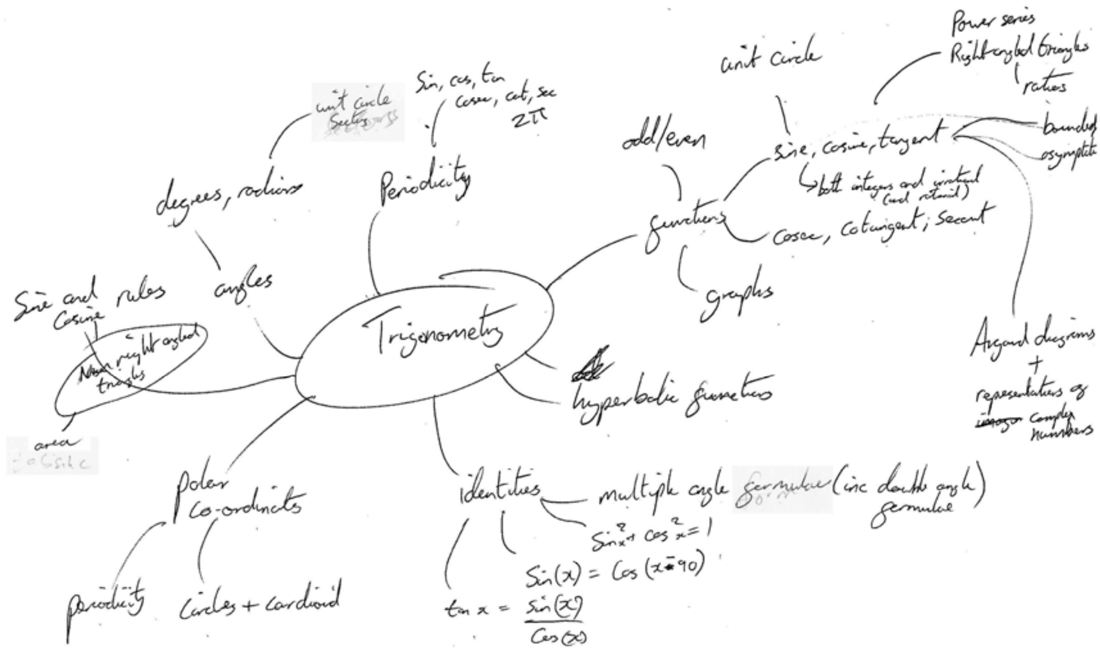


Figure 7.1: First concept map constructed by ST1.

Firstly it is interesting to note that the links given in ST1's concept map (see Figure 7.1) appear to consist of links mainly in the circle trigonometry context. In Figure 7.1, we see a collection of random facts, formulae and axioms. It should be noted that the notions of sine, cosine and tangent are not primary links to the concept of trigonometry. Working out from the central item 'Trigonometry'', this shows that most initials links are to Circle trigonometry. Only in the top right are there further links to triangle trigonometry. This shows us some indications that ST1's knowledge structure for the concept of trigonometry is directly link to circle trigonometry but triangle trigonometry involves a further step back. On the other hand, this is not consistent with the findings obtained from the interview transcriptions which show that ST1 has a strong link to the triangle trigonometry.

When he drew the concept map, ST1 started the map construction by linking trigonometry to functions (see Figure 7.1) then he continued to develop this link by stating the characteristics of functions and gave a few examples of it such as sine, cosine and tangent. Subsequently he linked sine, cosine and tangent to power series and right angled triangles. Later he drew the link of hyperbolic functions. Next he linked trigonometry to identities and gave a few formulae. After that he linked trigonometry to 'periodicity' which essentially is a characteristic of functions. Then ST1 created a link between trigonometry and angles and further develop it to include degrees and radians. Later he went back to his initial link (i.e. functions) and included the Argand diagrams which were representations of complex numbers. After that he drew a link for polar coordinates. Finally he included the link for sine and cosine rules.

ST1 included power series and complex numbers in his concept map which are essentially analytic trigonometry. However these links are not direct links. In fact, he put in the power series in the very early stage of the map construction. The complex numbers is added in a much later stage. In this case, I couldn't trace a systematic development in trigonometry through his sequence of constructing the concept map. It should be noted that he did include the unit circle in the concept map prior to the follow-up interview however during the follow-up interview it was obvious that he didn't link the unit circle to the graphical trigonometry to make sense of trigonometry. Meanwhile he didn't link the unit circle to the graphical trigonometry in his concept map.

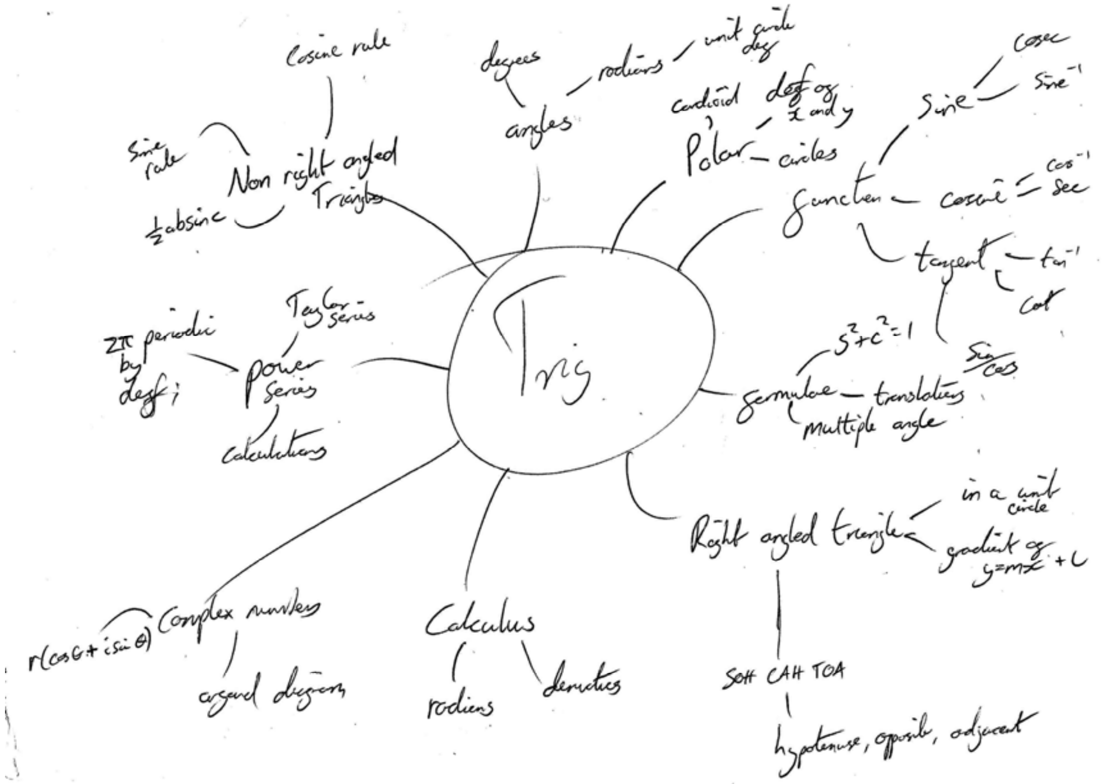


Figure 7.2: Second Concept Map constructed by ST1.

ST1 didn't refer to his first concept map while he was constructing the second concept map (see Figure 7.2) after the follow up interview. At first, he linked trigonometry to function and gave three examples of it which were sine, cosine and tangent. Then he drew the link for right angled triangle and included various stuffs such as SOH CAH TOA and unit circle etc. After that he linked trigonometry to calculus and further developed this link into radians and derivatives. ST1 then drew the link for angles. Next he developed the link for power series, subsequently he drew another link which was the complex numbers. Later he constructed the link for the non-right angled triangles. After that he related trigonometry to formulae. Lastly he drew the link of polar.

It is interesting to note that the second concept map contains more diverse primary links to the core concept of trigonometry. In the first concept map, ST1 didn't link the concept of right angle triangle directly to the concept of trigonometry whereas in the second concept map he did link it directly. These links could be triggered by the questions in the follow-up interviews. In the interview, he often used triangle trigonometry to make sense of most of the mathematics items therefore this had strengthen these links in his knowledge structure. He also linked power series and complex numbers directly to trigonometry which are concepts in analytic trigonometry. This might be a consequence of the follow-up interview because he was asked to state the relationships between those concepts (i.e. power series & complex numbers) and sine in this interview. ST1 has included a few concepts of circle trigonometry such as calculus, function and polar coordinates which are directly linked to trigonometry. Based on his second concept map (see Figure 7.2), he thinks that the 2 pi periodic of the sine curve is based on definition of power series. There is no indication of cross links between those primary links.

7.3 Concept maps of ST2.

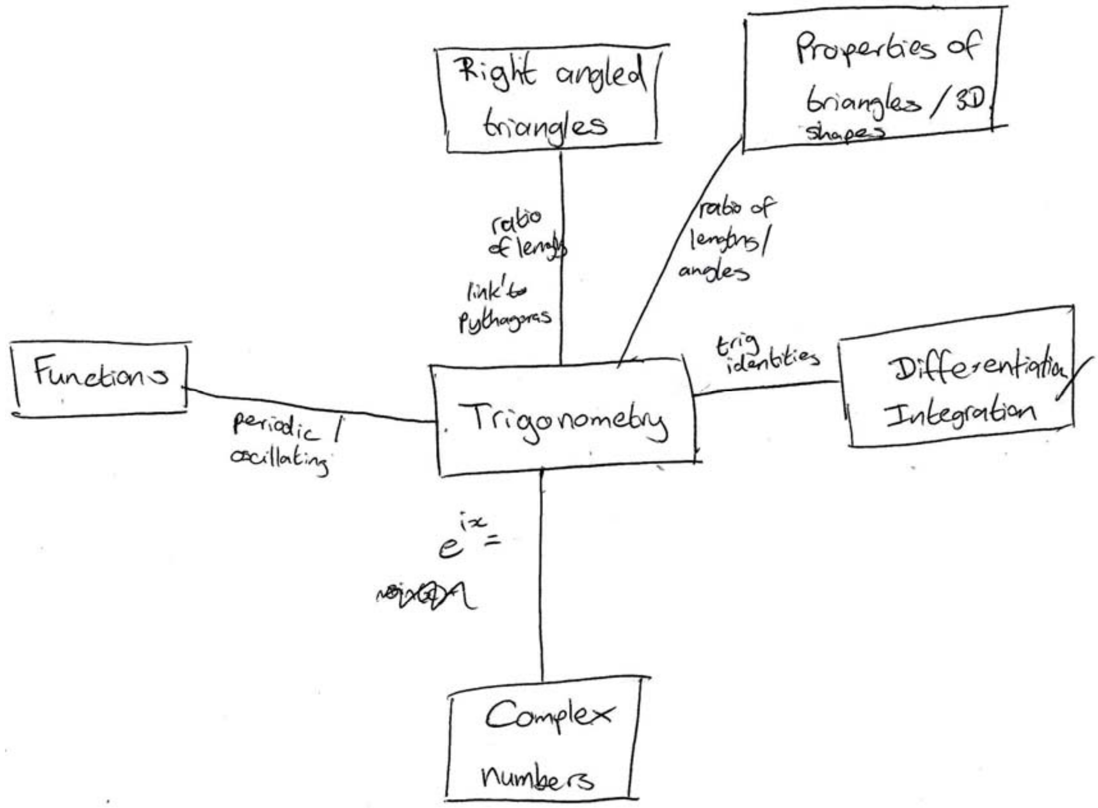


Figure 7.3: First concept map constructed by ST2.

The first concept map (see Figure 7.3) constructed by ST2 is quite sparse and simple. There are no cross links between the concepts. The first link she had drawn was right angled triangle. Then she linked the concept of trigonometry to differentiation and integration. After that she wrote the link for complex numbers followed by the link of properties of triangles. Lastly she drew the link for functions. She has included the concept of right-angled triangle in triangle trigonometry. In the context of circle trigonometry, she has included functions, differentiation and integration. Complex numbers, which is a concept in analytic trigonometry, is linked to the core concept of trigonometry. (She attempted to write the formula for e^{ix}, but she could not remember it.) The concept map has explanatory material on every link from the central concept of trigonometry but there are no secondary links to further detail. It should be noted that she didn't include the unit circle in this concept map.

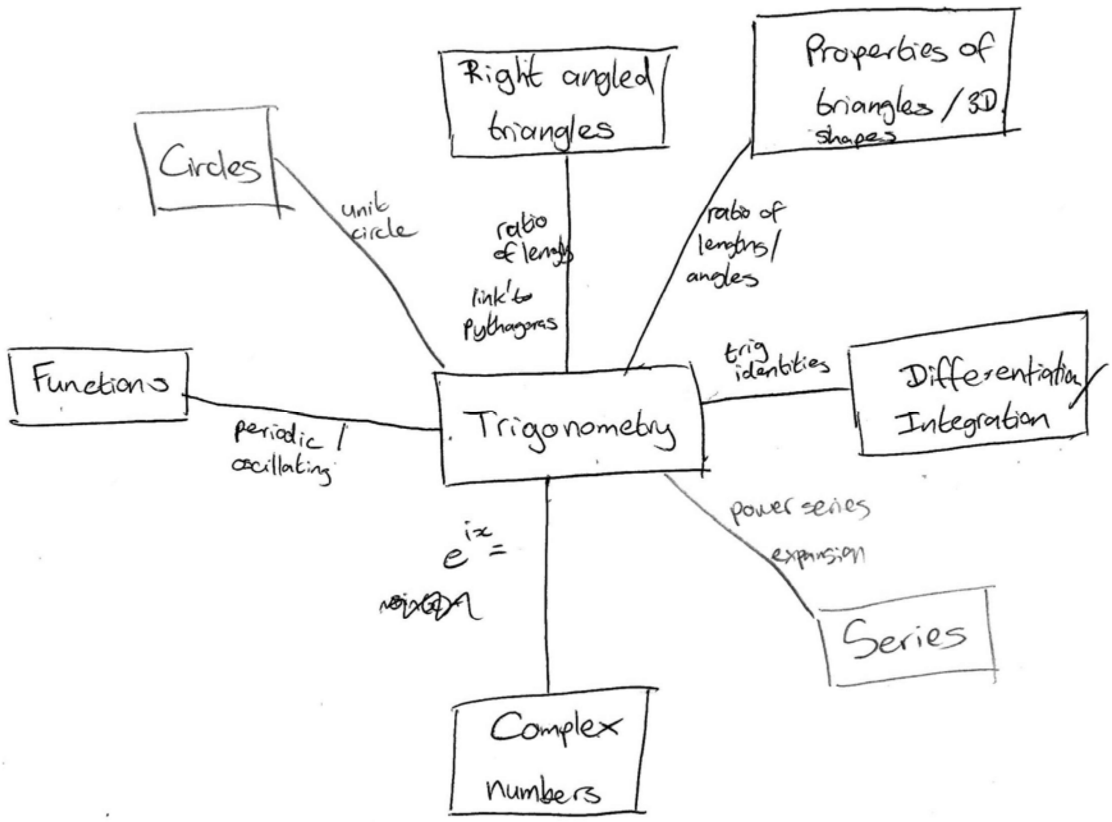


Figure 7.4: Second concept map constructed by ST2.

ST2 referred to her first concept map while constructing her second concept map after the follow-up interview. She only added two more concepts to the first concept map which were the unit circle and the power series expansion. She drew the link for series followed by the link of circles. The concepts which are written in pencil are the concepts added after the follow-up interview. The questions in the follow-up interview are most likely to have triggered these links. There are no cross links between those primary links. It seems like she has learnt most of the concepts as disparate facts with the focus on certain relationships (evident from the follow-up interview) so that she could operate more advance mathematical concepts.

7.4 Concept maps of ST3.

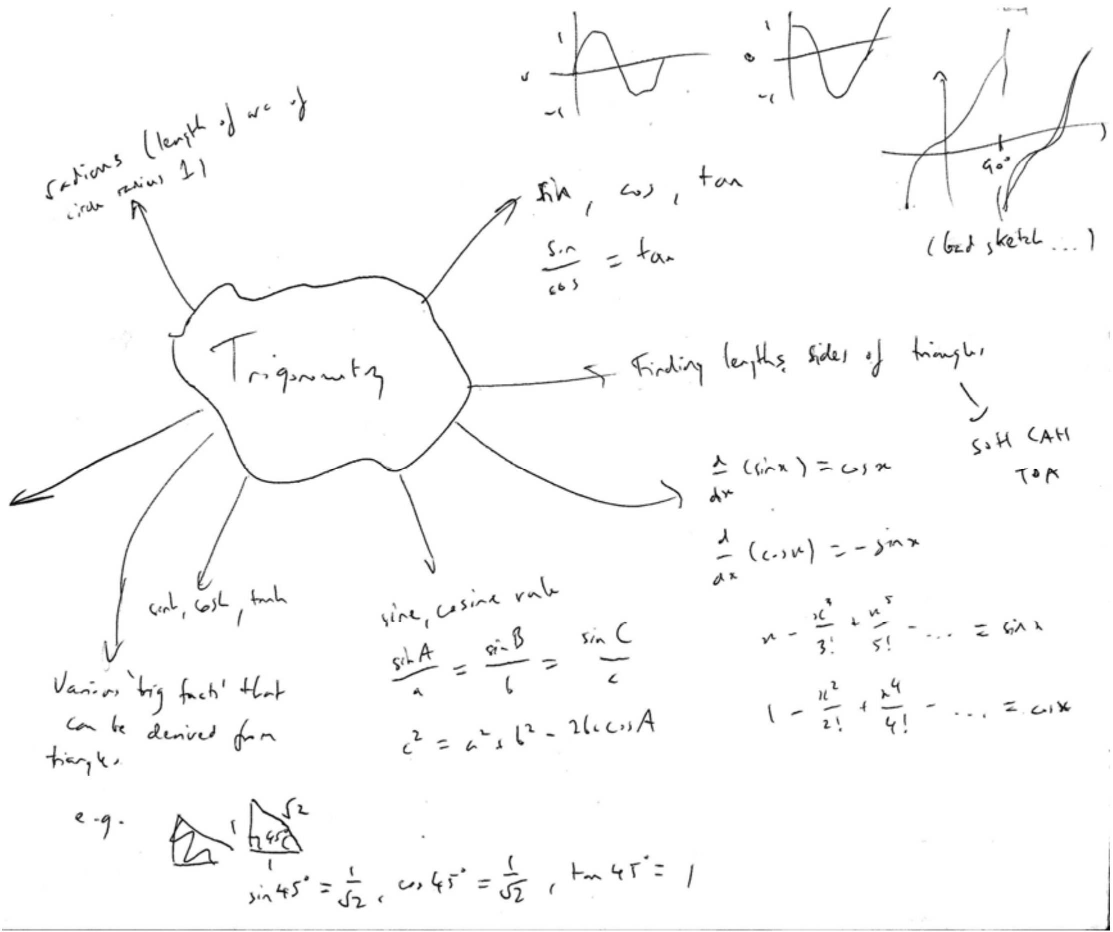


Figure 7.5: First concept map constructed by ST3.

ST3's first concept map involves the concepts of the three contexts of trigonometry (see Figure 7.5). He was drawing in a clockwise direction and started the map construction by writing sin, cos, tan then he drew the relevant graphs for them. After that he wrote the link of SOH, CAH, TOA followed by the link for sine rule and cosine rule. He then added the link for radians followed by the link for sinh, cosh and tanh. After that he drew the link for differentiation and power series. Lastly he drew the link for ''various ‘trig facts’ that can be derived from triangle’’. For the context of circle trigonometry, he drew the sine graph, cosine graph and tangent graph. He stated the Taylor series for sine and cosine in the concept map which is a concept in the analytic trigonometry.

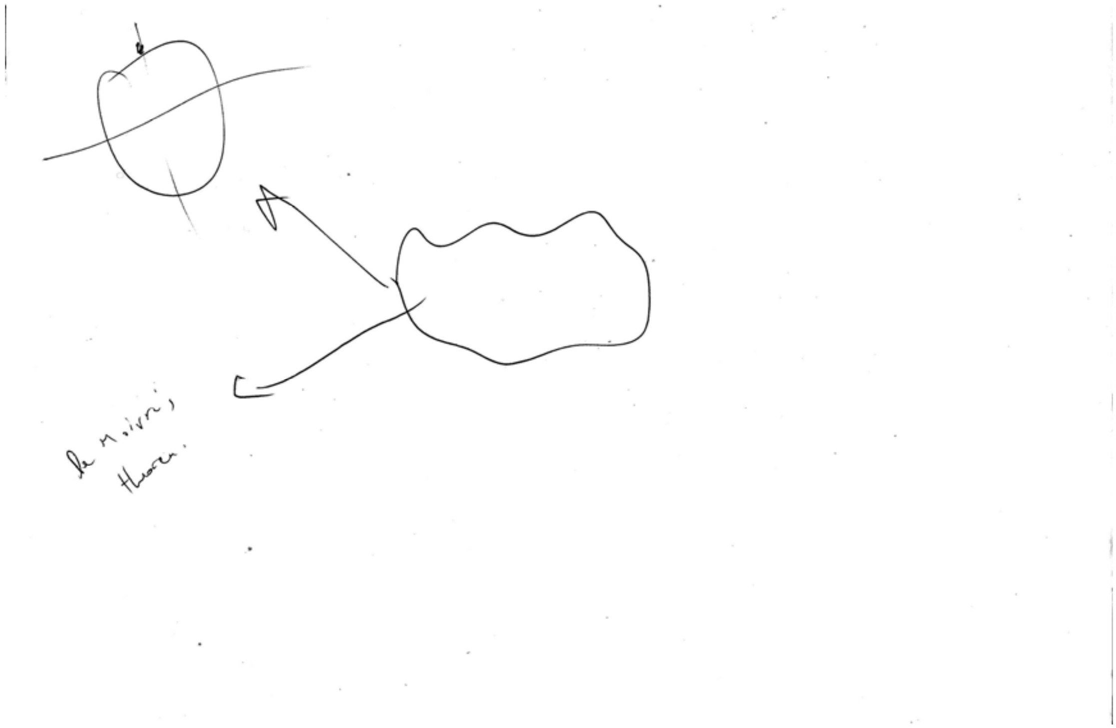


Figure 7.6: Elements added to the first concept map by ST3 as his second concept map.

ST3 didn’t draw the full second concept map after the follow-up interview. He only included two extra concepts into his first concept map which were the unit circle and the De Moivre’s theorem. Figure 7.6 shows the elements which he added to the first concept map. In short, Figure 7.6 should be read together with Figure 7.5 in order to have the full second concept map of ST3. There is no cross links between primary links. Apparently, he has strong links to triangle trigonometry and circle trigonometry in the sense that he draws those links at the very beginning stage. This is consistent with the findings obtained from the interview transcriptions.

7.5 Concept maps of ST4.

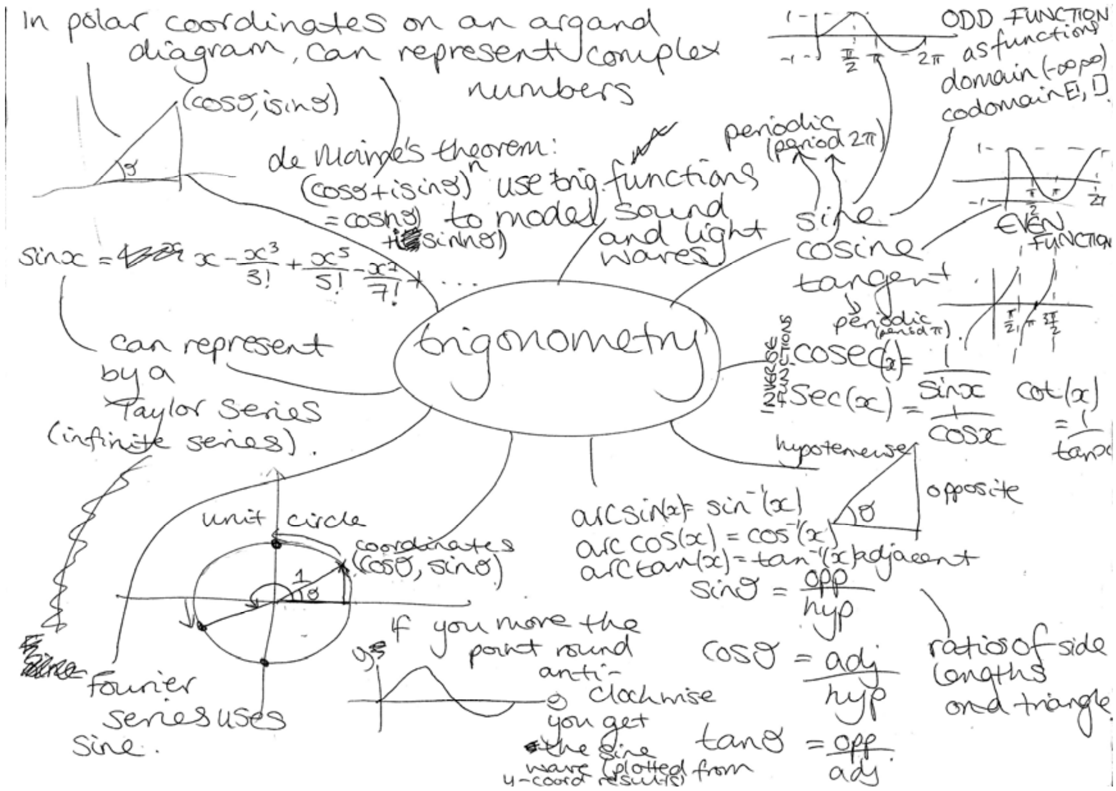


Figure 7.7: First concept map constructed by ST4.

The first concept map (see Figure 7.7) of ST4 consists of links related to the three distinct contexts of trigonometry. She started the construction of this concept map from circle trigonometry then gradually she linked the concept of trigonometry to triangle trigonometry. Lastly she linked it to analytic trigonometry. In general, she had constructed the concept map in the clockwise direction starting from the link "sine, cosine, tangent" (see Figure 7.7). The concept map is quite comprehensive in the sense that it covers quite a broad range of concepts. However the concept map doesn't have any inter-links between concepts of triangle trigonometry, circle trigonometry and analytic trigonometry. She drew the right angled triangle for triangle trigonometry. In circle trigonometry, she drew the unit circle, sine graph and cosine graph. For the case of analytic trigonometry, she linked it to the power series and the De Moivre's theorem. The concept map shows the different facets of trigonometry but it didn't explicitly explain and link the different concepts together.

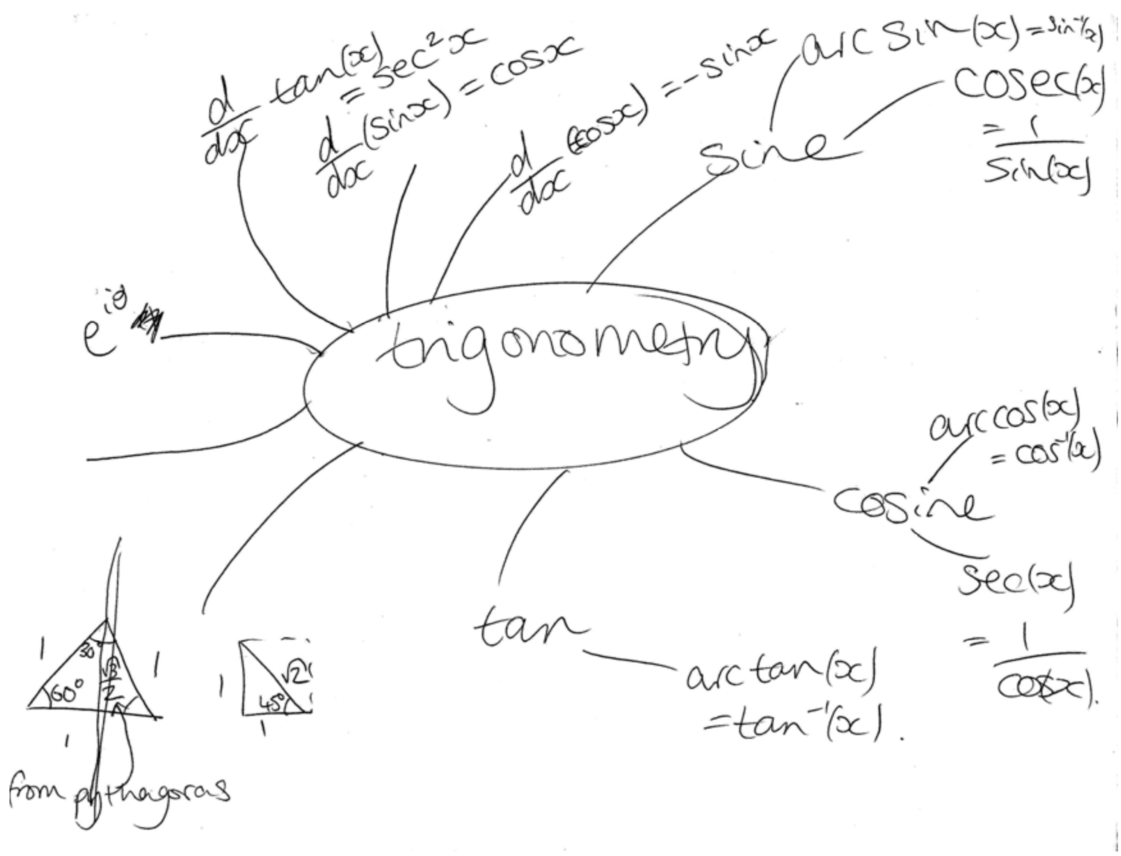


Figure 7.8: Elements added to the first concept map by ST4 as her second concept map.

Figure 7.8 shows the elements added to the first concept map by ST4 as her second concept map. In this case, Figure 7.8 should be read together with Figure 7.7 in order to have the full picture of ST4's second concept map. First she added the link for sine then she constructed another link which was cosine (see Figure 7.8). Later she linked trigonometry to tangent. After that she linked trigonometry to the right angled triangle (see Figure 7.8).

Next, she tried to write down the full Euler equation but she couldn't remember it. Lastly she put the links the differentiation of sine, cosine and tangent. Again, no interlinks between concepts are shown in the second concept map (see Figure 7.8).

7.6 Concept maps of ST5.

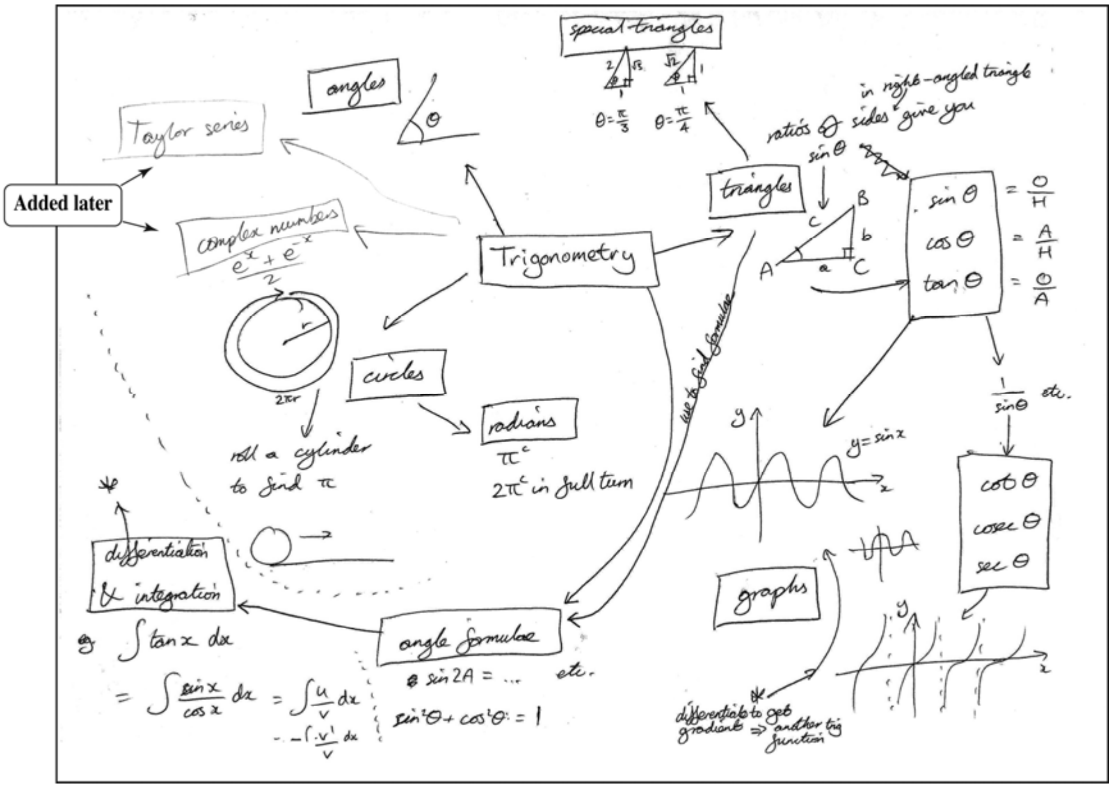


Figure 7.9: First and second concept map constructed by ST5.

ST5 constructed the first concept map (see Figure 7.9) with links to triangle trigonometry and circle trigonometry only. She started the construction with triangle trigonometry then gradually she moved to circle trigonometry by drawing the graphs of sine and tangent. It is interesting to note that she drew the graphs before relating the concept of trigonometry to circles and radians. Finally she linked the concept of trigonometry and the concept of triangle to the angle formulae. She also linked the angle formulae to differentiation and integration. After the follow-up interview, ST5 added two concepts to the concept map which were the Taylor series and complex numbers (see Figure 7.9). These two concepts are in the context of analytic trigonometry. This gives us an indication that the concepts in analytic trigonometry might not be immediately link to the ST5's knowledge structure of trigonometry unless she is triggered by certain events and stimuli. In fact, ST5's concept map is quite comprehensive and coherent in a general sense. She links differentiation to the gradient functions which essentially links back to the trigonometric functions. She also knows that when the trigonometry is put in the circle context then she will get the radians. On the other hand, the data gathered from the follow-up interview shows that she does not grasp the reason for using radians in calculus.

7.7 Summary.

In general, the concept maps presented in this chapter do show partly what the student teachers know about trigonometry but they were not focusing on the relationships between different concepts in trigonometry. These maps represent the evoked knowledge structures of the student teachers. Apparently most of the student teachers add the unit circle and the concepts in analytic trigonometry in particular the complex number and the power series after the follow-up interviews. This indicates that those concepts might not be directly link to their knowledge structures at the first instance. This might due to the fact that those concepts are seldom used by the student teachers therefore the links are not strong. Perhaps this is also related to the strength of the links or the uniqueness of human's thinking as well (McGowen & Tall, 1999).

On the other hand, the concept maps didn't always show the complex relationship between different contexts of trigonometry which the students teachers might have in their minds. Most student teachers apart from ST5 didn't show the complex relationships of different concepts in trigonometry through the concept maps. This might be related to the student teachers' skills and experiences in constructing concept maps. Some student teachers might not be familiar with presenting their knowledge structures by using concept map. Furthermore, without having a specific context hence certain relationships might not be focused. It should be noted that the concept maps should be interpreted together with the interview transcriptions in order to have a more coherent analysis.

Chapter 8
Mastery of Subject Matter Knowledge & Level of Confidence

8.1 Introduction.

This study concerns how a group of student teachers make sense of trigonometry. Humans make sense of mathematics through different combinations of perception, operation and reason. Meanwhile trigonometry is categorized into three distinct contexts namely triangle trigonometry, circle trigonometry and analytic trigonometry. The transition in different contexts involves supportive or problematic conceptions. Hence the mathematics items in the questionnaire are set in a way to see how the student teachers make sense of new contexts. Due to the specific and non-traditional features of these mathematics items, therefore it is interesting to explore the perceptions of student teachers regarding the importance of mastery of subject matter knowledge tested by those items. Additionally, it would be important to explore the emotions of student teachers in responding to those items in particular their level of confidence.

In general, this chapter is divided into two main sections. The first section discusses the student teachers' perceptions on the importance of mastery of subject matter knowledge tested by the mathematics items. Meanwhile the second section reports the student teachers' level of confidence in responding to the mathematics items. A summary is presented at the end of every section in order to answer the research questions as stated in section 4.2 (see research question no 9 and 10 in section 4.2 on page 63). Subject matter knowledge is subject-specific and is required by teachers to teach the subject effectively. These insights are gained through the part B of the questionnaire and the follow-up interviews. Research done by Rowland and Tsang (2005) on subject matter knowledge which involved 138 Hong Kong primary school mathematics teachers has showed that the SMK of the respondents was quite shallow. A test was used as one of the instruments to collect data regarding the SMK of the participants. In this survey, an interesting finding indicated that the teachers' perception of the SMK that is important in teaching was different from what they thought they would be able to solve (mathematics items). In reality, the respondents performed poorly with the test items that they considered important.

This chapter will begin with the analysis of the responses of 24 student teachers which were collected through the questionnaire. Then further insights were explored through the follow-up interviews. Follow-up interviews were conducted based on voluntary basis in order to supplement the quantitative data from the questionnaire.

8.2 Perceptions on the importance of mastery of subject matter knowledge.

In this section, the data collected from part B of the questionnaires will be reported. The range of coding used in this section is from -2 to 2, with -2 indicating "not important at all" and 2 indicating "very important".

Mastery of Subject Matter Knowledge indicated by the item is	Very important	Important	Neither important nor unimportant	Not important	Not at all important
Item 1	8 (33.3%)	15(62.5%)	1 (4.2%)	0(0%)	0(0%)
Item 2	9 (37.5%)	12(50.0%)	3(12.5%)	0(0%)	0(0%)
Item 3	3 (12.5%)	16(66.7%)	3 (12.5%)	2(8.3%)	0(0%)
Item 4	5 (20.8%)	15(62.5%)	4(16.7%)	0(0%)	0(0%)
Item 5	9 (37.5%)	13(54.2%)	2(8.3%)	0(0%)	0(0%)
Item 6	11(45.8%)	8(33.3%)	4(16.7%)	1(4.2%)	0(0%)
Item 7	5(20.8%)	15(62.5%)	3(12.5%)	0(0%)	1(4.2%)
Item 8	9(37.5%)	12(50.0%)	3(12.5%)	0(0%)	0(0%)
Item 9	5(20.8%)	11(45.8%)	7(29.2%)	1(4.2%)	0(0%)
Item 10	15(62.5%)	7(29.2%)	2(8.3%)	0(0%)	0(0%)
Item 11	10(41.7%)	10(41.7%)	3(12.5%)	0(0%)	0(0%)
Item 12 (a)	9(37.5%)	12(50.0%)	3(12.5%)	0(0%)	0(0%)
Item 12 (b)	7(29.2%)	13(54.2%)	4(16.7%)	0(0%)	0(0%)
Item 12 (c)	9(37.5%)	11(45.8%)	3(12.5%)	0(0%)	0(0%)
Item 13 (a)	8(33.3%)	13(54.2%)	3(12.5%)	0(0%)	0(0%)
Item 13 (b)	9(37.5%)	11(45.8%)	4(16.7%)	0(0%)	0(0%)
Item 14	7(29.2%)	13(54.2%)	4(16.7%)	0(0%)	0(0%)
Item 15 (a)	6(25.0%)	11(45.8%)	5(20.8%)	0(0%)	0(0%)
Item 15 (b)	5(20.8%)	10(41.7%)	6(25.0%)	0(0%)	1(4.2%)
Item 15 (c)	5(20.8%)	11(45.8%)	6(25.0%)	0(0%)	0(0%)
Item 15 (d)	4(16.7%)	10(41.7%)	8(33.3%)	0(0%)	0(0%)

Table 8.1: Responses for mastery of subject matter knowledge.

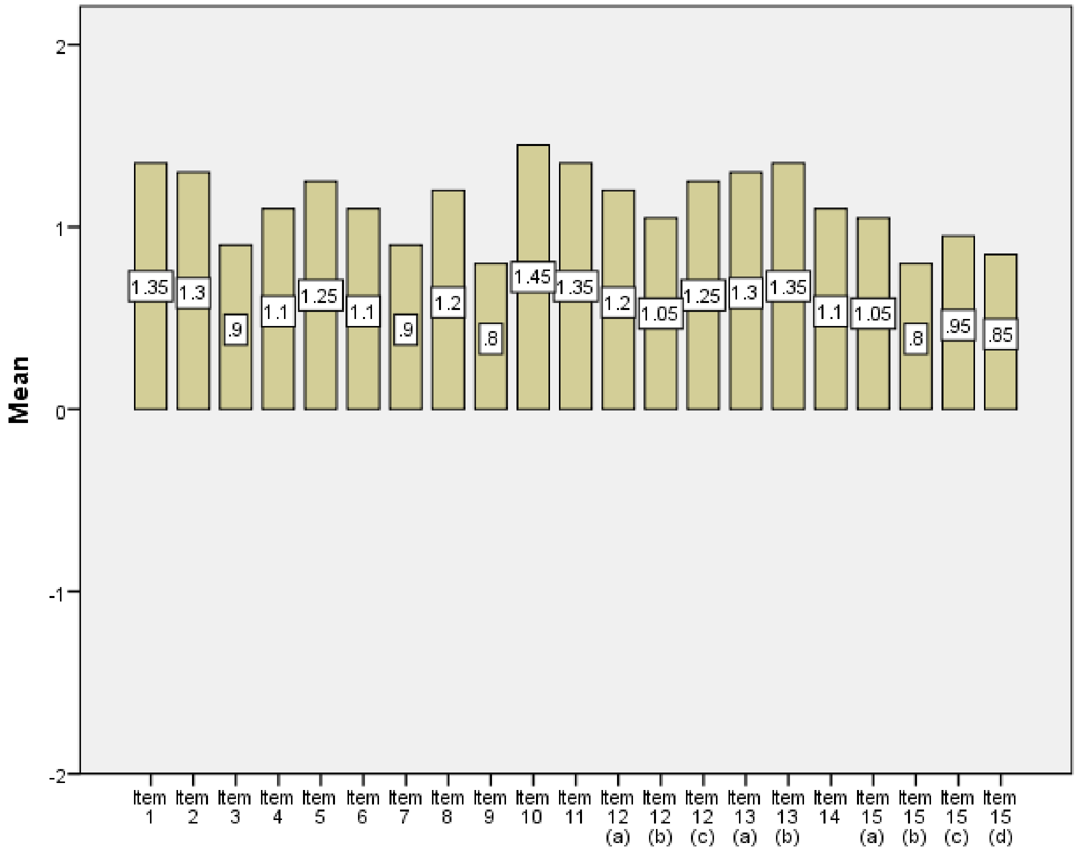


Figure 8.1: Mean values of mastery of SMK tested by the mathematics items.

According to Figure 8.1, it is evident most of the respondents felt that the mathematics items on the test were important and there is not much distinction between them. Based on Table 8.1, we can see that majority of the responses are in the category of important. Item 6 of the questionnaire asked ''What do Radians mean? Why do we need radians when we have degrees?''. This item shows interesting results in the sense that most of the respondents (45.8%) felt the SMK tested by this item were very important. Similarly, Item 10 which asked "What does dy/dx mean?" also shows an interesting result. In this case, there are 15 respondents (62.5%) felt this item was very important. Item 11 asked ''What would d/dx [sin x] mean? What is d/dx [sin x]? Explain why.'' There are 10 respondents (41.7%) felt that this item were very important. Meanwhile for Item 11, most of the respondent knew that the derivative of sin x was cos x but they didn't give an explanation of why it would be cos x. It was obvious that most of the respondents knew some of the ideas in calculus of trigonometry but they didn't grasp them. This result is interesting because most of the respondents could only answer part of the questions of item 11 and yet they felt that the SMK tested by this item was very important. Item 10 has the highest mean value which is 1.45 (see Figure 8.1). One of the sensible reasons in this case might be that the respondents have focused on a particular aspect for those items. For instance, Item 7 asked "For what values is sin x decreasing? Explain why is it decreasing for these values?", most of the respondents felt this item were important except for ST1 who felt that this item was not at all important. See below for ST1's explanation.

> KE: […] you think the knowledge is not important for item 7. Do you have any specific reason for that?
>
> ST1: I think… you probably need to know for what values it's decreasing but I think that's something you can just look at the graph or look at the function and work out, it's not something you need to know. I don't think I need to know why it's decreasing…
>
> KE: You don't think you need to know?… and then you don't think this is very important for a teacher of maths to know it?
>
> ST1: I don't think so.

As we can see, ST1 felt that Item 7 is not important at all because he felt that he didn't need to know for what values is sine x decreasing as he can always work it out from the graph. It was clear that he had focused on a particular aspect of this item. In order to understand why sin x is decreasing for certain values of x, learners must understand the unit circle. However ST1 didn't use the unit circle to make sense of it. He had used the gradient function for this aspect. ST1 stressed the importance of knowing the outlook of graphical trigonometry but it is not necessary to know why it is decreasing for certain values. This also infers that ST1 focuses on the graph. In fact, he didn't relate the sine graph and the unit circle in the follow-up interview. There is no evidence showing that he had linked the unit circle to the sine graph.

8.2.1 Items considered unimportant by a few student teachers.

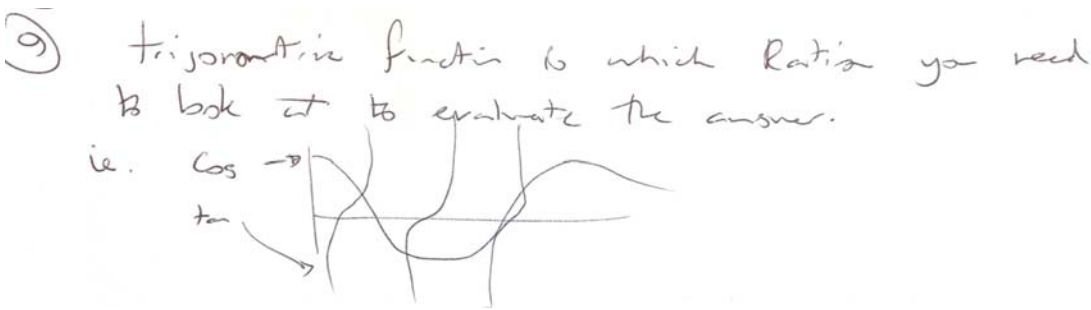
As shown in Table 8.1, there were 2 respondents (8.3%) who responded not important for item 3. This item asked ''How do you make sense of $\sin 200°$?'' A further analysis reveals that both these 2 respondents possessed a 2(i) degree. One respondent possessed a mathematics degree and the other respondent possessed a financial economics degree. The respondent with a mathematics degree responded in the context of circle trigonometry by saying 'I would identify the value on the graph and find the corresponding value' whereas the other respondent who possessed a financial economics responded to item 3 by saying 'it has a negative value'. Superficially it seems like there is no obvious relationship between the responses given. It may be hypothesise that they think this item is not important at all because they know a procedure to get this answer or they have a sense of what the answer would be.

As shown in Table 8.1, one respondent felt that item 6 was not important. Item 6 of the questionnaire asked ''What do Radians mean? Why do we need radians when we have degrees?''. A further analysis reveals that the only one respondent who responded not important for item 6 was a student teacher who possessed a 2(i) mathematics degree. Based on the answer script of this respondent, he/she knew that radians come from a unit circle but he/she didn't offer an explanation on how radians are easier to work with when using integration. This respondent felt that the ability of a mathematics teacher to be able to describe the radians and explaining the reason of using radians were not important. This also infers that this respondent thought that the conceptual understanding of radians for a mathematics teacher was not important.

Based on Table 8.1, one (4.2%) respondent felt that the SMK tested by item 15 (b) was not important at all. Item 15 (b) asked ''Explain any relationships between series and the concept of sine''. This respondent was ST1. In the follow-up interview, ST1 was asked to explain why he thought the mastery of SMK of this item was not important at all. Excerpt below shows his explanation.

> KE: […] for item 15(b) you were talking about not confident at all important. 15(b) is series
>
> ST1: The reason I said that is because as a secondary teacher that isn't part of the syllabus… ehem… yes so… yes you can show that… so you can say that is defined by series and that's used to give you accurate calculations but I don't think you can go into more detail than that.

ST1 felt that the importance of mastery of SMK was dependent to the school syllabus and dependent on what you can tell to the students. According to Figure 8.1, the mean value for item 15 (b) is 0. 8. This is the lowest mean value among the mathematics items of the questionnaire. Meanwhile item 9 asked ''What does 'trigonometric function' mean?'' Based on the table 8.1, one respondent (4.2%) responded not important for item 9. This respondent possessed a 2(ii) engineering degree. Below is his written answer for item 9



Apparently, this respondent was trying to link to the concept of ratio to the graphical trigonometry without mentioning the unit circle. This item also has the lowest mean which is 0.8 (see Figure 8.1).

8.2.2 Summary of mastery of subject matter knowledge.

Due to the limited amount of quantitative data, I couldn't explore the correlation between the variables and infer any conclusion from the data for the whole population. In general, most of the respondents regarded all the mathematics items in the questionnaire as either important or very important. The overall mean value for the whole questionnaire is 1.10. This also shows that the SMK tested by the mathematics items in the questionnaire is perceived as important on average. Some respondents regarded certain mathematics items as either not important or not at all important, this is because they had focused on different aspects of the items. The data collected from this section (Part B of the questionnaire) is not very informative.

8.3 Level of confidence.

This section reports the student teachers' level of confidence in responding to the mathematics items of the questionnaire. The purpose of this construct is to investigate the student teachers' emotions associated with the mathematics items. The data is collected through the part C of the questionnaire and the follow-up interview. Firstly, student teachers chose their responses ranging from not confident at all to very confident. Then follow-up interviews were conducted with selected student teachers on a voluntary basis in order to gain insights into why they had chosen certain responses. In this context, I am focusing on the exploration of the reasons why they felt not confident in responding to certain mathematics items. The range of coding used in this section is from -2 to 2, with -2 indicating ''not confident at all'' and 2 indicating ''very confident''. Skemp (1979) proposed a theory which linked emotions to mathematics. According to this theory, human emotions to mathematics are related to the goal state and anti-goal state. Meanwhile confidence is signal by the ability to achieve the goal state. The details of this theory and its relationship to other constructs in this study are presented in section 2.7.

8.3.1 Level of confidence in responding to the mathematics items.

What is your level of confidence in responding to the item	Not confident at all	Not confident	No opinion	Confident	Very confident
Item 1	1(4.2%)	9(37.5%)	5(20.8%)	6(25.0%)	3(12.5%)
Item 2	1(4.2%)	2(8.3%)	5(20.8%)	11(45.8%)	5(20.8%)
Item 3	1(4.2%)	7(29.2%)	9(37.5%)	5(20.8%)	2(8.3%)
Item 4	1(4.2%)	2(8.3%)	8(33.3%)	12(50.0%)	1(4.2%)

Item 5	1(4.2%)	2(8.3%)	8(33.3%)	10(41.7%)	3(12.5%)
Item 6	1(4.2%)	6(25.0%)	7(29.2%)	9(37.5%)	1(4.2%)
Item 7	1(4.2%)	6(25.0%)	7(29.2%)	5(20.8%)	5(20.8%)
Item 8	1(4.2%)	4(16.7%)	7(29.2%)	8(33.3%)	4(16.7%)
Item 9	3(12.5%)	4(16.7%)	9(37.5%)	7(29.2%)	1(4.2%)
Item 10	1(4.2%)	3(12.5%)	5(20.8%)	11(45.8%)	4(16.7%)
Item 11	1(4.2%)	5(20.8%)	8(33.3%)	7(29.2%)	3(12.5%)
Item 12 (a)	1(4.2%)	2(8.3%)	5(20.8%)	15(62.5%)	1(4.2%)
Item 12 (b)	1(4.2%)	3(12.5%)	5(20.8%)	14(58.3%)	1(4.2%)
Item 12 (c)	1(4.2%)	3(12.5%)	8(33.3%)	11(45.8%)	1(4.2%)
Item 13 (a)	1(4.2%)	2(8.3%)	9(37.5%)	9(37.5%)	3(12.5%)
Item 13 (b)	1(4.2%)	3(12.5%)	8(33.3%)	9(37.5%)	3(12.5%)
Item 14	2(8.3%)	4(16.7%)	8(33.3%)	9(37.5%)	1(4.2%)
Item 15 (a)	4(16.7%)	6(25.0%)	6(25.0%)	5(20.8%)	3(12.5%)
Item 15 (b)	4(16.7%)	7(29.2%)	8(33.3%)	2(8.3%)	3(12.5%)
Item 15 (c)	4(16.7%)	6(25.0%)	9(37.5%)	3(12.5%)	2(8.3%)
Item 15 (d)	5(20.8%)	7(29.2%)	6(25.0%)	4(16.7%)	2(8.3%)

Table 8.2: Level of confidence in responding to the mathematics items.

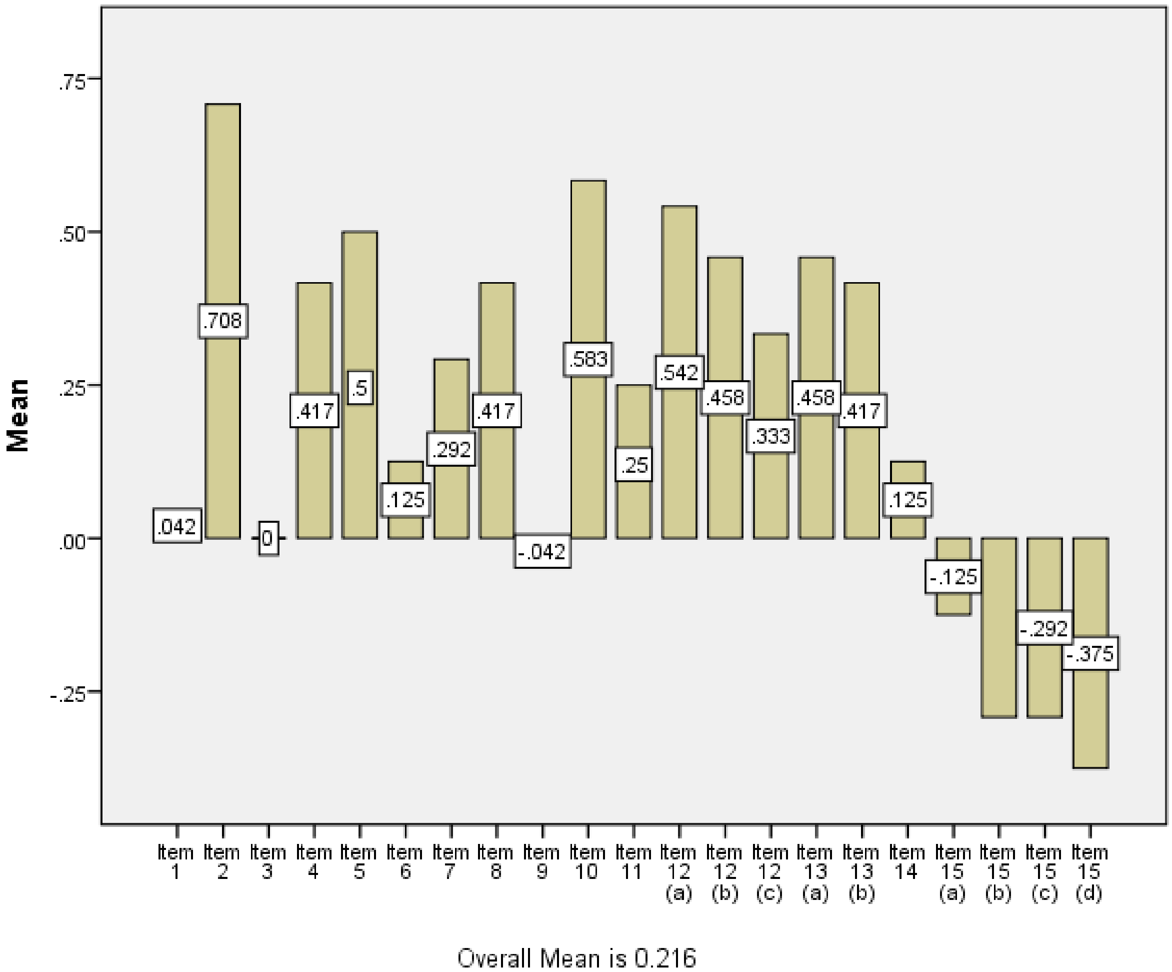


Figure 8.2: Mean values of level of confidence in responding to the mathematics items.

In this study, 24 student teachers responded to the questionnaire. A set of mixed responses was received for item 1 (see Table 8.2) and the mean value is 0.042 (see Figure 8.2). A further analysis reveals that 5 out of 9 respondents who responded not confident possess a mathematics degree. ST2 is one of the respondents who responded not confident with item 1 (see Table 8.3). The numbers in Table 8.3 indicates the coding used for 5 different categories of responses. In this case, the number 2 is used to represent the response very confident meanwhile 1 is for confident. The number 0 is for no opinion. -1 and -2 are for not confident and not confident at all respectively.

	Responses of				
	ST1	ST2	ST3	ST4	ST5
Item 1	1	-1	0	0	2
Item 2	2	1	0	1	1
Item 3	0	-1	0	0	2
Item 4	0	1	0	1	1
Item 5	1	1	0	1	2
Item 6	1	0	0	1	1
Item 7	2	1	0	-1	2
Item 8	1	-1	0	-1	1
Item 9	1	0	0	-2	-1
Item 10	2	1	0	1	2
Item 11	1	1	0	0	2
Item 12 (a)	1	1	0	1	1
Item 12 (b)	1	1	0	1	1
Item 12 (c)	0	1	0	0	0
Item 13 (a)	1	2	0	0	0
Item 13 (b)	1	2	0	0	0
Item 14	1	1	0	0	0
Item 15 (a)	1	-1	0	1	1
Item 15 (b)	0	-1	0	-1	0
Item 15 (c)	0	-1	0	0	0
Item 15 (d)	-1	-1	0	-2	1

Table 8.3: Level of confidence of the interviewees.

During the follow-up interview, ST2 explained why she has chosen this response.

> KE: According to your questionnaire, it seems like you are not confident with item 1, is there any specific reason for this?
>
> ST2: I think the problem is I have never had to really do it before and explain in your own words, you just get told what it is and how to use it. I don't know. I've never been asked to explain in my own words.

Based on the above excerpt, ST2 felt not confident with something that she never did before i.e. explain sine in her own words. This is a non-routine task for her. Apparently she was comfortable with accepting the descriptions or definitions told by the teachers. It seems like her learning experience in trigonometry is mainly procedural. Relating the mean value of item 1 which is 0.042 (see Figure 8.2) to the responses of item 1 (see Table 8.2), it should be noted that the total amount of responses on both camps are quite balance (i.e. very confident + confident VS not confident + not confident at all).

Based on Table 8.2, 11(45.8%) student teachers felt confident in answering Item 2 and the mean value for this item is 0.708 (see Figure 8.2). In short, this item asked the respondents to arrange $\sin 110°, \sin 250°, \sin 335°$ in ascending order. Most of the respondents got their answers by using the sine graph meanwhile the remaining respondents didn't offer an explanation. It is sensible to predict that most of the respondents are confident with this item because they can confidently get their answers through the sine graph.

As shown in Table 8.2, majority of the respondents (37.5%) responded no opinion for Item 3. This item asked "How do you make sense of $\sin 200°$?". ST2 was one of the respondents who felt not confident with this item (see Table 8.3). Excerpt below shows her explanation.

> KE: What about item 3?
>
> ST2: Probably the same thing I mean I could tell you what the value was, it's just difficult to explain how I made sense of it and how I came to see that in a way.

It is evident that ST2 felt not confident with Item 3 because she couldn't offer a convincing explanation of why $\sin 200°$ has certain value. In this case, her goal was to make sense of $\sin 200°$ but she couldn't achieve that goal therefore she felt not confident. The responses for Item 3 are very diverse and the mean value is 0 (see Figure 8.2).

Based on Table 8.2, majority of the respondents felt confident for Item 4, 5 and 6 meanwhile the mean values for these items are 0.417, 0.5 and 0.125 respectively (see Figure 8.2). Item 7 asked "For what values is $\sin x$ decreasing? Explain why it is decreasing for these values?". There are 7 (29.2%) respondents expressed no opinion (see Table 8.2) for Item 7. According to Table 8.3, ST4 responded not confident for this item and her explanation was as follow:

> ST4: So 7, I am confident that… that is true… that is decreasing from there but again I think I was using this (see Figure 8.3 below)… yeah… it's nice that it happens to work clockwise I was using it that way and I might have convinced myself more and as a proof, I wouldn't say it's very regular I am just using something that represents the same thing that I've just being saying so it would be the same as the graph…

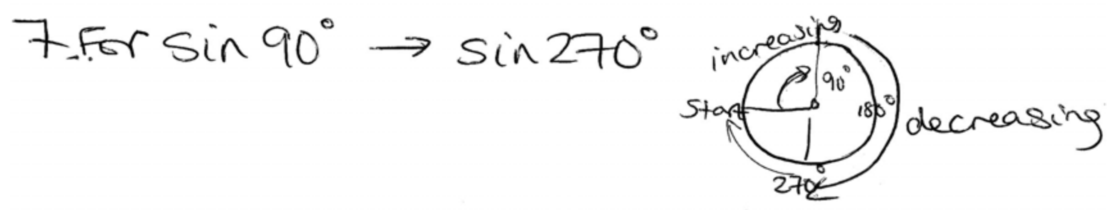


Figure 8.3: ST4's answer for Item 7.

Obviously she was confident that her answer for Item 7 was true but later she only realized that she had operated the unit circle in the opposite direction. As a result of this, she felt not confident for this item.

Item 8 asked the respondents to explain why $\sin\theta$ can never equal 2. There were 8 (33.3%) respondents expressed confident with this item (see Table 8.2). On the other hand, both ST2 and ST4 expressed not confident with this item (see Table 8.3). See excerpt below for their explanations.

> KE: Another one is item 8. Is there any specific reason for this?
>
> ST2: I've forgotten how to do it really I couldn't think of a reason why. I couldn't think about the series…
>
> KE: It means like something you feel you can't answer it so you feel not confident about it.
>
> ST2: Yeap.

It was evident that ST2 felt not confident with this item because she couldn't answer it. See the excerpt below for the explanation offered by ST4.

> KE: So what about item 8?
>
> ST4: Erm… 8… again I am using a proof there that I wouldn't consider a genuine proof… I am just trying to explain it in some way whereas… erm… I could have easily just drawn a graph to say that it's true… and that is pretty much what I am doing there… so in terms of explaining really why… I wouldn't say

> that is very rigorous again… so not confident in the explanation but the answer I know but not the explanation.

ST4's goal was to give a correct answer for Item 8. In this case, she felt not confident because she thought she still hasn't offer a rigorous explanation to justify her answer for this item.

According to Figure 8.2, the mean value for Item 9 is -0.042 which means the average response for this item is slightly tends to not confident. Item 9 asked "What does 'trigonometric function' mean?". Majority (37.5%) of the responses for this item are in the category of no opinion (see Table 8.2). Based on Table 8.3, ST4 expressed not confident at all for this item. Below was her explanation.

> ST4: Erm… 9… so ya… that's because I used the word injective and to me I wasn't confident that I had used the right word in terms of the mapping that I was trying to describe

ST4 was not confident at all because she thought she might have used the wrong word to answer Item 9. Meanwhile ST5 also expressed not confident with this item (see Table 8.3) and she offered the following explanation.

> KE: […] for item 9, you feel not confident to respond to this item, is there any specific reason?
>
> ST5: […] I think because it's wordy and because I think, You learn kind of competencies you learn how to calculate you learn things like that and you learn what it means in a general sense but and you had a picture of it in your head kind of thing but it's often you don't have you know actually explaining it to someone else in words is sometimes quite hard.

ST5's goal was to explain what is trigonometric function and she felt that this goal was difficult to achieve therefore she was not confident in responding to this item.

According to Table 8.2, the majority (45.8%) responses for Item 10 are confident. The mean value is 0.583 (see Figure 8.2). This item asked "What does dy/dx mean?". Meanwhile most of the respondents (33.3%) expressed no opinion for Item 11 (see Table 8.2). This item asked "What would d/dx [sin x] mean? What is d/dx [sin x]? Explain why.". The mean value for this item is 0.25 (see Figure 8.2). For Item 12 to 14, the majority responses are in the category of confident (see Table 8.2). These items are attached as appendix in this thesis.

As shown in Figure 8.2, the mean value for Item 15 (a) is -0.125. This item asked the respondents to explain any relationship between function and sine. A quarter of the respondents (25%) felt not confident with it (see Table 8.2). ST2 is one of the respondents in this group (see Table 8.3). She offered the following explanation.

> KE: […] for item 15(a), (b), (c), (d) you also feel not about them, is there any specific reason for this?
>
> ST2: I suppose I could understand. I've used like sine as a function and series and stuff but it was difficult explaining the relationship really especially because the series I've not used for a while… erm… the complex number of functions, I kind of knew the relationship but found it difficult to kind of explain it.

ST2 felt not confident for Item 15 because she felt difficult to verbalize those relationships. Meanwhile Item 15(b) asked the respondents to explain the relationship between series and sine. The mean value for this item is -0.292 (see Figure 8.2). ST4 expressed not confident with this item and gave her explanation as below.

ST4: Yeah… so even then saying the Mclarin series… I didn't really know…(she laughs)… what that would look like and I put taylor series so that's why I wasn't confident with that one.

ST4 wasn't confident because she wasn't able to move towards her goal state which was to provide the relationships between series and the concept of sine. In fact, ST4 gave a guess for the answer.

Item 15 (d) asked the respondents to explain any relationship between $y = mx$ and sine. Based on Figure 8.2, the mean value for this item is -0.375. Majority of the respondents (29.2%) had responded not confident to this item (see Table 8.2). Based on Table 8.3, ST1 and ST2 were part of this majority. Explanation of ST2 is given in the previous excerpt. Meanwhile ST1 gave her explanation as follow:

KE: […] Do you have any specific for this one (pointing to his response for item 15(d) of part c in the questionnaire)?

ST1: That's one of the things as soon as I took at it I had to think for a while how that was related and it took me a while to come up and I actually drew a graph then decided that… that calculation.

KE: Which means in comparison to other questions this one you took longer time to think about that.

ST1: Yeah… it's not something that I instantly calculated.

It should be noted that the ST1's level of confidence was related to his responding time to the mathematics item. The longer he took the less confidence he got for the mathematics items. As shown in Table 8.3, ST4 expressed not confident at all for this item and gave her explanation as follow:

KE: Alright… so the 15(d)? y equals to mx?

> ST4: Erm… for that one I think because I couldn't come up with an answer… it's hard to justify why… erm… so I think I was being a bit suspicious like if you'd asked me there must be something and I wondered why!

In general, ST4 felt not confident or not confident at all when she wasn't sure about her answers or explanations.

8.3.2 Summary for the Level of Confidence

As shown in Figure 8.2, we can notice that the respondents' level of confidence in responding to item 1, 3, 9 and 15 is lower than the rest of the items in the questionnaire. In this case, Item 1 asked "Describe sin x in your own words". Item 3 asked "How do you make sense of $\sin 200^\circ$?". Item 9 asked "What does 'trigonometric function' mean?". Item 15 asked the student teachers to explain any relationships between a set of given concepts (i.e function, series, complex numbers and y=mx) and sine. By relating the nature of these items to the respondents' explanations obtained from the follow-up interviews, it is evident that respondents have higher confidence to items which involve carrying out procedures compared to items which require explanations only.

In fact, the set of concepts given in Item 15 is highly diverse and my interest focused on how the student teachers may relate them in their evoked knowledge structure. As we can notice, item 15 is very different to other items because respondents are not given a specific context to work with and they are allowed to explain the relationships in any contexts. Hence, the student teachers have the lowest confidence in responding this item. The overall mean for level of confidence in responding to the mathematics items is 0.216.

Chapter 9

Student Teachers' Awareness of Learning Difficulties in Trigonometry

9.1 Introduction.

The research revealed certain aspects relating to the student teachers' awareness of the possible learning difficulties in trigonometry. This chapter considers data that arose in the follow-up interviews with the five student teachers.

9.2 Awareness of learning difficulties in trigonometry.

In general, two kinds of questions were asked in the follow-up interviews for this construct. The first kind of question is to gain insight into the awareness of own learning difficulties in trigonometry. The second type of question is to gain insight into the awareness of student teachers regarding the possible learning difficulties in trigonometry of secondary school students.

9.3 Student teachers' perceptions of their own learning difficulties.

9.3.1 Student teacher ST1.

> KE: […] do you have any difficulties in learning trigonometry? Can you think of any?
>
> ST1: I know, first time… when I was 13, 14… the time I met trigonometry… I… I don't think I grasped it at all and then when I came back to it the year after… I had to completely start again. So I had no… I was trying to multiply sine and x, I wasn't treating it as a function, I was actually treating it as a number… ehem… but since then… nothing specific […] for example when I saw sin 200, I thought it was sine multiplied by

> 200 so then I didn't grasp it as a ratio or anything I just grasped as a number…
>
> KE: […] Alright… and then basically do you have any problems in understanding or making sense of trigonometry? When you feel like very hard to understand? For example when you think you can't understand then you just learn the procedures.
>
> ST1: I definitely learned the procedure at 1st but now I think… I am still not… of angles between 0 and 90 I am happy with but above that then I just think about the graphs and then…so all my thoughts above 90 degrees will just shift to the graph and also the negative numbers, negative angles.
>
> KE: Ok… so these are the things that you feel it's quite difficult to make sense… right after the 90 degrees you shift into the graph?
>
> ST1: Yeah… that makes sense with the triangles make the triangles with the larger angles.

ST1 had problem in conceiving sin x as a function when he was 13 or 14. He thought that sin x should be grasped as a ratio. In the follow-up interview, he still thinks sine as the ratio of the opposite side to the hypotenuse of a right-angled triangle. This has impeded his learning of trigonometry in circle trigonometry where sine needs to be conceived as a number, consequently he used graphical trigonometry to make sense of trigonometry when the angles involved were greater than $90°$. ST1 said he should grasp $\sin 200°$ as a ratio however in the previous section of the follow-up interview he did express that he couldn't visualize a right-angled triangle with $\sin 200°$. If he couldn't visualize this right-angled triangle, then how is he going to get this ratio? This intimates a possible conflict in his mind without his awareness due to the changes of meaning between Euclidean geometry and modern Cartesian.

Relating ST1's learning experience of trigonometry to the trigonometry teaching trajectory in school, it is sensible to notice the confusion of ST1 in thinking of sine as a ratio or as number. The teaching of trigonometry starts from the introduction of sine as the ratio of opposite side to the hypotenuse of a right-angled triangle. Then in the unit circle, we need to think of sine as a number. ST1 has skipped the unit circle by using the graphical trigonometry solely when the angle involved is greater than $90°$. In this way, he could avoid the problematic conception that arises between the shifting from triangle trigonometry to circle trigonometry.

9.3.2 Student teacher ST2.

> KE: [...] do you have any difficulties in learning trigonometry?
>
> ST2: Erm... I hadn't really thought so like... erm... in doing trig questions... but maybe I don't have a general understanding of the relationships between things in reference to, I don't make sense certain points.

ST2 was aware that she didn't really understand trigonometry in particular the relationships between different parts of trigonometry. Apparently she has learnt trigonometry by rote without making sense of it. For instance, in the follow-up interview, she said she just accepted the range of sine x was between -1 and 1 without questioning why. There was one instance where she didn't want to think further for a question in the follow-up interview because she thought that she will not get the answer. According to Skemp's theory of goal and anti-goal, ST2 was trying to avoid her anti-goal because she sensed that she couldn't make sense of the mathematics statement that was being asked. Her goal is to be able to do the mathematics and accepts the mathematics facts without questioning why meanwhile deep inside her mind she could sense something that she couldn't make sense therefore she wants to avoid it. She realised she didn't grasp the concept of trigonometry.

9.3.3 Student teacher ST3.

> KE: Do you have any difficulties in learning trig?
>
> ST3: [...] I am one of these annoying people who are good at learning stuff and forgetting it when I don't need it anymore... I suppose I kind of learnt it if you like that's a really bad phrase I think... I am not the best at visualizing stuff always so I tend to think of it in term of facts rather than in terms of this is my concept if you like... I tend to be better more on the number and algebra side of things more than the concepts and understanding stuffs.

Apparently ST3 felt that he didn't have any particular difficulties in learning trigonometry. In fact, he does have problems in visualizing right-angled triangles with angles greater than $90°$ however this does not bother him because when the angles involved are greater than $90°$ then he will think of tracing a point on a rotating circle. He could link the perceptual ideas and symbolic ideas in a meaningful way. For analytic trigonometry, He didn't state the correct De Moivre's theorem. This is probably because he does not need De Moivre's theorem in most of the cases therefore he forgot about it. Another possible reason for this is because he didn't make sense of De Moivre's theorem. One may hypothesize that he could not see the relationships between Euler formula and De Moivre's theorem during the follow-up interview therefore he didn't derive it from Euler formula by using the exponential law.

9.3.4 Student teacher ST4.

> KE: […] do you have any difficulties in learning trig? What are the difficulties?
>
> ST4: I think that in terms of when I learnt it I didn't have any issues because you learned it in… in a very soh cah toa kind of way and when it comes to using it in the more complicated situations so as soon as you get to your infinite series… Fourier series that sort of things… then the understanding of it just seems to go… equally if you are trying to… in terms of learning it… learning by rote learning… I didn't find that difficult because it's quite simple diagramatic way of remembering pretty much everything… I mean I use my two little triangles for everything so for me I found that quite easy but I wouldn't necessarily automatically know the answer to every… so if somebody said what is the sine of 30 degrees I'd work it out rather than immediately responding.

ST4 felt that learning trigonometry was not difficult and she was referring to rote learning in this case. Furthermore she could derive most of the facts from the right-angled triangles. Her conception in triangle trigonometry becomes problematic in circle trigonometry when the angle involved is greater or equal to $90°$. For instance, she thought that the opposite side of the dynamic right-angled triangle could only get very close to the hypotenuse but will not coincide with it. Apparently she was not aware of her problematic conceptions therefore she couldn't sense these difficulties. She was aware of the difficulties in using trigonometry in particular analytic trigonometry because she couldn't understand it and she couldn't derive those analytic trigonometry concepts from the concepts she had met in triangle trigonometry.

9.3.5 Student teacher ST5.

> KE: [...] Do you have any difficulties in learning trigonometry?
>
> ST5: The last time I learned trigonometry it was easy, erm... I suppose the first time I learned trigonometry is that when you first see sine?....(inaudible)
>
> KE: So do you have any difficulties during that time?
>
> ST5: I don't remember having being any specific difficulties I suppose my inability to visualize 270 degrees triangle perhaps you could be considered a difficulty but I just think about well what is that equivalent to... so then I... you know draw the graph in my head and think ok so these are the points and it's closer to this therefore it does this...(inaudible).

ST5 realized she had difficulties in visualizing right-angled triangles when the angles involved were greater than $90°$. The follow-up interview shows that she knows there are relationships between trigonometry and Euler's numerical power series but she cannot remember them or grasp the detail of these relationships. Surprisingly she didn't see these as a kind of difficulties. In her first concept map (see Figure 7.9), she did not include the concepts of analytic trigonometry. After the follow-up interview, she added Taylor series and Complex numbers into her concept map. The follow-up interview might have triggered these links. Apparently she doesn't have strong and immediately links to the concepts of analytic trigonometry. This also explains her focus in trigonometry is not in analytic trigonometry therefore she couldn't identify the difficulties in analytic trigonometry in this instance.

9.4 Awareness of student's difficulties in learning trigonometry.

9.4.1 Student teacher ST1.

> KE: […] in your opinion, what kind of difficulties that a secondary school student might have in learning trig?
>
> ST1: Ehem… the same as me… not recognizing that it's a function that you can apply to numbers or apply to angles… ehem… and also using it… if you don't grasp that when using it as an inverse… using arc sin or arc cos, you can't grasp that either…

ST1 believed that students have difficulties in recognizing sine as a function. Apparently he thought that students will have difficulties on the compression from mathematics operations into an object which was a function.

9.4.2 Student teacher ST2.

> KE: […] in your opinion, what kind of difficulties that a secondary student might have in learning trigonometry?
>
> ST2: I suppose it's quite… it can be seen as quite abstract, I've just been taught these rules because sine equals opposite over hypotenuse, because sometimes I am not sure which angle we are talking about… we are talking about opposite and adjacent… erm… so there's that… all concepts that it's the ratios and the triangles sometimes… I am not too sure about.

ST2 felt that trigonometry is hard to understand. Sometimes she is confused with the angles and the sides of a triangle. In triangle trigonometry, the angle involved is always inscribes in a right-angled triangle whereas in circle trigonometry, the angle involved is not necessary inscribes in a right-angled triangle. This problematic conception caused great difficulty to some learners. Indeed the right-angled triangles involved in a unit circle are not always obvious. For instance, we couldn't see right-angled triangle for

$90°, 180°, 270°, 360°$ etc. She notices trigonometry is related to ratios and triangles sometimes. In this case, Micthelmore and White (1995) proposed the notion of abstract-apart to describe a phenomenon which indicates a mathematical idea is separated and apart from any context.

9.4.3 Student teacher ST3.

> KE: In your opinion, what kind of difficulties that a secondary school student might have in learning trig?
>
> ST3: Probably similar to me actually. It seems quite an abstract thing to them when it should be more… ehem… conceptual rather than heres a load of facts about trig.

ST3 felt that students might have difficulties in learning trigonometry because it was quite difficult to understand and conceptual. He sensed the needs to build a connective understanding between different parts of trigonometry rather just memorizing a lot of facts.

9.4.4 Student teacher ST4.

> KE: In your opinion, what kind of difficulties that a secondary school student might have in learning trig?
>
> ST4: Erm… probably the understanding of where it comes from… I suppose why… why you would necessarily care about anything beyond working out an angle… erm… though what I have come across with the current group is remembering the formula obviously you got sine rule and cosine rule area of a triangle that's not a right angle that sort of thing… erm… I guess they've not seen where it comes from and I think that kind of understanding maybe. It would help that there is a lot of background to it that I think maybe they have missed out at that age.
>
> KE: Are you talking about where the sine graph comes from? What do you mean?

> ST4: Erm… so I mean even where the sine rule and cosine rule come from if you are doing the area of a non-right angle triangle just to see that… if you split it up and you get those values from the right angles if you were to do it that way.
>
> KE: Ok… you mean how to derive those formulas?
>
> MC: Yeah… that's the sort of thing… I think that would help them… I think the difficulty is in remembering each one because they do… do it in quite a structured memory way.

ST4 felt that student might have difficulties in understanding trigonometry in particular building a connective understanding of it. She stressed the importance of deriving mathematics from known facts so that students can understand better. ST4 is referring to an actual experience in class (referring to 'the current group') and the problems that they have with triangles that are not right-angled. In the questionnaire, ST4 mentioned that she did not have any previous profession therefore we may hypothesize that she might be referring to a group of students during her teaching practice. She is concerned to make conceptual links e.g. splitting a triangle into two right angled triangles to calculate the area, while she notes that their difficulty is in remembering each formula 'in quite a structured memory way.' She seems like aware with the learning difficulties of the students. ST4 did link the triangle trigonometry and circle trigonometry in the follow-up interview. It is sensible to hypothesize that she is able to build these links partly because of her recent experience in teaching practice.

9.4.5 Student teacher ST5.

> KE: Ok. In your opinion, what kind of difficulties that a secondary school student might have in learning trigonometry?
>
> ST5: I think they need a link to real world stuff because trig… as soon as you go into trig suddenly it's all… you know… because

> yeah what is sine what is a triangle with 270 degrees and stuff like that it doesn't immediately make sense to them and you very soon leap off the page… you know… you leap off away from reality into just theoretical maths well it is this shape, why is it because it is and I think it's not rooted for a secondary school student it's not rooted in reality enough… erm… yeah.

ST5 felt that the problem of secondary school mathematics was not rooted in reality therefore students might have difficulties in making sense of it. She stressed the importance of realistic mathematics. She noticed some of the abstract ideas in circle trigonometry such as a triangle with $270°$. Apparently she could sense the problematic conceptions in shifting between triangle trigonometry and circle trigonometry which are due to the changes of meanings between Euclidean geometry and Modern Cartesian. ST5 might have noticed triangle trigonometry that works in the first quadrant but not in circle trigonometry outside the first quadrant.

9.5 Summary.

This summary is presented in a way to answer research question no. 11 of section 4.2. Most of the student teachers felt that trigonometry is hard to understand. It is rather easy to learn trigonometry by rote and do the computation. Meanwhile conceptual understanding in trigonometry is hard to achieve. This is because it involves three distinct contexts namely triangle trigonometry, circle and analytic trigonometry. In triangle trigonometry, students see it as ratio and proportion. In circle trigonometry, there is a dynamic trigonometry in the unit circle. Students need to see it as a number. There is also separately the gestalt vision of the graph and it's symmetry.

The shifting of triangle trigonometry to circle trigonometry needs compression. Students need to compress the ratio into a number.

It is interesting to note that ST1 still thinking trigonometry in terms of ratio of sides of a right angled triangle. This also explains why ST1 has difficulties in building a coherent links between triangle trigonometry and circle trigonometry. It is evident that most of the student teachers feel that secondary school students might have the same difficulties as they have in learning trigonometry. In general, they think that trigonometry is quite abstract in particular the concepts in circle trigonometry and analytic trigonometry such as visualising $\sin 270°$, infinite series, Fourier series etc. Students might easily get confused with the sides and the angles in a triangle especially in the circle trigonometry.

None of the student teachers has expressed explicitly the difficulties in coping with the changes of meanings between different contexts of trigonometry. However based on their excerpts, we can notice that some of them did sense the underlying difficulties in coping with the changing of contexts in trigonometry. In fact, most of the student teachers were not aware of their own problematic conceptions in making sense of trigonometry. All the student teachers didn’t build coherent links between circle trigonometry and analytic trigonometry. All these problems will definitely affect the teaching quality of the student teachers. The major issue is teachers need to be aware of the changes of meanings between the different contexts of trigonometry and recognize the supportive and problematic conceptions in making sense of a new context. This will definitely help the students to understand the mathematics better.

Chapter 10
Summary and Plans for Future Directions

10.1 Introduction.

This study is concerned how a group of student teachers make sense of trigonometry. The long term learning of trigonometry is a complex process in particular; this involves the dealing of changes of meanings across different contexts. In this study, I have focused on how student teachers cope with the transition of different contexts. A theoretical framework is proposed in chapter 3 in order to examine this. In this case, the formulated theoretical framework is based on the lens of mathematics cognition and the complexities in mathematics subject knowledge.

10.2 Main ideas and theoretical framework.

The theoretical framework proposed in this study is to examine how a group of student teachers make sense of trigonometry in particular; to explore how the student teachers cope with the changes of meanings as they learn more sophisticated trigonometry. There are three important ideas in this study. The first idea is about how humans make sense of mathematics through *perception, operation* and *reason* (see section 3.2 for details). The second idea is about the three distinct contexts in trigonometry namely triangle trigonometry, circle trigonometry (which involves graphical trigonometry) and analytic trigonometry (which involves power series and complex numbers). A detailed description of these contexts can be found in section 3.3 of this thesis. The third idea is about supportive and problematic conceptions in making sense of mathematics (see section 3.4 for details).

These three ideas are the essence of this theoretical framework. Student teachers make sense of trigonometry through *perception, operation* and *reason*. The long term learning of trigonometry is categorized into three distinct contexts namely triangle trigonometry, circle trigonometry and analytic trigonometry. As the student teachers shift into a new context, they have supportive and problematic conceptions which were rooted in the previous context(s) and these will either support (supportive conceptions) or impede generalization (problematic conceptions) in a new context.

10.3 Summary of the study.

This study has shown that the student teachers who have learnt advanced level university mathematics did not build coherent links across different contexts of trigonometry. This raises an important question of why the student teachers could not build these links. We hypothesize that this is about making sense of mathematics. As a consequence of this, it is sensible to say that learning more advanced level university mathematics does not necessarily contribute to the teaching of secondary mathematics because the student teachers couldn't use them in the teaching of school mathematics. Additionally, it is evident that the student teachers have compartmentalized the different contexts of trigonometry and they have developed personal ways of working in order to get on with their learning of trigonometry.

In order to understand how the student teachers make sense of trigonometry, a theoretical framework is proposed in this thesis. This involves investigating how the student teachers make coherent links across successive contexts in trigonometry, linking the ideas together through perception, operation and reason.

Based on the collected data, there is clear evidence showing that the student teachers operate in the three distinct contexts of trigonometry. For instance, most of the student teachers will make sense of trigonometry by using triangle trigonometry when the angle involved is less than 90° and switch to the graph immediately when the angle involved is more than 90°. A detail description of these contexts can be found in section 3.3 of this thesis.

The data also shows that the student teachers have their own preferences in working on different contexts to make sense of the mathematics items. For instance, some student teachers had used the unit circle (circle trigonometry) to make sense of why $\sin\theta$ can never equal 2 while others had used the right-angled triangle (triangle trigonometry). Furthermore, there was one student teacher who claimed to use the Taylor series (analytic trigonometry) to make sense of this, although the links he alluded to were not always clear.

The transition between different contexts of trigonometry is problematic for all the student teachers. For instance, most student teachers have difficulties in shifting between triangle trigonometry and circle trigonometry. This difficulty is mainly due to the changes of meaning between Euclidean geometry and Modern Cartesian. At the end, without linking triangle trigonometry and circle trigonometry in a coherent way, many student teachers tend to focus on using graphical trigonometry (using graphs without relating to the unit circle) when the angles involved are greater than 90 degrees. Meanwhile student teachers will use triangle trigonometry when the angles are smaller than 90 degrees. It is evident that when the angle is 90 degrees, student teachers have great difficulty in thinking of it. Some student teachers drew a weird figure for this (a triangle with two open ends). This clearly shows a supportive conception of triangle trigonometry with a problematic aspect when the angle is 90°.

The changes of meanings across different contexts will lead to the effect of either supportive or problematic conceptions in a new context. A supportive conception supports generalization whereas a problematic conception impedes generalization. One of the observed supportive conceptions with problematic aspects is that most student teachers couldn't see the triangle when the angle involved is more than $90°$. The development of circle trigonometry involves introducing new elements such as radians and functions. These elements provide the foundation for calculus. Most student teachers know they need to use radians in calculus but they did not grasp the reason for using it. Radian is a supportive conception in calculus with a problematic aspect relating to how it is introduced. Student teachers know they can use radians in all contexts of trigonometry but few grasp the reason why it must be used for advanced level mathematics.

Calculus is another area which is problematic for the student teachers. Some of them couldn't offer an explanation why the derivative of sin x is cos x. This indicates the student teachers know the concepts but they don't grasp them. Analytic trigonometry is problematic for all the student teachers. Some of the student teachers can't even state the Taylor series for sine and cosine. For instance student teacher ST1 perceives a Taylor series as a supportive conception with problematic aspects. He thought a Taylor series can be used to justify the characteristics of the sine graph and yet he didn't know the formula for the Taylor series. Superficially, the student teachers could link complex numbers to the concept of sine by stating De Moivre's theorem or the Argand diagram. However, to what extent they can link it to other aspects of trigonometry is not clear. In the same way respondent B, in the preliminary study, claimed to compute sin270° by substituting 3π/2 in the Taylor series, claiming that all the terms are zero except one, which may involve using the graph to see that sin270° is the same as cos0, linking together various subconscious ideas in imaginative ways. The subtle relationships within a student's mental schema can be a fruitful area of future research.

In general, the student teachers perceive that the subject matter knowledge tested by the mathematics items is either very important or important. Some student teachers perceive certain mathematics items are not important or not important at all. This is because these student teachers had focused on different aspects of the items and did not feel that it was important to remember facts that they could easily work out. Additionally, student teachers felt more confident in responding to mathematics items that involved computation instead of mathematics items that require explanations only.

Most of the student teachers feel that trigonometry is hard to understand but it is easy to rote learn it. Some of the student teachers can sense these difficulties in understanding trigonometry due to the changing of contexts. The transition from triangle trigonometry to circle trigonometry is difficult because it involves the changes of meaning between Euclidean geometry and Modern Cartesian. In this case, students need to compress the ratio into a number. The transition from circle trigonometry to analytic trigonometry is even more difficult because the concepts involved are so remote from the definitions of sine, cosine and tangent which are related to right-angled triangle. In this case, the relationships between Taylor series and sine are not obvious at all as the definition of sine is related to right angled triangle but Taylor series has nothing to do with right angled triangle.

10.4 Methodological considerations.

The sample involved in this study is very small. A questionnaire is distributed to a group of student teachers (24 persons) with an aim to collect a spectrum of responses notably from those who actually joined the follow-up interviews based on availability and voluntary basis. Hence there is no way this can reveal a whole picture but the evidence collected is strong and sould lay a good foundation for future investigations.

Additionally there are a few things that are different from my expectations and awareness at the start. In this case, the data clearly shows that the difference between circle trigonometry and graphical trigonometry. Student teachers tend to use graphical trigonometry without relating it to the unit circle. Furthermore, all the student teachers had used the visual symmetry of the graph rather than using the mathematics formula. This might be related to the nature of the questions of the questionnaire. We might hypothesize that the student teachers might use the mathematics formula when they want to get a numerical value.

This study also shows the complexity of the analytic trigonometry and how it is related to triangle trigonometry and circle trigonometry. Most of the student teachers know Taylor series as a representation of sine. It would be interesting to investigate how the respondents perceive Taylor series in relation to circle trigonometry in future research.

10.5 Suggestions for further research.

This study has highlighted the complications in making sense of trigonometry. It would be fruitful to investigate the transition from circle trigonometry to analytic trigonometry in future research. As we can notice, most student teachers know the concepts in analytic trigonometry but they don't grasp them. It is sensible to investigate whether the student teachers have coherent links between circle trigonometry and analytic trigonometry. Additionally, it would be interesting to see to what extent do the student teachers make sense of analytic trigonometry. For instance, how the respondents perceive Euler formula in relation to circle trigonometry.

This study is only a part of the effort to understand how student teachers make sense of mathematics. There is still a lot of things that need to be done in this area; in particular, to explore the problematic aspects of symbolism. As proposed in section 3.4, one of the problematic aspects in trigonometry is the formula $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. Students use this formula in calculus to solve problems. However the proof of this formula is problematic. The proof of the formula $\sin(\alpha+\beta)$ in triangle trigonometry is evident (see Figure 3.6 of chapter 3). The proof in circle trigonometry is very complicated and so far I have not seen any proof in circle trigonometry. At a later stage the proof in analytic trigonometry is simple and direct (see page 53).

Additionally, as we can notice from the follow-up interviews, all student teachers' had used graphical trigonometry to make sense of the mathematics statements. None of them has used a formula such as sin(a+b). In this case, it would be interesting to investigate the role of mathematics formulae in making sense of mathematics in future research.

10.6 Broader issues in Mathematics Education.

This research shows the need for a teacher as mentor to be aware of the students' current state in development to encourage them to make sense of new ideas. In this case, it is sensible to have a new aspect for teacher training which focuses on the supportive and problematic conceptions in a new context. Recognizing and acknowledging the supportive and problematic conceptions in a new context is of paramount importance so that teachers may be aware of the effect of the met-before that affects current learning. Problematic met-before or conception impedes future learning whereas supportive conception supports generalization. This raises the question whether learners feel that mathematics is abstract because of these problematic conceptions.

In the studying of trigonometry, it is not just about teaching what the students should know. Indeed, teachers should be concerned about what knowledge the students bring into the lesson as a result of their learning experiences in arithmetic and algebra. This raises a new thought on whether the learning of trigonometry can be improved by a new form of awareness in teaching much younger children. The American curriculum has focused on the ideas of ratio and proportion, this is an indication of the difficulties encounter in triangle trigonometry. Are there ways preparing for this teaching to make the concepts more meaningful? Blackett and Tall (1991), used interactive computer graphics to relate the visual model (triangle figures) to numerical data has helped students to improve their performances. The advancement of technology nowadays has allowed us to prepare better tools for the teaching and learning. For instance, students can operate on the Apple Ipad screen to have a better sense of proportion enactively.

Kidron and Tall (2013), show that power series might be understood more clearly by seeing polynomial approximation as a power series quickly approximating visually so that people can get a sense of how power series can quickly converge so that polynomials are good approximation to functions express as power series. This also shows the sophistication of making sense of mathematics which involves the blending of visual and symbolic representations. As we can see, all these are related to making sense of mathematics through perception, operation and reason. This shows the importance of making sense of mathematics so that more advanced ideas maybe understood meaningfully.

Trigonometry is a very important and interesting topic in mathematics because it holds a central position in linking together visual and symbolic ideas which later lead to the formal analytic ideas. If a learner has a coherent knowledge structure on trigonometry then at the later stage of learning, this learner should be able to reduce his/her cognitive burden by remembering less things and focusing on the relationships between different concepts to derive the things that he/she needs at a particular time. For instance, in trigonometry, we can derive most of the things from either sine or cosine. If we differentiate sine, we will get cosine. If we differentiate cosine, we will get negative sine. We can compute tangent by using sine divides by cosine. This shows the tight structure of mathematics knowledge. Tall (2011) used the term crystalline concept to emphasize this tight structure in mathematics knowledge. In this case, it is sensible to say we only need either sine or cosine in learning trigonometry as mathematics is highly structured. This indicates the importance of building a coherent knowledge structure.

There was a lot of mathematics education research studies conducted during the past decades. Different researchers are interested in different issues of mathematics education. Meanwhile, the main aim of all these research studies is to help learners to learn mathematics either explicitly or implicitly. There is no exception for this study. Hopefully this study can lay a good foundation for future research studies which will eventually help human to make sense of mathematics.

APPENDIX

Questionnaire

The questionnaire will remain anonymous unless you give your name for feedback purposes.

Please create a 6 digit random number for yourself as participant number. Please write down the same 6 digit number on your answer script.

Number:…………………………………Name(optional)…………………..

Degree Subject:…………………………Degree Class:……………………...

Graduated Institution:………………………………………………………

Previous Profession:……………………….Gender:…………………………

Part A: Please write down your response for the following questions in another piece of paper.

1. Describe $\sin x$ in your own words.
2. Please arrange the following values of sine in ascending order and explain your answer.
   a) $\sin 110°$
   b) $\sin 250°$
   c) $\sin 335°$
3. How do you make sense of $\sin 200°$?
4. What is the value of $\sin 270°$? Explain why $\sin 270°$ has this value?
5. What is sine over cosine? Does that mean anything?
6. What do Radians mean? Why do we need radians when we have degrees?
7. For what values is $\sin x$ decreasing? Explain why is it decreasing for these values?
8. Explain why sin θ can never equal 2.
9. What does 'trigonometric function'mean?
10. What does dy/dx mean?
11. What would d/dx [sin x] mean? What is d/dx [sin x]? Explain why.
12. Describe as fully as possible what you understood by the following terms:
    (a) $\sin 30°$
    (b) $\sin 120°$
    (c) $\tan 90°$
13. Explain your interpretation of the following terms
    (a) $\cos^{-1} 0.5$
    (b) $\sin^{-1} 2.5$

14. Which of the following are equivalent (where the angles are measured in degree)? Explain why they are equivalent.
    (a) $\sin x$
    (b) $\cos x$
    (c) $\tan x$
    (d) $\sin(x-90)$
    (e) $2\sin x$
    (f) $1+\sin x$
    (g) $\cos(x-90)$
    (h) $\sin 2x$
    (i) $\sin(x+1)$
    (j) $\sin(x+90)$
    (k) $\sin x+1$
    (l) $\sin x/\cos x$
15. For each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine:
    (a) function
    (b) series
    (c) complex number
    (d) $y=mx$

Part B: Comments on importance of mastery of subject matter knowledge.
Instruction: For each of the items 1 to 15 on page 1 and 2, indicate below how important you believe this knowledge to be for a teacher of mathematics. Choose <u>ONE</u> response only for each item.

Mastery of Subject Matter Knowledge indicated by the item is	Very important	Important	Neither important nor unimportant	Not important	Not at all important
Item 1					
Item 2					
Item 3					
Item 4					
Item 5					
Item 6					
Item 7					
Item 8					
Item 9					
Item 10					
Item 11					
Item 12 (a)					
Item 12 (b)					
Item 12 (c)					
Item 13 (a)					
Item 13 (b)					
Item 14					
Item 15 (a)					

Item 15 (b)					
Item 15 (c)					
Item 15 (d)					

Part C: Levels of confidence to respond to the mathematics items in the task.

Instruction: For each of the items 1 to 15 on page 1 and 2, indicate below your level of confidence in responding to the item. Choose only ONE response for each item.

What is your level of confidence in responding to the item	Not confident at all	Not confident	No opinion	Confident	Very confident
Item 1					
Item 2					
Item 3					
Item 4					
Item 5					
Item 6					
Item 7					
Item 8					
Item 9					
Item 10					
Item 11					
Item 12 (a)					
Item 12 (b)					
Item 12 (c)					
Item 13 (a)					
Item 13 (b)					
Item 14					
Item 15 (a)					
Item 15 (b)					
Item 15 (c)					
Item 15 (d)					

Transcript of ST1's follow-up interview.

KE: Item one sounds like this describe sine x in your own words so can you read your answer for item one for me?

ST1: The ratio of the opposite and hypotenuse in a right angled triangle.

KE: Do you have anything else that you want to add or you feel happy about this?

ST1: I know it's the ratio because… more because I've always been told is the ratio not because I have any kind of deep understanding of why… ehem… yeah… so I can't say something that I understand fully but it's just something that I know because I have been told it.

KE: It's alright. For the second item of this questionnaire it sounds like this please arrange the following values of sine in ascending order and explain your answer. You are given three options here sin 110 degrees, sin 250 degrees and sin 335 degrees. Can you read your answer for me for item 2?

ST1: So I drew a sine graph from just the positive from 0 degrees to 360 degrees such as one period… ehem… and I knew that it's highest value is 90 degrees and lowest value is 270 and I just approximated on and I worked out which one will be bigger by the difference between those values and the crossing points and the maximum and the minimum that I already knew.

KE: Ok. What about this one? What is this?

ST1: Go to the diagram… ehem… most positive to most negative so that is just me explaining why I chose this order.

KE: Ok, so your order is… you are talking about the biggest value is…

ST1: Sin 110 then sin 335.

KE: And the smallest is sin 250.

ST1: Yeap.

KE: For item 3 of the questionnaire it sounds like this how do you make sense of sin 200 degrees? Can you read your answer for me please for item 3.

ST1: Ehem… I put sin 200 is negative sine (200-180) which reading it back doesn't make much sense.

KE: Can I know because it seems like you crossed out this 180-20. why you crossed this out.

ST1: Ehem… I think I was trying to make sense of sin 200 being just taking the graph so having this first up to 90 defined by the… ehem… so defined by a ratio of the triangle and the rest of it just being a continuation of this… ehem… a continuation of the curve to make it 2 pi periodic so I tried to explain sin 200 is just this graph continued on 200.

KE: Do you want to draw? I can give you paper if you want to. You can always ask for paper from me.

ST1: I was just thinking that sin 200 would be about… ehem… so it's gonna be minus sin 20… ehem… yeah (he was writing on a piece of paper)… which is what I have got so for some reasons I put 200 minus 180.

KE: So basically when you are making sense of this you are trying to refer to the sine graph.

ST1: Yeah.

KE: Ok. Alright. Can you visualize this triangle for sin 200 degrees?

ST1: Ehem… no.

KE: You can't visualize this triangle.

ST1: No. I can't…(pause)… no it's never really a way I have thought about it.

KE: OK. Never mind. It's alright. Just want to understand what you are thinking. So for item 4 sounds this what is the value of sin 270 degrees and explain why sin 270 degrees has this value so can you read your answer for me?

ST1: Ehem… I have said sin 270 is minus 1 and I said I had no idea why and I just said is 2 pi periodic.

KE: OK. Basically you were trying to say sin 270 equals to negative 1 and then you have no idea why sin 270 equals to negative 1 and then why you written here 2 pi periodic.

ST1: Ehem… I was trying to justify the only real understanding why I said negative 1 purely because it's… because actually shift it along 180 degrees you are doing the same thing but negative… but again I can't see why it would be negative one more than because that's what the graph says.

KE: Ok. So you are just thinking about the graph at the moment. Alright. Shall we proceed or do you still want to think about this question.

ST1: Maybe we go to the next one.

KE: Ok. No problem.

KE: For item 5 of the questionnaire it sounds like this what is sine over cosine? Does that mean anything. Can you read your answer for me please?

ST1: I said it was tan x… sin x divided by cos x is equal to opposite over the hypotenuse divided by the adjacent over the hypotenuse which cancels down to give you opposite over adjacent so that gives you the ratio between those two sides.

KE: Alright. Ok. We proceed to item 6. Item 6 sounds like this what do radians mean why do we need radians when we have degrees. Can you read your answer for me please?

ST1: Radians are the measure of an angle, the length subtended by an arc of radius length.

KE: Radians are measure of an angle…ok.

ST1: Yeah… so I meant was if that's… ehem… so that was one radian… oh no… so if that's one that's just the length of the radius, that is the length of the radius and this is the length of the radius so that have been cut out from the circle (drawing on a piece of paper) and that going to be one radian.

KE: So this is what you are trying to say here?

ST1: Yup.

KE: Ok. So what about this (pointing to his answer for item 6)?

ST1: Ehem… so I said why we need it is the first thought came into my head was we need it for the calculus.

KE: So here you are trying to say that calculus does not work with degrees (pointing to his answer script and reading it aloud)…ok.

ST1: Yeah.

KE: Ok… and then?

ST1: I said these are a simplify form of measure so that it can be explained before irrational numbers are met. The Radians are not natural they're just the definition of angle measurement.

KE: So do you know why calculus doesn't work with degrees?

ST1: Ehem.

KE: Do you have any sense of why calculus doesn't work with degrees…do you have any idea about this?

ST1: These degrees are just a number that someone's put on it because 360 has a lot of factors whereas radians they kind of have their own natural place in mathematics they make sense but… ehem… is not something that I thought about but I have no conclusion about it.

KE: Do you want to think about it now? Or do you need more time?

ST1: I don't think I can.

KE: Let us look at item 7. Item 7 sounds like this for what values is sin x decreasing and then why it is decreasing for these values. Can you read your answer for me please?

ST1: I have just said dy by dx is equal to positive cos x.

KE: So dy by dx is equal to positive cos x. May I know how you came to this conclusion?

ST1: Ehem… when you differentiate you get the gradient function so whenever cos although I guess this is a bit of a circular argument but whenever cos is positive… eeerrr… sorry but what was the question again?

KE: Is this one… question 7 (pointing to item 7 in the questionnaire).

ST1: So when cos is negative then sin is decreasing so plot a negative between… eeerrr… pi over 2 radians and 3 pi over 2 so it's in that range so that sin x is decreasing.

KE: Alright so you are trying to understand these things by looking at the derivative of sin x?

ST1: hmmm.

KE: Y equals to sin x is for this y right (pointing at his answer script for confirmation)?

ST1: Yeah.

KE: Ok. Alright. Item 8 sounds like this explain why sin theta can never equal 2. Can you read your answer for item 8?

ST1: I put sin theta equals to 2 equals to opposite divided by hypotenuse… ammm... and if you just take those last two so two equals to over opposite hypotenuse and multiply the hypotenuse across so you have two hypotenuse equals to the opposite that implies that the hypotenuse is half the size of the opposite but in a triangle… in a right angled triangle… hypotenuse is always gonna be greater than … I guess greater the opposite of all the triangles.

KE: So you are doing some manipulation here and come out with this conclusion. What is this (pointing to his answer script)?

ST1: Hypotenuse is half of the opposite that is what that this telling us. The hypotenuse is greater than the opposite for all triangles.

KE: Hypotenuse half of the opposite, is it (pointing to his answer script for his confirmation) for all the triangles?

ST1: I was doing some computation based on what he said just now in order to get a clear picture of how he arrived at his conclusion.

KE: Item 9 sounds like this what does 'trigonometric function' mean?

ST1: I said is the family of functions defined by a combination of sine and cosine.

KE: Is there anything you want to add or comment?

ST1: Armmm… no… not really I think that is the way I reason it.

KE: We can proceed to item 10. Item 10 sounds like this what does dy by dx mean?

ST1: I put small change in y divided by a small change in x at an infinitesimal level this describes the gradient.

KE: I was trying to understand what did you mean infinitesimal?

ST1: Armm… when you… so if are looking at just say at quadratic… if you are looking at the gradient at this point… as you bring the point closer and closer towards it and you measure the tangent at that point it's going to get more and more accurate to the gradient at that point so it's only when you bring it to infinitesimal distance a tiny tiny distance away that… that becomes an accurate measure of the gradient at that point.

KE: Ok. In fact I didn't really get it. Can you explain it one more time? So this is the graph…

ST1: So at that point if you want to know the gradient of the tangent at that point then you start at the point from where we join those two up measure the gradient of that line and that should bring these points closer and closer together so the distance becomes smaller and smaller that's gonna give you an accurate representation of the gradient of the tangent of that point.

KE: Ok. So how does this term relate to the item here?

ST1: It's so infinitesimal is just when we're essentially on that point so we are measuring the gradient of one point now… ehem… or we are measuring at the gradient of two points that are so close together that they can't be separated.

KE: Ok. Alright. Item 11 sounds like this what would d by dx sin x mean what is d by dx sin x and then explain why so can you tell me your answer for item 11.

ST1: I have said what is the gradient of sin x that gives you the gradient of sin x… and the answer is d by dx of sin x is equal to cos x due to the definition of d by dx ie is the limit of h near to 0 (inaudible and he was reading from his answer script) written here …negative g of x + g of (x +h).

KE: What is the gradient of sin x?

ST1: So that's… I don't know why I wrote it like that but…so it's the gradient of sin x (he was pointing to the questionnaire to indicate that was his answer for 'what would d by dx sin x means').

KE: Is the gradient of sin x (pointing to the answer sheet with reference to questionnaire to confirm his answer)?

ST1: It's cos x (I was pointing to the questionnaire to get his confirmation for 'what is d by dx sin x').

ST1: Because of the definition of d by dx which is (pointing to his answer to 'explain why' and then interrupted by KE).

KE: In your opinion, what is a d by dx for you?

ST1: D by dx just by itself?

KE: Yeah… just by itself… just only this one (pointing to his answer sheet)?

ST1: For me that's just an operation when you apply it to a function and it gives you the gradient function of that original function.

KE: Ok. It seems like an operator.

ST1: Yeah… an operator.

KE: Ok. You were trying to say d by dx of sin x equals to cos x due to the definition of d by dx (pointing to his answer for item 11)... is the whole thing the idea of this limit (pointing to his answer to get confirmation)?

ST1: Yeah… so that's the definition of d by dx when it's applied to the function f.

KE: Ok.

ST1: Which is just… so in this case our function would be sin x.

KE: Alright. In your opinion what does limit mean?

ST1: Ehem… in limit… while limit is h tends to 0 is if you take… if you take the function either side you assess function either side of 0 and you slowly work your way in and limit is the number that when you… so again you're getting infinitesimally close to 0… then the limit of… so we just did the limit… so the h tends to 0… (inaudible)… when these two values are infinitesimally close either side to 0 and they are equal that will be the limit of h tends to 0.

KE: So that is your idea of… your perception about limit… alright… ok… we continue item 11. So basically can you define d by dx. What is the definition of d by dx?

ST1: I would say the limit is h tends to 0 of (-f(x)+f(x+h))/h.

KE: Alright. So for item 12 describe as fully as possible what you understood by the following terms so the first term is sin 30 degrees and then second term sin 120 degrees and then the third term is a tangent 90 degrees. What are your answers for these items?

ST1: Ehem… so for the first one sin 30…I say is the ratio of two sides when you the angle is 30.

KE: Ok.

ST1: Of the right angle triangle… for the second one I said it's a 2 pi periodic continuation based upon taylor series… so referring back to the power series definition.

KE: Alright… so 2 pi periodic continuation based upon Taylor series (pointing to his answer sheet to confirm his writing)… so what comes into your mind when you tried to make sense of this? I mean how do you link this into sin 120? What comes into your mind?

ST1:Ehem… probably the definition of sine given by the power series…although at the moment I can't remember exactly what that is.

KE: Ok… so you have a sense that sin 120 is related to the Taylor series.

ST1: Especially in terms of getting accurate kinds of answers.

KE: Ok… so you believe that Taylor series can give you an accurate answer for?

ST1: Yeah… accurate calculations for sine specific degrees.

KE: Ok… alright… what about 12(c) for tangent 90 degrees?

ST1: I have said it's infinite but I kind of disagree with myself… ehem… it doesn't exist because…… yeap so… so that is 90 degrees there… so as you approach from the left hand side it goes on to infinity which is probably why I said that but if you approach from the right hand side it will go down to negative infinity so the limit does not exist because they are not equal.

KE: So you think this one shouldn't be infinity?

ST1: No I don't think it would be… it just doesn't exist…with that one I refer it back to the triangles as well.

KE: You mean this one 12 (C)? You refer back to the triangle?

ST1: Yeah… so I kind of refer it both to the graph of the tangent and how it defines how it describes the triangle.

KE: You feel like this one should be doesn't exist?

ST1: Yeah.

KE: Ok. And then how do you make sense of it this angle cannot exist how do you arrive at this conclusion?

ST1: Ehem… so tan x is opposite over adjacent (writing on a piece of paper)… ehem… suppose that is x… the opposite… adjacent… so you were saying this angle here is 90 degrees… but then if that was 90 degrees then you've got two parallel lines and that can't possibly form a triangle (he was drawing a triangle on a piece of paper).

KE: Ok… so basically when you are trying to make sense of this you are thinking about a right angle triangle… ok… alright… item 13 sounds like this explain your interpretation of the following terms, so the first term is inverse cosine of 0.5.

ST1: What angles give you the sine value of 0.5, there are infinite possibilities because of the 2 pi periodic nature of sine.

KE: Ok… so what about 13 (b) inverse sine of 2.5?

ST1: I said impossible because the ratio of sides can never give you 2.5.

KE: So you are thinking about the triangle then you making sense of this?

ST1: I probably… probably the 1st thought is probably to the graph… and that bounded between minus 1 and 1 and then to explain why that happened.

KE: Ok… so you are thinking about the graph… you can notice from the graph it will not have 2.5 in the graph… alright so this is 13. Item 14 sounds like this which of the following are equivalent (where the angles are measured in degree)? Explain why they are equivalent. Can you read your answer for me please?

ST1: I put L and C.

KE: L and C… ok…

ST1: so tan x and sine over cos x.

KE: So you relate this one L and C…so you are trying to say L and C they are equivalent?

ST1: Yeap… because the reason described earlier so in question…where is it? Question 5… for the same reason.

KE: Ok… alright… which means that you are referring item 5?

ST1: Ehem.

KE: Ok… alright… and then what about this one (pointing to his answer for item 14)?

ST1: So I said I used the double angle formula on sin 2x which is (h) and I said it was equal to 2cosx sin x and just from how I recognize trigonometric functions… it's not equal to any of the other ones on there.

KE: So you remember this formula then you tried to match with other options... ok… so you couldn't find any match with the formula… ok… so what is this?

ST1: I was unsure about that… so I said I think that (g) (d) (a) (j) and (b) could be related… because… so for (g) for example cos of x minus 90 which is the cos graph moved right by 90 degrees along the x axis… ehem… so that would give you sin x, so (g) and (a) would be related.

KE: So you think (g) and (a) are related…ok.

ST1: So sin x minus 90 is sin moved right by 180 (he was drawing sin(x-90) on a piece of paper)…… so that is negative cos that one there…… so I can't see any… I can't really imagine that one… (j) that's moved the other way so that's equal to cos x so (j) and (b)… so that is (a) (g) (d) and I said (f) and (k).

KE: What is (f) and (k) here?… (f) is 1 + sin x, (k) is sin x + 1.

ST1: And I've said because of the commutativity of addition.

KE: Ok… based on commutativity of addition… alright… what is this?

ST1: So (i) sin of x + 1 is a translation of sin x left 1 and that didn't have any had no equal parts on that and 2 sin x I said is sin x stretched vertically… so there is no boundary between 2 and minus 2 and that didn't have any.

KE: So you are trying to say 2 sin x is equal to the sin x graph and then you stretched vertically… ok… item 15 basically sounds like this, for each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine. So basically the 1st one is function, can you read your answer for me please?

ST1: I've said sine is a function has values such the first value never repeated.

KE: Ok… has values such that the first value never repeated (pointing to his answer to confirm his writing).

ST1: So it was set theoretic definition of a function.

KE: Ok… sine is a function, pairs of values.

ST1: So the 1st value is never repeated.

KE: Ok… alright… and then for item 15 (b) what about series?

ST1: So I've said it can be defined by power series.

KE: Ok… and then what about complex number?

ST1: I've said r e to the power of I theta equals to r open bracket cos theta + I sin theta close bracket.

KE: Alright… what is e in this sense for you?

ST1: e is… ehem… the exponential… so an exponential function of 1 where at the exponential function… ehem… so is the number such that when raise the power of x and differentiated then gives you the same.

KE: Ok… alright… so what about the y =mx?

ST1: I've said you can form a triangle by the y=mx graph… dropping down the perpendicular to the x axis and they relate to the gradient of that line.

KE: Alright… so you are using y=mx is like a hypotenuse and then you construct a triangle on a Cartesian coordinate plane?

ST1: Yeap.

KE: Ok… now I am going to show you some responses of other students… to see what will you comment about them… for example… look at this one item 3 here… ok… item 3 here is… the question sounds like this what is the value of sin 270, explain why sin 270 has this value so a student gave me this response… any comments for this response… what do you think about this response?

ST1: I agree with that… it's probably something I would have said when I was doing my maths degree but now we have thought about the power series… how the power series is defined… I have never had a good understanding of that area definition of trigonometric functions, but I agree with that as a way of thinking about that.

KE: So basically you believe that by substituting into the Taylor expansion it will end up with this value (pointing to the response given)?

ST1: Yeah.

KE: Ok… the reason for this is due to the fact that when 3 pi over two is substituted into the Taylor expansion the terms end up being zero except for one term.

ST1: Ehem… ehem… without actually getting the power series and looking at them myself I am not 100 percent sure that is the only one term is not zero but I can't remember enough to argue that.

KE: Alright... but you believe that the Taylor series can help you to justify that?

ST1: Yeap.

KE: So you are not sure about whether it will end up being zero… ok.

ST1: hmm.

KE: But you feel that the Taylor series might help you to justify this thing… sin 3 pi over 2 is equal to negative 1… is that what you mean?

ST1: Hmm… if by Taylor series you mean the power series definition of sine then yeah.

KE: Ok… alright… and item 5 explain why sin theta can never equal 2 and then this guy gave this response (showing him the student's answer sheet)… so what do you think if you are a teacher? And then you want to mark them.

ST1: Ehem…… I think that's fine apart from…… I guess I haven't gone into much detail about why the bound…(inaudible)… I don't agree with this bit in the brackets here (subsequent terms are all smaller or equal to 1 so they can be bounded)… but the rest of it, I guess it makes sense.

KE: Alright… ok… you got a feeling it makes sense for you but your just not sure about of the terms you end up with.

ST1: I have to go through the power series and then check myself.

KE: Ok… ok… that's fine… based on your questionnaire, we look at part b here… for example like this item, item 7. It seems like item 7, just one response only you think the knowledge is not important for item 7. Do you have any specific reason for that?

ST1: I think… you probably need to know for what values it's decreasing but that's something you can just look at the graph or look at the function and work out, it's not something you need to know I don't think I need to know why it's decreasing.

KE: You don't think you need to know… and then you don't think this is very important for a teacher of maths to know it?

ST1: I don't think so.

KE: Alright… it's fine… for item 15 (b) you are talking about not at all important… 15 (b) is series.

ST1: The reason I said that is because as a secondary school teacher that isn't part of the syllabus… ehem… yes so… yes you can show that… so you can say that it is defined by series and that's used to give you accurate calculations but I don't think you can go into more detail than that.

KE: Ok. What A level you did last time?

ST1: I did maths, further maths, chemistry, psychology.

KE: Alright, maths, further maths, psychology, chemistry (to reconfirm his response).

ST1: I did an AS level in biology.

KE: The part c of the questionnaire seems like you got one item that you are not confident which is the the 15(d)… do you have any specific reason for this one?

ST1: That's one of the things as soon as I look at it I had to think for a while how that was related and it took me a while to come up and I actually drew a graph and then decided that that calculation.

KE: So which means that in comparison to other questions this one you took longer time to think about that?

ST1: Yeah… it's not something that I instantly calculated.

KE: Ok. Alright… basically do you have any difficulties in learning trigonometry? Can you think of any?

ST1: I know, first time… when I was 13 14… the first time, I met trigonometry… I… I don't think I grasped it at all and then when I came back to it the year after… I had to completely start again. So I had no… I was trying to multiply sin and x I wasn't treating it as a function, I was actually treating it as a number… ehem… but since then… nothing specific.

KE: Ok… alright… was that a secondary school at that time?

ST1: Yeap.

KE: Can you remember the kind of specific problem? Is it the procedure or you can't grasp the concept?

ST1: I didn't know that they were functions that worked on the x… so for example when I saw sin 200 I thought it was sin multiplied by 200 so then I didn't grasp it as a ratio or anything I just grasped it as a number… that works stuff out for you that you (inaudible).

KE: Ok… alright… other than this, do you have any other difficulties in the learning of trigonometry?

ST1: No… not anything I can think of.

KE: Another question now, in your opinion, what kind of difficulties that a secondary school student might have in learning trig?

ST1: Ehem… the same as me… not recognizing that it's a function that you can apply to numbers or apply to angles… ehem… and also using it… if you don't grasp that when using it as an inverse… using arc sin or arc cos you can't grasp that either… ehem.

KE: Have you taught this topic before in the school?

ST1: No. I haven't.

KE: Any other difficulties that you can think of that a secondary school student might have in learning trig?

ST1: I guess you've got to be quite proficient in rearranging formulae… so rearranging equations before you can start to… attempt the more difficult questions instead of just… instead of these types of questions which may involve multiple trigonometric functions or things like that.

KE: Alright, and then basically do you have any problems in understanding or making sense of trigonometry? When you feel like very hard to understand? For example when you think you can't understand then you just learn the procedures.

ST1: I definitely learned the procedure at 1st but now I think… I am still not… of angles between 0 and 90 I am happy with but above that then I just think about the graphs and then… so all my thoughts above 90 degrees will just shift to the graph and also the negative numbers, negative angles.

KE: Ok… so these are the things that you feel it's quite difficult to make sense… right after the 90 degrees you shift into the graph?

ST1: Yeah… that makes sense with the triangles make the triangles with the larger angles.

KE: Cool, so after 90 degrees you switch into the graph so alright. We are done with the questions.

Transcript of ST2's follow-up interview.

KE: Item 1 of the questionnaire. Describe sin x in your own words. Can you read your answer for me please?

ST2: Sin x is a function with range brackets and then -1 and 1.

KE: Is there anything you want to add or elaborate more?

ST2: hmmm… not really.

KE: Ok so you are quite happy with this one. Fine. No problem. Item 2 sounds like this please arrange the following values of sine in ascending order and explain your answer so you are given three sub items here. The 1st one is sin 110 degrees, sin 250 degrees and sin 335 degrees. Can you read your answer for me please?

ST2: I think I got… erm… what is it supposed to be in, ascending order? Erm… I've written (b)(c)(a) but on my graph it seems to be different. I think I meant to put (a)(b)(c).

KE: So what you are trying to say is… what is the largest value for these three?

ST2: Oh… no… no… no… the largest value was… (a).

KE: Ok… so you think the largest is sin 110. What about the smallest one?

ST2: Erm… erm… I think (b).

KE: (b) is the smallest one. Ok so you think (b) is the smallest one and then (c) is the middle one?

ST2: Yeap.

KE: Basically how did you arrive at your answer?

ST2: Ok. It looks like I drew a graph of the sine graph. Between 0 and 360 degrees… erm… so it's 1, -1 so it's 0… erm… and… I found that (a) I thought is the only positive one so that's why that one is the largest and then when looking at the negative ones… erm… which one is (b) again. Can I write on this?

KE: Ya.

ST2: (b) equals to sin 250.

KE: You can use paper if you want.

ST2: I found that it was minus 1 at 270… erm… sin 250 is 20 less than that and 335 is going to be 65 more than that but it won't go back to 0 yet. So by the symmetry I thought (b) is going to be the smallest. The most negative.

KE: Ok. So which means you used the graph to approximate the location of the point and to see which one is the biggest and the smallest. Ok so that is fine. Just now in item 1 you write down your description for sin x and then what is the relationship between your description in item 1 to the sine curve?

ST2: Erm… my graph is going between 1 and minus 1 on my axis so that showing the range… erm… and the fact well, that doesn't really matter much. This function is just the graph of it (pointing to her answer script).

KE: Let's us look at item 3, how do you make sense of sin 200 degrees?

ST2: Erm… to start off with I looked along on my graph to see where 200 was so sin x is 0 at 180 so 200 is just going to be a bit more negative so using the symmetry I had a look and thought it would be the same… erm… oh yes… so I said that sin 200 is 20 degrees greater than 180 so that would be the same if the negative value is 20 less than 180 and I've put evaluate this on a calculator (she was reading her own answer script).

KE: So your answer is this one (pointing to her answer)?

ST2: If I was to do it now, I would do a similar thing, but look at sin 200 and draw a horizontal line over here and say it was going to be the same as this one here.

KE: Draw a horizontal line to see the symmetry?

ST2: Ya.

KE: Ok. Alright. Can you visualize this triangle with sin 200 degrees?

ST2: Erm…(thinking for a while)… no.

KE: So can you draw the triangle?

ST2: No.

KE: That's fine. We move to item 4 so item 4 sounds like this what is the value of sin 270 degrees? Explain why sin 270 degrees has this value? Can you read your answer for me please?

ST2: Sin 270 degrees equals minus 1 and then I've drawn a unit circle on a graph. In a unit circle sin x is the y coordinate. When x equals 270, y coordinate is minus 1. Ok. You can also see it from the graph I drew (pointing to the graph that she drew for item 2).

KE: What about these few bits? Why you crossed out these bits?

ST2: I think that one might have been… I don't know what that one was a little right angle triangle with… it looks like a right angle triangle… there again it seems to be the unit circle. It's got maybe 60 degrees and I don't know what I was trying to calculate to be honest.

KE: Are you trying to draw a triangle with sin 270 degrees?

ST2: Erm… possibly… yeap… I am pretty sure that's what that would have been I just can't remember to be honest…yea.

KE: Can you visualize this triangle with sin 270 degrees?

ST2: Erm… looking at this angle from this line down to there so that would be 270 degrees… but I am not sure… can you draw a triangle with that? I am not sure.

KE: Alright. You also can't draw the triangle with this angle sin 270. For Item 5, item 5 sounds like this, what is sine over cosine? Does that mean anything? So can you read your answer for me please?

ST2: Sine over cosine equals tan.

KE: Is that anything you want to add or elaborate further?

ST2: Erm… not really.

KE: So we move to item 6, what do radians mean? Why do we need radians when we have degrees? So can you read your answer for me please?

ST2: I've put 180 degrees equals pi. Radians are another way of describing angles which are useful for trigonometric functions.

KE: I've just noticed this. What were you trying to write here?

ST2: Tan x is not defined for cos x equals 0. That is because I put sin x over cos x equals tan x and obviously you divided by 0 it's not going to be defined so that's why the tan graph looks like this.

KE: We continue for item 6, what do radians mean? Radians are another way of describing angles which are useful for trigonometric functions (I was reading her answer for item 6). Do you know why do we need radians when we have degrees?

ST2: Erm… it makes writing things easier I am not sure of the exact reason why we need them we are obviously writing pi it's a different kind of group that… like amounts of 180 degrees.

KE: Do you prefer to use degrees or radians?

ST2: Erm… probably in more advanced maths, I would prefer to use radians, but if I am for example looking at GCSE maths and normal triangles I would use degrees.

KE: Ok. Do you know what is 1 radian?

ST2: 1 radian would be… erm… I suppose a 180 degrees divided by pi… that is pi radian.

KE: Alright. For item 7, for what values is sin x decreasing? Explain why it is decreasing for these values? Can you read your answer for me please?

ST2: Sin x is decreasing for between 90 degrees and 270 degrees and then I've put in brackets 90 degrees plus 360 k and to 270 degrees plus 360k.

KE: Do you know why it's decreasing for these values?

ST2: Erm…… I can only describe it using this circle thing. So if I am trying to describe sin x is the y coordinate here of 0 if my angles there is going to be 0 if my angle increasing up to 90 degrees that's when the y coordinates 1 if I carry on to these two quadrants as the circles going round the y coordinate is getting smaller and smaller and getting more negative and then it's increasing.

KE: Ok. So you are thinking about the unit circle when you are reasoning this. Item 8 explain why sin theta can never equal 2, can you read your answer please?

ST2: But I am not sure, by definition?

KE: Do you wish to add further?

ST2: No. I don't know.

KE: Why you don't know? Is there any specific reason why you don't know?

ST2: I don't know. It's not something you should think about you just think it's between minus 1 and 1… and never really question why.

KE: Ok. For item 9, what does 'trigonometric function' mean?

ST2: I've put a trigonometric function is one which involves sin, cos, tan or inverses and hyperbolic functions.

KE: Ok. Anything you want to add to this? (she just shakes her head)…item 10 sounds like this what does dy by dx mean?

ST2: Dy by dx means the differential of y with respect to x

KE: Ok. What do you understand about differential?

ST2: Erm… it's to do with limits as you approaching the curve I am not sure how to describe it actually…… yeah.

KE: Do you want to think about it? You can ask for more time if you want to. (She shakes her head). Item 11 sounds like this what would d sin x by dx mean? What is d sin x by dx? Explain why.

ST2: Erm… d by dx sin x is the differential of sin x with respect to x. d by dx sin x is cos x. and it looks like I have started to try to draw an explanation why so I've got a coordinate on a curve which I assume is the sine curve and then I've picked another point a bit further along and I was going to try and look at the limits as this got closer, I think I forgotten how to do it. I didn't have time, I am not sure.

KE: So basically do you have any idea why it is (pointing to d sin x by dx) cos x?

ST2: No, I don't really.

KE: Do you want to think about it?

ST2: Erm… I don't think I will get there.

KE: Ok. For item 12, describe as fully as possible what you understood by the following terms you are given three terms here sin 30 degrees, sin 120 degrees and tan 90 degrees.

ST2: Ok. I drew some triangles for this, sin 30 I've got is a half.

KE: Ok, can you explain how you get this answer?

ST2: I drew a triangle, an equilateral triangle at 2 by 2 by 2 and then split into half so it went into two triangles that were, hypotenuse is 2, that length is 1 and that length was root 3. So that if it was an equilateral triangle that would still be 60 but bisecting that angle is going to be 30 and that is going to be 90 so I got a right angle triangle which is 30, 60 with these lines so I know that sine is the ratio the opposite side divided by the hypotenuse that's how I got half and then for (b) sin of 120 degrees, I've put equals sin 60 degrees I think I've put that one the graph looking back (she was turning her answer script to page 1)7…looking at the symmetry of the sin graph, I saw that sin of 60 was equal to sin 120 so then I used my triangle again to do the opposite divided by the hypotenuse.

KE: And then tangent?

ST2: I've put tangent is undefined. Tan is sin over cos and cos of 90 is 0 because sine divided by 0 is going to be undefined.

KE: Ok. Item 13, explain your interpretation of the following terms. You are given two terms inverse cos of 0.5 and inverse sine of 2.5.

ST2: Erm… I've put inverse cos of half, I've put that equals to x, so x equals inverse cos of half and then I kind of took the cosine of both sides so then cosine x equals half and then I did try to find x so this was a half and I think I found it was 60.

KE: Ok, so what about the second term inverse sin of 2.5?

ST2: I've put… again I've tried to do x equals to… x equals to the inverse sin 2.5 so I've put that is undefined.

KE: Can you explain why it is undefined. Is there any specific reason why x is undefined?

ST2: Erm…… I suppose for the same reason the range of sin is minus 1 to 1 so I thought if I could figure out if that would mean anything if it was 2.5.

KE: So you are thinking of this is out of the range of negative 1 and 1 so is undefined. Ok… item 14 which of the following are equivalent (where the angles are measured in degree)? Explain why they are equivalent so you are given 12 sub items here. Can you read your answer for me please?

ST2: I've put (f) and (k).

KE: (f) is 1 + sin x so you are linking this one and this one (pointing to her answer). Any specific reason for this one?

ST2: Because they are just been added, one sin x it doesn't really matter which way you add.

KE: Ok. What about the other one?

ST2: (c) and (l), because of the property I said earlier, tan x equals to sin x over cos x.

KE: Ok…and then?

ST2: (a) and (g).

KE: (a) and (g), so how do you make sense of (a) and (g)?

ST2: I looked at the sin graph and then I was thinking to do with transformation of graph so x minus 90 is going to be shifting the sine graph 90 degrees…(thinking out aloud)… the more I think about it I don't think I did it right there… I looked to sin 180 which was 0… erm… so I am sorry I must be doing the same as the cos I was transferring the cos one by 90 to the right, so I would be looking at what cos 180 was minus 1 so cos of 90 would now be minus 1 sorry that's not working… erm…… oh ok yeah… so basically I would be looking at transformation for the graphs if it's x minus something then you shifting to the right or I was looking at taking one value just for example 180 and doing cos of, I am saying that was x so cos of x minus 90 is 0 so my new graph which is sin x is the value of 180 is going to be 0, so if I just check.

KE: Ok, so you are using specific points to check the shifting of the graph. Ok so in this situation you are referring cos x minus 90 graph.

ST2: Yes.

KE: What is this? (b) and (j) is it? How do you make sense of this? You make sense of which one? Do you shift this one or the other one (pointing to her answer)?

ST2: Erm… I shifted that one, so I looked at the sin graph I shifted it 90 to the left and it looks like a cos graph.

KE: Alright, so you shifted this sin x plus 90 so you shifted the sin x graph to the left?

ST2: Yes.

KE: One more item here, for each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine you are given four concepts here the first one is function so can you read your answer for me please?

ST2: I've put sine is a function with a range of minus 1 and 1.

KE: Ok, for series?

ST2: Sin can be expressed as a power series.

KE: Do you know this power series? Can you write down the power series?

ST2: Erm… I am not sure… erm… is it either the Taylor series… erm… sorry I am getting mixed up I can't remember…… I can't quite remember I think it's got something to do with the power of x have alternating signs and divided by factorials.

KE: Ok. It's fine. What about complex number?

ST2: Erm… I've put argand diagrams.

KE: How did you relate with this?

ST2: Erm… so if you have got something like 3+2i or something you can write that on a graph with the real part and imaginary part (demonstrating her answer to me on her answer script).

KE: So what about the other one y equals to mx?

ST2: I didn't get anything for that one.

KE: Do you want to think about it?

ST2: No. Not really.

KE: Are there any questions you want to try again?

ST2: No.

KE: Now I am going to show one particular response from other respondent. This is a response from a PGCE student. This is the respond for item 3, what is the value for sin 270. Explain why sin 270 has this value so what do you think about this response.

ST2: Erm…(she was reading the response)… ok.

KE: Does this make sense to you?

ST2: Yeap… kind of… obviously I can't quite remember what a Taylor expansion is but if I could I am sure I will check it and that seems to be reasonable.

KE: Alright. Is there anything you want to comment on this response?

ST2: No.

KE: Another response from a student. This one is for item, explain why sin theta can never equal to 2 and then I got this response.

ST2: (she is reading the response).

KE: Does this make sense to you?

ST2: Not really… I don't know if that's a fact that they are bound on the Taylor expansion is 1 I might have to look into that, I don't know.

KE: Ok. Alright. That is fine. A few more questions. Basically do you have any difficulties in learning trigonometry?

ST2: Erm… I hadn't really thought so like… erm… in doing trig questions… but maybe I don't have a general understanding of the relationship between things in reference to I don't make sense certain points.

KE: Ok. So you were trying to say that the relationships between different parts of the trigonometry is quite hard for you. In your opinion what kind of difficulties that a secondary student might have in learning trigonometry?

ST2: I suppose it's quite… It can be seen as quite abstract, I've just been taught these rules because sine equals opposite over hypotenuse, because sometimes I am not sure which angle we are talking about… we are talking about opposite and adjacent… erm… so there's that… all concepts that it's the ratios and the triangles sometimes… I am not too sure about.

KE: Is there anything else you want to say?

ST2: No. I am ok.

KE: What difficulties do you have in understanding trigonometry?

ST2: Erm… I am not sure really, I kind of understand it while I was learning it but it's quite a long time since for example, so like proving different things Taylor series and stuff if I saw it I would hopefully understand it but it's just kind of remembering how it all links together really.

KE: Do these difficulties affect your learning?

ST2: Erm… I wouldn't say so… no.

KE: Do you have any confusion in trigonometry? Let say you are quite confuse why sometimes it works here and sometimes it doesn't work. Do you have this kind of confusion in learning trig?

ST2: I suppose the confusion is why certain things like that question said why can it not equal 2 and things like that that I don't really know or haven't seen proof of I have kind of just taken them to be facts without really thinking about them.

KE: You just take the fact and don't really think about it. Is there any specific reason that you never think about it?

ST2: Erm… I don't really know, I am sure you've got so much to do with these facts so you don't really think why.

KE: So you just take the facts and then use it to solve the problems?

ST2: Yeap.

KE: Regarding part B, you are asked to indicate whether the items that I asked in the questionnaire are important or not. For example in Item 3, you feel neither important nor unimportant, why do you have such perception? Is there any specific reason?

ST2: I suppose I probably put it's neither important nor unimportant, I think I left it as important to let people know what the value is but maybe not to be too important to actually be able to visualize the triangle with that degrees.

KE: So what about item 8?

ST2: I suppose, I put that because I thought if you just knew the values of sin theta can take you can probably do quite a lot of problems and calculations still that might not be necessary, you can understand why it's never equal to 2 but I suppose in higher level maths it is important to kind of…to see where is all coming from.

KE: One more item is the item 15(d).

ST2: I am not sure really possibly because I couldn't think of anything that was… erm… I don't know what I put for my answer explaining the relationships between it maybe I didn't think…it's probably more important for me to think of it likes as a function and as complex number I suppose.

KE: Lets us look at the part C, levels of confidence to respond to the mathematics items in the task. According to your questionnaire, it seems like you are not confident with item 1, is there any specific reason for this?

ST2: I think the problem is I have never had to really do it before and explain in your own words, you just get told what it is and how to use it. I don't know. I've never been asked to explain in my own words.

KE: What about item 3?

ST2: Probably the same thing I mean I could tell you what the value was it's just difficult to explain how I made sense of it and how I came to see that in a way.

KE: Another one is item 8, is there any specific reason for this?

ST2: I've forgotten how to do it really, I couldn't think of a reason why. I couldn't think about the series…(inaudible).

KE: It means that like something, you feel you can't answer it so you feel not confident about it?

ST2: Yeap.

KE: And also for item 15 (a), (b), (c), (d) you also feel not confident about it, is there any specific reason for these?

ST2: Erm… again I suppose I could understand… I've used like sine as a function and series and stuff but it was difficult explaining the relationship really especially because the series, I've have not use for a while… erm… the complex number of functions I kind of knew the relationship but found it difficult to kind of explain it.

KE: Yeap. We are done.

Transcript of ST3's follow-up interview.

KE: Describe sin x in your own words. Can you read your answer for me please?

ST3: Sin x is a trigonometric function. Given a right angled triangle with an angle x, sin x is the length of the opposite side in the triangle divided by the length of the hypotenuse.

KE: Ok. Is there anything you wish to add? Or you are comfortable with this?

ST3: Yeah. I am quite happy with it.

KE: Ok. And then item 2 sounds like this, please arrange the following values of sine in ascending order and explain your answer. You are given sin 110 degrees, sin 250 degrees, and sin 335 degrees. can you read your answer for me please?

ST3: Yeah… sure… I've just got 250 degrees, sin 335 and sin 110.

KE: Which means this is the biggest for you (pointing sin 250 in his answer script)?

ST3: Erm… erm…

KE: Which one is the biggest?

ST3: Sin 110.

KE: Ok… alright… so ascending order… so sin110 is the biggest.

ST3: I've used the graph.

KE: Ok… this is the biggest (pointing to sin 110) and 250 is the smallest… ok… can you explain how you get this answer?

ST3: Erm… I just sketched the sine curve… so you've got 180 in the middle… so 110's there so that's close to 1… so 250's close to 270 so it's going fairly close to minus 1, 335 is going to be a little bit bigger so that is the smallest ascending means (inaudible).

KE: Ok… so why crossed out these bits (pointing to his answer script for item 2)?

ST3: That's because that would be descending order… because that would be the biggest value and that would be the smallest value right (pointing to his answer for item 2), I just misread the question.

KE: Ok… basically what is the relationship between item 1 to the graph?

ST3: Erm… well I suppose in 1, I've talked about trigonometric function and of course you are only really going to have this part of the graph when you got the triangle stuff (pointing to his answer script) that's what the graph will look like so I've just used the graph.

KE: Ok… look at the other item… item 3… erm… item, how do you make sense of sin 200 degrees? Can you read your answer for me please?

ST3: Oh yea… so you get your sine curve so I just read the answer… so it's all to do with rotating a circle and tracing the position of the point as you do so I realize this is a shambolic explanation.

KE: Ok… maybe you can explain a bit your answer… how do you make sense of this bit… do you want to use paper?

ST3: No… I will be fine. I guess what I am trying to get at is you can get this graph by taking a circle and putting a pen on a point at the top and as you move the circle round you kind of get your sine curve…and that's how… that's what I think of when I think of sin 200 degrees when I get round my circle at 200 degrees… that's where I end up on the graph.

KE: Ok. Alright… and then one more thing is basically can you visualize this triangle with 200 degrees?

ST3: No.

KE: No… Ok… can you draw this triangle of sin 200 degrees?

ST3: Well, a triangle got 180 degrees in it so I would have trouble doing sin 200 degrees.

KE: Ok… which means you can't draw a triangle with sin 200 degrees… ok… alright… so the other thing is we look at item no 4 so what is the value of sin 270 degrees and explain why sin 270 degrees has this value? So can you give me your answer please?

ST3: Sin 270 degrees is equal to minus 1.

KE: Do you have any idea why?

ST3: Well, I suppose it's like (the inverse is the wrong word)…but it's gonna be the inverse of 90 degrees so it's not quite inverse but it's a bit outside it say the 90 degrees of the angle here… so that's your 90 degrees… and that's positive 1 the other bit is gonna be negative of it… so like with any angle whatever this is it's gonna be the negative of it.

KE: Ok. So basically same question, can you visualize this triangle with sin 270 degrees?

ST3: No.

KE: You couldn't draw this triangle?

ST3: No. I couldn't do that either.

KE: Ok. It's fine so item 5 what is sine over cosine, does that mean anything to you? Can you read your answer for me please?

ST3: Sine over cos is tan.

KE: Ok.

ST3: Erm… so tan of an angle is the opposite side over adjacent side of a triangle.

KE: Ok. Alright… so what does tangent mean?

ST3: Erm…

KE: You have answered it already… item 6, what do radians mean? Why do we need radians when we have degrees? Can you read your answer for me please?

ST3: A radian concerns the length of an arc of circle of radius 1. So x degrees equal the length of the part of the circumference I have attempted to highlight.

KE: Ok… so yeap… why do we need radians when we have degrees?

ST3: Erm…… I wouldn't say it's stuff to do with Fourier series…maybe even differentiation… erm… I can't remember… there is a good reason for this once you get further…at an advance level why you would want to use radian because degree doesn't work there is something but I can't remember what it is.

KE: Alright… basically do you prefer to use radians because you…??

ST3: If it was going to be something a bit fussy like measuring the angle in a triangle or something like 36.8 degrees or I would rather work in degrees than radians because I can't be bothered to translate that into pi or whatever it is.

KE: Ok.

ST3: But generally I prefer radian.

KE: Alright. Ok. Item 7 for what values is sin x decreasing? Explain why it is decreasing for these values. Can you read your answer for me please?

ST3: 90 degrees plus 360k is less than x which is less than 270 degrees plus 360k (I seems to prefer degrees in this occasion) that's where k is an integer… ehem… and it's decreasing because that is where the gradient of the graph of sine is negative.

KE: Ok… so you are using the gradient to make sense of this situation… let me see this is item 6… item 7… yeah… erm… how do you interpret gradient, for you what is a gradient?

ST3: It's like a line… like a gradient line… it's a curve… it's like this point here (drawing a gradient line to a curve) gradient and tangent line.

KE: Ok. Alright. This is item 7… Item 8, explain why sine theta can never equal 2.

ST3: Because for sine theta to equal 2 would imply that the length of the opposite side was longer than the hypotenuse which is impossible.

KE: So item 9… we go to item 9 what does "trigonometric function" mean?

ST3: Like I said it's a function involving cos, sin, tan those kind of function.

KE: Is there anything you want to add? Or you feel happy with this one?

ST3: Sorry. It's a bit waffle… yeah… I am fairly happy with that.

KE: Item 10 sounds like this what does dy by dx mean?

ST3: Dy by dx concerns the rate of change in y with respect to x.

KE: Ok. Item 11, what would d sin x by dx mean? What is d sin x by dx? Explain why.

ST3: I think I might have omitted the first part of the question but I've said d by dx of sin x is cos x. The gradient at (x, sin x) is cos x.

KE: How do you know that the gradient at (x, sin x) is cos x?

ST3: I suppose going back to the infinite series, we were talking about earlier you could derive it from there or anything you like. You could take the gradient lines and you can plot the gradient on a separate graph… or using Autograph… whatever you could get the gradient line moving around plotting the gradient of the slope at each point and you would get a cos graph.

KE: Alright. Ok… just now you said about Taylor series…so what about this part?

ST3: So yeah… is about the rate of change of that function at each point.

KE: Item 12 sounds like this describe as fully as possible what you understood by the following terms so you are given three terms here sin 30 degrees, sin 120 degrees, and tan 90 degrees.

ST3: Half for the first bit, I have written square root 3 over 2 for the second bit the third one is undefined because you can't get a right angled triangle where the right angle has an opposite and adjacent side because the right angle is always opposite the hypotenuse… so if you had that you would have two 90 degrees angles and you couldn't have a 0 degree end.

KE: What about this part? Sin 30 equals to half. How did you make sense of it? How did arrive at this answer?

ST3: I just imagining a right angled triangle, I just drew an equilateral triangle with side 2 and cut it into half down the middle so I got a one 2 root 3 triangle… I just used sin that opposite over hypotenuse which will give me 1 over 2.

KE: And what about this part? What about sin 120? How do you know it is sin 60?

ST3: Form the graph you just kind of reflect at 90 degrees, what if you had 80 degrees that gonna to be the same as 100 degrees, what if you had 70 degrees that is the same as 110.

KE: Oh I see is the symmetry.

ST3: Yeah…. the symmetry.

KE: And then this one (pointing to the triangle in his answer script)?

ST3: And again I've used the same triangle except there is one 2 and root 3 triangle is gonna have a 60 degrees angle because it's an equilateral triangle and 30 degrees angle which we were using before but now I've used the 60 degrees angle so it's opposite root 3 hypotenuse is 2 so is root 3 over 2.

KE: Let us… ehem… so item 13 explain your interpretation of the following terms inverse cos of 0.5 and you are given inverse sin 2.5 so can you read your answer for me please?

ST3: The inverse cos of nought point 5 is 60 degrees and 330 degrees and 60 +360k and 330+360k where k is an integer.

KE: Ok… and then inverse sin 2.5?

ST3: It's undefined because there is no x such that sin of x is equal to 2.5.

KE: Alright. Let us look at item 14 sounds like this, which of the following are equivalent where the angles are measured in degrees? explain why they are equivalent so you are given a few here… you can explain maybe…

ST3: Sure. Tan is equal to sin over cos.

KE: Yes. Just now you already mentioned this one.

ST3: Cos of x minus 90 is equal to sin x.

KE: How did you arrive at this answer?

ST3: I just visualized the graph, you know what the sin x looks like… the cos x is just going to shift so you get the same graph.

KE: Ok.

ST3: 1 plus sin x is equal to sin x plus 1.

KE: So what about this one, cos x equal to sin (x + 90)?

ST3: That's the same principle just shifting the graph you know what the sin… erggg… cos x looks like you know what your sin x graph looks like so you can just shift it along… the sin of nought and the sin of 90.

KE: Ok. Item 15, for each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine so the first one is a function.

ST3: I guess I just gave an example of function, (pointing at his answer script) I think that should say example rather than exam… the f(x) equals sin x is an example of a function.

KE: Alright… basically f(x) equals sin x… this is an example of function?

ST3: Yeap.

KE: Ok… so what about series?

ST3: I've listed the Taylor series.

KE: Yeap… you have listed the Taylor series. The third one complex number.

ST3: So I've linked that one with the De Moivre's theorem.

KE: Can you explain or give the De Moiver's theorem?

ST3: Ermmmm… that's the one with the sin x + cos x to the power of n, I can't remember the exact statement in the theorem right now… I think is that (he wrote it on his answer script)… I can't remember the exact wording… it's something like that… isn't it… if you ask me earlier of the day, I probably remember.

KE: And then for the last one y equals to mx?

ST3: Yeap… no… immediately came into my mind at the time.

KE: So do you want to add anything now? Any new idea?

ST3: I suppose is another function, isn't it? You could may be link it back to the function when it comes to it.

KE: Alright… now I am going to show some responses of a student…the question sounds like this basically what is the value of sin 270 degrees? Explain why sin 270 has this value? And then I got one response from a student (showing the response to him), what do you think about this response?

ST3: (he was reading)… (mumbling)… I think he would be wrong.

KE: Why you think it would be wrong?

ST3: That's not only one term that is zero when you put them into separate bits of the Taylor series… so it's a nice idea what he is suggesting that.

KE: You don't think he has answered it correctly. Alright, is that anything you want to comment?

ST3: (shaking his head to show he has nothing to add).

KE: Another response that I want to show you, let's see, explain why sine theta can never equal 2. Sine theta can never equal 2 because the bound on the Taylor expansion is 1, subsequent terms are all smaller or equal to 1 so they can be bounded (reading a student's response for ST3), so what do you think about this response?

ST3: (reading)… that is an interesting way of thinking about it… yeap… I suppose it depends what is means by subsequent terms are all less or equal to 1 because if it means the next term goes through less or equal to 1, I would be a little bit dubious about that because you could shove a million or something… as the first term (writing something on a piece of paper) so the first term is not going to be less or equal to 1 so the next term wouldn't be either… yeap… I can kind of see what they are getting at but I think they need to be more precise with they are saying… ehem… I mean because you've got the plus and the minus as well which is obviously going to balance it out so it's gonna be fine… but ya… sure…

KE: So you are little bit in doubt about this one. ok. Alright. It's fine. Do you mind if I ask you explain your understanding about the unit circle?

ST3: What do you mean? How the unit circle works?

KE: For example do you know how to derive the sine graph or not?

ST3: You can get the circle and as the circle moving along, you rotate the circle.

KE: So basically you are looking at the unit circle to generate the sin graph.

ST3: Yes.

KE: I just basically want to know how you are going to use the unit circle to generate the sine graph.

ST3: Well, you fix a point that's gonna start at the right place on a graph and you kind of imagine it's like a ball and you roll it along and it's kind of rotating around and around and as you've got this set. I've just picked this fixed point and as I roll it along, I keep this fix points like a pen (he was demonstrating on a pen). Do you see what I am saying?

KE: Yes. Let's look at part c of the questionnaire, basically you ticked no opinion. What makes you think that? Do you have any specific reason for that?

ST3: What were the questions asked again? Sorry, I ticked no opinion but what it was… was that by ticking no opinion…

KE: Confidence level in answering these questions

ST3: Yeah… sure… I suspect that what probably happen was… it was going on for quite a while so I just thought I tick some boxes… what happen at the end of the time.

KE: Do you feel confident in answering these questions?

ST3: Generally yes… more confident on some then others… yes…

KE: Do you have any difficulties in learning trig?

ST3: Ermmmm… I'm quite… I'm one of these annoying people who are good at learning stuff and forgetting it when I don't need it anymore… I suppose I kind of learnt it if you like that's a really bad phrase I think… I am not the best at visualizing stuff always so I tend to think of it in terms of facts rather than in terms of this is my concept if you like… I tend to be better more on the number and algebra side of things more than the concepts and understanding stuffs.

KE: Your difficulties are to visualize the things here… you are better at doing algebra and calculation?

ST3: Yes.

KE: In your opinion, what kind of difficulties that a secondary school student might have in learning trig?

ST3: Probably similar to me actually. It seems quite an abstract thing to them when it should be more… ehem… conceptual rather than heres a load of facts about trig.

KE: What difficulties you have in making sense of trigonometry?

ST3: I suppose as you highlighted earlier things like sin 270 to me they mean nothing… of course mathematically they do mean something I know. Nobody wouldn't be around otherwise wasting their time with it but for me in my brain it doesn't mean anything.

KE: Do these difficulties affect your learning?

ST3: No. No at all. I'm quite happy just to accept it and move on.

KE: Any confusions you have in making sense of trig?

ST3: No. Not really.

Transcript of ST4's follow-up interview.

ST4: Sin x is the sine function of the angle x. It is a ratio of the length of the side opposite the angle and the hypotenuse of the triangle. I've drew a little picture of the triangle (pointing to her answer script).

KE: Is there anything that you want to add?

ST4: Obviously when you develop all the other things there are lots of different things that come to mind so you can think of it not necessarily, think of the graph but may be that little thing (pointing to her answer script). The first thing is always the triangle for me.

KE: Ok… first thing always is the triangle. Got it. The second item sounds like this please arrange the following values of sine in ascending order and explain your answer so you are give sin 110 degrees, sin 250 degrees and sin 335 degrees. Can you read your answer for me please?

ST4: I put smallest is sin250 and then sin (referring to her answer script)… but I've written the same thing twice (reading and correcting her answer script). Ok these are my points. So (a) I put there so sin 110 I said was the largest because it's positive and (c) is in the middle which is the sin 335, I also remember that point there and then the lowest point is (b) (pointing to her paper) so I have done it mostly from the graph.

KE: Ok… mostly from the graph… sin 250, sin330 and sin 110.

ST4: I was trying to do it symmetrically… those points.

KE: Ok. Alright… what about these two figures (pointing to her answer script)?

ST4: These were in order to draw the graphs (pointing to the figures besides the graph). I was just double checking to get all the points in the right place.

KE: Ohhh.

ST4: I used the triangle to plot some of the key points on the graph if that makes sense.

KE: Ok. Why you crossed these bits (pointing to her answer script)?

ST4: I think originally I put 45 degrees and then I stopped and thought actually I don't think it is so I double checked and that is obviously not the maximum point so I knew that would be.

KE: Ok… so you draw these two figures in order to relate it to the graph?

ST4: Yeah… to double check them.

KE: To double check… how do you draw these figures and putting in these values?

ST4: Erm… so… triangle if there's two sides lengths 1… then by Pythagoras I work out the diagonal to be root 2 and I know that would be 45 because you cut the square into half diagonally so that angle is half of 90 then I do the opposite over hypotenuse.

KE: Ok…what about this one?

ST4: This one is… let me think… you have triangle that is length 1, 1, 1… if you cut it into half… then that lengths now half and then by Pythagoras that is root 3 over 2 (she was drawing the triangle) and that angle because it's an equilateral triangle must be 60 degrees.

KE: Ok… from there… what about these bits?

ST4: So that's the Pythagoras… so the hypotenuse squared minus that other side must be square of that and then I've got square root that of the length.

KE: Ok… so you take the square root.

ST4: And then I said (c) and (b) are both negative so they must be smaller (b) is more negative because of 250 is closer to 270 degrees which is the lowest point of the graph so that's the symmetry again so the lowest point is (pointing to her answer script)…

KE: So you are estimating the location of the points on the graph?

ST4: Yeah… but the reasoning is more that… erm… the lowest point would be there and numerically 270 is closer to 335 so symmetrically that must be a lower value… it's like decreasing (moving her hand to represent the sine graph).

KE: Ok. Alright… erm… do you see any relationship between your graph and your description in item 1? Can you see any relationship?

ST4: Yeah… so that's the ratio of lengths of sides the reason I come to lengths of sides it's the fact ratio and sides to begin with because I use them to do everything so here I've used them to draw the graph but there is no maybe direct relation in my mind to the graph…

KE: Ok… let us look at item 3. How do you make sense sin 200 degrees? Can you read your answer for me please?

ST4: Erm… sin 200 degrees is the negative of the value of sin 20 degrees… erm… because 200 degrees minus 180 degrees is half period of sine that's an odd function. I would consider this contextually in terms of direction. Sine theta is a measure of the circumference so I mean their height… I think… Negative answers would be below the dotted line so what I am saying there is 200 degrees would be more than 180… the value of the height would end up going downwards so that would be negative...I think that's what I mean by that…… so because it's half of the period of sine it's not going to give you the same value… erm… that's an odd function…so that should give you the negative.

KE: What do you understand about odd function?

ST4: I think… erm… it's to do with rotating if I remember rightly so if you rotate that sine round… you have to rotate it 180 degrees so it goes into itself once if that makes sense so you have to rotate it all the way round so it gets back to itself and with cos for example which I think I said was an even function… erm… I don't need to rotate, do I (inaudible)?

KE: It's alright. Just tell me what is in your mind.

ST4: So… yeah… I just because cos is reflected on y axis that makes it even because any value there is the same as minus so the cos of x is cos of minus x whereas sine would be positive… erm… where as sine is… sine of minus x would be minus sine of x.

KE: Ok… so these are your meanings for odd function and even function?

ST4: Yeah… I would more likely to think of it that way than transformation of the graph.

KE: Ok.

ST4: That's from that so… yep… I would do that from looking at the sketch mostly.

KE: Ok. Let's us talk about this figure. How do you make sense of this figure? It's written here sine theta is a measure of the circumference.

ST4: Erm… ya I don't mean that… so what I was trying to get at there was similar to this that I just did that value of the height when I said circumference, I was thinking when you go round you get the sine wave… so… what I was probably meaning was the y coordinate… erm… that was just a way to try and say that 200 degrees would be around here somewhere that would then be a negative value of y.

KE: Ok… so you are talking… if you are rotating first half of the circle you will have positive value and if you are rotating in the second half you will have negative value.

ST4: Yeap.

KE: I am trying to understand this figure. So what about this one? You are trying to rotate from here to here, is it (pointing to her answer script)?

ST4: Yeap… so I've drawn the arrow that way but I did mean that way… I assume that I was trying to do… so if we go around that way… may be I was going that way trying to… because it would still be negative at a time maybe I was doing from there so starting maybe at 0 there rather than here where I would in that case (inaudible).

ST4: And then to 180 but obviously I drew it the other way round.

KE: How do you know the first half is positive and the second half is negative?

ST4: Erm… to do with the height so in terms of the triangle… in terms of the triangle the y coordinate here is going to be a positive value because of the x axis when it comes down here so if you had… it's about 200 and your angle still measured all the way round there… erm… but here we've gone below the x axis so you have a negative y value… that's a long way of explaining it.

KE: Ok… basically it's alright to rotate it from here or there based on your opinion?

ST4: It comes out with the same answer but I should probably I think be using that one then you do get a value of… the triangle… so I think sine x if you drew it the other way would give you the same values if you went that way, cos wouldn't so in terms of sine if you go clockwise instead I think you get the same sign but not the same cos values… so in terms of an explanation… nothing wrong with that.

KE: Ok… erm… so can you visualize this triangle with sin200 degrees?

ST4: No… because it's needs to be a right angle triangle.

KE: Ok.

ST4: No… no.

KE: Can you draw this triangle?

ST4: No…

KE: Do you see any relationship between then definition of sine and the sine graph?

ST4: Between mine? Between what I've put there?

KE: Yup.

ST4: Not between… well for the first part I would say you've got a ratio of lengths you can get up to almost 90 degrees I suppose from that sort of thinking… erm… but you would have to then extrapolate for the rest of it.. so this when you get to larger function I'd use… erm… things like… it's an odd function and period of two pi but not really that closely linked..

KE: Ok. Let us look at item 4 in this questionnaire. Item 4 sounds like this what is the value of sin 270 degrees? Explain why sin270 degrees has this value. Can you read your answer for me please?

ST4: Sin of 270 degrees is equal to minus 1. Sine has a period of 360 degrees but is an odd function so sine of 270 degrees is equal to minus sine of 90 degrees which equals to minus 1… erm… sine 90 degrees I've done a little triangle for opposite it would be root two because of 1 and 1 triangle… erm… the hypotenuse is actually the same value because I've chosen the right angle to do that on.

KE: Ok. You know that sin 270 degrees equal to -1. Sine has a period of 360 degrees but as an odd function (reading her answer script) but as an odd function. How do you relate this bit to this bit (pointing to her answer script)?

ST4: So the 90 so if you do… I did a thing here you do 270 and take 180 degrees you got 90 degrees so I thought that would be easier than going to take 360 and doing negative 1 because then you can still do a triangle for it so because that is an 180 of the period I did it as odd function that's negative of that.

KE: What about this bit?

ST4: This bit… erm… I think I was just trying to as much as I know that the sine of 90 degrees is 1, I was trying to show that's how I thought it out.

KE: Alright. Ok. So basically can you visualise this triangle with sin270 degrees?

ST4: No.

KE: Can you draw the triangle?

ST4: No.

KE: So from here you were trying to confirm the value of sin 90 degrees (pointing to her answer script)?

ST4: Yeah… just to… because it was asking why where the 90 came from.

KE: From here you first relate to the odd function first so sin 270 degrees equals to negative sin90 degrees so now you are trying to confirm minus sin 90 degrees is -1 so you used this triangle and then …

ST4: Yeap.

KE: So maybe you can talk a little bit about this triangle (pointing to the triangle in her answer script for item 4). How do you put in the values or the lengths of the triangles?

ST4: So that's like that one again… the 1 and 1 by the Pythagoras it's square root two and because I wanted 90 degrees, I've never done that to a triangle before but I've used the right angle as my thing opposite the sine so opposite would be the square root of two and the hypotenuse is still the square root of two because that's the longest side… root two over root two is 1.

KE: Ok… Alright… so you are looking at this triangle… this angle?

ST4: Yeap… is the right angle.

KE: Ok… alright… thank you. We continue for item 5. Item 5 sounds like this, what is sine over cosine. Does that mean anything? Can you read your answer for me please?

ST4: Ok… I've got sine over cosine is equal to tan and I would see this from doing sine as opposite over hypotenuse so cosine at the same angle would be adjacent over hypotenuse and tan is the ratio of opposite over adjacent so I did sine over cos is opposite over hypotenuse over adjacent over hypotenuse so that would be opposite over adjacent.

KE: So you simplify it then you get the tan.

ST4: So they sort of match up.

KE: Ok… what does tangent mean in your own perception?

ST4: So… again going back to triangles, the ratio of lengths of opposite over adjacent.

KE: Ok… Do you have other meaning for tangent? Or this is the only one that you can think of?

ST4: In terms of meaning, I would think of a triangle… if I wanted to do stuff with it, I might look at the graph.

KE: Ok… alright… for item 6. Item 6 sounds like this, what do radians mean. Why do we need radians when we have degrees? Can you read your answer for me please?

ST4: Radians are a measure of angle. They represent the distance round the circumference of a circle of radius 1 and because the circumference is 2 pi r so 2 pi times radius the circumference of the circle of radius 1 is 2 pi times 1 which is 2 pi therefore 360 degrees what we would normally say is all the way round is 2 pi radians. We need radians to be able to differentiate trig… that's question 7.

KE: Alright… so ok… radians are measure of angle. They represent the distance round the circumference of a circle of radius 1 (reading her answer for item 6)… erm… can you explain a little bit of this figure (pointing to her figure in item 6)?

ST4: Erm… that's the circle of radius 1 so if we had an angle there and whatever it was in degrees we would know what it was in radians by the arc length.

KE: So which means you are giving a specific example in this case?

ST4: Yeap… so I've labelled that I just said that theta would then be called (pointing to her answer script).

KE: What about this bit? Because circumference is 2 pi r(reading her answer)? So you are trying to make sense of??

ST4: So all the way round would be two pi that's where…the circumference of the circle would be two pi the radians in general but the radius is 1 so it would be 2 pi times 1 which is just two pi which is the radians all the way round.

KE: So equals to 360?

ST4: Yeap.

KE: Maybe we can talk about this bit (pointing to her answer). We need radians to able to differentiate trig. Were you trying to say that we only able to use radians to differentiate trig?

ST4: No… erm… well we usually use radians I think because it is easier, you could do it in degrees, I don't quite know why radians confirm nicely… you could do it in degrees but you have to put in your factors of 360 they have all come out in this harder calculation… I think this is what I am going for.

KE: Ok… so you think we still can use degree but we need to put in that?

ST4: You put it that scale of factors.

KE: Ok… alright so do you know basically why we usually… erm… maybe you already told me just now because it is easier, no need to put in that scale factors and degrees you need to in that scale factors… erm… so this one is for this question (pointing to her answer script to get confirmation)?

ST4: Yeap.

KE: So let us move to item 7. Item 7 sounds like this, for what values is sin x decreasing? Explain why it is decreasing for these values? Can you read your answer?

ST4: Ok… so for sine of 90 degrees I've got sin 270 degrees, with the little diagram.

KE: So you are trying to tell me that sin x is decreasing from sin 90 degrees to sin 270 degrees?

ST4: Yeah.

KE: Ok… explain why it is decreasing for these values.

ST4: On here, I've started from this side going clockwise again… erm... so it's increasing because when we go here the height will be going up (inaudible)… if we start from this point if we go round we are increasing the y value then on that part of the circle you get to 90 degrees so then it starts going down so once we've reached the bottom at 270 degrees then it starts going up again.

KE: Ok. So now you think… which way you prefer actually?

ST4: I prefer this one because you can use cos but for sine you always get the same answer so that again would be increasing up until 90 decreasing round to bottom.

KE: Ok… which means that?

ST4: The y axis would be the same because it's oscillatory.

KE: So basically you can make sense in both sides… from this side is sensible and from that side is also sensible.

ST4: The reason that well I've used that to try and explain it but again I would look at the graph and then I would see that I know from there to there it's decreasing but that's just another way of saying it I suppose.

KE: Alright so you are consciously referring to the sine graph?

ST4: Yeap… I've convinced myself from that.

KE: This is just to confirm it but the first image is the sine graph?

ST4: Yeap.

KE: Ok. So what about this bit? Gradient is too shallow in degrees?

ST4: Actually that is question 6… erm… we need radians to differentiate… gradient is too shallow in decreasing degrees… oh no... hang on… in degrees… that's the other question… so what I mean by that is… erm… I guess that's to do with the scale factor thing in the end… erm… if you do it on autograph or something… you can do the plot sine function in degrees then the graph is actually quite shallow because you have gone 360 that way and one upwards so I think that's where is struggles for that my immediate thought was to do with that.

KE: Alright… when u said sin 90 degrees will decrease, the value will decrease from sin 90 degrees to sin 270 so this bit here say it's decreasing (pointing to her figure).

ST4: from 90.

KE: So you are talking about the value is decreasing.

ST4: the value of sine is…

KE: What about this one? The increasing (referring to her figure)?

ST4: Increasing is that arrow there…increasing then decreasing then increasing again (referring to her figure in item 7).

KE: So how do you make sense this part is increasing and this part is decreasing?

ST4: Erm… so it is our, y value so the height on the y axis is going up… up… up… up… and then the y values starts going down… down…

KE: Ok… so this is how you make sense of this?

ST4: Yeap.

KE: Alright. Ok… let's look at item 8. Explain why sine theta can never equal to 2. Can you read your answer for me please?

ST4: Erm… question 8… the max of sin theta is equal to 1… going back to the circle of radius 1… the max height is 1.

KE: Alright… so which means you are trying to tell me that the maximum is value for sine theta is 1 that's why can never equal to 2 and then when you try to make sense of this statement you are looking at the circle so…

ST4: So that's again I think…… the reason I was using the circle in the first place is… yeap… so I sort of assumed that it's true that the circle gives you the sine wave… and then said the maximum value of that would be 1.

KE: Are you trying to say that because the radius is 1 so the maximum is 1?

ST4: Yeap.

KE: Ok… let say if the radius of this is 2 (pointing to her unit circle in item 8), do you think is that any possibility that the maximum value will be 2?

ST4: Well not as sine… erm… well if that was 2… erm… so I think the reason I used the unit circle is because it gives those values for sine… erm… so I would probably justifying it more going back to the triangle as you do…… so…… opposite…… if you…… the biggest value you are going to get… because the hypotenuse is always longer than the opposite and the hypotenuse is always the longest… erm… so you are going to get the maximum value of sine… when the hypotenuse is or the opposite is as close to the hypotenuse as possible… so you want to shrink that triangle in… from that point which would cause that angle to get bigger... erm… and as you do that… that tends towards 1 because those two values are getting closer and closer… but that triangle doesn't actually exist.

KE: Ok… so basically are you trying to say that if let say this radius is 2 so you don't feel you can get a value over than 1? Or Is there any possibility that the maximum value of sine theta can equal to 2?

ST4: No… No… so erm… if you look at the ratio… because the hypotenuse is always bigger… erm… the maximum that can only tend towards possibly be as 1… it couldn't be 2… because the opposite it's shorter than the hypotenuse.

KE: Alright. So when you are trying to make sense of this you also refer back to the right angle triangle?

ST4: Yeap… I mean if I don't assume that… I am using a… I am assuming that… that is the unit circle because I know somewhere in my brain that gives me the sine… if that makes sense.

KE: Alright. We move to item 9. Item 9 sounds like this what does 'trigonometric function' mean.

ST4: Functions involving angles defined by ratios. A function is an injective mapping and in brackets I've written not sure about this!

KE: What do you mean injective mapping?

ST4: Ok… we had a bit of a debate about this afterwards as well… erm… so when I think of mapping usually think of potato shapes… so injective means you take…… may even erm… yeap… what I mean is you for the trig functions you can have many values that might go to the same one… so it is possible to do that… we couldn't work out whether it was injective or surjective. I don't remember that… erm… that's what I mean by injective at the time… I was trying to think of the right words(she was drawing a figure)… so a function can take things but you…… so you can map anything in the domain… you map everything in the domain (writing on a piece of paper).

KE: So what you are trying to say is like… erm… for different inputs you can have the same output?

ST4: Yeap.

KE: Ok… I got it… functions involving angles defined by ratios (reading her answer script). Is there anything that you wish to add? Or elaborate more for this item?

ST4: Erm… not really so I would say that… yeap… so other than pinning down whether I mean injective or not… erm… yeap… I think that's it.

KE: Ok no problem… item 10. Item 10 sounds like this what does dy by dx mean.

ST4: Dy by dx means differentiate y with respect to x and then I've put gradient next to it.

KE: Alright, so you know differentiate y with respect to x also means gradient?

ST4: Yeap… of the graph of x and y.

KE: Item 11, what would d sin x by dx mean? what is d sin x by dx? Explain why.

ST4: Dy by dx of sin x means differentiate sin x with respect to x… gradient of sine graphs in x… not sure what that means… basically the gradient of the sine graph… d by dx of sin x is equal to cos x… not sure why both have the same maximum and minimum and involve ratios of lengths that depend on each other.

KE: Ok… interesting response… may I know why you crossed out this bit?

ST4: I started writing about like area under the graph but then I was like no… that's integrating so I crossed out… I was over doing the explanation… so that is a sketch of sine x… I think I did a little sketch of that in order to use with that but then I realized we are just sticking to differentiation but then when it came to explaining I had a bit of a look at that to see if I could work out why cos x might come out of looking at multiple gradients as we round so the only way I could think of… maximum and minimum in term of gradient at 90 degrees for sin is horizontal so 0… so the 0 point would be at 90 on cos as oppose to up there so I could plot some points but I don't know exactly why.

KE: Ok… regarding this explanation… not sure why (reading her written response) both have same maximum and minimum so you are talking about sine and cos got the same maximum and minimum value?

ST4: Yes, but not that they are in the same values so…

KE: You realized they got the same maximum and minimum values.

ST4: Minus 1 to 1 but not in the same angle.

KE: Ok… and involve ratios of lengths that depend on each other (reading her answer).

ST4: So if you have got a triangle again… erm… so erm (drawing)… so obviously you have got your opposite, hypotenuse and adjacent… so sine of theta is going to be equal to… and… erm… cos of theta is adjacent over hypotenuse if that was a specific angle you can use the same triangle or you would have to use the same sort of ratios so that… erm… if you fix for example O and H to get a value for sine theta then A is determined by those because you couldn't have any other value of that so that depends on each other.

KE: Alright… so which means you are referring to a specific triangle?

ST4: For each value of theta.

KE: So that you can see that they are related to each other.

ST4: Yeap… that they have to be for that value… hihi (she laughs)… there is a relationship there… but I am not sure exactly why some of the middle values would necessarily be so… I could do… erm… (writing) so pi by 2 to 90 I know that (drawing on a piece of paper)… that is the gradient there… which will give me cos so that ends up translating to 0 on that graph that is cos…that is sine… (pointing to her drawing)and then again at the bottom there would be a 0 and 270 and here… it gets a bit difficult… but it's the in between values that I couldn't tell you why it is curved in between again… necessarily compare to…

KE: Which means you feel you don't know why you differentiate sin and you get cos but you have a sense that they are related ?

ST4: Yeap… I know that they are related but I couldn't explain why that turned out to be exactly cos.

KE: Alright. We move to item 12. Describe as fully as possible what you understood by the following terms . You are given three terms, the first term is sin 30 degrees, sin 120 degrees and tan90 degrees. Can you read your answer for me please?

ST4: So (a) sin 30 degrees I've put ratio of lengths… I've drawn little triangle that's my 60 triangle from earlier from casting an equilateral triangle… the other angle be 30 because that is a right angle triangle so again a ratio of lengths half over 1 the opposite over hypotenuse which gives you the half so I am still using that triangle and (b) sin 120 degrees I've put that equal to sin 60 degrees because I would have looked at the graph and have gone on… so 120 is about there and that's symmetric about the middle so I would have taken…… oh ya… so it's the same distance away from the 90 to 120 as it would be to 60 because there is 30 difference between them so I am saying it's symmetry there.

KE: And then you know that sin 60 degrees.

ST4: Sine 60 degrees from this one (inaudible) opposite is root 3 over 2 by hypotenuse.

KE: Alright. From here you get this answer?

ST4: Yeap.

KE: Ok… cool.

ST4: (c) tan 90 degrees is undefined so I had a little bit of a look at the triangle… erm… but tan is opposite over adjacent and although it's sort of work earlier when I did it with sine… and the opposite over adjacent actually gives you a value of sine so I think maybe you can't use the right angle in a right angle triangle to prove it… haha… that's all I can conclude from that… haha… but then I went to sine of 90 degrees which I know is 1 over cos of 90 degrees which I know is 0 and you can't divide by 0.

KE: Alright. In fact, you can't really imagine this tan 90 degrees in a triangle then after that you switch to the sine 90 and cos 90 to make sense of it. You know sin 90 is 1 and cos 90 is 0 so you just…

ST4: Yeah… because I think to have,… if you assume you can't use that right angle so you would have to have a right angle there and you'd have no end to them… does that make sense?… it would be like infinitely.

KE: Ok… cool…erm… basically how do you know sin 90 is 1?

ST4: Erm… what did I do earlier? See… when I drew that triangle I think I convinced myself from trying to do it with tan that you can't prove anything like that but again I would look at the graph and know that that is the maximum value which I guess you could do by differentiating but then you are assuming that you know sine goes to cos and that sort of thing.

KE: What about cos? How do you make sense cos 90 is 0?

ST4: Cos 90 again, I would probably just look at the graph and if I just wanted to work out the value because I actually spent a long time with these graphs I am pretty convinced that they are true if you know what I mean in terms of justifying it to myself I would look at the graph and check some points.

KE: Ok… let's us look at item 13. Item 13 sounds like this, explain your interpretation of the following terms so you have inverse cos of 0.5 another term is inverse sine of 2.5.

ST4: Erm… so the first one I've put is inverse cos of 0.5 is cos inverse is the inverse function for cos, i.e cos of cos inverse 0.5 equal 0.5 so cos inverse of 0.5 would give you an angle similarly the sine of sine inverse 2.5 would be equal to 2.5 so sine inverse of 2.5 would give you angle of theta for which sine theta is 2.5…… no such angle exists… see question 8… let's have a look… so I've said maximum there is 1.

KE: Alright… so you know the maximum of sine is 1.

ST4; Yeah… maximum… yeah… so sine theta can't be 2.5.

KE: Ok… so what about item 14. Item 14 sounds like this which of the following are equivalent (where the angles are measured in degrees)? Explain why they are equivalent. You are given a few items here. Maybe you can read your answer and explain your answer.

ST4: Yeah… so I've got L is equal to C because… L sine theta over cosine theta equals to tan theta and I showed that earlier… erm… I put A is equal to G so sine of x is equal to cos of x-90 from the graph… obviously that is a sketch… so the sine I know just thinking of values, 90 degrees 180 degrees, 270, 360 and I know cos would come from the top of there and have a low point there…like that so in terms of transforming the graph which shifted sine left for 45 degrees… do I mean that?....... No I mean 90 that is what I mean so the maximum point there is that 90 degrees for sine and the maximum point of cos to lighten the graphs up would be at 0 degrees so shifted 90 degrees to the left.

KE: so you shifted the sine graph to left?

ST4: Yeah… so it should be x+90… in terms of transforming... though I would now disagree with my statement.

KE: Alright… disagree with which statement?

ST4: Because I put cos minus 90… erm… I think when I did that I would have done it from the graph… yeah… I just got the transformation wrong there… equally I might check key points.

KE: So you think your answer for this part is not correct is it?

ST4: No… yeah.

ST4: I've put… erm… maybe trying to think back through the thought process… erm… if you look at cos… if you add 90 to sine… yeah… so if you did… if that was a minus like I originally said then sine would end up going to the right and the maximum and minimum points would be in the same place but I think the thought process there might have been that it flips… I am not sure about that and I might just have got that wrong… erm… so that wouldn't be G.

KE: Ok… you think this is correct?

ST4: Yeah… so sin x is cos x plus 90.

KE: What about this part?

ST4: So b cos of x is sin of x+90… so again I think that would be the other way round by the looks of things so cos of x would want to go to the right to find it… again I would put that as a minus because we are going to the right… which is of minus in brackets from the plus one graph… F, 1+sin x equals sin x + 1 and because it's not… erm… it's not in brackets so you could have done the addition in any order… rearrangement is what I've written there… so the rest aren't equivalent to any of the others.

KE: Ok. For item 15. For each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine.

ST4: (a) function… sine is a function of the values of angles. So your input is the angle and your output is the value.

KE: Ok… series?

ST4: Series… sine of a value can be written as a Maclaurin series which is the infinite series.

KE: Do you know this series?

ST4: I think what I mean by that was Taylor series or one of the two… this is what I am thinking of and I am pretty sure that is Taylor series not the Maclaurin series (she have written the series in her first concept map).

KE: Ok, so you are referring to this series?

MC: Yeah.

KE: So what about complex number?

ST4: So a complex number I've put cos theta plus I sin theta equals 1… and you can write sine in terms of e and i. So I think it's like theta…… oh ya… hang on… it's like e to the… something like this to pi i there is some two's and stuff involved over there… something like this… it's like e to the i to the e minus sign… (inaudible) to the something that's why I didn't write this down!!! Because I couldn't remember it. this one said de moivre's theorem and again I put that on there it's just something that comes to mind when I think of complex numbers and trig so cos theta plus I sin theta to the power of n it's the same as cos n theta plus I sin n theta… erm… which you can use to work out… so sin of 6 theta for example you can get that quicker by using that (slight interruption).

KE: Ok… hold on… alright so this is complex number and then the last one is y equals to mx.

ST4: Y equals mx is a function as far as I can think it has no close link to sine. So in terms of thinking y equals mx I would have to think of the grap with gradient m… erm… sine obviously periodic… it's got a maximum and minimum whereas that graph wouldn't… depending on your input I suppose but if you put the same inputs for x you would end up…

KE: So you don't really think there is any relationship between the y equals to mx other than this is a function and sine is a function?

ST4: Yeah…other than that I would look at the graphs for that (interruption).

KE: Sorry… sorry about that.

ST4: So ya I would look at the graph… to me sine has a lot of things different... it's a bit more straight forward… you've got fix gradients whereas sine obviously fluctuates because of the gradient.

KE: So you relate the graph and the equation of the straight line and you can't see any relationship between them.

ST4: No… other than the fact I would call them both functions.

KE: Ok… we are done this part... now I am going to show you an example… is this one… this one of the students' responses… erm… for this question… erm… look at item 3 what is the value of sin 270 explain why sin 270 has this value. This is the response (showing her the response). Can you make sense of this response?

ST4: Sin 270 degrees is equal to minus 1 the reason for this due to the fact that when 3 pi over 2 is substituted to the Taylor expansion the terms end up being zero except for one term…which gives sin 3 pi over 2 equals minus 1… ok… so…

KE: What do you think about this response?

ST4: So I am putting in 3 pi over 2 (I am assuming this is right) so I have hit a positive value there so I've 3 pi by 2 and take away I would end up getting lots of pi by 2…… I don't know where this 0 would come from that…

KE: So you are a bit doubt?

ST4: So if I am assuming that's the Taylor expansion that's what I know by it… I can't see where this series would come from... if you are taking powers of 3 pi by 2 because they would all be a value that's non zero to me.

KE: So you don't think it seems like reasonable?

ST4: Erm… no… but I am basing that on the fact that I am using that as what I am thinking of as that Taylor expansion… I don't know if they've got a… more physics based…(she laughs)… you know… like a series where that happens but I don't see where that comes from.

KE: Ok… let's us look at another response… item 5 sounds like this explain sine theta can never equal to 2.

ST4: Sin theta can never equal 2 because the bound on the Taylor expansion is 1 so the modulus of… so that is the sum of (reading the response).

KE: the sum of the terms.

ST4: Ah… ok… so subsequent terms are all be less than or equal to 1 so they can be bounded (reading the response).

KE: How do you make sense of this?

ST4: Erm… I think what they are going for… I mean they obviously know the Taylor expansion is bounded by 1 that is sort of a fact… I mean I've done quite a lot where I've got a fact but I can't justify it so they are saying that if all the terms are less than 1 you can bound it…erm…I would say you can have an infinite numbers of terms you can't necessarily bound it unless it's decreasing… erm… which sin theta… again ya… it's gonna get… erm… what's the maximum??…… yeah… I can't find a way to explain why it's true… erm… in terms of the bound bit, obviously I will believe them… the Taylor expansion is bounded by 1… erm… but the sum if all the terms are less than or equal to 1 they can be bounded, I would say they have to be decreasing in a certain way… to bound them like that as a group…

KE: Ok… yeap… this is done…look at this part (part c of the questionnaire). Is there any reason that make you feel not confident in responding to item 7, 8 and 15 (b)?

ST4: so 7, I am confident that… that is true… that is decreasing from there but again I think I was using this (pointing to her answer and drawing)… yeah… it's nice that it happens to work clockwise I was using it that way and I might have convinced myself more and as a proof, I wouldn't say it's very regular I am just using something that represents the same thing that I've just being saying so it would be the same as the graph…

KE: Ok so that's why you feel not very confident.

ST4: Yeah.

KE: So what about item 8?

ST4: Erm… 8… again I am using a proof there that I wouldn't consider a genuine proof… I am just trying to explain it in some way whereas… erm… I could have easily just drawn a graph to say that it's true… and that is pretty much what I am doing there… so in terms of explaining really why… I wouldn't say that is very rigorous again… so not confident in the explanation but the answer I know but not the explanation.

KE: Alright. The first thing in your mind you reflected on your graph. you make sense of… I know is 1 because I can see the graph as 1.

ST4: It's justifying what I wasn't confident with.

KE: What about this one which is series… relationship?

ST4: Again not confident that that representation really says anything more than a graph would.

KE: And then item 15 (b)? which is series.

ST4: Yeah… so even then saying the Maclaurin series… I didn't really know…(she laughs)… what that would look like and I put Taylor series so that's why I wasn't confident with that one.

KE: Alright. Is there any reason that you feel not confident at all with item 9 and item 15 (d)?

ST4: Erm… 9… so ya… that's because I used the word injective and to me I wasn't confident that I had used the right word in terms of the mapping that I was trying to describe.

KE: Alright… so the 15(d)? y equals to mx?

ST4: Erm… for that one I think because I couldn't come up with an answer… it's hard to justify why… erm… so I think I was being a bit suspicious like if you'd asked me there must be something and I wondered why!

KE: Alright. Basically do you have any difficulties in learning trig? What are the difficulties?

ST4: In learning it?

KE: Ya.

ST4: I think that in terms of when I learnt it I didn't have any issues because you learned it in…in a very soh cah toa kind of way and when it comes to using it in the more complicated situations so as soon as you get to your infinite series…Fourier series that sort of things… then the understanding of it just seems to go…equally if you are trying to… in terms of learning it… learning by rote learning… I didn't find that difficult because it's quite simple diagramatic way of remembering pretty much everything… I mean I use my two little triangles for everything so for me I found that quite easy but I wouldn't necessarily automatically know the answer to every… so if somebody said what is the sine of 30 degrees I'd work it out rather than immediately responding.

KE: In your opinion, what kind of difficulties that a secondary school student might have in learning trig?

ST4: Erm… probably the understanding of where it comes from… I suppose why… why you would necessarily care about anything beyond working out an angle… erm… though what I have come across with the current group is remembering the formula obviously you got sine rule and cosine rule area of a triangle that's not a right angle that sort of thing… erm… I guess they've not seen where it comes from and I think that kind of understanding maybe. it would help that there is a lot of background to it that I think maybe they have missed out to that age.

KE: Are you talking about where the sine graph comes from? What do you mean?

ST4: Erm… so I mean even where the sine rule and cosine rule come from if you are doing the area of a non-right angle triangle just to see that... if you split it up and you get those values from the right angles if you were to do it that way.

KE: Ok… you mean how to derive those formulas?

ST4: Yeah… that's the sort of thing… I think that would help them… I think the difficulty is in remembering each one because they do do it in quite a structured memory way.

KE: What difficulties you have in making sense of trigonometry?

ST4: Erm… I guess from this…anything beyond the triangles I take mostly for granted because I've learnt what the values are … erm… in making sense of why for example, a negative value, you know particular negative value, to me if it is all based in triangles in my mind then a negative value you have to kind of think of triangles that come down below the line that sort of thing so when it goes beyond 90 it's sort of extended by the graphs at that point.

KE: Alright… how do these difficulties affect your learning? I mean like beyond the 90 degrees you have difficulties right? How do these difficulties affect your learning?

ST4: Erm… I don't think it affects learning in terms of the values those…because obviously I only used the graph… I used properties and stuff like that… erm… I think the difficulty then comes in the more complicated uses of it so I mean even modelling it and things like that is to combine sine waves and stuff like that… they are quite complicated in terms of what cancels out I think.

KE: Ok… so you mean after 90 degrees you constantly refer to the sine graph so that you can use all those values…

ST4: Yeah… so that sort of slows you down a bit… as well I don't know in terms of learning it I suppose that is ok… if you think about like your arc sine and your cosec that sort of thing… erm… I haven't really learned the graphs as such I have been going through to work them out so that becomes a struggle… those sort of values can be quite difficult… erm… without using the sine properties.

KE: Ok… any confusion you have in making sense of trigonometry?

ST4: No, I mean that thing about… differentiating it… why you would do it in radians I understand that it works in radians and you would have to put in your scale factors but in terms of why it works in radians and not in degrees I haven't got my head around that yet… if that makes sense.

KE: Ok… it's done.

Transcript of ST5's follow-up interview.

KE: For Item 1 part (a) sounds like this, describe sine x in your own words, can you read your answer for me please?

ST5: The ratio of a over c in a right angled triangle. I labelled my a's and c differently to the concept map (pointing to her concept map).

KE: You also drew a triangle here?

ST5: Yeap… so it's the ratio of the opposite angle over hypotenuse.

KE: Ok. Is there anything you want to add? Or you are happy with this (pointing to her answer for item 1)?

ST5: No… I think so that's how I think of it always.

KE: Ok. Then we look at item 2. Item 2 sounds like this please arrange the following values of sine in ascending order and explain your answer. You are given sin 110 degrees, sin 250 degrees and sin 335 degrees. Can you read your answer for me please?

ST5: I have (b), (c) and then (a) so sin 250 degrees, sin 335 degrees and sin 110 degrees.

KE: So which one is the biggest you are talking about?

ST5: I think I've said (b) sin 250 degrees.

KE: Then the middle one is 335 degrees?

ST5: Oh no… it can't be… smallest (talking to herself)… ascending order… so the biggest one is sin 110.

KE: Ok, so the smallest is…?

ST5: The smallest is sin 250 degrees.

KE: Ok. Alright. Can you explain your answer?

ST5: Well… ya… I drew the graph and if you know where the important points are, the turning points... you know that it crosses at this point and then it crosses and then it turns at this point…and you can say… erm… so 110 is… first of all it's going to be positive, you know it's going to be there and so to work out the others… erm… you can say 335 degrees is closer to 360 degrees then it's only 25 away whereas 250 degrees is a while away from that…so you can say that this one is gonna be closer to the x axis so it's bigger then.

KE: So you approximated the location of the points on the sine graph?

ST5: Yes… because you know it's gonna be symmetrical about that bit so it's closer to the turning point. It's going to be lower down on that for that value.

KE: Alright. What is the relationship between your definition or description in item 1 to the sine curve?

ST5: Yeah… that's the thing because you don't actually do triangles of like… because whenever you draw a triangle like that… you always draw like an acute triangle… you know an acute angle here but actually you know they are big angles so in general… I don't know... I think of it. I suppose you could think of it as a really big like triangle but then you can't get a right angle triangle. So I suppose it is that but moved along so I think of this bit here like the bit between 0 and 90 degrees… erm… and then I just extrapolate it for the other values so I think of…yeah but it's not immediately I can see it's not immediately kind of accessible because these are such large values.

KE: Ok. So let us look at item 3 which sounds like this how do you make sense of sin 200 degrees.

ST5: I think of it as sin 180 plus 20 so it's 20 away from sin 180 degrees which is the point where it crosses.

KE: Ok. So what come into your mind was the sine graph when you are asked to make sense of sin 200 was it?

ST5: Erm…where are the turning points… so is it bigger than… is like when you are looking at angles and you've got… like I was doing bearings with my kids and so you could think of the compass points you know so you think… well this is 90 and this is 180 and stuff so if someone said what is 110 degrees or something then you think ok so it's bigger than 90 and it's smaller than 180 so it's got to be in this quadrant and it's closer to 90 so it's approximately about here… erm… so I can kind of think of that like that I think you can think of well it's bigger than 180 so it must be below the axis but it's smaller than 270 so it's not yet got to the turning point and is it closer to 180 or is it closer to 270.

KE: Can you visualize a triangle with sine 200 degrees?

ST5: No.

KE: So can you draw a triangle with this angle?

ST5: 200 degrees??

KE: Sin 200 degrees.

ST5: You can't because it's too big and so you'd get, it's a reflex angle and it doesn't make sense… because you know infinite… you know if you were on the curve space and the lines came back to meet you again or something but no… I can't.

KE: Ok, so for example like just now… seems like you are using the properties of the sine curve to make sense of a lot big angles.

ST5: Yeah.

KE: And then just now you mentioned about symmetry because you've used the symmetric property of the graph and then to approximate those values so basically how do you make sense of this symmetry? I mean why it is symmetry?

ST5: Well… you can do this I mean rolling a cylinder as well… if you've got a point on it like say you start at the bottom but if you then mapped that… erm… so this would give you a sine curve so if you do this and map the height of it then this would go… you know up and then at this point like half way you have a thing like this and your point would have reached the top so suddenly you are here… erm… and then you start going down again because like… you know… you are going up it… so that's why it is kind of even so that's why it's not squashed… you know one way or the other so you got that symmetry.

KE: Ok. I got it. For item 4, what is the value of sin 270 degrees? Explain why sin270 degrees has this value.

ST5: I said minus 1 because it's three quarters of a cycle.

KE: You always imagine a rolling cylinder on a surface?

ST5: Erm… I don't always imagine a cylinder, I tend to imagine the graph.

KE: The graph.

ST5: And I kind of know that derived from the rolling cylinder or whatever… but ya… I imagine a cycle of one single… sine cycle and then for if it's cos then it is slightly different but it's still one cycle and it's just it starts here instead it starts at 1.

KE: ok. Alright. Can you visualise this triangle with sin 270?

ST5: No.

KE: So you can't draw it as well?

ST5: No. I can't really visualize anything bigger than like even… even obtuse angles I always think of acute angles of… you know triangles if it were big… if it were you know a triangle like that I'd just in my head turn it around and look at this acute angle rather than looking at that obtuse angle because we are tuned by convention to have the base line on the bottom and so even… oh ya… no… the only thing I can visualize is acute angles to make these right angle triangle… make triangle in general not right angle… make it into acute angles.

KE: So your explanation for this is because of 3 over 4 of circle?

ST5: Of the cycle ya… the cycle through it… you are here and therefore you are down here.

KE: Alright. Ok. You are constantly referring to the sine curve?

ST5: Ya.

KE: Alright. For item 5, what is sine over cosine? Does that mean anything to you?

ST5: It's tan theta… and it's opposite over adjacent.

KE: Ok. We move to item 6, what do radians mean? Why do we need radians when we have degrees?

ST5: Well… degrees is just a measure of turn and angle is a measure of turn so you can… you can measure in an arbitrary kind of amount… 360 is a bit of random number but the Babylonian you know decided that 360 was useful because you can divide it into lots of things and so for that purpose if you are just doing like I don't know… polygons (no that's not good because I was about to say 5… divided into 5 is 72.5 which is not good)… ya… if you take like angles and you just divide them into normalish things then you talking about 30 of 60 or something which is perfectly fine but… but… because things like this… I don't know I think it is useful in talking about things in radians because then it's an obvious amounts of a whole rather than just an arbitrary number.

KE: What about the reason? Why do we need radians when we have degrees?

ST5: It's more exact like I said you can… erm… it's not like 30 degrees it isn't exact but I just… erm… I don't know I like pi. I grew up being a big geek so I memorized lots of digits of pi which of course is approximation but… because you stop somewhere but I like the rational transcendental going on forever nature of pi so I like pi as a number anyway and then… ya… so it's a good measure of all the way around it.

KE: Yeah… what about this bit (pointing her answer for item 6)?

ST5: If you have a cylinder which has a radius of 1 then then it's 2 pi all the way round so it's a good measure of general amounts of roundness, I don't know amount of turn.

KE: So basically is more like… erm… you feel like you like pi. Is there anything you want to add for this one (referring to item 6)?

ST5: I don't know… it's always… it's one of those questions like when people say what's a degree or are like what's an angle and you are like…it's this thing that I have always used but actually being able to describe what it is, is quite hard.

KE: Ok. So we move to item 7, for what values is sin x decreasing? Explain why it is decreasing for this these values?

ST5: Erm… it's decreasing from 90 degrees to 270 because it's between here and here so it goes down.

KE: So the first thing comes into your mind is the sine curve?

ST5: Yeah.

KE: Ok.

ST5: I'd draw it because if you just said that to me like when is it decreasing I'd have to draw it in order to see it… and what does it mean? It means… it's going down…if you had like this it'd mean it's on it's got to the top of the cylinder, you know the point when it's got to the top and now it's coming down but that doesn't explain what happens underneath but it's just it's, it's the down bit of the cycle.

KE: So when you are asked to derive the sine curve so what will you do? If let say you are asked to derive a sine curve so what would you do?

ST5: I'd… deriving it?

KE: Yeah.

ST5: I suppose I've done before on autograph and you know computer drawing programmey things where you have a point going round the circle and you link like you know the height on that and then as you so then you do this and so I think I'd do something like that again it's not necessarily a cylinder but it's more kind of, heres a circle if you are going as a constant speed and if you measure the height on it… I feel like I'm doing this whole like patting my head thing you know what I mean? if you are going round at a constant speed and link the curve… yeah… going straight up and down yeah and going along at a constant thingy instead of just doing this you know like this that's the shape that you get.

KE: Ok. Alright. So when are asked to derive the sine curve you will imagine like a unit circle, I mean…

ST5: Yeah.

KE: So this one… the Rolling cylinder, you only use it when you want to relate it to radians?

ST5: Yeah I think so… I think the rolling cylinder is great but it's on a surface so you can't go below it circumference and circle specifically but it's…

KE: You related this rolling cylinder to the unit circle?

ST5: No… because I don't think of r as being… because it has to be 1 I just think of take a cylinder any cylinder… and ya… no... I don't really relate it to the unit circle because it's moving whereas the unit circle stays there and the point goes around whereas on this it's more kind of the circle moves around and it's a point fixed on a circle rather a point that goes around the circle…yeah.

KE: I am interested with your rolling cylinder because I don't think like that way, that's why I am trying to understand your rolling cylinder and in what context you will think of the rolling cylinder? And then how do you reason a few properties of the sine concept.

ST5: Erm… I've seen it before in terms of like bicycles because you can get like on trains you know when you have trains and you have like all the wheels and they are linked with like a thing here and so… like a bar goes round and so as the wheel goes round the bar goes up and down up and down like you see it in all older movies sometimes that someone is standing on it so they go up and down… up and down so I think it is a bit like that so that's how I kind of think of it it's going forward but it's also going up and down… up and down at the same time because there is a fixed point where you are standing on the you know… goes up and down… up and down but moving forward at the same time that's why it's rolling like the cylinder, I mean, I'd rather draw that then just have it how it is I think when I first learned it I probably needed more props or more examples of real life connections whereas now, I just know the sine curve and I just remember the original sin always which is why I know like because how I know a sine curve goes to the origin and that's how I always remember that's a sine curve but a cos curve starts at 1 because the cos curve is the other one… I always remember the sine curve first and then I remember the cos one and it's like that but it starts at 1 instead.

KE: Ok… for what values sin x is decreasing, you mentioned between 90 and 270 because as x increases sin x decreases (reading her answer for item 7) so when you are writing this one (referring her answer to item 7), what is in your mind?

ST5: That's just as from here and then from 90 degrees onwards if you go this way then this goes down… that's all (pointing to her sine graph)… I don't know. I just look at the graph.

KE: Ok. You just look at the graph. Item 8, explain why sine theta can never equal 2, can you read your answer?

ST5: Erm… because sine theta is always between minus 1 and 1 by the definition of the function if you think of r equals to 1 as the unit circle, cylinder of radius 1 rolling on a surface you can't change size of the cylinder so it's always you can't suddenly have it above the point of 1 it would have to jump in the air.

KE: So you were trying to make sense through the rolling cylinder?

ST5: Yeah… the rolling cylinder, I mean you just… yeah, but mostly I think of the function as by definition it's between this and this so it's silly to think of it up here beyond that because it has to stay on that part.

KE: Ok. For item 9, what does 'trigonometric function' mean? so your answer is (reading her answer for item 9) function using sine, cos, tan etc, can be made by this (pointing to her answer)?

ST5: Yeah… e to the x plus… I couldn't remember the exact formulae because then when you get to… cosh and sinh… and the other one, tanh… that you start making them like e to the ix...plus e to the minus ix… erm… and that's one reason why… erm… I mean it's very… I can't remember when I first learnt it I think it was when I did pure further maths like p4 to six when I did it… but thinking of them like this suddenly makes it… you know… it works… you know… I don't immediately think of that but you know when you do that…you know when you do sin divided by cos then you do this and then there's a minus and there's a something… I don't know but then you assume that is tan because that's the definition of… you know… yeah, I can't remember the exact definitions… but I know you can… you know… make some more… it's like in physics where you have like… you know… you've got your electrons, protons and neutrons but theoretically you can make them all from little quartz kind of thing and if you have enough of these up ones and enough these down ones together then they make this thing when you actually do them you may arrange them in a different way then you get something which looks entirely separate like you see sine and it's different to cos but actually fundamentally it's made up of similar thing, it's just rearranged in a different way.

KE: Alright. So for item 10, what does dy by dx mean? So your answer is?

ST5: The derivative ie the gradient of a curve.

KE: Ok… so for item 11, what would d sin x by dx mean? What is d sin x by dx? And Explain why.

ST5: So it's cos so d by dx of sin x is cos because it's the gradient at… like you could look at the gradient at every point but if you plot the gradient of every point on a graph it will end up, it will be cos x and you can look at specifically things like you know… the maximum the minimum and say you know the max… the gradient is going to be zero and the minimum is going to be zero so those are those points where it's zero on the cos graph… erm… and like at the origin where it's going like that, like gradient 1 you can plot it at 1.

KE: Ok. So you are constantly, you are thinking of the gradient of every point on the sine graph then from that on you tried to get the cos graph?

ST5: Yeah… that gives you that.

KE: Ok… cool. What about this one? Do you have any specific meaning? What would d sin x by dx mean?

ST5: It's just the derivative of it and the gradient of it… I mean the derivative, differentiate it… yeah.

KE: Ok. No problem. Item 12, describe as fully as possible what you understood by the following terms, you are given sin 30, sin 120 and tangent 90 degrees. So ya… what is your answer for sin 30 degrees?

ST5: Because these others are like the special triangle that you draw… like if this (referring to her triangle in item 12) is equilateral triangle you will have 60 on each point so if you cut it down the middle, you have 60, 30, 90 triangle which is going to have 1, 2 because that was an equilateral triangle so this whole length would be 2 so you will get 1 which gives you root 3 over 2 and you can then say 30… ok so opposite over hypotenuse, that is the opposite, that is the hypotenuse so it gives you half.

KE: Ok. So what about sin 120?

ST5: Sin 120, if you look then on the graph again because it does not really make sense to say… sin 120… erm… sin 120 is like here and so you think how much is this from 180 (referring to her sine graph) and then you think well where would that be on here and so you can say well that's like 60 degrees… erm… so although 120 degrees is here because it's symmetrical you could think of it as the other end so then is 60 degrees sine 60 degrees so you can do the triangle again.

KE: Alright so you related sin 120 to sin 60 first then you used the same triangle to get the value?

ST5: Yeah.

KE: Ok. Alright. What about tangent 90 degrees?

ST5: Erm… it doesn't work.

KE: Because??

ST5: Because on a 2D plane… on a flat 2D plane… erm… flat is the important thing you could have it on a ball but you can't have two 90 degrees angles because they never meet and therefore like on the graph when you see it there is an asymptote because it never gets there.

KE: Alright. Ok. Then we move to item 13, explain your interpretation of the following terms, the first one you are given inverse cos of 0.5, the second is inverse sine of 2.5.

ST5: Alright, so the first one you want some angle for which… cos of it is a half so you work out you know…well, what's a half? You know a half from this special whatsits if it is something that you know… one of these values like a half or root three over two or root one over two like 0.8 or 0.86 or something if you see that then you can say ok… I know that it's got to be a special angle if it wasn't then you just… to me I would just put it into a calculator to you know… erm… you can say I suppose that this is a half and so I go along until I find it and that's fine when you wanting say they wanted to know how many values are there for which it equals a half then within this interval or something like that then you could say there's 1 because of this or there's 2 because it's between this and it's symmetrical but that wouldn't really help me work out what the answer should be.

KE: Hmmm… so you got two triangles here (pointing to her answers for item 13)?

ST5: Ya.

KE: So you were using this triangle here, right?

ST5: I think when I first drew them I couldn't remember which one was which.

KE: What about this triangle?

ST5: Erm… then the 45 degrees one (referring to her 45 degrees triangle that she drew)… so if you have like an isosceles triangle then you have one on each of these ends around 90 degrees angle then the hypotenuse has to be root 2.

KE: Ok, so you memorize this triangle?

ST5: No I work it out each time… I know that I want a triangle that's 45 degrees. I know that this is an equilateral triangle and this is an isosceles triangle and that's pretty much what I remember. I got to draw one and they are going to be as unity as I can make them so with this one it's obvious to say that they have two 1's and then this one is like occasionally when I start drawing it I draw that as 1 then I realize that I am going to have trouble so I make this one 1 and then that's gonna be 2 because it's an equilateral triangle and this is half of it but I have to I had to rewrite them each time.

KE: Alright. What about inverse sine of 2.5?

ST5: You can't have it because it doesn't… because by definition… it would be like this being up in the air again.

KE: Hmm… ok… alright… item 14, which of the following are equivalent (where the angles are measured in degree)? Explain why they are equivalent so you have sine x equals to cos x minus 90. How do you make sense of this?

ST5: Erm… apparently because it shifted 90 degrees to the right.

KE: You shifted what graph to 90 degrees?

ST5: You shift but like the rolling graph… if you kind of think of it like there is one rolling graph but It's just where you put it… so say the sine graph but then cos is left at and shifted… but then that's where I think I got… because I had answers… I did it wrong first time and I had to think about it like but it's it's… I think I can understand shifting but when you actually put x plus 90 or x minus 90 then you have to work out OK, what does that actually mean but I put it in because it's like but you have to remember the rules of… you know f of x plus 10 or something and x minus 10 which way does it go you know or f of 2x is it smaller or bigger or I mean I have to work this out again.

KE: So which means you were shifting the cos graph to the right?

ST5: I am shifting the side, I am starting with the sine graph and I am shifting that 90 degrees to the right.

KE: You shifted the sine graph 90 degrees to the right?

ST5: I think… erm… x minus… I don't know I probably work it out at the time.

KE: Ok.

ST5: (shifted the e)… erm… no ok… I am shifting the sine graph 90 degrees to the right I think.

KE: Ok… alright. What about this one (pointing her answer for item 14 (b))? This one is cos x equals to sin x plus 90.

ST5: Because it's like shifting the cos graph to the left.

KE: Tan x equals to sin x over cos x because…(looking at her answer for item 14 (c)) so you were using the operation and the formula to derive the thing… what about this one (pointing to the link which she erased on her answer script for item 14)?

ST5: No I got that wrong because I couldn't remember which way it was shifting so x minus 90 x plus 90, I just couldn't remember, I couldn't work it out in my head and I think I have to put in values to think actually this would be here not there kind of thing.

KE: Ok. Item 15, for each of the mathematical concepts listed below, please explain any relationships between it and the concept of sine. The first one is function. Your answer is sine is a function (reading her answer script). Alright. series, can have series involving sine for example sum of sine pi x. other than this, do you have any other idea of series?

ST5: Of series?

KE: Yes.

ST5: Erm… you can do oh… that is pi isn't it? Erm… if you do… erm… there is a way like Mclaurin series and you can come up with… erm… a thing for pi an answer for pi… by doing a basically by Taylor expansion but that's more to do with pi than to do with sine… erm…

KE: So can you remember that series?

ST5: Erm… I could probably derive it but I can't remember it there's like a plus… like it's bigger… you know… something over two plus something over three minus something over four plus something over five minus something I don't know… it's something like that it kind of alternates maybe pi over four is this plus this over this plus this over this and minus this over this plus… minus I don't know something like that I could just about manage to derive it if I had to but I don't want to… it's too much brain power.

KE: Alright… ok… it's fine… so what about complex numbers?

ST5: Complex numbers? Erm… usually like pi is minus 1… yea like sinhs and things, I couldn't remember the thingys but I know that theres… yea… erm… if you do like you get complex roots of things (I didn't put that one down, did I?) like if you think of like the circle and you are looking for roots you sometimes get like you know it's minus 1 but it's also like… erm… you know what I mean… so if this is like… erm… imaginary and this is real then whereas in if you just talking in real ones you might just get you know… minus 1 or you might not get roots but… so then it's about a round this unit circle again… finding you know this is something plus something like.

KE: Do you feel comfortable with complex number?

ST5: Yea… I think so I have known about them long enough but I mean I knew about them… I learn about them. I learnt about complex numbers when I was about 12 or 13… I don't know I only saw them at school until A level but it's just counting… I always thought of it as… the people nowadays they learn number lines so the real numbers always go side to side so you've got like 0 in the middle and you've got positive numbers and you know negative numbers but and then so imaginary numbers always going up and down instead whereas I learnt I always thought of it because we never learnt number lines erm…when I was at school that sounds like I am so old I am only 27! But yea I always thought of counting up and down… erm… and then imaginary numbers as well it's like counting sideways erm… so I always thought of it as that so when I got into when I actually learnt it formally I had to re-evaluate because they always talk about it in visual terms they always thought of it as that with the real on the horizontal axis.

KE: What do you think for example like I square equals to negative 1?

ST5: Yea… I think I am happy with like I am happy with a letter as a variable or as an answer or as multiple answers like a lot of people have trouble with especially as a continuation of… like first of all people just about to deal with x plus… x plus 4 equals 1 what is x… well… ok… that is not a good example you know 10… ok so x equals 6 and they can just about to deal with one answer and then you start getting like you know x square equals 4 well what is x it could be equal 2 or it could be minus 2 and that just blows some people's minds but I don't really it might have blown my mind at the time but I don't… I am happy manipulating letters, I think pi is fine.

KE: Ok. What about *y=mx*?

ST5: Erm… y equals mx is generally going to be a straight line that goes through the origin, m is the gradient and so I mean it could be the gradient of sine at that point if m is 1 I suppose… at the origin but otherwise it's a line there is a line that goes through the origin and diverges from sine very quickly.

KE: Ok. We are done with this part. Let's us look at part C so for item 9 you feel not confident to respond to this item, is there any specific reason?

ST5: So that was the test, which one was it (referring to her answer script)? Trigonometric function… erm yeah… I think because it's wordy and because I think, you learn kind of competencies you learn how to calculate you learn things like that and you learn what it means in a general sense but and you had a picture of it in your head kind of thing but it's often you don't have you know actually explaining it to someone else in words is sometimes quite hard.

KE: Alright. Now, I am to show you some responses of a student and let you to comment on them. Let's have a look (showing her the student's responses). The question is what is the value of sin 270 explain why sin 270 has this value.

ST5: Ohhh. The Taylor series.

KE: Yes. What do you think?

ST5: It is far more technical than mine… erm… yeah… I mean… I like that three pi over 2, I think it was three quarters of the way through that is 3 quarters long because of 2pi so that gives you 3 pi over 2… all the terms end up being zero except for one term… yeah… ohh no… it is true you can expand all of them and there is Taylor series and things…sine x… I forgotten about that I don't know I think it's more technical than mine it makes more sense than mine, there is a better reason than mine I got the same answer but… yeah… it's a better reason for why…

KE: So can you make sense of this (pointing the student's response)?

ST5: Yeah… I can't remember the actual expansion but I remember that there is one and then if you then input in 3 pi over 2 into that x and all the bits you will get x squared and that will be whatever erm… yeah… that everything goes to zero except for one… yeah… sounds right.

KE: Ok. The other response (showing her the other response of a student)... This is about why sine theta can never equal 2 and I got this response (showing her the response).

ST5: Ok… so it has to be smaller so they can be bounded (reading the response)… yeah but even if things are smaller than 1 individually bigger than… erm…

KE: What do you think about this one (pointing to the student's response)?

ST5: I mean that's certainly two, it is always… it was always smaller than one that's just saying that by definition is between but when it says the subsequent terms are all smaller than 1, they can be bounded… yes… but if you have like they because still like equal 1 over n but the sum there of…like things they can go up and still give you… yeah… I don't know I think that's as good a explanation as…

KE: So you are not really sure about this one?

ST5: Erm……… I just don't think it is a very good reason because I think if you said the subsequent terms… are all like the bound on the Taylor expansion as one… okay… but saying that because all the terms are smaller than one it doesn't mean that the whole thing is smaller than one… I daresay that it is smaller than one but I don't think that this is just saying that they are smaller than one… what you really need to say is that they are bounded not just that it is smaller than one so they can be bounded… one thing doesn't lead to the other… erm… yeah…

KE: Alright. A few more questions… do you have any difficulties in learning trigonometry?

ST5: The last time I learned trigonometry it was easy, erm… I suppose the first time I leaned trigonometry is that when you first see sine? (inaudible).

KE: So do you have any difficulties during that time?

ST5: I don't remember having being any specific difficulties, I suppose my inability to visualize 270 degrees triangle perhaps you could be considered a difficulty but I just think about well what is that equivalent to… so then I… you know draw the graph in my head and think ok so these are the points and it's closer to this therefore it does this…(inaudible).

KE: Ok. In your opinion, what kind of difficulties that a secondary school student might have in learning trigonometry?

ST5: I think they need a link to real world stuffs because trig… as soon as you go into trig suddenly it's all… you know… because yeah what is sine, what is a triangle with 270 degrees and stuff like that it doesn't immediately make sense to them and you very soon leap off the page… you know… you leap off away from reality into just theoretical maths well it is this shape, why is it because it is and I think it's not rooted for a secondary school student it's not rooted in reality enough… erm… yeah.

KE: Alright. What difficulties you have in understanding trigonometry?

ST5: Erm… isn't that the same as learning it?

KE: Yeah.

ST5: I don't know… no specific difficulties.

KE: It's similar to the previous questions. Do you have any confusion in trigonometry?

ST5: Erm…… they are interrelate and so sometimes you've got an integration you know tan x you have to see that is like… ok… so I should have use the angle formula and used equivalent series like that like you've got something like that erm… so certainly when I was first learning angle formula that you can get you can just get really… like you can go through and you think ok so that is equivalent and so you think… I will put that in that huge long thing and then when you get to the end and you are back where you started because you still got… I don't know… squareds and you want to get rid of your squareds and so you get confused that way… it's more just the practical doing of it that you can go round in circles.

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