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Slices of Quasifuchsian Space

by

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To grandpa Eraldo,
to all my family,
to Michele.

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[1] LaTeX 2_ε is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The style package *warwickthesis* was used.

Declarations

I hereby declare that the dissertation represents my own work and has not been previously submitted to this or any other institution for any degree, diploma or other qualification.

Parts of Chapter 1 and Chapter 2 are based on [28], written with my advisor Caroline Series during my period of study for this degree. Series' main contribution to the paper was the (combinatorial) proof of the Top Terms' Formula (Theorem 2.4.1), which I included in Appendix B. Therefore in Chapter 2 I present my (inductive) proof which inspired Series.

Parts of Chapter 1 and Chapter 3 are based on [27], written during my last year of Ph.D.

Abstract

In Chapter 1 we present the background material about curves on surfaces. In particular we define the Dehn–Thurston coordinates for the set $\mathcal{S} = \mathcal{S}(\Sigma)$ of free homotopy class of multicurves on the surface Σ. We also prove new results, like the precise relationship between Penner's and D. Thurston's definition of the twist coordinate and the formula for calculating the Thurston's symplectic form using Dehn–Thurston coordinates.

For Chapter 2, let Σ be a surface of negative Euler characteristic together with a pants decomposition $\mathcal{PC}$. Kra's plumbing construction endows Σ with a projective structure as follows. Replace each pair of pants by a triply punctured sphere and glue, or 'plumb', adjacent pants by gluing punctured disk neighbourhoods of the punctures. The gluing across the i^{th} pants curve is defined by a complex parameter $\mu_i \in \mathbb{C}$. The associated holonomy representation $\rho_{\underline{\mu}}\colon \pi_1(\Sigma) \longrightarrow PSL(2, \mathbb{C})$ gives a projective structure on Σ which depends holomorphically on the μ_i. In particular, the traces of all elements $\rho_{\underline{\mu}}(\gamma)$, where $\gamma \in \pi_1(\Sigma)$, are polynomials in the μ_i.

Generalising results proved in [24; 40] for the once and twice punctured torus respectively, we prove in Chapter 2 a formula giving a simple linear relationship between the coefficients of the top terms of $\mathrm{Tr}\, \rho_{\underline{\mu}}(\gamma)$, as polynomials in the μ_i, and the Dehn–Thurston coordinates of γ relative to $\mathcal{PC}$. We call this formula the Top Terms' Relationship.

In Chapter 3, applying the Top Terms' Relationship, we determine the asymptotic directions of pleating rays in the Maskit embedding of a hyperbolic surface Σ as the bending measure of the 'top' surface in the boundary of the convex core tends to zero. The Maskit embedding $\mathcal{M}$ of a surface Σ is the space of geometrically finite groups on the boundary of Quasifuchsian space for which the 'top' end is homeomorphic to Σ, while the 'bottom' end consists of triply punctured spheres, the remains of Σ when the pants curves have been pinched. Given a projective measured lamination $[\eta]$ on Σ, the pleating ray $\mathcal{P} = \mathcal{P}_{[\eta]}$ is the set of groups in $\mathcal{M}$ for which the bending measure $\mathrm{pl}^+(G)$ of the top component $\partial \mathcal{C}^+$ of the boundary of the convex core of the associated 3-manifold $\mathbb{H}^3/G$ is in the class $[\eta]$.

List of Symbols

The following table describes the significance of various symbols used throughout the thesis. The page on which each one is defined or first used is also given.

Symbol	Definition	Page
A_X	$A_X = \begin{pmatrix} 1 & X \\ 0 & -1 \end{pmatrix}$	112
$\mathbb{A}_{\epsilon,j}$	annulus $\mathbb{A}_{\epsilon,j}$ in $\mathbb{P}$	45
$\mathbb{A}$	(in general) annulus	7
B_X	$B_X = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$	112
b	number of boundary components of a surface	5
b_i, b_i^*	basepoints in Σ	56
b_0, b_0^*	basepoints in Δ_0, Δ_1	56
$B_i, B'_j, \mathcal{B}_i, \mathcal{B}'_j$	regions in $\mathbb{C}$	108
B	set of basepoints $B = \{b_1, b_1^*, \ldots, b_k, b_k^*\}$	56
$\hat{\mathbb{C}}$	Riemann sphere $\mathbb{C} \cup \{\infty\}$	32
$\mathcal{C}_M = \mathcal{C}(G)$	convex core of the 3–manifold $M = \mathbb{H}^3/G$	81
$\mathfrak{C}$	fixed charts $\mathfrak{C} = \{\Phi^b \vert b \in B\}$	38
$\mathrm{CH}(\Lambda)$	convex hull of $\Lambda(G)$	81
Dev	developing map $\mathrm{Dev}\colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$	37
$\mathbb{D}_*$	punctured disk	50
D_i	dual curves	6
d	complex distance $d = d_\alpha(\gamma_1, \gamma_2)$	91

$dbcc$	different boundary component connector arc	6
$E_\gamma(\mu_1,\ldots,\mu_\xi)$	$E_\gamma(\mu_1,\ldots,\mu_\xi) = w_1\cdots w_\xi \sum_{i=1}^{\xi} \frac{(q_i x_i + p_i)}{w_i}$	97
$\mathcal{F}(\Sigma)$	Fuchsian space of Σ	34
$f_{\underline{\mu}}$	marking homomorphism $f_{\underline{\mu}}\colon\ \Sigma \longrightarrow \Sigma(\underline{\mu})$	50
g	genus of a surface	5
$G_\eta(\theta)$	unique group in $\mathcal{M}$ with $\mathrm{pl}^+(G) = \theta\eta$	87
$\mathbb{H}$	hyperbolic 2–space $\mathbb{H} = \mathbb{H}^2$	2.1.1
$\mathbb{H}^3$	hyperbolic 3–space	2.1.1
$\mathbb{H}^\xi$	$\mathbb{H}^\xi = \mathbb{H}\times\ldots,\times\mathbb{H}$, product of ξ copies of $\mathbb{H}$	2.2
$\mathbb{H}^\mu$	$\mathbb{H}^\mu = \{z\in\mathbb{C} : \Im z > \Im\mu\}$	90
h_ϵ	loop in $\mathbb{P}$	43, 45
$h_{\infty,\mathbb{H}}$	horocycle $h_{\infty,\mathbb{H}} = \{z\in\mathbb{H} \mid \Im z = \frac{\Im\mu}{2}\}$	44
$h_{\infty,\mathbb{H}}$	horocycle $\Omega_\epsilon^{-1} h_{\infty,\mathbb{H}}$	45
$H_\epsilon, H_{\epsilon,+}$	strips $H_\epsilon \subset H_{\epsilon,+}$ in $\Delta\subset\mathbb{H}$	44, 45
$h(\Omega_{i_j})$	quantity in $\{0,1\}$	113
i	(in general) index in the set $\{1,\ldots,\xi\}$	6
$\mathbf{i}, \mathbf{i}_{DT}, \mathbf{i}_{PH}$	maps describing the Dehn–Thuston coordinates	8
$i(\alpha,\alpha')$	geometric interesection number between $\alpha,\alpha'\in\mathcal{S}$	5
$I_\beta(\gamma_1,\gamma_2)$	signed relative twist	93
$\hat{i}(\alpha,\alpha')$	algebraic interesection number between $\alpha,\alpha'\in\mathcal{S}$	11
j	(in general) index in the set $\{1,\ldots,k\}$	6
J	$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$	44
J_1, J_2, J_3	cyclic groups	106
$k = k(\Sigma)$	number of pairs of pants in Σ	6
$k(\Omega_{i_j})$	quantity in $\{0,1\}$	113
$\mathbb{L}$	$\mathbb{L} = \{z\in\mathbb{C} : \Im z < 0\}$	90
$\mathrm{lk}(\gamma_1,\ldots,\gamma_k)$	link of the curves $\gamma_1,\ldots,\gamma_k$	98
L_η	line approached by $\mathcal{P}_\eta$	101
l_σ^+	hyperbolic length of σ^+ in $\partial^+\mathcal{C}$	86

m	in general, a natural number $m \in \mathbb{N}$	82
$\mathcal{M}$	Maskit embedding	52
$M = M_G$	hyperbolic manifold $\mathbb{H}^3/G$ associated to G	34
$\mathrm{ML}(\Sigma)$, $\mathrm{PML}(\Sigma)$	space of (projective) measured laminations	20
$\mathrm{ML}_{\mathbb{Q}}(\Sigma)$	space of (projective) rational measured laminations	20
$\mathrm{MS}(\sigma)$	modular surface ossociated to $\sigma \in \mathcal{S}_0$	6
$O(\theta)$	$X(\sigma_i) = O(\theta)$ if $X \leqslant c\theta$ as $\theta \longrightarrow 0$	87
$O(1)$	universal bound independent of η	88
$\mathbb{P}$	standard triple punctured sphere $\mathbb{H}^2/\Gamma$	41
$\mathcal{PC}$	$\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$	6
$\mathcal{P}$	$\mathcal{P} = \{P_1, \ldots, P_k\}$	5
$\mathcal{P}_\eta$	pleaing ray associated to $[\eta] \in \mathrm{PML}(\Sigma)$	83
$\mathcal{P}_{\gamma_1,\ldots,\gamma_m}$	pleaing variety associated to $\gamma_1, \ldots, \gamma_m \in \mathcal{S}_0$	84
P_j	pair of pants	6
p_i	DT–twist parameter	10
$\hat{p}_i$	PH–twist parameter	15
$\mathrm{pl}^{\pm}$	pleating measured lamination on $\partial\mathcal{C}^{\pm}$ if $M \cong \Sigma \times \mathbb{R}$	82
q_i	length parameter	7
$\mathcal{QF}(\Sigma)$	Quasifuchsian space of Σ	34
$\mathcal{R}(\Sigma)$	representation variety of Σ	51
R	term of lower degree in Theorem 2.4.1	68
$\mathbf{R}$	involution of Σ	9
S_j	truncated surface $S_j \subset \mathbb{P}$	44
$\mathcal{S}_0 = \mathcal{S}_0(\Sigma)$	set of free homotopy classes of curves on Σ	5
$\mathcal{S} = \mathcal{S}(\Sigma)$	set of free homotopy classes of multicurve on Σ	5
$\mathfrak{S}_{\underline{\mu}}$	$\mathfrak{S}_{\underline{\mu}} = S_1 \sqcup \ldots \sqcup S_k$	45
sbcc	same boundary component connector arc	6
$\mathcal{ST}_i$	fundamental domains for J_i	106
$\hat{\mathcal{ST}}_1, \hat{\mathcal{ST}}_*$	fundamental set for J_i and a subregion	108
t_K	Kra's parameter	51

T_μ	$T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$	44
$\mathcal{T}(\Sigma)$	Teichmüller space of Σ	34
tw_i	(general) twist parameter	7
$\mathrm{tw}_\beta(\gamma, \mathrm{h})$	twist of the geodesic γ around β	93
$Tw_{\sigma,t}$	(right) Dehn twist homeomorphism	7
$Tw_\sigma(\gamma)$	(right) Dehn twist of γ about σ	7
W	$W = w_1 \cdots w_\xi$	102
w_i	$w_i = \frac{y_i}{Y}$	97
X_j	quantity $X_j = -(\mu_i + 2\alpha_j + 2\beta_j)$	112
X_j^*	quantity $X_j^* = X_j + h(\Omega_{i_j}) + k(\Omega_{i_{j-1}})$	113
x_i	$x_i = \Re\mu_i - 1$	97
y_i	$y_i = \Im\mu_i$	97
$\mathbf{x}^*$	$\mathbf{x}^* = (y_1, -x_1, \ldots, y_\xi, -x_\xi)$	99
Y	$Y = \|(y_1, \ldots, y_\xi)\| = (y_1^2 + \ldots + y_\xi^2)^{\frac{1}{2}}$	97

Table 1: List of notation (Latin letters)

Symbol	Definition	Page
γ_ϵ	path in P between $b(P)$ and $b^*(P)$ crossing λ_ϵ	58
$\partial_\epsilon(P_j), \partial_\epsilon(\mathbb{P})$	ϵ–boundary component of P_j, $\mathbb{P}$ and $\mathbf{P}$	42
$\Delta, \Delta_0, \Delta_1$	fundamental domain in $\mathbb{H}^2$ for Γ	41
$\Delta_0(P), \Delta_1(P)$	lift of the domain Δ_0, Δ_1 to Σ	45
ϵ	an element of the cyclically ordered set $\{0, 1, \infty\}$	41
ζ	map $\zeta\colon \mathbb{H}^2 \longrightarrow \mathbb{P}$	42
θ_γ	bending angle	82
η	general lamination in $\mathrm{ML}(\Sigma)$, often $\eta = \sum_{i=1}^m a_i\gamma_i$	82
$\vartheta(P, P'; \epsilon, \epsilon'; \sigma)$	path between adjacent pairs of pants	60
$\Theta_{\epsilon \longrightarrow \epsilon'}$	matrix $\rho_{\underline{\mu}}(\vartheta_{\epsilon \longrightarrow \epsilon'})$	62

Θ	matrix $\Theta = \Theta_{\infty \longrightarrow \infty}$	105
Γ	group $\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$	41
Γ'	group generated by Υ_∞, Υ'_0	105
$\Lambda(G)$	limit set of G	33
λ_ϵ	geodesic in Δ_0 joining $\epsilon + 1$ to $\epsilon + 2$	42
$\lambda(\gamma) = \lambda(\rho(\gamma))$	complex length of γ in $\mathbb{H}^3/\rho(\pi_1(\Sigma))$	85
$\underline{\mu}$	gluing parameter in $\mathbb{H}^\xi$	40
$\underline{\mu}^0$	fixed gluing parameter	49
ν	small positive constant	44
$\xi = \xi(\Sigma)$	complexity of the surface	5
Ξ_0, Ξ_1, Ξ_∞	matrices $\rho_{\underline{\mu}}(\gamma_\epsilon)$	58
$\pi_{1,2k}(\Sigma, B)$	fundamental groupoid of (Σ, B)	57
Π	$\Pi\colon\ \mathfrak{S}_{\underline{\mu}} \longrightarrow \mathfrak{S}_{\underline{\mu}}/\sim$	45
ρ	general holonomy map $\rho\colon\ \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$	37
$\rho_{\underline{\mu}}$	groupoid homomorphism $\rho_{\underline{\mu}}\colon\ \pi_{1,2k}(\Sigma, B) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$	38
$\Sigma = \Sigma_{g,b}$	surface of finite type with genus g and b cusps	5
$\tilde{\Sigma}$	universal cover of Σ	37
$\Sigma(\underline{\mu})$	(complex) projective structure on Σ	48
σ_i	pants curve	5
$\sigma^+ = \sigma_i^+$	geodesic representative of σ_i on $\partial\mathcal{C}^+$	86
$\tilde{\sigma}^+ = \tilde{\sigma}_i^+$	lift of σ_i^+ to $\partial\,\mathrm{CH}(\Lambda)^+$	89
$\boldsymbol{\tau}$	train track	21
υ_ϵ	path going around the boundary $\partial_\epsilon(P)$	58
Υ_ϵ	matrix $\rho_{\underline{\mu}}(\upsilon_\epsilon)$	59
Υ'_0	matrix $\Theta\Upsilon_0\Theta^{-1}$	105
Φ_j	map $\Phi_j\colon\ P_j \longrightarrow \mathbb{P}$	42
$\hat{\Phi}_j$	map $\hat{\Phi}_j\colon\ P_j \longrightarrow \Delta$	41
Ω^+	image of $\mathrm{Dev}_{\underline{\mu}}$	53
$\Omega(G)$	regular set of G	33

$\Omega^0(G)$	freely discontinuous set of G	33
Ω_ϵ	matrices defined in Equation 2.2	44
$\Omega_{\mathrm{Th}}(\gamma, \gamma')$	Thurston's symplectic form	22

Table 2: List of notation (Greek letters)

Introduction

In this thesis we mainly deal with Kleinian groups, which are discrete groups of isometries of the hyperbolic 3–space $\mathbb{H}^3$. In the upper–half–space model of $\mathbb{H}^3$ the orientation–preserving isometries are identified with the group $\mathrm{PSL}(2, \mathbb{C})$. Such groups also act by conformal automorphisms on the sphere at infinity $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \partial\mathbb{H}^3$. In the 1960s, Ahlfors and Bers studied Kleinian groups mainly analytically, in terms of their action on the Riemann sphere $\hat{\mathbb{C}}$. Thurston revolutionised the subject in the 1970s by taking a more topological viewpoint and showing that, in a certain sense, 'many' 3-manifolds are hyperbolic. He also introduced new concepts which helped a lot in understanding much more deeply the topic. In the last years, our understanding of Kleinian groups improved a lot with the proofs of three great conjectures: the Density Conjecture, the Ending Lamination Conjecture and the Tameness Conjecture. Combined, they give a remarkably complete picture of Kleinian groups.

Keen and Series in 1990s introduced the Pleating Coordinates Theory, which is a new tool in the study of the deformation spaces of holomorphic families of Kleinian groups. The key idea is to study these deformation spaces via the internal geometry of the associated hyperbolic 3–manifold, in particular, the geometry of the boundary of its convex core. This method allows one to relate combinatorial, analytical and geometrical data. One important result is to give algorithms enabling one to compute the exact position of the deformation space as a subset of $\mathbb{C}^n$. This answers a question posed by Bers in the late 1960s about the possiblity to compute explicitly these deformation spaces.

Important examples of Kleinian groups are Fuchsian and Quasifuchsian groups, which are groups such that the limit set is a circle or a topological circle, respectively. In this thesis we describe a slice on the boundary of the Quasifuchsian space, the Maskit slice, mainly using the Pleating Coordinates Theory. We modify the plumbing construction introduced by Kra which is an explicit construction of Kleinian groups in these slices.

Let Σ be a surface of negative Euler characteristic together with a pants decomposition $\mathcal{PC}$. Kra's plumbing construction endows Σ with a projective structure as follows. Replace each pair of pants by a triply punctured sphere and glue, or 'plumb', adjacent pants by gluing punctured disk neighbourhoods of the punctures. The gluing across the i^{th} pants curve is defined by a parameter $\mu_i \in \mathbb{C}$. The holonomy representation $\rho_{\underline{\mu}}\colon \pi_1(\Sigma) \longrightarrow PSL(2,\mathbb{C})$ associated with the projective structure on Σ that was described above depends holomorphically on the μ_i, and, by construction, the images of the pants curves are parabolic. In particular, the traces of all elements $\rho_{\underline{\mu}}(\gamma)$, for $\gamma \in \pi_1(\Sigma)$, are polynomials in the μ_i.

The main result of Chapter 2 is a relationship between the coefficients of the top terms of $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma)$, as polynomials in the μ_i, and the Dehn–Thurston coordinates of γ relative to $\mathcal{PC}$. This generalises results of Keen and Series [24] and Series [40] for the once and the twice punctured torus, respectively.

Our formula is as follows. Let $\mathcal{S} = \mathcal{S}(\Sigma)$ denote the set of homotopy classes of multicurves on Σ, and let $\sigma_1, \dots, \sigma_\xi$ be the pants curves defining $\mathcal{PC}$. The *Dehn–Thurston coordinates* of $\gamma \in \mathcal{S}$ are $\mathbf{i}(\gamma) = (q_i, p_i)$, for $i = 1, \dots, \xi$, where $q_i = i(\gamma, \sigma_i) \in \mathbb{N} \cup \{0\}$ is the geometric intersection number between γ and σ_i, and $p_i \in \mathbb{Z}$ is the twist of γ about σ_i. Our main result in Chapter 2 is the following.

Theorem A. *Let γ be a connected simple closed curve on Σ.*
If γ is not parallel to any of the pants curves σ_i, then, up to terms of lower degree, we have:

$$\operatorname{Tr}\rho_{\underline{\mu}}(\gamma) = \pm i^q 2^h \left(\mu_1^{q_1} \cdots \mu_\xi^{q_\xi} + \sum_{i=1}^{\xi} (p_i - q_i) \mu_1^{q_1} \cdots \mu_i^{q_i - 1} \cdots \mu_\xi^{q_\xi} \right),$$

where $q = \sum_{i=1}^{\xi} q_i > 0$ and $h = h(\gamma)$ is the total number of sbcc–arcs in γ, see Section 1.1 for the definition.

If $q_i = 0$ for all $i = 1, \dots, \xi$, then $\gamma = \sigma_i$ for some i, $\rho_{\underline{\mu}}(\gamma)$ is parabolic, and $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma) = \pm 2$.

In the literature, the definition of the twist parameters p_i is not uniquely determined, unlike the length parameters q_i. In particular, one of the main tools in our proof of Theorem A is the description of the relationship between Penner and Harer's definition [38] of p_i and Dylan Thurston's one [41], which we describe in the following result.

Theorem (Theorem 1.2.6)**.** *Suppose that two pairs of pants meet along a pants curve σ_i. Label their respective boundary curves (A, B, σ_i) and (C, D, σ_i) in clockwise* *order. Let q_i, $\hat{p}_i$ and p_i denote the length parameter, Penner and Harer's twist and Dylan Thurston's twist, respectively. Then*

$$\hat{p}_i = \frac{p_i + l(A, \sigma_i; B) + l(C, \sigma_i; D) - q_i}{2},$$

where $l(X, Y; Z)$ denotes the number of strands of $\gamma \cap P$ running from the boundary curve X to the boundary curve Y in the pair of pants P having boundary curves (X, Y, Z).

The idea comes from Appendix B of [41], where there is a similar statement (without proof). We correct and prove it in Section 1.2.3.

We want also to underline a very interesting result about the relationship between Thurston's product and the Dehn–Thurston coordinates for the set $\mathcal{S}$ of multicurves on Σ. Given a (generic birecurrent) train track $\boldsymbol{\tau} \subset \Sigma$, let $\mathbf{n}$, $\mathbf{n}' \in \mathcal{W}(\boldsymbol{\tau})$ be weightings on $\boldsymbol{\tau}$; see Section 1.3.3 for the definitions. Denote by $b_v(\mathbf{n})$ and $c_v(\mathbf{n})$ the weights of the left hand and right hand outgoing branches at v respectively. *Thurston's product* is defined by

$$\Omega_{\mathrm{Th}}(\mathbf{n}, \mathbf{n}') = \frac{1}{2} \sum_v b_v(\mathbf{n}) c_v(\mathbf{n}') - b_v(\mathbf{n}') c_v(\mathbf{n}).$$

In [40], Series relates Thurston's product to the Dehn–Thurston coordinates described above, but her proof works only for the case $\Sigma = \Sigma_{1,2}$, because she uses a particular choice of train tracks called *canonical train tracks*, see Keen–Parker–Series [23]. Our idea is to use the *standard train tracks* defined by Penner and Harer [38]. The Dehn-Thurston coordinates (using Penner and Harer's twist) give us a choice of a standard model and, using the relationship between Penner and Harer's and D. Thurston's twist (see Theorem 1.2.6), we prove the following result in Section 1.3.3.

Theorem (Theorem 1.3.3)**.** *Suppose that loops $\gamma, \gamma' \in \mathcal{S}$ belong to the same chart and that they are represented by coordinates $\mathbf{i}(\gamma) = (q_1, p_1, \ldots, q_\xi, p_\xi)$ and $\mathbf{i}(\gamma') = (q_1', p_1', \ldots, q_\xi', p_\xi')$. Then*

$$\Omega_{\mathrm{Th}}(\gamma, \gamma') = \frac{1}{2} \sum_{i=1}^{\xi} (q_i p_i' - q_i' p_i).$$

In addition, if γ, γ' are disjoint, then $\Omega_{\mathrm{Th}}(\gamma, \gamma') = 0$.

If the representation $\rho_{\underline{\mu}}$, constructed by Kra's plumbing construction, is free and discrete, then the resulting hyperbolic 3–manifold $M = \mathbb{H}^3/\rho_{\underline{\mu}}(\pi_1(\Sigma))$ lies on the

Maskit slice. The Maskit slice $\mathcal{M} = \mathcal{M}(\Sigma)$ is the space of geometrically finite groups on the boundary of Quasifuchsian space $\mathcal{QF}(\Sigma)$ for which the 'top' end is homeomorphic to Σ, while the 'bottom' end consists of triply punctured spheres, the remains of Σ when the pants curves have been pinched. We investigated $\mathcal{M}$ using the method of pleating rays. For this description we were inspired by Series' analysis of the case of the twice punctured torus; see [40].

Given a projective measured lamination $[\eta]$ on Σ, the *pleating ray* $\mathcal{P} = \mathcal{P}_\eta$ is the set of groups in $\mathcal{M}$ for which the bending measure $\mathrm{pl}^+(G)$ of the top component $\partial\mathcal{C}^+$ of the boundary of the convex core of the associated 3-manifold $\mathbb{H}^3/G$ is in the class $[\eta]$. We restrict to pleating rays for which $[\eta]$ is *rational*, that is, supported on multicurves. The Top Terms' Relationship discussed above enables us to find the asymptotic direction of $\mathcal{P}$ in $\mathcal{M}$ as the bending measure tends to zero in terms of natural parameters for the representation variety $\mathcal{R}(\Sigma)$ (see Section 2.2.5) and of the Dehn–Thurston coordinates for the support curves to $[\eta]$ relative to the pinched curves on the bottom side. This leads to a method of locating $\mathcal{M}$ in $\mathcal{R}$.

Bonahon and Otal describe precisely which laminations can be the pleating locus on $\partial\mathcal{C}^+$. In particular, $\eta = \sum_{i=1}^{\xi} a_i\delta_{\gamma_i}$ can be the pleating lamination on $\partial\mathcal{C}^+$ if and only if $q_i(\eta) > 0$, for all $i = 1, \dots, \xi$. We call such laminations *admissible*. See Section 3.2.2 for the definition of *non-exceptional* lamination. The main result of Chapter 3 is the following.

Theorem B. *Suppose that* $\eta = \sum_{i=1}^{m} a_i\delta_{\gamma_i}$ *is admissible and non-exceptional. Then, as the bending measure* $pl^+(G) \in [\eta]$ *tends to zero, the pleating ray* $\mathcal{P}_\eta$ *approaches the line*

$$\Re\mu_i = \frac{p_i(\eta)}{q_i(\eta)}, \qquad \frac{\Im\mu_1}{\Im\mu_j} = \frac{q_j(\eta)}{q_1(\eta)}.$$

One might also ask for the limit of the hyperbolic structure on $\partial\mathcal{C}^+(G)$ as the bending measure tends to zero. The following result answers this question.

Theorem C. *Let* $\eta = \sum_1^{\xi} a_i\delta_{\gamma_i}$ *be as above. Then, as the bending measure* $\mathrm{pl}^+(G) \in [\eta]$ *tends to zero, the induced hyperbolic structure of* $\partial\mathcal{C}^+$ *along* $\mathcal{P}_\eta$ *converges to the barycentre of the laminations* $\sigma_1, \dots, \sigma_\xi$ *in the Thurston boundary of* $\mathcal{T}(\Sigma)$*.*

Chapter 1

Curves on surfaces

In this chapter we recall the background material on surfaces which we will use later. In particular, in Section 1.1 we introduce some notation and basic facts about curves on surfaces. In Section 1.2 we discuss Dehn–Thurston coordinates, in particular we define different types of twist, following D. Thurston in Section 1.2.1, and Penner and Harer in Section 1.2.2. In Section 1.2.3 we explain the precise relationship between these two different definitions. In Section 1.3 we define Thurston's symplectic form and in Section 1.3.3 we describe an exact formula for calculating it using Dehn–Thurston coordinates.

1.1 Background material on surfaces

Suppose Σ is a surface of finite type, let $\mathcal{S}_0 = \mathcal{S}_0(\Sigma)$ denote the set of free homotopy classes of connected closed simple non-trivial non-boundary parallel curves on Σ. For simplicity we usually refer to elements of $\mathcal{S}_0$ as 'curves'. Let $\mathcal{S} = \mathcal{S}(\Sigma)$ be the set of free homotopy classes of multicurves on Σ, where a *multicurve* is a finite unions of disjoint curves in $\mathcal{S}_0$. The *geometric intersection number* $i(\alpha, \alpha')$ between multicurves $\alpha, \alpha' \in \mathcal{S}$ is the least number of intersections between representatives of the two homotopy classes, that is

$$i(\alpha, \alpha') = \min_{a \in \alpha,\ a' \in \alpha'} |a \cap a'|.$$

Now given a surface $\Sigma = \Sigma_{g,b}$ of finite type (with genus g and b boundary components) and negative Euler characteristic, choose a maximal set $\mathcal{PC} = \{\sigma_1, \dots, \sigma_\xi\}$ of homotopically distinct and non-boundary parallel curves in Σ called *pants curves*, where $\xi = \xi(\Sigma) = 3g - 3 + b$ is the complexity of the surface. These curves split the surface into $k = 2g - 2 + b$ (open) three-holed spheres $P_1, \dots, P_k$, called *pairs of pants*, obtained by removing three closed disjoint disks on the sphere. (Note that the boundary of P_i may include punctures of Σ.) We refer to both the set $\mathcal{P} = \{P_1, \dots, P_k\}$, and the set $\mathcal{PC}$, as *pants decompositions* of Σ. In the following, we will try to use, in general, the subindex i as $i \in \{1, \dots, \xi\}$ and the subindex j as $j \in \{1, \dots, k\}$. For example a general pants curve will be denoted σ_i and a general pair of pants P_j.

Let P_j to be a three-holed sphere embedded in Σ. Each pants curve $\sigma = \sigma_i$ is the common boundary of one or two pairs of pants whose union (including σ) we refer to as the *modular surface* associated to σ, and we denote $\mathrm{MS}(\sigma)$. If σ is the common boundary of one pair of pants, then $\mathrm{MS}(\sigma)$ is homeomorphic to a one-holed torus $\Sigma_{1,1}$, otherwise it is homeomorphic a four-holed sphere $\Sigma_{0,4}$.

Any hyperbolic pair of pants P is made by gluing two right angled hexagons along three alternate edges which we call its *seams*; see Fathi, Laudenbach and Poénaru [17]. In much of what follows, it is convenient to designate one of these hexagons as 'white' and one as 'black'.

Definition 1.1.1. A properly embedded arc in P, that is, an arc with its endpoints on ∂P, is called *sbcc*–arc (same boundary component connector) if it has both its endpoints on the same component of ∂P and *dbcc*–arc (different boundary component connector) otherwise.

See Figure 1.4 for an example of *sbcc*–arc. Note that in [28] the *sbcc*–arcs are called *scc* (same component connector) arc and the *dbcc*–arcs are called *dcc* (different component connector).

1.1.0.a Convention on dual curves

We shall need to consider *dual curves* to $\sigma_i \in \mathcal{PC}$, that is, curves which intersect σ_i minimally and which are completely contained in $\mathrm{MS}(\sigma_i)$, the union of the pants P, P' adjacent to σ_i along σ_i. The intersection number of such a connected curve with σ_i is 1 if $\mathrm{MS}(\sigma_i)$ a one-holed torus and 2 otherwise. In the first case, the curve is made by identifying the endpoints of a single *dbcc*–arc in the pair of pants adjacent to σ_i and, in the second, it is the union of two *sbcc*–arcs, one in each of the two pants whose union is $\mathrm{MS}(\sigma_i)$. We adopt a useful convention introduced in [41] which simplifies the formulae in such a way as to avoid the need to distinguish between these two cases. Namely, for those σ_i for which $\mathrm{MS}(\sigma_i)$ is $\Sigma_{1,1}$, we define the dual curve $D_i \in \mathcal{S}$ to be *two* parallel copies of the connected curve intersecting σ_i once, while if $\mathrm{MS}(\sigma_i)$ is $\Sigma_{0,4}$ we take a single copy. In this way we always have, by definition, $i(\sigma_i, D_i) = 2$, where $i(\alpha, \beta)$ is the geometric intersection number as above. Note that with this convention some dual curves are actually multicurves, but for simplicity we will continue to use the term 'dual curves'.

A *marking* on Σ is the specification of a fixed base (topological) surface Σ_0, together with a homeomorphism $f\colon \Sigma_0 \longrightarrow \Sigma$. There is a related notion of 'marking decomposition' which we will discuss in Section 1.2.1 below. Marking decompositions can be defined in various equivalent ways, for example by specifying the choice of dual curves. See Section 1.2.1 for details.

1.1.0.b Convention on twists

Our convention is always to measure twists to the right as positive. In particular, given a surface Σ, a curve $\sigma \in \mathcal{S}_0$ and $t \in \mathbb{R}$, the distance t Fenchel–Nielsen twist deformation around σ is the homeomorphism $Tw_{\sigma,t}\colon \Sigma \longrightarrow \Sigma$ defined in the following way. Let $\mathbb{A} = \mathbb{A}(\sigma) = \sigma \times [0,1]$ to be an annulus around σ. If we parameterise σ as $s \mapsto \sigma(s) \in \Sigma$ for $s \in [0,1)$, then the distance t twist, denoted $Tw_{\sigma,t}\colon \Sigma \longrightarrow \Sigma$, maps $\mathbb{A}$ to itself by $(\sigma(s), \theta) \mapsto (\sigma(s + \theta t), \theta)$ and is the identity elsewhere. You can extend this definition to a multicurve $\sigma \in \mathcal{S}$ by considering disjoin annuli around the curves in σ.

We denote by $Tw_{\sigma,1}(\gamma) = Tw_\sigma(\gamma)$ the *right* Dehn twist of the multicurve γ about the multicurve σ.

1.2 Dehn–Thurston coordinates

In this section we recall various ways to define Dehn–Thurston coordinates. In particular, we describe D. Thurston's and Penner and Harer's approaches, and the relationship between them.

Suppose we are given a surface Σ together with a pants decomposition $\mathcal{P}$, as defined in Section 1.1. Let γ be a multicurve in $\mathcal{S}$ (see Section 1.1 for the definition) and $q_i = i(\gamma, \sigma_i) \in \mathbb{Z}_{\geqslant 0}$, for $i = 1, \dots, \xi$. Notice that, if $\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}$ are pants curves which together bound a pair of pants embedded in Σ, then the sum $q_{i_1} + q_{i_2} + q_{i_3}$ of the corresponding intersection numbers is even. The $q_i = q_i(\gamma)$ are sometimes called the *length parameters* of γ.

To define the *twist parameter* $\mathrm{tw_i} = \mathrm{tw_i}(\gamma) \in \mathbb{Z}$ of γ about σ_i, we first have to fix a marking decomposition on Σ, for example by fixing a specific choice of dual curves D_i to the pants curve σ_i, see Section 1.2.1 below. Then, after applying an isotopy to move γ into a well-defined standard position relative to the marking, the twist $\mathrm{tw_i}$ is the signed number of times that γ intersects a short arc transverse to

σ_i. We make the convention that, if $i(\gamma, \sigma_i) = 0$, then $\mathrm{tw_i}(\gamma) \geqslant 0$ is the number of components in γ freely homotopic to σ_i.

There are various ways of defining the *standard position* of γ, leading to different definitions of the twist. The parameter $p_i(\gamma)$ which occurs in the statement of the Top Terms' Relationship (Theorem A) is the twist parameter defined by Dylan Thurston [41], however in the proof of the formula we find it convenient to use a slightly different definition: the twist parameter $\hat{p}_i(\gamma)$ described by Penner and Harer [38]. Both of these definitions are explained in detail below, as is the precise relationship between them, see Theorem 1.2.6. With either definition, a classical theorem of Dehn [12], see also [38] (p.12), asserts that the length and twist parameters uniquely determine γ:

Theorem 1.2.1. ***(Dehn's theorem)***
The map $\mathbf{i}\colon \mathcal{S}(\Sigma) \longrightarrow (\mathbb{Z}_{\geqslant 0} \times \mathbb{Z})^{\xi}$ *which sends a multicurve* $\gamma \in \mathcal{S}(\Sigma)$ *to the* (2ξ)*–vector* $(q_1(\gamma), \mathrm{tw}_1(\gamma), \ldots, \mathrm{q}_\xi(\gamma), \mathrm{tw}_\xi(\gamma))$ *is an injection. In addition, we have that the point* $(q_1(\gamma), \mathrm{tw}_1(\gamma), \ldots, \mathrm{q}_\xi(\gamma), \mathrm{tw}_\xi(\gamma))$ *is in the image of* $\mathbf{i}$ *(and hence corresponds to a multicurve) if and only if:*

(i) if $q_i = 0$*, then* $\mathrm{tw_i} \geqslant 0$*, for each* $i = 1, \ldots, \xi$*.*

(ii) if $\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}$ *are pants curves which together bound a pair of pants embedded in* Σ*, then the sum* $q_{i_1} + q_{i_2} + q_{i_3}$ *of the corresponding intersection numbers is even.*

We remark that, as a special case of (ii), the intersection number with a pants curve which bounds an embedded once–punctured torus or twice–punctured disk in Σ is even.

We will use the notation $\mathbf{i}_{DT}$ and $\mathbf{i}_{PH}$ when we consider the twist parameters $\mathrm{tw_i}$ to be the D. Thurston's twists parameters p_i and the Penner and Harer's twist parameters $\hat{p}_i$ respectively.

One can think of this theorem in the following way. Suppose given a multicurve $\gamma \in \mathcal{S}$, whose length parameters $q_i(\gamma)$ necessarily satisfy the parity condition (ii), then the $q_i(\gamma)$ uniquely determine $\gamma \cap P_j$ for each pair of pants P_j, $j = 1, \ldots, k$, in accordance with the possible arrangements of arcs in a pair of pants, see for example [38]. Now given two pants adjacent along the curve σ_i, we have $q_i(\gamma)$ points of intersection coming from each side and we have only to decide how to match them together to recover γ. The matching takes place in the cyclic cover of an annular neighbourhood of σ_i. The twist parameter $tw_i(\gamma)$ specifies which of the $\mathbb{Z}$ possible choices is used for the matching.

Remark 1.2.2. The name *Dehn–Thurston coordinates* refers to William Thurston who extended this parametrization to the set $\mathrm{ML}(\Sigma)$ of measured laminations on the surface Σ, see Theorem 3.1.1 of Penner [38]. (See Section 1.3.1 for the definition of $\mathrm{ML}(\Sigma)$.)

1.2.1 The DT-twist

In [41], Dylan Thurston gives a careful definition of the twist $\mathrm{tw_i}(\gamma) = \mathrm{p_i}(\gamma)$ of $\gamma \in \mathcal{S}$ which is essentially the 'folk' definition and the same as that implied in Fathi–Laudenbach–Poénaru [17]; see Section 1.2.1.e. He observes that this definition has a nice intrinsic characterisation, see Section 1.2.1.d. Furthermore, it turns out to be the correct definition for our formula in Theorem A.

1.2.1.a The marking decomposition

Given the pants decomposition $\mathcal{P}$ of a topological surface Σ, we note, following [41], that we can fix a *marking decomposition* on Σ in three equivalent ways. These are:

(a) an *involution*: an orientation–reversing map $\mathbf{R}\colon \Sigma \longrightarrow \Sigma$ so that for each $i = 1, \dots, \xi$ we have $\mathbf{R}(\sigma_i) = \sigma_i$;

(b) a *hexagonal decomposition*: this can be defined by a multicurve which meets each pants curve twice, decomposing each pair of pants into two hexagons;

(c) *dual curves*: for each i, a (multi)curve D_i so that $i(D_i, \sigma_j) = 2\delta_{ij}$.

The equivalence of these definition is easy and we will explain the ideas of the proof in the following paragraphs.

These characterisations are most easily understood in connection with a particular choice of hyperbolic metric h_0 on Σ. Recall that a pair of pants P in a hyperbolic surface is the union of two right angle hexagons glued along its seams, and that there is an orientation reversing symmetry of P which fixes the seams, see Fathi, Laudenbach and Poénaru [17]. The endpoints of exactly two seams meet each component of the boundary ∂P. Now let $\Sigma_* = (\Sigma, h_0)$ be a hyperbolic surface formed by gluing pants $P_1, \dots, P_k$ in such a way that the seams are exactly matched on either side of each common boundary curve σ_i. In this case the existence of the orientation reversing map $\mathbf{R}$ as in (a) and of the hexagonal decomposition as in (b) are clear and are clearly equivalent.

In order to understand the relationship between (a)–(b) and (c), we consider two cases:

- the modular surface $\mathrm{MS}(\sigma_i)$ associated to σ_i is homeomorphic to $\Sigma_{0,4}$;
- the modular surface $\mathrm{MS}(\sigma_i)$ associated to σ_i is homeomorphic to $\Sigma_{1,1}$.

In the first case, that is, if the modular surface associated to σ_i is made up of two distinct pairs of pants P and P', then, as explained above, the dual curve D_i to σ_i is obtained by gluing the two *sbcc*–arcs in P and in P' which run from σ_i to itself. Given a pair of pants P, there is a unique minimal length *sbcc*–arc in P which run from σ_i to itself and this arc intersects σ_i orthogonally. Each arc meets σ_i orthogonally so that in the metric Σ_* the two endpoints on each side of σ_i are exactly matched by the gluing. In the second case, that is, if the modular surface is a single pair of pants P, then the dual curve is obtained by gluing the minimal length single *dbcc*–arc in P which runs from σ_i to itself. Once again both ends of this arc meet σ_i orthogonally and in the metric Σ_* are exactly matched by the gluing. In this case, following the convention explained in Section 1.1.0.a, we take the dual curve D_i to be two parallel copies of the curve just described. Thus in all cases $i(D_i,\sigma_j)=2\delta_{ij}$ and furthermore the curves D_i are fixed by $\mathbf{R}$.

A general hyperbolic surface $\Sigma_h=(\Sigma,h)$ can be obtained from Σ_* by performing a Fenchel–Nielsen twist deformation $Tw_{\underline{\sigma},\underline{r}}\colon\ \Sigma_*\longrightarrow\Sigma_h$ about the multicurve $\underline{\sigma}=\sum_{i=1}^{\xi}\sigma_i$, where $\underline{r}=(r_1,\dots,r_\xi)\in\mathbb{R}^\xi$, see Section 1.1.0.b for definition. Clearly $Tw_{\underline{\sigma},\underline{r}}$ induces a reversing map, a hexagonal decomposition, and dual curves on the surface $Tw_{\underline{\sigma},\underline{r}}(\Sigma)$, showing that each of (a), (b) and (c) equivalently define a marking on an arbitrary surface Σ. This concludes the argument which explains the equivalence between (a), (b) and (c).

1.2.1.b The twist

Having defined the marking decomposition, we can now define the twist $p_i(\gamma)$ for any $\gamma\in\mathcal{S}$. Arrange, as above, the dual curves D_i to be fixed by $\mathbf{R}$, so that, in particular, if σ_i is the boundary of a single pair of pants P (and hence the modular surface $\mathrm{MS}(\sigma_i)\cong\Sigma_{1,1}$), then the two parallel components of the multicurve D_i are contained one in each of the two hexagons making up P. For each $i=1,\dots,\xi$, choose a small (open) annular neighbourhood $\mathbb{A}_i$ of σ_i, in such a way that the complement $\Sigma\setminus\cup_{i=1}^{\xi}\mathrm{Int}(\mathbb{A}_i)$ of the interiors of these annuli in Σ are closed pairs of pants $\hat{P}_1,\dots,\hat{P}_k$. Arrange γ in *D. Thurston's position*, that is so that its intersection with each $\hat{P}_i$ is fixed by $\mathbf{R}$ and so that it is transverse to D_i. Also push any component of γ parallel to any σ_i into $\mathbb{A}_i$.

If $q_i=i(\gamma,\sigma_i)=0$, define $p_i\geq 0$ to be twice the number of components of γ parallel to σ_i. Otherwise, $q_i=i(\gamma,\sigma_i)>0$. In this case, orient both $\gamma\cap\mathbb{A}_i$ and

$D_i \cap \mathbb{A}_i$ to run consistently from one boundary component of $\mathbb{A}_i$ to the other. (If $\mathrm{MS}(\sigma_i)$ is $\Sigma_{0,4}$, then the two arcs of $D_i \cap \mathbb{A}_i$ will be oriented in opposite directions relative to the connected curve D_i.) Then define

$$p_i = \hat{i}(\gamma \cap \mathbb{A}_i, D_i \cap \mathbb{A}_i),$$

where $\hat{i}(\alpha, \alpha')$ is the algebraic intersection number between the multicurves α and α', namely the sum of the indices of the intersection points of α and β, where an intersection point is of index $+1$ when the orientation of the intersection agrees with the orientation of Σ and -1 otherwise.

Note that this definition is independent of both the choice of the orientations of $\gamma \cap \mathbb{A}_i$ and $D_i \cap \mathbb{A}_i$, and the choice of the arrangement of γ in the pairs of pants adjacent to σ_i. Also, following the convention about dual curves in Section 1.1.0.a, the parameter p_i is always even. Two simple examples are illustrated in Figures 1.1 and 1.2.

1.2.1.c An alternative definition

The twist p_i can also be described in a slightly different way as follows. Lift $\mathbb{A}_i$ to its $\mathbb{Z}$–cover which is an infinite strip H. As shown in Figures 1.1 and 1.2, the lifts of $D_i \cap \mathbb{A}_i$ are arcs joining the two boundaries $\partial_0 H$ and $\partial_1 H$ of H. As explained in the previous section, we orient the lifts of both $\gamma \cap \mathbb{A}_i$ and $D_i \cap \mathbb{A}_i$ to run consistently from one boundary component of H to the other. They are equally spaced, like rungs of a ladder, in such a way that there are exactly two lifts in any period of the translation corresponding to σ_i. Any arc of γ enters H on one side and leaves on the other. Fix such a rung D_* say and number the strands of γ meeting $\partial_0 H$ in order as X_n, with $n \in \mathbb{Z}$, where X_0 is the first arc to the right of D_* and n increases moving to the right along $\partial_0 H$, relative to the orientation of the incoming strand of γ. Label the endpoints of γ on $\partial_1 H$ by X'_n, with $n \in \mathbb{Z}$ correspondingly, as shown in Figure 1.1. Since γ is simple, if X_0 is matched to X'_r, then X_n is matched to X'_{n+r} for all $n \in \mathbb{Z}$. Then it is not hard to see that $r = \frac{p_i}{2}$.

1.2.1.d Intrinsic characterisation

The intrinsic characterisation of the twist in [41] uses *Luo product* $\alpha \cdot \beta$, where $\alpha, \beta \in \mathcal{S}$ are multicurves on an oriented surface Σ. This is defined as follows [26; 41]:

- If $\alpha \cap \beta = \emptyset$, then $\alpha \cdot \beta = \alpha \cup \beta \in \mathcal{S}$.

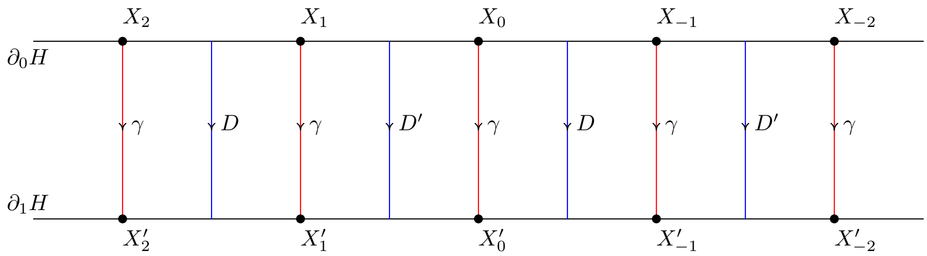

Figure 1.1: A curve γ with $p_i(\gamma) = 0$. The arcs D, D' together project to the dual curve D_i.

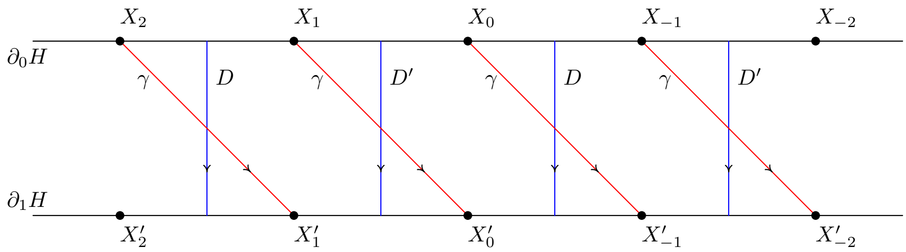

Figure 1.2: A curve γ with $p_i(\gamma) = -2$.

- Otherwise, arrange α and β in minimal position, that is, such that $i(\alpha \cap \beta) = |\alpha \cap \beta|$. In a neighbourhood of each intersection point $x_j \in \alpha \cap \beta$, replace $\alpha \cup \beta$ by the union of the two arcs which turn left from α to β relative to the orientation of Σ, see Figure 1.3. (In [26] this is called the *resolution* of $\alpha \cup \beta$ from α to β at x_j.) Then $\alpha \cdot \beta$ is the multicurve made up from $\alpha \cup \beta$ away from the points x_j, and the replacement arcs near each x_j.

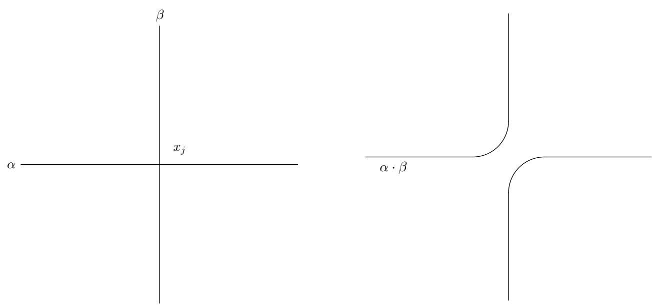

Figure 1.3: Luo product: the resolution of $\alpha \cup \beta$ at x_j.

Proposition 1.2.3 ([41] Definition 15)**.** *The function $p_i : \mathcal{S}(\Sigma) \longrightarrow \mathbb{Z}$ is the unique function such that, for all $\gamma \in \mathcal{S}$, we have:*

(i) $p_i(\sigma_j \cdot \gamma) = p_i(\gamma) + 2\delta_{ij}$*;*

(ii) p_i *depends only on the restriction of* γ *to the pairs of pants adjacent to* σ_i*;*

(iii) $p_i(\mathbf{R}(\gamma)) = -p_i(\gamma)$*, where* $\mathbf{R}$ *is the orientation reversing involution of* Σ *defined in Section 1.2.1.a.*

We call $p_i(\gamma)$ the *DT–twist parameter* of γ about σ_i. Property (i) fixes our convention noted above that the *right* twist is taken positive. Notice that $p_i(D_i) = 0$. We also observe:

Proposition 1.2.4. *Let* $\gamma \in \mathcal{S}$*. Then*

$$p_i\left(Tw_{\sigma_i}(\gamma)\right) = p_i(\gamma) + 2q_i.$$

1.2.1.e Relation to Fathi, Laudenbach and Poénaru's definition

In Fathi, Laudenbach and Poénaru [17], a multicurve $\gamma \in \mathcal{S}$ is parameterized by three non–negative integers (m_i, s_i, t_i). These are defined as the intersection numbers of γ with the three curves K_i, K_i' and K_i'', namely the pants curve σ_i, its dual curve D_i, and $Tw_{\sigma_i}(D_i)$, the right Dehn twist of D_i about σ_i, see Figure 4 on p. 62 in [17]. In particular:

- $m_i(\gamma) = i(\gamma, K_i) = i(\gamma, \sigma_i) = q_i(\gamma)$;
- $s_i(\gamma) = i(\gamma, K_i') = i(\gamma, D_i) = \dfrac{|p_i(\gamma)|}{2}$;
- $t_i(\gamma) = i(\gamma, K_i'') = i\left(\gamma, Tw_{\sigma_i}(D_i)\right) = \left|\dfrac{p_i(\gamma)}{2} - q_i(\gamma)\right|$.

As proved in [17], the three numbers m_i, s_i and t_i satisfy one of the three relations:

- $m_i = s_i + t_i$;
- $s_i = m_i + t_i$;
- $t_i = m_i + s_i$.

As it is easily verified by a case–by–case analysis, we have:

Theorem 1.2.5. *Each triple* (m_i, s_i, t_i) *uniquely determines and is determined by the parameters* q_i *and* p_i*. In fact,* $q_i = m_i$ *and* $p_i = 2sign(p_i)s_i$*, where*

$$sign(p_i) = \begin{cases} +1, & \text{if } m_i = s_i + t_i \ \text{ or } s_i = m_i + t_i; \\ -1, & \text{if } t_i = m_i + s_i. \end{cases}$$

Proof. If $sign(p_i) = -1$, then $\frac{p_i}{2} - q_i \leq 0$. So

$$t_i = |\frac{p_i}{2} - q_i| = -(\frac{p_i}{2} - q_i) = \frac{|p_i|}{2} + q_i = s_i + m_i.$$

If $sign(p_i) = +1$, then we have:

1. if $\frac{p_i}{2} \leqslant q_i$, then $t_i = |\frac{p_i}{2} - q_i| = q_i - \frac{|p_i|}{2} = m_i - s_i$;

2. if $q_i \leqslant \frac{p_i}{2}$, then $t_i = |\frac{p_i}{2} - q_i| = \frac{|p_i|}{2} - q_i = s_i - m_i$,

as we wanted to prove. □

1.2.2 The PH–Twist

We now summarise Penner and Harer's definition of the twist parameter following Section 1.2 of [38]. Instead of arranging the arcs of γ transverse to σ_i symmetrically with respect to the involution $\mathbf{R}$, we now arrange them to cross σ_i through a short closed arc $w_i \subset \sigma_i$. There is some choice to be made in how we do this, which leads to the difference with the definition of the previous section. It is convenient to think of w_i as contained in the two 'front' hexagons of the pairs of pants P and P' glued along σ_i, which we will also refer to as the 'white' hexagons, see Section 2.2.1.

Precisely, for each pants curve $\sigma_i \in \mathcal{PC}$, fix a short closed arc $w_i \subset \sigma_i$, called *window*, which we take to be symmetrically placed in the white hexagon of one of the adjacent pairs of pants P, midway between the two seams which meet $\sigma_i \subset \partial P$. See Figure 1.4 for details. For each σ_i, fix an annular neighbourhood $\mathbb{A}_i$ and extend w_i into a 'rectangle' $\mathfrak{R}_i \subset \mathbb{A}_i$ with one edge on each component of $\partial \mathbb{A}_i$ and 'parallel' to w_i and two edges arcs from one component of $\partial \mathbb{A}_i$ to the other. Let d_i be one of these two edges going from one component of $\partial \mathbb{A}_i$ to the other; see Figure 1.4. (See [38] for precise details.)

Now isotop $\gamma \in \mathcal{S}$ into *Penner and Harer standard position* as follows. Any component of γ homotopic to σ_i is isotoped into $\mathbb{A}_i$. Next, arrange γ so that it intersects each σ_i exactly $q_i(\gamma)$ times and, moreover, so that all points in $\gamma \cap \sigma_i$ are contained in the interior of w_i. We further arrange that all the twisting of γ occurs in $\mathbb{A}_i$. Precisely, isotop so that $\gamma \cap \partial \mathbb{A}_i \subset \partial \mathfrak{R}_i$, in other words, so that γ enters $\mathbb{A}_i$ across the interior of the edges of $\mathfrak{R}_i$ parallel to w_i. By pushing all the twisting into $\mathbb{A}_i$, we can also arrange that outside $\mathbb{A}_i$, any *dbcc*–arc of $\gamma \cap P$ does not cross any seam of P. The *sbcc*–arcs are slightly more complicated. Any such arc has both endpoints on the same boundary component, let say $\partial_0 P$. Give the white hexagon (the 'front' hexagon in Figure 1.4) the same orientation as the surface Σ. With this orientation, the two other boundary components $\partial_1 P$ and $\partial_\infty P$ are arranged as shown in Figure 1.4. We isotop the *sbcc*–arc so that outside $\mathbb{A}_i$ it loops round the *right hand* component $\partial_1 P$ ('right' relative to the incoming strand of the arc), without cutting the seam joining $\partial_0 P$ to $\partial_\infty P$, see Figure 1.4.

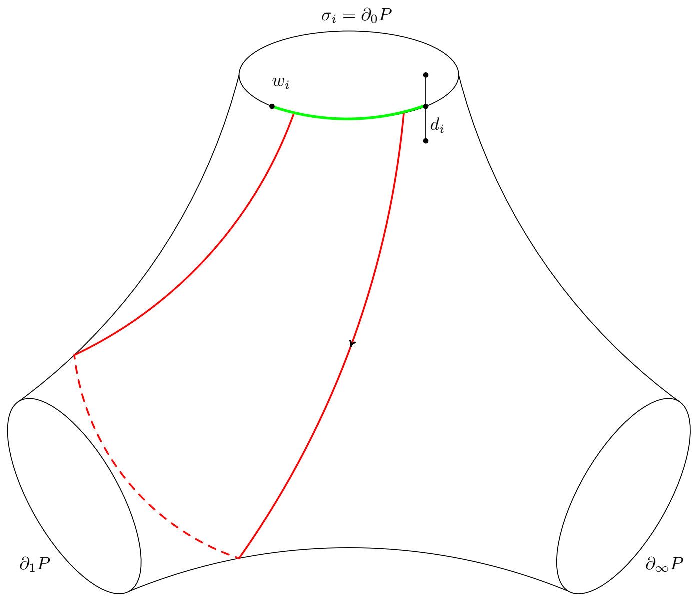

Figure 1.4: An *sbcc*–arc in Penner and Harer standard position, the window w_i and the arc d_i.

Having put γ into Penner and Harer standard position, we define the *Penner and Harer–twist* or *PH–twist* $\hat{p}_i(\gamma)$ as follows.

- If $q_i(\gamma) = i(\gamma, \sigma_i) = 0$, let $\hat{p}_i(\gamma) \geqslant 0$ be the number of components of γ which are freely homotopic to σ_i.

- If $q_i(\gamma) \neq 0$, let $|\hat{p}_i(\gamma)|$ be the minimum number of arcs of $\gamma \cap \mathbb{A}_i$ which intersect d_i, where the minimum is over all families of arcs properly embedded in $\mathbb{A}_i$, isotopic to $\gamma \cap \mathbb{A}_i$ by isotopies fixing $\partial\mathbb{A}$ pointwise. Take $\hat{p}_i(\gamma) \geqslant 0$ if some components of γ twist to the right in $\mathbb{A}_i$ and $\hat{p}_i(\gamma) \leqslant 0$ otherwise. (There cannot be components twisting in both directions since γ is embedded and, if there is no twisting, then $\hat{p}_i(\gamma) = 0$.)

1.2.2.a The dual curves in Penner and Harer position

As an example, we explain how to put the dual curves D_i into Penner and Harer standard position. This requires some care. For clarity, we denote one component

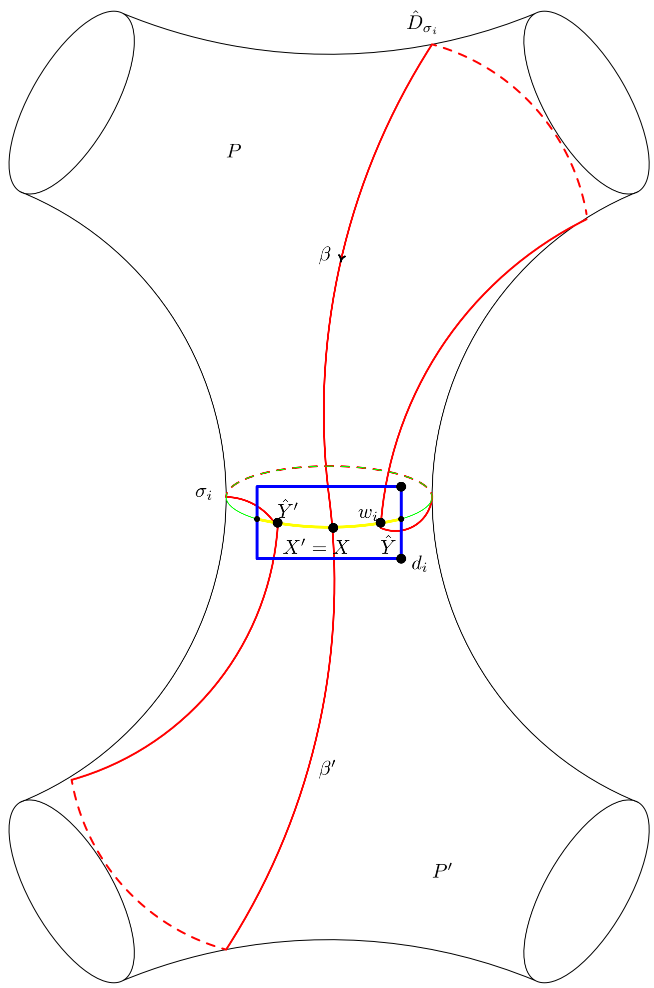

Figure 1.5: The dual curve D_i in Penner and Harer standard position and the window w_i on σ_i. The endpoints of d_i are on the boundary of the annulus $\mathbb{A}_i$ (not shown) around σ_i. The segment d_i is one edge of the rectangle $\mathfrak{R}_i$ which is drawn.

of the dual curve D_i by $\hat{D}_i$, so that in the case in which $\mathrm{MS}(\sigma_i)$ is $\Sigma_{1,1}$, we have $2\hat{D}_i = D_i$, while $\hat{D}_i = D_i$ otherwise.

If $\mathrm{MS}(\sigma_i)$ is $\Sigma_{1,1}$, then there is only one arc to be glued whose endpoints we can arrange to be in w_i. We simply take two parallel copies of this curve $\hat{D}_i$ so that $D_i = 2\hat{D}_i$ and $\hat{p}_i(D_i) = 0$.

If $\mathrm{MS}(\sigma_i)$ is $\Sigma_{0,4}$ then $D_i = \hat{D}_i$. In this case we have to match the endpoints of two *sbcc*–arcs $\beta \subset P$ and $\beta' \subset P'$, both of which have endpoints on σ_i. The arc β has one endpoint X in the front white hexagon of P, which we can arrange to be in w_i, and the other Y in the symmetrical position in the black hexagon. Label the endpoints of β' in a similar way. To get $\beta \cup \beta'$ into standard Penner and

Harer position, we have to move the back endpoints Y and Y' round to the front so that they also lie in w_i. Arrange P and P' as shown in Figure 1.5 with the white hexagons to the front. In Penner and Harer position, β has to curve round the right hand boundary component of P so that Y has to move to a point $\hat{Y}$ to the right of X along w_i in Figure 1.5. In P' on the other hand, β' has to loop round the right hand boundary component of P', so that Y' gets moved to a point $\hat{Y}'$ to the left of X' on w_i. Since X is identified to X', to avoid self–intersections, $\hat{Y}$ has to be joined to $\hat{Y}'$ by a curve which follows σ_i around the back of $P \cup P'$. By inspection, we see that $\hat{p}_i(\hat{D}_i) = -1$.

1.2.3 Relationship between the different definitions of twist

Our proof of Theorem A in Section 2.4 uses the explicit relationship between the above two definitions of the twist. The formula in Theorem 1.2.6 below appears without proof in [41]. We modify the statement and, for completeness, we also supply a proof. Note that this formula is not symmetric, as one would expect because of the non-symmetric definition of the PH–twist $\hat{p}_i$.

Suppose that two pairs of pants meet along $\sigma_i \in \mathcal{PC}$. Label their respective boundary curves (A, B, E) and (C, D, E) in clockwise order, where $E = \sigma_i$, see Figure 1.6. (Some of these boundary curves may be identified in Σ.)

Theorem 1.2.6. *(Appendix B of [41]) As above, let $\gamma \in \mathcal{S}$ and let $q_i = q_i(\gamma)$, $\hat{p}_i = \hat{p}_i(\gamma)$ and $p_i = p_i(\gamma)$ respectively denote its length parameter, its PH–twist and its DT–twist around σ_i. Then*

$$\hat{p}_i = \frac{p_i + l(A, E; B) + l(C, E; D) - q_i}{2},$$

where $E = \sigma_i$ and $l(X, Y; Z)$ denotes the number of strands of $\gamma \cap P$ running from the boundary curve X to the boundary curve Y in the pair of pants P having boundary curves (X, Y, Z).

Proof. Let $\gamma \in \mathcal{S}$. We use a case–by–case analysis to give a proof by induction on $n = q_i(\gamma)$. We shall assume that the modular surface $\mathrm{MS}(\sigma_i)$ is $\Sigma_{0,4}$, so that σ_i belongs to the boundary of two distinct pairs of pants $P = (A, B, E)$ and $P' = (C, D, E)$, and leave the case in which $\mathrm{MS}(\sigma_i)$ is $\Sigma_{1,1}$ to the reader. We begin with the cases $n = 1$ and $n = 2$, because $n = 2$ is useful for the inductive step.

When $n = 1$, the strand of γ which intersects σ_i must join one of the boundary components of P different from E, to one of the two boundary components of P' different from E. We have four cases corresponding to γ joining A or B to C or D.

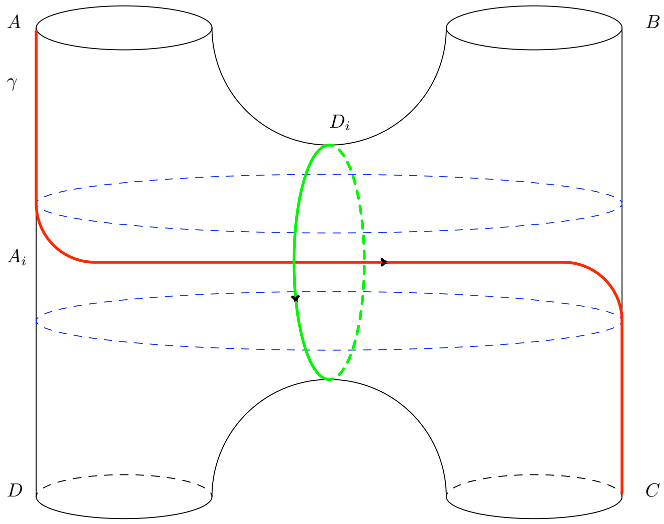

Figure 1.6: Case $n = 1$ when γ goes from A to C. E is the core curve of the annulus $\mathbb{A}_i$.

Figure 1.6 shows the case in which γ joins A to C. Without loss of generality we can consider the case $\hat{p} = 0$.

1. From A to C: $l(A, E; B) = 1$, $l(C, E; D) = 1$, $p = -1$ and $q = 1$;
2. From A to D: $l(A, E; B) = 1$, $l(C, E; D) = 0$, $p = 0$ and $q = 1$;
3. From B to C: $l(A, E; B) = 0$, $l(C, E; D) = 1$, $p = 0$ and $q = 1$;
4. From B to D: $l(A, E; B) = 0$, $l(C, E; D) = 0$, $p = 1$ and $q = 1$.

In all these cases the formula is true.

Now consider $n = 2$, so that $\gamma \cap \mathrm{MS}(\sigma_i)$ may have either one or two connected components. If there are two components, then each one was already analysed in the case $n = 1$, and the result follows by the additivity of the quantities involved.

If $\gamma \cap P$ is connected, we must have (in one of the pairs of pants P or P') a *sbcc*–arc which has both its endpoints on σ_i. Without loss of generality, we may suppose that this arc is in P. Choose an orientation on γ and call its initial and final points X_1 and X_2, respectively. The endpoints of this arc must be joined to the boundary components C or D of P'. Figure 1.7 illustrates the case in which X_1 is joined to D, while X_2 is joined to C. We will verify the formula in the following three cases (and again, without loss of generality, we assume $\hat{p} = 0$).

1. From C to C: $l(A, E; B) = 0$, $l(C, E; D) = 2$, $p = 0$ and $q = 2$;

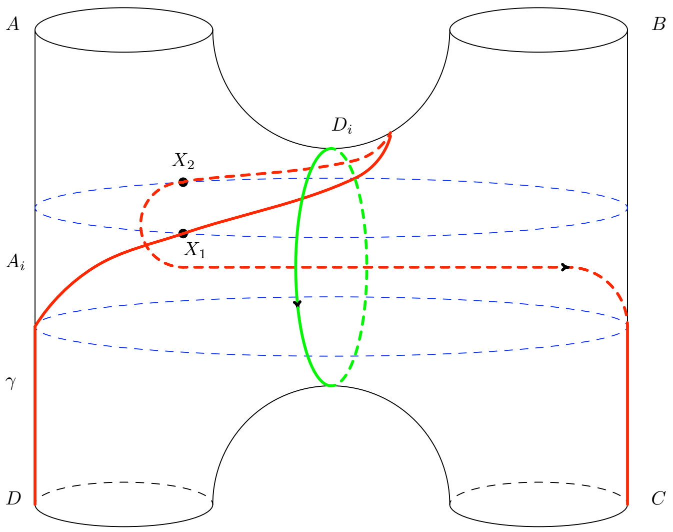

Figure 1.7: Case $n = 2$ when X_1 is joined to D and X_2 is joined to C (in D. Thurston's position).

2. From C to D: $l(A, E; B) = 0$, $l(C, E; D) = 1$, $p = 1$ and $q = 2$;

3. From D to D: $l(A, E; B) = 0$, $l(C, E; D) = 0$, $p = 2$ and $q = 2$.

Suppose now that the statement is true for any $n < q$, and let $\gamma \in \mathcal{S}$ such that $q_i(\gamma) = q$. If $\gamma \cap \mathrm{MS}(\sigma_i)$ is not connected, then each connected component intersects σ_i less then n times and the result follows from the inductive hypothesis and the additivity of the quantities involved.

If $\hat{\gamma} = \gamma \cap \mathrm{MS}(\sigma_i)$ is connected, then there is an arc which has both its endpoints on σ_i. Choose an orientation on γ. Without loss of generality, we can suppose that the first such arc is contained in P. Let X_1 and X_2 be its two ordered endpoints. Then X_2 splits $\hat{\gamma}$ into two oriented curves α and β, where α contains only one arc with both endpoints in σ_i, while β has $q-1$ arcs of this kind. Now we modify α and β in such a way that they become properly embedded arcs in $\mathrm{MS}(\sigma_i)$, that is, arcs with endpoints on $\partial(\mathrm{MS}(\sigma_i)) \subset \Sigma$. We do this by adding a segment for each one of α and β from X_2 to one of the boundary components C or D of P'. In order to respect the orientation of α and β, we add the segment twice, once with each orientation. This doesn't change the quantities involved. For example, suppose we add two segments from X_2 to C. This creates two oriented curves α' and β' in $\mathrm{MS}(\sigma_i)$ such that

$$\mathrm{tw_i}(\gamma) = \mathrm{tw_i}(\alpha \cup \beta) = \mathrm{tw_i}(\alpha' \cup \beta') = \mathrm{tw_i}(\alpha') + \mathrm{tw_i}(\beta')$$

and the conclusion now follows from the inductive hypothesis. $\square$

Remark 1.2.7. There is a nice formula for the number $l(X,Y,Z)$ in the above theorem. Given $a,b \in \mathbb{R}$, let $\max(a,b) = a \vee b$ and $\min(a,b) = a \wedge b$. Suppose that a pair of pants has boundary curves X,Y,Z and that $\gamma \in \mathcal{SC}$. Let $x = i(\gamma, X)$ and define y,z similarly. As above let $l(X,Y;Z)$ denote the number of strands of γ running from X to Y. Then we have

$$l(X,Y;Z) = 0 \vee \left(\frac{x+y-z}{2} \wedge x \wedge y \right),$$

see [41] p. 20.

1.3 Thurston symplectic form

In order to define Thurston's symplectic form, we adopt Penner and Harer's approach [38], which uses train tracks. Before defining train tracks, we recall, in Section 1.3.1, the notion of measured laminations following McMullen [34]. In Section 1.3.2, following Hamenstäd [19], we define train tracks, so as to be able to define the symplectic form, also called *Thurston's product*. In Section 1.3.3 we present an easy way to calculate this product using the Dehn–Thurston coordinates.

1.3.1 Measured laminations

Given a surface Σ with an hyperbolic structure, a *geodesic lamination* η on Σ is a closed set of pairwise disjoint complete simple geodesics on Σ called its *leaves*. A *transverse measure* on η is an assignment of a measure to each arc transverse to the leaves of η that is invariant under the push forward maps along the leaves of η. A *measured geodesic lamination* on Σ is a geodesic lamination together with a transverse measure. We define the *space of measured laminations* $\mathrm{ML}(\Sigma)$ to be the space of all homotopy classes of measured geodesic laminations on Σ with compact support. This space is called $\mathrm{ML}_0(\Sigma)$ in Penner and Harer [38]. An important subset of $\mathrm{ML}(\Sigma)$ is the space $\mathrm{ML}_{\mathbb{Q}}(\Sigma)$ of *rational geodesic laminations*, that is, measured geodesic laminations with support on multicurves. In addition, these definitions don't depend on the hyperbolic structure, but only on the topology of Σ, as discussed, for example, by Penner and Harer [38, Section 1.7].

Multiplying the transverse measure on a geodesic lamination by a positive constant, gives an action of $\mathbb{R}_+$ on $\mathrm{ML}(\Sigma)$. We can therefore define the set of *projective* *measured (geodesic) laminations* $\mathrm{PML}(\Sigma)$ on Σ as the quotient

$$\mathrm{PML}(\Sigma) = (\mathrm{ML}(\Sigma) \setminus 0)/\mathbb{R}_+,$$

where 0 is the empty lamination. Similarly you can also define the set of *rational projective measured (geodesic) laminations* $\mathrm{PML}_{\mathbb{Q}}(\Sigma)$.

There is a well defined topology on $\mathrm{ML}(\Sigma)$ which is defined as the *weak** topology. The interested reader can refer to Bonahon [6] for a more detailed discussion.

1.3.2 Train tracks and Thurston symplectic form

A *train track* on a surface Σ is an embedded 1–complex $\boldsymbol{\tau} \subset \Sigma$ such that:

i The edges (called *branches*) are smooth arcs and the tangent vectors at the endpoints are well–defined.

ii At any vertex v (called *switch*) the incident edges are mutually tangent and are divided into "incoming" and "outgoing" branches at v (according to their inward pointing tangent vectors at the switch).

iii Each closed curve component of $\boldsymbol{\tau}$ has a unique bivalent switch, and all other switches are at least trivalent.

iv The complementary regions of the train track have negative Euler characteristic (that is, they are different from disks with 0, 1 or 2 cusps at the boundary and different from annuli and once-punctured disks with no cusps at the boundary, where a cusp is a non-smooth point).

A train track is called *generic* if all switches are at most trivalent. In the case of a trivalent vertex, there is one incoming branch and two outgoing ones. A train track is called *maximal* if each complementary region is either a trigon or a once-punctured monogon.

Denote $\mathcal{B} = \mathcal{B}(\boldsymbol{\tau})$ the set of branches of $\boldsymbol{\tau}$. A function $w\colon \mathcal{B} \longrightarrow \mathbb{R}_{\geqslant 0}$ (resp. $w\colon \mathcal{B} \longrightarrow \mathbb{R}$) is a *transverse measure* (resp. *weighting*) for $\boldsymbol{\tau}$ if it satisfies the *switch condition*, that is for all switches v, we want $\sum_i w(e_i) = \sum_j w(E_j)$ where the e_i are the incoming branches at v and E_j are the outgoing ones.

A train track is called *recurrent* if it admits a transverse measure which is positive on every branch. A train track $\boldsymbol{\tau}$ is called *transversely recurrent* if every branch $b \in \mathcal{B}(\boldsymbol{\tau})$ is intersected by an embedded simple closed curve $c = c(b) \subset \Sigma$ which intersects $\boldsymbol{\tau}$ transversely and is such that $\Sigma - \boldsymbol{\tau} - c$ does not contain an embedded bigon, i.e. a disk with two corners on the boundary. (Here a corner is a point where

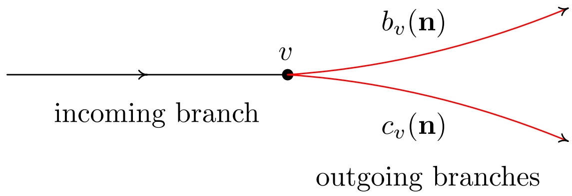

Figure 1.8: A switch v of a train track with one black incoming edge and two red weighted outgoing edges.

c cuts a branch of $\boldsymbol{\tau}$ transversally and a cusp is formed by two mutually tangent branches of $\boldsymbol{\tau}$ which meet at a switch, either both incoming or both outgoing.) A recurrent and transversely recurrent train track is called *birecurrent.*

A geodesic lamination (or a train track) λ is *carried by* a train track $\boldsymbol{\tau}$ if there is a map $F\colon\ \Sigma \longrightarrow \Sigma$ of class C^1 which is isotopic to the identity and which maps λ to $\boldsymbol{\tau}$ in such a way that the restriction of its differential dF to every tangent line of λ is non–singular. A generic transversely recurrent train track which carries a complete geodesic lamination is called *complete*, where we define a geodesic lamination to be *complete* if there is no geodesic lamination that strictly contains it.

Given a generic birecurrent train track $\boldsymbol{\tau} \subset \Sigma$, we define $\mathcal{V}(\boldsymbol{\tau})$ to be the collection of all (not necessary nonzero) transverse measures supported on $\boldsymbol{\tau}$ and let $\mathcal{W}(\boldsymbol{\tau})$ be the vector space of all weightings, that is assignments of (not necessary non-negative) real numbers, one to each branch of $\boldsymbol{\tau}$, which satisfy the switch conditions. By splitting, we can arrange $\boldsymbol{\tau}$ to be generic. Since Σ is oriented, we can distinguish the right and left hand outgoing branches. If $\mathbf{n}, \mathbf{n}' \in \mathcal{W}(\boldsymbol{\tau})$ are weightings on $\boldsymbol{\tau}$, then we denote by $b_v(\mathbf{n}), c_v(\mathbf{n})$ the weights of the left hand and right hand outgoing branches at v respectively; see Figure 1.8. *Thurston's product* $\Omega_{\mathrm{Th}}\colon\ \mathcal{W}(\boldsymbol{\tau}) \times \mathcal{W}(\boldsymbol{\tau}) \longrightarrow \mathbb{R}$ is defined as

$$\Omega_{\mathrm{Th}}(\mathbf{n}, \mathbf{n}') = \frac{1}{2} \sum_v \big(b_v(\mathbf{n})c_v(\mathbf{n}') - b_v(\mathbf{n}')c_v(\mathbf{n})\big)\,. \tag{1.1}$$

In Theorem 3.1.4 of Penner and Harer [38] it is proved that, if the train track $\boldsymbol{\tau} \subset \Sigma$ is complete, then the interior $\mathrm{int}\,(\mathcal{V}(\boldsymbol{\tau}))$ of $\mathcal{V}(\boldsymbol{\tau})$ can be thought of as a chart on the PIL manifold $\mathrm{ML}(\Sigma)$ of measured laminations (with compact support), as underlined by Penner and Harer in Section 3.1 of [38]. See Section 1.3.1 for the definition of $\mathrm{ML}(\Sigma)$. (PIL is short for *piecewise–integral–linear*, see [38, Section 3.1] for the definition.) In addition, in this case, we can identify $\mathcal{W}(\boldsymbol{\tau})$ with the tangent space to $\mathrm{ML}(\Sigma)$ at a point in $\mathrm{int}\,(\mathcal{V}(\boldsymbol{\tau}))$. The Thuston product Ω_{Th} defined above allows us to define a symplectic structure on the PIL manifold $\mathrm{ML}(\Sigma)$.

We now add a short remark which is not used in the following sections, but which could be useful for some readers. An *orientation* on a train track $\boldsymbol{\tau}$ is a specification of orientation for each branch of $\boldsymbol{\tau}$ such that incoming branches points towards outgoing ones, see page 31 of Penner and Harer for a more precise definition. It is interesting to note that, if $\boldsymbol{\tau}$ is oriented, then there is a natural map $h_{\boldsymbol{\tau}} \colon \mathcal{W}(\boldsymbol{\tau}) \longrightarrow \mathrm{H}_1(\Sigma; \mathbb{R})$, see Section 3.2 [38], which is related to Thurston's product by the following result. For a generalisation of this result to the case of an arbitrary (not necessarily orientable) train track $\boldsymbol{\tau} \in \Sigma$, see Section 3.2 [38].

Proposition 1.3.1 (Lemma 3.2.1 and 3.2.2 [38]). *For any train track $\boldsymbol{\tau}$, $\Omega_{\mathrm{Th}}(\cdot,\cdot)$ is a skew-symmetric bilinear pairing on $\mathcal{W}(\boldsymbol{\tau})$. In addition, if $\boldsymbol{\tau}$ is connected, oriented and recurrent, then for any $\mathbf{n}, \mathbf{n}' \in \mathcal{W}(\boldsymbol{\tau})$, $\Omega_{\mathrm{Th}}(\mathbf{n}, \mathbf{n}')$ is the homology intersection number of the classes $h_{\boldsymbol{\tau}}(\mathbf{n})$ and $h_{\boldsymbol{\tau}}(\mathbf{n}')$.*

As before, we add now a remark not important for the future results, but that we find really interesting and which could help in understanding the meaning of Ω_{Th}. We need to recall the last Proposition of Section 3.2 of [38] and some notation from Bonahon's work (see his survey paper [6] for a general introduction to the argument and for other further references). After rigorously defining the tangent space $T_\eta \operatorname{ML}(\Sigma)$ with $\eta \in \operatorname{ML}(\Sigma)$, Bonahon proves in [4] that we can interpret any tangent vector $v \in T_\eta \operatorname{ML}(\Sigma)$ as a geodesic lamination with a transverse Hölder distribution. Theorem 11 of Bonahon [5] shows that the space of Hölder distributions on a track $\boldsymbol{\tau}$ coincides with the space $\mathcal{W}(\boldsymbol{\tau})$, the vector space of all assignments of not necessary non-negative real numbers, one to each branch of $\boldsymbol{\tau}$, which satisfy the switch conditions. As a corollary of this result, one can prove that the space of Hölder distributions on a track $\boldsymbol{\tau}$ is finite dimensional. In particular, if $\boldsymbol{\tau}$ is maximal, then the real dimension of this space is $\dim_{\mathbb{R}}(\mathcal{W}(\boldsymbol{\tau})) = 2\xi(\Sigma_{g,b}) = 6g - 6 + 2b$. Bonahon also characterised which geodesic laminations with transverse distributions correspond to tangent vectors to $\operatorname{ML}(\Sigma)$. Notice that, if the lamination $\eta \in \operatorname{ML}(\Sigma)$ is carried by the track $\boldsymbol{\tau}$, we can locally identify $T_\eta \operatorname{ML}(\Sigma)$ with $\mathcal{W}(\boldsymbol{\tau})$.

Theorem 1.3.2 (Theorem 3.2.4 [38]). *For any surface Σ, Thurston's product is a skew-symmetric, nondegenerate, bilinear pairing on the tangent space to the PIL manifold* $\operatorname{ML}(\Sigma)$.

1.3.3 Thurston symplectic form and Dehn–Thurston coordinates

In this section we explain how to calculate Thurston symplectic form using Dehn–Thurston coordinates. We generalised an idea discussed by Series [40] in the case of the twice punctured torus.

In Proposition 4.3 of [40], Series describes a formula for calculating Thurston's product using a set of coordinates similar to the Dehn–Thurston coordinates described in Section 1.2, but her proof works only for the case $\Sigma = \Sigma_{1,2}$, since she uses a particular choice of train tracks, called *canonical train tracks*, see Keen–Parker–Series [23]. Our idea is to use the *standard train tracks* defined by Penner and Harer in Section 2.6 of [38]. See Figures 1.9 and 1.10. We are able to prove the following result.

Theorem 1.3.3. *Suppose that the curves $\gamma, \gamma' \in \mathcal{S}$ belongs to the same chart (and so are supported on a common standard train track). Then*

$$\Omega_{\mathrm{Th}}(\gamma, \gamma') = \frac{1}{2}\sum_{i=1}^{\xi}(q_i p_i' - q_i' p_i),$$

where $\mathbf{i}_{DT}(\gamma) = (q_1, p_1, \ldots, q_\xi, p_\xi)$ and $\mathbf{i}_{DT}(\gamma') = (q_1', p_1', \ldots, q_\xi', p_\xi')$ are the DT–coordinates of γ and γ'.
In addition, if γ, γ' are disjoint, then $\Omega_{\mathrm{Th}}(\gamma, \gamma') = 0$.

We consider this result really interesting in its own right and we believe it should have many further consequences.

The hypothesis that the curves $\gamma, \gamma' \in \mathcal{S}$ belongs to the same chart is necessary for the definition of Ω_{Th}. In fact, we define Ω_{Th} for weightings $\mathbf{n}, \mathbf{n}' \in \mathcal{W}(\boldsymbol{\tau})$, see Section 1.3.2.

Remark 1.3.4. Using Theorem 1.3.3 and recalling that $\dim_{\mathbb{R}}(\mathrm{ML}(\Sigma)) = 2\xi$, we can see that this symplectic form $\Omega_{\mathrm{Th}}(\cdot, \cdot)$ induces a map $\mathbb{R}^{2\xi} \longrightarrow \mathbb{R}^{2\xi}$ defined by $\mathbf{x} = (x_1, y_1, \ldots, x_\xi, y_\xi) \mapsto \mathbf{x}^* = (y_1, -x_1, \ldots, -y_\xi, x_\xi)$ such that

$$2\Omega_{\mathrm{Th}}(\mathbf{i}(\gamma), \mathbf{i}(\delta)) = \mathbf{i}(\gamma) \cdot \mathbf{i}(\delta)^*, \tag{1.2}$$

where $\cdot$ is the usual inner product on $\mathbb{R}^{2\xi}$ and where $\mathbf{i} = \mathbf{i}_{DT}$ is defined just after Theorem 1.2.1. We will use this map in Chapter 3.

The outline of the proof is the following. Given a multicurve $\gamma \in \mathcal{S}$, the Dehn-Thurston coordinates, using the length parameter q_i and Penner and Harer's twist $\hat{p}_i$, define a choice of standard weighted train tracks which carries γ, as discussed in Section 1.3.3.a below. Then, if we have a pair of multicurves $\gamma, \gamma' \in \mathcal{S}$ supported on a common standard train track, we can calculate Thurston's product $\Omega_{\mathrm{Th}}(\gamma, \gamma')$ using the definition in Equation (1.1). Finally, using the relationship between the PH–twist and the DT–twist, as described by Theorem 1.2.6, one can prove Theorem 1.3.3, which will be very important in the proof of the main theorems of Chapter

3. The pants decomposition $\mathcal{PC}$ (used in the definition of the Dehn–Thurston coordinates $\mathbf{i}$) decomposes Σ in pieces which are annuli $\mathbb{A}_i$ around the pants curves and pairs of pants P_j in the complement of these annuli. In particular, the sum of Thurston's products in the annuli $\mathbb{A}_i$ give us $\sum_{i=1}^{\xi}(q_i\hat{p}'_i - q'_i\hat{p}_i)$, using Penner and Harer's twists, while the sum over the pairs of pants give us some other terms, so that the total sum give us the results that we want, that is the product $\sum_{i=1}^{\xi}(q_i p'_i - q'_i p_i)$. Note that in this final sum we are using DT–twist.

We now explain the detailed proof.

1.3.3.a Standard models

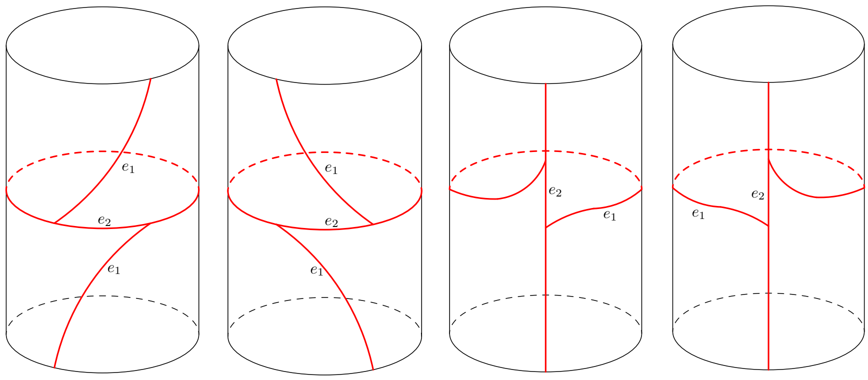

Figure 1.9: Types of track in a annulus: type $+2$, -2, $+1$ and -1.

Given a pants decomposition $\mathcal{PC} = \{\sigma_1, \dots, \sigma_\xi\}$ and a multicurve γ such that $\mathbf{i}_{PH}(\gamma) = (q_1, \hat{p}_1, \dots, q_\xi, \hat{p}_\xi)$ (read with respect to $\mathcal{PC}$), Penner and Harer describe exactly how to construct a train track $\boldsymbol{\tau}$ which carries γ. In particular, we can decompose Σ in pieces which are annuli $\mathbb{A}_i$ around the pants curves and pairs of pants P_i in the complement of these annuli. Then, according to the inequalities satisfied by the coordinates q_i and $\hat{p}_i$, we can write down a train track for each $\mathbb{A}_i$ and P_i. On the annuli there are 4 (maximal) types of tracks and on the pairs of pants there are the same number of (maximal) types of tracks. In detail, for the annulus $\mathbb{A}_i$ around the pants curve σ_i, we have the following tracks (see Figure 1.9):

1. type *+2*, if $|\hat{p}_i| \geqslant q_i$ and $\hat{p}_i \geqslant 0$;
2. type *−2*, if $|\hat{p}_i| \geqslant q_i$ and $\hat{p}_i \leqslant 0$;
3. type *+1*, if $q_i \geqslant |\hat{p}_i|$ and $\hat{p}_i \geqslant 0$;
4. type *−1*, if $q_i \geqslant |\hat{p}_i|$ and $\hat{p}_i \leqslant 0$.

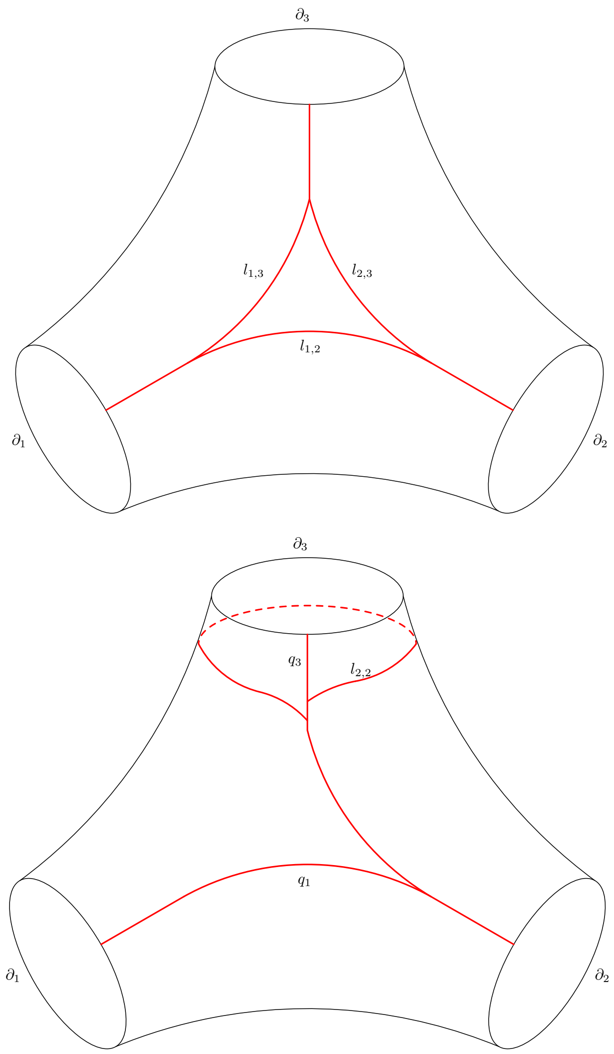

Figure 1.10: Types of track in a pair of pants: type 0 and 2.

For the pair of pants $P_{i,j,k}$ such that the boundary curves are σ_i, σ_j and σ_k (which we don't require to be distinct), we have four possibilities (see Figure 1.10):

1. type *0*, if $q_i \leqslant q_j + q_k$, $q_j \leqslant q_i + q_k$ and $q_k \leqslant q_i + q_j$;
2. type *1*, if $q_i \geqslant q_j + q_k$;
3. type *2*, if $q_j \geqslant q_i + q_k$;
4. type *3*, if $q_k \geqslant q_i + q_j$.

Notice that we discussed only the maximal tracks. There are also subtracks which are carried by more than one maximal track at the same time. In fact, that is the case when the coordinates satisfy more than one type of inequalities in the description above (that happen when the coordinates satisfies the equalities instead of the inequalities). In this case one chooses one of the maximal tracks which carries the subtrack.

Figures 1.9 and 1.10 describe the weights for some of the edges of the track in an annulus and in a pair of pants, respectively. The other edges are determined by the switch conditions. For the annulus we have weights e_1 and e_2, while for the pair of pants we have weights $l_{i,j}$. We can write the value of these weights using the Dehn–Thurston coordinates (with Penner and Harer's twist). In detail, for the annulus $\mathbb{A}_i$ we have

$$e_1(\sigma_i) = \begin{cases} q_i, & \text{if type } +2 \\ q_i, & \text{if type } -2 \\ \hat{p}_i, & \text{if type } +1 \\ |\hat{p}_i| = -\hat{p}_i, & \text{if type } -1 \end{cases}$$

and

$$e_2(\sigma_i) = \begin{cases} \hat{p}_i - q_i, & \text{if type } +2 \\ |\hat{p}_i| - q_i = -\hat{p}_i - q_i, & \text{if type } -2 \\ q_i - \hat{p}_i, & \text{if type } +1 \\ q_i - |\hat{p}_1| = q_i + \hat{p}_i, & \text{if type } -1, \end{cases}$$

while for the pair of pants $P_{i,j,k}$ with boundary curves $\sigma_i, \sigma_j, \sigma_k$, we have that

$$\begin{cases} l_{a,b}(\sigma_i, \sigma_j, \sigma_k) = \frac{q_a+q_b-q_c}{2}, & \text{if } \{a,b,c\} = \{i,j,k\}, \\ l_{a,a}(\sigma_i, \sigma_j, \sigma_k) = \frac{q_a-q_b-q_c}{2}, & \text{if } \{a,b,c\} = \{i,j,k\}. \end{cases}$$

See also Figure 2.6.1 and 2.6.2 of Penner and Harer [38].

1.3.3.b Proof of Theorem 1.3.3

Now we have all the tools we need in order to prove Theorem 1.3.3 which is the main result of this section. The proof is a bit technical, so we describe some examples in Section 1.3.3.c where the reader can see an application of the ideas behind the following proof.

Proof of Theorem 1.3.3. If $\gamma, \gamma' \in \mathcal{S}$ belong to the same chart, then they define the same type of standard tracks in each annulus and pairs of pants. If $\mathbf{i}_{PH}(\gamma) = (q_1, \hat{p}_1, \ldots, q_\xi, \hat{p}_\xi)$ and $\mathbf{i}_{PH}(\gamma') = (q'_1, \hat{p}'_1, \ldots, q'_\xi, \hat{p}'_\xi)$ are the Dehn–Thurston coordinates of the multicurves γ and γ', where $\hat{p}_i$ and $\hat{p}'_i$ are the PH–twists, we can associate weights on the edges of this standard tracks in a natural way, see Penner and Harer [38] or Section 1.3.3.a.

Using the standard train tracks associated to the multicurves γ and γ' and using the definition of Thurston's product Ω_{Th} (that is, Equation (1.1)), you can see that in the annulus $\mathbb{A}_i$ about σ_i the product is $\Omega_{\mathrm{Th}}(\gamma,\gamma')|_{\mathbb{A}_i} = \alpha_i = q_i\hat{p}'_i - q'_i\hat{p}_i$ for all types of track. On the other hand, in the pair of pants with boundary curves $\sigma_i, \sigma_j, \sigma_k$ we have different value for different types. In particular, for a track on pair of pants $P_{i,j,k}$ of type 0 we have $\Omega_{\mathrm{Th}}(\gamma,\gamma')|_{P_{i,j,k}} = \beta_{i,j,k} = q'_i q_k - q'_k q_i + q'_k q_j - q_k q'_j + q'_j q_i - q_j q'_i$, while if $P_{i,j,k}$ is of type 1 we have $\Omega_{\mathrm{Th}}(\gamma,\gamma')|_{P_{i,j,k}} = \beta_{i,j,k} = q'_i q_k - q'_k q_i$. Types 2 and 3 are similar to type 1.

Now, let P be a pair of pants with boundary curves σ_1, σ_2 and σ_3 and consider the sum $\sum_{i=1}^{3}(q_i p'_i - q'_i p_i)$, where p_i and p'_i are the DT–twists. If you apply the formula of Theorem 1.2.6 to p_i and p'_i, you can check that you get exactly the terms $2(\alpha_1 + \alpha_2 + \alpha_3 + \beta_{1,2,3}) = 2\Omega_{\mathrm{Th}}(\gamma,\gamma')$ for any type of standard track in P. This proves our result. □

1.3.3.c Examples

In this section we give two examples in order to explain better our method in the proof of Theorem 1.3.3. The second example also explains why we need to use the DT–twist in the formula.

Example 1 In the first simple example, let's consider a four punctured sphere $\Sigma = \Sigma_{0,4}$ with a pants decomposition $\mathcal{PC} = \{\sigma = \sigma_1\}$ and two multicurves γ and γ' on Σ defined by $\mathbf{i}_{PH}(\gamma) = (q,\hat{p}) = (2,-1)$ and $\mathbf{i}_{PH}(\gamma') = (q',\hat{p}') = (2,0)$. See Figure 1.5 for a picture of γ. Note that we are using Penner and Harer's definition of the Dehn–Thurston coordinates (in fact, that is the meaning of the subscript PH in $\mathbf{i}_{PH}$). In this case, we can split Σ in two pairs of pants P and P' and one annulus $\mathbb{A}$. The coordinates $\mathbf{i}_{PH}(\gamma)$ and $\mathbf{i}_{PH}(\gamma')$ tells us that in P and P' we have train track of type 1, while in $\mathbb{A}$ we have a train track of type -1. Since in the pair of pants the multicurves don't meet the other boundary components, we have only two switches $v \in P$ and $v_1 \in P'$ and in $\mathbb{A}$ we have two other switches v_1 (in the upper part of the picture), v'_1 (in the lower part). At v and v we have two (inward) half-edges with weight $l_{1,1} = \frac{q}{2}$, while at v_1 and v'_1 we have two inward edges with weights $q - |\hat{p}| = q + \hat{p}$ and $|\hat{p}| = -\hat{p}$ respectively.

Using the definition of Ω_{Th}, we get:

$$2\Omega_{\mathrm{Th}}(\gamma,\gamma') = 2\{\Omega_{\mathrm{Th}}|_{v}(\gamma,\gamma') + \Omega_{\mathrm{Th}}|_{v_1}(\gamma,\gamma') + \Omega_{\mathrm{Th}}|_{v'_1}(\gamma,\gamma') + \Omega_{\mathrm{Th}}|_{v'}(\gamma,\gamma')$$

$$= (l_{1,1}l'_{1,1} - l'_{1,1}l_{1,1}) + [(-\hat{p})(q' + \hat{p}') - (-\hat{p}')(q + \hat{p})]$$
$$+ [(-\hat{p})(q' + \hat{p}') - (-\hat{p}')(q + \hat{p})] + (l_{1,1}l'_{1,1} - l'_{1,1}l_{1,1})\}$$
$$= 2(-q'\hat{p} + q\hat{p}').$$

You can now apply Theorem 1.2.6 to see that $\hat{p} = \frac{p-q}{2}$ and $\hat{p}' = \frac{p'-q'}{2}$ and hence $\Omega_{\mathrm{Th}}(\gamma, \gamma') = \frac{1}{2}(qp' - pq')$, as we wanted.

According to this example, one might think that a formula similar to Theorem 1.3.3 could work for PH–coordinates as well. The next example explains why this is not the case.

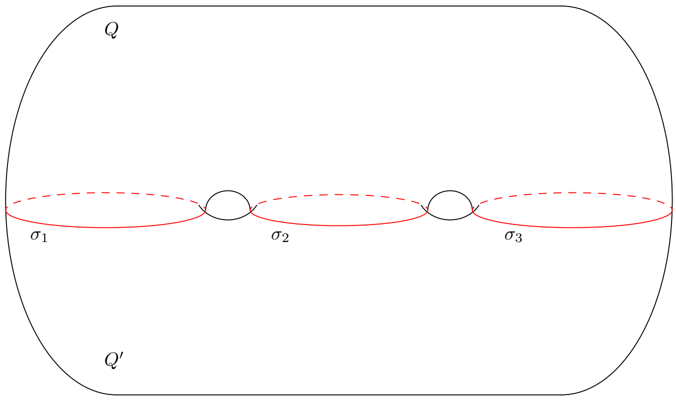

Figure 1.11: Pants decomposition for $\Sigma_{2,0}$.

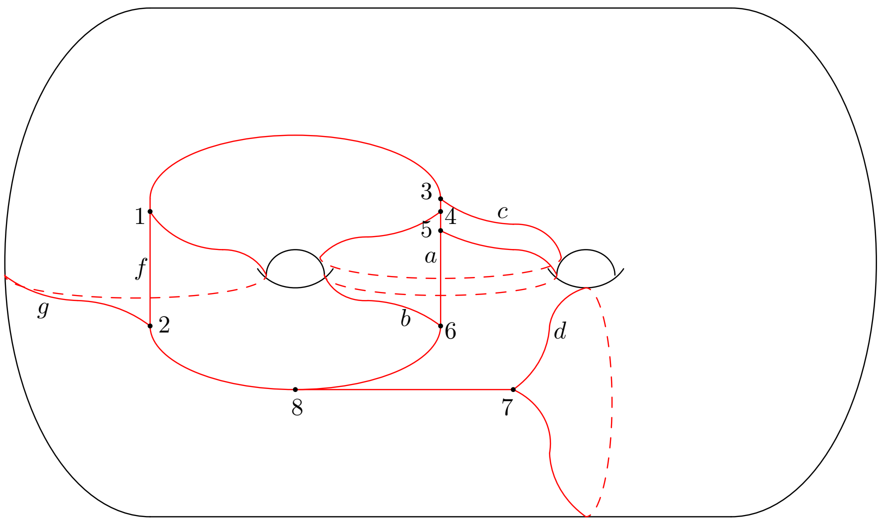

Figure 1.12: Train track on $\boldsymbol{\tau}$ on $\Sigma_{2,0}$ with 8 switches.

Example 2 For this second example, let's consider a genus two surface $\Sigma = \Sigma_{2,0}$ with the pants decomposition $\mathcal{PC} = \{\sigma_1, \sigma_2, \sigma_3\}$ described in Figure 1.11. The pants decomposition $\mathcal{PC}$ splits Σ into two pair of pants Q and Q' and into three annuli $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$ around the pants curves $\sigma_1, \sigma_2, \sigma_3$ respectively. Consider two multicurves γ and γ' on Σ defined by $\mathbf{i}_{PH}(\gamma) = (q_1, \hat{p}_1, q_2, \hat{p}_2, q_3, \hat{p}_3) = (2, -1, 0, 0, 0, 0)$ and $\mathbf{i}_{PH}(\gamma') = (q'_1, \hat{p}'_1, q'_2, \hat{p}'_2, q'_3, \hat{p}'_3) = (1, 0, 1, -1, 0, 0)$. They are supported by the train track $\boldsymbol{\tau}$ described in Figure 1.12. In detail, we have a (subtrack of a) maximal track of type 1 in Q and Q' and a maximal track of type -1 in $\mathbb{A}_1$ and $\mathbb{A}_2$.

On $\boldsymbol{\tau}$ there are 8 switches which we call $v_1, \ldots, v_8$ and 12 edges some of which are called $e_a, \ldots, e_g$; see Figure 1.12. In the figure the edge with labelling e_J is denoted J, where $J \in \{a, b, c, d, f, g\}$. The weighting $w \colon \mathcal{B}(\boldsymbol{\tau}) \longrightarrow \mathbb{R}$ on the set of branches $\mathcal{B}(\boldsymbol{\tau})$ is determined by the PH–coordinates $\mathbf{i}_{PH}(\gamma)$ and $\mathbf{i}_{PH}(\gamma')$. In particular, we have the following:

- $w(e_a) = q_2 - |\hat{p}_2| = q_2 + \hat{p}_2$;
- $w(e_b) = |\hat{p}_2| = -\hat{p}_2$;
- $w(e_c) = w(e_d) = \frac{q_1 - q_2}{2}$;
- $w(e_f) = q_1 - |\hat{p}_1| = q_1 + \hat{p}_1$;
- $w(e_g) = |\hat{p}_1| = -\hat{p}_1$.

The weights on the other edges are determined by the switch conditions.

As before, we calculate $\Omega_{\mathrm{Th}}(\gamma, \gamma')$ using the definition, that is Equation (1.1). In particular, we calculate the value at each switch:

- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_1} = (-\hat{p}_1)(q'_1 + \hat{p}'_1) - (-\hat{p}'_1)(q_1 + \hat{p}_1) = -\hat{p}_1 q'_1 + \hat{p}'_1 q_1$;
- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_2} = (-\hat{p}_1)(q'_1 + \hat{p}'_1) - (-\hat{p}'_1)(q_1 + \hat{p}_1) = -\hat{p}_1 q'_1 + \hat{p}'_1 q_1$;
- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_3} = (\frac{q_1 - q_2}{2})(\frac{q'_1 + q'_2}{2}) - (\frac{q'_1 - q'_2}{2})(\frac{q_1 + q_2}{2}) = \frac{q'_2 q_1 - q'_1 q_2}{2}$;
- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_4} = (q_2)(\frac{q'_1 - q'_2}{2}) - (q'_2)(\frac{q_1 - q_2}{2}) = \frac{q'_1 q_2 - q'_2 q_1}{2}$;
- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_5} = (-\hat{p}_2)(q'_2 + \hat{p}'_2) - (-\hat{p}'_2)(q_2 + \hat{p}_2) = -\hat{p}_2 q'_2 + \hat{p}'_2 q_2$;
- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_6} = (-\hat{p}_2)(q'_2 + \hat{p}'_2) - (-\hat{p}'_2)(q_2 + \hat{p}_2) = -\hat{p}_2 q'_2 + \hat{p}'_2 q_2$;
- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_7} = (\frac{q_1 - q_2}{2})(\frac{q'_1 - q'_2}{2}) - (\frac{q'_1 - q'_2}{2})(\frac{q_1 - q_2}{2}) = 0$;
- $2\Omega_{\mathrm{Th}}(\gamma, \gamma')|_{v_8} = (q_2)(q'_1 - q'_2) - (q'_2)(q_1 - q_2) = q_2 q'_1 - q'_2 q_1$.

Hence

$$2\Omega_{\mathrm{Th}}(\gamma,\gamma') = 2(-\hat{p}_1 q_1' + \hat{p}_1' q_1) + 2(-\hat{p}_2 q_2' + \hat{p}_2' q_2) + q_2 q_1' - q_2' q_1.$$

Note that this shows that the formula of Theorem 1.3.3 doesn't work for Penner and Harer's definition of the twist.

Using Theorem 1.2.6, we can see that $\hat{p}_1 = \frac{p_1+q_2-q_1}{2}$ and $\hat{p}_2 = \frac{p_2+q_2-q_2}{2} = \frac{p_2}{2}$. So we have:

$$\begin{aligned}2\Omega_{\mathrm{Th}}(\gamma,\gamma') &= [-(p_1+q_2-q_1)(q_1') + (p_1'+q_2'-q_1')(q_1)] + [-(p_2)(q_2') + (p_2')(q_2)] + q_2 q_1' - q_2' q_1 \\ &= q_1 p_1' - q_1' p_1 + q_2 p_2' - q_2' p_2,\end{aligned}$$

as we wanted to prove.

Chapter 2

Top Terms' Relationship

In this chapter we explain in detail the gluing construction mentioned in the Introduction. We recall some background material on 2 and 3–manifolds in Section 2.1. In Section 2.2 we describe the gluing construction inspired by Kra's plumbing construction. We explain the relationship with Kra's ideas in Section 2.2.4 and with the Maskit slice in Section 2.2.5. In Section 2.3 we discuss how to calculate the image of some paths on the surface which are fundamental in the inductive proof of the Top Term Relationship described in Section 2.4.

2.1 Background material about structures on 3–manifolds

In this section we recall some basic definitions that we need later. In particular, we introduce the notion of Kleinian groups and Teichmüller space in Section 2.1.1, the notion of groupoid in Section 2.1.2 and the notion of complex projective structure in Section 2.1.3.

2.1.1 Kleinian groups and Teichmüller space

A *Kleinian group* G is a discrete subgroup of the orientation–preserving isometries of the hyperbolic 3–space $\mathbb{H}^3$. In the upper–half–space model of $\mathbb{H}^3$ the orientation–preserving isometries are identified with the group $\mathrm{PSL}(2, \mathbb{C})$, so that a Kleinian group can be considered as a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Such a group also acts by conformal automorphisms on the sphere at infinity $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Discreteness implies points in $\mathbb{H}^3$ have finite stabilisers and discrete orbits under the group G, but the orbit $G \cdot x$ of a point $x \in \mathbb{H}^3$ typically accumulates on $\hat{\mathbb{C}}$. The set of accumulation points of $G \cdot x$ in $\hat{\mathbb{C}}$ for one point $x \in \mathbb{H}^3$ (and hence for all) is called the limit set of G, and its complement in $\hat{\mathbb{C}}$ is called the domain of discontinuity. Since we will use this notion later (in particular in Section 2.2.5), we need to be more precise. We follow Maskit [32].

Let $\mathbb{X}$ be a topological space and let G be a group of homeomorphism of $\mathbb{X}$ into itself. We say that the action of G at a point $x \in \mathbb{X}$ is:

- *discontinuous*, if there is a neighbourhood U of x so that $gU \cap U \neq \emptyset$ for all but finitely many $g \in G$;
- *freely discontinuous*, if there is a neighbourhood U of x so that $gU \cap U \neq \emptyset$ for all non-trivial $g \in G$.

We denote $\Omega^0(G)$ the set of points at which the action is freely discontinuous. A point x is a *limit point* for the Kleinian group G if there is a point $z \in \Omega^0(G)$ and a sequence $\{g_m\}$ of distinct elements of G with $g_m z \longrightarrow x$. As remarked by Maskit, this definition doesn't depend on the point z chosen. The set of limit points is called the *limit set* of G, and is usually denoted $\Lambda(G)$, while the set of points at which the action is discontinuous is called *regular set* or *domain of discontinuity* or *ordinary set* of G and is usually denoted $\Omega(G)$.

Clearly a freely discontinuous action is also discontinuous, and, if G is a torsion free group, also the converse is true, see Proposition 2.1.2.

Maskit defines a group $G \subset \mathrm{PSL}(2, \mathbb{C})$ to be *Maskit–Kleinian* if there is a point $z \in \hat{\mathbb{C}}$ at which the action is freely discontinuous. The following result is true.

Proposition 2.1.1 (Proposition C.3 of Chapter II in [32])**.** *Any Maskit–Kleinian group G is discrete (that is Kleinian).*

The converse to Proposition 2.1.1 is not true, see Remark C.4 of Maskit [32]. For example totally degenerate groups are Kleinian, but not Maskit–Kleinian. Maskit also proves the following theorem.

Theorem 2.1.2 (Theorem E.6, Proposition E.8 of Chapter II in [32])**.** *For any Maskit–Kleinian group G, then $\hat{\mathbb{C}}$ is the disjoint union of $\Lambda(G)$ and $\Omega(G)$, and $\Omega(G) \setminus \Omega^0(G)$ is a discrete subset of $\Omega(G)$ which consists of fixed points of elliptic elements of G.*

A Kleinian group whose limit set consists of at most two points is called an *elementary group*; otherwise it is called *non-elementary*. Maskit proves the following fact for Maskit–Kleinian groups, but the result is still true for Kleinian groups.

Proposition 2.1.3 (Proposition E.3, Proposition E.4 in Chapter V in [32])**.** *If G is a non-elementary Maskit–Kleinian group, then the limit set $\Lambda(G)$ is the closure of the fixed points of the loxodromic elements of G. In addition, if E is a non-empty G-invariant closed set, then E contains $\Lambda(G)$.*

Since all the Kleinian groups that we will study are torsion-free, assume, from now on, that all Kleinian groups are also torsion-free. In that case the sets $\Omega(G)$ and $\Omega^0(G)$ coincide.

An example of Kleinian groups are the Fuchsian groups. A *Fuchsian group* is a discrete subgroup of the orientation–preserving isometries $\text{Isom}^+(\mathbb{H})$ of the hyperbolic 2–space $\mathbb{H} = \mathbb{H}^2$. Using the half-space model for $\mathbb{H}$, we have that $\text{Isom}^+(\mathbb{H}) = \text{PSL}(2, \mathbb{R})$. We define the *Fuchsian space* $\mathcal{F}(\Sigma)$ as the space of marked groups $G \simeq \pi_1(\Sigma)$ such that G is Fuchsian.

Since we need the following results in the proof of Theorem 2.2.5, we are going to discuss Fuchsian groups a bit more deeply. The limit set $\Lambda(G)$ of a Fuchsian group G acting on $\mathbb{H}$ is contained in $\partial\mathbb{H} = \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In particular, Theorem 3.4.6 of Katok [22] tells us that either $\Lambda(G) = \partial\mathbb{H}$ or $\Lambda(G)$ is a perfect nowhere dense subset of $\partial\mathbb{H}$. Hence, we can give the following definition.

Definition 2.1.4. A Fuchsian group G is called:

(i) *of the first kind*, if $\Lambda(G) = \partial\mathbb{H}$;

(ii) *of the second kind*, if $\Lambda(G)$ is a perfect nowhere dense subset of $\partial\mathbb{H}$.

Let's recall also another interesting result about Fuchsian groups of the first kind.

Theorem 2.1.5 (Theorem 4.5.2 of [22])**.** *If a Fuchsian group G has fundamental region of finite area, then G is of the first kind.*

Another example of Kleinian groups are the Quasifuchsian groups. A Kleinian group is *Quasifuchsian* if $\Lambda(G)$ is a topological circle. If G is Quasifuchsian (and torsion free), then the associated manifold $M_G = \mathbb{H}^3/G$ is homeomorphic to the product of such a surface Σ with the open interval $(-1, 1)$, and $\Omega(G)$ has exactly two simply connected G–invariant components $\Omega^{\pm}$. See Marden [29]. We denote $\mathcal{QF}(\Sigma)$ the *Quasifuchsian space*, i.e the space of marked groups $G \simeq \pi_1(\Sigma)$ such that G is Quasifuchsian.

Given an oriented surface Σ of negative Euler characteristic, we define the Teichmüller space $\mathcal{T}(\Sigma)$ following McMullen [34]. The *Teichmüller space* $\mathcal{T}(\Sigma)$ is the space of marked complex structures on Σ. In detail, it consists of equivalence classes $[(f, X)]$ of pairs (f, X), where X is a Riemann surface of finite area and $f\colon \text{int}(\Sigma) \longrightarrow X$ is an orientation preserving homeomorphism. Two pairs (f_1, X_1) and (f_2, X_2) are equivalent in $\mathcal{T}(\Sigma)$ if there is a conformal map $g\colon X_1 \longrightarrow X_2$ such that $g \circ f_1$ is isotopic to f_2. The space $\mathcal{T}(\Sigma)$ is a finite–dimensional complex manifold, diffeomorphic to a ball in $\mathbb{R}^{2\xi}$.

Using the Uniformisation Theorem, the Teichmüller space can also be defined as the space of marked hyperbolic structure of finite area on Σ. In detail, it consists of equivalence classes $[(f, X)]$ of pairs (f, X), where X is a complete hyperbolic surface of finite area and $f\colon \operatorname{int}(\Sigma) \longrightarrow X$ is a homeomorphism. The canonical orientation of X is required to agree with the given orientation of Σ. Two pairs (f_1, X_1) and (f_2, X_2) represent the same point in $\mathcal{T}(\Sigma)$ if there is a isometry $g\colon X_1 \longrightarrow X_2$ such that $g \circ f_1$ is isotopic to f_2. Sometimes people refer to the Teichmüller space in this second setting as the *Fricke space*, see Imayoshi and Taniguchi [20] and Aramayona [1].

Using the Cartan–Hadamard Theorem, which we will recall as Theorem 2.1.6, we can identify the Teichmüller space $\mathcal{T}(\Sigma)$ with the Fuchsian space $\mathcal{F}(\Sigma)$. Since there are diffrerent version of the theorem, we recall here the statement we are referring to.

Theorem 2.1.6 (Cartan–Hadamard Theorem)**.** *Let X be a connected surface equipped with a hyperbolic structure, and suppose that the natural path-metric on X is complete. Then X is isometric to $\mathbb{H}/\Gamma$, where Γ is a Fuchsian group acting freely on $\mathbb{H}$.*

We refer to Theorem 3.8 of Aramayona [1] for a proof and for further references. Note that, if the hyperbolic structure is of finite area, then the Fuchsian group is of the first kind, see Definition 2.1.4.

2.1.2 Fundamental groupoid

In this section, we will define the notion of 'groupoid'. We will follow Brown [8], [9].

Definition 2.1.7. A *groupoid* consists of:

- A set $\mathcal{OB}$ of *objects*.
- For each $x, y \in \mathcal{OB}$, a (possibly empty) set $\mathcal{MOR}(x, y)$ of *morphisms* (or *arrows*) from x to y; $\mathcal{MOR} = \cup_{(x,y)\in\mathcal{MOR}(x,y)}$ is the set of all morphisms.
- Two maps $s, t\colon \mathcal{MOR} \longrightarrow \mathcal{OB}$, called *source* and *target*, and a map $i\colon \mathcal{OB} \longrightarrow \mathcal{MOR}$ such $s \circ i = t \circ i = \mathrm{Id}$.

If $f, g \in \mathcal{MOR}$ such that $t(f) = s(g)$, then a *product* fg exists and satisfies $s(fg) = s(f)$ and $t(fg) = t(g)$. In addition:

- this product is *associative*;
- the elements $i(x)$, where $x \in \mathcal{OB}$, act as *identities*;

- each element $f \in \mathcal{MOR}(x, y)$ has an *inverse* $f^{-1} \in \mathcal{MOR}(y, x)$ with $s(f^{-1}) = t(f)$, $t(f^{-1}) = s(f)$, $ff^{-1} = i \circ s(f)$, $f^{-1}f = i \circ t(f)$.

Now as an example of groupoid we will define the fundamental groupoid of a surface Σ with respect to a subset $B \subset \Sigma$. We will use this notion later.

Definition 2.1.8 (Fundamental groupoid)**.** Given a topological space Y with a fixed subset $B \subset Y$, then we define the *fundamental groupoid associated to the pair* (Y, B), denoted $\pi_1(Y, B)$, as the groupoid such that:

- $\mathcal{OB} = B$;
- For each $x, y \in \mathcal{OB}$, $\mathcal{MOR}(x, y)$ is the set of homotopy classes, relative to the endpoints, of the paths in Y from x to y.
- The map $s, t \colon \mathcal{MOR} \longrightarrow \mathcal{OB}$ sends $\gamma \in \mathcal{MOR}$ to its initial and final points, while $i \colon \mathcal{OB} \longrightarrow \mathcal{MOR}$ maps $x \in \mathcal{OB}$ to the homotopy class of the trivial loop in Y based at x.

One can then check that the remaining hypotheses in the definition of groupoid are satisfied.

2.1.3 Complex projective structure

In this section we will define the notion of marked complex projective structure on a surface Σ and we will also explain how to define the associated developing map and holonomy representation. In addition, in Section 2.1.3.b, we will explain also how to define the notion of groupoid holonomy representation.

A *(complex) projective structure* on a surface Σ is a (G, X)–structure on Σ, where $G = \mathrm{PSL}(2, \mathbb{C})$ and $X = \hat{\mathbb{C}}$, that is a (complex) projective structure consists of a (maximal) atlas of charts with values in $\hat{\mathbb{C}}$ and Möbius transition functions. More precisely, a (complex) projective structure on a surface Σ is a (maximal) covering of Σ by open sets $\{U_i : i \in I\}$ and maps $\Phi_i : U_i \longrightarrow V_i \subset \hat{\mathbb{C}}$ such that:

1. Φ_i is a homeomorphism from U_i onto its image V_i;
2. for all pairs $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, if W is a connected component of $U_i \cap U_j$, then $\Phi_i \circ \Phi_j^{-1}|_{\Phi_j(W)}$ is the restriction of some $g \in \mathrm{PSL}(2, \mathbb{C})$.

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are *equivalent* or *compatible* if the union $\mathcal{A} \cup \mathcal{A}'$ still defines an atlas of complex projective charts. In addition, in each equivalence class there is a unique maximal atlas, where a *maximal atlas* of charts is an atlas not properly contained in any other atlas.

We can define the *space of marked (complex) projective structure* $\mathcal{P}(\Sigma)$ as the set of equivalence classes $[(f, Z)]$ of pairs (f, Z), where Z is Σ with a (complex) projective structure and $f\colon \operatorname{int}(\Sigma) \longrightarrow Z$ is an orientation preserving homeomorphism. Two pairs (f_1, Z_1) and (f_2, Z_2) are equivalent in $\mathcal{P}(\Sigma)$ if there is an orientation-preserving diffeomorphism $g\colon Z_1 \longrightarrow Z_2$ such that $g \circ f_1$ is isotopic to f_2. The space $\mathcal{P}(\Sigma)$ is a finite–dimensional complex manifold, diffeomorphic to a ball in $\mathbb{R}^{4\xi}$, where $\xi = \xi(\Sigma)$ is the complexity of the surface, see Section 1.1. See Dumas [14] for a detailed discussion of this notion.

2.1.3.a Developing map and holonomy representation

Associated to every complex projective structure (and, more generally, to any (G, X)–structure) on a surface Σ, there is the pair (Dev, ρ), where:

- ρ is a homomorphism $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$, called the *holonomy representation*;
- Dev is a map $\mathrm{Dev}\colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ from the universal covering space $\tilde{\Sigma}$ of Σ to the Riemann sphere $\hat{\mathbb{C}}$, called the *developing map*, equivariant with respect to ρ.

A projective structure on Σ lifts to a projective structure on the universal cover $\tilde{\Sigma}$. Then, as summarised by Dumas [14], a developing map can be constructed by analytic continuation starting from any base point x_0 in $\tilde{\Sigma}$ and any chart defined on a neighbourhood U of x_0. Another chart (defined on U') that overlaps U can be adjusted by a Möbius transformation so as to agree on the overlap, in such a way that one can define a map from $U \cup U'$ to $\hat{\mathbb{C}}$. Continuing with this method one defines a map on successively larger subsets of $\tilde{\Sigma}$. The simple connectivity of $\tilde{\Sigma}$ is essential here, as nontrivial homotopy classes of loops in the surface create obstructions to unique analytic continuation of a projective class. See Section B.1 of Benedetti and Petronio [2] or Section 3.2.1 of Aramayona [1] for a detailed discussion in the analogous case of a hyperbolic structure.

In terms of the projective structure, the holonomy representation $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ is described as follows; see Epstein [15] for a more precise description, or also the next subsection. A path γ in Σ passes through an ordered chain of simply connected open sets $U_0, \ldots, U_n$ such that $U_i \cap U_{i+1}$ is connected and non-empty for every $i = 0, \ldots, n-1$. This gives us the overlap maps $R_i = \Phi_i \circ \Phi_{i+1}^{-1}$ for $i = 0, \ldots, n-1$. The sets V_i and $R_i(V_{i+1})$ overlap in $\hat{\mathbb{C}}$ and hence the developing image of $\tilde{\gamma}$ in $\hat{\mathbb{C}}$ passes through, in order, the sets $V_0, R_0(V_1), R_0R_1(V_2) \ldots, R_0 \cdots R_{n-1}(V_n)$. If γ is closed, we have $U_n = U_0$ so that $V_0 = V_n$. Then, by definition, the holonomy of the homotopy class $[\gamma]$ is $\rho([\gamma]) = R_0 \cdots R_{n-1} \in \mathrm{PSL}(2,\mathbb{C})$. In the next section we will see that, actually, this method shows that we can also define a groupoid holonomy representation associated to a projective structure.

The map Dev is defined up to post-composition with elements of $\mathrm{PSL}(2,\mathbb{C})$, while ρ is defined up to conjugation by elements of $\mathrm{PSL}(2,\mathbb{C})$. So $\mathrm{PSL}(2,\mathbb{C})$ acts on the sets of pairs (Dev,ρ) in the following way: given $A \in \mathrm{PSL}(2,\mathbb{C})$, then we have

$$A \cdot (\mathrm{Dev}, \rho(\cdot)) = (A \circ \mathrm{Dev}, A\rho(\cdot)A^{-1}).$$

As described by Dumas [14], we can define the space of equivalence classes of complex projective structures on Σ in a different way, using developing maps and holonomy representations associated to the projective structures. It consists of pairs (Dev,ρ), where two pairs (Dev_1,ρ_1) and (Dev_2,ρ_2) represent the same point in $\mathcal{P}(\Sigma)$ if there exists $A \in \mathrm{PSL}(2,\mathbb{C})$ such that $\mathrm{Dev}_2 = A \circ \mathrm{Dev}_1$ and $\rho_2(\cdot) = A \circ \rho_1(\cdot) \circ A^{-1}$. Note that to define an element of $\mathcal{P}(\Sigma)$ we need also to fix a marking on each projective structure.

2.1.3.b Groupoid holonomy representation

Now we want to define a holonomy map $\rho\colon \pi_1(\Sigma, B) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ from the fundamental groupoid $\pi_1(\Sigma,B)$ of Σ to $\mathrm{PSL}(2,\mathbb{C})$ which agrees with the usual definition of holonomy map, when restricted to the fundamental group $\pi_1(\Sigma,b)$ (where $b \in B$). This definition is essentially the one described by Epstein [15] (and summarised in the previous section). For completeness we will repeat his arguments, following [15] very closely.

Any element of $\pi_1(\Sigma,B)$ is the homotopy class (relative to the endpoints) of a path joining two points in the base set B. For each base point b in B, fix a preferred germ of a chart. Recall that, given a point x of a topological space X, and two maps $f, g\colon X \longrightarrow Y$ (where Y is any set), then f and g define the same *germ* at x if there is a neighbourhood U of x such that, restricted to U, the maps f and g are equal. We denote the germ of the fixed chart at $b \in B$ as $\Phi^b\colon U^b \longrightarrow V^b \subset \hat{\mathbb{C}}$, and we denote the collection of all these fixed (germs of) charts as $\mathfrak{C} = \{\Phi^b | b \in B\}$. We are going to define the map

$$\rho^{\mathfrak{C}}\colon \pi_1(\Sigma,B) \longrightarrow \mathrm{PSL}(2,\mathbb{C}).$$

Let $\gamma\colon [0,1] \longrightarrow \Sigma$ be a path such that $\gamma(0) = b_1$ and $\gamma(1) = b_2$, where $b_1, b_2 \in B$. Choose a partition $0 = t_1 \leq t_2 \ldots \leq t_n = 1$ and charts $\Phi_i\colon U_i \longrightarrow V_i \subset \hat{\mathbb{C}}$ for $i = 1,\ldots,n-1$ such that $\gamma[t_i,t_{i+1}] \subset U_i$ for $i = 1,\ldots,n-1$ and such that $U_j \cap U_{j+1}$

is connected and non-empty for every $j = 1, \dots, n-2$. Let $\Phi_0 = \Phi^{b_1}$, $\Phi_n = \Phi^{b_2}$ and let $t_0 = t_1 = 0$, $t_{n+1} = t_n = 1$. Let $R_{i-1} \in \mathrm{PSL}(2,\mathbb{C})$ such that $\Phi_{i-1} = R_{i-1}\Phi_i$ near $\gamma(t_{i-1})$, for $i = 1, \dots, n$. We define $h(\gamma, \Phi^{b_1}, \Phi^{b_2}) = R_0 \cdots R_{n-1} \in \mathrm{PSL}(2,\mathbb{C})$.

Now we want to show that $h(\gamma, \Phi^{b_1}, \Phi^{b_2})$ is independent of the choices involved. We want to prove the following claims:

1. If $0 = t_1 \leq t_2 \ldots \leq t_n = 1$ is fixed and we change Φ_i to Φ'_i, then $h(\gamma, \Phi^{b_1}, \Phi^{b_2})$ doesn't change.

2. Given the partition $0 = t_1 \leq t_2 \ldots \leq t_n = 1$, any finer partition of the unit interval defines the same element $h(\gamma, \Phi^{b_1}, \Phi^{b_2})$.

For the first claim, we can assume that only one chart changes, because then we can conclude by induction. Let $g \in \mathrm{PSL}(2,\mathbb{C})$ be the element such that $\Phi_i = g\Phi'_i$ on $\gamma[t_{i-1}, t_i]$. Then $R'_{i-1} = R_{i-1}g$ and $R'_i = g^{-1}R_i$. This shows that $h(\gamma, \Phi^{b_1}, \Phi^{b_2})$ doesn't change.

For the second claim, we can assume that one single point T is added, with $t_{i-1} \leq T \leq t_i$. Again, if more points are added we can conclude by induction. The computation of $h(\gamma, \Phi^{b_1}, \Phi^{b_2})$ can be done by using the same chart $\Phi_i \colon U_i \longrightarrow V_i$ for both the intervals $[t_{i-1}, T]$ and $[T, t_i]$. The transition function in $\mathrm{PSL}(2,\mathbb{C})$ associated with T is the identity, so that the computation of $h(\gamma, \Phi^{b_1}, \Phi^{b_2})$ doesn't change.

From these two claims, we can see that $h(\gamma, \Phi^{b_1}, \Phi^{b_2})$ is well defined. In addition, we can prove the following properties:

(i) $h(\gamma^{-1}, \Phi^{b_2}, \Phi^{b_1}) = h(\gamma, \Phi^{b_1}, \Phi^{b_2})^{-1}$.

(ii) If γ and γ' are two paths in Σ such that $\gamma(1) = \gamma'(0)$, then $h(\gamma\gamma', \Phi^{b_1}, \Phi^{b_3}) = h(\gamma, \Phi^{b_1}, \Phi^{b_2})h(\gamma', \Phi^{b_2}, \Phi^{b_3})$, where $\gamma(1) = \gamma'(0) = b_2$, $\gamma(0) = b_1$, $\gamma'(1) = b_3$.

(iii) If $g, g' \in \mathrm{PSL}(2,\mathbb{C})$, then $h(\gamma, g\Phi^{b_1}, g'\Phi^{b_2}) = gh(\gamma, \Phi^{b_1}, \Phi^{b_2})(g')^{-1}$.

(iv) If γ and γ' are homotopic, keeping the endpoints fixed, then $h(\gamma, \Phi^{b_1}, \Phi^{b_2}) = h(\gamma', \Phi^{b_1}, \Phi^{b_2})$.

The properties *(i)*, *(ii)* and *(iii)* follow easily from the definition. To prove property *(iv)*, the idea is that a very small change in γ can be dealt with without changing the charts $\Phi \colon U_i \longrightarrow V_i$. For the proof of this point, see also the proof of Proposition B.1.3 of Benetti and Petronio [2].

These fact tells us that, given the choice of the fixed (germs of) charts $\mathfrak{C} = \{\Phi^b | b \in B\}$, then the map

$$\rho^{\mathfrak{C}} \colon \pi_1(\Sigma, B) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$$

defined by $\rho^{\mathfrak{C}}([\gamma]) = h(\gamma, \Phi^{b_1}, \Phi^{b_2})$, where $s([\gamma]) = b_1$ and $t([\gamma]) = b_2$, is well-defined and is a homomorphism of groupoids. Recall that s and t are the source and target maps introduced in Definition 2.1.8. We call this map the *groupoid holonomy homomorphism.* If we don't fix the set $\mathfrak{C} = \{\Phi^{b_1}, \ldots, \Phi^{b_k^*}\}$, that is, if we allow the set $\mathfrak{C}$ to be exchanged with a different set $\mathfrak{C}'$, then the map $\rho\colon\ \pi_1(\Sigma, B) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ is well defined, up to left and right multiplication by elements of $\mathrm{PSL}(2,\mathbb{C})$, see point (*iii*) above.

Similarly, fixing a point $b \in B$ and its germ $\Phi^b \subset \mathfrak{C}$ of a chart at b, we see that the map

$$\rho^{\Phi^b} = \rho^{\mathfrak{C}}|_{\pi_1(\Sigma,b)}\colon\ \pi_1(\Sigma, b) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$$

defined by $\rho^{\Phi^b}([\gamma]) = h(\gamma, \Phi^b, \Phi^b)$ is well-defined (and it belongs to the conjugacy class of the holonomy map described in Section 2.1.3.a). If we don't fix the chart Φ^b (among the germs of charts based at b), then the map $\rho\colon\ \pi_1(\Sigma, b) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ is well defined, up to conjugation by elements of $\mathrm{PSL}(2,\mathbb{C})$ which fix b, see again point (*iii*) above. Similarly, if we don't fix the base point b (among the base points in B), then, by property (*i*) and (*ii*) above, then the map $\rho\colon\ \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ is well defined, up to conjugation by elements of $\mathrm{PSL}(2,\mathbb{C})$.

2.2 The gluing construction

In this section we discuss how to endow Σ with a projective structure coming from Kra's plumbing construction.

As explained in the Introduction, the representations which we consider are holonomy representations of projective structures on Σ, chosen so that the holonomies of all the loops $\sigma_i \in \mathcal{PC}$ determining the pants decomposition $\mathcal{P}$ are parabolic. The interior of the set of free, discrete, and geometrically finite representations of this form is called the *Maskit embedding* of Σ, see Section 2.2.5 below.

The construction of the projective structure on Σ is based on Kra's plumbing construction [25], see Section 2.2.4. However it is convenient to describe it in a somewhat different way. We will refer to our construction as the *gluing construction.* The idea is to manufacture Σ by gluing triply punctured spheres across punctures. There is one triply punctured sphere for each pair of pants $P \in \mathcal{P}$, and the gluing across the pants curve σ_i is implemented by a specific projective map depending on a parameter $\mu_i \in \mathbb{H}$. The μ_i are the parameters of the resulting holonomy representation $\rho = \rho_{\underline{\mu}}\colon\ \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$, where $\underline{\mu} = (\mu_1, \ldots, \mu_\xi) \in \mathbb{H}^\xi$. (Note that $\mathbb{H}^\xi = \mathbb{H} \times \ldots, \times \mathbb{H}$ is the product of ξ copies of $\mathbb{H}$.) We will call these parameters the *gluing parameters.*

More precisely, we first fix an identification of the interior of each pair of pants P_j to a standard triply punctured sphere $\mathbb{P}$. We endow $\mathbb{P}$ with the projective structure coming from the unique hyperbolic metric on a triply punctured sphere. The gluing is carried out by deleting open punctured disk neighbourhoods of the two punctures in question and gluing horocyclic annular collars round the resulting two boundary curves, see Figure 2.1.

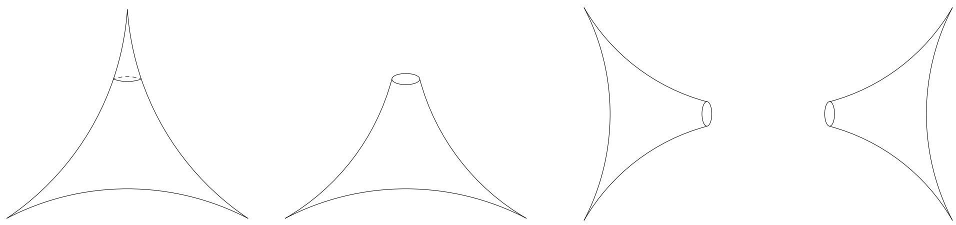

Figure 2.1: Deleting horocyclic neighbourhoods of the punctures and preparing to glue.

2.2.1 The gluing

To describe the gluing in detail, first recall (see for example [37] p. 207) that any triply punctured sphere is homeomorphic to the standard triply punctured sphere $\mathbb{P} = \mathbb{H}/\Gamma$, where

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

Fix a standard fundamental set Δ for Γ as shown in Figure 2.2, so that the three punctures of $\mathbb{P}$ are naturally labelled $0, 1, \infty$. (A *fundamental set* for the group G is a subset of $\Omega(G)$ which contains exactly one point from each equivalence class of points of $\Omega(G)$; see p.32 of Maskit [32].) In detail let Δ be defined by the following:

$$\Delta = \{z \in \mathbb{H} : -1 < \Re z \leq 1, |z - \frac{1}{2}| \geq \frac{1}{2}, |z + \frac{1}{2}| > \frac{1}{2}\}.$$

From now on, we will use ϵ to denote element of the cyclically ordered set $\{0, 1, \infty\}$. Let Δ_0 be the (closed) ideal triangle with vertices $\{0, 1, \infty\}$, and Δ_1 be the interior of its reflection in the imaginary axis. We sometimes refer to Δ_0 as the white triangle and Δ_1 as the black. The set Δ is the union of Δ_0 and Δ_1.

With our usual pants decomposition $\mathcal{P}$, we define bijections

$$\hat{\Phi}_j \colon\ P_j \longrightarrow \Delta$$

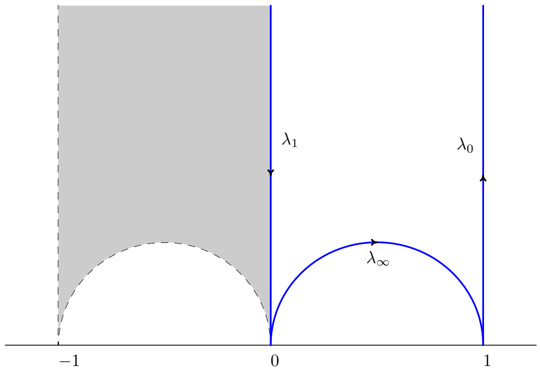

Figure 2.2: The standard fundamental set for Γ. The white triangle Δ_0 is unshaded and the black one Δ_1 is shaded.

from each (open) pair of pants P_j to the fundamental set Δ by

$$\hat{\Phi}_j := (\zeta|_\Delta)^{-1} \circ \Phi_j,$$

where

$$\Phi_j \colon\ P_j \longrightarrow \mathbb{P}$$

is the homeomorphism which identifies P_j to $\mathbb{P}$, and where

$$\zeta \colon\ \mathbb{H} \longrightarrow \mathbb{P} = \mathbb{H}/\Gamma$$

is the natural quotient map (which is a bijection when restricted to Δ). Note that $\hat{\Phi}_j$ restricts to a homeomorphism between P_j minus the seams to Δ minus $\lambda_1 \cup \lambda_0 \cup \lambda_\infty$, where λ_ϵ is the geodesic joining $\epsilon+1$ and $\epsilon+2$ with ϵ in the cyclically ordered set $\{0, 1, \infty\}$.

This identifications induce a labelling of the three boundary components of P_j as $0, 1, \infty$ in some order, fixed from now on. We denote the boundary of P_j labelled $\epsilon \in \{0, 1, \infty\}$ by $\partial_\epsilon P_j$. The identifications also induce a colouring of the two right angled hexagons whose union is P_j, one being white and one being black. We will call $\partial_\epsilon \mathbb{P}$ the boundary component of $\mathbb{P}$ corresponding to $\partial_\epsilon P$ under the identification $\Phi\colon\ P \longrightarrow \mathbb{P}$.

Suppose that the pants $P, P' \in \mathcal{P}$ are adjacent along the pants curve σ meeting along boundaries $\partial_\epsilon P$ and $\partial_{\epsilon'} P'$. (If $P = P'$ then clearly $\epsilon \neq \epsilon'$.) The gluing across σ is described by a complex parameter μ with $\Im\mu > 0$, called the *gluing parameter*, as already defined above. We first discuss the gluing in the case $\epsilon = \epsilon' = \infty$.

Arrange the pants with P on the left as shown in Figure 2.3. (Note that the

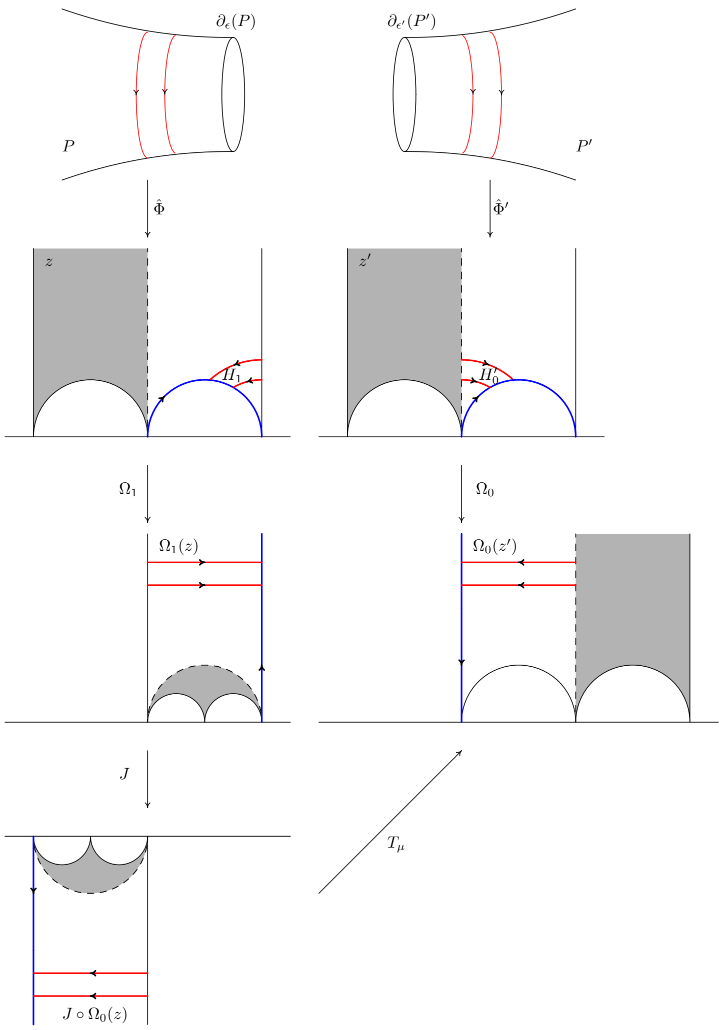

Figure 2.3: The gluing construction when $\epsilon = 1$ and $\epsilon' = 0$. Only the parts of H_1 and H_0' in Δ_0 and Δ_0' are shown.

illustration in the figure explains the more general case $\epsilon = 1$ and $\epsilon' = 0$.)

Take two copies $\mathbb{P}, \mathbb{P}'$ of $\mathbb{P}$. Each of these is identified with $\mathbb{H}/\Gamma$ as described above. We refer to the copy of $\mathbb{H}$ associated to $\mathbb{P}'$ as $\mathbb{H}'$ and denote the natural parameters in $\mathbb{H}, \mathbb{H}'$ by z, z' respectively. Let ζ and ζ' be the projections $\zeta\colon \mathbb{H} \longrightarrow \mathbb{P}$ and $\zeta'\colon \mathbb{H}' \longrightarrow \mathbb{P}'$ respectively.

Let $h_\infty = h_\infty(\mu)$ be the loop on $\mathbb{P}$ which lifts to the horocycle

$$h_{\infty,\mathbb{H}} = \{z \in \mathbb{H} | \Im z = \frac{\Im \mu}{2}\}$$

on $\mathbb{H}$. For a small positive number $\nu > 0$, we define

$$H_\infty = H_\infty(\mu,\nu) = \{z \in \mathbb{H} | \frac{\Im\mu - \nu}{2} < \Im z < \frac{\Im\mu + \nu}{2}\} \subset \mathbb{H}$$

to be the horizontal strip which projects to the annular neighbourhood $\mathbb{A}_\infty = \mathbb{A}_\infty(\mu)$ of $h_\infty \subset \mathbb{P}$. Let $S \subset \mathbb{P}$ be the surface $\mathbb{P}$ with the projection of the horocyclic neighbourhood $\{z \in \mathbb{H} | \Im z \geq \frac{\Im\mu+\nu}{2}\}$ of ∞ deleted. Note that S is open. Define h'_∞, S' and $\mathbb{A}'_\infty$ in a similar way. We are going to glue S to S' by matching $\mathbb{A}_\infty$ to $\mathbb{A}'_\infty$ in such a way that h_∞ is identified to h'_∞ with orientation reversed, see Figure 2.3. The resulting homotopy class of the loop h_∞ on the glued up surface (the quotient of the disjoint union of the surfaces S_j by the attaching maps across the $\mathbb{A}_i = \mathbb{A}(\sigma_i)$) is in the homotopy class of σ. To keep track of the marking on Σ, we do the gluing on the level of the $\mathbb{Z}$–covers of S, S' corresponding to h_∞, h'_∞, that is, we actually glue the strips H_∞ and H'_∞. See Section 2.2.3 for a detailed discussion of the marking.

As shown in Figure 2.3, the deleted punctured disks are on opposite sides of h_∞ in S and h'_∞ in S'. Thus we first need to reverse the direction in one of the two strips H_∞ and H'_∞. Set

$$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

We reverse the direction in H_∞ by applying the map $J(z) = -z$ to $\mathbb{H}$. We then glue H_∞ to H'_∞ by identifying $z \in H_\infty$ to $z' = T_\mu J(z) \in H'_\infty$. Since both J and T_μ commute with the conjugacy classes (with respect to the action of Γ) of the holonomies $z \mapsto z+2$ and $z' \mapsto z'+2$ of the curves h_∞, h'_∞, this identification descends to a well defined identification of $\mathbb{A}_\infty$ with $\mathbb{A}'_\infty$, in which the 'outer' boundary $\zeta(\{z \in \mathbb{H} | \Im z = \frac{\Im\mu+\nu}{2}\})$ of $\mathbb{A}_\infty$ is identified to the 'inner' boundary $\zeta'(\{z' \in \mathbb{H} | \Im z' = \frac{\Im\mu-\nu}{2}\})$ of $\mathbb{A}'_\infty$. In particular, h_∞ is glued to h'_∞ reversing orientation.

Now we treat the general case in which P and P' meet along punctures with arbitrary labels $\epsilon, \epsilon' \in \{0,1,\infty\}$. As above, let $\Delta_0 \subset \mathbb{H}$ be the ideal 'white' triangle with vertices $0,1,\infty$. Notice that there is a unique orientation preserving symmetry Ω_ϵ of Δ_0 which sends the vertex $\epsilon \in \{0,1,\infty\}$ to ∞:

$$\Omega_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \Omega_\infty = \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

Let h_ϵ be the loop on $\mathbb{P}$ which lifts to the horocycle

$$h_{\epsilon,\mathbb{H}} = \Omega_\epsilon^{-1}(\{z \in \mathbb{H} | \Im z = \frac{\Im \mu}{2}\})$$

on $\mathbb{H}$, so that h_ϵ is a loop round $\partial_\epsilon(\mathbb{P})$ in $\mathbb{P}$. Also let $H_\epsilon = H_\epsilon(\mu, \nu)$ be the region in $\mathbb{H}$ defined by

$$H_\epsilon = \Omega_\epsilon^{-1}(\{z \in \mathbb{H} | \frac{\Im \mu - \nu}{2} < \Im z < \frac{\Im \mu + \nu}{2}\}) = \Omega_\epsilon^{-1}(H_\infty).$$

The strip H_ϵ projects to annular neighbourhoods $\mathbb{A}_\epsilon = \mathbb{A}_\epsilon(\mu)$ of $h_\epsilon \subset \mathbb{P}$. Define $h'_{\epsilon'}$, $H'_{\epsilon'}$ and $\mathbb{A}'_{\epsilon'}$ in a similar way.

To do the gluing, first move ϵ and ϵ' to ∞ using the maps Ω_ϵ and $\Omega_{\epsilon'}$ and then proceed as before. Thus the gluing identifies $z \in H_\epsilon$ to $z' \in H'_\epsilon$ by the formula

$$\Omega_{\epsilon'}(z') = T_\mu \circ J\left(\Omega_\epsilon(z)\right), \tag{2.3}$$

see Figure 2.3.

Finally, we carry out the above construction for each pants curve $\sigma_i \in \mathcal{PC}$. To do this, we need to ensure that the annuli corresponding to the three different punctures of a given $\mathbb{P}_j$ are disjoint. (Note that, for example, the condition $\Im \mu_i > 2$, for all $i = 1, \ldots, \xi$, ensures that the three curves h_0, h_1 and h_∞ associated to the three punctures of P_j are disjoint in $\mathbb{P}$.) Under this condition, we can clearly choose $\nu > 0$ so that their annular neighbourhoods $\mathbb{A}_{\epsilon,j,+} \subset S_j$ are disjoint. In what follows, we shall usually write h and H for h_ϵ and H_ϵ provided the subscript is clear from the context.

Hence, we are defining the quotient $\mathfrak{S}_{\underline{\mu}}/\sim$ (homeomorphic to Σ), where $\mathfrak{S}_{\underline{\mu}} = S_1 \sqcup \ldots \sqcup S_k$ is the disjoint union of the truncated surfaces $S_j \subset \mathbb{P}$ defined above and the equivalence relation $\sim$ is given by the attaching maps along the annuli $\mathbb{A}_\epsilon(\sigma_i)$. Let $\Pi\colon \mathfrak{S}_{\underline{\mu}} \longrightarrow \mathfrak{S}_{\underline{\mu}}/\sim$ be the quotient map. In Section 2.2.2 we will see that this quotient is endowed with a complex projective structure $\Sigma(\underline{\mu})$ coming from our gluing construction.

Remark 2.2.1. In the above construction, we glued a curve exiting from the white triangles $\Delta_0(P)$ to one entering the white triangle $\Delta_0(P')$, where we denote $\Delta_0(P_j) = \hat{\Phi}_j^{-1}(\Delta_0) \subset P_j$ and $\Delta_1(P_j) = \hat{\Phi}_j^{-1}(\Delta_1) \subset P_j$ the white and the black hexagons in P_j, respectively. On the other hand, suppose we wanted to glue the two black triangles $\Delta_1(P)$ and $\Delta_1(P')$. This can be achieved, when gluing $\partial_\infty(P)$ to $\partial_\infty(P')$, by replacing the parameter μ with $\mu - 2$. However, following our recipe, it is not possible to glue a curve exiting a white triangle to a curve entering a black one, because the black triangle is to the right of both the outgoing and incoming lines, while the white triangle is to the left.

Remark 2.2.2. In our construction we require the parameter $\underline{\mu}$ to be in $\mathbb{H}^{\xi}$, but the gluing construction makes sense also for gluing parameters $\underline{\mu} \in \mathbb{L}^{\xi}$, where $\mathbb{L} = \{z \in \mathbb{C} | \Im z < 0\}$. In fact, if in Theorem 2.2.5 we substitute the hypothesis $\underline{\mu} \in \mathbb{H}^{\xi}$ with $\underline{\mu} \in \mathbb{L}^{\xi}$, we get 3–manifolds where the pants curves $\sigma_1, \ldots, \sigma_{\xi}$ are pinched in the top components Ω^+, rather than in the bottom one Ω^-.

2.2.1.a Independence of the direction of the travel

The recipe for gluing two pairs of pants apparently depends on the direction of travel across their common boundary. The following lemma shows that, in fact, the gluing in either direction is implemented by the same recipe and uses the same parameter μ.

Lemma 2.2.3. *Let pants P and P' be glued across a common boundary σ, and suppose the gluing used when travelling from P to P' is implemented by* (2.3) *with the parameter μ. Then the gluing when travelling in the opposite direction from P' to P is also implemented by* (2.3) *with the same parameter μ.*

Proof. Using the maps Ω_ϵ if necessary, we may, without loss of generality, suppose that we are gluing the boundary $\partial_\infty P$ to $\partial_\infty P'$. (Note that $\Omega_\infty = \mathrm{Id}$ by Equation (2.2).) By definition, to do this we identify the horocyclic strip $H \subset \mathbb{H}$ to the strip $H' \subset \mathbb{H}'$ using the map $T_\mu \circ J$.

Fix a point $X \in h$. The gluing sends X to $T_\mu J(X) \in h'$. The gluing in the other direction, that is, from P' to P, reverses orientation of the strips to be glued and is done using a translation $T_{\mu'}$, say. To give the same gluing we must have $T_{\mu'} J T_\mu J(X) = X$. This gives $\mu' - (-X + \mu) = X$, which reduces to $\mu = \mu'$, as claimed. □

2.2.2 Projective structure $\Sigma(\underline{\mu})$ and holonomy representation

The gluing construction described in Section 2.2.1 gives a way to define a marked complex projective structure on Σ. We describe the projective structure in this section and we deal with the marking in Section 2.2.3. The idea is the following. First, we define a complex projective structure on each truncated surface $S_j \subset \mathbb{P}$, where $j = 1, \ldots, k$, and then, we describe why the attaching maps allow us to define a complex projective structure on the quotient $\mathfrak{S}_{\underline{\mu}} / \sim$ (homeomorphic to Σ) defined in Section 2.2.1.

We recall some basic facts about complex projective structures; see Dumas [14].

1. If Z is a complex projective structure on Σ and $\Sigma' \subset \Sigma$ is an open subset, then the restriction of Z to Σ' defines a complex projective structure on Σ'.

2. If X is preserved by a group Γ of Möbius transformations acting freely and properly discontinuously, then the quotient surface $Y = X/\Gamma$ has a natural projective structure in which the charts are local inverses of the covering $X \longrightarrow Y$.

3. A Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ gives rise to a projective structure on the quotient surface $\mathbb{H}/\Gamma$;

4. If $\pi \colon X \longrightarrow Y$ is a covering map, then a complex projective structure on Y lifts to a complex projective structure on X.

The first and third facts allow us to define a complex projective structure on each $S_j \subset \mathbb{P} = \mathbb{H}/\Gamma$, with $j = 1, \ldots, k$; see Section 2.2.1 for the definitions.

Now, suppose $\sigma_i = \partial_\epsilon P_j \cap \partial_{\epsilon'} P_{j'}$, we discuss the gluing of $S = S_j$ and $S' = S_{j'}$ along the annuli $\mathbb{A} = \mathbb{A}_\epsilon(\sigma_i) \subset S$ and $\mathbb{A}' = \mathbb{A}_{\epsilon'}(\sigma_i) \subset S'$, that is we discuss the complex projective structure on $S \sqcup S'/\sim$, where the equivalence relation $\sim$ is given by the attaching maps along the annuli $\mathbb{A}$ and $\mathbb{A}'$.

Recall that there are strips $H = H_\epsilon = \Omega_\epsilon^{-1}(H_\infty)$ and $H' = H_{\epsilon'} = \Omega_{\epsilon'}^{-1}(H_\infty)$ in $\mathbb{H} \subset \hat{\mathbb{C}}$ such that $\zeta(H) = \mathbb{A} \subset S$ and $\zeta'(H') = \mathbb{A}' \subset S'$, where $\zeta \colon \mathbb{H} \longrightarrow \mathbb{P} = \mathbb{H}/\Gamma$ and $\zeta' \colon \mathbb{H} \longrightarrow \mathbb{P}' = \mathbb{H}/\Gamma$. So $\zeta \sqcup \zeta' \colon H \sqcup H' \longrightarrow \mathbb{A} \sqcup \mathbb{A}'$. Let $\pi_H \colon H \sqcup H' \longrightarrow H \sqcup H'/\sim$ and $\pi_{\mathbb{A}} \colon \mathbb{A} \sqcup \mathbb{A}' \longrightarrow \mathbb{A} \sqcup \mathbb{A}'/\sim$. With abuse of notation let's denote $\zeta \sqcup \zeta' \colon (H \sqcup H'/\sim) \longrightarrow (\mathbb{A} \sqcup \mathbb{A}'/\sim)$. We can see that $\pi_{\mathbb{A}} \circ (\zeta \sqcup \zeta') = (\zeta \sqcup \zeta') \circ \pi_H$.

Note that $V \subset H \sqcup H'/\sim$ is open if and only if $\pi_H^{-1}(V) \subset H \sqcup H'$ is open. Note also that $\pi_H^{-1}(V) = V_1 \sqcup V_2$, where $V_1 = \pi_H^{-1}(V) \cap H$ and $V_2 = \pi_H^{-1}(V) \cap H'$, see Figure 2.4. Note that in the figure $\mathbb{H}$ and $\mathbb{H}'$ are superimposed so that $\Delta_0 = \Delta_0'$ and the set V_1 and V_2 are in different positions with respect to Δ_0 and Δ_0' even though they correspond to the same set V in $S \sqcup S'/\sim$. Using the first fact above, we can see that there are natural complex projective structures on H and H', respectively, where the charts are the inclusion maps i. We define a complex projective structure on $H \sqcup H'/\sim$ as follows. Let $\mathcal{V} = \{V_i\}$ be a covering of $H \sqcup H'/\sim$, we define two sets of charts on V. The charts $\psi^1 = \psi^1_V \colon V \longrightarrow H \subset \hat{\mathbb{C}}$ are defined by $V \overset{\pi^{-1}}{\longmapsto} V_1 \overset{i}{\mapsto} H$ and the charts $\psi^2 = \psi^2_V \colon V \longrightarrow H' \subset \hat{\mathbb{C}}$ defined by $V \overset{\pi^{-1}}{\longmapsto} V_2 \overset{i}{\mapsto} H'$.

The transition maps are given by $\psi^1 \circ (\psi^2)^{-1} = \Omega_\epsilon^{-1} J^{-1} T_{\mu_i}^{-1} \Omega_{\epsilon'} \subset \mathrm{PSL}(2, \mathbb{C})$ for every $V \in \mathcal{V}$. This defines a complex projective structure on $H \sqcup H'/\sim$ which descends to a complex projective structure on $\mathbb{A} \sqcup \mathbb{A}'/\sim$ (see the second fact above). This defines a complex projective structure on $S \sqcup S'/\sim$.

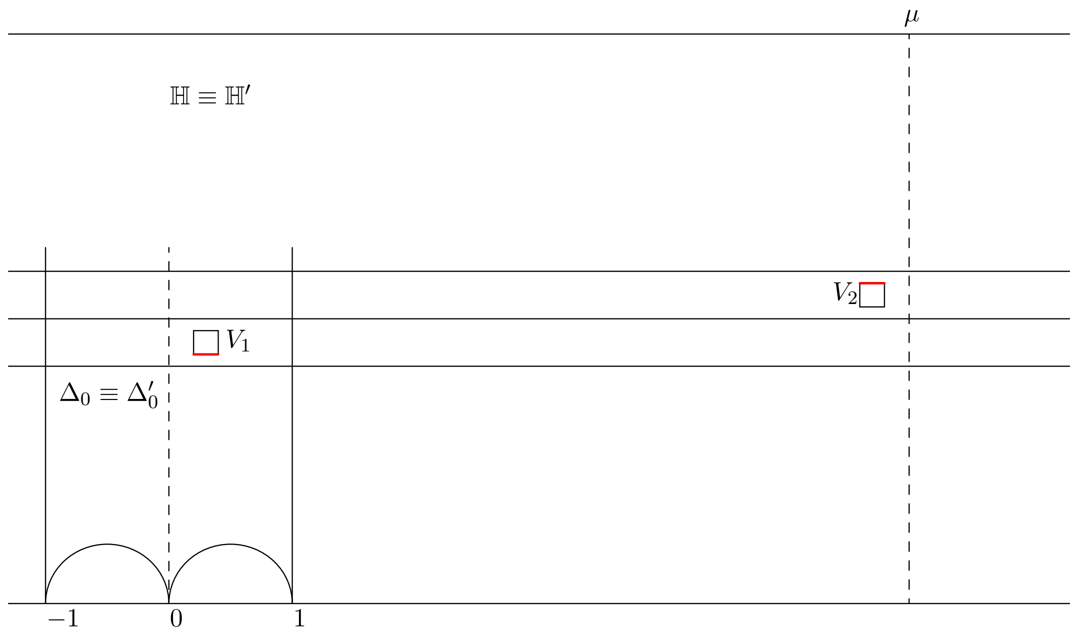

Figure 2.4: The open sets $V_1 \subset H_\epsilon$ and $V_2 \subset H_{\epsilon'}$ in case $H = H' = H_\infty$. In the figure $\mathbb{H}$ and $\mathbb{H}'$ are superimposed so that $\Delta_0 = \Delta_0'$. The set V_1 and V_2 are in different positions with respect to Δ_0 and Δ_0' even though they correspond to the same set V in $S \sqcup S'/\sim$.

Carrying out the above construction for each pants curve $\sigma_i \in \mathcal{PC}$ and considering the unique maximal atlas in the equivalence class of this atlas, we can define a complex projective structure on the quotient $\mathfrak{S}_{\underline{\mu}}/\sim$ (homeomorphic to Σ), which we denote by $\Sigma(\underline{\mu})$, where $\underline{\mu} = (\mu_1, \dots, \mu_\xi) \in \mathbb{H}^\xi$. Note that this projective structure is obtained by the gluing procedure described in Section 2.2.1 with parameter μ_i along curve σ_i.

Then, as described in Section 2.1.3, we can consider the developing map

$$\mathrm{Dev}_{\underline{\mu}} \colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$$

and the holonomy representation

$$\rho_{\underline{\mu}} \colon \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$$

associated to this complex projective structure, where $\underline{\mu} = (\mu_1, \dots, \mu_\xi) \in \mathbb{H}^\xi$. Both these maps are well defined up to the action of $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{Dev}_{\underline{\mu}}$ is equivariant with respect to $\rho_{\underline{\mu}}$.

As a consequence of the construction, we note the following fact which underlies the connection with the Maskit embedding (Section 2.2.5), and which (together with the definition of the twist parameter in the case $q_i = 0$, see Section 1.2) proves the final statement of Theorem A.

Lemma 2.2.4. *Suppose that $\gamma \in \pi_1(\Sigma)$ is a loop homotopic to a pants curve σ_i. Then, for any $\underline{\mu} \in \mathbb{H}^\xi$, we have that $\rho_{\underline{\mu}}(\gamma)$ is parabolic and that $\operatorname{Tr} \rho_{\underline{\mu}}(\gamma) = \pm 2$.*

2.2.3 Marking on $\Sigma(\underline{\mu})$

To complete the description of the marked complex projective structure, we have to specify a marking on $\Sigma(\underline{\mu})$, that is a homeomorphism $f_{\underline{\mu}} : \Sigma \longrightarrow \Sigma(\underline{\mu})$ from a fixed topological surface Σ to the surface $\Sigma(\underline{\mu})$. Endow Σ with a marking decomposition, as described in Section 1.2.1.a, that is a pants decomposition $\mathcal{PC} = \{\sigma_1, \dots, \sigma_\xi\}$ and a set of dual curves $\{D_1, \dots, D_\xi\}$.

We first describe this marking for the particular case $\Sigma(\underline{\mu}^0)$, where $\underline{\mu}^0 = (\mu_1^0, \dots, \mu_\xi^0)$ is defined by $\Re\mu_j^0 = 1$, for all $j = 1, \dots, \xi$, and then we see how to deal with all the other cases. The imaginary part of μ_j^0 is not important. (For definiteness, you can fix it to be $\Im\mu_j^0 = 5$.) In particular, we describe a marking decomposition for $\Sigma(\underline{\mu}^0)$, so that we can define the homeomorphism $f_{\underline{\mu}^0} : \Sigma \longrightarrow \Sigma(\underline{\mu}^0)$ by sending the pants curves σ_i and the dual curves D_i on Σ to the corresponding curves in $\Sigma(\underline{\mu}^0)$.

Recall that $\lambda_\epsilon \subset \Delta_0$ is the unique oriented geodesic from $\epsilon + 1$ to $\epsilon + 2$, where ϵ is in the cyclically ordered set $\{0, 1, \infty\}$, see Figure 2.2 and Section 2.2.2. The lines λ_ϵ project to the seams of $\mathbb{P}$. We call λ_0 (from 1 to ∞) and λ_1 (from ∞ to 0) respectively the *incoming* and the *outgoing* strands (coming into and going out from the puncture) at ∞, and refer to their images under the maps Ω_ϵ in a similar way. For $\mu \in \mathbb{C}$, let $X_\infty(\mu) = 1 + \Im\mu/2$ be the point at which the incoming line λ_0 meets the horizontal horocycle $\{z \in \mathbb{H} | \Im z = \Im\mu/2\}$ in $\mathbb{H}$, and let $Y_\infty(\mu) = \Im\mu/2$ be the point at which the outgoing line λ_1 meets the same horocycle. Also define $X_\epsilon(\mu) = \Omega_\epsilon^{-1}(X_\infty)$ and $Y_\epsilon(\mu) = \Omega_\epsilon^{-1}(Y_\infty)$. Now pick a pants curve σ and, as usual, let $P, P' \in \mathcal{P}$ be its adjacent pants in Σ, to be glued across boundaries $\partial_\epsilon P$ and $\partial_{\epsilon'} P'$. Let $X_\epsilon(P, \mu), X_\epsilon(P', \mu)$ be the points corresponding to $X_\epsilon(\mu), X_\epsilon(\mu)$ under the identifications $\zeta|_\Delta$ and $\zeta'|_{\Delta'}$ of Δ, Δ' with $\mathbb{P}$, and similarly for $Y_\epsilon(P, \mu), Y_\epsilon(P', \mu)$. The base structure $\Sigma(\underline{\mu}^0)$ is the one in which the identification (2.3) matches the point $X_\epsilon(P, \mu)$ on the incoming line across $\partial_\epsilon\mathbb{P}$ to the point $Y_{\epsilon'}(P', \mu)$ on the outgoing line to $\partial_{\epsilon'}\mathbb{P}'$. Referring to the gluing equation (2.3), we see that this condition is fulfilled precisely when $\Re\mu = 1$. We define the structure on Σ by specifying $\Re\mu_i = 1$ for $i = 1, \dots, \xi$. The imaginary part of μ_i is unimportant for the above condition to be true. This explains our choice for $\underline{\mu}^0$.

Now note that the reflection $z \mapsto -\bar{z}$ of $\mathbb{H}$ induces an orientation reversing isometry of $\mathbb{P}$ which fixes its seams; with the gluing matching the seams, as above, this extends, in an obvious way, to an orientation reversing involution of Σ. Following (a) of Section 1.2.1.a, this specification is equivalent to a specification of a marking decomposition on Σ. This gives us a way to define the marking $f_{\underline{\mu}^0} \colon \Sigma \longrightarrow \Sigma(\underline{\mu}^0)$.

Finally, we define a marking on the surface $\Sigma(\underline{\mu})$. After applying a suitable stretching to each pants in order to adjust the lengths of the boundary curves, we can map $\Sigma(\underline{\mu}^0) \longrightarrow \Sigma(\underline{\mu})$ using a map which is the Fenchel–Nielsen twist $Tw_{\sigma_i, \Re\mu_i - 1}$ on an annulus around $\sigma_i \in \mathcal{PC}, i = 1, \dots, \xi$ and the identity elsewhere, see Section 1.2.1.a for the definition of $Tw_{\sigma,t}$. This gives a well defined homotopy class of homeomorphisms $f_{\underline{\mu}} \colon \Sigma \longrightarrow \Sigma(\mu)$. Notice that the stretch map used above depends on $\Im\mu_i$.

With this description, it is easy to see that $\Re\mu_i$ corresponds to twisting about σ_i; in particular, $\mu_i \mapsto \mu_i + 2$ is a full right Dehn twist about σ_i. The imaginary part $\Im\mu_i$ corresponds to vertical translation and has the effect of scaling the lengths of the σ_i.

2.2.4 Relation to Kra's plumbing construction

Kra in [25] uses essentially the above construction to manufacture surfaces by gluing triply punctured spheres across punctures, a procedure which he calls *plumbing*. Plumbing is based on so called 'horocyclic coordinates' in punctured disk neighbourhoods of the punctures which have to be glued.

Given a puncture ϵ on a triply punctured sphere $\mathbb{P}$, let $\zeta \colon \mathbb{H} \longrightarrow \mathbb{P}$ be the natural projection, normalised so that ϵ lifts to $\infty \in \mathbb{H}$, and so that the holonomy of the loop round ϵ is, as above, $\varsigma \mapsto \varsigma + 2$. Let $\mathbb{D}_*$ denote the punctured unit disk $\{z \in \mathbb{C} : 0 < z < 1\}$. The function $f \colon \mathbb{H} \longrightarrow \mathbb{D}_*$, given by $f(\varsigma) = e^{i\pi\varsigma}$, is well defined in a neighbourhood N of ∞ and is a homeomorphism from an open neighbourhood of ϵ in $\mathbb{P}$ to an open neighbourhood of the puncture in $\mathbb{D}_*$. Choosing another puncture ϵ' of $\mathbb{P}$, we can further normalise so that ϵ' lifts to 0. Hence f maps the part of the geodesic from ϵ' to ϵ contained in N to the interval $(0, r)$, for suitable $r > 0$. These normalisations (which depend only on the choices of ϵ and ϵ') uniquely determine f. Kra calls the natural parameter $z = f(\varsigma)$ in $\mathbb{D}_*$, the *horocyclic coordinate* of the puncture ϵ relative to ϵ'.

Now suppose that $\hat{z}$ and $\hat{z}'$ are horocyclic coordinates for distinct punctures in distinct copies $\mathbb{P}_{\hat{z}}$ and $\mathbb{P}_{\hat{z}'}$ of $\mathbb{P}$. Denote the associated punctured disks by $\mathbb{D}_*(\hat{z})$ and $\mathbb{D}_*(\hat{z}')$. To plumb across the two punctures, first delete punctured disks $\{0 < \hat{z} < r\}$ and $\{0 < \hat{z}' < r'\}$ from $\mathbb{D}_*(\hat{z})$ and $\mathbb{D}_*(\hat{z}')$ respectively. Then glue the remaining surfaces along the annuli

$$\mathbb{A}(\hat{z}) = \{\hat{z} \in \mathbb{D}_*\colon\ r < \hat{z} < s\} \ \text{ and } \ \mathbb{A}(\hat{z}') = \{\hat{z}' \in \mathbb{D}_*\colon\ r' < \hat{z}' < s'\}$$

by the formula $\hat{z}\hat{z}' = t_K$. (To avoid confusion we have written t_K for Kra's parameter $t \in \mathbb{C}$.) It is easy to see that this is essentially identical to our construction; the difference is simply that we implement the gluing in $\mathbb{H}$ and $\mathbb{H}'$ without first mapping to $\mathbb{D}_*(\hat{z})$ and $\mathbb{D}_*(\hat{z}')$. Our method has the advantage of having a slightly simpler formula and also of respecting the twisting around the puncture, which is lost under the map f.

The precise relation between our coordinates $z, z' \in \mathbb{H}$ in Section 2.2.1 and the horocyclic coordinates $\hat{z}, \hat{z}'$ is:

$$z = f^{-1}(\hat{z}) = -\frac{i}{\pi}\log\hat{z}, \quad z' = f^{-1}(\hat{z}') = -\frac{i}{\pi}\log\hat{z}'.$$

The relation

$$\hat{z}\hat{z}' = t_K$$

translates to

$$\log\hat{z}' + \log\hat{z} = \log t_K$$

which, modulo $2\pi i\mathbb{Z}$, is exactly our relation

$$z' = -z + \mu.$$

Hence we deduce that

$$\mu = -\frac{i}{\pi}\log t_K,$$

or equivalently

$$t_K = \exp(i\pi\mu).$$

2.2.5 Relation to the Maskit embedding of Σ

As usual, let $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$ be a pants decomposition of Σ. We have constructed a family of projective structures on Σ, to each of which is associated a natural holonomy representation $\rho_{\underline{\mu}}\colon \pi_1(\Sigma) \longrightarrow PSL(2,\mathbb{C})$. We want to prove that our construction, for suitable values of the parameters, gives exactly the *Maskit embedding* of Σ; see Figure 2.5.

For the definition of this embedding we need to recall the definition of the *representation variety* $\mathcal{R}(\Sigma)$ of Σ. For us $\mathcal{R} = \mathcal{R}(\Sigma)$ will be the set of non-elementary

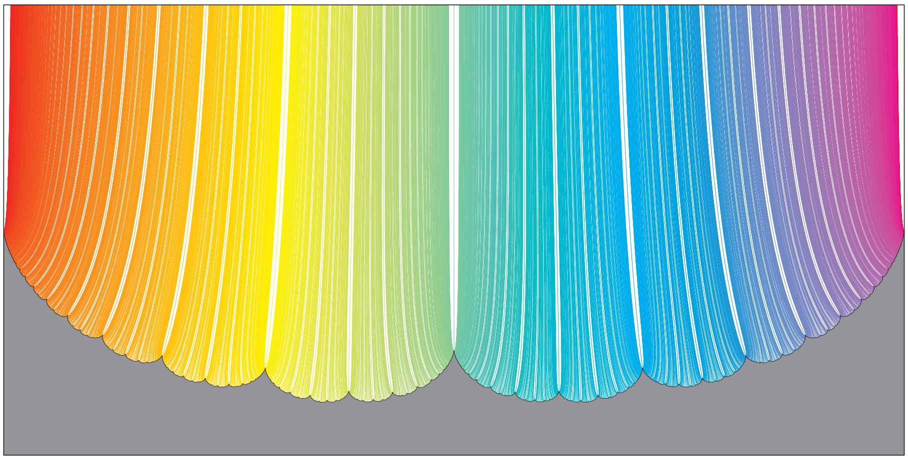

Figure 2.5: The Maskit embedding $\mathcal{M}(\Sigma_{1,1})$ for the once punctured torus $\Sigma_{1,1}$. Reproduced, with permission, from [37] published by Cambridge University Press.

representations $\rho\colon \pi_1(\Sigma) \longrightarrow PSL(2,\mathbb{C})$ modulo conjugation in $PSL(2,\mathbb{C})$. More precisely, $\mathcal{R}(\Sigma)$ is defined as the GIT (short for Geometric Invariant Theory) quotient

$$\mathcal{R}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, C))// \,\mathrm{PSL}(2,\mathbb{C}),$$

where $\mathrm{PSL}(2,\mathbb{C})$ acts on the space $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2, C))$ of homomorphisms from $\pi_1(\Sigma)$ to $\mathrm{PSL}(2,\mathbb{C})$ by conjugation. See Kapovich [21].

For the definition of the Maskit embedding we follow [40], see also [31]. Let $\mathcal{M} \subset \mathcal{R}$ be the subset of representations for which:

(i) the group $G = \rho\,(\pi_1(\Sigma))$ is discrete (Kleinian) and ρ is an isomorphism;

(ii) the images of σ_i, $i = 1, \dots, \xi$, are parabolic;

(iii) all components of the regular set $\Omega(G)$ are simply connected and there is exactly one invariant component $\Omega^+(G)$;

(iv) the quotient $\Omega(G)/G$ has $k+1$ components (where $k = 2g-2+n$ if $\Sigma = \Sigma_{(g,n)}$), $\Omega^+(G)/G$ is homeomorphic to Σ and the other components are triply punctured spheres.

In this situation, see for example [30] (Section 3.8), the corresponding 3–manifold $M_G = \mathbb{H}^3/G$ is topologically $\Sigma \times (0,1)$. Moreover G is a geometrically finite cusp group on the boundary (in the algebraic topology) of the set of Quasifuchsian representations of $\pi_1(\Sigma)$. The 'top' component $\Omega^+(G)/G$ of the conformal boundary may be identified to $\Sigma \times \{1\}$ and is homeomorphic to Σ. On the 'bottom' component $\Omega^-(G)/G$, identified to $\Sigma \times \{0\}$, the pants curves $\sigma_1, \dots, \sigma_\xi$ have been pinched, making $\Omega^-(G)/G$ a union of k triply punctured spheres glued across punctures corresponding to the curves σ_i. The conformal structure on $\Omega^+(G)/G$, together with the pinched curves $\sigma_1, \ldots, \sigma_\xi$, are the *end invariants* of M_G in the sense of Minsky's Ending Lamination Theorem. Since a triply punctured sphere is rigid in the hyperbolic space $\mathbb{H}^3$, the conformal structure on $\Omega^-(G)/G$ is fixed and independent of ρ, while the structure on $\Omega^+(G)/G$ varies. It follows from standard Ahlfors–Bers theory, using the measurable Riemann mapping theorem (see again [30] Section 3.8), that there is a unique group corresponding to each possible conformal structure on $\Omega^+(G)/G$. Formally, the *Maskit embedding* of the Teichmüller space of Σ is the map $\mathcal{T}(\Sigma) \longrightarrow \mathcal{R}$ which sends a point $X \in \mathcal{T}(\Sigma)$ to the unique group $G \in \mathcal{M}$ for which $\Omega^+(G)/G$ has the marked conformal structure X.

For the proof, we need to use results discussed in Section 2.1.1 about Maskit–Kleinian and Fuchsian groups.

Theorem 2.2.5. *Suppose that $\underline{\mu} \in \mathbb{H}^\xi$ is such that the associated developing map $\mathrm{Dev}_{\underline{\mu}} \colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ is an embedding. Then the holonomy representation $\rho_{\underline{\mu}}$ is a group isomorphism and $G = \rho_{\underline{\mu}}(\pi_1(\Sigma)) \in \mathcal{M}$.*

Proof. Since the developing map $\mathrm{Dev}_{\underline{\mu}} \colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ is an embedding, then $\rho_{\underline{\mu}}$ acts discontinuously on $\Omega^+ = \mathrm{Dev}(\tilde{\Sigma})$, so by using the definitions of Section 2.1.1, we can see that the group $G = \rho_{\underline{\mu}}(\pi_1(\Sigma))$ is Maskit–Kleinian, and hence Kleinian, by Proposition 2.1.1. This tells us also that $\Omega^+ \subset \Omega(G)$. By construction (see Lemma 2.2.4), the holonomy of each of the curves $\sigma_1, \ldots, \sigma_\xi$ is parabolic. This proves (i) and (ii).

The set Ω^+ is a simply connected G–invariant set contained in $\Omega(G)$. The simply connectivity follows from the fact that the developing map $\mathrm{Dev}_{\underline{\mu}}$ is an embedding and $\tilde{\Sigma}$ is simply connected. Now, consider its closure $\overline{\Omega^+}$, that is, consider the accumulation points of the set Ω^+. First note that since $\overline{\Omega^+} = \Omega^+ \cup \partial\Omega^+$ is closed and G–invariant, then, by the second statement of Proposition 2.1.3, it contains the limit set $\Lambda(G)$. Now, since $\Omega^+ \subset \Omega(G)$ and since, by Theorem 2.1.2, $\hat{\mathbb{C}}$ is the disjoint union of $\Lambda(G)$ and $\Omega(G)$, then $\Lambda(G) \subset \partial\Omega^+$.

To prove that $\Lambda(G) \supset \partial\Omega^+$ we use an idea used by Thurston to prove that the space of marked complex projective structures is homeomorphic to the product $\mathcal{T}(\Sigma) \times \mathrm{ML}(\Sigma)$, where $\mathcal{T}(\Sigma)$ is the Teichmüller space of Σ and $\mathrm{ML}(\Sigma)$ is the space of measured laminations. Since $\hat{\mathbb{C}}$ is the ideal boundary of hyperbolic space $\mathbb{H}^3$, we can consider the boundary of the hyperbolic convex hull of $\hat{\mathbb{C}} - \Omega^+$ and denote it $\mathrm{Pl} = \mathrm{Pl}_{\underline{\mu}}$. Moreover, there is a retraction map $r \colon \Omega^+ \longrightarrow \mathrm{Pl}$ and observe that $\partial\mathrm{Pl} = \partial\Omega^+$. Then Pl is a convex pleated plane in $\mathbb{H}^3$ invariant under the action of $G = \rho(\pi_1(\Sigma))$ by isometries. If you equip Pl with the path metric, then it is a complete hyperbolic 2–manifold (see Theorem 1.12.1 of Epstein and Marden [16]). By this isometry, the action of G on Pl corresponds to a discontinuous action on $\mathbb{H}$ by a Fuchsian group $G' \in \mathcal{F}(\Sigma)$. Moreover, we claim that our construction tells us also that there is a fundamental region for the action of G on Pl which has finite area (and hence the same it is true for the action of G' on $\mathbb{H}$), see Lemma 2.2.6 below. So, using Theorem 2.1.5, this finite area region tells us that G' is of the first kind and hence that $\Lambda(G') = \partial\mathbb{H}$. In addition, the isometry from Pl to $\mathbb{H}$ extends continuously to the boundaries $\partial\mathrm{Pl} \subset \partial\mathbb{H}^3$ and $\partial\mathbb{H}$, see Theorem 3.6 of Minsky [35]. This tells us that the limit set $\Lambda(G)$ is the boundary of the pleated plane Pl, that is $\Lambda(G) = \partial\Omega^+$. So Ω^+ is a connected component of the regular set $\Omega(G)$ of G, and its boundary $\partial\Omega^+$ is the limit set $\Lambda(G)$.

Now let $P \in \mathcal{P}$, and let $\tilde{P}$ be a lift of P to the universal cover $\tilde{\Sigma}$. The boundary curves $\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}$ of P lift, in particular, to three curves in $\partial\tilde{P}$ corresponding to elements $\gamma_{i_1}, \gamma_{i_2}, \gamma_{i_3} \in \pi_1(\Sigma)$ such that $\gamma_{i_1}\gamma_{i_2}\gamma_{i_3} = id$ and such that $\rho(\gamma_{i_j})$ is parabolic for $j = 1, 2, 3$. These generate a subgroup $\Gamma(\tilde{P})$ of $SL(2, \mathbb{R})$ conjugate to Γ, see Section 2.2.1. Thus the limit set $\Lambda(\tilde{P})$ of $\Gamma(\tilde{P})$ is a round circle $C(\tilde{P})$.

Without loss of generality, fix the normalisation of G such that $\infty \in \Omega^+(G)$. Since $\Omega^+(G)$ is connected, it must be contained in the component of $\hat{\mathbb{C}} \setminus \Lambda(\tilde{P})$ which contains ∞. Since $\Lambda(G) = \partial\Omega^+(G)$, we deduce that $\Lambda(G)$ is also contained in the closure of the same component, and hence that the open disk $D(\tilde{P})$ bounded by $C(\tilde{P})$ and not containing ∞, contains no limit points. (In the terminology of [24], $\Gamma(\tilde{P})$ is peripheral with peripheral disk $D(\tilde{P})$.) It follows that $D(\tilde{P})$ is precisely invariant under $\Gamma(\tilde{P})$ and hence that $D(\tilde{P})/G = D(\tilde{P})/\Gamma(\tilde{P})$ is a triply punctured sphere.

Thus $\Omega(G)/G$ contains the surface $\Sigma(G) = \Omega^+(G)/G$ and the union of k triply punctured spheres $D(\tilde{P})/\Gamma(\tilde{P})$, one for each pair of pants in $\mathcal{P}$. Thus the total hyperbolic area of $\Omega(G)/G$ is at least $4\pi k$. Now Bers' area inequality [3], see also Theorem 4.6 of Matsuzaki–Taniguchi [33], states that

$$\mathrm{Area}(\Omega(\mathrm{G})/\mathrm{G}) \leq 4\pi(\mathrm{T} - 1),$$

where T is the minimal number of generators of G. In our case $T = 2g + b - 1$. Since $k = 2g + b - 2$, we have

$$4\pi(2g + b - 2) \leq \mathrm{Area}(\Omega(G)/G) \leq 4\pi(T - 1) = 4\pi(2g + b - 2).$$

We deduce that $\Omega(G)$ is the disjoint union of $\Omega^+(G)$ and the disks $D(\tilde{P})$, with

$P \in \mathcal{P}$. This completes the proof of (iii) and (iv). $\square$

Lemma 2.2.6. *In the above setting, there is a fundamental region of finite area for the action of G on* Pl *(endowed with the path metric).*

Proof of the claim. In order to prove the lemma we will show that a fundamental region $r(E)$ for the action of G on Pl can be constructed as the union of k regions $r(E_j)$, each of which is compact or of finite area. These regions $r(E_j)$ correspond to the lifts (by the retraction map r) to Pl of the images E_j (under $(\zeta|_\Delta)^{-1}$) of the truncated surfaces S_j corresponding to the pair of pants P_j, as we are going to explain.

Suppose that $\partial P_j = \{s_{i_1}, s_{1_2}, s_{1_3}\}$ and let $\hat{\Phi}_j\colon\, P_j \longrightarrow \Delta$, so that $s_{i_1} = \partial_\infty(P_j)$, $s_{i_2} = \partial_0(P_j)$ and $s_{i_3} = \partial_1(P_j)$. Then consider the region E_j obtained by removing from Δ the horoballs of heights $\frac{\Im\mu_{i_j}}{2}$, that is:

- $\{z \in \mathbb{C} | \Im z > \frac{\Im\mu_{i_1}}{2}\}$ around ∞;
- $\{z \in \mathbb{C} : |z - i\frac{1}{\Im\mu_{i_2}}| < \frac{1}{\Im\mu_{i_2}}\}$ around 0;
- $\{z \in \mathbb{C} : |z - (1 + i\frac{1}{\Im\mu_{i_3}})| < \frac{1}{\Im\mu_{i_3}}\}$ around 1;
- $\{z \in \mathbb{C} : |z - (-1 + i\frac{1}{\Im\mu_{i_3}})| < \frac{1}{\Im\mu_{i_3}}\}$ around -1,

where μ_{i_j} is the gluing parameter corresponding to the pants curve σ_{i_j} for $j = 1, 2, 3$, if $\sigma_{i_j} \notin \partial\Sigma$, and where $\mu_{i_j} := \infty$ (and $\frac{1}{\Im\mu_{i_j}} := 0$) if $\sigma_{i_j} \in \partial\Sigma$. If $\sigma_i = \partial_\epsilon(P_j) \in \partial\Sigma$, let $\mathcal{H}_i$ be a horoball of height 1 around ϵ. Then let $E = E_1 \cup \ldots \cup E_k$ be the union of the regions E_j under the attaching maps defined in Section 2.2.1 and let $\mathcal{H} = \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_b$ be the union of the regions $\mathcal{H}_i$ for $i = 1, \ldots, b$, where b is the number of boundary components of Σ. The set E is a fundamental region for the action of G on $\hat{\mathbb{C}}$.

We want to show that $r(E)$ has finite area. The set $E' = E \setminus (E \cap \mathcal{H})$ is a compact set contained in the interior of Ω^+ and so its image $r(E')$ under r remains compact and contained in the interior of Pl, and hence has finite area. On the other hand, for each horoball $\mathcal{H}_i$, with $i = 1, \ldots, b$, based at the parabolic fixed point $x = \mathrm{Fix}(g) \in \partial\Omega^+$ of $g \in G$, we have that $\mathcal{H}_i \setminus \{x\}$ is contained in the interior of Ω^+ (since it is the image, under $\langle g \rangle$, of $\mathcal{H}_i \cap \Delta$), and $r(\mathcal{H}_i)/G$ is a punctured disk (since $r(\mathcal{H}_i)/G = r(\mathcal{H}_i)/\langle g \rangle$, because $r(\mathcal{H}_i)$ is precisely invariant under the parabolic element g). Hence it has finite area. So the claim is proved. $\square$

This gives an alternative viewpoint on our main result: we are finding a formula for the leading terms of the trace polynomials in the parameters μ_i of simple curves on Σ under the Maskit embedding of $\mathcal{T}(\Sigma)$. This was the context in which the result was presented in [24; 40], see also Section 2.3.4.a.

2.3 Calculation of paths

In this section we discuss how to compute the holonomy $\rho_{\underline{\mu}}\colon \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$.

In order to prove Theorem A of the Introduction, we need to describe $\rho_{\underline{\mu}}$ in a concrete way, so we need to compute the holonomy images of the generators of the fundamental groupoid of the surface Σ, see Section 2.1.2 and 2.3.1. Thus, in Section 2.3.2, we study paths contained in one pair of pants, then in Section 2.3.3, we specify a particular path joining one hexagon to the next, and, finally, in Section 2.3.4, we compute the holonomy representations of some paths in the one holed torus and in the four times punctured sphere, as an example.

2.3.1 Groupoid holonomy map

In Section 2.1.3.b we explained how, given a projective structure Z on Σ, given any subset $B \subset \Sigma$ and chosen germs of charts $\Phi^b\colon U^b \longrightarrow V^b$ for every $b \in B$, it is possible to define a groupoid holonomy map (associated to Z)

$$\rho^{\mathfrak{C}}\colon \pi_1(\Sigma, B) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$$

from the fundamental groupoid $\pi_1(\Sigma, B)$ (see Definition 2.1.8) to $\mathrm{PSL}(2,\mathbb{C})$, where $\mathfrak{C} = \{\Phi^b | b \in B\}$. In addition, when we restrict this map to the fundamental group $\pi_1(\Sigma, b)$ (where $b \in B$) we get the usual holonomy map $\rho\colon \pi_1(\Sigma, b) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ well defined up to conjugation (if we don't ask the germ Φ^b to be fixed).

In our case, we will choose as base set B on the surface Σ the barycentres of the hexagons in the pair of pants P_j, as we are going to explain. In Section 2.2 we described the maps $\hat{\Phi}_j\colon P_j \longrightarrow \Delta \subset \mathbb{H}$ from each pair of pants $P_j \in \mathcal{P}$ to the standard fundamental set $\Delta \subset \mathbb{H}$. Recall that Δ is the union of the white hexagon Δ_0 and the black one Δ_1, and that $\Delta_0(P_j) = \hat{\Phi}_j^{-1}(\Delta_0) \subset P_j$ and $\Delta_1(P_j) = \hat{\Phi}_j^{-1}(\Delta_1) \subset P_j$ are the white and the black hexagons in P_j, respectively. Also let $b_j = \hat{\Phi}_j^{-1}(b_0)$ and $b_j^* = \hat{\Phi}_j^{-1}(b_0^*)$, where $b_0 = \frac{1+i\sqrt{3}}{2} \in \Delta_0$ and $b_0^* = \frac{-1+i\sqrt{3}}{2} \in \Delta_1$ are the barycentres of the white and the black triangles Δ_0 and Δ_1, respectively. These points $B = \{b_1, b_1^*, \ldots, b_k, b_k^*\}$ will serve as base points in Σ.

We can now define the fundamental groupoid $\pi_{1,2k}(\Sigma, B)$ of Σ and explain how to define a holonomy map from this fundamental groupoid into $\mathrm{PSL}(2,\mathbb{C})$.

Definition 2.3.1. The fundamental groupoid $\pi_{1,2k}(\Sigma, B)$ is the fundamental groupoid associated to the topological surface Σ with base space $B = \{b_1, b_1^*, \ldots, b_k, b_k^*\}$, see Definition 2.1.8.

Note also that in our description of the gluing we have fixed, for each base point in B, a preferred germ of a chart. For any $b \in B$ this germ of a chart Φ^b is the germ of the map $\hat{\Phi}_j\colon P_j \longrightarrow \Delta$ described in Section 2.2.1. Let $\mathfrak{C} = \{\Phi^{b_1}, \Phi^{b_1^*}, \ldots, \Phi^{b_k^*}\}$. Then the discussion in Section 2.1.3.b shows that the groupoid holonomy transformation

$$\rho_{\underline{\mu}}^{\mathfrak{C}}\colon \pi_{1,2k}(\Sigma, B) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$$

is well defined and agrees with the standard definition of the holonomy map, when considering the restriction of it to any fundamental group $\pi_1(\Sigma, b) \subset \pi_{1,2k}(\Sigma, B)$ for any $b \in B$. So, from now on, we will denote both these maps as $\rho_{\underline{\mu}}$.

From now on, unless differently specified, the holonomy map will be the homomorphism of groupoids $\rho_{\underline{\mu}}\colon \pi_{1,2k}(\Sigma, B) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$. In addition, if $(X, B') \subset (\Sigma, B)$, where $B = \{b_1, b_1^*, \ldots, b_k, b_k^*\}$, we will consider often, with abuse of notation, $\rho_{\underline{\mu}}\colon \pi_1(X, B') \longrightarrow \mathrm{PSL}(2, \mathbb{C})$, omitting the composition with the injective map $\pi_1(X, B') \hookrightarrow \pi_{1,2k}(\Sigma, B)$.

Any curve $\gamma \in \pi_1(\Sigma)$ which intersects the pants curves $\sigma_1, \ldots, \sigma_\xi$ in total $q = q(\gamma)$ times passes through a sequence of pants $P_{i_1}, \ldots, P_{i_q} = P_{i_1}$ and can therefore be written as a product $\prod_{j=1}^{q} \kappa_j \vartheta_j$, where $\kappa_j \in \pi_1(P_{i_j}; b_{i_j})$ is a path in P_{i_j} with both its endpoints in the base point $b_{i_j} = b(P_{i_j})$ and $\vartheta_j = \vartheta(P_{i_j}, P_{i_{j+1}})$ is a path from b_{i_j} to $b_{i_{j+1}}$ across the boundary σ_{i_j} between P_{i_j} and $P_{i_{j+1}}$. We describe the holonomy of paths with endpoints in one pair of pants P in Section 2.3.2, and of paths with endpoints in two adjacent pair of pants P and P', in Section 2.3.3.

2.3.2 Paths in a pair of pants

In this section we consider a pair of pants $P_j = P$ with the base points $b_j = b$ and $b_j^* = b^*$, as described in Section 2.3, and we calculate the holonomy of the paths in P joining b and b^* and joining b to itself. As usual, we identify P with Δ so that the components of its boundary are labelled $0, 1, \infty$ in some order. Recall that in Section 2.2.3 we defined the three lines $\lambda_\epsilon \subset \mathbb{H}$ as the unique oriented geodesics going from $\epsilon + 1$ to $\epsilon + 2$, where ϵ is in the cyclically ordered set $\{0, 1, \infty\}$, see Figure 2.2. Orient each of the three boundary curves $\partial_\epsilon(P)$ consistently. We denote by $\upsilon_\epsilon \in \pi_1(P; b)$ the loop based at b and freely homotopic to the oriented loop $\partial_\epsilon(P)$. To calculate the holonomy of υ_ϵ, we begin by noting the holonomies of the three homotopically distinct paths γ_ϵ in P, with $\epsilon \in \{0, 1, \infty\}$, joining b to b^*, see Figure 2.6.

2.3.2.a The paths γ_ϵ

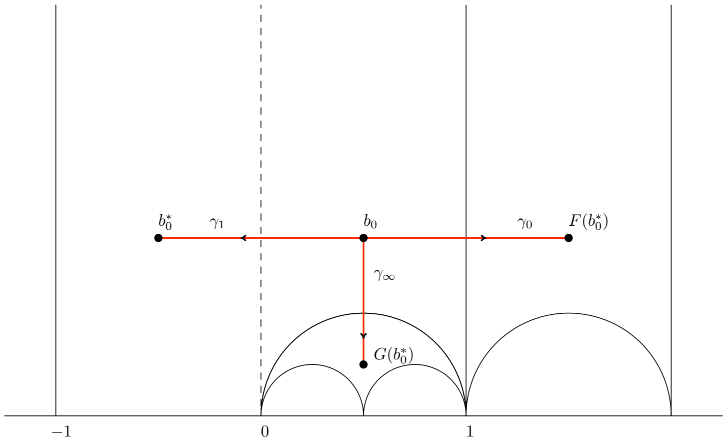

Figure 2.6: Paths between b_0 and b_0^* where $F = \rho(\gamma_0)$ and $G = \rho(\gamma_\infty)$.

Each path γ_ϵ is determined by the geodesic λ_ϵ which it crosses. Thus γ_0 connects b and b^* crossing λ_0, and so on. For any $\underline{\mu} \in \mathbb{H}^\xi$, let $\rho_{\underline{\mu}}\colon \pi_{1,2k}(\Sigma, B) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$.

Lemma 2.3.2. *For any $\underline{\mu} \in \mathbb{H}^\xi$, the holonomies of the above paths γ_ϵ, with $\epsilon \in \{0,1,\infty\}$, are:*

$$\rho_{\underline{\mu}}(\gamma_0) = \Xi_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}; \quad \rho_{\underline{\mu}}(\gamma_1) = \Xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \rho_{\underline{\mu}}(\gamma_\infty) = \Xi_\infty = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Proof. This result is clear from Figure 2.6, where you can see the lift $\hat{\gamma}_\epsilon$ of the paths $\Phi(\gamma_\epsilon) \subset \mathbb{P} = \mathbb{H}/\Gamma$ to $\mathbb{H}$, where $\Phi\colon P \longrightarrow \mathbb{P}$. In fact, $\hat{\gamma}_1$ connects b_0 in Δ_0 to b_0^* in Δ_1, so $\rho_{\underline{\mu}}(\gamma_1) = \mathrm{Id}$. On the other hand, $\hat{\gamma}_0$ connects $b_0 \in \Delta_0$ to $F(b^*) \in F(\Delta_1)$, where $F = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and, similarly $\hat{\gamma}_\infty$ connects $b_0 \in \Delta_0$ to $G(b^*) \in G(\Delta_1)$, where $G = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. So the result is proved. $\square$

2.3.2.b The paths υ_ϵ

In this section we describe the holonomy of the loop $\upsilon_\epsilon \in \pi_1(P; b)$, that is the loop based at b and freely homotopic to the oriented loop $\partial_\epsilon(P)$. To calculate the holonomy of υ_ϵ, we use the paths γ_ϵ discussed in Section 2.3.2.a. See Figure 2.7 for a picture of the loop υ_0 in P.

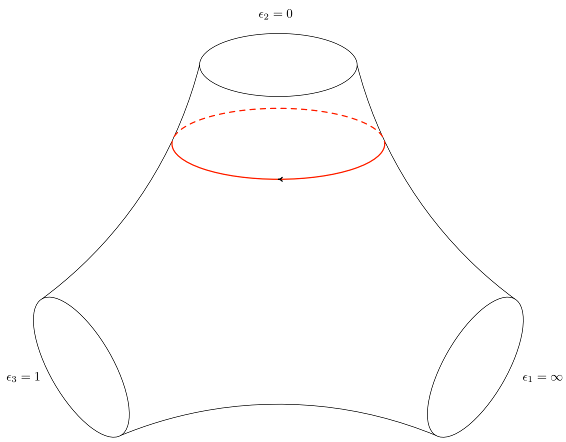

Figure 2.7: The loop υ_0 homotopic to $\partial_0 P$.

Lemma 2.3.3. *For any $\underline{\mu} \in \mathbb{H}^\xi$, the holonomy image of the loop $\upsilon_\epsilon \in \pi_1(P; b)$, where $\epsilon \in \{0, 1, \infty\}$, is the following:*

(i) $\rho_{\underline{\mu}}(\upsilon_0) = \Upsilon_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$;

(ii) $\rho_{\underline{\mu}}(\upsilon_1) = \Upsilon_1 = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$;

(iii) $\rho_{\underline{\mu}}(\upsilon_\infty) = \Upsilon_\infty = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

Proof. To calculate the holonomy $\rho_{\underline{\mu}}(\upsilon_0)$ of the loop υ_0 round the boundary $\partial_0 P$, we have to go from b to b^* crossing λ_∞ and then go from b^* to b crossing λ_1. Hence, as illustrated in Figure 2.7, we have to go along the path γ_∞ and then along the path γ_1^{-1}. Thus we find:

$$\rho_{\underline{\mu}}(\upsilon_0) = \rho_{\underline{\mu}}(\gamma_\infty \gamma_1^{-1}) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Similarly, to calculate the holonomy of $\rho_{\underline{\mu}}(\upsilon_1)$, we have to go from b and b^* crossing λ_0 and then return from b^* to b crossing λ_∞. This means going along γ_0 and then along γ_∞^{-1}. Thus the holonomy is:

$$\rho_{\underline{\mu}}(\upsilon_1) = \rho_{\underline{\mu}}(\gamma_0 \gamma_\infty^{-1}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}.$$

Finally, to calculate the holonomy of $\rho_{\underline{\mu}}(\upsilon_\infty)$, we have to go from b to b^* crossing

λ_1 and return from b^* to b crossing λ_0. Hence we have to go along the path γ_1 and then along the path γ_0^{-1}, so the holonomy is:

$$\rho_{\underline{\mu}}(\upsilon_\infty) = \rho_{\underline{\mu}}(\gamma_1\gamma_0^{-1}) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

□

As a check, we verify that

$$\rho_{\underline{\mu}}(\upsilon_0)\rho_{\underline{\mu}}(\upsilon_\infty)\rho_{\underline{\mu}}(\upsilon_1) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} = \mathrm{Id}$$

in accordance with the relation $\upsilon_0\upsilon_\infty\upsilon_1 = \mathrm{id}$ in $\pi_1(P;b)$.

Remark 2.3.4 (Holonomy of the *sbcc*–arc). The *sbcc*–arcs in the pair of pants P correspond to paths going around a certain boundary components of P. In particular, the *sbcc*–arcs starting at the boundary component $\partial_\epsilon(P)$ correspond (up to the change of orientation) to the path $\upsilon_{\epsilon+1}$ and, hence, its holonomy image is the matrix $\Upsilon_{\epsilon+1}$ or $\Upsilon_{\epsilon+1}^{-1}$.

2.3.3 Paths ϑ between adjacent pants

In this section we describe the holonomy of the path $\vartheta = \vartheta(P, P')$ between two pairs of pants, say P and P', with initial point $b = b(P) \subset \Delta_0(P)$ and ending point $b' = b(P') \subset \Delta_0(P')$. This path, together with the paths γ_ϵ and υ_ϵ, defines a set of generators for the fundamental groupoid $\pi_{1,2k}(\Sigma, B)$.

If $\partial_\epsilon P$ is glued to $\partial_{\epsilon'} P'$, then there is an obvious path $\vartheta = \vartheta(P, P'; \epsilon, \epsilon'; \sigma)$ on Σ from b to b' crossing the pants curve $\sigma = \sigma_i$. As discussed in Section 2.2.2, we will identify Σ with the quotient $\mathfrak{S}_{\underline{\mu}}/\sim$ defined in Section 2.2.1. In particular, $P \cup P'$ is identified with $S \sqcup S'/\sim$, where $S, S' \subset \mathbb{P}$ are the truncated surfaces defined in Section 2.2.1 and where $\sim$ is the equivalence relation given by the attaching maps along the annuli $\mathbb{A} \subset S$ and $\mathbb{A}' \subset S'$.

We can describe ϑ by defining, first, a path in $\mathbb{H}$ and a path in $\mathbb{H}'$, which project to a path in $S \sqcup S'/\sim$, see Section 2.2.1 and 2.2.2.

Recall that

$$H_\infty = H_\infty(\mu, \nu) = \{z \in \mathbb{H} | \frac{\Im\mu - \nu}{2} < \Im z < \frac{\Im\mu + \nu}{2}\} \subset \mathbb{H},$$

where $\mu = \mu_i \in \mathbb{H}$ is the gluing parameter associated to the pants curve $\sigma = \sigma_i$ and $\nu > 0$ is a small positive number, and $H_\epsilon = \Omega_\epsilon^{-1}(H_\infty)$, where the matrices Ω_ϵ were

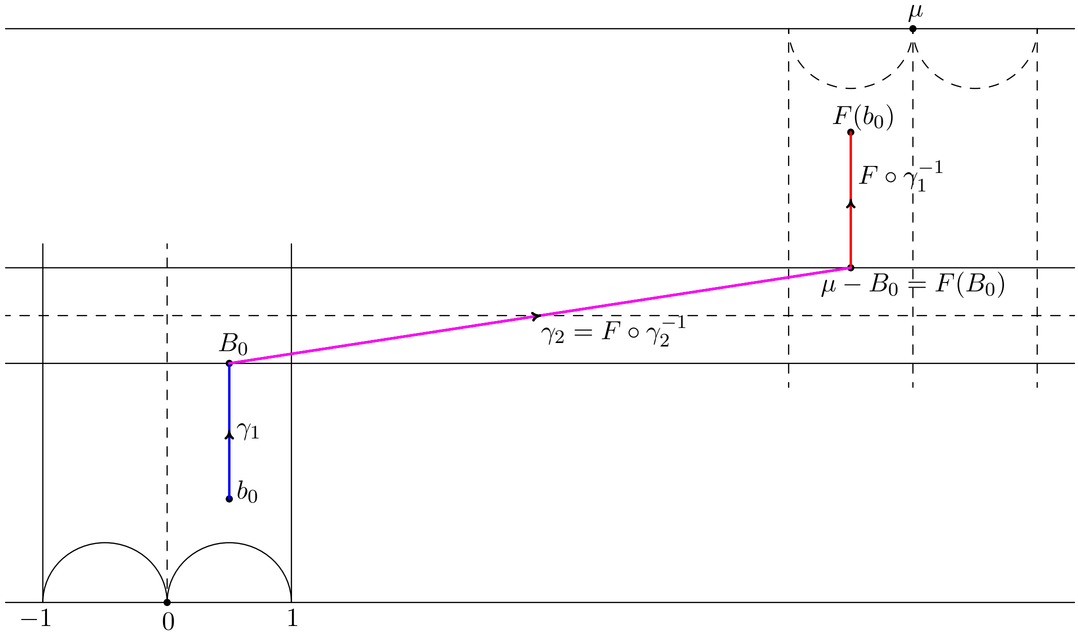

Figure 2.8: The path δ_1 between b_0 and B_0, the path $\delta_2 = F(\delta_2')^{-1}$ between B_0 and $\mu - B_0$ and the path $F(\delta_1^{-1})$ between $\mu - B_0$ and $F(b_0)$, where $F = J^{-1}T_\mu^{-1}$, in the case $H = H' = H_\infty$.

defined in Section 2.2.1. Recall also that $S \subset \mathbb{P}$ is the surface $\mathbb{P}$ with the projection of the horocyclic neighbourhood $\Omega_\epsilon^{-1}\left(\{z \in \mathbb{H} | \Im z \geq \frac{\Im\mu+\nu}{2}\}\right)$ of ϵ deleted.

Consider in $\mathbb{H}$ the straight lines $\delta_1 = \delta_{1,\infty}$ between $b_0 = \frac{1+i\sqrt{3}}{2}$ and $B_0 = \frac{1+i(\Im\mu-2\nu)}{2}$ and $\delta_2 = \delta_{2,\infty}$ between B_0 and $\mu - B_0 = \mu - \left(\frac{1+i(\Im\mu-2\nu)}{2}\right)$; see Figure 2.8. If we concatenate these two paths, we have a path $\delta_\infty = \delta_1 \cdot \delta_2$ in $\mathbb{H}$ which is contained in the complement of the open horoball $\{z \in \mathbb{H} | \Im z > \frac{\Im\mu+\nu}{2}\}$ around ∞. Similarly, let also $\delta_\epsilon = \Omega_\epsilon^{-1}(\delta_\infty)$ and $\delta_{i,\epsilon} = \Omega_\epsilon^{-1}(\delta_i)$ with $\epsilon = 0, 1, \infty$ and $i = 1, 2$. Define the path $\delta'_{\epsilon'}$ and $\delta'_{i,\epsilon'}$ in $\mathbb{H}'$ in a similar way, where $\epsilon' = 0, 1, \infty$ and $i = 1, 2$.

Let $\zeta\colon \mathbb{H} \longrightarrow \mathbb{H}/\Gamma$ and let $\zeta'\colon \mathbb{H} \longrightarrow \mathbb{H}/\Gamma$. Then the path δ_ϵ descends to a path $\delta_{\epsilon,S} = \zeta(\delta_\epsilon) \subset S \subset \mathbb{P}$ and the path δ'_ϵ descends to a path $\delta'_{\epsilon',S'} = \zeta'(\delta'_{\epsilon'}) \subset S' \subset \mathbb{P}'$. Let also $\delta_{i,\epsilon,S} = \zeta(\delta_{i,\epsilon}) \subset S \subset \mathbb{P}$ and $\delta'_{i,\epsilon',S'} = \zeta'(\delta'_{i,\epsilon'}) \subset S' \subset \mathbb{P}'$ with $\epsilon, \epsilon' = 0, 1, \infty$ and $i = 1, 2$.

Note that in the strip $H \sqcup H'/\sim$, where $H = H_\epsilon$ and $H' = H_{\epsilon'}$ we have that the projection of the path δ_2 and of the path $(\delta_2')^{-1}$ (that is, δ_2' with reversed orientation) coincide, see Section 2.2.2 for understanding the identification $\sim$ in detail. In fact, we have that

$$\delta_{2,\epsilon} = (\delta'_{2,\epsilon})^{-1} \subset H \sqcup H'/\sim,$$

as you can see from the fact that

$$\Omega_{\epsilon}^{-1}(B_0) = \Omega_{\epsilon'}^{-1}(\mu - B_0) \text{ and } \Omega_{\epsilon}^{-1}(\mu - B_0) = \Omega_{\epsilon'}^{-1}(B_0) \text{ in } H \sqcup H'/\sim .$$

This tells us that $\delta_{\epsilon,S} \cup (\delta'_{\epsilon',S'})^{-1}$ defines a path in $S \sqcup S'/\sim$.

With these definitions and using the notation of Section 2.2, we can see that $\vartheta(P, P'; \epsilon, \epsilon'; \sigma)$ is defined as the following union

$$\vartheta(P, P'; \epsilon, \epsilon'; \sigma) = \delta_{\epsilon,S} \cup (\delta'_{\epsilon',S'})^{-1} \subset S \sqcup S'/\sim .$$

Unless needed for clarity, we refer to all these paths as $\vartheta(P, P')$ or $\vartheta(P, P'; \epsilon, \epsilon')$.

For finding the holonomy and the developing image of $\vartheta(P, P')$, we need to glue δ_ϵ with $F \circ (\delta'_{\epsilon'})^{-1}$, where $F = \Omega_{\epsilon}^{-1} J^{-1} T_{\mu}^{-1} \Omega_{\epsilon'}$ is the transition function defined in Section 2.2.2. See Sections 2.1.3.a and 2.1.3.b for the definition of the developing map and of the groupoid holonomy map. Figure 2.8 shows the developing image $\mathrm{Dev}_{\underline{\mu}}(\vartheta(P, P'; \infty, \infty; \sigma_i))$ of $\vartheta(P, P'; \epsilon, \epsilon'; \sigma_i)$. Now referring to the gluing equation (2.3) and to the description of the groupoid holonomy map of Section 2.1.3.b, we see that the holonomy of $\vartheta(P, P')$ is given by the following formula:

$$\rho_{\underline{\mu}}\left(\vartheta(P, P'; \epsilon, \epsilon'; \sigma)\right) = \Theta_{\epsilon \longrightarrow \epsilon'} = \Omega_{\epsilon}^{-1} J^{-1} T_{\mu}^{-1} \Omega_{\epsilon'}, \tag{2.4}$$

since the path $\vartheta(P, P'; \epsilon, \epsilon'; \sigma)$ can be covered with only two charts and the only transition map is the map $\Omega_{\epsilon}^{-1} J^{-1} T_{\mu}^{-1} \Omega_{\epsilon'}$, where $\mu = \mu_i$.

Remark 2.3.5. Note that if you look at Figure 2.8 as a picture of two superimposed copies of $\mathbb{H}$, say $\mathbb{H}$ and $\mathbb{H}'$, as we did in Figure 2.4, then the point $B_0 \in \mathbb{H}$, which is the starting point of δ_2, and the point $\mu - B_0 = F(B_0) \in \mathbb{H}'$, which is the starting point of $F \circ \delta'_2$ (where $F = \Omega_{\epsilon}^{-1} J^{-1} T_{\mu}^{-1} \Omega_{\epsilon'}$), project to the same point in $S \sqcup S'/\sim$ under the two different charts ψ^1 and ψ^2 defined in Section 2.2.2 exactly when the surfaces S and S' are glued by a Fenchel–Nielsen twist $Tw_{\sigma_i, \Re\mu_i - 1}$. This agrees with the description of the marking we made in Section 2.2.3. In fact, in that section we first defined the marking for the surface $\Sigma(\underline{\mu}^0)$, where $\underline{\mu}^0 = (\mu_1^0, \ldots, \mu_\xi^0)$ is defined by $\Re\mu_i^0 = 1$, for all $i = 1, \ldots, \xi$, where the seams of S and the seams of S' match. Then, for defining the marking on the general surface $\Sigma(\underline{\mu})$, we use a Fenchel–Nielsen twist $Tw_{\sigma_i, \Re\mu_i - 1}$ on the annulus around $\sigma_i \in \mathcal{PC}$, for all $i = 1, \ldots, \xi$.

As already noted in Lemma 2.2.3, the gluing parameters μ are independent of the direction of travel (from P to P' or vice versa). From (2.4) we have

$$\rho_{\underline{\mu}}\left(\vartheta(P', P; \epsilon', \epsilon)\right) = \Omega_{\epsilon'}^{-1} J^{-1} T_{\mu}^{-1} \Omega_{\epsilon}$$

so that

$$\rho_{\underline{\mu}}(\vartheta(P', P; \epsilon', \epsilon)^{-1}) = \Omega_\epsilon^{-1} T_\mu J \Omega_{\epsilon'}.$$

Using the identities $J^{-1} = -J$, $T_\mu^{-1} = T_{-\mu}$ and $T_\mu J = J T_{-\mu}$ this gives

$$\rho_{\underline{\mu}}\left(\vartheta(P', P; \epsilon', \epsilon)^{-1}\right) = -\rho_{\underline{\mu}}\left(\lambda(P', P; \epsilon', \epsilon)\right)^{-1}, \tag{2.5}$$

as one would expect. That fact will be particularly important for our proof in Appendix B.

2.3.3.a Types of crossing

Table 2.1: The holonomy images $\Theta_{\epsilon \longrightarrow \epsilon'}$

Type of crossing	Matrix of crossing	Calculation
$0 \longrightarrow 0$	$\Omega_0^{-1} J^{-1} T_\mu^{-1} \Omega_0$	$\Theta_{0\longrightarrow 0} = i \begin{pmatrix} 1 & 0 \\ 2-\mu & -1 \end{pmatrix}$
$0 \longrightarrow 1$	$\Omega_0^{-1} J^{-1} T_\mu^{-1} \Omega_1$	$\Theta_{0\longrightarrow 1} = i \begin{pmatrix} 1 & -1 \\ 1-\mu & \mu-2 \end{pmatrix}$
$0 \longrightarrow \infty$	$\Omega_0^{-1} J^{-1} T_\mu^{-1} \Omega_\infty$	$\Theta_{0\longrightarrow \infty} = i \begin{pmatrix} 0 & 1 \\ 1 & 1-\mu \end{pmatrix}$
$1 \longrightarrow 0$	$\Omega_1^{-1} J^{-1} T_\mu^{-1} \Omega_0$	$\Theta_{1\longrightarrow 0} = i \begin{pmatrix} 2-\mu & -1 \\ 1-\mu & -1 \end{pmatrix}$
$1 \longrightarrow 1$	$\Omega_1^{-1} J^{-1} T_\mu^{-1} \Omega_1$	$\Theta_{1\longrightarrow 1} = i \begin{pmatrix} 1-\mu & \mu-2 \\ -\mu & \mu-1 \end{pmatrix}$
$1 \longrightarrow \infty$	$\Omega_1^{-1} J^{-1} T_\mu^{-1} \Omega_\infty$	$\Theta_{1\longrightarrow \infty} = i \begin{pmatrix} 1 & 1-\mu \\ 1 & -\mu \end{pmatrix}$
$\infty \longrightarrow 0$	$\Omega_\infty^{-1} J^{-1} T_\mu^{-1} \Omega_0$	$\Theta_{\infty\longrightarrow 0} = i \begin{pmatrix} 1-\mu & -1 \\ -1 & 0 \end{pmatrix}$
$\infty \longrightarrow 1$	$\Omega_\infty^{-1} J^{-1} T_\mu^{-1} \Omega_1$	$\Theta_{\infty\longrightarrow 1} = i \begin{pmatrix} -\mu & \mu-1 \\ -1 & 1 \end{pmatrix}$
$\infty \longrightarrow \infty$	$\Omega_\infty^{-1} J^{-1} T_\mu^{-1} \Omega_\infty$	$\Theta_{\infty\longrightarrow \infty} = i \begin{pmatrix} 1 & -\mu \\ 0 & -1 \end{pmatrix}$

In this section we describe all the types of crossing from one pair of pants to the next which can appear in our representation, that is we list the holonomy $\Omega_{\epsilon \longrightarrow \epsilon'}$ of the paths $\vartheta(P, P'; \epsilon, \epsilon')$. When a curve crosses a pants curve from the side labelled ϵ to the side labeled ϵ', where $\epsilon, \epsilon' \in \{0, 1, \infty\}$, we write $\epsilon \longrightarrow \epsilon'$. In Table 2.1 there is a summary of all the different matrices which describe these paths. Recall from Equation (2.4) that $\rho_{\underline{\mu}}(\vartheta(P, P'; \epsilon, \epsilon'; \sigma)) = \Theta_{\epsilon \longrightarrow \epsilon'} = \Omega_\epsilon^{-1} J^{-1} T_\mu^{-1} \Omega_{\epsilon'}$, where we defined the matrices J and T_μ in Equation (2.1) and the matrices Ω_ϵ in Equation (2.2).

2.3.4 Examples of calculations of paths

To conclude this section, we look at the special cases of the once punctured torus and the four times punctured sphere.

2.3.4.a The once punctured torus

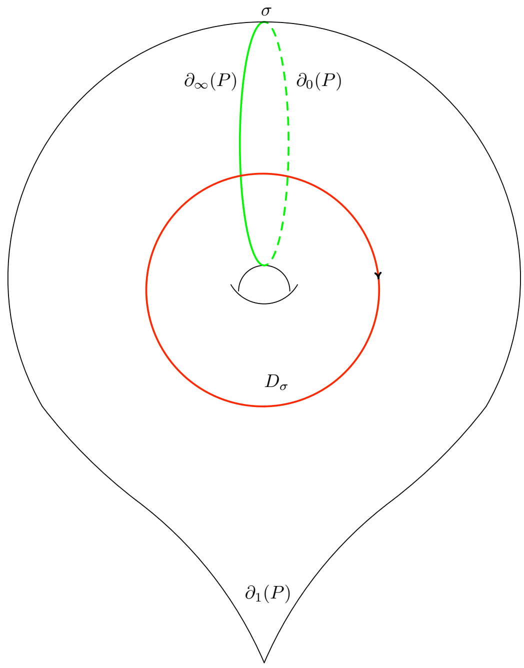

Figure 2.9: Gluing construction for the once punctured torus.

The once punctured torus $\Sigma_{1,1}$ is decomposed into one pair of pants by cutting along a single pants curve σ. To determine the projective structure on $\Sigma_{1,1}$ following our construction, we take a pair of pants P and glue the boundaries $\partial_\infty P$ and $\partial_0 P$, so that the remaining boundary $\partial_1 P$ becomes the puncture on $\Sigma_{1,1}$, see Figure 2.9. In this case $\underline{\mu} = \mu \in \mathbb{H}$ because $\xi(\Sigma_{1,1}) = 1$. To find the representation $\rho_\mu \colon \pi_1(\Sigma_{1,1}) \longrightarrow PSL(2, \mathbb{C})$, it is sufficient to compute the holonomy of σ and of its dual curve D_σ.

To do the gluing, take two copies of P and, following the notation in Section 2.2, label the copy on the left in the figure P, and that on the right, P'. We identify P with the standard triply punctured sphere $\mathbb{P}$ by the homeomorphism $\Phi \colon P \longrightarrow \mathbb{P}$, so that the universal covers $\tilde{P}, \tilde{P}'$ are identified with copies $\mathbb{H}, \mathbb{H}'$ of the upper half plane $\mathbb{H}$ with coordinates z, z' respectively. The cusps to be glued are labelled $\epsilon = \infty$ and $\epsilon' = 0$. We first apply the standard symmetries $\Omega_\epsilon, \Omega_{\epsilon'}$ which carry $\epsilon = \infty$ and

$\epsilon' = 0$ to ∞. Referring to (2.2), we see that $\Omega_\infty(z) = z$ and $\Omega_0(z') = 1 - \frac{1}{z'}$.

According to the choices made in Section 1.2.1.a, the dual curve D_σ to σ is the curve $\vartheta(P, P'; \infty, 0)$ joining $b(P) \in P$ to $b(P') \in P'$. By (2.7) in Section 2.3 and by the formulae (2.2) for the standard symmetries, we have:

$$\begin{aligned}\rho_\mu(D_\sigma) &= \Omega_\infty^{-1} J^{-1} T_\mu^{-1} \Omega_0 \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & -\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} \mu - 1 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

Since clearly we have $\rho_\mu(\sigma) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, this is enough to specify the representation $\rho_\mu\colon\ \pi_1(\Sigma_{1,1}) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$.

The original motivation for studying the representations in this paper came from the study of the Maskit embedding of the once punctured torus, see [24] and Section 2.2.5. The Maskit embedding for $\Sigma_{1,1}$ is described in [24] as the representation $\rho'_{\mu_K}\colon\ \pi_1(\Sigma_{1,1}) \longrightarrow PSL(2, \mathbb{C})$ given by

$$\rho'_{\mu_K}(\sigma) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho'_{\mu_{\mathrm{K}}}(\mathrm{D}_\sigma) = -\mathrm{i} \begin{pmatrix} \mu_K & 1 \\ 1 & 0 \end{pmatrix}.$$

This agrees with the above formula setting $\mu_K = \mu - 1$.

2.3.4.b The four holed sphere $\Sigma_{0,4}$

We decompose $\Sigma_{0,4}$ into two pairs of pants P and P' by cutting along the curve σ, and label the boundary components as shown in Figure 2.10, so that the boundaries to be glued are both labelled ∞. In the figure, P is the upper of the two pants and P' the lower one. We shall calculate the holonomy of the dual D_σ to σ in two different ways, first we put D_σ in the standard Penner and Harer position and second we put it in the symmetrical D. Thurston's position. As it is to be expected, the two calculations give the same result. Also in this case, $\underline{\mu} = \mu \in \mathbb{H}$ because $\xi(\Sigma_{0,4}) = 1$.

The loop D_σ in Penner and Harer standard position. If we put the loop D_σ in Penner and Harer standard position, as illustrated in Figure 1.5, and as described in Section 1.2.2, we see that it is the concatenation of the paths:

1. $\vartheta(P, P'; \infty, \infty)$ from $b_0(P)$ to $b_0(P')$;
2. $\upsilon_0(P')$;

3. $\upsilon_\infty(P')$;

4. $\vartheta(P', P; \infty, \infty)$ from $b_0(P')$ to $b_0(P)$;

5. $\upsilon_0^{-1}(P)$.

Thus, using the calculations in Sections 2.3.3 and 2.3.2, we have

$$\begin{aligned}\rho_\mu(D_\sigma) &= \rho_\mu\left(\vartheta(P, P'; \infty, \infty) \cdot \upsilon_0(P') \cdot \upsilon_\infty(P') \cdot \vartheta(P', P; \infty, \infty) \cdot \upsilon_0^{-1}(P)\right) \\ &= \begin{pmatrix} i & -i\mu \\ 0 & -i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -i & i\mu \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4\mu^2 + 6\mu - 3 & 2\mu^2 - 4\mu + 2 \\ -4\mu + 4 & 2\mu - 3 \end{pmatrix},\end{aligned}$$

giving

$$\operatorname{Tr} \rho_\mu(D_\sigma) = -4\mu^2 + 8\mu - 6.$$

Now $q(D_\sigma) = 2$ and $p(D_\sigma) = 0$ (see Section 1.2.1), and the number h of *sbcc*–arcs in D_σ is 2. Thus, as Theorem A predicts, we have that

$$\operatorname{Tr} \rho(D_\sigma) = \pm i^2 2^2 (\mu + (0 - 2)/2)^2 + R,$$

where R represents terms of degree at most 0 in μ. So our result is in accordance with the statement of Theorem A.

The loop D_σ in symmetrical D. Thurston standard position. As usual, we take as base points the barycentres $b(P)$ and $b^*(P)$ of the 'white' and the 'black' hexagons respectively in P and the same base points $b(P')$ and $b^*(P')$ in P'. Also denote $\vartheta^*(P, P'; \infty, \infty)$ the path $\mathbf{R}(\vartheta(P, P'; \infty, \infty))$ from $b_0^*(P)$ to $b_0^*(P')$ through the black hexagons, where $\mathbf{R}$ is the orientation reversing symmetry of $\Sigma(\mu)$ as defined in Section 1.2.1.

From Figure 2.10, we see that D_σ is the concatenation of the paths:

1. $\vartheta(P, P') = \vartheta(P, P'; \infty, \infty)$ from $b_0(P)$ to $b_0(P')$;

2. $\gamma_\infty(P')$ in P' from $b_0(P')$ to $b_0^*(P')$;

3. $\vartheta^*(P, P') = \vartheta^*(P', P; \infty, \infty)$ from $b_0^*(P')$ to $b_0^*(P)$;

4. $\gamma_\infty^{-1}(P)$ in P from $b_0^*(P)$ to $b_0(P)$.

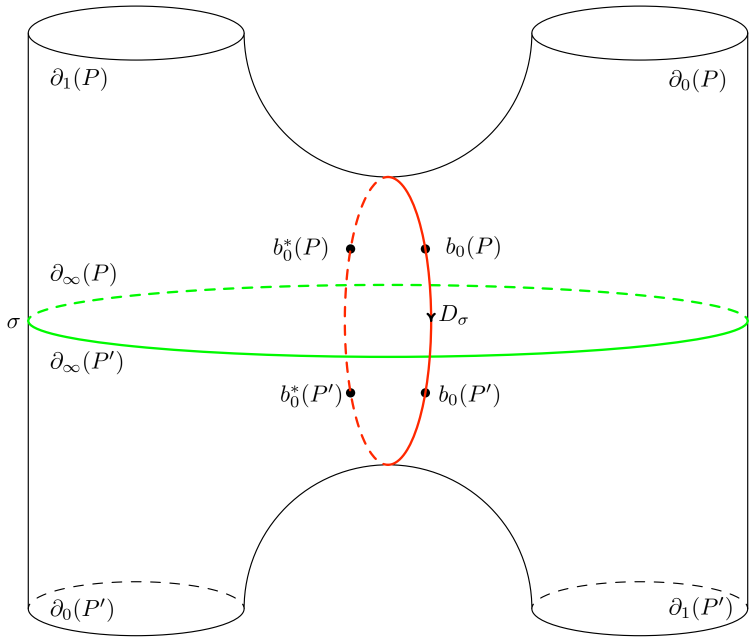

Figure 2.10: The loop D_σ in its symmetrical DT–position in the four holed sphere.

Thus

$$\rho_\mu(D_\sigma) = \rho_\mu\left(\vartheta(P, P';)\right) \cdot \rho_\mu\left(\gamma_\infty(P)\right) \cdot \rho_\mu\left(\vartheta^*(P', P)\right) \cdot \rho_\mu\left(\gamma_\infty^{-1}(P)\right).$$

Following Remark 2.2.1 and Equation (2.5), we have:

$$\rho_\mu\left(\vartheta^*(P', P)\right) = \rho_{\mu-2}\left(\vartheta(P', P)\right) = \rho_{\mu-2}\left(\vartheta^{-1}(P, P')\right) = -\rho_{\mu-2}\left(\vartheta(P, P')\right).$$

So, from the discussion in Section 2.3.3, we have $\rho_\mu\left(\vartheta^*(P', P)\right) = \begin{pmatrix} -i & i(\mu-2) \\ 0 & i \end{pmatrix}$. Thus, referring also to the calculations of Section 2.3.2, we see that

$$\begin{aligned} \rho_\mu(D_\sigma) &= \begin{pmatrix} i & -i\mu \\ 0 & -i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -i & i(\mu-2) \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -4\mu^2+6\mu-3 & 2\mu^2-4\mu+2 \\ -4\mu+4 & 2\mu-3 \end{pmatrix}. \end{aligned}$$

Hence $\mathrm{Tr}\left(\rho_\mu(D_\sigma)\right) = -4\mu^2 + 8\mu - 6$ as before.

2.4 Proof of Top Terms' Relationship

In this final section we discuss the Top Terms' Relationship, which is stated in the Introduction as Theorem A. In the statement of Theorem A we use the notion of *sbcc*–arcs; see Section 1.1 for the definition.

Theorem 2.4.1. *Let γ be a connected simple closed curve on Σ and let $\underline{\mu} = (\mu_1, \dots, \mu_\xi) \in \mathbb{H}^\xi$. If γ is not parallel to any of the pants curves σ_i, then $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma)$ is a polynomial in $\mu_1, \cdots, \mu_\xi$ whose top terms are given by:*

$$\begin{aligned}\operatorname{Tr}\rho_{\underline{\mu}}(\gamma) &= \pm i^q 2^h \Big(\mu_1 + \frac{(p_1 - q_1)}{q_1}\Big)^{q_1} \cdots \Big(\mu_\xi + \frac{(p_\xi - q_\xi)}{q_\xi}\Big)^{q_\xi} + R,\\ &= \pm i^q 2^h \left(\mu_1^{q_1} \cdots \mu_\xi^{q_\xi} + \sum_{i=1}^{\xi} (p_i - q_i)\mu_1^{q_1} \cdots \mu_i^{q_i - 1} \cdots \mu_\xi^{q_\xi}\right) + R,\end{aligned}$$

where:

- $q = \sum_{i=1}^{\xi} q_i > 0$;
- *$h = h(\gamma)$ is the total number of sbcc–arcs in $\gamma \setminus \cup_{i=1}^{\xi}(\gamma \cap \sigma_i)$ in the standard representation of γ relative to $\mathcal{P}$;*
- *R represents terms with total degree in $\mu_1 \cdots \mu_\xi$ at most $q - 2$ and of degree at most q_i in the variable μ_i.*

If $q = 0$, then $\gamma = \sigma_i$ for some i, $\rho_{\underline{\mu}}(\gamma)$ is parabolic, and $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma) = \pm 2$.

In [28] we gave a combinatorial proof of this theorem that we include here in Appendix B. Before finding that proof, we tried to prove the result using induction on the total intersection number $q = \sum_{i=1}^{\xi} q_i > 0$ of the curve γ with the pants curves and a particular decomposition of the trace $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma)$ into 'base blocks'. Since the idea of the combinatorial proof in Appendix B was Series', we decided to discuss in this chapter only the calculations which inspired that proof, see Section 2.4.1. These calculations correspond to the step cases of the inductive proof we wanted to use. Unfortunately, the method breaks down because the Dehn–Thurston coordinates (using D. Thurston's twist) are not well defined for arcs between the base points in B. In fact, if γ_{i_j} is an arc crossing a pants curve $\sigma = \partial_\epsilon P \cap \partial_{\epsilon'} P'$, we need to know the boundary component across which γ_{i_j} entered P and the one across which it left P' in order to define $p(\gamma_{i_j})$. So the proof reduces to the one described in the Apppendix. We give a sketch of this proof in Section 2.4.2.

2.4.0.c Generators of $\pi_{1,2k}(\Sigma, B)$ and degree of the trace polynomial

Recall that we consider each curve $\gamma \in \pi_1(\Sigma)$ as a product of elements of the fundamental groupoid $\pi_{1,2k}(\Sigma, B)$. Since we think it could help the reader, we recall here the different generators of $\pi_{1,2k}(\Sigma, B)$, specifying for each one the holonomy image and the section where we have defined them:

- The path $\gamma_\epsilon \in P$ which joins b and b^* in P crossing $\lambda_\epsilon(P)$. Its holonomy image is $\rho_{\underline{\mu}}(\gamma_\epsilon) = \Xi_\epsilon$, see Section 2.3.2.a for the definition.
- The path $\upsilon_\epsilon \in P$ which goes around the boundary component $\partial_\epsilon(P)$. Its holonomy image is $\rho_{\underline{\mu}}(\upsilon_\epsilon) = \Upsilon_\epsilon$, see Section 2.3.2.b for the definition.
- The path $\vartheta(P, P'; \epsilon, \epsilon')$ going from $b(P)$, in the pair of pants P, to $b(P')$, in the pair of pants P'. Its holonomy image is $\rho_{\underline{\mu}}(\vartheta(P, P'; \epsilon, \epsilon')) = \Theta_{\epsilon \longrightarrow \epsilon'} = \Omega_\epsilon^{-1} J^{-1} T_\mu^{-1} \Omega_{\epsilon'}$, see Section 2.3.3 for the definition.

Recall that the definition of the map Ω_ϵ is given in Equation (2.2), while the definitions of the map J and T_μ are given in Equation (2.1). Recall also that the matrices $\Theta_{\epsilon \longrightarrow \epsilon'}$ are given in Table 2.1, and that, as underlined in Remark 2.3.4, an *sbcc*–arc in the pair of pants P starting at the boundary component $\partial_\epsilon(P)$ corresponds (up to the change of orientation) to the path $\upsilon_{\epsilon+1}^{\pm}$ and, hence, its holonomy image is the matrix $\Upsilon_{\epsilon+1}$ or $\Upsilon_{\epsilon+1}^{-1}$.

Now suppose that γ is a curve on Σ. Although not logically necessary, we can greatly simplify our description by arranging γ in standard Penner and Harer position, so that it always passes from one pants to the next through the white hexagons $\Delta_0(P_j)$. Suppose, as in Section 2.2.2, that γ passes through a sequence of pants $P_{i_1}, \dots, P_{i_n}$. We may as well assume that γ starts at the base point $b(P_{i_1})$ of P_{i_1}. Given our identification Φ_{i_1} of P_{i_1} with $\mathbb{P}$, there is a unique lift $\tilde{b}(P_{i_1})$ of $b(P_{i_1})$ to Δ_0 and hence there is a unique lift $\tilde{\gamma}$ of $\gamma \cap P_{i_1}$ to $\mathbb{H}$ starting at $\tilde{b}(P_{i_1})$. This path exits Δ_0 either across one of its three sides, or across that part of a horocycle which surrounds one of the three cusps $0, 1, \infty$ contained in Δ_0. In the first case, the holonomy is given by the usual action of the group Γ on $\mathbb{H}$, where Γ is the triply punctured sphere group as in Section 2.2.1. (This is explained in detail in Section 2.3.2.) In the second case, we have a precise description of the gluing across the boundary annuli, giving a unique way to continue $\tilde{\gamma}$ into a lift of P_{i_2}. In this case we continue in a new chart in which the lift of P_{i_2} is identified with $\Delta \subset \mathbb{H}$, as described before.

The following result applies to an arbitrary connected loop on Σ.

Proposition 2.4.2. *Let $\gamma \in \pi_1(\Sigma)$ be a (connected) curve and suppose that the total intersection number of γ with the pants curves σ_i is $\sum_i i(\gamma, \sigma_i) = q$. Then for any $\underline{\mu} = (\mu_1, \dots, \mu_\xi) \in \mathbb{H}^\xi$, the trace $\operatorname{Tr} \rho_{\underline{\mu}}([\gamma])$ is a polynomial $p = p(\underline{\mu}) \in \mathbb{C}[\mu_1, \dots, \mu_\xi]$ in $\mu_1, \dots, \mu_\xi$ of maximal total degree q and of maximal degree $q_k(\gamma) = i(\gamma, \sigma_k)$ in the parameter μ_k.*

Proof. Suppose the boundary $\partial_\epsilon P$ of one pair of pants P is glued to the boundary $\partial_{\epsilon'} P'$ of another pair P' along a pants curve σ. As before let's denote $\hat{\Phi} \colon P \longrightarrow \Delta \subset \mathbb{H}$ and $\hat{\Phi}' \colon P' \longrightarrow \Delta' \subset \mathbb{H}'$ the identifications of P and P' with Δ and Δ' respectively. Then the map $\hat{\Phi} \circ (\hat{\Phi}')^{-1} \colon \Delta' \longrightarrow \Delta$, which glues the horocycle labelled ϵ' in Δ'_0 to the horocycle labelled ϵ in Δ_0, is $\Omega_\epsilon^{-1} J^{-1} T_\mu^{-1} \Omega_{\epsilon'}$, where, as usual, the maps Ω_ϵ and $\Omega_{\epsilon'}$ are the standard maps taking ϵ and ϵ' to ∞, see Equation 2.2 for the definition of the matrices Ω_ϵ. This fact is explained in Section 2.2; in particular, see Equation 2.3 and Figure 2.3. Thus, with the notation of Section 2.2.2, the overlap map $R = \hat{\Phi} \circ (\hat{\Phi}')^{-1}$ is

$$\Omega_\epsilon^{-1} J^{-1} T_\mu^{-1} \Omega_{\epsilon'}. \tag{2.6}$$

Any curve $\gamma \in \pi_1(\Sigma)$ which intersects the pants curves $\sigma_1, \dots, \sigma_\xi$ in total q times passes through a sequence of pants $P_{i_1}, \dots, P_{i_q} = P_{i_1}$ and can therefore be written as a product $\prod_{j=1}^q \kappa_j \vartheta_j$ where $\kappa_j \in \pi_1(P_{i_j}; b_{i_j})$ is a path in P_{i_j} with both its endpoints in the base point $b_{i_j} = b(P_{i_j})$ and $\vartheta_j = \vartheta(P_{i_j}, P_{i_{j+1}})$ is a path from b_{i_j} to $b_{i_{j+1}}$ across the boundary σ_{i_j} between P_{i_j} and $P_{i_{j+1}}$. See Section 2.3.2 and 2.3.3 for a detailed description of these paths. In particular, we have $\rho_{\underline{\mu}}(\vartheta_j) = \Omega_{\epsilon_j}^{-1} J^{-1} T_{\mu_{i_j}}^{-1} \Omega'_{\epsilon_{j+1}}$.

It follows that the holonomy of γ is a product

$$\rho_{\underline{\mu}}([\gamma]) = \prod_{j=1}^q \rho_{\underline{\mu}}(\kappa_j) \Omega_{\epsilon_j}^{-1} J^{-1} T_{\mu_{i_j}}^{-1} \Omega'_{\epsilon_{j+1}}, \tag{2.7}$$

from which the result follows. □

It is clear from this formula that $\operatorname{Tr} \rho_{\underline{\mu}}([\gamma])$ is an invariant of the free homotopy class of a closed curve $\gamma \in \pi_1(\Sigma)$, because changing the base point of the path γ changes the above product by conjugation, as already discussed in Section 2.1.2.

2.4.0.d Independence from the labelling

In this section we want to prove that the trace $\operatorname{Tr} \rho_{\underline{\mu}}(\gamma)$ doesn't depend on the labelling chosen for the boundary components of the pairs of pants P_j.

The idea behind the independence from the labelling is the following. If you write $\rho_{\underline{\mu}}(\gamma)$ as a product of matrices coming from the images of the generators of the fundamental groupoid $\pi_{1,2k}(\Sigma, B)$ for a particular choice of labelling, then you can get the matrix decomposition of a new one just by adding blocks of type $\Omega_\epsilon\Omega_\epsilon^{-1}$, as we are about to explain. This adding operation, of course, does not change the trace which we are calculating.

Lemma 2.4.3. *The trace is independent from the labelling $\{0, 1, \infty\}$ chosen for the boundary components of the pairs of pants in the pants decompositions $\mathcal{P}$.*

Proof. We consider a path in P entering from the boundary $\partial_\epsilon(P)$ and exiting from the boundary $\partial_{\epsilon'}(P)$. We want to prove that, if we change the labelling in P, than the holonomy of γ remains the same. The holonomy of γ (while crossing from $\partial_\epsilon(P)$ to $\partial_\epsilon(P'))$ depends on ϵ and ϵ' only for the following pieces:

- $\Omega_\epsilon\Upsilon^\alpha_{\epsilon'}\Omega^{-1}_{\epsilon'}$, if $\epsilon \neq \epsilon'$; or
- $\Omega_\epsilon\Upsilon_{\epsilon+1}\Upsilon^\alpha_\epsilon\Omega^{-1}_\epsilon$, if $\epsilon = \epsilon'$ and, hence, the paths create an *sbcc*–arc based at $\partial_\epsilon(P)$,

where $\alpha \in \mathbb{Z}$ is the PH–twist of this piece of path.

So we want to prove that

- $\cdots\Omega_\epsilon\Upsilon^\alpha_{\epsilon'}\Omega^{-1}_{\epsilon'}\cdots = \pm\cdots\Omega_\varpi\Upsilon^\alpha_{\varpi'}\Omega^{-1}_{\varpi'}\cdots$, where $\epsilon, \epsilon', \varpi, \varpi' \in \{0, 1, \infty\}$ and $\epsilon \neq \varpi$, $\epsilon \neq \epsilon'$, $\varpi \neq \varpi'$; or
- $\cdots\Omega_\epsilon\Upsilon_{\epsilon+1}\Upsilon^\alpha_\epsilon\Omega^{-1}_\epsilon\cdots = \pm\cdots\Omega_\varpi\Upsilon_{\varpi+1}\Upsilon^\alpha_\varpi\Omega^{-1}_\varpi\cdots$, where $\epsilon, \varpi \in \{0, 1, \infty\}$ and $\epsilon \neq \varpi$.

We discuss the first situation in the case $\epsilon = 0, \epsilon' = 1$ and $\varpi = 1$. In this situation we have $\varpi' = \infty$. Since

$$\Omega_0\Upsilon^\alpha_1\Omega^{-1}_1 = \Omega_1\Omega^{-1}_1\Omega_0\Upsilon^\alpha_1\Omega^{-1}_1\Omega_\infty\Omega^{-1}_\infty,$$

then we only need to prove that

$$\Omega^{-1}_1\Omega_0\Upsilon^\alpha_1\Omega^{-1}_1\Omega_\infty = \pm\Upsilon^\alpha_\infty.$$

We have that:

$$\begin{aligned}\Omega_1\Omega_0^{-1}\Upsilon^\alpha_1\Omega_1\Omega^{-1}_\infty &= (-1)^\alpha\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2\alpha+1 & -2\alpha\\ 2\alpha & -2\alpha+1\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}\\ &= (-1)^\alpha\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 2\alpha\\ -1 & -1\end{pmatrix}\end{aligned}$$

$$= (-1)^{\alpha+1} \begin{pmatrix} 1 & -2\alpha \\ 0 & 1 \end{pmatrix} = \pm \Upsilon_\infty^\alpha,$$

as we wanted to prove. The other cases can be analysed in a similar way. □

2.4.1 Case of simple closed curves

In Section 2.4.1.a and Section 2.4.1.b we prove Theorem 2.4.1 for simple closed curves $\gamma \in \pi_1(\Sigma)$ with $q(\gamma) = 1$ and $q(\gamma) = 2$, respectively. We do our calculations for a particular choice of labelling of the pants curves, but our result does not depend on the choice, as proved in Lemma 2.4.3.

2.4.1.a Case $q = 1$

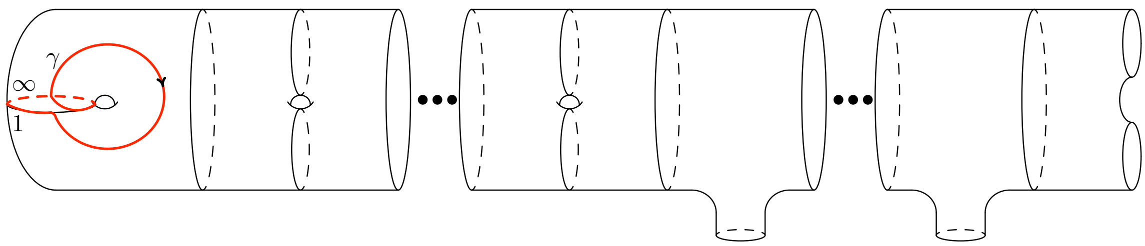

Figure 2.11: Case $q = 1$

Let γ be a curve such that $q = \sum_{i=1}^{\xi} q_i = 1$, then, without loss of generality, we can suppose $q = q_i = 1$. In this case the curve γ is contained in the modular surface $\mathrm{MS}(\sigma_i)$ associated to the pants curve σ_i and $\mathrm{MS}(\sigma_i)$ is a once holed torus, see Section 1.1 for the definition of modular surface. Let's choose the labelling of Figure 2.11. We can do that because of Lemma 2.4.3. Let, as usual, $\underline{\mu}$ be a vector in $\mathbb{H}^\xi$. Suppose that $\gamma = \upsilon_1^{-\alpha}\vartheta(P, P; 1, \infty; \sigma_i)$, where $P = \mathrm{MS}(\sigma_i)$. With this assumption, Penner and Harer's twist is $\hat{p}_i = \alpha$.

Using the results of Section 2.3.2, we have the following:

$$\begin{aligned}
\rho_{\underline{\mu}}(\gamma) &= \Upsilon_1^{-\alpha}\Theta_{1\longrightarrow\infty} \\
&= i(-1)^\alpha \begin{pmatrix} -2\alpha+1 & 2\alpha \\ -2\alpha & 2\alpha+1 \end{pmatrix} \begin{pmatrix} 1 & 1-\mu_i \\ 1 & -\mu_i \end{pmatrix} \\
&= i(-1)^\alpha \begin{pmatrix} 1 & -\mu_i - 2\alpha + 1 \\ 1 & -\mu_i - 2\alpha \end{pmatrix}.
\end{aligned}$$

So

$$\mathrm{Tr}\left(\rho_{\underline{\mu}}(\gamma)\right) = i(-1)^{-\alpha+1}(\mu_i + 2\alpha - 1).$$

The relationship between Penner and Harer's definition and D. Thurston's one, tells us that $\hat{p}_i = \frac{p_i+1-1}{2}$, see Theorem 1.2.6. Using this fact and the fact that $\hat{p}_i = \alpha$, we can see that the DT–twist is $p_i = 2\alpha$. In addition, the total number of *sbcc*–arcs is $h = 0$. So we have:

$$\mathrm{Tr}\left(\rho_{\underline{\mu}}(\gamma)\right) = \pm i^q 2^h (\mu_i + p_i - q_i),$$

as we wanted to prove.

2.4.1.b Case $q = 2$

Let γ be a curve such that $q = \sum_{i=1}^{\xi} q_i = 2$, then there are three possibilities:

(i) γ meets two different pants curves (suppose $q = q_i + q_j = 2$);

(ii) γ meets only one pants curve which is the boundary of two different pairs of pants (suppose $q = q_i = 2$ and $\mathrm{MS}(\sigma_i) \cong \Sigma_{0,4}$);

(iii) γ meets only one pants curve which is the boundary of only one pair of pants (suppose $q = q_i = 2$ and $\mathrm{MS}(\sigma_i) \cong \Sigma_{1,1}$).

We prove these cases using the trace relations of Appendix A. Let $\underline{\mu} \in \mathbb{H}^{\xi}$.

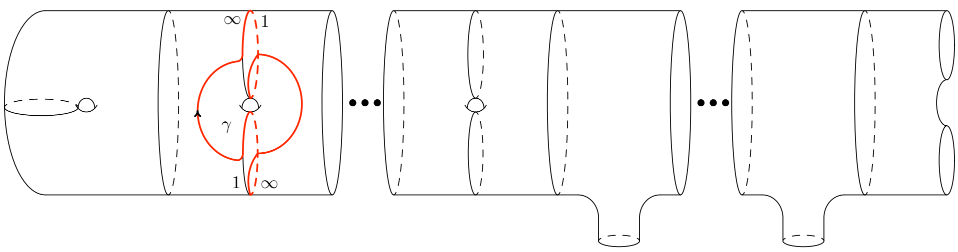

Figure 2.12: Case $q = 2$ (i)

Case $q = 2$ (i) We have to prove Theorem 2.4.1 for simple closed curves γ with $q(\gamma) = 2$ and which meet two different pants curves. In Figure 2.12 you can see a curve γ that meets two different pants curves. Without loss of generality, we can suppose $q = q_i + q_j = 2$. In this case, using Lemma 2.4.3, let's choose the labelling of Figure 2.12. Suppose that $\mathrm{MS}(\sigma_i) = P \cup P'$, and assume

$$\gamma = \upsilon_{\infty}^{-\alpha}(P)\vartheta(P, P'; \infty, 1; \sigma_i)\upsilon_{\infty}^{-\alpha}(P')\vartheta(P', P; \infty, 1; \sigma_j).$$

In this case we have that the PH–twist are $\hat{p}_i = \alpha$ and $\hat{p}_j = \beta$.

Using the results of Section 2.3.2, we have the following:

$$\rho_{\underline{\mu}}(\gamma) = \Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow 1}(\mu_i)\Upsilon_\infty^{-\beta}\Theta_{\infty\longrightarrow 1}(\mu_j),$$

where $\Theta_{\infty\longrightarrow 1}(\mu_i) = \begin{pmatrix} -\mu_i & \mu_i - 1 \\ -1 & 1 \end{pmatrix}$ and $\Theta_{\infty\longrightarrow 1}(\mu_j) = \begin{pmatrix} -\mu_j & \mu_j - 1 \\ -1 & 1 \end{pmatrix}$. Using Proposition A.1, we have that

$$\mathrm{Tr}(\rho_{\underline{\mu}}(\gamma)) = \mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(C),$$

where

$$A = \Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow 1}(\mu_i), \quad B = \Upsilon_\infty^{-\beta}\Theta_{\infty\longrightarrow 1}(\mu_j)$$

and

$$C = AB^{-1} = \Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow 1}(\mu_i)\Theta^{-1}_{\infty\longrightarrow 1}(\mu_j)\Upsilon_\infty^{\beta}.$$

Then we have:

$$\begin{aligned} A &= \Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow 1}(\mu_i) \\ &= \begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\mu_i & \mu_i - 1 \\ -1 & 1 \end{pmatrix} \\ &= i\begin{pmatrix} -\mu_i - 2\alpha & \mu_i + 2\alpha - 1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

So $\mathrm{Tr}(A) = -i(\mu_i + 2\alpha - 1)$. Similarly $\mathrm{Tr}(B) = -i(\mu_j + 2\beta - 1)$. Finally, the matrix C satisfies:

$$\begin{aligned} C &= \Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow 1}(\mu_i)\Theta^{-1}_{\infty\longrightarrow 1}(\mu_j)\Upsilon_\infty^{\beta} \\ &= -\begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\mu_i & \mu_i - 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 - \mu_j \\ 1 & -\mu_j \end{pmatrix}\begin{pmatrix} 1 & -2\beta \\ 0 & 1 \end{pmatrix} \\ &= -\begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \mu_i - \mu_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2\beta \\ 0 & 1 \end{pmatrix} \\ &= -\begin{pmatrix} 1 & \mu_i - \mu_j + 2\alpha - 2\beta \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence we have:

$$
\begin{aligned}
\operatorname{Tr}(\rho_{\underline{\mu}}(\gamma)) &= \operatorname{Tr}(A)\operatorname{Tr}(B) - \operatorname{Tr}(C) \\
&= \operatorname{Tr}(A)\operatorname{Tr}(B) + R \\
&= -[\mu_i\mu_j + (2\beta-1)\mu_i + (2\alpha-1)\mu_j] + R,
\end{aligned}
$$

where R represents terms with total degree 0. Using Theorem 1.2.6, we have that the DT–twists are $p_i = 2\alpha$ and $p_j = 2\beta$. So we have $p_i - q_i = 2\alpha - 1$ and $p_j - q_j = 2\beta - 1$. Since the total number of *sbcc*–arcs is 0, we have that

$$
\operatorname{Tr}(\rho_{\underline{\mu}}(\gamma)) = \pm i^q 2^h \left(\mu_i\mu_j + (p_j - q_j)\mu_i + (p_i - q_i)\mu_j\right) + R,
$$

as we wanted to prove.

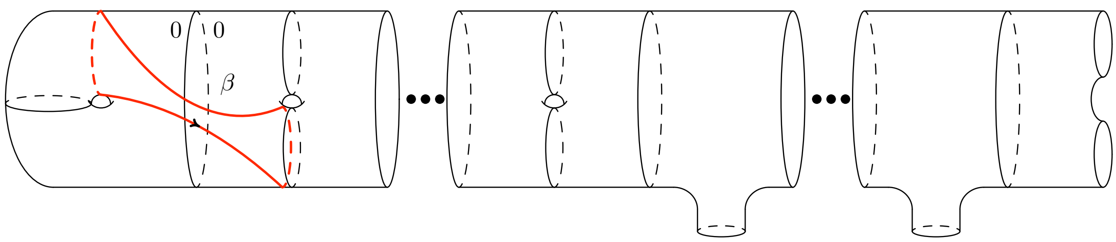

Figure 2.13: Case $q = 2$ (ii)

Case $q = 2$ **(ii)** We have to prove Theorem 2.4.1 for simple closed curves γ with $q(\gamma) = 2$ and which meet only one pants curve which is the boundary of two different pair of pants. In that case, without loss of generality, we can suppose $q = q_i = 2$ and $\mathrm{MS}(\sigma_i) \cong \Sigma_{0,4}$, where $\mathrm{MS}(\sigma_i)$ is the modular surface associated to σ_i. In this case let's choose the labelling of Figure 2.13, thanks to Lemma 2.4.3.

Suppose that $\mathrm{MS}(\sigma_i) = P \cup P'$, and let γ be

$$
\gamma = \upsilon_1^{\pm}(P)\upsilon_0^{-\alpha}(P)\vartheta(P, P'; 0, 0; \sigma_i)\upsilon_1^{\pm}(P')\upsilon_0^{-\beta}(P')\vartheta(P', P; 0, 0; \sigma_i).
$$

In this situation we have that the PH–twist is $\hat{p}_i = \alpha + \beta$.

Using the calculations of Section 2.3.2, we have the following:

$$
\rho_{\underline{\mu}}(\gamma) = \Upsilon_1^{\pm}\Upsilon_0^{-\alpha}\Theta_{0\longrightarrow 0}\Upsilon_1^{\pm}\Upsilon_0^{-\beta}\Theta_{0\longrightarrow 0}.
$$

Note that the matrices $\Upsilon_1^{\pm}$ correspond to the *sbcc*–arcs starting at $\partial_0(P)$ and $\partial_0(P')$.

We do the calculation in the case $\rho_{\underline{\mu}}(\gamma) = \Upsilon_1\Upsilon_0^{-\alpha}\Theta_{0\longrightarrow 0}\Upsilon_1^{-1}\Upsilon_0^{-\beta}\Theta_{0\longrightarrow 0}$, but the other cases are similar. Using Proposition A.1, we have that

$$\operatorname{Tr}(\rho_{\underline{\mu}}(\gamma)) = \operatorname{Tr}(A)\operatorname{Tr}(B) - \operatorname{Tr}(C),$$

where

$$A = \Upsilon_1\Upsilon_0^{-\alpha}\Theta_{0\longrightarrow 0}, \;\; B = \Upsilon_1^{-1}\Upsilon_0^{-\beta}\Theta_{0\longrightarrow 0}$$

and

$$C = AB^{-1} = \Upsilon_1\Upsilon_0^{-\alpha}\Theta_{0\longrightarrow 0}\Theta_{0\longrightarrow 0}^{-1}\Upsilon_0^{\beta}\Upsilon_1.$$

In particular we have:

$$\begin{aligned}
A &= \Upsilon_1\Upsilon_0^{-\alpha}\Theta_{0\longrightarrow 0} \\
&= i\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2-\mu_i & -1 \end{pmatrix} \\
&= i\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\mu_i - 2\alpha + 2 & -1 \end{pmatrix} \\
&= i\begin{pmatrix} 1-4\alpha-2\mu_i & -2 \\ -\mu_i - 2\alpha & -1 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
B &= \Upsilon_1^{-1}\Upsilon_0^{-\beta}\Theta_{0\longrightarrow 0} \\
&= i\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\beta & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2-\mu_i & -1 \end{pmatrix} \\
&= i\begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\mu_i - 2\beta + 2 & -1 \end{pmatrix} \\
&= i\begin{pmatrix} -3+4\beta+2\mu_i & 2 \\ +3\mu_i+6\beta-2 & 3 \end{pmatrix}.
\end{aligned}$$

So $\operatorname{Tr}(A) = -2i(\mu_i + 2\alpha)$ and $\operatorname{Tr}(B) = 2i(\mu_i + 2\beta)$. Finally, the matrix

$$C = \Upsilon_1\Upsilon_0^{-\alpha}\Theta_{0\longrightarrow 0}\Theta_{0\longrightarrow 0}^{-1}\Upsilon_0^{\beta}\Upsilon_1 = \Upsilon_1\Upsilon_0^{-\alpha+\beta}\Upsilon_1$$

doesn't contain the parameter μ_i. Hence we have:

$$
\begin{aligned}
\operatorname{Tr}(\rho_{\underline{\mu}}(\gamma)) &= \operatorname{Tr}(A)\operatorname{Tr}(B) - \operatorname{Tr}(C) \\
&= \operatorname{Tr}(A)\operatorname{Tr}(B) + R \\
&= 4(\mu_i^2 + (2\alpha + 2\beta)\mu_i + 4\alpha\beta) + R \\
&= 4(\mu_i^2 + (2\alpha + 2\beta)\mu_i) + R',
\end{aligned}
$$

where R and R' represent terms with total degree 0. In addition, using Theorem 1.2.6, we have that the DT–twist is $p_i = 2\alpha+2\beta+2$ and $p_i - q_i = 2\alpha+2\beta$. Since the total number of *sbcc*–arcs is 2, we have that $\operatorname{Tr}(\rho_{\underline{\mu}}(\gamma)) = \pm i^q 2^h \left(\mu_i^q + (p_i - q_i)\mu_i^{q-1}\right) + R$, as we wanted to prove.

Case $q = 2$ **(iii)** We have to prove Theorem 2.4.1 for simple closed curves γ with $q(\gamma) = 2$ and which meet only one pants curve which is the boundary of only one pair of pants In this situation, without loss of generality, we can suppose $q = q_i = 2$ and $\operatorname{MS}(\sigma_i) \cong \Sigma_{1,1}$, where $\operatorname{MS}(\sigma_i)$ is the modular surface associated to σ_i. In this case, we are in a case similar to Figure 2.11, but with the curve γ going twice around the handle. We choose the labelling of Figure 2.11. We can do that, because of Lemma 2.4.3.

Let $\operatorname{MS}(\sigma_i) = P$ and assume

$$
\gamma = \upsilon_1^{-\alpha}\vartheta(P, P; 1, \infty; \sigma_i)\upsilon_1^{-\beta}\vartheta(P, P; 1, \infty; \sigma_i).
$$

In this case, the PH–twist is $\hat{p}_i = \alpha + \beta$.

Similarly to the previous cases we have $\rho_{\underline{\mu}}(\gamma) = \Upsilon_1^{-\alpha}\Theta_{1\longrightarrow\infty}\Upsilon_1^{-\beta}\Theta_{1\longrightarrow\infty}$. Using Proposition A.1, we have that $\operatorname{Tr}(\rho_{\underline{\mu}}(\gamma)) = \operatorname{Tr}(A)\operatorname{Tr}(B) - \operatorname{Tr}(C)$, where $A = \Upsilon_1^{-\alpha}\Theta_{1\longrightarrow\infty}$, $B = \Upsilon_1^{-\beta}\Theta_{1\longrightarrow\infty}$ and $C = AB^{-1} = \Upsilon_1^{-\alpha}\Theta_{1\longrightarrow\infty}\Theta_{1\longrightarrow\infty}^{-1}\Upsilon_1^{\beta}$.

By Section 2.4.1.a, we have that $\operatorname{Tr}(A) = i(-1)^{-\alpha+1}(\mu_i + 2\alpha - 1)$ and $\operatorname{Tr}(B) = i(-1)^{-\beta+1}(\mu_i + 2\beta - 1)$, and, similarly to case $q = 2$ (ii), the matrix C doesn't contain the parameter μ_i.

So

$$
\operatorname{Tr}\left(\rho_{\underline{\mu}}(\gamma)\right) = (-1)^{\alpha+\beta+1}\left(\mu_i^2 + (2\alpha + 2\beta - 2)\mu_i\right) + R,
$$

where R is a term with total degree 0. So, using Theorem 1.2.6, we have that the DT–twist is $p_i = 2\alpha + 2\beta$. Since the total number of *sbcc*–arcs is 0, we have that $\operatorname{Tr}(\rho_{\underline{\mu}}(\gamma)) = \pm i^q 2^h \left(\mu_i^q + (p_i - q_i)\mu_i^{q-1}\right) + R$ as we wanted to prove.

2.4.2 Inductive proof of Theorem 2.4.1

Using the discussion made in Section 2.3, we can see that any simple closed curve $\gamma \in \pi_1(\Sigma)$ can be written as a product $\gamma = \gamma_{i_1} \dots \gamma_{i_q} \in \pi_{1,k}(\Sigma, B')$, where γ_{i_j} is a paths with endpoints in the set $B' = \{b_1, \dots, b_k\} \subset B$ consisting of base points b_j in the white hexagons $\Delta_0(P_j)$. (See Section 2.3.1 for the definition of the base points.) Note that the number of such arcs γ_{i_j} is $q = q(\gamma) = \sum_{i=1}^{\xi} q_i(\gamma)$ and note also that $q(\gamma_{i_j}) = 1$ for all $j = 1, \dots, q$.

In particular, each path γ_{i_j} is of the form:

$$\gamma_{i_j} = \begin{cases} \upsilon_\epsilon^{-\alpha} \vartheta(\epsilon, \epsilon') & \text{if } h = 0 \\ \upsilon_{\epsilon+1}^{\pm 1} \upsilon_\epsilon^{-\alpha} \vartheta(\epsilon, \epsilon') & \text{if } h = 1, \end{cases}$$

where $h = h(\gamma_{i_j})$ is the number of *sbcc*–arcs in γ_{i_j} and where υ_ϵ and $\vartheta(\epsilon, \epsilon')$ were defined in Section 2.3.4. Note that any term $\gamma_{i_j} = \upsilon_{\epsilon+1}^{\pm 1} \upsilon_\epsilon^{-\alpha} \vartheta(\epsilon, \epsilon')$ corresponds to an *sbcc*–arc of γ and it should be preceeded by a factor $\vartheta(\epsilon', \epsilon)$. Let $\hat{p}(\gamma_{i_j}) = \alpha$ be the PH–twist of γ_{i_j}. Then

$$\hat{p}_i(\gamma) = \sum_{\gamma_{i_j} \cap \sigma_i \neq \emptyset} \hat{p}(\gamma_{i_j}). \tag{2.8}$$

Using the discussion of Section 2.3 and the fact that $\Upsilon_\epsilon = \Omega_\epsilon^{-1} \Upsilon_\infty \Omega_\epsilon$, we can see the following:

$$\rho_{\underline{\mu}}(\gamma_{i_j}) = \begin{cases} \Upsilon_\epsilon^{-\alpha} \Theta_{\epsilon \longrightarrow \epsilon'} = \Omega_\epsilon^{-1} \Upsilon_\infty^{-\alpha} \Theta_{\infty \longrightarrow \infty} \Omega_{\epsilon'} & \text{if } h = 0 \\ \Upsilon_{\epsilon+1}^{\pm 1} \Upsilon_\epsilon^{-\alpha} \Theta_{\epsilon \longrightarrow \epsilon'} = \Omega_\epsilon^{-1} \Omega_1 \Upsilon_\infty^{\pm 1} \Omega_0 \Upsilon_\infty^{-\alpha} \Theta_{\infty \longrightarrow \infty} \Omega_{\epsilon'} & \text{if } h = 1, \end{cases}$$

Note that the matrix $\Upsilon_\infty^{-\alpha} \Theta_{\infty \longrightarrow \infty}$ corresponds to the matrix A_X defined in Appendix B with $X = -(\mu_i + 2\alpha)$.

Unfortunately, there is no canonical way to define the DT–twist $p(\gamma_{i_j})$ of γ_{i_j}, so the idea of an inductive proof breaks down. Using Theorem 1.2.6, we can define the DT–twist for arcs with endpoints on the pants curve. We believe that a different proof of Theorem 2.4.1 can be found by defining a new fundamental groupoid which contains $\pi_{1,2k}(\Sigma, B)$. (In particular, in the new fundamental groupoid one needs to add one base point for each pants curve.) Using this new groupoid, we think that one can give an inductive proof of Theorem 1.2.6. Anyway, we decided not to do this, because that method would introduce new definitions and would make the material more difficult to understand.

Now we explain the idea of the combinatorial proof of Appendix B. The main thought is to cut the product $\rho_{\underline{\mu}}(\gamma) = \rho_{\underline{\mu}}(\gamma_{i_1}) \dots \rho_{\underline{\mu}}(\gamma_{i_q})$ in a different way (that is, after each piece $\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} = A_X$) and to use the trace formula (iv) of Theorem A.1. In detail, we find a new decomposition

$$\rho_{\underline{\mu}}(\gamma) = \hat{\gamma}_{i_1} \dots \hat{\gamma}_{i_q},$$

where

$$\hat{\gamma}_{i_j} = \begin{cases} \Omega_{\epsilon'}\Omega_\epsilon^{-1}\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} & \text{if } h = 0 \\ \Omega_1\Upsilon_\infty^{\pm 1}\Omega_0\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} & \text{if } h = 1, \end{cases}$$

Notice that, in the first case, $\epsilon \neq \epsilon'$, so $\Omega_{\epsilon'}\Omega_\epsilon^{-1} \in \{\Omega_0, \Omega_1\}$. Supposing that $q(\hat{\gamma}_{i_j}) = q_i(\hat{\gamma}_{i_j}) = 1$, we have three types of base blocks:

$$\hat{\gamma}_{i_j} = \begin{cases} \Omega_1\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} = \Omega_1 A_X = i\begin{pmatrix} 0 & 1 \\ 1 & -\mu_i - 2\alpha + 1 \end{pmatrix} & \text{if } h = 0, \\ \Omega_0\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} = i\begin{pmatrix} 1 & -\mu_i - 2\alpha + 1 \\ 1 & -\mu_i - 2\alpha \end{pmatrix} & \text{if } h = 0, \\ \Omega_0\Upsilon_\infty^{\pm 1}\Omega_1\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} = i\begin{pmatrix} 1 - 4\alpha - 2\mu_i & -2 \\ -\mu_i - 2\alpha & -1 \end{pmatrix} & \text{if } h = 1, \end{cases} \tag{2.9}$$

where $X = -(\mu_i + 2\alpha)$, see Appendix B.

Each factor $\hat{\gamma}_{i_j}$ corresponds to a crossing $\Theta_{\infty\longrightarrow\infty}$ of a pants curve, and it is preeceeded and followed by a matrix Ω_1 or Ω_0. For each factor $\hat{\gamma}_{i_j}$, we can describe the twist parameter $p(\gamma_{i_j})$, as we are about to explain, and we have

$$p_i(\gamma) = \sum_{\gamma_{i_j} \cap \sigma_i \neq \emptyset} p(\gamma_{i_j}). \tag{2.10}$$

In fact, if γ_{i_j} is an arc crossing a pants curve $\sigma = \partial_\epsilon P \cap \partial_{\epsilon'} P'$, we need to know the boundary component across which γ_{i_j} entered P and the one across which it left P' in order to define $p(\gamma_{i_j})$ using Theorem 1.2.6. Knowing that, we can define $p(\gamma_{i_j})$.

For example, if $\hat{\gamma}_{i_j} = \Omega_0\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty}$ is part of the product

$$\rho_{\underline{\mu}}(\gamma) = \dots \Theta_{\infty\longrightarrow\infty}\hat{\gamma}_{i_j}\Omega_1\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} \dots,$$

then $p(\hat{\gamma}_{i_j}) = 2\alpha + 1$, while if

$$\rho_{\underline{\mu}}(\gamma) = \dots \Theta_{\infty\longrightarrow\infty}\hat{\gamma}_{i_j}\Omega_0\Upsilon_\infty^{-\alpha}\Theta_{\infty\longrightarrow\infty} \dots,$$

then $p(\hat{\gamma}_{i_j}) = 2\alpha$. The reason for that is the following. Using the above factorisation and Lemma 2.4.3, we can suppose that γ_{i_j} cuts the pants curve $\sigma_i = \partial_\infty(P) \cap \partial_\infty(P')$, and so the terms Ω_ϵ before and after $\Upsilon_\infty^{-\alpha}\Theta_{\infty \longrightarrow \infty}$ tell us that γ enters P from $\partial_0(P)$ and leaves P' from $\partial_0(P)$ (in the first case) or leaves P' from $\partial_1(P)$ (in the second case). We can then use Theorem 1.2.6 and see the value of $p(\hat{\gamma}_{i_j})$ in the two cases. Table 2.2 summarise how to define $p(\hat{\gamma}_{i_j})$.

Table 2.2: Types of blocks

Factor $\hat{\gamma}_{i_j}$ of $\rho_{\underline{\mu}}(\gamma)$	$p(\hat{\gamma}_{i_j})$
$\ldots \Omega_0 \Upsilon_\infty^{-\alpha}\Theta_{\infty \longrightarrow \infty}\Omega_0 \ldots$	2α
$\ldots \Omega_0 \Upsilon_\infty^{-\alpha}\Theta_{\infty \longrightarrow \infty}\Omega_1 \ldots$	$2\alpha - 1$
$\ldots \Omega_1 \Upsilon_\infty^{-\alpha}\Theta_{\infty \longrightarrow \infty}\Omega_0 \ldots$	$2\alpha + 1$
$\ldots \Omega_1 \Upsilon_\infty^{-\alpha}\Theta_{\infty \longrightarrow \infty}\Omega_1 \ldots$	2α

In the Appendix we will prove Proposition B.4 and Theorem B.5 by induction and this will prove Theorem 2.4.1.

Chapter 3

Asymptotic behaviour

As explained in the Introduction, in this chapter we extend to a general hyperbolic surface $\Sigma_{g,b}$ the results proved by Series in [40] for the case of a twice punctured torus $\Sigma_{1,2}$. Before explaining our result, we give, in Section 3.1, some definitions about 3–manifolds and pleating rays.

3.1 Three manifolds and pleating rays

In this background section, we recall some known material related to hyperbolic 3–manifolds. Since we are in the same setting as Series' paper [40], we follow her description.

Let M be a hyperbolic 3–manifold, that is a complete 3-dimensional Riemannian manifold of constant curvature -1 such that the fundamental group $\pi_1(M)$ is finitely generated. We exclude the somewhat degenerate case that $\pi_1(M)$ has an abelian subgroup of finite index, that is $\pi_1(M)$ is an elementary Kleinian group. An important subset of M is its convex core $\mathcal{C}_M = \mathcal{C}$. We say that a subset M' of M is *convex* when, given any two points in M', then the geodesic segment between them is entirely contained in M'.

Definition 3.1.1. The *convex core* $\mathcal{C}_M$ of a hyperbolic 3–manifold M is the smallest, non-empty, closed, convex subset of M such that the inclusion of $\mathcal{C}_M$ into M is a homotopy equivalence.

Given a hyperbolic 3–manifold $M = \mathbb{H}^3/G$, we can also define the convex core as the quotient $\mathrm{CH}(\Lambda)/G$, where $\mathrm{CH}(\Lambda)$ is the convex hull of the limit set $\Lambda = \Lambda(G)$ of G, see Section 2.1.1 for the definition of limit set.

The boundary $\partial\mathcal{C}_M$ of the convex core is a surface of finite topological type whose geometry was described by W. Thurston [42]. If M is geometrically finite, then

Thurston proved that there is a natural homeomorphism between the components of $\partial\mathcal{C}_M$ and the components of the conformal boundary Ω/G of M. Each component F of $\partial\mathcal{C}_M$ inherits a hyperbolic structure from M. In addition, Thurston proved that each such component F is a pleated surface, that is a hyperbolic surface which is totally geodesic almost everywhere and such that the locus of points where it fails to be totally geodesic is a geodesic lamination; see [16] for a detailed discussion on the pleated structure of the boundary of the convex core.

Given a topological surface $\Sigma = \Sigma_{g,b}$, a pleated surface is defined in the following way.

Definition 3.1.2. A *pleated surface* with topological type Σ in a hyperbolic 3–dimensional manifold M is a map $f\colon\, \Sigma \longrightarrow M$ such that:

(i) the path metric obtained by pulling back the hyperbolic metric of M by f is a hyperbolic metric m on Σ;

(ii) there is an m-geodesic lamination λ such that f sends each leaf of λ to a geodesic of M and is totally geodesic on $\Sigma - \lambda$.

In this case, we say that λ is the *bending (or pleated) lamination*, while the images of the complementary components of λ are called *flat pieces*.

The bending lamination of each component of $\partial\mathcal{C}_M$ carries a natural transverse measure, called the *bending measure* (or *pleating measure*). In the case M is homeomorphic to $\Sigma \times \mathbb{R}$, there are two components $\partial^+\mathcal{C}_M$ and $\partial^-\mathcal{C}_M$ of $\partial\mathcal{C}_M$ and we denote $\mathrm{pl}^{\pm} \in \mathrm{ML}(\Sigma)$ the respective pleating measure on each one.

We deal with manifolds for which the bending lamination is *rational*, that is, supported on closed curves. As explained in Section 1.3.1, the subset of rational measured laminations is denoted $\mathrm{ML}_{\mathbb{Q}}(\Sigma) \subset \mathrm{ML}(\Sigma)$ and consists of measured laminations of the form $\eta = \sum_{i=1}^{m} a_i\delta_{\gamma_i}$, where the curves $\gamma_i \in \mathcal{S}(\Sigma)$ are disjoint and non-homotopic, $a_i \geq 0$, and δ_{γ_i} represents the transverse measure which gives weight 1 to each intersection with γ_i. For simplicity, we write $\eta = \sum_{i=1}^{m} a_i\delta_{\gamma_i} = \sum_{i=1}^{m} a_i\gamma_i \in \mathrm{ML}_{\mathbb{Q}}(\Sigma)$. If $\sum_{i=1}^{m} a_i\gamma_i$ is the bending measure of a pleated surface Σ, then a_i is the angle between the flat pieces adjacent to γ_i, also denoted θ_{γ_i}. In particular, $\theta_{\gamma_i} = 0$ if and only if the flat pieces adjacent to γ_i are in a common totally geodesic subset of $\partial\mathcal{C}$. We take the term pleated surface to include the case in which a closed leaf γ of the bending lamination maps to the fixed point of a rank one parabolic cusp of M. In this case, the image pleated surface is cut along γ and thus may be disconnected. Moreover, the bending angle between the flat pieces adjacent to γ is $\theta_\gamma = \pi$. See discussion in Series [40] or Choi–Series [11].

Bonahon and Otal [7] described exactly which laminations can appear as bending laminations in the boundary $\partial\mathcal{C}_M$ of the convex core. We discuss only the case we are interested in. We recall that a set of curves $\{\gamma_1,\dots,\gamma_m\}$ in a surface Σ is said to *fill* the surface if, for any $\gamma \in \mathcal{S}(\Sigma)$, there exists $j \in \{1,\dots,m\}$ such that $i(\gamma,\gamma_j) \neq 0$. Equivalently, the set of curves $\{\gamma_1,\dots,\gamma_m\}$ in Σ *fills* the surface Σ if all the components of $\Sigma \setminus \cup_{i=1}^m \gamma_i$ are discs or once punctured discs.

Theorem 3.1.3 (Theorem 1 of [7])**.** *Suppose that M is 3–manifold homeomorphic to $\Sigma \times (0,1)$, and that $\eta^+ = \sum_{i=1}^m a_i^+\gamma_i^+, \eta^- = \sum_{j=1}^n a_j^-\gamma_j^+ \in \mathrm{ML}_{\mathbb{Q}}(\Sigma)$. Then there exists a geometrically finite group G such that $M = \mathbb{H}^3/G$ and such that the bending measures on the two components $\partial\mathcal{C}^{\pm}(G)$ of $\partial\mathcal{C}(G)$ equal $\eta^{\pm}$ respectively, if and only if $a_i^+, a_j^- \in (0,\pi]$ for all i and j and the set $\{\gamma_i^+,\gamma_j^- | i=1,\dots,m, j=1,\dots,n\}$ fills up Σ (i.e. if $i(\eta^+,\gamma)+i(\eta^-,\gamma) > 0$ for every $\gamma \in \mathcal{S}$). If such a structure exists, it is unique.*

Specialising now to the Maskit embedding $\mathcal{M} = \mathcal{M}(\Sigma)$, let $\rho\colon\ \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ be a holonomy representation such that the image $G = \rho(\pi_1(\Sigma))$ is in $\mathcal{M}$. The boundary $\partial\mathcal{C}(G)$ of the convex core has $k+1$ components: one is $\partial^+\mathcal{C}$ and the others are k triply punctured spheres whose union we denote $\partial^-\mathcal{C}$. $\partial^+\mathcal{C}$ faces Ω^+/G and is homeomorphic to Σ. The induced hyperbolic structures on the components of $\partial^-\mathcal{C}$ are rigid, while the structure on $\partial^+\mathcal{C}$ varies. Recall that we called $\mathrm{pl}^+(G) \in \mathrm{ML}(\Sigma)$ the bending lamination of $\partial^+\mathcal{C}$. Following the discussion above, we view $\partial^-\mathcal{C}$ as a single pleated surface with bending lamination $\pi(\sigma_1+\ldots+\sigma_\xi)$, indicating that the triply punctured spheres are glued across the annuli whose core curves $\sigma_1,\dots,\sigma_\xi$ correspond to the parabolics $\rho(\sigma_i) \in G$.

Corollary 3.1.4. *A lamination $\eta \in \mathrm{ML}_{\mathbb{Q}}(\Sigma)$ is the bending measure of a group $G \in \mathcal{M}$ if and only if $i(\eta,\sigma_1),\dots,i(\eta,\sigma_\xi) > 0$. If such a structure exists, it is unique.*

Definition 3.1.5. A lamination $\eta \in \mathrm{ML}_{\mathbb{Q}}(\Sigma)$ is called *admissible* if

$$i(\eta,\sigma_1),\dots,i(\eta,\sigma_\xi) > 0.$$

3.1.0.a Pleating rays

In the set $\mathrm{PML} = \mathrm{PML}(\Sigma)$ of projective measured laminations on Σ, we denote the projective class of $\eta = a_1\gamma_1+\ldots+a_m\gamma_m \in \mathrm{ML}(\Sigma)$ by $[\eta]$. See Section 1.3.1 for the definition of ML and PML. The *pleating ray* $\mathcal{P} = \mathcal{P}_{[\eta]}$ of $\eta \in \mathrm{ML}$ is the set of groups $G \in \mathcal{M}$ for which $\mathrm{pl}^+(G) \in [\eta]$. The ray $\mathcal{P}_{[\eta]}$ depends only on the projective class of η, but, to simplify the notation, we write $\mathcal{P}_\eta$ for $\mathcal{P}_{[\eta]}$. In addition, $\mathcal{P}_\eta$ is non-empty if and only if η is admissible. In particular, we write $\mathcal{P}_\gamma$ for the ray $\mathcal{P}_{[\delta_\gamma]}$. As $\mathrm{pl}^+(G)$ increases, $\mathcal{P}_\eta$ limits to a geometrically finite group $\overline{G} = G_{\mathrm{cusp}}(\eta)$ in the algebraic closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ at which at least one of the support curves to η is parabolic; in particular, at $\overline{G}$ we have that that $\mathrm{pl}^+(\overline{G}) = \theta(a_1\gamma_1 + \ldots + a_m\gamma_m)$ with $\max\{\theta a_1, \ldots, \theta\alpha_m\} = \pi$. We write $\overline{\mathcal{P}_\eta} = \mathcal{P}_\eta \cup G_{\mathrm{cusp}}(\eta)$.

Similarly, for disjoint non-homotopic curves $\gamma_1, \ldots, \gamma_m \in \mathcal{S}_0$ (with, of course, $m \leq \xi$), we define the *pleating variety* $\mathcal{P}_{\gamma_1,\ldots,\gamma_m}$ of $\gamma_1, \ldots, \gamma_m$ to be the set of groups $G \in \mathcal{M}$ for which $\mathrm{pl}^+(G) = \sum_{i=1}^m a_i\gamma_i$ with $a_i > 0$ for all i. Thus $\mathcal{P}_{\gamma_1,\ldots,\gamma_m}$ is the union of the pleating rays $\mathcal{P}_\eta$ with $\eta = \sum_{i=1}^m a_i\gamma_i$ with $a_i > 0$. If γ_i is admissible, then the ray $\mathcal{P}_{\gamma_i}$ is contained in the boundary of $\mathcal{P}_{\gamma_1,\ldots,\gamma_m}$; note that γ_i may not be admissible even though $\eta = \sum_{i=1}^m a_i\gamma_i$ is. In Section 5 of [40] the interested reader can find some computations of pleating varieties in the case $\Sigma_{1,2}$. We write $\overline{\mathcal{P}}_{\gamma_1,\ldots,\gamma_m} = \cup_\eta \overline{\mathcal{P}}_\eta$ where the union is over $\eta = \sum a_i\gamma_i$ with $a_i \geq 0$.

From now on, when we will write $\eta = \sum_{i=1}^m a_i\gamma_i$, we will always assume $a_i > 0$, unless differently specified.

The following key lemma is proved in Proposition 4.1 of Choi and Series [11], see also Lemma 4.6 of Keen and Series [24]. The essence is that the complex length of the curves in a bending line of the boundary $\partial\mathcal{C}(G)$ of the convex core can't have rotational part, because the two flat pieces of $\partial\mathcal{C}(G)$ on either side of a bending line are invariant under translation along the bending line.

Lemma 3.1.6. *If the axis of $g \in G$ is a bending line of $\partial\mathcal{C}(G)$, then* $\mathrm{Tr}(g) \in \mathbb{R}$.

Notice that the lemma applies even when the bending angle θ_γ along γ vanishes. Thus, if $\{\gamma_1, \cdots, \gamma_m, \gamma_{m+1}, \cdots, \gamma_\xi\}$ is a pants decomposition of Σ and if $G \in \overline{\mathcal{P}}_{\gamma_1,\ldots,\gamma_m}$, then we have $\mathrm{Tr}\, g \in \mathbb{R}$ for any $g \in G$ whose axis projects to a curve γ_i, $i = 1, \ldots, \xi$.

In order to compute pleating rays, we need the following result which is a special case of Theorems B and C of [11], see also [24]. Recall that a codimension-p submanifold $N \hookrightarrow \mathbb{C}^n$ is called *totally real* if it is defined locally by equations $\Im f_i = 0$ for $i = 1, \ldots, p$, where f_i are local holomorphic coordinates for $\mathbb{C}^n$. As usual, if γ is a bending line, we denote its bending angle by θ_γ.

We recall the definition of complex length, see e.g. [11] for details. Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Re z > 0\}$. If $A \in \mathrm{PSL}(2,\mathbb{C})$, then $\mathrm{Tr}\, A$ is well defined up to multiplication by ± 1.

Definition 3.1.7. The *complex length* $\lambda(A) \in \mathbb{C}_+/2i\pi$ of a loxodromic element $A \in \mathrm{PSL}(2,\mathbb{C})$ is defined by $\mathrm{Tr}\, A = 2\cosh\frac{\lambda(A)}{2}$. Given a representation $\rho \in \mathcal{R}(\Sigma)$

(that is, $\rho\colon \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$) and given a curve $\gamma \in \mathcal{S}_0$, then we define $\lambda(\gamma) = \lambda(\rho(\gamma))$.

By construction, $\mathcal{P}_{\gamma_1,\ldots,\gamma_m} \subset \mathcal{M} \subset \mathcal{R}(\Sigma)$. Recall that the representation variety $\mathcal{R}(\Sigma)$ was defined in Section 2.2.5.

Theorem 3.1.8. *The complex lengths $\lambda(\gamma_1), \ldots, \lambda(\gamma_\xi)$ are local holomorphic coordinates for $\mathcal{R}(\Sigma)$ in a neighbourhood of $\mathcal{P}_{\gamma_1,\ldots,\gamma_\xi}$. Moreover $\mathcal{P}_{\gamma_1,\ldots,\gamma_\xi}$ is connected and is locally defined as the totally real submanifold $\Im \operatorname{Tr} \gamma_i = 0$ of $\mathbb{R}$, where $i = 1, \ldots, \xi$. Any ξ–tuple $(f_1, f_2, \ldots, f_\xi)$, where f_i is either the hyperbolic length $\Re\lambda(\gamma_i)$ or the bending angle θ_{γ_i}, are global coordinates on $\mathcal{P}_{\gamma_1,\ldots,\gamma_\xi}$.*

As explained by Series [40], this result extends to $\overline{\mathcal{P}}_{\gamma_1,\ldots,\gamma_\xi}$, except that one has to replace $\Re\lambda(\gamma_i)$ by $\operatorname{Tr} \gamma_i$ in a neighbourhood of a point for which γ_i is parabolic. In fact, as discussed in Section 3.1 of [11] and in Section 3 of [40], complex length and traces are interchangeable except at cusps (where traces must be used) and points where a bending angle vanishes (where complex length must be used). The parameterisation by lengths or angles extends to $\overline{\mathcal{P}}_{\gamma_1,\ldots,\gamma_\xi}$.

Notice that the above theorem gives a local characterisation of $\overline{\mathcal{P}}_{\gamma_1,\ldots,\gamma_\xi}$ as a subset of the representation variety $\mathcal{R}(\Sigma)$ and not just as a subset of $\mathcal{M}$. In other words, to locate $\mathcal{P}$, one does not need to check whether nearby points lie *a priori* in $\mathcal{M}$; it is enough to check that the traces remain real and away from 2 and that the bending angle on one or other of θ_{γ_i} does not vanish. As we shall see, this last condition can easily be checked by requiring that further traces be real valued.

3.2 Asymptotic behaviour of pleating rays

As explained in the Introduction, an important result of the Pleating Coordinates Theory introduced in 1990s by Keen and Series is to give algorithms enabling one to compute the exact position of the deformation spaces of holomorphic families of Kleinian groups as subset of the representation variety $\mathcal{R}$. Theorem B from the Introduction, which describes the direction of the pleating rays in the Maskit embedding $\mathcal{M} = \mathcal{M}(\Sigma)$ of $\Sigma = \Sigma_{g,b}$ as the bending measure tends to zero, enables us to compute the location of the slice $\mathcal{M}(\Sigma)$ as a subset of $\mathcal{R}(\Sigma)$. As explained by Series [40] in the case of the twice punctured sphere, the idea is the following. The pleating rays are real 1–submanifolds of $\mathcal{M}$ along which the projective class of the bending measure of the component $\partial^+\mathcal{C}$ of the convex hull boundary is supported on a fixed (subset of) a pants decomposition of Σ. In general, the pleating ray is a connected nonsingular branch of the real algebraic variety along which the traces of the support curves take real values; see Theorem 3.1.8. The main result of this paper identifies the correct branch by determining its direction as the parameters of the representation tend to infinity, equivalently as the bending measure tends to zero; see Theorem B. Then, the slice $\mathcal{M}$ can then be plotted by following these real trace branches until one of the supporting curves becomes parabolic.

The aim of this section is to prove Theorem B and C from the Introduction. As already observed by Series [40], almost all the results of Section 6 of [40] generalise straightforwardly, but for Section 7 of [40] some non-trivial extensions are needed. In Section 3.2.1 we restate the theorems of Section 6 of [40] and we gave a sketch of the ideas used in the proofs. We refer to the original paper for a more detailed discussion. Also all the results of Section 7 of [40] still remain true in our more general context, but we need to discuss the results more deeply in order to generalise them. We will do that in Section 3.2.2.Note also that we correct some misprints in [40].

3.2.1 Geometry of the top boundary component $\partial^+\mathcal{C}(G)$ of the convex core

Recall that in Chapter 2, given a pants decomposition $\mathcal{PC} = \{\sigma_1, \dots, \sigma_\xi\}$ on a surface Σ and given a vector $\underline{\mu} = (\mu_1, \dots, \mu_\xi) \in \mathbb{H}^\xi$, we constructed a complex projective structure $\Sigma(\underline{\mu})$ with associated developing map $\mathrm{Dev}_{\underline{\mu}}\colon \tilde{\Sigma} \longrightarrow \hat{\mathbb{C}}$ and holonomy representation $\rho_{\underline{\mu}}\colon \pi_1(\Sigma) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$. Let $G = G_{\underline{\mu}} = \rho_{\underline{\mu}}(\pi_1(\Sigma))$. In particular, in Theorem 2.2.5 we proved that, if $\mathrm{Dev}_{\underline{\mu}}$ is an embedding, then $G_{\underline{\mu}} \in \mathcal{M}$, where $\mathcal{M}$ is the Maskit slice, see Section 2.2.5 for the definition. In particular, for groups $G_{\underline{\mu}} \in \mathcal{M}$, the pants curves $\sigma_1, \dots, \sigma_\xi$ have been pinched on the 'bottom' component $\Omega^-/G_{\underline{\mu}}$ of $\Omega(G_{\underline{\mu}})/G_{\underline{\mu}}$.

The key idea for proving Theorems B and C from the Introduction is to understand the geometry of the top component $\partial^+\mathcal{C}(G)$ of the convex core for groups $G \in \mathcal{P}_\eta \subset \mathcal{M}$, pleated along $\mathrm{pl}^+(G) = \theta\eta$, as the bending angle $\theta \longrightarrow 0$. Recall that the definition of $\mathcal{M}$ depends on the choice of a pants decomposition $\mathcal{PC} = \{\sigma_1, \dots, \sigma_\xi\}$, which tells us the curves which are pinched in the bottom surface of the associated manifold. Before stating the results, we need to fix some notation. We use Series' notation, so that the interested reader can refer to her paper [40] more easily. Given a group $G \in \mathcal{P}_\eta \subset \mathcal{M}$ and given an index $i \in \{1, \dots, \xi\}$, let $\sigma^+ = \sigma_i^+$ denote the geodesic representative of σ_i on $\partial^+\mathcal{C}$ and let $l_\sigma^+ = l_{\sigma_i}^+$ be its hyperbolic length in the hyperbolic structure on $\partial^+\mathcal{C}$. We show that $l_\sigma^+ \longrightarrow 0$ as $\theta \longrightarrow 0$, while σ^+ becomes asymptotically orthogonal to the bending lines. From this we deduce results on the asymptotic behaviour of the parameters μ_i.

In the case of the twice puncture torus $\Sigma_{1,2}$, Series uses, as generators of the

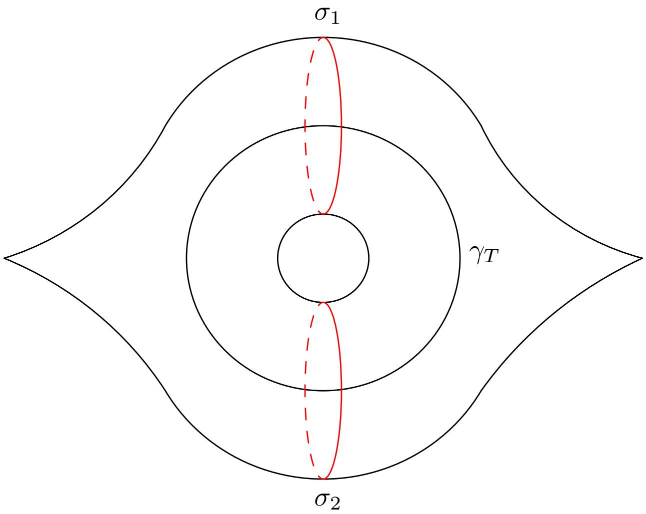

Figure 3.1: The generators $\{\sigma_1, \sigma_2, \gamma_T\}$ of $\pi_1(\Sigma_{1,2})$.

free group $\pi_1(\Sigma_{1,2})$ of rank three the curves $\{\sigma_1, \sigma_2, \gamma_T\}$, where $\{\sigma_1, \sigma_2\}$ are a pants decomposition and the curve γ_T plays the role of dual curve for both σ_1 and σ_2. See Figure 3.1. In our situation the role of the curve γ_T is played by the dual curves $D_1, \ldots, D_\xi$.

Moreover, the main points (in the generalisation of the results of Section 6 [40]) at which the following differs from [40] are the proof of Proposition 3.2.7 (see Section 3.2.3) and of Proposition 3.2.16. In particular, in the first proof we need to use the gluing construction described in Section 2.2, while in the second one we need to use the definition of DT–twist defined in section 1.2.1.b.

Notation 3.2.1. Given a quantity $X = X(\sigma_i)$ which depends on the pants curves $\sigma_i \in \mathcal{PC}$, we write $X(\sigma_i) = O(\theta^e)$, meaning that $X \leqslant c\theta^e$ as $\theta \longrightarrow 0$ for some constant $c > 0$, where e is an exponent (usually $e = 0, 1, \frac{1}{2}$).

Remark 3.2.2 (Section 6.1 of [40]). The estimates below all depends on the lamination η. So, more precisely, one has $X \leqslant c(\eta)\theta^e$. However it is easily seen, by following through the arguments, that the dependence on η is always of the form $X(\sigma_i) \leqslant cq^e\theta^e$, where $q = i(\sigma_i, \eta)$ and where, now, c is a universal constant independent of η. The dependence of the constants on η is not important for our argument, but it may be useful elsewhere.

The main result in Section 6 of [40] is Proposition 6.1, which we generalise as:

Theorem 3.2.3. *Let* $\eta = \sum_{i=1}^{m} a_i\delta_{\gamma_i}$ *be an admissible rational measured lamination on the surface* $\Sigma = \Sigma_{g,b}$ *and let* $G = G_\eta(\theta)$ *be the unique group in* $\mathcal{M}$ *with* $\mathrm{pl}^+(G) =$

$\theta\eta$. Then, as $\theta \longrightarrow 0$, we have:

$$\Re\mu_i = -\frac{p_i(\eta)}{q_i(\eta)} + O(1) \quad \text{and} \quad \Im\mu_i = \frac{4 + O(\theta)}{\theta q_i(\eta)},$$

where $O(1)$ denotes a universal bound independent of η.

From this theorem we can see that $\frac{\Re\mu_i}{\Im\mu_i} = O(\theta)$. So, using the fact that $\frac{\Re\mu_i}{\Im\mu_i} = \cot(\mathrm{Arg}(\mu_i)) = \tan\left(\frac{\pi}{2} - \mathrm{Arg}(\mu_i)\right)$, we can deduce the following result.

Corollary 3.2.4. *With the same hypothesis as Theorem 3.2.3, as $\theta \longrightarrow 0$, we have that:*

$$\left|\mathrm{Arg}(\mu_i) - \frac{\pi}{2}\right| = O(\theta) \quad \text{and} \quad \frac{\Im\mu_i}{\Im\mu_j} = \frac{q_j}{q_i} + O(\theta).$$

The proof of Theorem 3.2.3 relies on two other main results:

- Proposition 3.2.5 (which is a generalisation of Proposition 6.6 and 6.11 of [40]) for the asymptotic behaviour of the imaginary part $\Im\mu_i$ of the parameters μ_i;
- Proposition 3.2.16 (which is a generalisation of Proposition 6.14 of [40]) for the real part $\Re\mu_i$ of μ_i.

When we deal with the imaginary part $\Im\mu_i$, the main idea is to estimate the lengths $l^+_{\sigma_i}$. The following result is a generalisation of Proposition 6.6 and 6.11 of [40]. The second claim shows, in particular, that $l^+_{\sigma_i} \longrightarrow 0$ as $\theta \longrightarrow 0$.

Proposition 3.2.5. *Given an admissible lamination η, suppose $G = G_{\underline{\mu}}$ is the unique group in $\mathcal{M}$ such that $\mathrm{pl}^+(G) = \theta\eta$, where $\underline{\mu} = (\mu_1, \dots, \mu_\xi) \in \mathbb{H}^\xi$. Then, along the pleating ray $\mathcal{P}_\eta$, we have that, as $\theta \longrightarrow 0$:*

(i) $\Im\mu_i(1 - O(\theta)) \leqslant \frac{4}{l^+_{\sigma_i}} \leqslant \Im\mu_i(1 + O(\theta))$;

(ii) $\theta i(\eta, \sigma_i)(1 - O(\theta)) \leqslant l^+_{\sigma_i} \leqslant \theta i(\eta, \sigma_i)(1 + O(\theta))$.

In order to explain the main ideas behind the proof of this result, we will summarise Series' method, and we will explain how to adapt her results to our case. The first lemma proved by Series in this context is Proposition 6.3 of [40], where she estimates the length of a piecewise geodesic arc in $\mathbb{H}^3$; see also Theorem 4.2.10 of Canary, Epstein and Green [10]. For completeness, we recall the statement of this result. See also Figure 3.2.

Lemma 3.2.6 (Proposition 6.3 of [40])**.** *Let λ be a piecewise geodesic arc in $\mathbb{H}^3$ with endpoints P and P', and let $\hat{\lambda}$ be the $\mathbb{H}^3$ geodesic joining P to P'. Suppose that*

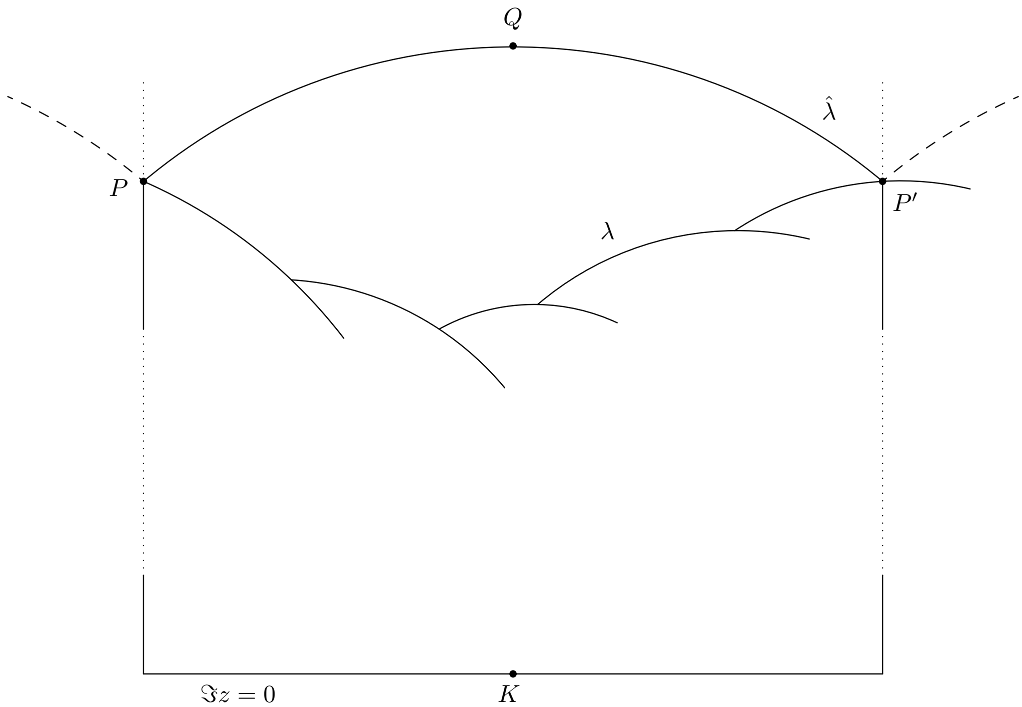

Figure 3.2: The piecewise geodesic λ in $\mathbb{H}^3$.

for all $X \in \lambda$ the angle between PX and λ is bounded in modulus by $a \in (0, \pi/4)$. Then $l(\hat{\lambda}) \geq (\cos a) l(\lambda)$ for all $X \in \lambda$, where $l(\hat{\lambda})$ and $l(\lambda)$ are the lengths of $\hat{\lambda}$ and λ in $\mathbb{H}^3$, respectively.

In order to be able to apply Series' proofs also in our case, we need to prove Proposition 3.2.7 below, which follows from our gluing construction; see Section 2.2.1 and Figure 2.3. A similar argument is proved in Proposition 2.1 of Series [40]; see also Appendix 1 of [40].

Without loss of generality, since our holonomy representation is defined up to conjugation, we can assume that $\rho_{\underline{\mu}}(\sigma_i) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \Upsilon_\infty$. (We will try to use the same notation as Chapter 2.) Recall that we denoted $\sigma^+ = \sigma_i^+$ the piecewise geodesic representative of σ_i on $\partial^+\mathcal{C}$ and recall that $\partial^+\mathcal{C} = \partial^+ \operatorname{CH}(\Lambda)/G$, where $\operatorname{CH}(\Lambda)$ is the convex hull of the limit set $\Lambda = \Lambda(G)$ of G. Let $\tilde{\sigma}^+ = \tilde{\sigma}_i^+$ be the lift of σ_i^+ to $\partial^+ \operatorname{CH}(\Lambda)$ invariant under translation $\varsigma \mapsto \varsigma - 2$. In particular, in Figure 3.2, the piecewise geodesic $\tilde{\sigma}^+$ corresponds to λ.

We also suppose that $\sigma = \sigma_i$ is the common pants curve of the two pairs of pants P and P' such that $\partial P = \{\sigma, \sigma_{i_2}, \sigma_{i_3}\}$ and $\partial P' = \{\sigma, \sigma_{i_4}, \sigma_{i_5}\}$, where σ_{i_j} are pants curves. Let μ_{i_j} be the gluing parameters associated to σ_{i_j} with $j = 2, \ldots, 5$, if

$\sigma_{i_j} \in \mathcal{PC}$, or let $\mu_{i_j} := \infty$ (and $\frac{1}{\mu_{i_j}} := 0$), if $\sigma_{i_j} \in \partial\Sigma$.

Proposition 3.2.7. *In the above setting (that is, assuming that $\rho_{\underline{\mu}}(\sigma_i) = \Upsilon_\infty$, $\sigma_i = P \cap P'$, $\partial P = \{\sigma, \sigma_{i_2}, \sigma_{i_3}\}$ and $\partial P' = \{\sigma, \sigma_{i_4}, \sigma_{i_5}\}$) and assuming $\Im\mu_i > 1$, than the lines $\{z \in \mathbb{C} | \Im z = 0\}$ and $\{z \in \mathbb{C} | \Im z = \Im\mu_i\}$ are contained in the limit set $\Lambda(G_{\underline{\mu}})$. In addition, if $\Im\mu_{i_j} > 4$ for all $j = 2, \dots, 5$, then the limit set $\Lambda(G_{\underline{\mu}})$ is contained in the union of two strips*

$$\{z \in \mathbb{C} | 0 \leq \Im z \leq \frac{1}{2}\} \cup \{z \in \mathbb{C} | \Im\mu_i - \frac{1}{2} \leq \Im z \leq \Im\mu_i\}.$$

Since the proof is quite long and not required for the remaining discussion of the section, we decided to postpone it to Section 3.2.3.

With this result, most of Series' proofs generalise straightforwardly to our case, as we are going to describe.

By applying Lemma 3.2.6, Series proves Proposition 6.4 of [40]. In our situation, the corresponding result is the following, whose proof is the same as that of [40].

Proposition 3.2.8. *Under the hypothesis of Proposition 3.2.5 we have that, as $\theta \longrightarrow 0$, then:*

$$l^+_{\sigma_i} \leqslant \theta i(\eta, \sigma_i)(1 + O(\theta)). \tag{3.1}$$

Note that this proposition corresponds to the second inequality of Proposition 3.2.5 (*ii*).

Again with the hypothesis of Proposition 3.2.5, one can deduce the following generalisation of Corollary 6.5 of [40].

Lemma 3.2.9. *With the hypothesis of Proposition 3.2.5, then $1/\Im\mu_i \leq O(\theta)$, as $\theta \longrightarrow 0$. Moreover the groups $G_\eta(\theta)$ have no algebraic limit as $\theta \longrightarrow 0$.*

The main idea to prove the first claim of Lemma 3.2.9 is that the horizontal lines $\{z \in \mathbb{C} : \Im z = 0\}$ and $\{z \in \mathbb{C} : \Im z = \Im\mu_i\}$ are contained in the limit set Λ, and that the half planes $\mathbb{L} = \{z \in \mathbb{C} : \Im z < 0\}$ and $\mathbb{H}^{\mu_i} = \{z \in \mathbb{C} : \Im z > \Im\mu_i\}$ are contained in Ω^-; see Proposition 3.2.7. For the second claim of Lemma 3.2.9 it is enough to show that, as $\theta \longrightarrow 0$, then the trace of some element becomes infinite. In particular, we consider the trace of the dual curve D_i. Looking at the calculations made in Section 2.3.4, we can see that:

$$\mathrm{Tr}(D_i) = \begin{cases} \pm(\mu_i - 1) & \text{if } \mathrm{MS}(\sigma_i) \cong \Sigma_{1,1}, \\ \pm(4\mu_i^2 - 8\mu_i + 6) & \text{if } \mathrm{MS}(\sigma_i) \cong \Sigma_{0,4}, \end{cases}$$

where $\mathrm{MS}(\sigma_i)$ is the modular surface associated to the pants curve σ_i. (See Section 1.1 for the definition of $\mathrm{MS}(\sigma_i)$.) This gives the conclusion because, from the first claim, we see that, as $\theta \longrightarrow 0$, then $\Im\mu_i \longrightarrow \infty$.

With the same hypothesis as above, given a bending line γ of the pleating lamination η which intersects σ_i, then Series proves Proposition 6.8 of [40]. The corresponding result is the following.

Proposition 3.2.10. *Let $\eta \in \mathrm{ML}_{\mathbb{Q}}(\Sigma)$ be an admissible lamination and let $G = G_{\underline{\mu}}$ be the unique group in $\mathcal{M}$ such that $\mathrm{pl}^+(G) = \theta\eta$, where $\underline{\mu} = (\mu_1, \ldots, \mu_\xi) \in \mathbb{H}^\xi$. Normalise G so that $\rho_{\underline{\mu}}(\sigma_i) = \Upsilon_\infty$ and let γ be a bending line of the pleating lamination η which intersects σ_i. Then, as $\theta \longrightarrow 0$, there is a lift $\tilde{\gamma}$ of γ with endpoints $\gamma^\pm$ such that $|\Re(\gamma^+ - \gamma^-)| \leq 2$ and $\Im\mu_i - 1 < |\Im(\gamma^+ - \gamma^-)| < \Im\mu_i$.*

Any lift of a bending line satisfying these conditions is called *good*. The idea of the proof is to consider again the setting of Corollary 6.5 and the position of the limit set $\Lambda(G)$ in terms of the parameters μ_i. The proof of Proposition 6.8 of [40] generalises to our case. In particular, we use Proposition 3.2.7. In fact, Lemma 3.2.9 tells us that, as $\theta \longrightarrow 0$, then the hypothesis for the second claim of Proposition 3.2.7 is verified as $\theta \longrightarrow 0$.

An easy corollary of the existence of good lifts and of Proposition 3.2.9 is the following. See Corollary 6.9 of [40].

Corollary 3.2.11. *Let $\tilde{\gamma}$ be a good lift of a bending line γ and set $\gamma^+ - \gamma^- = 2re^{i\alpha}$, where, without loss of generality, we take $\Im(\gamma^+ - \gamma^-) > 0$. Then, as $\theta \longrightarrow 0$, we have*

$$r = \frac{\Im\mu_i}{2}(1 + O(\theta)) \quad \textit{and} \quad |\pi/2 - \alpha| = O(\theta).$$

The next result follows from the cross-ratio formula for the complex distance D between geodesics in $\mathbb{H}^3$. Since there are some typos in Series' arguments (see Proposition 6.10 of [40]), we restate the formula with a proof. You can see Series [39] for a detailed discussion about the complex distance $d = d_\alpha(\gamma_1, \gamma_2)$, where α is the common perpendicular between the geodesics γ_1 and γ_2.

Lemma 3.2.12 (Cross-ratio formula for complex distance between geodesics)**.** *Let z_1 and z_2 and w_1 and w_2 be endpoints of two oriented geodesic γ_1 and γ_2 in $\mathbb{H}^3$, and let $d = d_\alpha(\gamma_1, \gamma_2)$ be the complex distance between the two geodesics γ_1 and γ_2. Then*

$$[z_1, z_2, w_1, w_2] := \frac{z_1 - z_2}{z_1 - w_1} \cdot \frac{w_2 - w_1}{w_2 - z_2} = -\frac{1}{\sinh^2(\frac{d}{2})}.$$

Proof. If two oriented geodesics γ_1 and γ_2 in $\mathbb{H}^3$ are at complex distance d, we can conjugate them in such a way that the endpoints z_1, z_2 of the first move to 1, -1 and the endpoints w_1, w_2 of the second one move to e^d, $-e^d$. Then a simple substitution gives the result. $\square$

Applying Lemma 3.2.12 to the endpoints $z_1 = \gamma^+, z_2 = \gamma^-, w_1 = \gamma^+ - 2, w_2 = \gamma^- - 2$ and using Corollary 3.2.11, we obtain:

Proposition 3.2.13. *Let $\tilde{\gamma}$ be a good lift of a bending line γ of the pleating lamination η which intersects $\tilde{\sigma}_i$. Then the complex distance d between $\tilde{\gamma}$ and $\Upsilon_\infty(\tilde{\gamma})$ satisfies*

$$\sinh^2(\frac{d}{2}) = -\frac{1}{r^2 e^{2i\alpha}}.$$

These results are now sufficient for completing the proof of claim (i) in Proposition 3.2.5, see the proof of Proposition 6.6 of [40] for the details. In particular, for the lower bound one should use Lemma 3.2.6, 3.2.9 and the existence of good lifts, while for the lower bound one should apply Corollary 3.2.11 and Proposition 3.2.13.

So now we only need to prove the first inequality of claim (ii) of Proposition 3.2.5. This result is a direct consequence of the fact that, asymptotically, $\tilde{\sigma}^+$ becomes orthogonal to the bending lines.

To explain in detail what that means, let's fix the same conventions as in the discussion preceding Proposition 3.2.8. Let P' and $P(= P' - 2)$ be the points in $\mathbb{H}^3$ at which the piecewise geodesic $\tilde{\sigma}^+$ meets the lifted bending lines $\tilde{\gamma}$ and $\Upsilon_\infty(\tilde{\gamma})$ respectively. Let Q be the highest point of the geodesic segment joining P and P', and let K be the footpoint of the perpendicular from Q to $\mathbb{C}$, as shown in Figure 3.2. Then we have the following result.

Lemma 3.2.14. *In the above setting, as $\theta \longrightarrow 0$, the angle $\angle PKQ$ satisfies $\angle PKQ = O(\sqrt{\theta})$.*

Again, the proof of Proposition 6.12 of Series [40] works in our more general situation. In particular the proof uses Proposition 3.2.6 and 3.2.9.

This result is used in the proof of Proposition 6.13 of [40] in order to show that, along the pleating variety $\mathcal{P}_\eta$, the curve $\tilde{\sigma}^+$ is asymptotically orthogonal to the bending lines as $\theta \longrightarrow 0$. More precisely, a generalisation of Series' result to our situation is the following fact.

Proposition 3.2.15. *Suppose that $\tilde{\sigma}^+$ meets an (oriented) bending line $\tilde{\gamma}$ at a point P so that the acute angle between $\tilde{\sigma}^+$ and $\tilde{\gamma}$ is $\psi(P)$, then, as $\theta \longrightarrow 0$, we have that $|\psi(P) - \pi/2| \leq O(\sqrt{\theta})$.*

The proof of this result follows from Series' proof of Proposition 6.13 [40]. In particular, we should use Proposition 3.2.14 and 3.2.10.

These results are now sufficient to prove the claim (ii) of Proposition 3.2.5. Again, a central role in this estimate is played by Lemma 3.2.6. See the proof of Proposition 6.11 of Series [40].

Hence to complete the proof of Proposition 3.2.3, it remains to bound $\Re\mu_i$. The result one wants to prove is the following.

Proposition 3.2.16. *Let $\eta \in \mathrm{ML}_{\mathbb{Q}}$ be admissible and suppose that the multicurve $\gamma \in \mathcal{S}$ is contained in the support of η and $q_i(\gamma) \neq 0$. Then, if $G = G(\mu_i) \in \mathcal{P}_\eta$, we have $\Re\mu_i = -p_i(\gamma)/q_i(\gamma) + O(1)$ and hence, in particular, $|\operatorname{Arg}\mu_i - \pi/2| = O(\theta)$ as $\theta \longrightarrow 0$.*

To prove this bound, Series uses the concept of 'twist of one geodesic around another' following Minsky [36]. Suppose given a hyperbolic metric h on the surface Σ. The *twist* $\mathrm{tw}_\beta(\gamma, \mathrm{h})$ *of a curve* γ *about another curve* β is defined as follows. Let p be an intersection point of γ with β. Let P be a lift to $\mathbb{H} = \mathbb{H}^2$ of p and let $\tilde{\gamma}, \tilde{\beta}$ be the lifts of γ, β through P. Orient $\tilde{\gamma}, \tilde{\beta}$ with positive endpoints Z, W respectively on $\partial\mathbb{H}$ so that the anticlockwise arc from Z to W does not contain the other two endpoints. Let R be the footpoint of the perpendicular from Z to $\tilde{\beta}$. Let t be the oriented distance PR, where $t > 0$ if R follows P in the positive direction along $\tilde{\beta}$, and $t \leq 0$ otherwise. One verifies, see Lemma 3.1 of [36], that $t/l_\beta(h)$ is independent, up to an additive error of 1, of the choices made, including the choice of p. So one can define $\mathrm{tw}_\beta(\gamma, \mathrm{h}) = \inf t/l_\beta(h)$, where we take the infimum over all possible choices of lifts as above.

The twist $\mathrm{tw}_\beta(\gamma, \mathrm{h})$ is independent of the orientation of β and γ but depends on the choice of the hyperbolic metric $h \in \mathcal{T}$, where $\mathcal{T}$ is the Teichmüller space of Σ. Hovewer Minsky proved in Lemma 3.5 of [36] that, given γ_1 and γ_2 in $\mathcal{S}$, then $\mathrm{tw}_\beta(\gamma_1, \mathrm{h}) - \mathrm{tw}_\beta(\gamma_2, \mathrm{h})$ is independent of $h \in \mathcal{T}$, up to a bounded additive error of 1. See also Lemma 6.16 of [40]. So Series defines the *signed relative twist* of γ_1, γ_2 with respect to β to be

$$I_\beta(\gamma_1, \gamma_2) = \inf_{h \in \mathcal{T}} \mathrm{tw}_\beta(\gamma_1, \mathrm{h}) - \mathrm{tw}_\beta(\gamma_2, \mathrm{h}),$$

and she describes a useful way of computing it, which is the following.

Lemma 3.2.17 (Lemma 6.17 of [40])**.** *Let $\gamma_1, \gamma_2 \in \mathcal{S}$ and let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be lifts of γ_1, γ_2 to $\mathbb{H}$ which cut the fixed axis $\tilde{\beta}$ corresponding to β. Let $b \in \Gamma$ be the primitive element whose axis is $\tilde{\beta}$ and whose attracting fixed point is the positive endpoint of*

$\tilde{\beta}$, *where* Γ *is the Fuchsian group uniformising* h*. Then* $\mathrm{tw}_\beta(\gamma_1, \mathrm{h}) - \mathrm{tw}_\beta(\gamma_2, \mathrm{h})$ *is equal, in magnitude, to the number of times the images* $b^n(\tilde{\gamma}_1)$*, with* $n \in \mathbb{Z}$*, intersect* $\tilde{\gamma}_2$*, up to a bounded additive error of* 1*. The sign is negative, if* $b(\tilde{\gamma}_1)$ *follows* $\tilde{\gamma}_1$ *in the positive direction along* $\tilde{\gamma}_2$*, and positive otherwise.*

Series proves Proposition 6.14 of [40] by computing $I_{\sigma_i}(\gamma, \gamma_T)$ in two different ways, where $\gamma_T \in \mathcal{S}$ is the curve corresponding to the generator $T \in \pi_1(\Sigma)$; see Figure 3.1. Recall that she analyses the case $\Sigma = \Sigma_{1,2}$. Adapting Series' method to our case, we prove Proposition 3.2.16 by computing $I_{\sigma_i}(\gamma, D_i)$ in two different ways, where D_i is the dual curve to the pants curve σ_i. Recall that the twist parameter $p_i(\gamma)$ we are using in this thesis is half the twist parameter Series uses.

(i) Using combinatorial ideas related to the definition of the Dehn–Thurston coordinates $p_i(\gamma)$ and $q_i(\gamma)$ and using Lemma 3.2.17, one can prove, following ideas of Lemma 6.18 of [40], that $I_{\sigma_i}(\gamma, D_i) = -\frac{p_i(\gamma)}{2q_i(\gamma)} + O(1)$; see Proposition 3.2.18.

(ii) Using properties related to the position of the limit set and of a fundamental domain for the action of the group $G = G_{\underline{\mu}}$ in the regular domain $\Omega(G)$, one can prove, following ideas of the proof of Proposition 6.14 of [40], that $I_{\sigma_i}(\gamma, D_i) = \frac{\Re\mu_i}{2} + O(1)$; see the proof of Proposition 3.2.16.

Since Series' proofs need to be changed slightly, we will prove Proposition 3.2.16 in detail. First, we have the following result.

Proposition 3.2.18. *Suppose that the multicurve* $\gamma \in \mathcal{S}(\Sigma)$ *has Dehn–Thurston coordinates* $\mathbf{i}_{DT}(\gamma) = (q_1(\gamma), p_1(\gamma), \ldots, q_\xi(\gamma), p_\xi(\gamma))$ *and suppose that* $q_i(\gamma) \neq 0$*. Then* $I_{\sigma_i}(\gamma, D_i) = -\frac{p_i(\gamma)}{2q_i(\gamma)} + O(1)$*.*

Proof. For this proof, we will use the Dehn–Thurston coordinates, see Section 1.2. In particular, we will use the D. Thurston twist, see Section 1.2.1.

Given $\sigma = \sigma_i$, let $\mathbb{A} = \mathbb{A}_i$ be a small annular neighbourhood of σ. Then, by definition of D_i, we have that $D_i \cap \mathbb{A}$ consists of two arcs δ and δ'. Put γ in D. Thurston position. Then $\gamma \cap \mathbb{A}$ is made of $q_i = q_i(\gamma)$ connected components and each connected component intersects δ either m or $m+1$ times, where $m \in \mathbb{Z}$. By the definition of the twist we have that $mq_i \leq \frac{|p_i|}{2} \leq (m+1)q_i$, that is $m = \left[\frac{|p_i|}{2q_i}\right] + O(1)$, where $[\cdot]$ is the integer part.

Consider a hyperbolic structure $\Sigma_* = (\Sigma, h_0)$ on Σ such that σ_i and D_i are perpendicular and let $\Gamma_0 \in \mathcal{F}(\Sigma)$ be the associated Fuchsian group. Choose lifts $\tilde{\sigma}$, $\tilde{D}$, $\tilde{\gamma}$ and $\tilde{\mathbb{A}}$ of σ, D, γ and $\mathbb{A}$, respectively, to the covering space $\mathbb{H}$, such that

$\tilde{\sigma} \subset \tilde{\mathbb{A}}$, $\tilde{D} \cap \tilde{\sigma} \neq \emptyset$ and $\tilde{\gamma} \cap \tilde{\sigma} \neq \emptyset$. Let $f \in \Gamma_0$ be the primitive element whose axis is $\tilde{\sigma}$ and whose attracting fixed point is the positive endpoint of $\tilde{\sigma}$. By Lemma 3.2.17 we have that $I_{\sigma_i}(\gamma, D_i)$ is equal, in magnitude, to the number of times the images $f^n(\tilde{\gamma})$ intersect $\tilde{D}$, up to a bounded additive error of 1 (where $n \in \mathbb{Z}$). We can see that this number is, up to a bounded additive error of 1, the number of times that a connected component of $\gamma \cap \mathbb{A}$ cuts the arc δ in $\mathbb{A} \subset \Sigma$, that is, the number m defined above. This proves that $|I_{\sigma_i}(\gamma, D_i)| = \left|\frac{p_i(\gamma)}{2q_i(\gamma)}\right| + O(1)$.

Finally, in order to understand the sign of $I_{\sigma_i}(\gamma, D_i)$, consider the metric Σ_*, which we choose so that σ_i and D_i are perpendicular. Then $\mathrm{tw}_{\sigma_\mathrm{i}}(\mathrm{D_i}, \Sigma_*) = 0$, while $\mathrm{tw}_{\sigma_\mathrm{i}}(\gamma, \Sigma_*)$ is negative if $p_i > 0$ and positive if $p_i < 0$, that is the sign of $\mathrm{tw}_{\sigma_\mathrm{i}}(\gamma, \Sigma_*)$ is the sign of $-p_i$. This concludes the proof. □

These results are now enough to prove Proposition 3.2.16 and, hence, also Proposition 3.2.3.

Proof of Proposition 3.2.16. In Proposition 3.2.18 we showed that $I_{\sigma_i}(\gamma, D_i) = -\frac{p_i(\gamma)}{2q_i(\gamma)} + O(1)$. We will now show that $I_{\sigma_i}(\gamma, D_i) = \frac{\Re\mu_i}{2} + O(1)$ which will prove the result.

As done before, normalize $G_{\underline{\mu}}$ so that $\rho_{\underline{\mu}}(\sigma_i) = \Upsilon_\infty$ and let $\tilde{\sigma}^+ = \tilde{\sigma}_i^+$ be the lift of σ_i to $\partial^+ \mathrm{CH}(\Lambda)$ which is invariant under $z \mapsto z - 2$. Consider the region $\mathcal{ST}'$ defined in the proof of Proposition 3.2.7. (We refer to the notation used in Section 3.2.3. See especially Figure 3.4.) We can consider a good lift $\tilde{\gamma}$ of γ, which certainly intersects $\tilde{\sigma}^+$. It follows from the usual ping-pong theorem methods, that there is a lift $\tilde{D} = \tilde{D}_i$ of D_i to $\partial^+ \mathrm{CH}(\Lambda)$ which cuts the side of $\mathcal{ST}'$ between B_2 and B_3' and the side between B_4 and B_5, see Section 3.2.3.

By Lemma 3.2.17, $I_{\sigma_i}(\gamma, \mathbb{D}_i)$ is, up to sign, the number of images $\Upsilon_\infty^n(\tilde{\gamma})$ of $\tilde{\gamma}$ which cut $\tilde{D}_i$. Hence, since $\tilde{\gamma}$ is a good lift, we can see that $|I_{\sigma_i}(\gamma, D_i)| = \left|\frac{\Re\mu_i}{2}\right| + O(1)$. Using Lemma 3.2.17 and the orientation described in Figure 3.4, we can see that $I_{\sigma_i}(\gamma, D_i) = \frac{\Re\mu_i}{2} + O(1)$. Hence we have

$$\Re\mu_i = -\frac{p_i(\gamma)}{q_i(\gamma)} + O(1)$$

and the result follows. □

Remark 3.2.19. Note that, following through the arguments of the proofs of Propositions 3.2.18 and 3.2.16, we can see that $\Re\mu_i$ agrees with $-\frac{p_i(\gamma)}{q_i(\gamma)}$ up to a bounded additive error of 4. Unfortunately, that doesn't improve the results we want to prove in the remaining part of this chapter.

These results let us prove also Theorem C from the Introduction. We follow Series' proof very closely. Let's recall the statement of the theorem.

Theorem 3.2.20 (Theorem C)**.** *Let $\eta = \sum_1^\xi a_i \delta_{\gamma_i}$ be as above. Then, as the bending measure $\mathrm{pl}^+(G) \in [\eta]$ tends to zero, the induced hyperbolic structure of $\partial^+\mathcal{C}$ along $\mathcal{P}_\eta$ converges to the barycentre of the laminations $\sigma_1, \dots, \sigma_\xi$ in the Thurston boundary of $\mathcal{T}(\Sigma)$.*

Proof. Let $\eta = \sum_1^\xi a_i \delta_{\gamma_i}$ be an admissible lamination and let $G = G_\eta(\theta)$ be the unique group for which $\mathrm{pl}^+(G) = \theta\eta$. Let $h(\theta)$ denote the hyperbolic structure of $\partial^+\mathcal{C}(G)$. Let $l^+_{\sigma_i}$ be the hyperbolic length of the geodesic representative of σ_i on the hyperbolic surface $\partial^+\mathcal{C}(G)$. By Proposition 3.2.5 (ii), we have that, as $\theta \longrightarrow 0$, then $l^+_{\sigma_i} \longrightarrow 0$, for all $i = 1, \dots, \xi$. So the limit of the hyperbolic structures $h(\theta)$ in $\mathrm{PML}(\Sigma)$ is in the linear span of $\delta_{\sigma_1}, \dots, \delta_{\sigma_\xi}$. We want to prove that the limit is the barycentre $\sum_{i=1}^\xi \delta_{\sigma_i}$.

Let $\delta, \delta' \in \mathcal{S}$. Since $\sigma_1, \dots, \sigma_\xi$ are a maximal set of simple curves on Σ, the thin part of $h(\theta)$ is eventually contained in collars $\mathbb{A}_i$ around σ_i of approximate width $\log(\frac{1}{l^+_{\sigma_i}})$ and the lengths of δ, δ' outside the collars $\mathbb{A}_i$ are bounded (with a bound depending only on the combinatorics of δ, δ' and hence the canonical coordinates $\mathbf{i}(\delta), \mathbf{i}(\delta')$). By Proposition 3.2.16, the twisting around $\mathbb{A}_i$ is bounded. We deduce that, for any curve δ transverse to σ_i, we have:

$$l^+_\delta = 2\sum_{i=1}^{\xi} q_i(\delta)\log(\frac{1}{l^+_{\sigma_i}}) + O(1), \tag{3.2}$$

see, for example, Proposition 4.2 of Diaz and Series [13]. By Theorem 3.2.3 and by Proposition 3.2.5 (i) we have $\frac{l^+_{\sigma_i}}{l^+_{\sigma_j}} \longrightarrow \frac{q_j(\eta)}{q_i(\eta)}$, and, since η is admissible, $q_i(\xi) > 0$ for $i = 1, \dots, \xi$. Thus $\frac{\log l^+_{\sigma_i}}{\log l^+_{\sigma_j}} \longrightarrow 1$. Hence, by factorizing both in the numerator and the denominator the term $2\log(\frac{1}{l^+_{\sigma_1}})$, we get the following:

$$\frac{l^+_\delta}{l^+_{\delta'}} \longrightarrow \frac{\sum_{i=1}^\xi q_i(\delta)}{\sum_{i=1}^\xi q_i(\delta')} = \frac{i(\delta, \sum_1^\xi \delta_{\sigma_i})}{i(\delta', \sum_1^\xi \delta_{\sigma_i})}.$$

The result follows from the definition of convergence to a point in $\mathrm{PML}(\Sigma)$. $\square$

3.2.2 Asymptoticity results for the pleating rays

The next results are the key tools for the proof of Theorem B. We need to fix more notation. Suppose that γ is a bending line of $\partial\mathcal{C}^+(G)$ for a group $G = G_{\underline{\mu}} \in \mathcal{P}_\eta$.

The Top Terms' Formula (that is, Theorem 2.4.1), together with the condition $\operatorname{Tr}\gamma \in \mathbb{R}$ of Lemma 3.1.6, determines the asymptotic behaviour of the parameters $\underline{\mu} \in \mathcal{P}_{\underline{\mu}}$ as the bending measure tends to 0, in terms of the canonical coordinates $\mathbf{i}(\gamma)$ of γ. For the remaining of this Chapter $\mathbf{i}(\gamma)$ will always denote the vector $\mathbf{i}_{DT}(\gamma) = (q_1(\gamma), p_1(\gamma), \ldots, q_\xi(\gamma), p_\xi(\gamma))$ (read using D. Thurston twist), see Section 1.2. In particular, for $\underline{\mu} = (\mu_1, \ldots, \mu_\xi) \in \mathbb{C}^\xi$, set

$$\mu_i - 1 = x_i + iy_i, \quad Y = \|(y_1, \ldots, y_\xi)\| = (y_1^2 + \ldots + y_\xi^2)^{\frac{1}{2}} \text{ and } \quad w_i = \frac{y_i}{Y}.$$

Define

$$\begin{aligned} E_\gamma(\mu_1, \ldots, \mu_\xi) &= w_2 \cdots w_\xi(q_1 x_1 + p_1) + \ldots + w_1 \cdots w_{\xi-1}(q_\xi x_\xi + p_\xi) \\ &= w_1 \cdots w_\xi \sum_{i=1}^{\xi} \frac{(q_i x_i + p_i)}{w_i}, \end{aligned}$$

where $w_i > 0$, for $i = 1, \ldots, \xi$.

The reason why we introduce this expression is the following result, which generalises Proposition 7.1 of [40]. Again Series' proof extends clearly to our case, but we will repeat it here, since it will help the reader to understand the meaning of $E_\gamma(\mu_1, \ldots, \mu_\xi)$.

Proposition 3.2.21. *Suppose that $\eta \in \mathrm{ML}_{\mathbb{Q}}$ is an admissible lamination, that $G_{\underline{\mu}} \in \mathcal{P}_\eta$ has bending measure $\mathrm{pl}^+(G) = \theta\eta$, where $\underline{\mu} = (\mu_1, \ldots, \mu_\xi) \in \mathbb{H}^\xi$, and that γ is a bending line of η. Then, as $\theta \longrightarrow 0$, we have*

$$E_\gamma(\mu_1, \ldots, \mu_\xi) = O(\theta).$$

Proof. Suppose first that $q_i = q_i(\gamma) > 0$ for all $i = 1, \ldots, \xi$. Set $a_i = -\frac{p_i(\gamma)}{q_i(\gamma)}$. By Theorem 2.4.1 we have

$$\operatorname{Tr}\rho_{\underline{\mu}}(\gamma) = \pm i^q 2^h \Big(\mu_1 - a_1 - 1\Big)^{q_1} \cdots \Big(\mu_\xi - a_\xi - 1\Big)^{q_\xi} + R, \tag{3.3}$$

where:

- $q = \sum_{i=1}^{\xi} q_i(\gamma) > 0$;
- $h = h(\gamma)$ is the total number of *sbcc*–arcs of γ in the complement of the pants curves;
- R represents terms with total degree in $\mu_1 \cdots \mu_\xi$ at most $q-2$ and of degree at most q_i in the variable μ_i

By Proposition 3.2.16, $x_i - a_i = O(1)$ and by Corollary 3.2.4

$$\left|\frac{q_i(\eta)}{q_j(\eta)} - \frac{w_j}{w_i}\right| = O(\theta). \tag{3.4}$$

(Notice that the terms in (3.4) involve $q_i(\eta)$ as opposed to $q_i = q_i(\gamma)$ in (3.3).)

Note that, by Lemma 3.2.9, $Y \longrightarrow \infty$ as $\theta \longrightarrow 0$.

Hence, arranging the terms of (3.3) in order of decreasing powers of Y, and recalling that $i^{2q} = \pm 1$, we get the following

$$\pm\left(\operatorname{Tr}\rho_{\underline{\mu}}(\gamma)\right)2^{-h} =$$

$$Y^q w_1^{q_1}\dots w_\xi^{q_\xi} + iY^{q-1}w_1^{q_1}\dots w_\xi^{q_\xi}\left(\sum_{i=1}^{\xi}\frac{q_i(x_i - a_i)}{w_i}\right) + O(Y^{q-2}).$$

Note that

$$w_1^{q_1}\dots w_\xi^{q_\xi}\left(\sum_{i=1}^{\xi}\frac{q_i(x_i - a_i)}{w_i}\right) = w_1^{q_1-1}\dots w_\xi^{q_\xi-1}E_\gamma(\mu_1,\dots,\mu_\xi).$$

By Lemma 3.1.6, $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma) \in \mathbb{R}$. Hence from the definition of E_γ we have that

$$w_1^{q_1-1}\dots w_\xi^{q_\xi-1}E_\gamma(\mu_1,\dots,\mu_\xi) = O(1/Y).$$

From this and using (3.4), we deduce that

$$E_\gamma(\mu_1,\dots,\mu_\xi) = O(1/Y).$$

Since $1/Y = O(\theta)$ by Lemma 3.2.9, this proves the result, if $q_i = q_i(\gamma) > 0$ for all $i = 1,\dots,\xi$..

We still have to deal with the case that there exists a non-empty subset I in $\{1,\dots\xi\}$ such that $q_i = q_i(\gamma) = 0$ for all $i \in I$. Then in the trace polynomyal $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma)$ the factors $\left(\mu_i - a_i - 1\right)^{q_i} = 1$ for all $i \in I$. The result then follows by similar reasoning to the above. Note, infact, that we need to use Equation (3.4) only for the q_j which appear in $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma)$, that is the q_j such that $j \in \{1,\dots\xi\}\setminus I$. □

Our aim is to locate the pleating ray $\mathcal{P}_\eta$, where $\eta = \sum_{i=1}^m a_i\gamma_i$. If $G \in \mathcal{P}_\eta$, then $\partial^+\mathcal{C}(G) - \{\gamma_1,\dots,\gamma_m\}$ is flat, so that, not only $\gamma_1,\dots,\gamma_m$, but also any curve $\zeta \in \operatorname{lk}(\gamma_1,\dots,\gamma_m)$, is a bending line for G, where $\operatorname{lk}(\gamma_1,\dots,\gamma_m)$ denotes the set of (isotopy classes of) curves in Σ disjoint from $\gamma_1\cup\dots\cup\gamma_m$. The names comes from the fact that it is the link of the simplex $(\gamma_1,\dots,\gamma_m)$ in the complex of curves of Σ.

(Note that this set in [40] is called *wheel.*) One can think of it as the set of all curves $\zeta \in \mathcal{S}_0 = \mathcal{S}_0(\Sigma)$ disjoint from $\gamma_1, \ldots, \gamma_m$. Thus $\underline{\mu} = (\mu_1, \ldots, \mu_\xi)$ is constrained by the equations:

$$\Im \operatorname{Tr} \gamma_i = \Im \operatorname{Tr} \zeta = 0 \quad \forall i = 1, \ldots, m, \ \forall \zeta \in \operatorname{lk}(\gamma_1, \ldots, \gamma_m),$$

and hence, using Proposition 3.2.21, it is constrained by the equations:

$$\begin{cases} E_{\gamma_i}(\mu_1, \ldots, \mu_\xi) = O(\theta) \\ E_\zeta(\mu_1, \ldots, \mu_\xi) = O(\theta), \end{cases}$$

for all $\zeta \in \operatorname{lk}(\gamma_1, \ldots, \gamma_m)$ and for $i = 1, \ldots, m$.

We would like to solve these equations simultaneously for $\mu_1, \ldots, \mu_\xi$. Following the analysis in Section 7 of [40], we note that, for any curve $\omega \in \mathcal{S}$, we have:

$$E_\omega(\mu_1, \ldots, \mu_\xi) = \mathbf{i}(\omega) \cdot \mathbf{u},$$

where $\cdot$ is the scalar product in $\mathbb{R}^\xi$, where, as usual, $\mathbf{i}(\omega) = (q_1(\omega), p_1(\omega), \ldots, q_\xi(\omega), p_\xi(\omega))$ and where

$$\begin{aligned} \mathbf{u} \; &= \; (u_{11}, u_{12}, \ldots, u_{\xi 1}, u_{\xi 2}) \; = \; w_1 \cdots w_\xi \left(\frac{x_1}{w_1}, \ \frac{1}{w_1}, \ \ldots, \ \frac{x_\xi}{w_\xi}, \ \frac{1}{w_\xi}\right) \\ &= (w_2 \cdots w_\xi x_1, \ \ w_2 \cdots w_\xi, \ \ldots, \ \ w_1 \cdots w_{\xi-1} x_\xi, \ \ w_1 \cdots w_{\xi-1}), \end{aligned}$$

with $x_i = \Re \mu_i - 1, \ w_i = \frac{\Im \mu_i}{Y}$ as above. We will use linear algebra and Thurston's symplectic form Ω_{Th} to approximately solve the equations:

$$\begin{cases} \mathbf{i}(\gamma_i) \cdot \mathbf{u} = O(\theta) \\ \mathbf{i}(\zeta) \cdot \mathbf{u} = O(\theta), \end{cases}$$

for all $\zeta \in \operatorname{lk}(\gamma_1, \ldots, \gamma_m)$ and for $i = 1, \ldots, m$. See Section 1.3 for the definition of Ω_{Th} and recall the last claim of Theorem 1.3.3, which says that, if γ, γ' are disjoint, then $\Omega_{\mathrm{Th}}(\gamma, \gamma') = 0$. As already underlined in Equation (1.2), Thurston's symplectic form induces a map $\mathbb{R}^{2\xi} \longrightarrow \mathbb{R}^{2\xi}$ defined by

$$\mathbf{x} = (x_1, y_1, \ldots, x_\xi, y_\xi) \mapsto \mathbf{x}^* = (y_1, -x_1, \ldots, y_\xi, -x_\xi)$$

and such that

$$2\Omega_{\mathrm{Th}}(\mathbf{i}(\gamma), \mathbf{i}(\delta)) = \mathbf{i}(\gamma) \cdot \mathbf{i}(\delta)^*, \tag{3.5}$$

for all $\gamma, \delta \in \mathcal{S}_0$ and where $\cdot$ is the usual inner product on $\mathbb{R}^{2\xi}$. So Theorem 1.3.3 tell us that $\mathbf{i}(\gamma)^*$ is orthogonal not only to $\mathbf{i}(\gamma)$, but also to all curves in $\mathrm{wh}(\gamma)$.

We need the following Lemma, which generalises Lemma 7.2 of [40]. See Section 1.3.3.a for a definition of *standard* train tracks. Recall also the definition of $\mathrm{lk}(\gamma_1, \ldots, \gamma_m)$ given at page 98.

Lemma 3.2.22.

(i) Suppose that $\{\gamma_1, \ldots, \gamma_\xi\}$ is a pants decomposition of Σ. Then γ_i are supported on a common standard train track and $\mathbf{i}(\gamma_i)$ are independent vectors in $\mathbf{i}(\mathcal{S}(\Sigma)) \subset (\mathbb{Z}_+ \times \mathbb{Z})^\xi$.

(ii) Given any set of disjoint curves simplex $\mathbf{g} = \{\gamma_1, \ldots, \gamma_m\}$ in Σ, we can find $2\xi - 2m$ curves $D_1, \ldots, D_{2\xi-2m} \in \mathrm{lk}(\gamma_1, \ldots, \gamma_m)$ such that the vectors $\mathbf{i}(\gamma_1), \ldots, \mathbf{i}(\gamma_m)$, $\mathbf{i}(D_1) \ldots, \mathbf{i}(D_{2\xi-2m})$ are independent vectors in $\mathbf{i}(\mathcal{S}(\Sigma)) \subset (\mathbb{Z}_+ \times \mathbb{Z})^\xi$.

Proof. (i): Recall that the standard train tracks are defined with respect to the pants decomposition $\mathcal{PC} = \{\sigma_1, \ldots, \sigma_\xi\}$ for Σ. The pants decomposition $\mathcal{PC}$ decomposes the surface Σ into pieces which are annuli $\mathbb{A}_1, \ldots, \mathbb{A}_\xi$ around the pants curves and pairs of pants $P_1, \ldots, P_k$ in the complement of these annuli, where $\xi = \xi(\Sigma_{g,b}) = 3g - 3 + b$ and $k = -\chi(\Sigma_{g,b}) = 2g - 2 + b$.

Recall that the space $\mathrm{ML}(\Sigma)$ of measured laminations on Σ is an open piecewise integral linear (PIL) ball of dimension 2ξ, where the maximal charts are the spaces $\mathcal{V}(\boldsymbol{\tau})$, where $\boldsymbol{\tau}$ is a maximal train track in Σ and $\mathcal{V}(\boldsymbol{\tau})$ is the collection of all (not necessary nonzero) transverse measures supported on $\boldsymbol{\tau}$, see Section 1.3.2 and Section 1.3.3.a for the definitions.

The disjointness of the curves $\gamma_1, \ldots, \gamma_\xi$ tells us that the train track $\boldsymbol{\tau'} = \cup_{i=1}^{\xi} \gamma_i$ is carried by a common standard train track, say $\boldsymbol{\tau} \subset \Sigma$. In fact, you can look at how these curves intersect the above annuli and pairs of pants. (Of course the choice of $\boldsymbol{\tau}$ may not be unique.) Then, as described by Penner [38] in Proposition 2.2.4, the incidence matrix describes the linear inclusion $\mathcal{V}(\boldsymbol{\tau'}) \subset \mathcal{V}(\boldsymbol{\tau})$.

Since the curves γ_i are linearly independent vectors in $\mathcal{V}(\boldsymbol{\tau'})$, then, using the inclusion above, they are linearly independent vectors in $\mathcal{V}(\boldsymbol{\tau})$. Finally, since the map $\mathbf{i}$ restricted to any chart is a linear inclusion, we have that the vectors $\mathbf{i}(\gamma_i)$ are independent vectors in $\mathbf{i}(\mathcal{S}(\Sigma)) \subset (\mathbb{Z}_+ \times \mathbb{Z})^\xi$, as we wanted.

(ii): First, note that $\xi(\Sigma_{\mathbf{g}}) = \xi(\Sigma) - m = \xi - m = \xi'$, where $\Sigma_{\mathbf{g}} = \Sigma \setminus \{\gamma_1, \ldots, \gamma_m\}$. This can be proved by induction on m. (The case $m = 1$ can be discussed by dealing with the case γ separating and non-separating.)

Second, recall that the space $\mathrm{ML}(\Sigma_{\mathbf{g}})$ of measured laminations on $\Sigma_{\mathbf{g}}$ is an open piecewise integral linear (PIL) ball of dimension $2\xi'$. Note also that the set $\mathrm{ML}_{\mathbb{Q}}(\Sigma_{\mathbf{g}})$ of rational measured laminations in $\Sigma_{\mathbf{g}}$ is a dense subset of $\mathrm{ML}(\Sigma_{\mathbf{g}})$. Note that the boundary curves of Σ aren't contained in $\mathrm{ML}(\Sigma)$.

We will show our result by constructing a chart $\mathcal{V}(\boldsymbol{\eta})$ of dimension $2\xi - m$ such that $\boldsymbol{\eta}$ carries the curves $\gamma_1, \ldots, \gamma_m$ and is carried by a standard train track $\boldsymbol{\tau}$ and by using the density of $\mathrm{ML}_{\mathbb{Q}}(\Sigma_{\mathbf{g}})$ into $\mathrm{ML}(\Sigma_{\mathbf{g}})$ explained above.

The idea is the following. Consider a maximal lamination λ' in $\Sigma_{\mathbf{g}}$ (where maximal means that the complement of λ' in $\Sigma_{\mathbf{g}}$ is made of pieces which are triangles or once punctured monogons.) Let λ be the lamination defined by $\lambda := \lambda' \sqcup \gamma_1 \ldots \sqcup \gamma_m$. Then λ is carried by a standard train track $\boldsymbol{\tau}$.

By splitting, we can find a train track $\boldsymbol{\eta}$ which is carried by $\boldsymbol{\tau}$ and such that the curves $\gamma_1, \ldots, \gamma_m$ are disjointly embedded in $\boldsymbol{\eta}$ and the lamination λ' is carried by $\boldsymbol{\eta}$. So the curves $\gamma_1, \ldots, \gamma_m$ are independent vectors in the chart $\mathcal{V}(\boldsymbol{\eta})$ and there exists a subtrack $\boldsymbol{\eta}'$ of $\boldsymbol{\eta}$ contained in $\Sigma_{\mathbf{g}}$ which carries λ' (and so of dimension $2\xi' = 2\xi - 2m$). By the density of $\mathrm{ML}_{\mathbb{Q}}(\Sigma_{\mathbf{g}})$ into $\mathrm{ML}(\Sigma_{\mathbf{g}})$, we can find curves $D_1, \ldots, D_{2\xi-2m} \subset \Sigma_{\mathbf{g}}$ (and so in $\mathrm{lk}(\gamma_1, \ldots, \gamma_m)$) carried by $\boldsymbol{\eta}'$ and such that $\gamma_1, \ldots, \gamma_m, D_1, \ldots, D_{2\xi-2m}$ are linearly independent vectors in $\mathcal{V}(\boldsymbol{\eta})$.

Now, as before, the incidence matrix describes the linear inclusion $\mathcal{V}(\boldsymbol{\eta}) \subset \mathcal{V}(\boldsymbol{\tau})$. So the curves $\gamma_1, \ldots, \gamma_m, D_1, \ldots, D_{2\xi-2m}$ are linearly independent vectors in $\mathcal{V}(\boldsymbol{\tau})$. Finally, since the map $\mathbf{i}$ restricted to any chart is a linear inclusion, we have that the vectors $\mathbf{i}(\gamma_1), \ldots, \mathbf{i}(\gamma_m)$, $\mathbf{i}(D_1) \ldots, \mathbf{i}(D_{2\xi-2m})$ are independent vectors in $\mathbf{i}(\mathcal{S}(\Sigma)) \subset (\mathbb{Z}_+ \times \mathbb{Z})^\xi$, as we wanted. $\square$

Now we can state precisely Theorem B of the Introduction and prove it. Before doing that, let's define the exceptional laminations.

Definition 3.2.23. A geodesic lamination $\eta = \sum_{i=1}^m a_i \delta_{\gamma_i}$ is *exceptional* if the matrix $(q_i(\gamma_j))_{\substack{i=1,\ldots,\xi \\ j=1,\ldots,m}}$ has rank strictly less than m.

Theorem 3.2.24 (Theorem B). *Suppose that $\eta = \sum_{i=1}^m a_i \gamma_i$ is admissible and not exceptional (with $m \leqslant \xi$). Let $\mathbf{i}(\eta) = (q_1(\eta), p_1(\eta), \ldots, q_\xi(\eta), p_\xi(\eta))$. Let also*

$$L_\eta : [0, \infty) \longrightarrow \mathbb{H}^\xi$$

be the line defined by $t \mapsto (z_1(t), \ldots, z_\xi(t))$, where

$$z_j(t) = -\frac{p_j}{q_j} + 1 + it\frac{q_1}{q_j}.$$

Let $(\mu_1(\theta), \dots, \mu_\xi(\theta)) \in \mathbb{H}^\xi$ *be the point corresponding to the group* $G = G_\eta(\theta)$ *with* $\mathrm{pl}^+(G) = \theta\eta$*, so that the pleating ray* $\mathcal{P}_\eta$ *is the image of the map* $\theta \mapsto (\mu_1(\theta), \dots, \mu_\xi(\theta))$ *for a suitable range of* $\theta > 0$*. Then* $\mathcal{P}_\eta$ *approaches* L_η *as* $\theta \longrightarrow 0$ *in the sense that, if* $t(\theta) = \frac{4}{\theta q_1}$*, then, for* $i = 1, \dots, \xi$*, we have*

$$\begin{cases} |\Re\mu_i(\theta) - \Re z_i(t(\theta))| = O(\theta) \\ |\Im\mu_i(\theta) - \Im z_i(t(\theta))| = O(1). \end{cases}$$

Proof of Theorem 3.2.24. We use the previous notation, that is we write $\mu_i(\theta) - 1 = \mu_i - 1 = x_i + iy_i$, $Y = \|(y_1, \dots, y_\xi)\|$, and $w_i = \frac{y_i}{Y}$, where the dependence on θ is clear. By Theorem 3.2.3, we have $y_i - \frac{4}{\theta q_i} = O(1)$. On the other hand, with $t = t(\theta)$ as in the statement of the theorem, we find $\Im z_i(t) = t\frac{q_1}{q_i} = \frac{4}{\theta q_i}$. Thus, for $i = 1, \dots, \xi$, we have that, as $\theta \longrightarrow 0$, then:

$$|\Im\mu_i(\theta) - \Im z_i(t(\theta))| = O(1),$$

as we wanted to prove.

Now, let's deal with the coordinates $x_i = \Re\mu_i(\theta) - 1$. Given $\gamma_1, \dots, \gamma_m$, let $D_1, \dots, D_{2\xi-2m}$ be the curves defined by Lemma 3.2.22. Then we have that $\mathbf{i}(\gamma_1), \dots, \mathbf{i}(\gamma_m)$, $\mathbf{i}(D_1), \dots, \mathbf{i}(D_{2\xi-2m})$ are linearly independent. If $(\mu_1, \dots, \mu_\xi) \in \mathcal{P}_\eta$, then the curves $\gamma_1, \dots, \gamma_m, D_1, \dots, D_{2\xi-2m}$ are all bending lines of $G_{\underline{\mu}}$, where $\underline{\mu} = (\mu_1, \dots, \mu_\xi)$. Since $D_1, \dots, D_{2\xi-2m}$ lie in a flat piece, it follows from Lemma 3.1.6 that

$$\Im\,\mathrm{Tr}(\gamma_i) = \Im\,\mathrm{Tr}(D_j) = 0$$

for $i = 1, \dots, m$ and $j = 1, \dots, 2\xi - 2m$. So, by Proposition 3.2.21, it follows that

$$E_\zeta(\mu_1, \dots, \mu_\xi) = O(\theta) \ \text{ as } \ \theta \longrightarrow 0$$

for $\zeta \in \{\gamma_1, \dots, \gamma_m, D_1, \dots, D_{2\xi-2m}\}$. Defining $W = w_1 \cdots w_\xi$ and regarding these as equations in $\mathbb{R}^{2\xi}$ for a parameter $\mathbf{u} \in \mathbb{R}^{2\xi}$, where

$$\mathbf{u} \ = \ (u_{11}, u_{12}, \dots, u_{\xi 1}, u_{\xi 2}) \ = \ W(\frac{x_1}{w_1}, \ \frac{1}{w_1}, \ \dots, \ \frac{x_\xi}{w_\xi}, \ \frac{1}{w_\xi}),$$

we have that, for $\zeta \in \{\gamma_1, \dots, \gamma_m, D_1, \dots, D_{2\xi-2m}\}$,

$$\mathbf{i}(\zeta) \cdot \mathbf{u} = O(\theta). \tag{3.6}$$

On the other hand, by Theorem 1.3.3, we have:

$$\Omega_{\mathrm{Th}}(\gamma_i,\zeta)=0 \text{ for } i=1,\dots,m, \text{ for } \zeta\in \mathrm{lk}(\gamma_1,\dots,\gamma_m)\cup\{\gamma_1,\dots,\gamma_m\}.$$

Hence, using Equation (3.5), we have

$$\mathbf{i}(\zeta)\cdot\mathbf{i}(\gamma_i)^*=0 \text{ for } i=1,\dots,m, \text{ for } \zeta\in\{\gamma_1,\dots,\gamma_m,D_1,\dots,D_{2\xi-2m}\}. \tag{3.7}$$

Now $\mathbf{i}(\gamma_1),\dots,\mathbf{i}(\gamma_m)$, $\mathbf{i}(D_1)\dots,\mathbf{i}(D_{2\xi-2m})$ are independent, and using the definition of $\mathbf{i}(\cdot)^*$, we can see that $\mathbf{i}(\gamma_1)^*\dots,\mathbf{i}(\gamma_m)^*$ are independent too. In addition, by Equation (3.7), we have that $\mathbf{i}(\gamma_i)^*$ is perpendicular to $\mathbf{i}(\zeta)$ for all $i=1,\dots,m$ and $\zeta\in\{\gamma_1,\dots,\gamma_m,D_1,\dots,D_{2\xi-2m}\}$. So the vectors $\mathbf{i}(\gamma_1),\dots,\mathbf{i}(\gamma_\xi)$, $\mathbf{i}(D_{m+1})\dots,\mathbf{i}(D_\xi)$, $\mathbf{i}(\gamma_1)^*\dots,\mathbf{i}(\gamma_m)^*$ form a basis of $\mathbb{R}^{2\xi}$. Hence we can write

$$\mathbf{u}(\theta)=\lambda_1(\theta)\mathbf{i}(\gamma_1)^*+\ldots+\lambda_m(\theta)\mathbf{i}(\gamma_m)^*+\nu(\theta)\mathbf{v}(\theta), \tag{3.8}$$

where $\mathbf{v}=\mathbf{v}(\theta)$ is in the linear span of $\mathbf{i}(\gamma_1),\dots,\mathbf{i}(\gamma_m)$, $\mathbf{i}(D_1)\dots,\mathbf{i}(D_{2\xi-2m})$ and $||\mathbf{v}||=1$.

Using (3.6), we find that $\mathbf{u}\cdot\mathbf{v}=O(\theta)$ (where the constants depend on $\mathbf{i}(\gamma_1),\dots,\mathbf{i}(\gamma_m)$, $\mathbf{i}(D_1)\dots,\mathbf{i}(D_{2\xi-2m})$). Then $\mathbf{v}\cdot\mathbf{i}(\gamma_i)^*=0$ for $i=1,\dots,m$, gives $\nu(\theta)=O(\theta)$. Equating the two sides of (3.8) gives

$$\begin{aligned} u_{i1}&=\frac{Wx_i}{w_i}=\lambda_1 p_i(\gamma_1)+\cdots+\lambda_m p_i(\gamma_m)+O(\theta),\\ u_{i2}&=\frac{W}{w_i}=-\lambda_1 q_i(\gamma_1)-\cdots-\lambda_m q_i(\gamma_m)+O(\theta). \end{aligned} \tag{3.9}$$

So we proved $\mathbf{u}$ approximately belongs to the m–dimensional subspace generated by $\mathbf{i}(\gamma_1)^*,\dots,\mathbf{i}(\gamma_m)^*$. Now we want to prove $\mathbf{u}$ is approximately parallel to the vector $\mathbf{i}(\eta)^*$, that is $(\lambda_1,\dots,\lambda_m)$ is proportional to $(a_1,\dots,a_m)$.

By Corollary 3.2.4, we have:

$$\left|\frac{y_i}{y_j}-\frac{a_1q_j(\gamma_1)+\cdots+a_mq_j(\gamma_m)}{a_1q_i(\gamma_1)+\cdots+a_mq_i(\gamma_m)}\right|=O(\theta). \tag{3.10}$$

Note that $\frac{y_i}{y_j}=\frac{u_{j2}}{u_{i2}}$.

Defining new variables

$$v_{i2}=-\lambda_1q_i(\gamma_1)-\cdots-\lambda_mq_i(\gamma_m),$$

we have, by (3.9), that $v_{i2} = u_{i2} + O(\theta)$. So we have:

$$\left|\frac{v_{j2}}{v_{i2}} - \frac{y_i}{y_j}\right| = O(\theta) \tag{3.11}$$

Hence, Equation (3.10) and Equation (3.11) give the following:

$$\left|\frac{v_{j2}}{v_{i2}} - \frac{a_1 q_j(\gamma_1) + \cdots + a_m q_j(\gamma_m)}{a_1 q_i(\gamma_1) + \cdots + a_m q_i(\gamma_m)}\right| = O(\theta), \tag{3.12}$$

that is

$$\left|\frac{\lambda_1 q_j(\gamma_1) + \cdots + \lambda_m q_j(\gamma_m)}{\lambda_1 q_i(\gamma_1) + \cdots + \lambda_m q_i(\gamma_m)} - \frac{a_1 q_j(\gamma_1) + \cdots + a_m q_j(\gamma_m)}{a_1 q_i(\gamma_1) + \cdots + a_m q_i(\gamma_m)}\right| = O(\theta) \tag{3.13}$$

Since this is true for all $i, j = 1, \ldots, \xi, \quad i \neq j$, and since the $(\xi \times m)$ matrix $(q_r(\gamma_s))_{\substack{r=1,\ldots,\xi \\ s=1,\ldots,m}}$ has maximal rank (because of the hypothesis that η is not exceptional), then we can conclude the following:

$$\left|\frac{\lambda_i}{\lambda_j} - \frac{a_i}{a_j}\right| = O(\theta), \;\; \forall i, j = 1, \ldots, m, \;\; i \neq j,$$

that is $\mathbf{u} = \alpha \mathbf{i}(\eta)^* + O(\theta)$ for some $\alpha > 0$, as we wanted to prove. This concludes the proof because $x_i = \frac{u_{i1}}{u_{i2}}$. □

Remark 3.2.25 (The exceptional case). There is a natural generalisation of Series' arguments (see Section 7.2 of [40]) in order to discuss the exceptional case. As underlined by Series in Remark 7.6 of [40], in the exceptional case one can only prove that the pleating ray $\mathcal{P}_\eta$ is close to some line in the pleating variety $\mathcal{P}_{\gamma_1,\ldots,\gamma_m}$ (but not necessary to L_η).

3.2.3 Proof of Proposition 3.2.7

In this section we are going to prove Proposition 3.2.7. We recall the statement.

Proposition (Proposition 3.2.7)**.** *Assuming that $\rho_{\underline{\mu}}(\sigma_i) = \Upsilon_\infty$, $\sigma_i = P \cap P'$, $\partial P = \{\sigma, \sigma_{i_2}, \sigma_{i_3}\}$ and $\partial P' = \{\sigma, \sigma_{i_4}, \sigma_{i_5}\}$ and assuming $\Im\mu_i > 1$, than the lines $\{z \in \mathbb{C} | \Im z = 0\}$ and $\{z \in \mathbb{C} | \Im z = \Im\mu_i\}$ are contained in the limit set $\Lambda(G_{\underline{\mu}})$. In addition, if $\Im\mu_{i_j} > 4$ for all $j = 2, \ldots, 5$, then the limit set $\Lambda(G_{\underline{\mu}})$ is contained in the union of two strips, that is, in the region*

$$\{z \in \mathbb{C} | 0 \leq \Im z \leq \frac{1}{2}\} \,\cup\, \{z \in \mathbb{C} | \Im\mu_i - \frac{1}{2} \leq \Im z \leq \Im\mu_i\}.$$

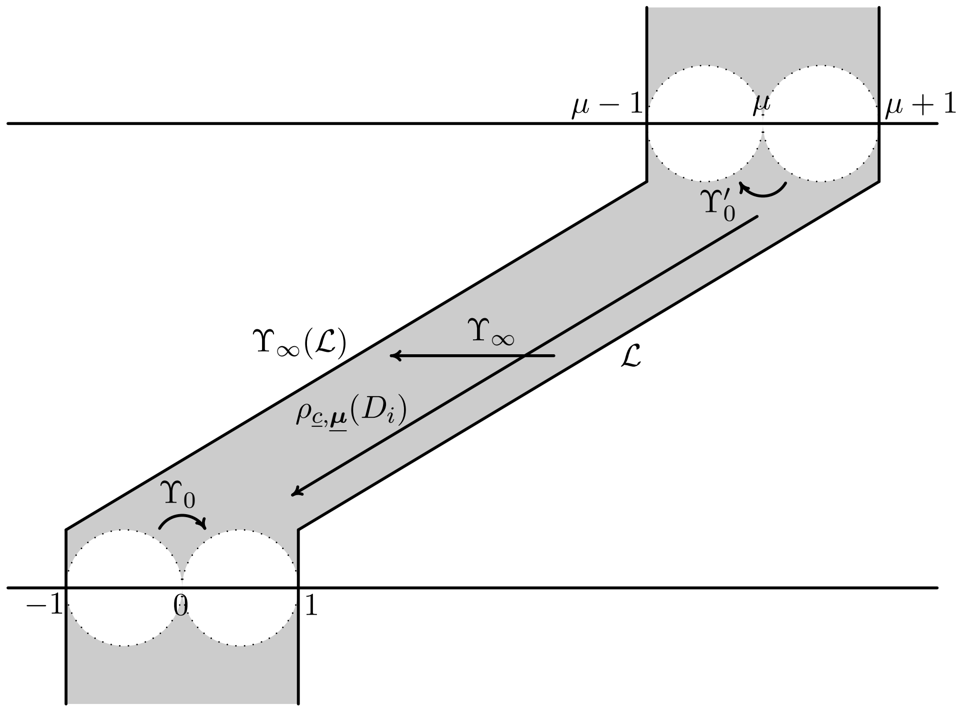

Figure 3.3: The fundamental domain $\mathcal{ST}$ for $\langle \Upsilon_\infty, \Upsilon_0, \Upsilon_3 \rangle$.

Recall that μ_{i_j} is the gluing parameters associated to σ_{i_j} with $j = 2, \dots, 5$, if $\sigma_{i_j} \in \mathcal{PC}$, or $\mu_{i_j} := \infty$ (and $\frac{1}{\mu_{i_j}} := 0$), if $\sigma_{i_j} \in \partial\Sigma$.

Proof. For the first claim, without loss of generality, we assume that $\sigma_i = \sigma$ is the common pants curve of the two pairs of pants P and P' and that σ corresponds to $\partial_\infty(P)$ and $\partial_\infty(P')$. Consider the group

$$\Gamma = \langle \Upsilon_\infty, \Upsilon_0 \rangle,$$

where $\Upsilon_\infty = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $\Upsilon_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Let $\Theta = \Theta_{\infty \longrightarrow \infty} = i \begin{pmatrix} 1 & -\mu \\ 0 & -1 \end{pmatrix}$, where $\mu = \mu_i$ is the gluing parameter associated to the pants curve $\sigma = \sigma_i$. (The notation agrees with that used in Chapter 2.) Note that $\Theta \Upsilon_\infty \Theta^{-1} = \Upsilon_\infty^{-1}$. Let Γ' be the group obtained by conjugating the group Γ with the matrix Θ, that is

$$\Gamma' = \langle \Upsilon_\infty, \Upsilon_0' \rangle,$$

where $\Upsilon_0' = \Theta \Upsilon_0 \Theta^{-1} = \begin{pmatrix} 1 - 2\mu & 2\mu^2 \\ -2 & 1 + 2\mu \end{pmatrix}$ is the parabolic transformation with fixed point at $\Theta(0) = \mu$.

Let $\mathcal{L}$ be the piecewise straight line obtained as the disjoint union of the lines $\{z \in \mathbb{C} | \Re z = 1, \Im z < \frac{1}{2}\}$ and $\{z \in \mathbb{C} | \Re z = \Re \mu + 1, \Im z > \Im \mu - \frac{1}{2}\}$, together with the straight segment between the point $1 + \frac{1}{2}i$ and $\mu + 1 - \frac{1}{2}i$; see Figure 3.3. The strip $\mathcal{ST}_1$ between $\mathcal{L}$ and $\Upsilon_\infty(\mathcal{L})$ is a fundamental domain for the group $J_1 = \langle \Upsilon_\infty \rangle$ acting on $\hat{\mathbb{C}}$.

A fundamental domain for the group $J_2 = \langle \Upsilon_0 \rangle$ acting on $\hat{\mathbb{C}}$ is

$$\mathcal{ST}_2 = \{z \in \mathbb{C} : |z + \frac{1}{2}| > \frac{1}{2}, |z - \frac{1}{2}| > \frac{1}{2}\},$$

and a fundamental domain for the group $J_3 = \langle \Upsilon_0' \rangle$ acting on $\hat{\mathbb{C}}$ is

$$\mathcal{ST}_3 = \{z \in \mathbb{C} : |z - \mu + \frac{1}{2}| > \frac{1}{2}, |z - \mu - \frac{1}{2}| > \frac{1}{2}\}.$$

Since, by hypothesis, $\Im \mu > 1$, then the union of the closures of the intersections $\mathcal{ST}_i \cap \mathcal{ST}_j$, with $i \neq j$ and $i, j = 1, 2, 3$, is the whole of $\hat{\mathbb{C}}$. Moreover the boundaries of the $\mathcal{ST}_i$ only intersect at parabolic fixed points. In this situation by the Klein-Maskit Combination Theorem, see Theorem A.13 of Maskit [32], we can see that $\mathcal{ST} = \mathcal{ST}_1 \cap \mathcal{ST}_2 \cap \mathcal{ST}_3$ is a fundamental domain for the group $\Gamma *_{\langle \Upsilon_\infty \rangle} \Gamma' = \langle \Upsilon_\infty, \Upsilon_0, \Upsilon_0' \rangle$. (In the statement of the Theorem A.13, Maskit uses the notion of fundamental set, see Section 2.2.1 for the definition.) The region $\mathcal{ST}$ has 3 connected components: one in $\mathbb{L} = \{z \in \mathbb{C} : \Im z < 0\}$, one in $\mathbb{H}^\mu = \{z \in \mathbb{C} : \Im z > \Im \mu\}$ and one in the strip $\{z \in \mathbb{C} : 0 < \Im z < \Im \mu\}$; see Figure 3.3.

The first claim now follows easily. In fact, $\Lambda(\Gamma) = \hat{\mathbb{R}} = \{z \in \mathbb{C} | \Im z = 0\} \cup \{\infty\}$ and $\Lambda(\Gamma') = \{z \in \mathbb{C} | \Im z = \Im \mu\} \cup \{\infty\}$. Hence, since $\Gamma, \Gamma' \subset G_{\underline{\mu}}$, then the lines $\{z \in \mathbb{C} | \Im z = 0\} \cup \{\infty\}$ and $\{z \in \mathbb{C} | \Im z = \Im \mu\} \cup \{\infty\}$ are contained in the limit set $\Lambda(G_{\underline{\mu}})$.

We now want to show that the disks $\mathbb{L} = \{z \in \mathbb{C} : \Im z < 0\}$ and $\mathbb{H}^\mu = \{z \in \mathbb{C} : \Im z > \Im \mu\}$ are contained in the regular set $\Omega(G_{\underline{\mu}})$. This follows from the the proof of Theorem 2.2.5. In fact, the groups Γ and Γ' correspond to the groups denoted $\Gamma(\tilde{P})$ in that proof. In particular, that proof shows that Γ and Γ' are peripheral groups, using the terminology of [24]. Now, since we proved above that the lines $\{z \in \mathbb{C} | \Im z = 0\} \cup \{\infty\}$ and $\{z \in \mathbb{C} | \Im z = \Im \mu\} \cup \{\infty\}$ are contained in the limit set $\Lambda(G_{\underline{\mu}})$, this shows that the peripheral disks of Γ and Γ' are $\mathbb{L}$ and $\mathbb{H}^\mu$, respectively. Hence $\mathbb{L}$ and $\mathbb{H}^\mu$ are contained in $\Omega(G_{\underline{\mu}})$. (Actually, they are contained in Ω^-, following the notation of Section 2.2.5.)

Finally, for the last claim, recall that we are assuming that σ is the common pants curve of the two pairs of pants P and P' such that $\partial P = \{\sigma, \sigma_{i_2}, \sigma_{i_3}\}$ and

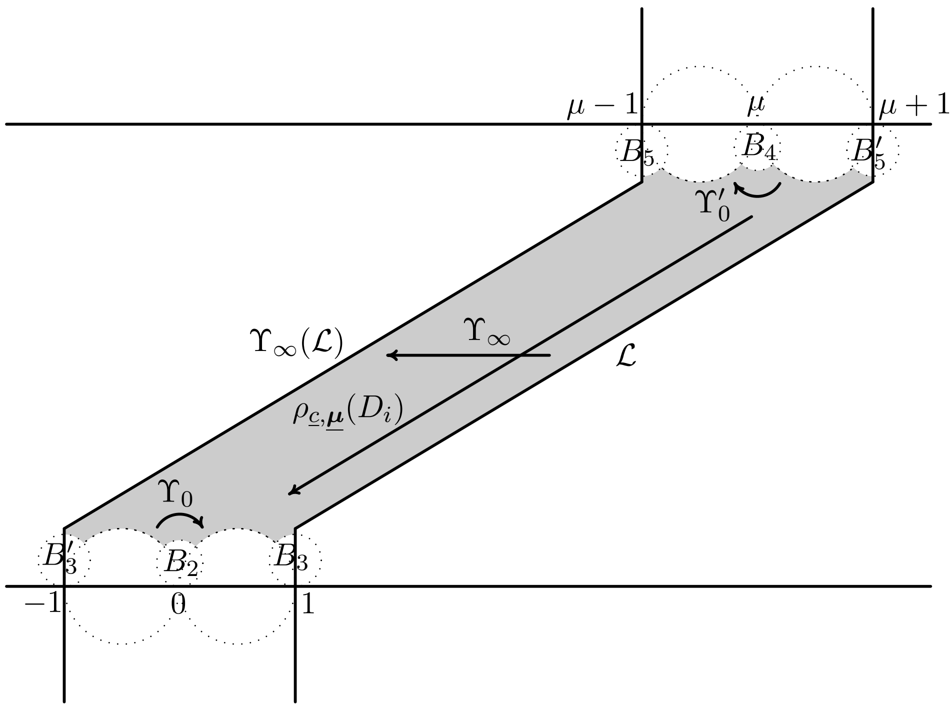

Figure 3.4: The region $\mathcal{ST}'$ contained in $\Omega(G_{\underline{\mu}})$.

$\partial P' = \{\sigma, \sigma_{i_4}, \sigma_{i_5}\}$. Let μ_{i_j} be the gluing parameters associated to the pants curves σ_{i_j} (or let $\mu_{i_j} = \infty$ if σ_{i_j} is an element of $\partial\Sigma$). By hypothesis we assumed that $\Im\mu_{i_j} > 4$ for all $j = 1, \dots, 4$. Recall also that the horocycle $\{z \in \mathbb{C} | \Im z = r\}$ (with $r \in \mathbb{R}_+$) based at ∞ corresponds, using Ω_ϵ (see Equation (2.2)), to the horocycles:

- $\{z \in \mathbb{C} : |z - i\frac{1}{2r}| < \frac{1}{2r}\}$ based at 0;
- $\{z \in \mathbb{C} : |z - (1 + i\frac{1}{2r})| < \frac{1}{2r}\}$ based at 1.

So the horocycle $\{z \in \mathbb{C} | \Im z = \frac{\Im\mu}{2}\}$ based at ∞ corresponds, for example, to the horocycle $\{z \in \mathbb{C} : |z - i\frac{1}{\Im\mu}| < \frac{1}{\Im\mu}\}$ based at 0. Hence, supposing $\partial_0(P) = \sigma_{i_2}$, $\partial_1(P) = \sigma_{i_3}$, $\partial_0(P') = \sigma_{i_4}$, $\partial_1(P') = \sigma_{i_5}$, we define the following horocycles:

- $B_2 = \{z \in \mathbb{C} : |z - i\frac{1}{\Im\mu_{i_2}}| < \frac{1}{\Im\mu_{i_2}}\}$;
- $B_3 = \{z \in \mathbb{C} : |z - (1 + i\frac{1}{\Im\mu_{i_3}})| < \frac{1}{\Im\mu_{i_3}}\}$;
- $B_3' = \{z \in \mathbb{C} : |z - (-1 + i\frac{1}{\Im\mu_{i_3}})| < \frac{1}{\Im\mu_{i_3}}\}$;
- $B_4 = \{z \in \mathbb{C} : |z - (\mu - i\frac{1}{\Im\mu_{i_4}})| < \frac{1}{\Im\mu_{i_4}}\}$;
- $B_5 = \{z \in \mathbb{C} : |z - (\mu - 1 - i\frac{1}{\Im\mu_{i_5}})| < \frac{1}{\Im\mu_{i_5}}\}$;

- $B'_5 = \{z \in \mathbb{C} : |z - (\mu + 1 - i\frac{1}{\Im\mu_{i_5}})| < \frac{1}{\Im\mu_{i_5}}\}$.

Let also $\mathcal{B}_j = \{z \in \mathbb{C} : z \notin B_j\}$ for $j = 2, 3, 4, 5$ and, similarly $\mathcal{B}'_i = \{z \in \mathbb{C} : z \notin B'_i\}$ for $i = 3, 5$. We can now define the following region:

$$\mathcal{ST}' = \mathcal{ST}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3 \cap \mathcal{B}'_3 \cap \mathcal{B}_4 \cap \mathcal{B}_5 \cap \mathcal{B}'_5 \cap \{z \in \mathbb{C} : 0 < |\Im z| < \Im\mu\}.$$

If $\Im\mu_{i_j} > 4$ for all $j = 1, \ldots, 4$, then the strip

$$\mathcal{ST}_* = \mathcal{ST}_1 \cap \{z \in \mathbb{C} : \frac{1}{2} < |\Im z| < \Im\mu - \frac{1}{2}\}$$

is contained in $\mathcal{ST}'$.

We claim that $\mathcal{ST}'$ is contained in $\Omega^+ \subset \Omega(G_{\underline{\mu}})$, where Ω^+ is the image of the developing map $\mathrm{Dev}_{\underline{\mu}}$; see Section 2.2.5 for the definition of Ω^+.

This claim implies that the strip $\{z \in \mathbb{C} : \frac{1}{2} < |\Im z| < \Im\mu - \frac{1}{2}\}$ is contained in $\Omega(G_{\underline{\mu}})$ because the translates of $\mathcal{ST}_*$ by $\rho_{\underline{\mu}}(\sigma_i) = \Upsilon_\infty$ cover it. (Actually one has to consider the strip

$$\widehat{\mathcal{ST}}_* = \widehat{\mathcal{ST}}_1 \cap \{z \in \mathbb{C} : \frac{1}{2} < |\Im z| < \Im\mu - \frac{1}{2}\},$$

where $\widehat{\mathcal{ST}}_1$ is the fundamental set associated to J_1, that is $\widehat{\mathcal{ST}}_1 = \mathcal{ST}_1 \cup \mathcal{L}$.)

This fact proves the last part of Proposition 3.2.7 because, since we already know that $\mathbb{L}$ and $\mathbb{H}^\mu$ are contained in $\Omega(G_{\underline{\mu}})$, we can now see that the limit set $\Lambda(G_{\underline{\mu}})$ should be contained in the strips $\{z \in \mathbb{C} | 0 \le \Im z \le \frac{1}{2}\}$ and $\{z \in \mathbb{C} | \Im\mu_i - \frac{1}{2} \le \Im z \le \Im\mu_i\}$, as we wanted to prove.

The claim follows easily from our construction. In fact, the region $\mathcal{ST}'$ is exactly the union of the two regions which are images of the truncated regions $S \subset \mathbb{P}$, $S' \subset \mathbb{P}$ under the maps $(\zeta'|_\Delta)^{-1} \colon S' \longrightarrow \mathbb{H}$ and $\Theta \circ (\zeta|_\Delta)^{-1} \colon S \longrightarrow \mathbb{H}$, where $\zeta \colon \mathbb{H} \longrightarrow \mathbb{P}$, $\zeta' \colon \mathbb{H} \longrightarrow \mathbb{P}'$ and $\Theta = \Theta_{\infty \longrightarrow \infty} = i \begin{pmatrix} 1 & -\mu \\ 0 & -1 \end{pmatrix}$, see Section 2.2.1 for the definition of ζ and S and see Table 2.1 for the definition of Θ. $\square$

Appendix A

Trace identities

We recall here some useful trace identities that are originally due to Fricke and Klein [18]. See also Mumford–Series–Wright [37, pag. 191–192].

Proposition A.1. *Given* $A, B \in \mathrm{SL}(2, \mathbb{C})$*, then:*

(i) $\mathrm{Tr}(A^{-1}) = \mathrm{Tr}(A)$*;*

(ii) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ *and* $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$*;*

(iii) $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$*;*

(iv) $\mathrm{Tr}(AB) + \mathrm{Tr}(A^{-1}B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$*;*

(v) $\mathrm{Tr}(A^2) + \mathrm{Tr}(\mathrm{Id}) = [\mathrm{Tr}(A)]^2$.

Proof.

- Identity (i) follows easily from the fact that, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}),$$

  then

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- Equalities (ii) are an easy calculation.

- Identity (iii) follows easily from (ii) by replacing B with BA^{-1}.

- The proof of (iv) follows easily from the Cayley–Hamilton Theorem, which says that, given $A \in \mathrm{SL}(2,\mathbb{C})$, then $A^2 - \mathrm{Tr}(A)A + 1 = 0$. So, multiplying on the right by $A^{-1}B$, we get

$$AB + A^{-1}B = (\mathrm{Tr}(A))\, B,$$

from which (iv) follows easily by taking traces of both sides.

- Equality (v) is a particularly case of (iv).

□

Note that in Mumford–Series–Wright [37] the identity (iv) is called *grandfather identity.*

In Section 2.3.2.b we calculated the holonomy of the paths υ_ϵ in a pair of pants. The matrices corresponding to the holonomy images of these paths belong to the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ spanned by the matrices $\Upsilon_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\Upsilon_\infty = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. In particular, since it is useful for the calculations of Section 2.4, we calculate the k-th power of the matrices Υ_0, Υ_∞ and $\Upsilon_1 = \Upsilon_\infty^{-1}\Upsilon_0^{-1} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$, for any integer $k \in \mathbb{Z}$. This result could be easily proved by induction, so we omit the proof.

Proposition A.2. *For any integer k, the k-th power of the matrices Υ_0, Υ_∞ and Υ_1 is given by:*

1. $\Upsilon_0^k = \begin{pmatrix} 1 & 0 \\ 2k & 1 \end{pmatrix}$;

2. $\Upsilon_\infty^k = \begin{pmatrix} 1 & -2k \\ 0 & 1 \end{pmatrix}$;

3. $\Upsilon_1^k = (-1)^k \begin{pmatrix} 2k+1 & -2k \\ 2k & -2k+1 \end{pmatrix}$.

Appendix B

Combinatorial proof of Top Terms' Relationship

In this section we prove the Top Terms' Relationship (Theorem 2.4.1) using a combinatorial method which covers all cases, without having to discuss the base cases separately, as you need to do with the inductive proof of Section 2.4. Note that this proof is the one included in the article [28] and was Series' idea.

Our method is to show that the product of matrices forming the holonomy always takes a special form and then to give a case-by-case proof. Fix $\underline{\mu} \in \mathbb{H}^{\xi}$.

First, consider the holonomy representation of a typical path. Let $\gamma \in \mathcal{S}_0$ be a simple loop on Σ. We suppose γ is in Penner and Harer standard position, so that it always cuts σ_{i_j} in the arc $w_{i_j} \subset \sigma_{i_j}$. Starting from the base point in some pants P, it crosses, in order, pants curves σ_{i_j}, with $j = 1, \dots, q(\gamma)$. If the boundaries glued across σ_{i_j} are $\partial_\epsilon P$ and $\partial_{\epsilon'} P'$, then, by equation (2.6), the contribution to the holonomy product $\rho_{\underline{\mu}}(\gamma)$ is

$$\Omega_\epsilon^{-1} J^{-1} T_{\mu_{i_j}}^{-1} \Omega'_{\epsilon'},$$

where $\mu_{i_j} = \mu_i$ whenever $\sigma_{i_j} = \sigma_i \in \mathcal{PC}$.

A single positive twist around $\partial_\epsilon P$ immediately before this boundary component contributes $\rho_{\underline{\mu}}(v_\epsilon^{-1}) = \Omega_\epsilon^{-1} \rho_{\underline{\mu}}(v_\infty^{-1}) \Omega_\epsilon$ (because v_ϵ twists in the *negative* direction round $\partial_\epsilon P$, see Figure 2.7), while a single positive twist around $\partial_{\epsilon'} P'$ after the crossing contributes $\rho_{\underline{\mu}}(v_{\epsilon'}) = \Omega_{\epsilon'}^{-1} \rho_{\underline{\mu}}(v_\infty) \Omega_{\epsilon'}$. Thus if, in general, γ twists α_j times around $\partial_\epsilon P = \sigma_{i_j}$ immediately before the crossing and β_j times after, the total contribution to the holonomy is

$$\Omega_\epsilon^{-1} \rho_{\underline{\mu}}(v_\infty)^{-\alpha_j} J^{-1} T_{\mu_i}^{-1} \rho_{\underline{\mu}}(v_\infty)^{\beta_j} \Omega'_{\epsilon'}, \tag{B.1}$$

where $\sigma_{i_j} = \sigma_i \in \mathcal{PC}$.

From Sections 2.2.1 and 2.3.2 we have

$$J^{-1}T_{\mu}^{-1} = \begin{pmatrix} i & -i\mu \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \rho_{\underline{\mu}}(\upsilon_\infty) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

For variables X, Y, write $A_X = \begin{pmatrix} 1 & X \\ 0 & -1 \end{pmatrix}$ and $B_Y = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$. We calculate

$$\rho_{\underline{\mu}}(\upsilon_\infty)^{-\alpha_j} J^{-1} T_{\mu_i}^{-1} \rho_{\underline{\mu}}(\upsilon_\infty)^{\beta_j} = iA_{X_j},$$

where $X_j = -(\mu_i + 2\alpha_j + 2\beta_j)$, from which we note in particular that, as expected, which side of the crossing the twists occur makes no difference to the final product; see also Lemma 2.4.3.

Proposition B.1. *(i) Suppose that γ contains no sbcc–arcs. Then, for every $\underline{\mu} \in \mathbb{H}^\xi$, we have that $\rho_{\underline{\mu}}(\gamma)$ is of the form $\pm i^q \Pi_{i=1}^q A_{X_j}\Omega_{i_j}$, where $\Omega_{i_j} = \Omega_0$ or Ω_1 for all j. If the term A_{X_j} corresponds to the crossing of a pants curve $\sigma_{i_j} = \sigma_i$, with α_j twists before the crossing and β_j after, then $X_j = -(\mu_i + 2\alpha_j + 2\beta_j)$.*

(ii) If γ contains sbcc–arcs, then, for every $\underline{\mu} \in \mathbb{H}^\xi$, the holonomy image $\rho_{\underline{\mu}}(\gamma)$ takes the same form as above, with an extra term

$$A_{X_j}\Omega_1 B_{\pm 2}\Omega_0 A_{X_j}$$

inserted for each sbcc–arc which crosses σ_{i_j} twice in succession.

(iii) In all cases, the total PH–twist of γ about $\sigma_i \in \mathcal{PC}$ is $\hat{p}_i(\gamma) = \sum_{\sigma_{i_j}=\sigma_i}(\alpha_j + \beta_j)$.

Proof. As computed above we have

$$\rho_{\underline{\mu}}(\upsilon_\infty)^{-\alpha} J^{-1} T_{\mu}^{-1} \rho_{\underline{\mu}}(\upsilon_\infty)^{\beta} = i \begin{pmatrix} 1 & -(\mu + 2\alpha + 2\beta) \\ 0 & -1 \end{pmatrix} = iA_{-(\mu+2\alpha+2\beta)}.$$

It follows that the holonomy $\rho_{\underline{\mu}}(\gamma)$ is a concatenation of q terms of the form $\Omega_\epsilon^{-1} A_{-(\mu+2\alpha+2\beta)}\Omega_{\epsilon'}$, one for each crossing of a pants curve σ_{i_j}. If γ contains no *sbcc*–arcs, then it enters and leaves each pants P across distinct boundary components, say $\partial_{\epsilon_1}P$ and $\partial_{\epsilon_2}P$ respectively. Then the corresponding adjacent terms in the concatenated product are then

$$\ldots \Omega_{\epsilon_1}\Omega_{\epsilon_2}^{-1} \ldots,$$

where $\epsilon_1 \neq \epsilon_2$, from which (i) easily follows.

We also note that, regardless of how the twists are organised (that is, before or after the crossings), the sum $\sum_{j=1}^{q_i(\gamma)}(\alpha_j + \beta_j)$ of the coefficients in the terms corresponding to crossings of the pants curve σ_i is equal to $\hat{p}_i(\gamma)$, the i^{th} PH–twisting number of the curve γ with respect to the pants decomposition $\mathcal{PC}$. This proves (iii).

Now suppose that γ contains some *sbcc*–arcs. Suppose that γ cuts a curve σ_{i_j} twice in succession entering and leaving a pants P across the boundary $\partial_\infty P$. Since γ is in PH–form, after crossing $\partial_\infty P$ it goes once around $\partial_0 P$ in either the positive or negative direction and then returns to $\partial_\infty P$, see Figure 1.5. Since γ is simple, the twisting around σ_{i_j} is the same on the inward and the outward journeys. The term in the holonomy is therefore

$$A_{X_j}\rho_{\underline{\mu}}(\upsilon_0)^{\pm 1}A_{X_j} = A_{X_j}\Omega_0^{-1}\rho_{\underline{\mu}}(\upsilon_\infty)^{\mp 1}\Omega_0 A_{X_j} = A_{X_j}\Omega_0^{-1}B_{\pm 2}\Omega_0 A_{X_j}$$

as claimed.

If, more generally, γ enters and leaves P across $\partial_\epsilon P$, then this entire expression is multiplied on the left by Ω_ϵ^{-1} and on the right by Ω_ϵ. By the same discussion as in (i), this leaves the form of the holonomy product unchanged. The contribution to the twist about σ_{i_j} is calculated as before. $\square$

We are now ready for our combinatorial proof of Theorem A. Suppose first that $\gamma \in \mathcal{S}_0$ contains no *sbcc*–arcs. If

$$\rho_{\underline{\mu}}(\gamma) = \Pi_{i=1}^{q} A_{X_j}\Omega_{i_j},$$

define $X_j^* = X_j + h(\Omega_{i_j}) + k(\Omega_{i_{j-1}})$, where $\Omega_{i_0} := \Omega_{i_q}$, and

$$h(\Omega_{i_j}) = \begin{cases} 1 & \text{if } i_j = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad k(\Omega_{i_j}) = \begin{cases} 0 & \text{if } i_j = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Thus:

$$\begin{aligned}
&\Omega_0 A_X \Omega_0 \to X^* = X + 1\\
&\Omega_0 A_X \Omega_1 \to X^* = X\\
&\Omega_1 A_X \Omega_0 \to X^* = X + 2\\
&\Omega_1 A_X \Omega_1 \to X^* = X + 1.
\end{aligned}$$

Remark B.2. Replacing $\rho_{\underline{\mu}}(\gamma)$ by $\rho_{\underline{\mu}}(\gamma)^{-1}$ leaves the occurrences of the above blocks unchanged. The entire matrix product is multiplied by $(-1)^q$. This is because $A_X^{-1} = -A_X$ and, for example,

$$(\Omega_0 A_X \Omega_1)^{-1} = \Omega_1^{-1} A_X^{-1} \Omega_0^{-1} = -\Omega_0 A_X \Omega_1.$$

Now, given the path of some $\gamma \in \mathcal{S}_0$, consider a crossing for which $\sigma_{i_j} = \sigma_i$. Let $\Omega_{i_{j-1}} A_{X_j} \Omega_{i_j}$ be the corresponding terms in $\rho_{\underline{\mu}}(\gamma)$ (where $\Omega_{i_{j-1}}$ is associated to the crossing of the previous pants curve $\sigma_{i_{j-1}}$). Let p_j and $\hat{p}_j$ be the respective contributions from this j^{th} crossing to the DT– and PH–twist coordinates of γ, so that the total twists p_i and $\hat{p}_i$ about σ_i are obtained by summing over all crossings for which $\sigma_{i_j} = \sigma_i$ are $p_i = \sum_{\sigma_{i_j}=\sigma_i} p_j$ and, likewise, $\hat{p}_i = \sum_{\sigma_{i_j}=\sigma_i} \hat{p}_j$.

For any variable $a_j \in \mathbb{R}$ which depends on the j^{th} crossing, define a_j^* according to the same rule as X^* above, in other words $a_j^* = a_j + h(\Omega_{i_j}) + k(\Omega_{i_{j-1}})$. We have:

Lemma B.3. *Suppose that γ contains no sbcc–arcs and, as usual, let p_j and $\hat{p}_j$ be the contributions to the DT– and PH–twists of γ corresponding to the j^{th} crossing of a pants curve $\sigma_i \in \mathcal{PC}$. Then $(-2\hat{p}_j)^* = -p_j + 1$.*

Proof. This is verified using Theorem 1.2.6, together with the fact that γ is assumed to be in PH–standard form.

Consider a crossing for which $\sigma_{i_j} = \sigma_i$ with corresponding term $\Omega_{i_{j-1}} A_{X_j} \Omega_{i_j}$ in $\rho_{\underline{\mu}}(\gamma)$. Suppose, for example, that the relevant term in the holonomy is $\Omega_0 A_{X_j} \Omega_0$, so that, by definition, $a_j^* = a_j + 1$ for any variable a_j. Without loss of generality, we may suppose that σ_{i_j} is the gluing of $\partial_\infty P$ to $\partial_\infty P'$ as shown in Figure 1.6. The first Ω_0 means that there is an arc from D to E, and the second Ω_0 means there is an arc from E to A. The formula of Theorem 1.2.6 therefore gives a contribution of $2\hat{p}_j = p_j + 0 + 1 - 1 = p_j$. Thus $(-2\hat{p}_j)^* := -2\hat{p}_j + 1 = -p_j + 1$, as claimed.

Similarly, consider the sequence $\Omega_1 A_{X_j} \Omega_0$. In this case, $(-2\hat{p}_j)^* = -2\hat{p}_j + 2$. From Theorem 1.2.6 we find $2\hat{p}_j = p_j + 1 + 1 - 1 = p_j + 1$, so $(-2\hat{p}_j)^* = -2\hat{p}_j + 2 = -p_j + 1$.

The other two possible sequences are similar. $\square$

In the case of no *sbcc*-arcs, Theorem 2.4.1 is an immediate corollary of this lemma and the following proposition:

Proposition B.4. *Suppose that γ contains no sbcc–arcs, then*

$$\operatorname{Tr}\left(\rho_{\underline{\mu}}(\gamma)\right) = \pm i^q \operatorname{Tr}\left(\Pi_{j=1}^q A_{X_j} \Omega_{i_j}\right) = \pm i^q \left(\Pi_{j=1}^q X_j^*\right) + R,$$

where R denotes terms of degree at most $q-2$ in the X_j.

Proof of Theorem A. (No *sbcc*–arcs case.) By Proposition B.1, if $\sigma_{i_j} = \sigma_i$ then $X_j = -(\mu_i + 2\alpha_j + 2\beta_j)$. There are $q_i(\gamma)$ such terms X_j. Thus the top order term of $\operatorname{Tr}\rho_{\underline{\mu}}(\gamma)$ is $\mu_1^{q_1} \dots \mu_\xi^{q_\xi}$, with coefficient $\pm i^q$, in accordance with the result of Theorem A.

Now the contribution to the PH–twist corresponding to the j^{th} crossing is $\hat{p}_j = \alpha_j + \beta_j$. Thus:

$$\begin{aligned}
\Pi_{j=1}^q X_j^* &= \Pi_{j=1}^q [-(\mu_i + 2\alpha_j + 2\beta_j)]^* \\
&= \Pi_{j=1}^q [-(\mu_i + 2\hat{p}_j)]^* \\
&= \Pi_{j=1}^q [-\mu_i + (-2\hat{p}_j)^*] \\
&= \Pi_{j=1}^q [-\mu_i - p_j + 1] \\
&= (-1)^q \Pi_{j=1}^q (\mu_i + p_j - 1),
\end{aligned}$$

where we used Lemma B.3 to evaluate $(-2p_j)^*$. Hence the coefficient of the term $\mu_1^{q_1} \dots \mu_i^{q_i - 1} \dots \mu_\xi^{q_\xi}$ is

$$\pm i^q \sum_{\sigma_{i_j} = \sigma_i} (p_j - 1) = \pm i^q (p_i - q_i),$$

which is exactly the coefficient in Theorem A. □

Proof of Proposition B.4. We prove this by induction on the length q of the product $\Pi_{j=1}^q A_{X_j} \Omega_{i_j}$. If $q = 1$, with respect to the cyclic ordering we see either the block $\Omega_0 A_X \Omega_0$ or the block $\Omega_1 A_X \Omega_1$, so that $X^* = X + 1$. In both cases we check directly that $\operatorname{Tr} A_X \Omega_0 = \operatorname{Tr} A_X \Omega_1 = X + 1$.

The case $q = 2$ corresponds to a product $A_{X_1} \Omega_\epsilon A_{X_2} \Omega_{\epsilon'}$. Hence there are four possibilities corresponding to $\epsilon = \pm 1$ and $\epsilon' = \pm 1$. These cases can be checked either by multiplying out or by using the trace identity (iv) of Proposition A.1, which we rewrite as

$$\operatorname{Tr}(AB) = \operatorname{Tr}(A) \operatorname{Tr}(B) - \operatorname{Tr}(AB^{-1}). \tag{B.2}$$

For example, if $\epsilon = \epsilon' = 0$, then

$$\begin{aligned}
\operatorname{Tr}(A_{X_1} \Omega_0 A_{X_2} \Omega_0) &= \operatorname{Tr}(A_{X_1} \Omega_0) \operatorname{Tr}(A_{X_2} \Omega_0) + \operatorname{Tr} A_{X_1} A_{X_2} \\
&= (X_1 + 1)(X_2 + 1) + 2 = X_1^* X_2^* + 2,
\end{aligned}$$

where we used the relation $A_X^{-1} = -A_X$.

If $\epsilon = 0, \epsilon' = 1$ then

$$\operatorname{Tr}(A_{X_1}\Omega_0 A_{X_2}\Omega_1) = \operatorname{Tr}(A_{X_1}\Omega_0)\operatorname{Tr}(A_{X_2}\Omega_1) - \operatorname{Tr}\left(A_{X_1}\Omega_0\Omega_1^{-1}A_{X_2}^{-1}\right).$$

The first term on the right hand side is $(X_1+1)(X_2+1)$ and the last term reduces to

$$-\operatorname{Tr}(A_{X_2}^{-1}A_{X_1}\Omega_1) = -X_2 + X_1 - 1.$$

Hence

$$\operatorname{Tr}(A_{X_1}\Omega_0 A_{X_2}\Omega_1) = X_1X_2 + 2X_2 + 2 = (X_1+2)X_2 + 2 = X_1^* X_2^* + 2.$$

The other two cases with $q = 2$ are similar (or can be obtained from these by replacing γ with γ^{-1}).

Now we do the induction step. Suppose the result true for all products of length less than q. We split into three cases:

(i) Ω_0 appears 3 times consecutively;

(ii) Ω_0 appears at most 2 times consecutively;

(iii) Ω_0 and Ω_1 appear alternately.

In *case (i)*, after cyclic permutation, the product is of the form

$$A_{X_1}\Omega_0 A_{X_2}\Omega_0 A_{X_3}\Omega_0 \ldots \Omega_{i_q}.$$

We apply (B.2), splitting the product as

$$(A_{X_2}\Omega_0) \times (A_{X_3}\Omega_0 \ldots \Omega_{i_q} A_{X_1}\Omega_0).$$

Considering the first term of this split product alone, A_{X_2} is still preceded and followed by Ω_0. Likewise, taking the second term alone, A_{X_1} and A_{X_3} are still preceded and followed by the same values of Ω_i as they were before and nothing else has changed. Thus the induction hypothesis gives

$$\operatorname{Tr}(A_{X_2}\Omega_0) = X_2^* \quad \text{and} \quad \operatorname{Tr}(A_{X_3}\Omega_0 \ldots \Omega_{i_q} A_{X_1}\Omega_0) = X_3^* \ldots\ X_q^* X_1^*.$$

Now consider the remaining term coming from (B.2):

$$\operatorname{Tr}[A_{X_2}\Omega_0(A_{X_3}\Omega_0 \ldots \Omega_{i_q} A_{X_1}\Omega_0)^{-1}] = \operatorname{Tr}[A_{X_2}A_{X_1}^{-1}\Omega_{i_q}^{-1} \ldots A_{X_3}^{-1}].$$

Cyclically permuting, the three terms $A_{X_3}^{-1}, A_{X_2}, A_{X_1}^{-1}$ combine to give a single term $A_{X_3+X_2+X_1}$, so that the trace has degree at most $q-2$ in the variables $X_3 + X_2 + X_1, X_4, \dots, X_q$. Putting all this together proves the claim.

In *Case (ii)*, suppose first $q \geq 4$. Thus after cyclic permutation the product is of the form:

$$A_{X_1}\Omega_0 A_{X_2}\Omega_0 A_{X_3}\Omega_1 A_{X_4} \dots \Omega_1.$$

We apply (B.2) splitting as

$$(A_{X_1}\Omega_0 A_{X_2}\Omega_0 A_{X_3}\Omega_1) \times (A_{X_4} \dots A_{X_q}\Omega_1).$$

Taking each of these subproducts separately, we see that again the terms Ω_i preceding and following each A_X are unchanged. So the induction hypothesis gives

$$\mathrm{Tr}(A_{X_1}\Omega_0 A_{X_2}\Omega_0 A_{X_3}\Omega_1) = X_1^* X_2^* X_3^*$$

and

$$\mathrm{Tr}(A_{X_4} \dots A_{X_q}\Omega_1) = X_4^* \dots X_q^*.$$

Moreover, we note

$$A_{X_1}\Omega_0 A_{X_2}\Omega_0 A_{X_3}\Omega_1 \Omega_1^{-1} A_{X_q}^{-1} \dots A_{X_4}^{-1}$$

is of degree at most $q-2$ in the variables $X_4 + X_1, X_2, X_3 + X_q, X_5, \dots, X_{q-1}$. The result follows.

The case $q = 3$ is dealt with by splitting

$$A_{X_1}\Omega_0 A_{X_2}\Omega_0 A_{X_3}\Omega_1 \quad \text{as} \quad (A_{X_3}\Omega_1 A_{X_1}\Omega_0) \times A_{X_2}\Omega_0,$$

using the previously considered case $q = 2$, and noting that

$$A_{X_3}\Omega_1 A_{X_1}\Omega_0 \Omega_0^{-1} A_{X_2}^{-1}$$

has degree 1 in the variable $X_1 + X_2 + X_3$. (Recall that $A_X^{-1} = -A_X$.)

Finally, in *case (iii)* we split

$$A_{X_1}\Omega_0 A_{X_2}\Omega_1 A_{X_3}\Omega_0 A_{X_4} \dots \Omega_1 \quad \text{as} \quad (A_{X_1}\Omega_0 A_{X_2}\Omega_1) \times (A_{X_3}\Omega_0 A_{X_4} \dots \Omega_1)$$

and the argument proceeds in a similar way to that before. □

Now we add in the effect of having *sbcc*–arcs, that is we deal with the case $h > 0$.

Theorem B.5. *Suppose that a matrix product of the form in Theorem B.4 is modified by the insertion of h blocks $A_{X_j}\Omega_0^{-1}B_{Y_r}\Omega_0 A_{X_j}$, $r = 1, \ldots, h$ for variables $Y_r \in \mathbb{C}$. Then its trace is*

$$\pm \left(\Pi_{j=1}^{h} Y_k\right)\left(\Pi_{j=1}^{q} X_j^*\right) + R,$$

where R denotes terms of degree at most $q-2$ in the X_j.

Proof. We first check the case $q = 1, h = 1$ by hand. (Note that such a block cannot be the holonomy matrix of a simple closed curve.) We have:

$$A_X\Omega_1 B_Y\Omega_0 = \begin{pmatrix} X(1+Y) - (X+1) & * \\ * & 1 \end{pmatrix},$$

hence $\mathrm{Tr}(A_X\Omega_1 B_Y\Omega_0) = XY$. Since the term $\Omega_0 A_X\Omega_1$ contributes the factor X and the term B_Y contributes Y, this fulfills our hypothesis.

Now work by induction on h. Suppose the result holds for products

$$\Omega_u A_{X_1}\Omega_{i_1}A_{X_2}\Omega_{i_2}\ldots A_{X_s}$$

containing at most $h-1$ terms of the form B_{Y_r} and consider a product

$$\Omega_u A_{X_1}\Omega_{i_1}A_{X_2}\Omega_{i_2}\ldots\Omega_v A_X\Omega_1 B_{Y_h}\Omega_0 A_X.$$

There are four possible cases:

(i) $u = 1, v = 0$;

(ii) $u = 1, v = 1$;

(iii) $u = 0, v = 0$;

(iv) $u = 0, v = 1$.

Case (i): Consider the extra contribution to the trace resulting from the additional block $A_X\Omega_1 B_{Y_h}\Omega_0 A_X$. The first occurrence of A_X appears in a block $\Omega_0 A_X\Omega_1$ which, according to what we want to prove, should contribute a factor X. Likewise, the block $\Omega_0 A_X\Omega_1$ containing the second occurrence of A_X should contribute X, and the term B_Y should contribute Y. Thus it is sufficient to show that

$$\begin{aligned}\mathrm{Tr}(\Omega_1 A_{X_1}\Omega_{i_1}A_{X_2}\Omega_{i_2}\ldots A_{X_s}\Omega_0 A_X\Omega_1 B_Y\Omega_0 A_X) &= \\ \pm X^2 Y\,\mathrm{Tr}(\Omega_1 A_{X_1}\Omega_{i_1}A_{X_2}\Omega_{i_2}\ldots A_{X_s}) + R,&\end{aligned}$$

where R denotes terms of total degree at most 2 less then the first term in the X_j.

Splitting the product as

$$(\Omega_1 A_{X_1} \Omega_{i_1} A_{X_2} \Omega_{i_2} \dots \Omega_0 A_X) \times (\Omega_1 B_Y \Omega_0 A_X)$$

and using (B.2), we see that the second factor contributes XY and the first factor, containing the sequence $\Omega_0 A_X \Omega_1$, contributes X. The remaining term

$$(\Omega_1 A_{X_1} \Omega_{i_1} A_{X_2} \Omega_{i_2} \dots \Omega_0 A_X) \times (\Omega_1 B_Y \Omega_0 A_X)^{-1}$$

coming from (B.2) has, as usual, degree in the X_j lower by 2. This proves the claim in this case.

Case (ii): Again splitting the product as

$$(\Omega_1 A_{X_1} \Omega_{i_1} A_{X_2} \Omega_{i_2} \dots \Omega_1 A_X) \times (\Omega_1 B_Y \Omega_0 A_X),$$

the first split factor contains the block $\Omega_1 A_X \Omega_1$ which contributes a factor $(X+1)$ to the trace. The second split factor contributes XY.

In the unsplit product we have from the first occurrence of A_X the block $\Omega_1 A_X \Omega_1$, which contributes a factor $X+1$, and, from the second A_X, the block $\Omega_0 A_X \Omega_1$, which contributes X, again proving our claim.

Case (iii): This can be done by inverting the previous case. Alternatively, splitting again as

$$(\Omega_0 A_{X_1} \Omega_{i_1} A_{X_2} \Omega_{i_2} \dots \Omega_0 A_X) \times (\Omega_1 B_Y \Omega_0 A_X),$$

the first split factor contains the block $\Omega_0 A_X \Omega_0$, which contributes a factor $X+1$, while the second split factor contributes, as usual, XY.

In the unsplit product we have, from the first A_X, the block $\Omega_0 A_X \Omega_1$ which contributes X, and, from the second A_X, the block $\Omega_0 A_X \Omega_0$ which contributes $X+1$, again proving our claim.

Case (iv): Again split as

$$(\Omega_0 A_{X_1} \Omega_{i_1} A_{X_2} \Omega_{i_2} \dots \Omega_1 A_X) \times (\Omega_1 B_Y \Omega_0 A_X).$$

The first split factor, containing $\Omega_1 A_X \Omega_0$, contributes $X+2$ and the second split factor contributes XY.

In the unsplit product we have, from the first A_X, the term $\Omega_1 A_X \Omega_1$, which contributes $X+1$, and from the second A_X the term $\Omega_0 A_X \Omega_0$, which contributes

$X+1$. The induction still works because, up to terms of lower degree, $X(X+2)Y = (X+1)(X+1)Y$. $\square$

Remark B.6. Not all the cases discussed above are realisable as the holonomy representations of simple connected loops γ. For example, the cases

$$v = 1, u = 1, Y = +2 \quad \text{and} \quad v = 0, u = 0, Y = -2$$

produce non–simple curves.

Proof of Theorem A. This follows from Proposition B.4 and Theorem B.5 by setting $Y_j = \pm 2$.

For the final statement of the theorem, see Lemma 2.2.4. $\square$

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